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Set-independence graphs of vector spaces and partial quasigroups

Research Article

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Abstract: As a generalization of independence graphs of vector spaces and groups, we introduce the notions of set-independence graphs of vector spaces and partial quasigroups. The former are characterized for finite-dimensional vector spaces over finite fields. Further, we prove that every finite simple graph is isomorphic to either the independence graph of a partial quasigroup or an induced subgraph of the latter. We also prove that isomorphic partial quasigroups give rise to isomorphic set-independence graphs. As an illustrative example, all finite graphs of order $n \leq 5$ are identified with the independence graph of a partial quasigroup of the same order.

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1. Introduction

Since the original manuscript of Cayley [5] concerning the graphical representation of groups by means of the so-called *Cayley graphs*, a wide amount of authors have dealt with the graphical representation of distinct types of algebraic structures so that known concepts and results of the latter may be translated to the language of graph theory. See in this regard the more recent proposals for magmas [4], groups [10], rings [7], finite fields [12], vector spaces [8] or algebras [3].

Of particular interest for the aim of this paper, it is remarkable the description of graphs whose adjacency relation derives from the independence among elements of a given algebraic structure. The relevance of (linear) independence as a general concept in abstract algebra was already put in emphasis

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by Marczewski [13, 14]. In order to translate this concept to the language of graph theory, Esmkhani [6] introduced the independence graph of a vector space $\mathbb{V}$ as the simple graph $\Gamma_{\mathbb{V}}$ having all its non-zero vectors as vertices, where two vertices are adjacent if and only if they are linearly independent. Some of its graph invariants were determined by Esmkhani himself and also by Lanong and Dutta [8]. More recently, Lucchini [10] has introduced the independence graph of a finite group G as the simple graph $\Gamma(G)$ whose vertices are the elements of G and such that two distinct vertices v and w are adjacent if and only if there exists a minimal generating set of G containing both elements v and w.

This paper delves into this topic by generalizing the notion of independence graphs of vector spaces and groups to set-independence graphs of vector spaces and partial quasigroups. It is organized as follows. In Section 2, we describe some preliminary concepts and results on graph theory and partial quasigroups that are used throughout the manuscript. Then, the concepts of set-independence graphs of vector spaces and partial quasigroups are respectively introduced and studied in Sections 3 and 4. Particularly, this type of graphs is completely characterized in case of dealing with finite-dimensional vector spaces over finite fields. Concerning partial quasigroups, it is proved that every finite simple graph is isomorphic to either the independence graph of a partial quasigroup or an induced subgraph of the latter. Moreover, it is shown that isomorphic partial quasigroups give rise to isomorphic set-independence graphs.

2. Preliminaries

We start with some notations on graph theory and some preliminary concepts and results on partial quasigroups that are used throughout the paper. See [2] for more details about these topics.

2.1. Graph theory

All the graphs in this paper are finite and simple. Let $G = (V(G), E(G))$ be a graph with set of vertices $V(G)$ and set of edges $E(G)$. A *spanning subgraph* $H \subseteq G$ is any subgraph of G such that $V(H) = V(G)$. If $V(H) \subset V(G)$, then the subgraph is *proper*, which we denote $H \subset G$.

The subset of vertices that are adjacent to a vertex $v \in V(G)$ is its *neighborhood* $N_G(v)$. Its cardinality is the *degree* $\deg_G(v)$ of the vertex v. The graph G is *r-regular* if all its vertices have equal degree r. It is (r_1, r_2)-regular if its vertices have degree either r_1 or r_2. Further, the *chromatic index* $\chi'(G)$ is the least number of colors that are required to color each edge of the graph G so that no two edges being incident on the same vertex share the same color.

An *independent set* of G is any set of vertices within G such that no two of them are adjacent. A graph is *complete m-partite* if its set of vertices can be partitioned into $m \geq 2$ disjoint independent sets so that every pair of vertices in distinct sets are adjacent. From here on, let $K_{n_1,\ldots,n_m}$ denote the complete m-partite graph whose i^{th} independent set is formed by n_i vertices, for all $i \leq m$. The *Turán graph* $T(n, m)$, with $m \geq 2$ and n two positive integers, is a complete m-partite graph of order n such that there are $(n \bmod m)$ independent sets containing each one of them $\lceil n/m \rceil$ vertices, and each one of the remaining independent sets contains $\lfloor n/m \rfloor$ vertices.

The following three results concerning the connectivity of the independence graphs described in the introductory section are known. The first one was established by Esmkhani [6] and the last two ones by Lucchini [10].

Theorem 2.1. *Let $\mathbb{V}$ be an n-dimensional vector space over the finite field $\mathbb{F}_q$, with q a prime power. Then, the independence graph $\Gamma_{\mathbb{V}}$ is edgeless if and only if $n = 1$. Otherwise, it is a connected graph of size $(q^n - 1)(q^n - q)/2$. It is complete if and only if $q = 2$.*

Lemma 2.2. *Let $\Gamma(G)$ be the independence graph of a finite group G. A vertex $v \in V(\Gamma(G))$ is isolated if and only if either it generates G or it belongs to the intersection of all the maximal subgroups of G.*

Theorem 2.3. *Let G be a finite group. Then, the subgraph $\Delta(G)$ of the independence graph $\Gamma(G)$ that is induced by the non-isolated vertices of the latter is connected.*

2.2. Partial quasigroups

A *partial quasigroup* $(Q, \cdot)$ (or simply Q if there is no risk of confusion) of order n is any finite set Q of n elements that is endowed with a partial product $\cdot$ so that, for each pair of elements $a, b \in Q$, there exists at most one element $x \in Q$ and at most one element $y \in Q$ such that $a \cdot x = b$ and $a \cdot y = b$. It is a *quasigroup* if both equations have exactly one solution. Every associative quasigroup constitutes indeed a group. Further, the multiplication table of a partial quasigroup of order n is a *partial Latin square* of the same order. That is, an $n \times n$ array with entries in Q so that each symbol occurs at most once per row and at most once per column. If no empty cell exists within this array, then it is a *Latin square*. It constitutes the multiplication table of a quasigroup.

