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Interfaces with Other Disciplines

Design of water quality policies based on proportionality in multi-issue problems with crossed claims

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ABSTRACT

Water pollutants can be classified into three categories, each of which includes several classifications of substances. In this paper, we present a methodology based on bankruptcy models to determine the emission limits of polluting substances belonging to more than one category. We model the problem as a multi-issue allocation problem with crossed claims and introduce the constrained proportional awards rule to obtain the emission limits. This rule is based on the concept of proportionality and extends the proportional rule for bankruptcy problems. We also provide an axiomatic characterization of this rule. Moreover, this allocation rule is illustrated by means of a numerical example based on real-world data. Finally, managerial and policy implications of this approach for water pollution control are given.

1. Introduction

Water is necessary to the survival of all life forms, which is why it is referenced in almost all reports prepared by international institutions and specialized organizations such as the United Nations (UN), the World Health Organization (WHO) and the Food and Agriculture Organization (FAO). Water is an essential good for economic development, health and the environment. Three primary challenges related to water include fresh water accessibility, fresh water management and water pollution. The solutions to these problems are directly or indirectly included in many of the *Sustainable Development Goals* (SDGs) promoted by the UN.[1] This paper addresses the problem of water pollution, and in particular, the design of water quality policies as a means of water pollution control.

We would like to emphasize the importance of water pollution control. In fact, water pollution presents a serious threat to human health, survival of ecosystems and the biodiversity of the planet. The contamination of freshwater causes numerous diseases, and reduces the availability of an already scarce resource that is essential for both agriculture and human consumption. Therefore, proper management of water pollution control in a certain region is imperative for the survival of the region and the development of its economic activity (Goel, 2009; Helmer & Hespanhol, 1997).

Goel (2009) defines a water pollutant as follows: "*A water pollutant can be defined as a physical, chemical or biological factor causing aesthetic or detrimental effects on aquatic life and on those who consume the water*". Nesaratnam (2014) divides water pollutants into several categories: benzenoids, oxygen-demanding wastes, and eutrophic nutrients. These water pollutants come from different sources, generally, resulting from human activity and having different effects on water quality.

Toxic benzenoids such as *benzene*, *ethylbenzene*, *toluene*, and *xylenes*, including *phenols*, are poisonous to any living organism and are able to cause serious disease in humans. These aromatic hydrocarbons have a low boiling point and are abundant in petroleum representing its most dangerous fraction. Furthermore, hydrocarbons, once incorporated into a given organism, are very stable, being able to pass through many members of the food chain without being altered. They are therefore transferred through the entier food chain, a situation analogous to that of heavy metals and pesticides (see, Tomlinson, 1971 for details about benzenoid compounds). Moreover, these substances, particularly phenols, can also negatively affect the presence of dissolved oxygen (DO) in water, which is essential to the development of the life of animals and

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[1] https://www.un.org/sustainabledevelopment/.

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plants. A body of water is classified as contaminated when the DO concentration falls below the level necessary to maintain a normal biota for such water. The main cause of deoxygenation of water is the presence of oxygen-demanding waste substances. These are compounds that are easily degraded or decomposed due to bacterial activity in the presence of oxygen. Regarding pollutants that are oxygen-demanding wastes, we find the already mentioned *phenols*, *ammoniacal compounds*, *oxidizable inorganic substances* (OIS), and overall *biodegradable organic compounds* (BOC) (Riffat, 2013).

Related to the last group of water pollutants, eutrophication is the process by which a body of water becomes excessively enriched with nutrients that in turn induce excessive growth of aquatic plants and algae. The most evident effect of eutrophication is the creation of dense blooms of noxious and smelly phytoplankton that reduce water clarity and damage water quality. However, there are other more dangerous effects impacting life in aquatic ecosystems. Apart from natural causes, eutrophication is caused by the actions of humans, resulting from the discharge of detergents, fertilizers or wastewaters containing nitrates or phosphates in an aquatic system. Particularly relevant are nitrogen eutrophics such as *nitrates* and *ammoniacal compounds* (see, for instance, Ansari et al., 2011 for details about the eutrophication problem).

