RESEARCH ARTICLE | MARCH 23 2010

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Chaos 20, 013124 (2010)
https://doi.org/10.1063/1.3339819

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Existence of homoclinic connections in continuous piecewise linear systems

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(Received 23 July 2009; accepted 9 February 2010; published online 23 March 2010)

Numerical methods are often used to put in evidence the existence of global connections in differential systems. The principal reason is that the corresponding analytical proofs are usually very complicated. In this work we give an analytical proof of the existence of a pair of homoclinic connections in a continuous piecewise linear system, which can be considered to be a version of the widely studied Michelson system. Although the computations developed in this proof are specific to the system, the techniques can be extended to other piecewise linear systems. © *2010 American Institute of Physics*. [doi:10.1063/1.3339819]

The occurrence of a homoclinic orbit to a saddle-focus equilibrium satisfying certain eigenvalue condition assures the appearance of complex dynamics (Shil'nikov, 1965). Unfortunately, the proof of the existence of such an orbit is generically a difficult task and numerical techniques are often used. Arneodo, Coullet, and Tresser, in 1982, realized that piecewise linear systems gave a good chance of proving the existence of those dynamical objects and that Shil'nikov's result could be extended to this class of systems [see Arneodo *et al.* (1982) and Tresser (1984)]. In fact, as it is well known nowadays, piecewise linear systems are able to reproduce most of the dynamical behavior exhibited by general nonlinear systems. Furthermore, they are also becoming an important tool in the understanding of a wide range of dynamical phenomena in several areas of physics, engineering, and sciences in general. In this work, we present alternative conditions to those established in Arneodo *et al.* (1982) for the existence of a homoclinic connection in piecewise linear systems. Moreover, we give a complete analytical proof of the existence of a symmetrical pair of such connections in a continuous piecewise linear system which can be considered to be a version of the widely studied Michelson system.

I. INTRODUCTION

Homoclinic connections are orbits that are biasymptotic, for $t \to \pm\infty$, to the same equilibrium point. The existence of a homoclinic connection to a saddle-focus equilibrium point usually forces a complex dynamical behavior in a neighborhood of such connection. For instance, the celebrated works of Shil'nikov (Shil'nikov, 1965; Shil'nikov, 1970) assure, under certain eigenvalue ratio condition, the existence of infinitely many periodic orbits of saddle type accumulating to the homoclinic cycle.

An exhaustive recent revision of homoclinic connections for autonomous vector fields has been carried out in Homburg and Sandstede (2009). That work deals with the dynamic behavior related to the existence of homoclinic and heteroclinic orbits, the bifurcations of global connections, and the main analytical and geometric techniques used in their study. Other good works about theoretical and numerical aspects related to global connections are the pair of books (Shil'nikov *et al.*, 1998; 2001) and the survey (Champneys and Kuznetsov, 1994) which is more focused on the detection and continuation of global connections.

A large list of references about homoclinic connections and their bifurcations can also be found in these four previously cited works. Nevertheless, we would like to add here a short list of references about different topics regarding homoclinic cycles. For instance, several analyses of periodic motions near homoclinic connections (both in phase and parameter space) appear in Belyakov (1974); (1981); (1984), Gaspard *et al.* (1984), and Glendinning and Sparrow (1984).

The works (Devaney, 1976; 1978; Champneys, 1998; 1999) are devoted to global connections in reversible and Hamiltonian systems. The particular case of the restricted three-body problem is considered in Gómez *et al.* (1988).

Homoclinic connections and their bifurcations have also been reported and studied in nonsmooth systems (Arneodo *et al.*, 1982; Tresser, 1984) and partial differential equations (Feroe, 1981; Coullet, Riera, and Tresser, 2004; and Coullet, Toniolo, and Tresser, 2004). In fact, there are applications in many fields of science where homoclinic orbits have a special relevance (Gaspard *et al.*, 1993).

The principal problem in the study of homoclinic orbits is that a rigorous proof of its existence is generally a difficult task. One of the approaches of this problem is based on

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finding local degeneracies whose unfoldings may exhibit global bifurcations. Two classical works about the appearance of homoclinic connections from a nilpotent singularity of a planar vector field are Bogdanov (1976) and Takens (1974). Regarding global connections in $\mathbb{R}^3$, Ibáñez and Rodriguez proved the existence of homoclinic orbits to saddle-focus equilibria in the three-parameter unfolding of a nilpotent singularity of codimension three (Ibañez and Rodriguez, 2005). Some recent works (Wilczak, 2005; Wilczak, 2006) have been devoted to a different approach, which consist on the derivation of computer-assisted proofs for the existence of global connections.

Regarding piecewise linear systems, there are a lot of works about the existence of homoclinic cycles (Arneodo *et al.*, 1981; Arneodo *et al.*, 1982; Chua *et al.*, 1986; Matsumoto *et al.*, 1985; Matsumoto *et al.*, 1988; Medrano-T. *et al.*, 2005; and Medrano-T. *et al.*, 2006). In many of them, authors require numerical arguments to show that existence. In others (Llibre *et al.*, 2007), authors start from a degenerate situation to avoid any numerical dependence. In the present work we consider a different strategy which can be also used in a generic case.

