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Switching Model Predictive Control of canals characterized by large operating conditions

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Abstract: This paper presents a switched linear system representation of water canal dynamics to incorporate different operating modes, which arise due to the occurrence of extreme weather phenomena such as flooding and drought episodes. To guarantee the stability during mode switching, a proper analysis on permanence regions–given by a collection of equilibrium states–for the switched linear system is presented. The permanence region is computed within a compact set, which depends on an adequate level region for the canals. A suitable algorithm is used to formulate an asymptotic stable Model Predictive Control (MPC) that steers and maintains the states of the system inside the target region indefinitely in a feasible manner. This strategy is successfully tested in a simulation using a realistic model of a canal.

Keywords: Switched Linear Systems, Stable Regions, Time-Delay Systems, Model Predictive Control, Water Systems

1. INTRODUCTION

Free-surface open-channel systems are large-scale networks consisting of rivers, canals and waterways. As these are mainly used for freight transportation, the main management objective consists of guaranteeing the navigability, which requires keeping water levels within predefined limits. However, the fact that these systems are strongly affected by extreme weather phenomena complicates the achievement of this objective. In the context of climate change, severe and long-lasting drought and record rainfall are increasingly expected worldwide.

Model predictive control (MPC) is one of the most popular approaches for real-time control of free-surface systems due to its simplicity and versatility, a fact that can be testified by the extense body of literature on the topic (Ocampo-Martinez et al., 2013; Fele et al., 2014). However, extreme weather episodes may render this simplification no longer valid, as these lead to time-varying average flows along the water courses, and therefore time-varying operating points. Linear parameter-varying (LPV) models, which describe a nonlinear system that can be modeled as parameterized linear systems (each designed at a different operating point), emerge as an approach to take large operating ranges into account. LPV models have been successfully employed in open-channel modeling and control (Bolea and Puig, 2016). When multiple operating modes must be considered for canal management purposes as a result of multiple operating points, the use of LPV or nonlinear models is not strictly required. A good alternative is to rely on switching systems (Duviella et al., 2007), in which a linear model is identified for each operating point within the operational range of the canal. Switching between modes can then be considered when the operating conditions of the canal change. Advantages of this modeling strategy are the use of a bank of simpler linear models and the fact that it is supported by the theory on switching systems. However, in order to formulate a stabilizing control strategy for switched systems, the characterization of general invariance regions becomes a necessity. From set control theory it is known that invariance regions play an important role in the controllability and stability analysis (Blanchini and Miani, 2015). General invariant sets are the only ones that can be formally stabilized by an MPC strategy (Rawlings et al., 2017). For a switched linear system where the signal must switch mode in finite time (i.e., mode switching is mandatory), the only formal invariant set is the origin, and this point usually lacks applicability. More general regions can be found in the literature: for instance, Anderson et al. (2021); Perez et al. (2022) assess this problem with different characterizations and computations of a particular Permanence Sets (PS), i.e., General Control Invariant Sets for switched systems.

In this work, a PS is proposed inside a target region $\mathcal{T}$ outside of the origin. The PS is a subset of the output space of the model for the water canal dynamics. The dynamics of the mode are modeled by a delay switched linear system (with switches affecting only the output variable). The result is used to formulate an stable MPC which is tested in several scenario simulations.

2. DYNAMICAL MODEL AND SET-CONTROL ANALYSIS

2.1 Control Model of Canals

The dynamics of a canal, i.e., part of a water course between two control structures, are accurately described by the Saint-Venant equations, a set of partial nonlinear differential equations that are not well suited for real-time control. Under simplifying assumptions and considering a unique operating point, the Saint-Venant equations can be simplified and linearized, and the

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Integrator Delay Zero (IDZ) model can be obtained. Moreover, a discrete-time version of the IDZ model can be obtained in state-space form (Segovia et al., 2019). By considering several operating points and switching dynamics, the state-space representation is given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B^{\star}u(k-n), \\ y(k) &= C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k) + D^{\star}_{\sigma(k)}u(k-n), \end{aligned} \quad (1)$$

with $x(0) = x_0$, where $x(k) \in \mathcal{X} \subset \mathbb{R}^2$ is the state of the system $[m^3]$, $u(k) \in \mathcal{U} \subset \mathbb{R}^2$ is the control input (gate control actions at both ends of the canal) $[m^3/s]$, $y(k) \in \mathcal{Y} \subset \mathbb{R}^2$ is the output (water levels at both ends of the canal) $[m]$, n is the delay (in samples) and $\sigma(k) \in \Sigma := \{1, 2, \ldots, q\}$ is the switching signal [1] that selects the mode $\sigma(k)$ at time $k \in \mathbb{N}$, among $q > 1$ possible values. Note that only matrices $C_{\sigma(k)}$ and $D_{\sigma(k)}$ depend on the operating mode, which means that the switched signal $\sigma(k)$ only affects the output $y(k)$. Matrices $B^{\star}$ and $D^{\star}_{\sigma(k)}$ appear as a consequence of the delayed effect of a control action at the opposite end. Furthermore, the transport delay of the canal n can be defined as the minimum time required for an action (e.g., controlled discharge, disturbance, etc.) to propagate from one end of the canal to the other end.