Currently, there is a certain concern about other contaminants present in the water of which little is known, these contaminants are called emerging contaminants. Among these, many products can be found, such as pharmaceuticals and personal care products, nanomaterials, fire retardants, pesticides, plasticizers, surfactants, disinfection byproducts, antibiotic-resistant bacteria, microplastics, and genes (see, for instance, Geissen et al., 2015).

The European Union (EU) has promoted and implemented different environmental policies to protect water quality. Thus, EU directives have specified emission limit values for water and set standards on how to monitor, report, and manage water quality (see, for instance, European Parliament & the Council of the European Union, 2000; European Parliament & the Council of the European Union, 2006; European Parliament & the Council of the European Union, 2008; European Parliament & the Council of the European Union, 2013). Steinebach (2019) analyses the effectiveness of EU policies in the quality of the national water resources of the member states over a period of 23 years (1990–2012). In the case of Spain, there is also a legal body to control different aspects of water management, including water quality (Royal Decree, 9/2008). All previous legal frameworks direct the responsibility for the management and control of water to the regional and local authorities closest to the water resource. Consequently, on the one hand, the legislation establishes limits (of immission) on the parameters indicating contamination in water bodies, whose values vary according to the use of those bodies (bathing, purification, etc.). On the other hand, the discharges are those that are carried out directly or indirectly to the body waters, whatever their nature. Finally, local and regional administrations can legislate the maximum concentrations (of emission) of pollutants in discharges. Therefore, it is of great social and economic interest to generate tools that help local and regional authorities to design policies to control water quality.

The most common in the literature is to find regulations that limit the discharge of certain substances but do not consider their common and uncommon effects. Therefore, it is possible that several substances whose effects are the same are discharged into the water within their limits but produce very high concentrations of substances that cause an undesired effect in the water. We also find in the literature publications related to total discharge limits but without distinguishing by their effects on water. Therefore, in the literature, we find that either the discharges of a certain substance or the total discharges may be limited, however, we have not found that the problem of limiting the discharge of substances as a whole is addressed. Differentiating by substances and considering their multiple effects on water quality is important. Thus, in this paper, we exam the situation in which a certain authority responsible for the water quality of the region is interested in controlling water pollution. They may be particularly interested in limiting the concentration of the three categories of water pollutants mentioned above. On the one hand, for each of these categories of pollutants, certain levels of concentration are fixed to maintain a reasonable degree of water. On the other hand, the substances mentioned above are monitored and there are maximum concentration limits for them. The relationship between the categories of pollutants and the substances in each category is shown in Fig. 1. Thus, the problem to be solved by the authority is how to allocate new thresholds to the substances taking into account the limits fixed for each category of pollutants. Therefore, the authority faces an allocation problem with certain special characteristics. One way to solve the problem is to resort to solutions that can be found in the literature on allocation problems, or based on them to introduce new solutions adapted to the problem. In this paper, we use the allocation model introduced in Acosta et al. (2022), which is the one that best fits the situation described. For this model, we introduce a new solution based on the concept of proportionality that adapts to the structure of the described allocation problem. We carry out an axiomatic analysis of the proposed solution to show that it has good properties and illustrate its application with a numerical example based on real data. Finally, managerial and policy implications of the approach proposed in this work are suggested.

2. Literature review

2.1. OR literature related to water management problems

The applications of operational research (OR) to environmental management problems have been increasing since the first studies of the 1970s, see, for example, the review by Bloemhof-Ruwaarda et al. (1995), and the references therein. This review highlighted the potential of OR to solve environmental management problems or to include environmental elements in optimization problems. More recently, Mishra (2020) insisted on the impact of OR in environmental management. Many applications of game theory to environmental management problems can be found in the literature (see, for example, Dinar et al., 2008; Hanley & Folmer, 1998).