Recently, in work (Carmona *et al.*, 2008), the proof of the existence of a reversible T-point heteroclinic cycle has been given in a continuous piecewise linear system. A T-point is a global bifurcation that organizes a rich periodic and aperiodic behavior [see Glendinning and Sparrow (1986), Bykov (1993); (1999); (2000), and Fernández-Sánchez *et al.* (2002)]. This global bifurcation has generically codimension two but in the presence of a symmetry or reversibility this codimension can be reduced. The methods used in Carmona *et al.* (2008) are based on the explicit integration of the flow in each linear region of the space of variables and the construction of a system of equations and inequalities that have to be fulfilled by such kind of global bifurcation. Similar ideas are developed here for the case of a homoclinic connection. These techniques can be extended to other piecewise linear systems [for instance, the Chua's circuit (Matsumoto, 1984)] taking into account that it is important to obtain suitable expressions for the solution in each zone of linearity.

The system studied in work (Carmona *et al.*, 2008),

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = 1 - y - c|x|, \end{cases} \tag{1.1}$$

where $c>0$, can be considered as a continuous piecewise linear version of the well-known Michelson system (Kuramoto and Tsuzuki, 1976; Michelson, 1986; Freire *et al.*, 2002; and Webster and Elgin, 2003). In fact, the equations of system (1.1) can be obtained from the Michelson system performing a simple linear change of variables followed by the change of x^2 to $|x|$. Moreover, both systems are volume preserving and time reversible with respect to the involution $\mathbf{R}(x,y,z)=(-x,y,-z)$. Some other dynamical aspects of the Michelson system also remain in its piecewise linear version (Carmona *et al.*, 2008).

System (1.1) is formed by two linear systems separated by the plane $\{x=0\}$, called separation plane, and it can be written in a matricial form as

$$\dot{\mathbf{x}} = \begin{cases} A_+\mathbf{x} + \mathbf{e}_3 & \text{if } x \geq 0 \\ A_-\mathbf{x} + \mathbf{e}_3 & \text{if } x \leq 0 \end{cases} \tag{1.2}$$

with

$$A_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -1 & 0 \end{pmatrix}, \quad A_- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & -1 & 0 \end{pmatrix} \quad \text{and}$$

$$\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In the half-space $\{x<0\}$, the system has exactly one equilibrium point $\mathbf{p}_-=(-1/c,0,0)^T$, which is a saddle-focus point. Let $\lambda>0$ and $\alpha \pm i\beta$ be the eigenvalues of the Jacobian matrix at $\mathbf{p}_-$. This clearly implies that

$$c = \lambda(1+\lambda^2), \quad \alpha = -\frac{\lambda}{2}, \quad \beta = \frac{\sqrt{4+3\lambda^2}}{2}. \tag{1.3}$$

By the reversibility with respect to $\mathbf{R}$, there exists exactly one saddle-focus equilibrium $\mathbf{p}_+=(1/c,0,0)^T$ in the half-space $\{x>0\}$ whose eigenvalues are given by $-\lambda$ and $-\alpha \pm i\beta$.

Using the expression of the parameter c given in Eq. (1.3), system (1.1) can be written as

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = 1 - y - \lambda(1+\lambda^2)|x|, \end{cases} \tag{1.4}$$

and the parameter $\lambda>0$ can be chosen as the fundamental parameter of the family.

In the particular case of piecewise linear systems with two zones, homoclinic connections can be classified attending to the number of intersections with the separation plane. It is obvious that the number of intersections between any homoclinic connection of system (1.4) and the separation plane $\{x=0\}$ has to be greater than one. So, we say that a homoclinic connection of system (1.4) is direct if it intersects the separation plane $\{x=0\}$ at exactly two points.

The analytical proof of the existence of a pair of direct homoclinic connections will be the main goal of this work, as it is summarized in the following theorem.

Theorem 1.1: *There exists a value $\lambda_h>1/2$ such that the piecewise linear version (1.4) of the Michelson system has, for $\lambda=\lambda_h$, two direct homoclinic connections, which are symmetric with respect to the involution* $\mathbf{R}$.

Note that due to the reversibility, if there exists a homoclinic connection Γ of system (1.4), then a new homoclinic connection which can be mapped onto Γ by $\mathbf{R}$ also exists. Thus, it is only necessary to prove the existence of a direct homoclinic connection Γ to the equilibrium $\mathbf{p}_-$.

In Fig. 1 the pair of homoclinic connections of system (1.4) given by Theorem 1.1 is shown. It is important to remark that the proof of Theorem 1.1 is partially based on

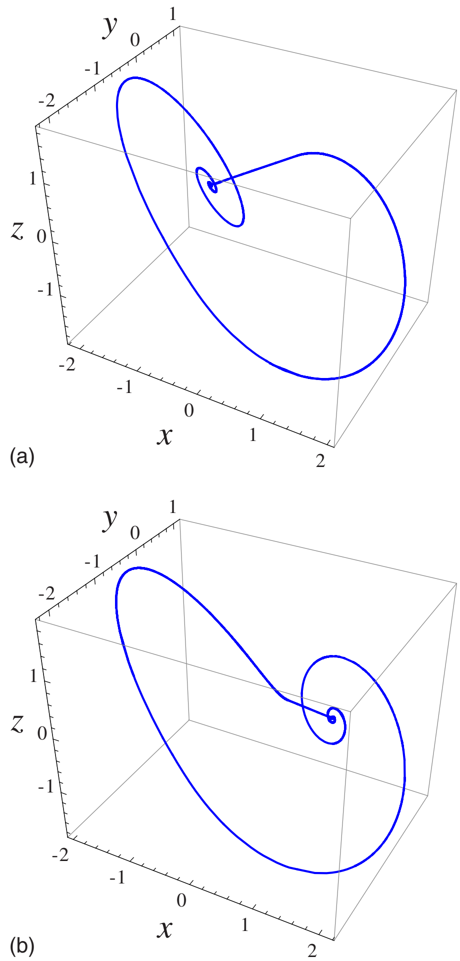

FIG. 1. (Color online) Direct homoclinic orbit to (a) $\mathbf{p}_-$ and (b) $\mathbf{p}_+$.

some results of Carmona *et al.* (2008). In that work the boundary value 1/2 mentioned in the statement of the theorem was chosen for the sake of simplicity of the handmade calculations and it does not have any dynamical meaning. In fact, some numerical computations allow to obtain $\lambda_h \approx 0.660\,759\,953$.