The so-called *switching path*, $\boldsymbol{\sigma}(T) := \{\sigma(k)\}_{k=0}^{T}$, with $T \in \mathbb{N}$, is a sequence that indicates how weather phenomena such as flooding or drought periods affect the consideration of the appropriate operating point.

2.2 Set-Control

Delayed terms in Eq. (1) represent how the control affects the dynamics of the water levels. However, it can be easily shown that the delayed system (1) is equivalent to the 'undelayed' [2] linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k), \end{aligned} \quad (2)$$

with $x(k) \in \mathcal{X} \subset \mathbb{R}^n$, $u(k) \in \mathcal{U} \subset \mathbb{R}^m$, $y(k) \in \mathcal{Y} \subset \mathbb{R}^n$ and $\sigma(k) \in \Sigma = \{1, \cdots, q\}$.

The equivalence of Eqs. (1) and (2) is possible by an extension of the dimension of the system (see details in Section 6.1).

From now on, in order to facilitate the comprehension of the analysis, the *undelayed* linear system (2) will be considered in the results of this section. The next definition presents several concepts of PS for system (2).

Definition 1 (Control Equilibrium Set)**.** *The control equilibrium set,* $\mathcal{X}_s \subseteq \mathcal{X}$ *of system* (2) *is given by all the admissible states* $x_s \in \mathcal{X}$ *for which there is* $u_s \in \Sigma \times \mathcal{U}$ *that fulfills* $x_s = Ax_s + Bu_s$.

According to the system, the switching signal only affects the outputs, so the analysis of set-control will be focused on the output space $\mathcal{Y}$. The equilibrium sets $\mathcal{X}_s$ and $\mathcal{U}_s$ define the equilibrium set for the outputs for every $\sigma \in \Sigma$ by

$$\mathcal{Y}_s^{\sigma} = C_{\sigma}\mathcal{X}_s \oplus D_{\sigma}\mathcal{U}_s, \quad (3)$$

where $\oplus$ represents the Minkowski sum. The control equilibrium outputs for all modes is given by

$$\mathcal{Y}_s = \bigcup_{\sigma \in \Sigma} \mathcal{Y}_s^{\sigma}.$$

Next, the concept of invariant sets and controllable sets will be presented. Prior to that, note the dynamics of the outputs based on the previous step:

$$\begin{aligned} y(k+1) &= C_{\sigma}x(k+1) + D_{\sigma}u(k+1) \\ &= C_{\sigma}(Ax(k) + Bu(k)) + D_{\sigma}u(k+1) \\ &= \underbrace{C_{\sigma}[A \quad B]}_{F_{\sigma}} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + D_{\sigma}u(k+1) \\ &= F_{\sigma}[x(k) \;\; u(k)]' + D_{\sigma}u(k+1). \end{aligned}$$

Then, the concepts of control invariant sets and controllable sets can be defined as follows.

Definition 2 (Control Invariant Set for Outputs)**.** *A nonempty set* $\Omega \subset \mathcal{Y}$ *is said to be a* control invariant set *for the switched system* (2) *if for every state* $x \in \mathcal{X}$ *and input* $u \in \mathcal{U}$ *such that* $y = C_{\sigma}x + D_{\sigma}y \in \Omega$, *for some* $\sigma \in \Sigma$, *there is a pair* $(\bar{\sigma}, \bar{u}) \in \Sigma \times \mathcal{U}$ *such that* $F_{\bar{\sigma}}[x \;\; u]' + D_{\bar{\sigma}}\bar{u} \in \Omega$.

Remark 3. *Note that the control equilibrium set* $\mathcal{Y}_s$ *is a particular case of a control invariant set for the output.*

Definition 4 (Controllable Set of Mode σ for the Output)**.** *Consider a nonempty set* $\Omega \subset \mathcal{Y}$, *the one step controllable set to* Ω *for mode* σ *is given by*

$$\mathcal{S}(\sigma, \Omega) = \{y = Ax + Bu : \exists\, \bar{u} \in \mathcal{U} \text{ s.t. } F_{\sigma} \begin{bmatrix} x \\ u \end{bmatrix} + D_{\sigma}\bar{u} \in \Omega\}.$$

The k*-step controllable set to* Ω *for mode* σ *for the output can be defined iteratively by* $\mathcal{S}^k(\sigma, \Omega) := \mathcal{S}(\sigma, \mathcal{S}^{k-1}(\sigma, \Omega))$ *for* $k \geq 1$, *with* $\mathcal{S}^0(\sigma, \Omega) := \Omega$.

