Two qubits of a *W* state violate Bell's inequality beyond Cirel'son's bound

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It is shown that the correlations between two qubits selected from a trio prepared in a *W* state violate the Clauser-Horne-Shimony-Holt inequality more than the correlations between two qubits in any quantum state. Such a violation beyond Cirel'son's bound is smaller than the one achieved by two qubits selected from a trio in a Greenberger-Horne-Zeilinger state [A. Cabello, Phys. Rev. Lett. **88**, 060403 (2002)]. However, it has the advantage that all local observers can know from their own measurements whether or not their qubits belong to the selected pair.

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I. INTRODUCTION

The Bell inequality [1] proposed by Clauser, Horne, Shimony, and Holt (CHSH) [2], points out that in any local-realistic theory, that is, in any theory in which the local variables of a particle determine the results of local experiments on this particle, the absolute value of a combination of four correlations is bound by 2,

$$|C(A,B)-mC(A,b)-nC(a,B)-mnC(a,b)|\leqslant 2. \quad (1)$$

In inequality (1), A and a are two observables taking values -1 or 1 on particle i, and B and b are two observables taking values -1 or 1 on a distant particle j; m and n can be either -1 or 1.

The CHSH inequality (1) is violated for certain quantum states and certain choices of the observables A, a, B, and b [2]. Therefore, the conclusion is that no local-realistic theory can reproduce the predictions of quantum mechanics [1].

Later on, Cirel'son [3] showed that, according to quantum mechanics, for any two-qubit system *prepared in a quantum state*, the absolute value of the combination of correlations appearing in the CHSH inequality (1) is bound by $2\sqrt{2}$ (*Cirel'son's bound*). This bound is also the maximum violation predicted by quantum mechanics for the two-qubit singlet state (or any other two-qubit Bell state) [2]. Indeed, this is the violation of Bell's inequality traditionally tested in real experiments involving systems of two qubits prepared in a quantum state [4–9].

However, as shown in Ref. [10], according to quantum mechanics the CHSH inequality can be violated beyond Cirel'son's bound. The reason is the following. Bell's inequalities are derived assuming local realism, without any mention to quantum mechanics. Therefore, when searching for violations of a Bell's inequality, one is not restricted to studying correlations between ensembles of systems *prepared in a quantum state*; instead, one can study any ensemble of systems, irrespective of whether such an ensemble is meaningful in quantum mechanics or not (i.e., irrespective of whether it can be described by a quantum state or not). For instance, one can consider trios of qubits prepared in a certain quantum state and then assume local realism to select a pair of qubits in each trio, and calculate, using quantum mechanics, the correlations between these two qubits. The whole procedure makes sense and can be translated into real experiments as long as one can obtain the required correlations and probabilities for the two selected qubits from the data obtained in a real experiment with three qubits prepared in a quantum state.

In Ref. [10], a violation of the CHSH inequality (1) beyond Cirel'son's bound is presented for certain subensembles of two qubits of an ensemble of trios prepared in a Greenberger-Horne-Zeilinger (GHZ) state [11]. In this paper, we shall investigate whether a violation beyond Cirel'son's bound could be found for pairs of qubits selected from trios prepared in a *W* state [12].

The structure of the paper is as follows. In Sec. II, we will find that, for a certain choice of observables, the *W* state violates the CHSH inequality (1) beyond Cirel'son's bound. The observables used in Sec. II do not provide the maximum achievable violation using a *W* state. In Sec. III, we explain the reason behind this choice of observables. In addition, the violation in Sec. II is smaller that the one obtained in Ref. [10] using a GHZ state. However, in Sec. IV, we will see that there are some reasons that make the violation provided by the *W* state more interesting than that provided by the GHZ state. Finally, in Sec. V, we discuss how to obtain the required probabilities for the two selected qubits from the data obtained in a real experiment with three qubits.

II. THE *W* STATE VIOLATES THE CHSH INEQUALITY BEYOND CIREL'SON'S BOUND

Let us consider three distant qubits 1,2,3, prepared in the *W* state

$$|W\rangle=\frac{1}{\sqrt{3}}(|+--\rangle+|-+-\rangle+|--+\rangle), \quad (2)$$

where $\sigma_z|\pm\rangle=\pm|\pm\rangle$. For each three qubits prepared in the *W* state (2), we are going to concentrate our attention on *two* of them, namely, *those two in which, if we had measured* σ_z, *we would have obtained the result* -1. These two qubits will be called i and j hereafter, while the corresponding third qubit (the one in which, if we had measured σ_z, we would

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have found the result (1) will be called k. In quantum mechanics, the result of measuring σ_z is not predefined and therefore this prescription for choosing pairs is meaningless. However, the prescription makes sense in a local-realistic theory.