Two partial quasigroups $(Q, \cdot)$ and $(Q', *)$ are *isomorphic* if there exists a bijection (called *isomorphism*) $f : Q \to Q'$ such that, for each pair of elements $a, b \in Q$ such that $a \cdot b \in Q$, it is $f(a) * f(b) = f(a \cdot b)$. Equivalently, both partial Latin squares describing Q and Q' coincide up to a same permutation of their rows, columns and symbols.

Let $(Q, \cdot)$ be a (partial) quasigroup and let $T \subseteq Q$. If $(T, \cdot)$ is also a (partial) quasigroup, then it is a *(partial) subquasigroup* of Q. It is *maximal* if no proper (partial) subquasigroup of Q contains T strictly. A (partial) quasigroup without proper (partial) subquasigroups is a *(partial) monoquasigroup*. The smallest (partial) subquasigroup of Q containing a subset $T \subseteq Q$ is the *(partial) subquasigroup generated by* T, which we denote $\langle T \rangle_Q$ (or simply $\langle T \rangle$ if there is no risk of confusion). If $\langle T \rangle_Q = Q$, then T is a *generating set* of Q. It is *minimal* if no proper subset of T is a generating set of Q. We term *non-generating subset* of Q to any of its subsets that does not constitute a generating set of Q. Finally, a partial quasigroup is *monogenic* if it is generated by a single element. Every partial monoquasigroup is monogenic, because each one of its single elements generates the partial quasigroup itself.

3. Set-independence graphs of vector spaces

In this section, we introduce the set-independence graph of a vector space $\mathbb{V}$ as a natural generalization of the notion described by Esmkhani [6]. To this end, let $\mathbb{V}^*$ denote the set formed by all the non-zero vectors within $\mathbb{V}$, and let $\text{Span}_{\mathbb{V}}(S)$ denote the linear span except for the zero vector of any subset $S \subseteq \mathbb{V}$. We introduce the S*-independence graph* of $\mathbb{V}$ as the simple graph $\Gamma_S(\mathbb{V})$ having $\mathbb{V}^*$ as its set of vertices, and such that two distinct vertices v and w are adjacent if and only if $\text{Span}_{\mathbb{V}}(S \cup \{v\}) \neq \text{Span}_{\mathbb{V}}(S \cup \{w\})$. If $S = \emptyset$, then it constitutes the independence graph introduced by Esmkhani. That is, $\Gamma_\emptyset(\mathbb{V}) = \Gamma_{\mathbb{V}}$. In addition, we call *set-independence graph* of $\mathbb{V}$ to any of its S-independence graphs, whatever the subset $S \subseteq \mathbb{V}^*$ is. We characterize this type of graphs for any n-dimensional vector space $\mathbb{V}$ over the finite field $\mathbb{F}_q$, with q a prime power. Particularly, we are interested in generalizing Theorem 2.1. To this end, if $m = \dim_{\mathbb{V}}(\text{Span}_{\mathbb{V}}(S))$, with $S \subseteq \mathbb{V}$, then we say that $\Gamma_S(\mathbb{V})$ is an (m, n, q)*-independence graph*. Our first result establishes the coincidence, up to isomorphism, of every (m, n, q)-independence graph.

Lemma 3.1. *Every pair of (m, n, q)-independence graphs $\Gamma_S(\mathbb{V})$ and $\Gamma_{S'}(\mathbb{V})$ are isomorphic. Moreover, if $\text{Span}_{\mathbb{V}}(S) = \text{Span}_{\mathbb{V}}(S')$, then $\Gamma_S(\mathbb{V}) = \Gamma_{S'}(\mathbb{V})$.*

Proof. Let $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ be two bases of the vector space $\mathbb{V}$ such that $\{e_1, \ldots, e_m\}$ and $\{e'_1, \ldots, e'_m\}$ are respective bases of the vector subspaces $\text{Span}_{\mathbb{V}}(S)$ and $\text{Span}_{\mathbb{V}}(S')$. Then, let f be the linear transformation on $\mathbb{V}$ that is linearly described from $f(e_i) = e'_i$, for all $i \leq n$. This map f constitutes a bijection on the set $\mathbb{V}^*$ such that, for each pair of vectors $v, w \in \mathbb{V}^*$, we have that $w \in \text{Span}_{\mathbb{V}}(S \cup \{v\})$ if and only if $f(w) \in \text{Span}_{\mathbb{V}}(S' \cup \{f(v)\})$. That is, f constitutes an isomorphism between both set-independence graphs $\Gamma_S(\mathbb{V})$ and $\Gamma_{S'}(\mathbb{V})$. In the particular case in which $\text{Span}_{\mathbb{V}}(S) = \text{Span}_{\mathbb{V}}(S')$, we have that $\text{Span}_{\mathbb{V}}(S \cup \{v\}) = \text{Span}_{\mathbb{V}}(S' \cup \{v\})$, for all $v \in \mathbb{V}$. So, the last assertion holds readily from the condition of adjacency in any set-independence graph. □

Lemma 3.1 enables us to focus on subsets of generators of the vector space $\mathbb{V}$. In this regard, let $\{e_1, \ldots, e_n\}$ be a basis of this vector space. Then, let $S_m := \{e_1, \ldots, e_m\}$, with $1 \leq m \leq n$, and $S_0 := \emptyset$.