On the one hand, ReVelle (2000) reviews the challenging OR problems in environmental management, indicating five areas of focus: (1) water management, (2) water quality management, (3) solid waste management, (4) cost allocation for environmental infrastructures, and (5) air quality management. On the other hand, Liu et al. (2011) highlight three problems in water management: pollution, water governance and access rights to water in practice. In addition, they propose the use of ethical principles in addressing water resource management problems. These principles are closely related to those that are common in game theory. Therefore, it seems reasonable that game theory plays a relevant role in the analysis of the problems mentioned. Thus, Dinar & Hogarth (2015) provide an exhaustive review on applications of game theory to water (resource) management. In many of these issues, there are problems in the allocation of water resources to uses, water resources to regions, waste discharges or wastewater treatment costs, among others. In general, allocation problems describe situations in which a resource (or resources) must be distributed among a set of agents. These problems are of great interest in many settings; for this reason, the literature on the matter is extensive.

In Helmer & Hespanhol (1997), water pollution control is highlighted as one of the most relevant problems in water resource management. Moreover, different aspects and principles of water quality management are analysed. As for the allocation problems

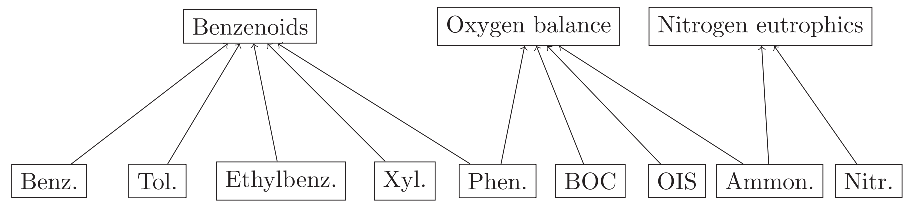

Fig. 1. Relationship between families of pollutants and substances.

that we find in the literature on water pollution control, most are related either to the allocation of waste discharges in water or to the allocation of costs in wastewater treatment or a combination of both. Bogardi & Szidarovszky (1976) present a game theoretical model based on oligopoly games to determine the amount of water to be treated by each polluter. Niksokhan et al. (2009) and Nikoo et al. (2011, 2012) study how to reallocate the treatment costs of discharges to various polluters when they cooperate. To do so, they combine the use of optimization models, non-cooperative games and cooperative games. Once the reallocation of the treatment costs is determined, a trading discharge permit policy associated with it is obtained. Poorsepahy-Samian et al. (2012) present a methodology for water and pollution discharge permit allocation in a shared river. This methodology is based on linear optimization, cooperative game theory and minimax regret theory to determine the best water and discharge permit allocation strategies. Bai et al. (2019) use the indicators water environmental carrying capacity (WECC)[2] and total emission pollutant control (TEPC)[3] to allocate emission pollutant permits to polluters in a two-step procedure by applying the principle of equal proportion reduction and the reduction potential of agents. Chen et al. (2019) use a nonlinear optimization model to allocate emission pollutant permits to point and non-point pollution sources.[4] Xie et al. (2022) use DEA and non-cooperative game theory to allocate wastewater discharge permits among polluters. In almost all previous works, the proposed allocation methodologies are illustrated through numerical examples inspired by real-world data. Moreover, in Bai et al. (2019) and Chen et al. (2019), emission permits are differentiated by more than one pollutant (or pollution parameters of water quality). On the other hand, Ni and Wang (2007), Dong et al. (2012), Alcalde-Unzu et al. (2015, 2021); Gómez-Rúa (2012, 2013) and Li et al. (2021), among others, study different methods to allocate the costs of cleaning a polluted river among pollutant agents. These allocation methods are mainly analysed from a game theoretical perspective, and therefore, ethical principles are used as suggested by Liu et al. (2011).