Section 6.2 shows a method to compute the controllable set $\mathcal{S}(\sigma, \Omega)$ for a convex set Ω.

2.3 Main Results: Characterization of Feasible PS

Invariant and equilibrium sets outside of the origin for switched systems—which switches mode after a finite amount of time—are hard to characterize. However, the problem was simplified in Perez et al. (2022) by proposing the concept of Permanence Sets (PS). This is a more general concept that allows to find a stabilizing region for the control switched system. Generally, equilibrium, invariant and cyclic set are particular cases of a PS. In short, a PS Ω, can be defined as follows.

Consider a set $\mathcal{T}$ such that $\Omega \subset \mathcal{T}$, where every point $y(0) \in \Omega$ satisfies the following condition:

$$\forall y(0) \in \Omega \;\; \exists\, u(k) \in \mathcal{U} \text{ such as } y(k) \in \mathcal{T}, \forall k > 0.$$

Note that the trajectory of the output $y(k)$ can remain inside the target set $\mathcal{T}$ if it starts inside a particular region (the PS Ω) inside $\mathcal{T}$. The concept of PS is properly defined next.

Definition 5 (Control Permanence Set—CPS)**.** *Consider a target set* $\mathcal{T} \subset \mathcal{Y}$. *A nonempty set* $\Omega \subset \mathcal{T}$ *is said to be a* control permanence set *of* $\mathcal{T}$ *for the switched system* (2)*, if for every* $y(0) \in \Omega$ *there exists an infinite sequence of inputs* $\mathbf{u} := \{u(k)\}_{k=0}^{\infty}$ *and a switching path* $\boldsymbol{\sigma} = \{\sigma(k)\}_{k=0}^{\infty}$ *such that* $y(k) \in \mathcal{T}$ *for every* $k \geq 0$.

Note that if $\Omega \equiv \mathcal{T}$ on the above definition, the Control Permanence Set is a Control Invariant Set.

Next, an algorithm is proposed to characterize a Control Permanence set for the outputs. Consider a value M_{σ} for every $\sigma \in \Sigma$,

[1] In this study the switching signal σ is not a control variable, and represents different environmental conditions (like drought or rains).

[2] The term *undelayed* is used to stress the fact that the delay is hidden by the extension of the system dimension.

which represents the maximum amount of time the switched system (2) must remain in mode σ before switching to another mode (this parameter is defined by the application). Then, the following algorithm computes a Control Permanence Set of the target set $\mathcal{T}$.

Algorithm 1 Output Control Permanence Set construction

Require: $System$, $k_\sigma \leq M_\sigma, \mathcal{T}$,
1: Compute $\bar{\mathcal{Y}}_s^j := \mathcal{Y}_s^j \cap \mathcal{T}, \forall j \in \Sigma$
2: Compute $\mathcal{S}^{k_j}(j, \bar{\mathcal{Y}}^j), \forall j \in \Sigma \triangleright$ in every of the k_2 steps $\cap\mathcal{T}$
3: $\bar{\mathcal{Y}}^j = \bar{\mathcal{Y}}_s^j \cap \bigcup_{i\in\Sigma, i\neq j} \mathcal{S}^{k_i}(i, \bar{\mathcal{Y}}_s^i) \cap \mathcal{T}, j \in \Sigma$
4: Compute $\mathcal{S}^{k_j}(j, \bar{\mathcal{Y}}_s^j), \forall j \in \Sigma \triangleright$ in every of the k_2 steps $\cap\mathcal{T}$
5: Is $\bar{\mathcal{Y}}_s^j \subset \bigcup_{i\in\Sigma, i\neq j} \mathcal{S}^{k_i}(i, \bar{\mathcal{Y}}_s^i)$ for all $j \in \Sigma$?
6: If step 5 is false $\rightarrow$ return to step 3.
7: If step 5 is true $\rightarrow$ stop. $\Omega = \bigcup_{i\in\mathcal{N}} \bar{\mathcal{Y}}_s^i$ for $\mathcal{N} \subset \Sigma$ for all $i \in \mathcal{N}$

Proposition 6. *If Algorithm 1 converges to a nonempty set $\Omega \subset \mathcal{Y}$, then Ω is a Control Permanence Set of $\mathcal{T}$ for switched system* (2).