From Lemma 3.1, every set-independence graph of $\mathbb{V}$ is isomorphic to a set-independence graph $\Gamma_{S_m}(\mathbb{V})$, with $0 \leq m \leq n$. It does not depend on the selected basis. In what follows, we see how the triple (m, n, q) identifies, up to isomorphism, any set-independence graph of a vector space. Thus, for instance, Figure 1 illustrates the (m, n, q)-independence graphs, for $0 \leq m \leq n$ and $(n, q) \in \{(2, 3), (3, 2)\}$. In this figure, and from now on, we label $l_1 \dots l_n$, with $l_i \in \mathbb{F}_q$, for all $i \leq n$, the vertex within the set-independence graph under consideration that is associated to the non-zero vector $\sum_{i=1}^{n} l_i e_i$. Figure 1 also illustrates the following results.

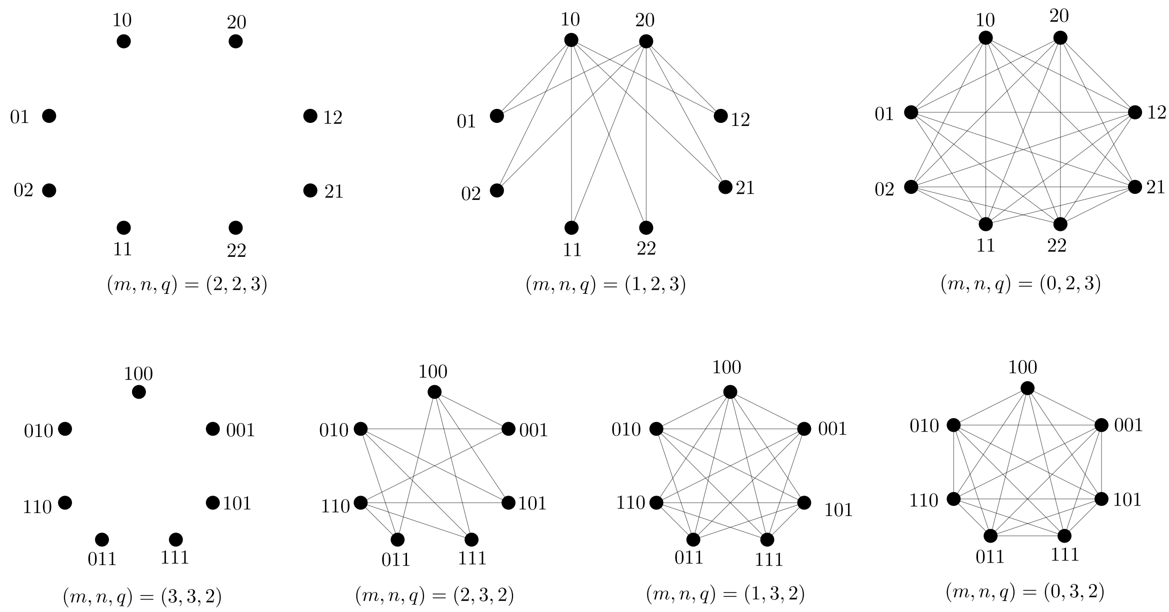

Figure 1. S_m**-independence graphs of** n**-dimensional vector spaces over** $\mathbb{F}_q$**.**

Proposition 3.2. *Every (m, n, q)-independence graph satisfies that:*

1. *It has order $q^n - 1$ and size $(q^n - q^m) \cdot \left(q^m - 1 + \frac{q^n - q^{m+1}}{2}\right)$. In particular, it is edgeless if $n \in \{1, m\}$.*
2. *It is complete if and only if $n > 1$ and $(m, q) = (0, 2)$.*
3. *It is $(q^n - q)$-regular, if $m = 0$, and $(q^n - q^m, q^m + q^n - q^{m+1} - 1)$-regular, if $1 < m < n$.*

Proof. From Lemma 3.1, it is enough to focus on $\Gamma_{S_m}(\mathbb{V})$. Its set of vertices is $\mathbb{V}^*$, which contains $q^n - 1$ non-zero vectors. In addition, if $v \in \mathbb{V}^*$, then

$$N_{\Gamma_{S_m}(\mathbb{V})}(v) = \begin{cases} \mathbb{V}^* \setminus \mathrm{Span}_{\mathbb{V}}(S_m), & \text{if } v \in \mathrm{Span}_{\mathbb{V}}(S_m), \\ \mathrm{Span}_{\mathbb{V}}(S_m) \cup (\mathbb{V}^* \setminus \mathrm{Span}_{\mathbb{V}}(S_m \cup \{v\})), & \text{otherwise.} \end{cases}$$

Hence, the size and regularity of $\Gamma_{S_m}(\mathbb{V})$ hold readily, because

$$\deg_{\Gamma_{S_m}(\mathbb{V})}(v) = \begin{cases} q^n - q^m, & \text{if } v \in \mathrm{Span}_{\mathbb{V}}(S_m), \\ q^m + q^n - q^{m+1} - 1, & \text{otherwise.} \end{cases}$$

It has size $\binom{q^n - 1}{2}$ (and hence, it is complete) if and only if $n > 1$ and $(m, q) = (0, 2)$. □

Proposition 3.3. *Let $n > 1$ be a positive integer. Then,* $\Gamma_{S_n}(\mathbb{V}) \subset \Gamma_{S_{n-1}}(\mathbb{V}) \subset \ldots \subset \Gamma_{S_1}(\mathbb{V}) \subset \Gamma_{\mathbb{V}}$. *Moreover,*

1. $\Gamma_{S_n}(\mathbb{V})$ *is edgeless;*
2. $\Gamma_{S_m}(\mathbb{V})$ *is isomorphic to the complete* $\frac{q^{n-m}+q-2}{q-1}$*-partite graph* $K_{q^m-1,\, q^{m+1}-q^m,\ldots,\, q^{m+1}-q^m}$*, for every positive integer $m < n$; and*
3. $\Gamma_{\mathbb{V}}$ *is isomorphic to the Turán graph* $T\left(q^n-1, \frac{q^n-1}{q-1}\right)$.