The main objective of previous works was not to determine the emission limits of pollutants, but to allocate emission limits (or permits) to the polluting agents. Nevertheless, from the emission permits, the limit of pollutant emissions could be determined. However, in most, only one of the categories of pollutants, a particular polluting substance or total pollution is considered, and only in some are the emission limits of polluting substances differentiated, without considering that they may have more than one negative effect on water quality as shown in Fig. 1. In this work, on the contrary, the main objective is to allocate emission limits to each polluting substance, paying special attention to the fact that they can have more than one negative effect on water quality. Moreover, the problem that we address would correspond to a pre-analysis phase, which should be considered in a prior step to the problem addressed in the aforementioned works, that is, the emission limits are set and then the emission permits are allocated. The approach of considering the interactions of pollutants has already been suggested by Endres (1985) and Kuosmanen & Laukkanen (2011), who show the importance of this approach for avoiding inefficient environmental policies. Therefore, the approach used in this work is interesting and fills a gap in the literature on the control of pollutant emissions into water.

2.2. Proportionality in the literature related to allocation problems

A particular allocation problem arises in situations where there is a perfectly divisible resource over which there is a set of agents who have rights or demands, but the resource is not sufficient to honour them. This problem is known as the bankruptcy problem and was first formally analysed in O'Neill (1982) and Aumann & Maschler (1985). Since then, it has been extensively studied in the literature and many allocation rules have been defined (see Thomson, 2019, for a detailed inventory of rules). In the literature we also find works that apply this bankruptcy problem model to study the water allocation problem (Wickramage et al., 2020) and the problem of allocation of pollution discharge permits in rivers (Aghasian et al., 2019; Moridi, 2019). However, for other environmental problems, see, for example, Giménez-Gómez et al. (2016), Gutiérrez et al. (2018), and Duro et al. (2020) which analyse the CO_2 allocation problem.

The base bankruptcy model does not always fit all problems, which is the reason there are different extensions of the classical bankruptcy model. Some of them are the following: Young (1994) and Moulin (2000) study bankruptcy problems in the indivisible goods case. An application of the discrete bankruptcy model to the apportionment problem in proportional electoral systems is given in Sánchez-Soriano et al. (2016). Pulido et al. (2008, 2002) introduce bankruptcy problems with references and claims to study allocation problems in university management. Gozalvez et al. (2012), and Lucas-Estañ et al. (2012) present bankruptcy problems with claims given by a discrete nonlinear function of the resource to analyse radio resource allocation problems in telecommunications. Habis & Herings (2013) and Koster & Boonen (2019) study bankruptcy problems in which the estate and the claims are stochastic values. An interesting extension of bankruptcy problems is multi-issue allocation problems (Calleja et al., 2005). These describe situations in which there is a (perfect divisible) resource that can be distributed among several issues, and a (finite) number of agents that have claims on each of those issues, such that the total claim is above the available resource. This problem is also solved by means of allocation rules, and there are different ways to do so (see, for example, Borm et al., 2005; Calleja et al., 2005; González-Alcón et al., 2007; Izquierdo & Timoner, 2016). However, the situation described in Fig. 1 does not fit a multi-issue allocation problem, as referred to in the previous paragraph, but a

[2] "Water environmental carrying capacity (WECC) is a comprehensive evaluation index for the state of the water environment system and is usually used to reflect the bearing capacity of the water environment system under the impacts of human activities." Bai et al. (2019).

[3] "Total emission pollutant control (TEPC) is a commonly used water environment management strategy aiming to improve water quality by controlling a total load of water pollutants within the range of a given WECC." Bai et al. (2019).

[4] See, for example, https://www.watereducation.org/aquapedia-background/point-source-vs-nonpoint-source-pollution.

multi-issue allocation problem with crossed claims, as introduced by Acosta et al. (2022). These describe situations in which there are several (perfect divisible) resources and a (finite) set of agents who have claims on them, but only one claim (not a claim for each resource) with which one or more resources are requested. The total claim for each resource exceeds its availability. Therefore, in this work, we use multi-issue allocation problems with crossed claims to allocate emission limits to pollutants. To the best of our knowledge, there are no applications of this model of bankruptcy problems to water pollution control.