Proof. Assume Algorithm 1 converges to a nonempty set Ω. Consider $y(0) \in \Omega$, then $y(0) \in \bar{\mathcal{Y}}_s^j \cap \mathcal{S}^{k_i}(i, \bar{Y}_s^i)$ for some $j, i \in \mathcal{N} \subset \Sigma$ with $j \neq i$. Then, $y(0) \in \mathcal{S}^{k_i}(i, \bar{Y}_s^i)$, which means that $y(0) \in \bar{\mathcal{Y}}_s^j$ can be feasibly driven in k_i steps to $\bar{\mathcal{Y}}_s^i$ without leaving set $\mathcal{T}$. Furthermore, every point of $\bar{\mathcal{Y}}_s^i$ belongs to a set $\mathcal{S}^{k_\ell}(\ell, \bar{Y}_s^\ell)$ for some $\ell \in \mathcal{N}$. Then, the same reasoning can be applied to this point, can be proved iteratively so that $y(k) \in \mathcal{T}$ for all $k \geq 0$. □

There is no guarantee that Algorithm 1 converges to a nonempty set for every possible condition (as it will be shown in the following simulation scenarios). Nevertheless, parameter k_σ can be tuned to avoid situations in which Algorithm 1 converges to an empty set. This is based on the statement that the size of set $\mathcal{S}^{k_\sigma}(\sigma, \cdot)$ increases with the value k_σ (although at the expense of increasing the computational cost). Another alternative to increase the set $\mathcal{S}^{k_\sigma}(\sigma, \Omega)$ is by increasing the size of Ω, so the control equilibrium sets $\mathcal{Y}_s^j$ can be replaced by control invariant sets for each mode.

The following simulation results are based on an identified switched linear model with three modes $\sigma = 1, 2, 3$ that account for normal, rainy and dry conditions. The target set for the output (see Fig. 3) is given by

$$\mathcal{T} = \{(y_1, y_2) : 2.3 \leq y_1 \leq 3,\ 2 \leq y_2 \leq 3\}.$$

For the simulation results depicted in Fig. 1, the following three different equilibrium sets $\mathcal{Y}_s^i$ are selected: $\mathcal{Y}_1 = \{y \in \mathcal{Y} : A_1 x \leq b_1\}$, $\mathcal{Y}_2 = \{y \in \mathcal{Y} : A_2 x \leq b_2\}$, $\mathcal{Y}_3 = \{y \in \mathcal{Y} : A_3 x \leq b_3\}$, with

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$b_1 = [-2.22, -2.42, 2.57, 2.57]'$, $b_3 = [-2.22, -2.67, 2.75, 2.7]'$

Note that $\mathcal{Y}_s^1 = \mathcal{Y}_s^2$ (violet set in Fig. 1). Fig. 1 also shows $\mathcal{Y}_s^3$ for the next step, which is the intersection of the current $\mathcal{Y}_s^3$ with $\mathcal{S}^{k_1}(\sigma, \mathcal{Y}_s^1) \cup \mathcal{S}^{k_2}(\sigma, \mathcal{Y}_s^2)$. Values $k_1 = k_2 = 70$ and $k_3 = 130$

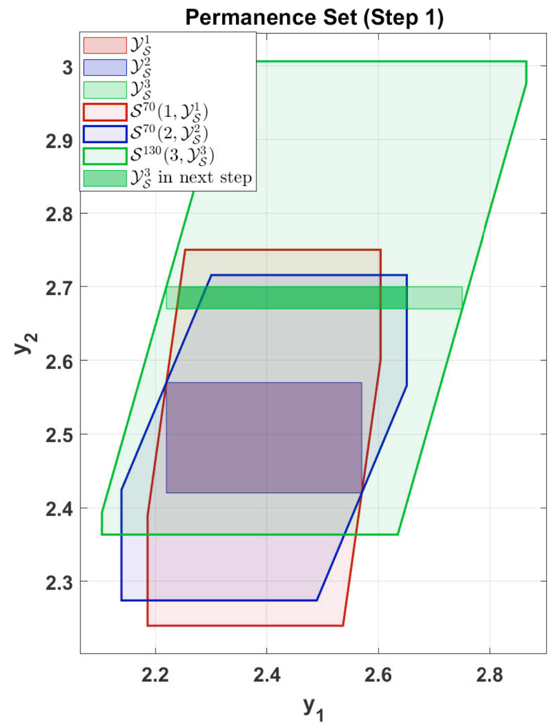

Fig. 1. Equilibrium sets $\mathcal{Y}_s^i$ for $i = 1, 2, 3$, and their corresponding controllable sets $\mathcal{S}^{k_i}(i, \mathcal{Y}_s^i)$ inside a target set $\mathcal{T}$. The first step of Algorithm 1 reduces the size of the set $\mathcal{Y}_s^3$ (darker green).

amount of steps are chosen for each case as the minimum values such that Algorithm 1 converges to a nonempty set. Fig. 2 shows the results after 3 steps of the algorithm. As it can be easily seen in the figure Step 5 of the Algorithm 1 is fulfilled. Therefore, the result $\Omega = \bar{\mathcal{Y}}_s^1 \cup \bar{\mathcal{Y}}_s^3$ (note that $\bar{\mathcal{Y}}_s^1 = \bar{\mathcal{Y}}_s^2$) is the PS for this problem. This region will be used afterwards in the canal control simulation.