Proof. From Proposition 3.2, all the graphs have equal order and $\Gamma_{S_n}(\mathbb{V})$ is edgeless. In addition, for each non-negative integer $m < n$, $\Gamma_{S_{m+1}}(\mathbb{V})$ is the proper spanning subgraph of $\Gamma_{S_m}(\mathbb{V})$ that results after removing those edges $vw \in E(\Gamma_{S_m}(\mathbb{V}))$ such that $\mathrm{Span}_{\mathbb{V}}(S_{m+1} \cup \{v\}) = \mathrm{Span}_{\mathbb{V}}(S_{m+1} \cup \{w\})$. Further, the neighborhoods described in the proof of Proposition 3.2 enable us to partition the set of vertices $\mathbb{V}^*$ into $\frac{q^{n-m}+q-2}{q-1}$ disjoint parts. The first one coincides with $\mathrm{Span}_{\mathbb{V}}(S_m)$. The remaining ones are defined so that two non-zero vectors $v, w \in \mathbb{V}^* \setminus \mathrm{Span}_{\mathbb{V}}(S_m)$ belong to the same part if and only if $w \in \mathrm{Span}_{\mathbb{V}}(S_m \cup \{v\})$. As such, there are $1 + \frac{q^n-q^m}{q^{m+1}-q^m} = \frac{q^{n-m}+q-2}{q-1}$ disjoint parts. Assertion (b) holds readily because every pair of vertices belonging to two distinct parts of the just described partition are adjacent. Finally, every non-zero vector $v \in \mathbb{V}^*$ satisfies that $N_{\Gamma_{\mathbb{V}}}(v) = \mathbb{V}^* \setminus \mathrm{Span}_{\mathbb{V}}(v) = N_{\Gamma_{\mathbb{V}}}(w)$, for all $w \in V^* \cap \mathrm{Span}_{\mathbb{V}}(v)$. So, since $|\mathbb{V}^* \cap \mathrm{Span}_{\mathbb{V}}(v)| = q-1$, for all $v \in \mathbb{V}^*$, the independence graph $\Gamma_{\mathbb{V}}$ is a complete $\frac{q^n-1}{q-1}$-partite graph, with each part containing $q-1$ vertices. It is isomorphic to the Turán graph $T\left(q^n-1, \frac{q^n-1}{q-1}\right)$. □

The following result follows readily from Lemma 3.1 and Proposition 3.3. (The case $n = n' = 1$ is trivial.)

Theorem 3.4. *An (m, n, q)-independence graph and an (m', n', q')-independence graph are isomorphic if and only if either $n = n' = 1$ or $(m, n, q) = (m', n', q')$.*

4. Set-independence graphs of partial quasigroups

A remarkable characteristic of every set-independence graph of a finite-dimensional vector space over a finite field is its regularity. It arises from the fact that all the vectors within a vector space play the same role with respect to the notion of linear independence. As Lucchini has recently revealed [10], it is not so in case of dealing with independence graphs of finite groups. In this section, Lucchini's proposal is generalized in a natural way by introducing the notion of set-independence graph of a partial quasigroup. Regularity is not going to be either a main property of this new family of graphs.

Let Q be a partial quasigroup of order n and let $S \subseteq Q$. We introduce the S-independence graph of this partial quasigroup as the simple graph $\Gamma_S(Q)$ having the elements of Q as vertices, and such that two distinct vertices v and w are adjacent if and only if there exists a minimal generating set of Q containing the set $S \cup \{v, w\}$. In addition, we denote $\Delta_S(Q)$ the subgraph of $\Gamma_S(Q)$ that is induced by its non-isolated vertices. Moreover, a *set-independence graph* of Q is defined as any of its S-independence graphs, whatever the subset $S \subseteq Q$ is. Finally, we term the graph $\Gamma_\emptyset(Q)$ the *independence graph* of the partial quasigroup Q. In case of dealing with a group, this graph coincides with Lucchini's independence graph. That is, $\Gamma_\emptyset(Q) = \Gamma(Q)$, whenever Q is a group. Due to it, we denote respectively by $\Gamma(Q)$ and $\Delta(Q)$ the independence graph of any partial quasigroup Q and the subgraph induced by its non-isolated vertices. In what follows, we prove that, for every finite simple graph G, there exists a partial quasigroup Q such that either G is isomorphic to $\Gamma(Q)$, or the subgraph of G induced by its non-isolated vertices is isomorphic to $\Delta(Q)$. A preliminary lemma is necessary to this end.

Lemma 4.1. *The following assertions hold for every partial quasigroup Q.*

1. *A vertex $v \in V(\Gamma(Q))$ is isolated if and only if either $\langle\{v\}\rangle = Q$ or v is within the intersection of all the maximal partial subquasigroups of Q.*
2. *Let S and S' be two subsets of Q. Then:*
   (a) *If $S \subseteq S'$, then $\Gamma_{S'}(Q) \subseteq \Gamma_S(Q)$. As a consequence, every set-independence graph of Q is a spanning subgraph of $\Gamma(Q)$.*
   (b) *$\Gamma_{S\cup S'}(Q) \subseteq \Gamma_S(Q) \cap \Gamma_{S'}(Q)$.*

Proof. The first statement follows similarly to the proof of Lemma 2.2 (see [10, Lemma 4]). So, let us focus on the second statement. If S and S' are two subsets of Q such that $S \subseteq S'$, then, for each edge $vw \in E(\Gamma_{S'}(Q))$, there exists a minimal generating set of the partial quasigroup Q containing the set $S' \cup \{v, w\}$. Since it also contains the set $S \cup \{v, w\}$, the set-independence graph $\Gamma_{S'}(Q)$ is a spanning subgraph of $\Gamma_S(Q)$. The consequence in (a), as well as assertion (b), follow readily. □

Lemma 4.1 enables us to describe a family of partial quasigroups whose set-independence graphs are all of them edgeless. The following constructive example illustrates this fact.