In allocation problems, the concept of proportionality is put into practice with the well-known proportional rule. This rule has been extensively studied in the literature from many different points of view and for many allocation models. Focusing on bankruptcy models and their extensions to the multi-issue case, the proportional rule has been characterized in the context of bankruptcy problems in Chun (1988) and de Frutos (1999). In both papers, non-manipulability plays a central role in the axiomatic characterization of the proportional rule. For multi-issue allocation problems, Ju et al. (2007) and Moreno-Ternero (2009) introduce two different definitions of proportional rule following two different approaches. Moreover, Ju et al. (2007) and Bergantiños et al. (2010) provide characterizations of both proportional rules. Again, in both approaches, non-manipulability is an essential property. In this paper, we introduce a definition of the proportional rule, which we call the constrained proportional awards rule, for multi-issue allocation problems with crossed claims and provide a characterization of it. Once again, non-manipulability is used. In this paper, to solve the allocation problem faced by the authority responsible for the water quality, we introduce the constrained proportional awards rule for multi-issue allocation problems with crossed claims that naturally extends the proportional rule for single issue allocation problems. This rule is characterized axiomatically by using five properties: *Pareto efficiency*, *equal treatment of equals*, *guaranteed minimum award*, *consistency*, and *non-manipulability by splitting*. The first one says that there is no feasible allocation in which at least one of the claimants receive more. *Equal treatment of equals* states that equal agents must receive the same. *Guaranteed minimum award* establishes that a claimant should not receive less than what she would receive in the worst case, if the issues were distributed separately. *Consistency* requires that if a subset of agents leave the problem respecting what had been allocated to those who remain, then what those agents receive in the new problem is the same as what they received in the original problem. Finally, *non-manipulability by splitting* states that it is not profitable to split one agent into several agents. Therefore, we fill a gap in the literature on proportional distributions in allocation problems in line with previous studies.

In summary, in this paper, we address two gaps in the literature, one applied and another theoretical. On the one hand, we propose a methodology based on bankruptcy models to allocate emission limits to pollutants, taking into account that their effects on water quality can be several, and therefore, there are interactions among them. This approach agrees with the works of Endres (1985) and Kuosmanen & Laukkanen (2011). To the best of our knowledge, this approach is novel in the field of water pollution control policies. On the other hand, we introduce and axiomatically analyse a new solution based on the concept of proportionality for multi-issue allocation problems with crossed claims that are an extension of bankruptcy problems.

The rest of the paper is organized as follows. Section 3 presents multi-issue allocation problems with crossed claims (MAC) and the concept of rules for these problems. In Section 4, the constrained proportional awards rule for multi-issue bankruptcy problems with crossed claims is defined. In Section 5, we present several properties that are interesting in the context of MAC problems. In Section 6, we characterize the constrained proportional awards rule. Section 7 includes an application of the constrained proportional awards rule to the management of water pollution control. Section 8 concludes.

3. Multi-issue allocation problems with crossed claims

Before starting with the description of the model used in this work, some mathematical notation is provided. Given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, $a \oplus b \in \mathbb{R}^{n+m}$ denotes the concatenation of the two vectors. Given $a, b \in \mathbb{R}^n$, $a \leq b$ means that $a_i \leq b_i, \forall i = 1, \ldots, n$; and $a < b$ means that $a_i \leq b_i, \forall i = 1, \ldots, n$ with at least a strict inequality. Given a set S, $|S|$ denotes its cardinal. Given two sets S, T, $S \subset T$ includes the possibility that $S = T$.