On the other hand, the saddle index of the saddle-focus equilibria is $-\alpha/\lambda = 1/2$ for every value λ. From the point of view of the stability of the periodic orbits close to the homoclinic connection, this can be considered to be a limit case [see Glendinning and Sparrow (1984) and Ovsyannikov and Shil'nikov (1987)]. Moreover, due to the piecewise-linear character of system (1.1) it is not possible to use other quantities which can be interesting for smooth systems. For example, when the saddle index is 1/2, the sign of the integral of the divergence of the vector field over a homoclinic orbit has an important role for the dynamics in a neighborhood of the homoclinic orbit [see Gonchenko and Shil'nikov (2007)]. In the case of system (1.1), this quantity vanishes and thus, no conclusions can be obtained in this way.

The rest of the paper is organized as follows. In Sec. II we describe the basic geometric elements of the problem. Section III is devoted to the proof of Theorem 1.1, which is divided into two parts. In Sec. IV we deal with other global connections and show some numerical results.

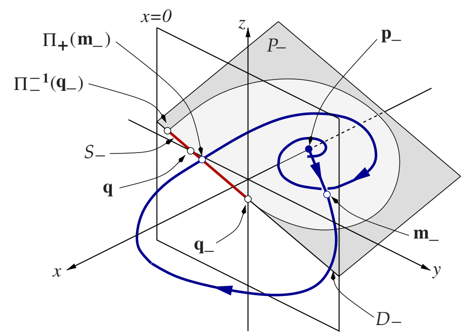

FIG. 2. (Color online) Direct homoclinic connection to $\mathbf{p}_-$ and some geometric elements of the flow.

II. SOME GEOMETRIC ELEMENTS OF THE FLOW

In this section we describe the behavior of the flow crossing the plane $\{x=0\}$ and the basic elements of the linear dynamics locally contained in the half-spaces $\{x<0\}$ and $\{x>0\}$. For every point $\mathbf{p}=(x_\mathbf{p},y_\mathbf{p},z_\mathbf{p})^T \in \mathbb{R}^3$ we denote by $\mathbf{x}_\mathbf{p}(t;\lambda)=(x_\mathbf{p}(t;\lambda),y_\mathbf{p}(t;\lambda),z_\mathbf{p}(t;\lambda))^T$ the solution of the system (1.4) with parameter λ and initial condition $\mathbf{x}_\mathbf{p}(0;\lambda)=\mathbf{p}$. The corresponding orbit is denoted by $\gamma_\mathbf{p}$.

If $x_\mathbf{p}=0$ and $y_\mathbf{p}>0$, then the orbit $\gamma_\mathbf{p}$ crosses transversally the plane $\{x=0\}$ with $x_\mathbf{p}(-t;\lambda)<0$ and $x_\mathbf{p}(t;\lambda)>0$ for $t>0$ small enough. If $x_\mathbf{p}(t;\lambda)$ vanishes in $(0,+\infty)$, then we define the flying time $t_\mathbf{p}^+$ as the positive value such that $x_\mathbf{p}(t_\mathbf{p}^+;\lambda)=0$ and $x_\mathbf{p}(t;\lambda)>0$ in $(0,t_\mathbf{p}^+)$. In such a case, we define the Poincaré map Π_+ at the point $\mathbf{p}$ as $\Pi_+(\mathbf{p}) = \mathbf{x}_\mathbf{p}(t_\mathbf{p}^+;\lambda)$. Note that the Poincaré map Π_+ only depends on the linear system $\dot{\mathbf{x}}=A_+\mathbf{x}+\mathbf{e}_3$ given in system (1.2).

If $x_\mathbf{p}=0$ and $y_\mathbf{p}<0$, then the orbit $\gamma_\mathbf{p}$ crosses transversally the plane $\{x=0\}$ with $x_\mathbf{p}(-t;\lambda)>0$ and $x_\mathbf{p}(t;\lambda)<0$ for $t>0$ small enough. If $x_\mathbf{p}(t;\lambda)$ vanishes in $(0,+\infty)$, then we define the flying time $t_\mathbf{p}^-$ as the positive value such that $x_\mathbf{p}(t_\mathbf{p}^-;\lambda)=0$ and $x_\mathbf{p}(t;\lambda)<0$ in $(0,t_\mathbf{p}^-)$. In such a case, we define the Poincaré map Π_- at point $\mathbf{p}$ as $\Pi_-(\mathbf{p})=\mathbf{x}_\mathbf{p}(t_\mathbf{p}^-;\lambda)$. This map only depends on the linear system $\dot{\mathbf{x}}=A_-\mathbf{x}+\mathbf{e}_3$.