Example 4.2. *Let $Q = \{1, \dots, n\}$ and let $(Q, \cdot)$ be the partial quasigroup defined so that $v \cdot v = v + 1$, for all $v \in Q \setminus \{n\}$, and $n \cdot n = 1$. It is so that the partial quasigroup Q is generated by any of its single elements. The first assertion of Lemma 4.1 implies, therefore, that the independence graph of Q is edgeless. As a consequence, the third assertion of the same lemma, implies that every set-independence graph of Q is edgeless.* ◁

Now, we prove constructively that every finite simple graph of positive size is isomorphic to an induced subgraph of the independence graph of a partial quasigroup.

Proposition 4.3. *Let G be a simple graph of order n, with $m < n - 1$ isolated vertices. Then, the following statements hold.*

1. *If $m \geq \chi'(G)$, then there exists a partial quasigroup Q of order n such that G is isomorphic to the independence graph $\Gamma(Q)$.*
2. *Otherwise, if $m < \chi'(G)$, then there exists a partial quasigroup Q of order $n + \chi'(G) - m$ such that the subgraph of G induced by its non-isolated vertices is isomorphic to the induced subgraph $\Delta(Q)$ of the independence graph $\Gamma(Q)$.*

Proof. Firstly, we enumerate the vertices in G from $1 + \rho$ to $n + \rho$, where $\rho = \max\{0, \chi'(G) - m\}$, so that the first m vertices are the isolated ones. Let $Q = \{1, \dots, n + \rho\}$ and let $c : E(G) \to \{1, \dots, \chi'(G)\}$ be an edge coloring of G. (That is, no two edges being incident on the same vertex share image.) Then, let $(Q, \cdot)$ be the partial quasigroup that is defined so that, for every pair of positive integers $v, w \leq n$,

$$\begin{cases} 1 \cdot v = v + 1, & \text{if } v < n, \\ v \cdot v = 1, & \text{if } 1 < v \leq m + \rho, \\ v \cdot w = c(vw), & \text{if } m + \rho < v < w \text{ and } vw \in E(G). \end{cases}$$

It is well-defined because the map c is an edge coloring. In particular, the partial quasigroup Q is generated by each one of its single first $m + \rho$ elements. Thus, the first assertion of Lemma 4.1 implies that the vertices 1 to $m + \rho$ are all of them isolated in the incidence graph $\Gamma(Q)$. In addition, every edge $vw \in E(G)$ is such that $\{v, w\}$ constitutes itself a minimal generating set of Q. So, by definition, $vw \in \Gamma(Q)$. The result holds because there does not exist any other minimal generating set of Q. □

In order to illustrate the constructive proof of Proposition 4.3, let us consider the pair of simple graphs in Figure 2, whose vertices and edges are already labeled according to the mentioned proof.

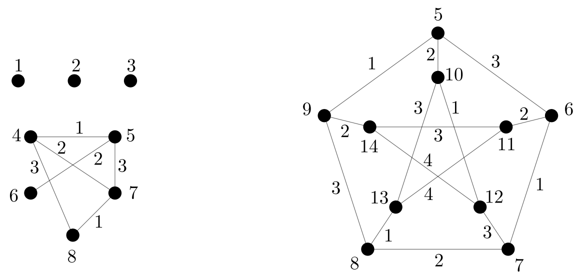

Figure 2. A pair of simple graphs illustrating Proposition 4.3.

It is so that the left graph coincides with the incidence graph of the partial quasigroup of order eight with multiplication table

2	3	4	5	6	7	8	
	1						
		1					
				1		2	3
					2	3	
							1

The right graph is the Petersen graph, whose chromatic index is four. The constructive proof of Proposition 4.3 shows that this graph is isomorphic to the induced graph $\Delta(Q)$ of the partial quasigroup Q of order 14 having multiplication table

2	3	4	5	6	7	8	9	10	11	12	13	14	
	1												
		1											
			1										
					3			1	2				
						1				2			
							2				3		
								3				1	
													2
											1	3	
												4	3
													4

Proposition 4.3, together with Example 4.2, enables us to ensure our previously announced result.

Theorem 4.4. *Let G be a finite simple graph. Then, there exists a partial quasigroup Q such that either G is isomorphic to $\Gamma(Q)$, or the subgraph of G induced by its non-isolated vertices is isomorphic to $\Delta(Q)$.*

Properties of independence graphs of non-associative quasigroups may differ from those ones described by Lucchini for groups. To illustrate it, a preliminary lemma is required.

Lemma 4.5. *The following assertions hold for every partial quasigroup Q.*

1. *Every edge of $\Gamma(Q)$ containing a vertex $v \in Q$ is an edge of the set-independence graph $\Gamma_{\{v\}}(Q)$.*
2. *If S is a non-empty subset of Q, then the graph $\Delta_S(Q)$ is connected. More specifically, the following two assertions hold.*

(a) If S is not a minimal generating set of the partial subquasigroup $\langle S\rangle$, then the set-independence graph $\Gamma_S(Q)$ is edgeless.

(b) If S is a minimal generating set of the partial subquasigroup $\langle S\rangle$, then $\Delta_S(Q)$ is the subgraph of $\Gamma(Q)$ that is induced by the vertices of all the minimal generating sets of Q containing S. If T is one of these sets, then it induces a $|T|$-clique within $\Gamma_S(Q)$.

Particularly, if S is a minimal generating set of Q, then $vw \in E(\Gamma_S(Q))$ if and only if $v, w \in S$.

Proof. Let $vw \in E(\Gamma(Q))$. By definition, there exists a minimal generating set of Q containing the set $\{v, w\}$. But it also implies that $vw \in E(\Gamma_{\{v\}}(Q))$. Thus, the first statement holds. Now, in order to prove the second statement, it is enough to prove both assertions (a) and (b). The first one holds straightforwardly from the adjacency of $\Gamma_S(Q)$. So, let us suppose that S is a minimal generating set of $\langle S\rangle$. If there does not exist any generating set of Q containing S, then $\Delta_S(Q)$ is the null graph and hence, it is connected. Otherwise, the adjacency of $\Gamma_S(Q)$ implies that every minimal generating set T of Q containing S induces a $|T|$-clique within $\Gamma_S(Q)$. In particular, every non-isolated vertex in $\Gamma_S(Q)$ belongs to S or it is adjacent to all the vertices of S. □

Let us illustrate Lemma 4.1 for a non-associative quasigroup.