We consider a situation where there is a finite set of issues $I = \{1, 2, \ldots, m\}$ such that there is a perfectly divisible amount e_i of each issue i. Let $E = (e_1, e_2, \ldots, e_m)$ be the vector of available amounts of issues. There is a finite set of claimants $N = \{1, 2, \ldots, n\}$ such that each claimant j claims c_j. Let $c = (c_1, c_2, \ldots, c_n)$ be the vector of claims. However, each claimant claims to different sets of issues, in general. Thus, let α be a set-valued function that associates with every $j \in N$ a set $\alpha(j) \subset I$. In fact, $\alpha(j)$ represents the issues to which claimant j asks for. Furthermore, $\sum_{j:i \in \alpha(j)} c_j > e_i$, for all $i \in I$, otherwise, those issues could be discarded from the problem because they do not impose any limitation, and so the allocation would be trivial. Therefore, a multi-issue allocation problem with crossed claims (MAC in short) is defined by a 5-tuple (I, N, E, c, α), and the family of all these problems is denoted by $\mathcal{MAC}$.

A rule for $\mathcal{MAC}$ is a mapping R that associates with every $(I, N, E, c, \alpha) \in \mathcal{MAC}$ a unique vector $R(I, N, E, c, \alpha) \in \mathbb{R}^N$ such that:

1. $0 \leq R_j(I, N, E, c, \alpha) \leq c_j$, for all $j \in N$.
2. $\sum_{j \in N: i \in \alpha(j)} R_j(I, N, E, c, \alpha) \leq e_i$, for all $i \in I$.

These two requirements are standard in the literature on allocation problems (see Thomson, 2019), and state that the allocation is feasible. Therefore, the vector $R(I, N, E, c, \alpha)$ represents an allocation to the claimants that is simultaneously feasible for all issues.

Given $(I, N, E, c, \alpha) \in \mathcal{MAC}$, we define the following two sets:

- $\mathcal{I} = \{i \in I : e_i > 0\}$ is the set of active issues;
- $\mathcal{N} = \{j \in N : c_j > 0 \text{ and } e_i > 0, \forall i \in \alpha(j)\}$ is the set of active claimants.

4. The constrained proportional awards rule for $\mathcal{MAC}$ problems

The proportional rule (Aristotle, 1908, 4th Century BD) is perhaps the most important rule for solving allocation problems in general,[5] and bankruptcy problems in particular. This rule simply divides the resource in proportion to the claims. Formally, for a one-issue allocation problem (I, N, E, c), where $I = \{1\}$, the proportional rule (PROP) is defined as follows:

$$PROP_j(I, N, E, c) = \frac{c_j}{C} E, \quad j \in N. \tag{1}$$

The question in $\mathcal{MAC}$ problems is what "in proportion to the claims" means. In the context of one-issue allocation problems, "in proportion to the claims" means that all claimants receive the same amount for each unit of claim. How to extrapolate this to the MAC situations. To answer this question, we introduce the constrained proportional awards rule as the result of an iterative process in which the available amount of at least one of the issues is

[5] See, for instance, Algaba et al. (2019a) who present two solutions belonging to the family of proportional solutions for the problem of sharing the profit of a combined ticket for a transport system.

fully distributed in each step and so on while possible. This rule is formally defined below.

Definition 1. Let $(I, N, E, c, \alpha) \in \mathcal{MAC}$, the *constrained proportional awards (CPA) rule* for (I, N, E, c, α), $CPA(I, N, E, c, \alpha)$, is defined by means of the following iterative procedure:

- Initialization (Step 0)
  1. $\mathcal{I}^1 = \mathcal{I}$, $\mathcal{N}^1 = \mathcal{N}$.
  2. For each $i \in I$, $e_i^1 = e_i$, and for each $j \in N$, $c_j^1 = c_j$.
- General step (Step s)
  1. $\mathcal{I}^s$, $\mathcal{N}^s$.
  2. For each $i \in \mathcal{I}^s$, we calculate the greatest $\lambda_i^s \in [0,1]$, so that $\lambda_i^s \sum_{j \in \mathcal{N}^s : i \in \alpha(j)} c_j^s \le e_i^s$, and take $\lambda^s = \min\{\lambda_i^s : i \in \mathcal{I}^s\}$.
  3. We allocate to each claimant $j \in \mathcal{N}^s$, $a_j^s = \lambda^s c_j^s$, and $a_j^s = 0$ otherwise.
  4. We update the active issues, $\mathcal{I}^{s+1}$, and the active claimants, $\mathcal{N}^{s+1}$. If $\mathcal{I}^{s+1} = \varnothing$ or $\mathcal{N}^{s+1} = \varnothing$, then the process ends, and