If $\mathbf{p}$ belongs to the z-axis, i.e., $x_\mathbf{p}=0$ and $y_\mathbf{p}=0$, then $\mathbf{p}$ is called a contact point of the flow of system (1.4) with the plane $\{x=0\}$ because the vector field at this point is tangent to the plane. Following Llibre and Teruel (2004), the first coordinate of the Taylor expansion of $\mathbf{x}_\mathbf{p}(t;\lambda)-\mathbf{p}$ at $t=0$ is

$$\mathbf{e}_1^T(\mathbf{x}_\mathbf{p}(t;\lambda)-\mathbf{p}) = z_\mathbf{p}\frac{t^2}{2} + \frac{t^3}{3!} + \mathbf{e}_1^T\mathbf{x}_\mathbf{p}^{(4)}(\xi;\lambda)\frac{t^4}{4!}.$$

Hence, if $z_\mathbf{p}<0$, then orbit $\gamma_\mathbf{p}$ is locally contained in the half-space $\{x\le 0\}$; if $z_\mathbf{p}>0$, then $\gamma_\mathbf{p}$ is locally contained in the half-space $\{x\ge 0\}$; and if $z_\mathbf{p}=0$, then $\gamma_\mathbf{p}$ crosses the plane $\{x=0\}$ from the half-space $\{x<0\}$ to the half-space $\{x>0\}$.

Now we describe the basic elements of the linear dynamics in the half-space $\{x<0\}$. All this information is summarized in Fig. 2. The elements in the other half-space can be obtained using the involution $\mathbf{R}$.

The unstable manifold $W^u(\mathbf{p}_-)$ of $\mathbf{p}_-$ contains the half-line $\mathcal{L}_-=\{\mathbf{p}_- - \mu(1,\lambda,\lambda^2)^T : -1/(\lambda+\lambda^3)\le\mu<\infty\}$ generated by the eigenvector associated with the eigenvalue λ of the matrix A_-. The half-line and the plane $\{x=0\}$ intersect at the point

$$\mathbf{m}_- = \left(0, \frac{1}{1+\lambda^2}, \frac{\lambda}{1+\lambda^2}\right)^T.$$

The stable two-dimensional manifold $W^s(\mathbf{p}_-)$ is locally contained in the half-plane

$$\mathcal{P}_- = \{\lambda(1+\lambda^2)x + \lambda^2 y + \lambda z = -1 : x \le 0\},$$

which is called the focal half-plane of $\mathbf{p}_-$. This half-plane is obtained from the eigenvectors associated with the complex eigenvalues of A_-. The half-plane $\mathcal{P}_-$ and the separation plane $\{x=0\}$ intersect along the straight line

$$\mathcal{D}_- = \{x=0, \lambda^2 y + \lambda z = -1\}.$$

Let us emphasize that not every point in $\mathcal{D}_-$ belongs to the stable manifold $W^s(\mathbf{p}_-)$. The intersection point of $\mathcal{D}_-$ and the z-axis is $\mathbf{q}_- = (0,0,-1/\lambda)^T$. Since $\mathbf{q}_-$ is a contact point, the orbit $\gamma_{\mathbf{q}_-}$ is tangent to the separation plane $\{x=0\}$ at $\mathbf{q}_-$. Thus, the segment $\mathcal{S}_- \subset \mathcal{D}_-$ with end points $\mathbf{q}_-$ and $\Pi_-^{-1}(\mathbf{q}_-)$ is contained in $W^s(\mathbf{p}_-)$.

III. EXISTENCE OF A DIRECT HOMOCLINIC CONNECTION TO $\mathbf{p}_-$

A direct homoclinic orbit to $\mathbf{p}_-$ has to intersect the plane $\{x=0\}$ at $\mathbf{m}_-$, since it corresponds to the linear one-dimensional manifold of $\mathbf{p}_-$. On the other side, this orbit also has to belong to the two-dimensional manifold of $\mathbf{p}_-$, that is, it has to intersect segment $\mathcal{S}_-$. Thus, when the condition $\Pi_+(\mathbf{m}_-) \in \mathcal{S}_-$ holds, a direct homoclinic connection to $\mathbf{p}_-$ exists in system (1.4). In fact, the existence of such homoclinic connection can be derived from conditions

$$\overline{\mathbf{q}\mathbf{q}_-} \subset \mathcal{S}_- \tag{3.1}$$

and

$$\Pi_+(\mathbf{m}_-) \in \overline{\mathbf{q}\mathbf{q}_-}, \tag{3.2}$$

where $\mathbf{q} = (0, -1/\lambda^2, 0)$ is the intersection point of the straight lines $\mathcal{D}_-$ and $\mathcal{D}_+$, see Fig. 2.

As a corollary of Proposition 3.3 in Carmona *et al.* (2008), it follows that there exists a value $\lambda^* \in (0, 1/2)$ such that for every $\lambda \ge \lambda^*$ condition (3.1) is satisfied. On the other hand, since the orbit through $\mathbf{m}_-$ cannot intersect the focal plane $\mathcal{P}_+$, it is easy to conclude that $\Pi_+(\mathbf{m}_-) \in \overline{\mathbf{q}\mathbf{q}_-}$ if and only if $\Pi_+(\mathbf{m}_-) \in \mathcal{D}_-$. In other words, conditions (3.1) and (3.2) are equivalent to the existence of $t_h > 0$ and $\lambda_h > 1/2$, such that $\mathbf{x}_{\mathbf{m}_-}(t_h, \lambda_h) \in \mathcal{D}_-$ and $x_{\mathbf{m}_-}(t, \lambda_h) > 0$ for every $t \in (0, t_h)$. It is obvious that if such a pair (t_h, λ_h) exists, then $\mathbf{x}_{\mathbf{m}_-}(t_h, \lambda_h)$ has to satisfy the system

$$\begin{cases} x_{\mathbf{m}_-}(t,\lambda) = 0 \\ \lambda^2 y_{\mathbf{m}_-}(t,\lambda) + \lambda z_{\mathbf{m}_-}(t,\lambda) + 1 = 0, \end{cases} \tag{3.3}$$

obtained by integrating, for $x > 0$, system (1.4) with initial condition $\mathbf{x}(0,\lambda) = \mathbf{m}_-$.