Example 4.6. *Let Q be the non-associative quasigroup of multiplication table*

2	3	4	5	1
4	2	5	1	3
5	1	3	2	4
3	5	1	4	2
1	4	2	3	5

Let $v \in Q$. Since $\langle\{v\}\rangle = Q$ if and only if $v = 1$, and Q has no proper maximal subquasigroups, Lemma 4.1 imply that the vertex 1 is isolated in every set-independence graph of Q. Moreover, it is the only isolated vertex in $\Gamma(Q)$. So, unlike groups of prime power order (which are edgeless because of Lemma 2.2 and the fact that each one of its single elements generates the group under consideration), the independence graph of a quasigroup of prime power order can have positive size.

Figure 3 shows the set-independence graphs of Q having more than one edge. This figure illustrates in particular the first statement of Lemma 4.5. Notice also from this figure and the second statement of that lemma that $E(\Gamma_{\{v,w\}}) = \{vw\}$, for every pair of distinct vertices $v, w \in Q \setminus \{1\}$. Any other set-independence graph of this quasigroup is edgeless.

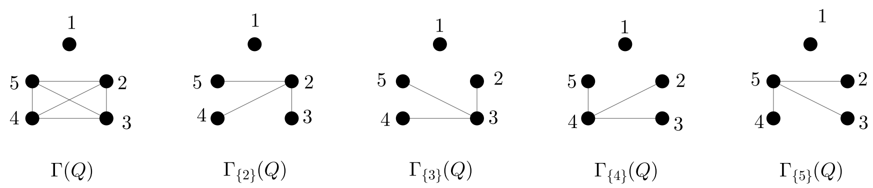

Figure 3. Set-independence graphs of the quasigroup described in Example 4.6.

Every set-independence graph $\Gamma_{\{v\}}(Q)$ in Example 4.6 coincides with the star (maybe edgeless) formed by all the edges within $\Gamma(Q)$ that are incident with v. The following result shows that this property only happens for certain partial quasigroups. Here, and from now on, let $d(Q)$ and $m(Q)$ respectively denote the smallest and largest sizes of any minimal generating set of a partial quasigroup

Q. Thus, for instance, the quasigroup Q described in Example 4.6 satisfies that $(d(Q), m(Q)) = (1, 2)$. Particularly, $d(Q) = 1$ if and only if Q is monogenic.

Proposition 4.7. *Let Q be a partial quasigroup of order n. Then, the following assertions hold.*

1. *$d(Q) = m(Q) = 1$ if and only if every set-independence graph of Q is edgeless.*
2. *$d(Q) = m(Q) = n$ if and only if every set-independence graph of Q is complete.*
3. *If $m(Q) = 2$, then every set-independence graph $\Gamma_{\{v\}}(Q)$ is the star (maybe edgeless) that is formed by all the edges within $\Gamma(Q)$ that are incident with the vertex v. In such a case, $E(\Gamma_{\{v,w\}}) = \{vw\}$, for every pair of distinct vertices $v, w \in Q$ such that $\langle \{v\} \rangle \neq Q \neq \langle \{w\} \rangle$. Any other set-independence graph of Q is edgeless.*

Proof. The first two statements follow readily from the definition of set-independence graph. Further, the first assertion of the third statement holds from the same definition, together with the first statement of Lemma 4.5. The last consequence follows readily from the second statement of that lemma. □

Proposition 4.7 allows to characterize the set-independence graphs of every partial quasigroup of order two. (Notice that the case $n = 1$ is trivial.) A previous lemma is required to this end.

Lemma 4.8. *Let Q be a partial quasigroup of order n. Then, $m(Q) = n$ if and only if $d(Q) = n$.*

Proof. By definition, $d(Q) \leq m(Q)$. If $d(Q) < m(Q) = n$, then there exists a minimal generating set of Q that is formed by $d(Q) < n$ distinct elements. But then, the partial quasigroup Q would not be a minimal generating set of itself, which contradicts the fact that $m(Q) = n$. □

Proposition 4.9. *Every set-independence graph of a partial quasigroup Q of order two is either edgeless, whenever Q is monogenic, or isomorphic to the complete graph K_1, otherwise.*

Proof. From Lemma 4.8, it must be $d(Q) = m(Q)$. Then, the result holds readily from Proposition 4.7 once it is noticed that $d(Q) = m(Q) = 1$, if the partial quasigroup Q is monogenic, and $d(Q) = m(Q) = 2$, otherwise. □

Example 4.10. *The following first two partial Latin squares are multiplication tables of monogenic partial quasigroups. From Proposition 4.9, all their set-independence graphs are edgeless. (The first one is a monoquasigroup, but it is not so the second one.) The last two partial Latin squares correspond to non-monogenic quasigroups. Their set-independence graphs is the complete graph K_1.*

2	
	1

2	1
1	

1	
	2

	1

◁

For a partial quasigroup Q of order three, Lemma 4.8 implies that $(d(Q), m(Q)) \in \{(1, 1), (1, 2), (2, 2), (3, 3)\}$. So, Proposition 4.7 characterizes all its set-independence graphs, except for the independence graph corresponding to $m(Q) = 2$. It is described in the following result.