Fig. 3 shows initial equilibrium sets $\mathcal{Y}_s^i$, $i = 1, 2, 3$, for which the Algorithm converges to an empty set. In this case, it is not possible to drive the system from one equilibrium $\mathcal{Y}_s^i$ to another $\mathcal{Y}_s^j$ with $i \neq j$ with only one mode. The proposed MPC will not be able to stabilize these equilibrium sets without leaving the target set $\mathcal{T}$. A possible control strategy in this scenario is to switch mode in the region given by the intersection $\mathcal{S}^{70}(2, \mathcal{Y}_s^2) \cap \mathcal{S}^{130}(3, \mathcal{Y}_s^3)$ (intersection of the blue and green region in Fig. 3), or in the region given by the intersection $\mathcal{S}^{70}(2, \mathcal{Y}_s^2) \cap \mathcal{S}^{70}(1, \mathcal{Y}_s^1)$ (intersection of the blue and red sets in Fig. 3). However, investigation of this kind of strategy is left for future research.

3. CONTROL DESIGN

This section presents an MPC (based on the results of Segovia et al. (2019)) that feasibly drives and maintains the output levels of the switched system of the canal inside a given target region $\mathcal{T} \subset \mathcal{Y}$. The results presented in Fig. 2 are considered to design a target reference for the output according to two different modes (normal and rainy conditions).

In this work, the MPC minimizes the distance between the predicted output y_k at time k with a given reference $y_k^{ref}(\sigma) \in \bar{\mathcal{Y}}_s^\sigma$. This is, when mode $\sigma(k)$ is active at time k, the reference

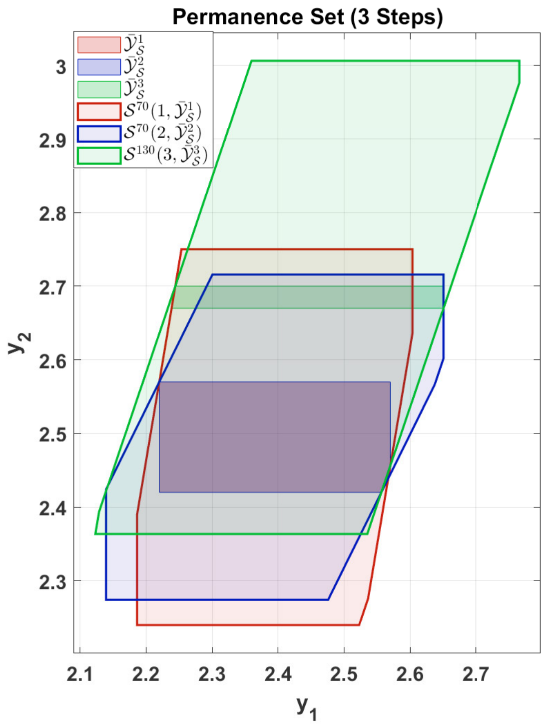

Fig. 2. Final equilibrium sets $\bar{\mathcal{Y}}_s^i$ for $i = 1, 2, 3$. Note that every point of every $\bar{\mathcal{Y}}_s^j$ belongs at least to one controllable set $\mathcal{S}^{k_i}(i, \bar{\mathcal{Y}}_s^i)$, with $j \neq i$.

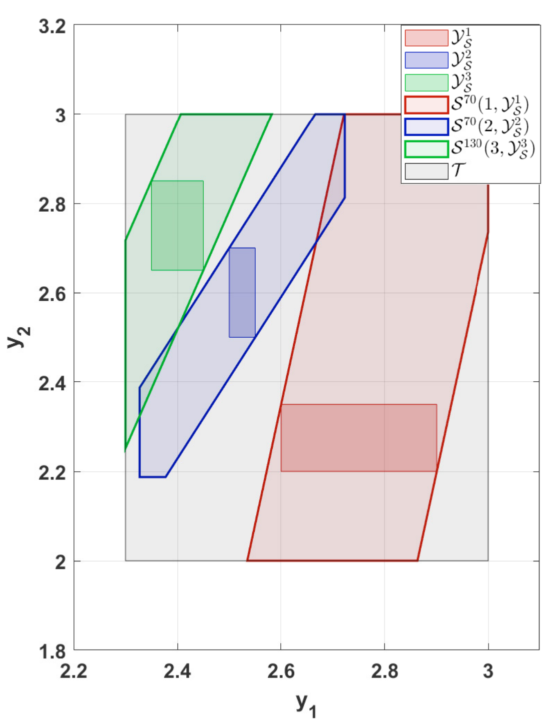

Fig. 3. Scenario where Algorithm 1 converges to an empty set.

is defined by $y_k^{ref} := y_k^{ref}(\sigma)$, so the reference follows the switching path $\boldsymbol{\sigma} = \{\sigma(k)\}$ with $k \geq 0$.

Remark 7. *The following MPC formulation is based on a punctual reference that switches according to the switching path $\boldsymbol{\sigma} = \{\sigma(k)\}$ with $k \geq 0$. However, a relaxation term in the cost function is incorporated so as to provide robustness to the controller.*

The cost function of the MPC is defined next.