For reasons that will be explained in Sec. III, we are interested in the correlations when we choose $A=Z_i$, $a=X_i$, $B=Z_j$, and $b=X_j$, where Z_q and X_q are the spin of qubit q along the z and x directions, respectively. In addition, the particular CHSH inequality (1) we are interested in is the one in which $m=n=x_k$, where x_k is one of the possible results, -1 or 1 (although we do not know which one), of measuring X_k. With this choice we obtain the following CHSH inequality:

$$|C(Z_i,Z_j)-x_kC(Z_i,X_j)-x_kC(X_i,Z_j)-C(X_i,X_j)|\leqslant 2, \tag{3}$$

which holds for any local-realistic theory, regardless of the particular value, either -1 or 1, of x_k.

The next step is to use quantum mechanics to calculate the four correlations appearing in inequality (3) for the subensemble of two qubits i and j taken from three qubits prepared in the W state (2).

For the subensemble of two qubits i and j defined above,

$$C(Z_i,Z_j)=1, \tag{4}$$

because, for the W state (2),

$$P_{Z_1Z_2Z_3}(1,-1,-1)=\frac{1}{3}, \tag{5}$$

$$P_{Z_1Z_2Z_3}(-1,1,-1)=\frac{1}{3}, \tag{6}$$

$$P_{Z_1Z_2Z_3}(-1,-1,1)=\frac{1}{3}, \tag{7}$$

where $P_{Z_1Z_2Z_3}(1,-1,-1)$ means the probability of qubit 1 giving the result 1, and qubits 2 and 3 giving the result -1 when measuring σ_z on all three qubits.

By the definition of qubits i and j,

$$C(Z_i,X_j)=-x_k, \tag{8}$$

because, for the W state (2),

$$P_{Z_1X_2X_3}(-1,1,-1)+P_{Z_1X_2X_3}(-1,-1,1)=0, \tag{9}$$

$$P_{X_1Z_2X_3}(1,-1,-1)+P_{X_1Z_2X_3}(-1,-1,1)=0, \tag{10}$$

$$P_{X_1X_2Z_3}(1,-1,-1)+P_{X_1X_2Z_3}(-1,1,-1)=0. \tag{11}$$

Analogously, using Eqs. (9)–(11),

$$C(X_i,Z_j)=-x_k. \tag{12}$$

Finally, for the W state (2),

$$P_{X_1X_2X_3}(1,1,1)=\frac{3}{8}, \tag{13}$$

$$P_{X_1X_2X_3}(-1,-1,-1)=\frac{3}{8}, \tag{14}$$

$$P_{X_1X_2X_3}(1,1,-1)=\frac{1}{24}, \tag{15}$$

$$P_{X_1X_2X_3}(-1,-1,1)=\frac{1}{24}, \tag{16}$$

$$P_{X_1X_2X_3}(1,-1,1)=\frac{1}{24}, \tag{17}$$

$$P_{X_1X_2X_3}(-1,1,-1)=\frac{1}{24}, \tag{18}$$

$$P_{X_1X_2X_3}(-1,1,1)=\frac{1}{24}, \tag{19}$$

$$P_{X_1X_2X_3}(1,-1,-1)=\frac{1}{24}. \tag{20}$$

From Eqs. (13) and (14), the contribution of cases $x_1=x_2=x_3=1$ is cancelled by the contribution of cases $x_1=x_2=x_3=-1$; from Eqs. (15) and (16), the contribution of cases $x_1=x_2=-x_3=1$ is cancelled by the contribution of cases $x_1=x_2=-x_3=-1$, etc. Therefore, irrespective of whether i and j are qubits 1 and 2, or 1 and 3, or 2 and 3, we conclude that

$$C(X_i,X_j)=0. \tag{21}$$

Correlations (4), (8), (12), and (21) violate the CHSH inequality (3). The violation (3 vs 2) goes beyond Cirel'son's bound ($2\sqrt{2}$).

III. WHY *X* AND *Z*?

A particular type of local-realistic theories are those in which the only local experiments whose results are assumed to be predetermined are those which satisfy the criterion for "elements of reality" proposed by Einstein, Podolsky, and Rosen (EPR): *"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity"* [13].

As can be easily checked, the violation reported in Sec. II is not the maximal violation of the CHSH inequality (3) for two qubits in the W state (2). For instance, considering local spin observables on plane x-z and assuming $A=B$ and $a=b$, we find a maximum violation of 3.046 [by choosing $A=\cos(0.628)\sigma_x-\sin(0.628)\sigma_z$ and $a=\cos(1.154)\sigma_x+\sin(1.154)\sigma_z$]. Why then have we chosen $A=Z_i$, $a=X_i$, $B=Z_j$, and $b=X_j$? The reason is that these observables are not only local observables but, for the W state (2), they also satisfy EPR's criterion of elements of reality.