Now, the proof of condition (3.2) is divided into two parts. First, we establish that system (3.3) has a solution (t_h, λ_h) with $t_h > 0$ and $\lambda_h > 1/2$. Second, we check that $x_{\mathbf{m}_-}(t, \lambda_h) > 0$ for every $t \in (0, t_h)$.

After some algebra, system (3.3) leads to the following equivalent system:

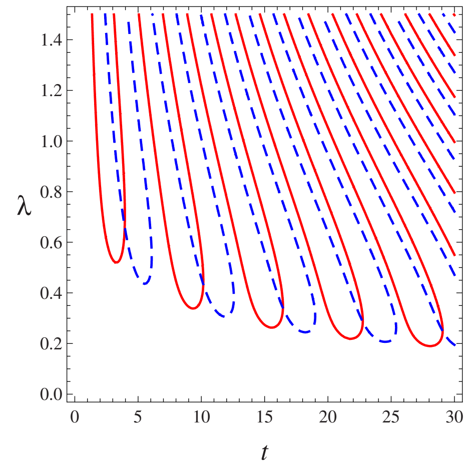

FIG. 3. (Color online) Dashed curves are given by equation $E_1(t,\lambda)=0$ and solid ones correspond to equation $E_2(t,\lambda)=0$.

$$\begin{cases} E_1(t,\lambda) = 0 \\ E_2(t,\lambda) = 0, \end{cases} \tag{3.4}$$

where

$$E_1(t,\lambda) = 2\lambda^2 e^{(3\lambda/2)t}[\sqrt{4+3\lambda^2}\cos(\beta t) - 3\lambda \sin(\beta t)] + \sqrt{4+3\lambda^2}[(1+\lambda^2) - (1+3\lambda^2)e^{\lambda t}], \tag{3.5}$$

$$E_2(t,\lambda) = 2\lambda^2 e^{(3\lambda/2)t}[\sqrt{4+3\lambda^2}\cos(\beta t) + \lambda \sin(\beta t)] + \sqrt{4+3\lambda^2}(1+\lambda^2)e^{\lambda t}, \tag{3.6}$$

and β is defined in expression (1.3).

The curves defined by the equations of system (3.4) are shown in Fig. 3. It is possible to see that they intersect in several points. This is a numerical evidence of the existence of solutions (t,λ), with $t > 0$ and $\lambda > 0$, for this system. In what follows, an analytical proof of the existence of the first intersection point (corresponding to the smallest value of $t > 0$) is derived.

Taking into account the relative position of the curves given by the equations of system (3.4) it is convenient to manipulate these equations to get a more suitable system. Adding $e^{\lambda t}(1+\lambda^2)$ times Eq. (3.5) to $[e^{\lambda t}(1+3\lambda^2) - (1+\lambda^2)]$ times Eq. (3.6) and dividing by $2\lambda^2 e^{(3\lambda/2)t}$ gives

$$E(t,\lambda) = \sqrt{4+3\lambda^2}[2(1+2\lambda^2)e^{\lambda t} - (1+\lambda^2)]\cos(\beta t) - \lambda[2e^{\lambda t} + 1 + \lambda^2]\sin(\beta t) = 0. \tag{3.7}$$

From Eq. (3.4) the trigonometric functions are given by

$$\sin(\beta t) = -\frac{\sqrt{4+3\lambda^2}[2(1+2\lambda^2)e^{\lambda t} - (1+\lambda^2)]e^{(-3\lambda/2)t}}{8\lambda^3},$$
$$\cos(\beta t) = -\frac{(1+\lambda^2+2e^{\lambda t})e^{(-3\lambda/2)t}}{8\lambda^2}. \tag{3.8}$$

Note that both functions are strictly negative for $t > 0$ and $\lambda > 0$.

It is now obvious that

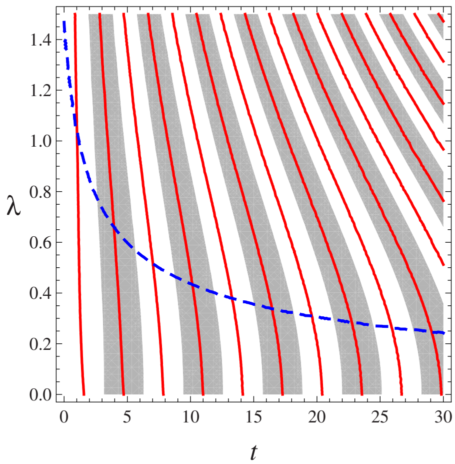

FIG. 4. (Color online) Solid curves are given by Eq. (3.7) and the dashed one corresponds to Eq. (3.9). The set where $\sin(\beta t)<0$ is shaded.

$$\sin^2(\beta t)+\cos^2(\beta t) = \frac{e^{-3\lambda t}}{64\lambda^4}\left[\frac{(4+3\lambda^2)[2(1+2\lambda^2)e^{\lambda t}-(1+\lambda^2)]^2}{\lambda^2} + (1+\lambda^2+2e^{\lambda t})^2\right]$$

or, equivalently,

$$\frac{e^{-3\lambda t}}{64\lambda^4}\left[\frac{(4+3\lambda^2)[2(1+2\lambda^2)e^{\lambda t}-(1+\lambda^2)]^2}{\lambda^2} + (1+\lambda^2+2e^{\lambda t})^2\right]-1=0.$$

Simplifying this equation gives

$$p(t,\lambda)=-16\lambda^6e^{3\lambda t}+(1+\lambda^2)^2[4(1+3\lambda^2)e^{2\lambda t} -2(2+3\lambda^2)e^{\lambda t}+1+\lambda^2] =0. \tag{3.9}$$

Note that a solution (t,λ) of system (3.4) also satisfies the system given by Eqs. (3.7) and (3.9). However, as it is established in Lemma 3.1, another condition is necessary for the converse to be true: the sinus function in Eq. (3.8) must always be negative for $t>0$ and $\lambda>0$.