3.1 Operational objectives

- Keep water levels as close to the reference levels as possible with sub-cost $J_k^{(1)}$:

$$J_k^{(1)} = \left(y_k - y_k^{ref}\right)^{\mathsf{T}} \left(y_k - y_k^{ref}\right), \tag{4}$$

  with $y_k^{ref} \in \bar{\mathcal{Y}}_s^{\sigma}$ for some $\bar{\mathcal{Y}}_s^{\sigma} \subset \Omega$, where Ω is a PS for $\mathcal{T}$. The reference levels at time instant k, i.e., y_k^{ref}, depend on the current mode $\sigma(k)$, i.e., $y_k^{ref} = y_k^{ref}(\sigma(k))$ for the active mode $\sigma(k)$ at time k.
- In order to robustify the controller, keep the levels within the navigation rectangle and avoid temporal unfeasibility due to operating mode switches and disturbances, a soft constraint on navigation bounds is included. This is achieved by means of the decision variable $\boldsymbol{\alpha}_k$, which must be penalized to ensure that the water levels are outside the bounds as little as possible:

$$J_k^{(2)} = \boldsymbol{\alpha}_k^{\mathsf{T}} \boldsymbol{\alpha}_k. \tag{5}$$

- Minimize control effort:

$$J_k^{(3)} = u_k^{\mathsf{T}} u_k. \tag{6}$$

- Ensure smoothness of control actions to extend the useful life of actuators:

$$J_k^{(4)} = \Delta u_k^{\mathsf{T}} \Delta u_k, \tag{7}$$

  with $\Delta u_k = u_k - u_{k-1}$.

The multi-objective function $J(y_0; \mathbf{y}^{ref}, \mathbf{u}, \boldsymbol{\sigma})$ is then given by

$$J(y_0; \mathbf{y}^{ref}, \mathbf{u}, \boldsymbol{\sigma}) := \sum_{k=1}^{H_p} \sum_{j=1}^{4} \beta^j \, J_k^j \,, \tag{8}$$

where y_0 is the initial output, $\mathbf{y}^{ref} = \{y_k^{ref}\}$ is the reference sequence, $\mathbf{u} = \{u_k\}$ are the predicted inputs, and $\boldsymbol{\sigma} = \{\sigma_k\}$ are the known signals, with $k = 1, \cdots, H_p$, with H_p the prediction horizon. The value β^j is the weight of the j-th objective.

3.2 MPC design

The problem to be solved at time instant k and $i \in \{k, ..., k + H_p - 1\}$ is given by

$$\min_{\{u_{i|k}\}_{i=k}^{k+H_p-1}} J(y_{k|k}; \mathbf{y}^{ref}, \mathbf{u}, \boldsymbol{\sigma}) \tag{9a}$$

subject to:

$$x_{i+1|k} = A x_{i|k} + B u_{i|k} + B^{\star} u_{i-n|k}, \tag{9b}$$
$$y_{i|k} = C_{\sigma(k)} x_{i|k} + D_{\sigma(k)} u_{i|k} + D^{\star}_{\sigma(k)} u_{i-n|k}, \tag{9c}$$
$$u_{i|k} \in \mathcal{U}, j = k - n, ..., k + H_p - 1 \tag{9d}$$
$$x_{i|k} \in \mathcal{X}, j = k - n, ..., k + H_p - 1 \tag{9e}$$
$$\underline{y}_{i|k} - \boldsymbol{\alpha}_{i|k} \leq y_{i|k} \leq \overline{y}_{i|k} + \boldsymbol{\alpha}_{i|k}, \tag{9f}$$
$$\boldsymbol{\alpha}_{i|k} \geq 0, \tag{9g}$$

$$x_{k|k} = \hat{x}_k, \tag{9h}$$

$$d_{j|k} = \hat{d}_j,\ j \in \{k-n, ..., k+H_p-1\} \tag{9i}$$

$$u_{l|k} = u_l^{MPC},\ l \in \{k-n, ..., k-1\}, \tag{9j}$$

where i is the time instant along the prediction horizon, j and l span time intervals that differ from the one described by i, and the pairs $\{\underline{y}_{i|k}, \overline{y}_{i|k}\}$ and $\{\underline{u}_{i|k}, \overline{u}_{i|k}\}$ denote lower and upper output and input bounds, respectively. Note the temporal dependency of the bounds: this is connected to the existence of several modes, which may be characterized by different bounds. Moreover, $\hat{x}_k$ and $\hat{d}_j$ represent the initial state and the disturbance estimates [3], as their values are not directly available for measurement. On the one hand, $\hat{x}_k$ is determined as the solution of the moving horizon estimator (MHE) proposed in Segovia et al. (2019, Eq. (27)) (its formulation is not included in this paper due to space limitations). Finally, u_l^{MPC} denotes past control inputs injected into the system, and which still have an effect on the water levels at time k.