From Eqs. (5)–(7), it can be immediately seen that z_1, z_2, and z_3 are elements of reality, since any of them can be predicted with certainty from spacelike separated measurements of σ_z on the other two qubits. In addition, from Eqs. (9)–(11), it can easily be seen that, if $z_i=-1$ then, with certainty, $x_j=x_k$. Therefore, if $z_i=-1$, then by measuring x_j (x_k) one can predict x_k (x_j) with certainty. Therefore, if $z_i=-1$, then x_j and x_k are elements of reality. If $z_i=1$ then, using Eqs. (5)–(7), it can immediately be seen that $z_j=-1$. Therefore, following the previous reasoning, x_i and x_k are elements of reality (although x_i could have ceased to be an element of reality after measuring σ_z on particle i). In conclusion, for trios of qubits in the *W* state (2), z_1, z_2, z_3, x_1, x_2, and x_3 are EPR elements of reality and thus, according to EPR, they should have predefined values -1 or 1 before any measurement.

The violation of the CHSH inequality (3) presented in Sec. II is thus not only a proof of the impossibility of local hidden variables, but also proves a more powerful result: the apparently mild condition proposed by EPR is inconsistent with quantum mechanics.

IV. WHY *W*?

As was shown in Ref. [10], two qubits belonging to a three-qubit system in a GHZ state can provide a higher violation (4 vs 2, instead of 3 vs 2) of the CHSH inequality (3), even using observables that satisfy EPR's criterion of elements of reality. Why then use a *W* state?

One reason is because a test of the violation of Bell's inequalities beyond Cirel'son's bound could be achieved in practice in the near future. Sources of *W* states based on parametric down-converted photons are now available for real experiments [14] and some new proposals to prepare *W* states via cavity quantum electrodynamics have recently been presented [15].

Another reason is because this violation beyond Cirel'son's bound is, in one sense, surprising. The *W* state is the genuine three-qubit entangled state whose entanglement has the highest robustness against the loss of one qubit [12]. In particular, from a single copy of the reduced density matrix for any two qubits belonging to a three-qubit *W* state, one can always obtain by means of a filtering measurement a state that is arbitrarily close to a Bell state. Therefore, one might think that any two qubits belonging to a *W* state will not lead to a higher violation of the CHSH inequality (3) than that for two qubits in a Bell state, and thus it is of interest to realize that this is not the case.

There is, however, another subtler reason for preferring the *W* state instead of the GHZ state for a test of violation of Bell's inequalities beyond Cirel'son's bound. Any test of this kind requires a prescription for selecting a pair of qubits from each trio prepared in a quantum state. Such a prescription assumes local realism. In the violation of the CHSH inequality (3) presented in Sec. II, this prescription is simple: qubits *i* and *j* are those two in which, if we had measured σ_z, we would have obtained the result -1. However, in the violation of the CHSH inequality (3) using a GHZ state described in Ref. [10], the prescription is not so simple: there, qubits *i* and *j* are either those two in which, if we had measured σ_z, we would have obtained the result -1, or any two, if we had obtained the result 1 for all three qubits if we had measured σ_z. This means that, for the *W* state, any local observer could know whether or not his qubit belonged to the selected pair just by measuring σ_z, while for the GHZ state, the fact that whether or not a qubit belongs to the selected pair cannot be decided with certainty from a measurement on that qubit, but requires knowledge of the results of measurements on the other two qubits. From the perspective of local realism, for the *W* state, one of the elements of reality carried by each qubit determines whether or not it belongs to the selected pair; while for the GHZ state, this information is not local since it is distributed among distant elements of reality.

V. EXPERIMENTAL CH INEQUALITY

The result in Sec. II opens the possibility of using sources of three-qubit *W* states [14,15] to experimentally test the CHSH inequality. The main advantage of an experiment like this (or that proposed in Ref. [10]) is that it will admit a direct comparison with the dozens of previous experiments with two qubits [4–9] and thus goes beyond any previous experiments to test local realism using sources of three qubits [16,17] inspired by proofs of Bell's theorem without inequalities [11] or by Bell's inequalities for three qubits [18,19].

However, in any real experiment using three qubits, the experimental data consist of the number of simultaneous detections by three detectors $N_{ABC}(a,b,c)$ for various observables *A*, *B*, and *C*. This number is assumed to be proportional to the corresponding joint probability, $P_{ABC}(a,b,c)$. Therefore, in order to make inequality (3) useful for real experiments, it would be convenient to translate it into the language of joint probabilities.