     $$CPA_j(I, N, E, c, \alpha) = \sum_{h=1}^{s} a_j^h, \forall j \in N.$$

     Otherwise, the available amounts of issues and the claims are updated:

     $$e_i^{s+1} = e_i^s - \lambda^s \sum_{j \in N : i \in \alpha(j)} c_j^s, \forall i \in I, \text{ and } c_j^{s+1} = c_j^s - \lambda^s c_j^s, \forall j \in N,$$

     and we go to Step $s+1$

The application of the iterative procedure described in Definition 1 generates a succession of problems $(I^s, N^s, E^s, c^h, \alpha)$, $s = 1, 2, \ldots$, such that $I^s = I, N^s = N, E^s \ge E^{s+1}, c^s \ge c^{s+1}, s = 1, 2, \ldots$ Moreover, for each of those problems, an allocation $a^s \in \mathbb{R}^n_{\ge 0}$ is obtained. Therefore, $CPA_j(I, N, E, c, \alpha) = \sum_{s=1}^{+\infty} a_j^s, \forall j \in N$.

Note that if $\mathcal{I}^s \neq \varnothing$ and $\mathcal{N}^s \neq \varnothing$, then for each $i \in \mathcal{I}^s$, $\lambda_i^s > 0$. Moreover, if $\lambda_i^s < 1$, then λ_i^s is such that $\lambda_i^s \sum_{j \in \mathcal{N}^s : i \in \alpha(j)} c_j^s = e_i^s$, therefore, $i \notin \mathcal{I}^{s+1}$ and $\{j \in \mathcal{N}^s : i \in \alpha(j)\} \cap \mathcal{N}^{s+1} = \varnothing$. If $\lambda_i^s = 1$, we have two alternatives: either (1) $\{j \in \mathcal{N}^s : i \in \alpha(j)\} = \varnothing$, in which case it does affect neither the next set of active issues nor the next set of active claimants; or (2) $\{j \in \mathcal{N}^s : i \in \alpha(j)\} \neq \varnothing$, in which case $\{j \in \mathcal{N}^s : i \in \alpha(j)\} \cap \mathcal{N}^{s+1} = \varnothing$, and i will belong to $\mathcal{I}^{s+1}$ or not depending on whether $\sum_{j \in \mathcal{N}^s : i \in \alpha(j)} c_j^s < e_i^s$.

According to the above, $\lambda^s > 0$. If $\lambda^s < 1$, then $\mathcal{I}^{s+1} \subsetneq \mathcal{I}^s$ and $\mathcal{N}^{s+1} \subsetneq \mathcal{N}^s$. If $\lambda^s = 1$, then $\mathcal{N}^{s+1} = \varnothing$, and the process ends. Therefore, in each step at least the available amount of one issue is distributed in its entirety, except maybe in the last step. This implies that the process ends in a finite number of steps, at most $|I|$. Thus, $CPA_j(I, N, E, c, \alpha) = \sum_{s=1}^{r} a_j^s, \forall j \in N$, where $r \le |I|$. Accordingly, CPA is well-defined and always leads to a single point.

Note that when we have a one-issue allocation problem, then it is easy to check that we obtain PROP. Thus, this definition extends PROP to the context of MAC.