Remark 8. *A terminal constraint set and a terminal cost can be added to* (9) *to stabilize the system (Mayne et al., 2000). On the one hand, the terminal constraint set $\mathcal{X}_{H_p}$ can be defined as an invariant ellipsoidal set given by (Conte et al., 2013)*

$$\mathcal{X}_{H_p} = \left\{ x_{k+H_p|k} \in \mathbb{R}^{n_x} \,\middle|\, x_{k+H_p|k}^{\intercal} Q x_{k+H_p|k} \leq 1 \right\}. \tag{10}$$

On the other hand, the terminal cost is expressed as $\left\| x_{k+H_p|k} \right\|_Q^2$, and Q is defined as the corresponding LQR gain (Limon et al., 2008).

4. CASE STUDY

A canal with physical parameters as in Table 1 is used to test the control strategy. Two different operating modes—normal and rainy conditions—are considered. A 24-hour simulation is designed as follows: the system starts in normal mode; after eight hours, a rainy episode that lasts four hours is considered, after which normal mode is considered until the end of the simulation. Moreover, a sampling time equal to thirty minutes is considered.

Uncontrolled offtakes for irrigation purposes at the upstream and downstream ends of the canal are regarded as system disturbances, and have an additive effect on both states and outputs. These flows, which are assumed to be unmeasurable but bounded (maximum value of 2 m^3/s), complicate the realization of operational objectives. The approach followed in this paper is to consider worst-case upstream and downstream disturbances, i.e., equal to 2 m^3/s, during the prediction step, although random offtake values within the bounds are simulated. This mismatch is compensated by the closed-loop nature of the control strategy, although it is worth noting that the resulting optimal control inputs may be over-conservative.

The control problem is designed in Matlab R2020b (64 bits), using the Gurobi 9.2 optimization package [4] as solver and YALMIP (Löfberg, 2004) as parser. The closed-loop solution for a certain reference $y_k^{ref} = y_k^{ref}(\sigma(k))$ and the mode switch given by $\sigma = 1$ and $\sigma = 2$ is depicted in Figure 4. It can be realized that the control strategy is able to steer the water levels to the time-varying references with minimal error despite imperfect disturbance knowledge. Moreover, the water levels are kept within the navigation rectangle at all times, and the control inputs (not provided due to space limitations) are kept within admissible bounds.

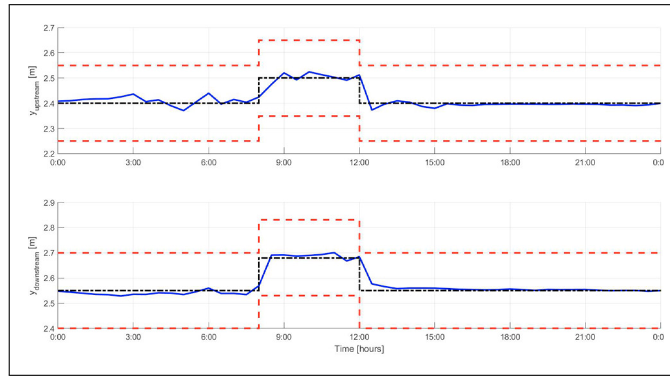

Fig. 4. Simulation results for a canal with two operating modes: level bounds (dashed red line), reference values (black dash-dotted line) and water levels (blue solid line)

In order to quantify the performance of the controller, the tracking indices given in Segovia et al. (2018, Eq. (17)) are considered. These indices are equal to 95.87% and 98.33% for the upstream and downstream water levels, respectively, which allows to conclude that the tracking performance is guaranteed.

5. CONCLUSIONS

In this paper, a switching MPC for canals with stability and controllability properties during the all switching instant was proposed. The canal was modelled on the basis of a linear model and considering three operating points. The MPC was designed to control the canal levels at the boundaries for large operating ranges while guaranteeing controllability under changes of modes. Several simulations for realistic cases of study demonstrate the satisfactory performance of the control strategy.

6. APPENDIX

6.1 Equivalent 'Undelayed' System

In what follows, the equivalence between Eq. (1) and Eq. (2) will be demonstrated. W.l.o.g. it can be assumed that $n = 1$ (the same reasoning can be applied for $n > 1$). Eq. (1) is equivalent to

$$\begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & A \end{bmatrix} \begin{bmatrix} x(k-1) \\ x(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B^{\star} & B \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k) \end{bmatrix},$$

$$\begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & C_\sigma(k) \end{bmatrix} \begin{bmatrix} x(k-1) \\ x(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ D_\sigma^{\star}(k) & D_\sigma(k) \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k) \end{bmatrix}.$$

Defining $\bar{x}(k) = \begin{bmatrix} x(k-1) \\ x(k) \end{bmatrix}$, $\bar{u}(k) = \begin{bmatrix} u(k-1) \\ u(k) \end{bmatrix}$, $\bar{y}(k) = \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}$ and matrices

[3] Disturbances of additive nature are not explicitly considered in the model, but they are in the simulation scenario (more details in Section 4).