In Fig. 4 the curves given by Eqs. (3.7) and (3.9) and the sign of the sinus function in Eq. (3.8) are shown. Comparing with Fig. 3, note that there exist intersection points between the curves which are not solutions of system (3.4).

Lemma 3.1: For $t>0$ and $\lambda>0$, system (3.4) is equivalent to the system

$$\begin{cases} E(t,\lambda)=0\\ p(t,\lambda)=0\\ \sin(\beta t)<0. \end{cases} \tag{3.10}$$

Proof: The first part of the equivalence, that is, the proof that a solution (t,λ) of system (3.4) with $t>0$ and $\lambda>0$ also satisfies the system (3.10), is direct.

For the other implication, let us consider the system

$$\begin{cases} -\sqrt{4+3\lambda^2}[2(1+2\lambda^2)e^{\lambda t}-(1+\lambda^2)]X \\ \quad +\lambda[2e^{\lambda t}+(1+\lambda^2)]Y=0\\ X^2+Y^2-1=0, \end{cases} \tag{3.11}$$

which represents the intersection in coordinates (X,Y) of a straight line with positive slope containing the origin and the unit circle. Obviously, system (3.11) has a unique solution with negative second coordinate.

Note that

$$(X_1,Y_1)=\left(-\frac{(2e^{\lambda t}+1+\lambda^2)e^{(-3\lambda/2)t}}{8\lambda^2}, -\frac{\sqrt{4+3\lambda^2}[2(1+2\lambda^2)e^{\lambda t}-(1+\lambda^2)]e^{(-3\lambda/2)t}}{8\lambda^3}\right)$$

is a solution of system (3.11) whose second coordinate is negative for $t>0$ and $\lambda>0$. On the other hand, if (t,λ) is a solution of system (3.4) with $t>0$ and $\lambda>0$, then

$$(X_2,Y_2)=(\sin(\beta t),\cos(\beta t))$$

is also a solution of system (3.11) whose second coordinate is negative.

Therefore, we conclude that $(X_1,Y_1)=(X_2,Y_2)$ with $t>0$ and $\lambda>0$. Since this equality corresponds to system (3.8), which is equivalent to system (3.4), the lemma holds.

Now let us prove that system (3.10) has at least a solution.

Lemma 3.2: System (3.3) has a solution (t_h,λ_h) in the open set

$$\Omega=\left\{(t,\lambda)\in\mathbb{R}^2: \frac{2\pi}{\sqrt{4+3\lambda^2}}<t<\frac{4\pi}{\sqrt{4+3\lambda^2}}, \frac{1}{2}<\lambda<\sqrt{3}\right\}.$$

Proof: From Lemma 3.1 it is known that systems (3.3) and (3.10) are equivalent for $t>0$ and $\lambda>0$.

Since the third condition of system (3.10) is satisfied for every $(t,\lambda)\in\Omega$, it is only necessary to show that system

$$\begin{cases} E(t,\lambda)=0\\ p(t,\lambda)=0 \end{cases} \tag{3.12}$$

has solution in Ω. This is, as it is going to be proved, a consequence of Poincaré–Miranda theorem (Kulpa, 1997), which can be considered as an n-dimensional extension of Bolzano theorem.

The change of variables $\mu=\lambda^2$, $\tau=\sqrt{4+3\lambda^2}t/2$ transforms system (3.12) into the system

$$\begin{cases} \tilde{E}(\tau,\mu)=E\left(\frac{2\tau}{\sqrt{4+3\mu}},\sqrt{\mu}\right)=0\\ \tilde{p}(\tau,\mu)=p\left(\frac{2\tau}{\sqrt{4+3\mu}},\sqrt{\mu}\right)=0, \end{cases} \tag{3.13}$$

and Ω into $\tilde{\Omega}=(\pi,2\pi)\times(1/4,3)$.

From the definition of E it is obvious that $\tilde{E}(\pi,\mu)>0$ and $\tilde{E}(2\pi,\mu)<0$ for $\mu\geq 0$. Thus, function $\tilde{E}$ takes different signs at the vertical sides of the boundary of $\tilde{\Omega}$.

In order to analyze the sign of function $\tilde{p}$ at the horizontal sides of the boundary of $\tilde{\Omega}$, let us define

$$P(s,\mu) = -16\mu^3 s^3 + (1+\mu)^2[4(1+3\mu)s^2 - 2(2+3\mu)s + 1 + \mu], \tag{3.14}$$

which corresponds to function $\tilde{p}$ when

$$s = \exp\left(\sqrt{\mu}\frac{2\tau}{\sqrt{4+3\mu}}\right) \geq 1. \tag{3.15}$$

Since the derivative of $P(s,3)$ with respect to s is negative in $\mathbb{R}$ and $P(1,3)<0$, we have $P(s,3)<0$ for $s \geq 1$. Therefore, $\tilde{p}(\tau,3)<0$ for every $\tau \in [\pi, 2\pi]$.

For the last side of the rectangle, straightforward computations show that the derivative of $P(s,1/4)$ is positive in $[1,27]$. Taking into account that $P(1,1/4)=\frac{259}{64}$ it follows that $P(s,1/4)$ is positive for every $s \in [1,27]$.