Taking into account that

$$P_{Z_iZ_j}(-1,-1)=\frac{1}{4}[1-C(Z_i)-C(Z_j)+C(Z_i,Z_j)], \tag{22}$$

$$P_{Z_iX_j}(-1,-x_k)=\frac{1}{4}[1-C(Z_i)-x_kC(X_j)+x_kC(Z_i,X_j)], \tag{23}$$

$$P_{X_iZ_j}(-x_k,-1)=\frac{1}{4}[1-x_kC(X_i)-C(Z_j)+x_kC(X_i,Z_j)], \tag{24}$$

$$P_{X_iX_j}(x_k,x_k)=\frac{1}{4}[1+x_kC(X_i)+x_kC(X_j)+x_k^2C(X_i,X_j)], \tag{25}$$

where $C(Z_i)$ is the mean of the results of measuring σ_z on qubit *i*, and assuming physical locality [i.e., assuming that $C(Z_i)$ is independent of whether σ_z or σ_x is measured on qubit *j*, that is, assuming that the value of $C(Z_i)$ is the same in Eqs. (22) and (23), etc.], the CHSH inequality (3) between correlations can be transformed into a Clauser-Horne (CH) inequality [20] between joint probabilities,

$$-1 \leqslant P_{Z_iZ_j}(-1,-1) - P_{Z_iX_j}(-1,-x_k) - P_{X_iZ_j}(-x_k,-1) - P_{X_iX_j}(x_k,x_k) \leqslant 0. \tag{26}$$

As can be easily checked, the bounds l of the CHSH inequality (3) are transformed into the bounds $(l-2)/4$ of the corresponding CH inequality (26). Therefore, the local-realistic bound in the CH inequality (26) is 0 and Cirel'son's bound is $(\sqrt{2}-1)/2 \approx 0.207$.

For qubits i and j of a system in the W state (2),

$$P_{Z_iZ_j}(-1,-1) = 1, \tag{27}$$

$$P_{Z_iX_j}(-1,-x_k) = 0, \tag{28}$$

$$P_{X_iZ_j}(-x_k,-1) = 0, \tag{29}$$

$$P_{X_iX_j}(x_k,x_k) = \frac{3}{4}. \tag{30}$$

Therefore, probabilities (27)–(30) violate the CH inequality (26). Such a violation (0.25 vs 0) is beyond the corresponding Cirel'son's bound (0.207).

On the other hand, since we do not know which ones are qubits i and j, we cannot obtain the four joint probabilities (27)–(30) just by performing measurements on two qubits. Therefore, we must show how the joint probabilities of qubits i and j are related to the probabilities of the three qubits.

As can easily be seen from the definition of qubits i and j,

$$P_{Z_iZ_j}(-1,-1) = P_{Z_1Z_2Z_3}(1,-1,-1) + P_{Z_1Z_2Z_3}(-1,1,-1) + P_{Z_1Z_2Z_3}(-1,-1,1) + P_{Z_1Z_2Z_3}(-1,-1,-1). \tag{31}$$

Therefore, in order to experimentally obtain $P_{Z_iZ_j}(-1,-1)$, we must measure the four probabilities in the right-hand side of Eq. (31). In the W state (2), the first three probabilities in the right-hand side of Eq. (31) are expected to be 1/3 and the fourth is expected to be zero.

On the other hand, $P_{Z_iX_j}(-1,-x_k)$ and $P_{X_iZ_j}(-x_k,-1)$ are both less than or equal to

$$P_{Z_1X_2X_3}(-1,1,-1) + P_{Z_1X_2X_3}(-1,-1,1 + P_{X_1Z_2X_3}(1,-1,-1) + P_{X_1Z_2X_3}(-1,-1,1) + P_{X_1X_2Z_3}(1,-1,-1) + P_{X_1X_2Z_3}(-1,1,-1). \tag{32}$$

Therefore, in order to experimentally obtain $P_{Z_iX_j}(-1,-x_k)$ and $P_{X_iZ_j}(-x_k,-1)$, we must measure (using three different setups) all six probabilities in sum (32). In the W state (2), each of these six probabilities is expected to be zero.

Finally,

$$P_{X_iX_j}(x_k,x_k) = P_{X_1X_2X_3}(1,1,1) + P_{X_1X_2X_3}(-1,-1,-1). \tag{33}$$

Therefore, in order to experimentally obtain $P_{X_iX_j}(x_k,x_k)$, we must measure the two probabilities in the right-hand side of Eq. (33). In the W state (2), each of them is expected to be 3/8.

VI. CONCLUSIONS

Two qubits selected from a trio prepared in a W state violate the CHSH inequality, or the corresponding CH inequality, more than two qubits prepared in *any* quantum state. Such violations beyond Cirel'son's bound are smaller than those achieved by two qubits selected from a trio in a GHZ state [10]. However, for the W state the argument is simpler, since all local observers can know from their own measurements whether or not their qubits belong to the selected pair.

The importance of these arguments relies on the fact that they suggest how to use sources of three-qubit quantum entangled states to experimentally reveal violations of the familiar two-qubit Bell inequalities beyond those obtained using sources of two-qubit quantum states.

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