Proposition 4.11. *Let $(Q, \cdot)$ be a partial quasigroup of order three such that $m(Q) = 2$. If $d(Q) = 1$, then the independence graph $\Gamma(Q)$ is formed by an isolated vertex and an edge connecting the other two vertices. Otherwise, if $d(Q) = 2$, then $vw \in E(\Gamma(Q))$ if and only if $\{v, w\}$ is a generating set of Q.*

Proof. Since $m(Q) = 2$, there is a minimal generating set $\{u_1, u_2\}$ of the partial quasigroup Q, with $u_1 \neq u_2$. Then, by definition, $u_1u_2 \in E(\Gamma(Q))$. Let $u_0 \in Q \setminus \{u_1, u_2\}$. If $\langle \{u_0\} \rangle = Q$, then $d(Q) = 1$ and thus, the first statement of Lemma 4.1 implies that u_0 is an isolated vertex of $\Gamma(Q)$. Otherwise, if $\langle \{u_0\} \rangle \neq Q$, then $d(Q) = m(Q) = 2$. Thus, every minimal generating set of Q is formed by two distinct elements, and the result holds. □

Figure 4 shows that every graph of order three is the independence graph of a partial quasigroup of the same order. For each graph, we show the possible pairs of parameters $(d(Q), m(Q))$ for which one such a partial quasigroup Q exists.

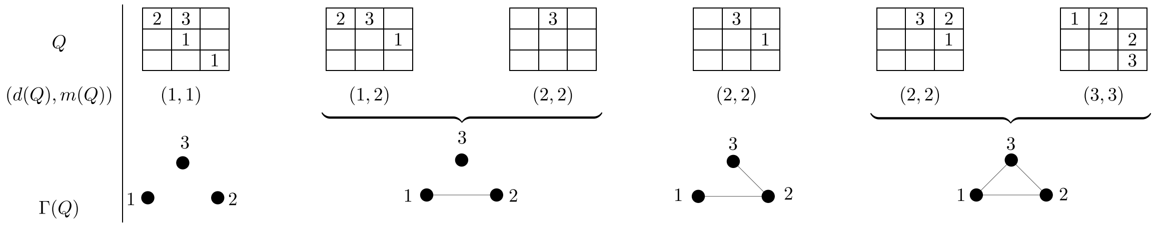

Figure 4. **Independence graphs of partial quasigroups of order three.**

Further, in order to make easier the study of set-independence graphs of partial quasigroups, the following lemma enables us to focus on representatives of their isomorphism classes.

Lemma 4.12. *Let Q and Q' be two isomorphic partial quasigroups, and let $S \subseteq Q$. If f is an isomorphism between Q and Q', then the set-independence graphs $\Gamma_S(Q)$ and $\Gamma_{f(S)}(Q')$ are isomorphic.*

Proof. It follows readily from the fact that the map f constitutes itself an isomorphism between the graphs $\Gamma_S(Q)$ and $\Gamma_{f(S)}(Q')$. Notice to this end that this map f preserves the generating sets of both graphs $\Gamma_S(Q)$ and $\Gamma_{f(S)}(Q')$, and $f(S \cup \{v, w\}) = f(S) \cup \{f(v), f(w)\}$, for all $v, w \in Q$. □

Figure 5 illustrates the independence graphs of the five isomorphism classes of quasigroups of order three. Only the first representative is associative, and hence, a group. No quasigroup of order three is related to an independence graph of size one or two. However, from Lemma 4.5, every graph of order three is isomorphic to a set-independence graph of the fifth quasigroup in Figure 5.

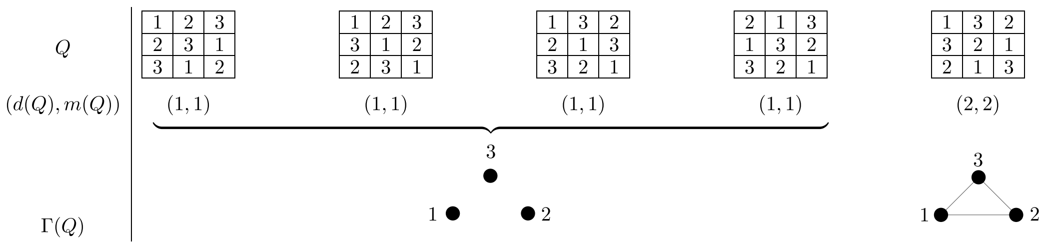

Figure 5. **Independence graphs of quasigroups of order three.**

We finish our study by showing that every graph of order $n \in \{4, 5\}$ is the independence graph of a partial quasigroup of the same order. It is illustrated in Figures 6 and 7. Notice that, except for the edgeless graph, all the graphs are associated to a partial quasigroup Q sucht that $m(Q) = 2$. In this way, Proposition 4.7 describes all their associated set-independence graphs.

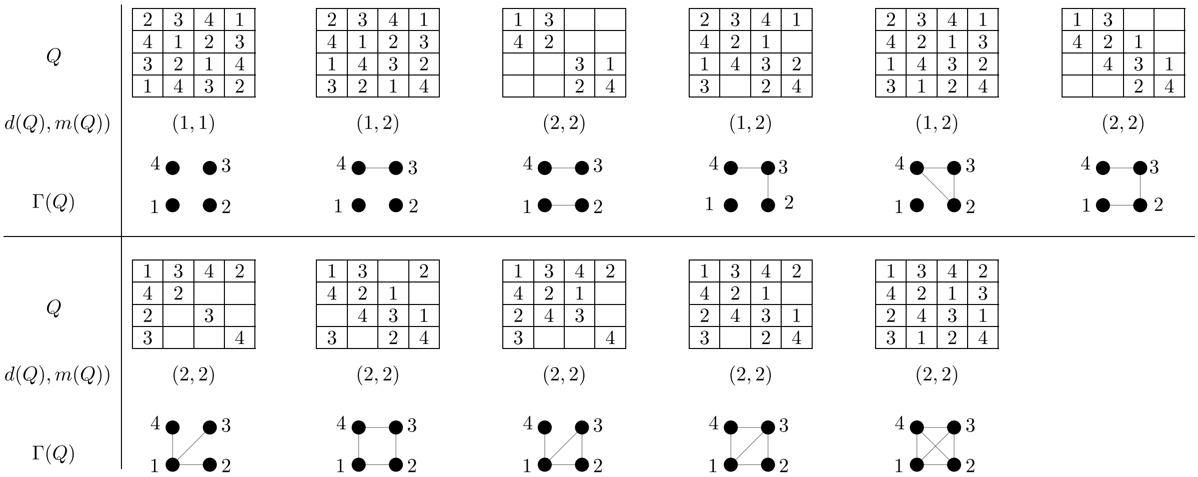

Figure 6. Independence graphs of quasigroups of order four.