Note that from Eq. (3.15), if $\mu=1/4$ and $\tau \in [\pi,2\pi]$, then $s \in [1,27]$. Thus, $\tilde{p}(\tau,1/4)$ is positive for $\tau \in [\pi,2\pi]$. The lemma is followed by the Poincaré–Miranda theorem.

At this moment we have proved that there exists a point $(t_h,\lambda_h) \in \Omega$ such that $\mathbf{x}_{\mathbf{m}\_}(t_h,\lambda_h) \in \mathcal{D}_-$. For condition (3.2) to be fulfilled it is also necessary to prove that $x_{\mathbf{m}\_}(t,\lambda_h)>0$ for every $t \in (0,t_h)$. The next result deals with this inequality.

Lemma 3.3: If $(t_h,\lambda_h) \in \Omega$ is a solution of system (3.10), then $x_{\mathbf{m}\_}(t,\lambda_h)>0$ for every $t \in (0,t_h)$.

Proof: According to the equations of system (1.4), the derivative with respect to t of function $x_{\mathbf{m}\_}(t,\lambda_h)$ is given by $y_{\mathbf{m}\_}(t,\lambda_h)$. By integrating this system for $x>0$, we obtain

$$\begin{aligned}\dot{x}_{\mathbf{m}\_}(t,\lambda_h) &= y_{\mathbf{m}\_}(t,\lambda_h)\\ &= c_1 e^{-\lambda_h t} + e^{(\lambda_h/2)t}\left[c_2 \cos\left(\frac{\sqrt{4+3\lambda_h^2}}{2}t\right) + c_3 \sin\left(\frac{\sqrt{4+3\lambda_h^2}}{2}t\right)\right],\end{aligned} \tag{3.16}$$

where

$$c_1 = \frac{1}{1+3\lambda_h^2} > 0, \quad c_2 = \frac{2\lambda_h^2}{(1+3\lambda_h^2)(1+\lambda_h^2)} > 0,$$

$$c_3 = \frac{2\lambda_h(2+3\lambda_h^2)}{(1+3\lambda_h^2)(1+\lambda_h^2)\sqrt{4+3\lambda_h^2}} > 0.$$

On the one hand, note that $x_{\mathbf{m}\_}(0,\lambda_h)=0$ and $\dot{x}_{\mathbf{m}\_}(0,\lambda_h)>0$. On the other hand, let us assume that $(t_h,\lambda_h) \in \Omega$ is a solution of system (3.10). Therefore, $x_{\mathbf{m}\_}(t_h,\lambda_h)=0$. Substituting Eq. (3.8) in Eq. (3.16) it is obvious that

$$\dot{x}_{\mathbf{m}\_}(t_h,\lambda_h) = y_{\mathbf{m}\_}(t_h,\lambda_h) = \frac{-2+e^{-\lambda_h t_h}}{2\lambda_h^2} < 0.$$

Let us also assume that there exists a value $\hat{t} \in (0,t_h)$ such that $x_{\mathbf{m}\_}(\hat{t},\lambda_h)=0$. Then, $y_{\mathbf{m}\_}(t,\lambda_h)$ must vanish in at least three values in $(0,t_h)$, that is, the equation

$$h(\tau) = e^{(3\lambda_h/\sqrt{4+3\lambda_h^2})\tau}\left[\frac{c_2}{c_1}\cos(\tau) + \frac{c_3}{c_1}\sin(\tau)\right] = -1,$$

which is obtained from $y_{\mathbf{m}\_}(t,\lambda_h)=0$, has to vanish in at least three values in $(0,2\pi)$.

Since $h(0)=c_2/c_1>0$, equation $h(\tau)=0$ must have at least three solutions in $(0,2\pi)$, what is not possible. Thus, function $x_{\mathbf{m}\_}(t,\lambda_h)$ cannot vanish in $(0,t_h)$ and the proof is concluded.

IV. OTHER GLOBAL CONNECTIONS

In Sec. III, the existence of a pair of direct homoclinic connections, which are symmetric with respect to the involution $\mathbf{R}$, has been proved for $\lambda=\lambda_h \approx 0.660\,759\,953$. The first step of this proof is the analysis of the solutions of the system (3.10). Those solutions are the intersections of the solid and dashed curves in Fig. 4 which lie in the shadow regions. Besides the first intersection, which corresponds to the value λ_h, we can observe that other intersections exist.

The second intersection point corresponds to $(t_H,\lambda_H) \approx (10.154\,021\,01, 0.433\,912\,36)$. It can be also proved that a pair of direct homoclinic connections, which are symmetric with respect to the involution $\mathbf{R}$, exists for λ_H. Remember that the existence of an intersection point is not the only condition that has to be fulfilled to assure the existence of a homoclinic connection; it is also necessary to check that the orbit with initial condition $\mathbf{m}_-$ does not intersect the separation plane for $t \in (0,t_H)$ and $\mathbf{x}_{\mathbf{m}\_}(t_H,\lambda_H)$ belongs to $\mathcal{S}_-$. As a comparison to the first pair of homoclinic orbits, these sec-