[4] https://www.gurobi.com/

Table 1. Physical data of the test canal

LNL $[m]$	NNL $[m]$	HNL $[m]$	Length $[m]$	Width $[m]$	Side slope $[m/m]$	Bottom slope $[m/m]$	Manning coeff. $[s/m^{1/3}]$	Average flow $[m^3/s]$
$\{2.25, 2.40\}$ $\{2.35, 2.53\}$	$\{2.40, 2.55\}$ $\{2.50, 2.68\}$	$\{2.55, 2.70\}$ $\{2.65, 2.83\}$	26720	20	0	0	0.035	6 10

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ B^\star & B \end{bmatrix},$$

$$\bar{C}_{\sigma(k)} = \begin{bmatrix} 0 & 1 \\ 0 & C_{\sigma(k)} \end{bmatrix}, \quad \bar{D}_{\sigma(k)} = \begin{bmatrix} 0 & 0 \\ D^\star_{\sigma(k)} & D_{\sigma(k)} \end{bmatrix},$$

Eq. (1) is equivalent to

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \\ \bar{y}(k) &= \bar{C}_{\sigma(k)}\bar{x}(k) + \bar{D}_{\sigma(k)}\bar{u}(k), \end{aligned} \tag{11}$$

with $\bar{x}(k) \in \bar{\mathcal{X}} = \mathcal{X} \times \mathcal{X}$, $\bar{u}(k) \in \bar{\mathcal{U}} = \mathcal{U} \times \mathcal{U}$. It can be noted that Equations (11) and. (2) are equivalent.

6.2 Controllable Set Characterization for the Output

Direct manipulation of Eq. (1) leads to

$$\begin{aligned} y(k+1) &= C_{\sigma(k)}x(k+1) + D_{\sigma(k)}u(k+1) \\ &\quad + D^\star_{\sigma(k)}u(k+1-n) \\ &= C_{\sigma(k)}\Big(Ax(k) + Bu(k) + B^\star u(k-n)\Big) \\ &\quad + D_{\sigma(k)}u(k+1) + D^\star_{\sigma(k)}u(k+1-n), \end{aligned} \tag{12}$$

and, by assuming C_σ non-singular for all $\sigma \in \Sigma$, then

$$x(k) = C^{-1}_{\sigma(k)}\Big(y(k) - D_{\sigma(k)}u(k) - D^\star_{\sigma(k)}u(k-n)\Big).$$

Thus, $y(k+1)$ can be expressed as a function that depends on the previous outputs and inputs as follows:

$$\begin{aligned} y(k+1) &= E^{-1}_{\sigma(k)}y(k) + \Big(C_{\sigma(k)}B - E_{\sigma(k)}D_{\sigma(k)}\Big)u(k) \\ &\quad + D_{\sigma(k)}u(k+1) \\ &\quad + (C_{\sigma(k)}B^\star - E_{\sigma(k)}D^\star_{\sigma(k)})u(k-n) \\ &\quad + D^\star_{\sigma(k)}u(k+1-n), \end{aligned} \tag{13}$$

with $E_{\sigma(k)} = C_{\sigma(k)}AC^{-1}_{\sigma(k)}$. Since $u(k), u(k-n), u(k+1-n) \in \mathcal{U}$, and by assuming matrix $E_{\sigma(k)}$ is non-singular for all $\sigma \in \Sigma$, (14) can be transformed into

$$\mathcal{Y}^+ = E^{-1}_{\sigma(k)}\mathcal{Y} \oplus F_{\sigma(k)}\mathcal{U}, \tag{14}$$

and

$$\mathcal{Y} = E_{\sigma(k)}(\mathcal{Y}^+ \ominus F_{\sigma(k)}\mathcal{U}), \tag{15}$$

with $F_{\sigma(k)} = C_{\sigma(k)}(B - B^\star) + E_{\sigma(k)}(D_{\sigma(k)} - D^\star_{\sigma(k)}) + D_{\sigma(k)} + D^\star_{\sigma(k)}$.

If the set $\Omega := \mathcal{Y}^+$ and $\mathcal{S}(\sigma, \Omega) := \mathcal{Y}$, and $\bar{F}_{\sigma(k)} = -F_{\sigma(k)}$ in Eq. (15), then

$$\mathcal{S}(\sigma, \Omega) = E_{\sigma(k)}(\Omega \oplus \bar{F}_{\sigma(k)}\mathcal{U}).$$

The k-step controllable set to Ω, $\mathcal{S}^k(\sigma, \Omega)$ for $k > 1$, can be computed iteratively by $\mathcal{S}^k(\sigma, \Omega) := \mathcal{S}(\sigma, \mathcal{S}^{k-1}(\sigma, \Omega))$ for $k \geq 1$, with $\mathcal{S}^0(\sigma, \Omega) := \Omega$.

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