5. Conclusion and further work

In this paper, we have generalized the concept of independence graphs of vector spaces and groups to set-independence graphs of vector spaces and partial quasigroups. In particular, we have characterized all the set-independence graphs of vector spaces over finite fields, which constitute all of them regular spanning subgraphs of Turán graphs.

Unlike the regularity of the latter, set-independence graphs of partial quasigroups arise as a promising approach for relating both types of combinatorial structures. It is so that Lemma 4.12 has shown that isomorphic partial quasigroups give rise to isomorphic set-independence graphs.

We also propose some open questions for further work on this topic. Our first two questions refer to the minimal and maximal sizes of any minimal generating set of a partial quasigroup. They generalize the open problems that were respectively introduced by Lucchini [9] (see also [11] and the references therein) and Apisa and Klopsch [1].

Problem 5.1. *Obtain the largest size of any minimal generating set of a partial quasigroup.*

Problem 5.2. *Let c be a non-negative integer. Characterize all partial quasigroups Q such that $m(Q) - d(Q) \leq c$.*

Our third question deals with the existence of partial quasigroups having associated at least one set-independence graph isomorphic to a given graph.

Problem 5.3. *Let G be a simple graph of order n. Does there exist a partial quasigroup Q of the same order n and a subset $S \subseteq Q$ such that the set-independence graph $\Gamma_S(Q)$ is isomorphic to G? In case of an affirmative answer, which are the lowest and largest sizes of Q and S for which one such a partial quasigroup exists? Moreover, which are these values in case of fixing either Q or S?*

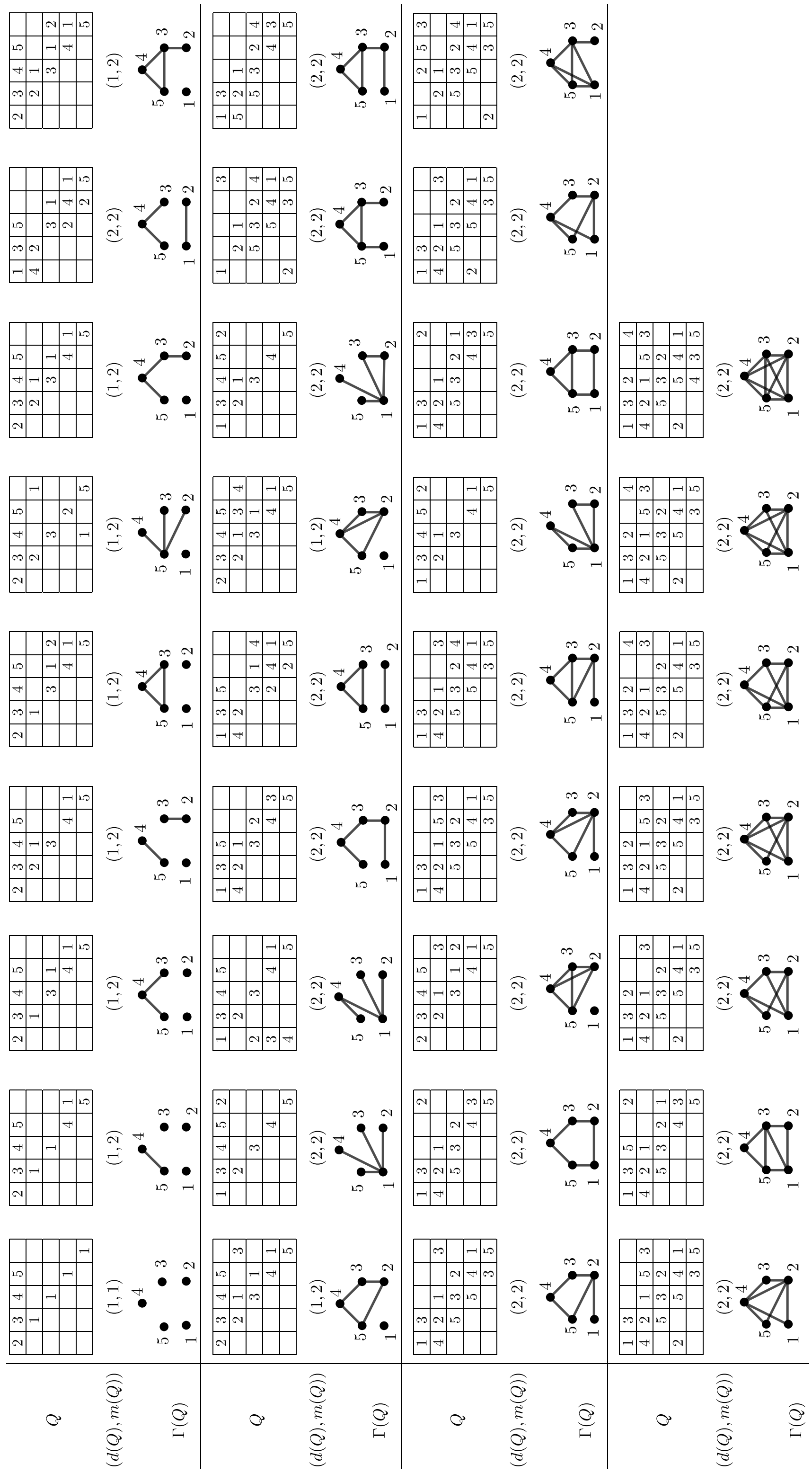

Figure 7. Independence graphs of quasigroups of order five.

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