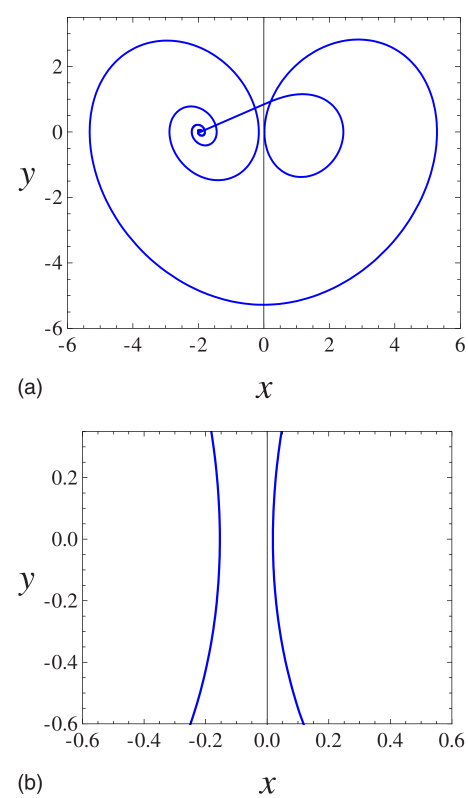

FIG. 5. (Color online) (a) Projection onto the plane xy of a direct homoclinic orbit with a second loop to $\mathbf{p}_-$. (b) Zoom of (a), where it is clear that the second loop does not intersect $x=0$.

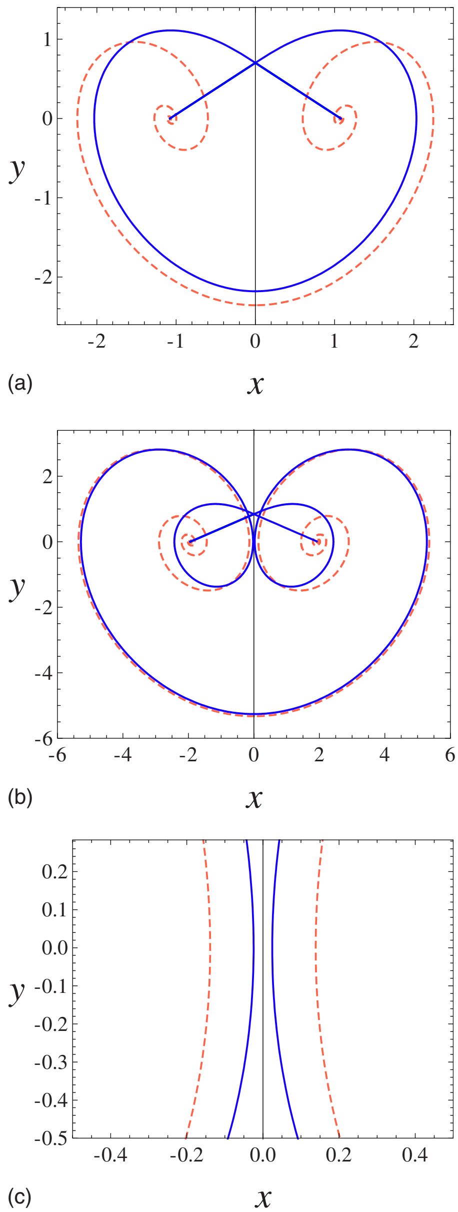

FIG. 6. (Color online) Projections onto the plane xy of the direct T-point heteroclinic cycles which exist for (a) $\lambda \approx 0.651\,535\,56$ and (b) $\lambda \approx 0.433\,278\,34$. In each figure, the solid curve corresponds to the one-dimensional manifolds of the equilibria while the dashed one is a transversal intersection of the two-dimensional manifolds. (c) Zoom of (b), where it is clear that the small loops do not intersect $x=0$.

ond homoclinic connections give an extra loop around the one-dimensional manifold of the other equilibrium. The homoclinic connection to $\mathbf{p}_-$ is shown in Fig. 5.

Regarding the remainder intersection points in Fig. 4, they do not correspond to real direct homoclinic connections: although each one of them is a solution (t', λ') of system (3.10), the orbit with initial condition $\mathbf{m}_-$ intersects the separation plane for values of $t \in (0, t')$.

This behavior is similar for reversible T-point heteroclinic cycles in system (1.4). In Carmona *et al.* (2008), the existence of a "direct" reversible T-point heteroclinic cycle was proved for $\lambda \approx 0.651\,535\,56$. This cycle is called direct in the sense that its heteroclinic orbit corresponding to the one-dimensional manifolds has exactly three intersections with the separation plane (which is the minimum possible number of intersections), while the heteroclinic orbit corresponding to the two-dimensional manifolds has only one intersection. Moreover, the existence of another direct reversible T-point heteroclinic cycle can be proved for $\lambda \approx 0.433\,278\,34$. This cycle has two extra loops around the one-dimensional manifolds of the equilibria, see Fig. 6.

A first step in the proof of the existence of these reversible T-point heteroclinic cycles is the analysis of the existence of solution of a system analogous to Eq. (3.10) [given by Eqs. (4.3) and (4.6) in Carmona *et al.* (2008)]. Besides the values of λ given in the previous paragraph, there exist other solutions of the system which, as the homoclinic case, do not correspond with real reversible T-point heteroclinic cycles.

ACKNOWLEDGMENTS

We would like to thank the editor and the referee for their useful comments and suggestions. The first three authors are partially supported by the Ministerio de Ciencia y Tecnología, Plan Nacional I+D+I, in the frame of the projects under Grant Nos. MTM2006-00847 and MTM2007-64193 and by the Consejería de Educación y Ciencia de la Junta de Andalucía (Grant Nos. TIC-0130, EXC/2005/FQM-872, P08-FQM-03770, and MTM2009-07849). The fourth author is partially supported by the MCYT (Grant No. MTM2005-06098-C02-1), by UIB (Grant No. UIB2005/6), and by CAIB (Grant No. CEH-064864).

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