From the application of the iterative process to calculate CPA, we can consider the chains of active issues and active claimants in the application of the procedure to calculate $CPA(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$:

$$\mathcal{I}^1 \supsetneq \mathcal{I}^2 \supsetneq \ldots \supset \mathcal{I}^r, \text{ and } \mathcal{N}^1 \supsetneq \mathcal{N}^2 \supsetneq \ldots \supsetneq \mathcal{N}^r$$

Furthermore, we can associate with each pair of sets $\mathcal{I}^s$ and $\mathcal{N}^s$ a number ρ^s, $\rho^s \in [0,1]$, which represents the proportion of claims obtained by claimants in $\mathcal{N}^s$ but not in $\mathcal{N}^{s+1}$. Moreover, by construction $\rho^s < \rho^{s+1}$. Thus, we have that

$$0 < \rho^1 < \rho^2 < \ldots < \rho^r \le 1.$$

These $\rho's$ represent the accumulative proportion of the claims allocated to the claimants, i.e., what part of their claims they have received up to a given step of the iterative procedure. In this way, this procedure is reminiscent of the constrained equal awards rule (CEA) in bankruptcy problems, but instead of using the principle of egalitarianism, the principle of proportionality is used hence, the name of *constrained proportional awards* rule. Therefore, not all claimants receive the same proportion of their claims, but the rule tries to keep the proportionality as much as possible restricted to (1) the relation between the available amounts of issues and the total claims to them and (2) the goal of allocating as much as possible of all available amounts of issues.

Finally, note that if a problem can be separated into two disjointed problems, it is the similar to calculating CPA for the entire problem for each, then pasting the results. This is established in the following proposition.

Proposition 1. *Given $(I, N, E, c, \alpha) \in \mathcal{MAC}$, if there are problems $(I_1, N_1, E_1, c^1, \alpha_1)$, $(I_2, N_2, E_2, c^2, \alpha_2) \in \mathcal{MAC}$, such that $I_1 \cup I_2 = I$, $N_1 \cup N_2 = N$, $E_1 \oplus E_2 = E$, $c^1 \oplus c^2 = c$, and $\alpha_1(j) = \alpha(j), \forall j \in N_1$ and $\alpha_2(j) = \alpha(j), \forall j \in N_2$, so that $\left(\bigcup_{j \in N_1} \alpha(j)\right) \bigcap \left(\bigcup_{j \in N_2} \alpha(j)\right) = \varnothing$, then*

$$CPA(I, N, E, c, \alpha) = CPA(I_1, N_1, E_1, c^1, \alpha_1) \oplus CPA(I_2, N_2, E_2, c^2, \alpha_2).$$

Proof. The proof follows from the fact that since there are no crossed demands between the two subproblems, they do not affect each other and, therefore, the results are independent of each other. □

5. Properties

In this section, we present several properties that are interesting in the context of MAC problems. These properties are related to efficiency, fairness, consistency, and manipulability.

First, we introduce two concepts related to two claimant comparisons. In MAC situations, claimants are characterized by two elements: their claims and the issues to which they claim. Therefore, both should be considered when establishing comparisons among them.

Definition 2. Let $(I, N, E, c, \alpha) \in \mathcal{MAC}$, and two claimants $j, k \in N$, we say they are *homologous*, if $\alpha(j) = \alpha(k)$; and we say that they are *equal*, if they are homologous and $c_j = c_k$.

Next, we give a set of properties that are very natural and reasonable for an allocation rule in MAC situations.

The first property relates to efficiency. In allocation problems, it is desirable for resources to be fully distributed, but in MAC situations, this is not always possible (see Acosta et al., 2022). Therefore, a weaker version of that is considered in which only is required that there is no feasible allocation in which at least one of the claimants receives more. This is established in the following axiom.

Axiom 1 (PEFF). Given a rule R, it satisfies *Pareto efficiency*, if for every problem $(I, N, E, c, \alpha) \in \mathcal{MAC}$, there is no a feasible allocation $a \in \mathbb{R}^N_+$ such that $a_j \ge R_j(I, N, E, c, \alpha), \forall j \in N$, with at least one strict inequality.

Note that *PEFF* implies that at least the available amount of one issue is fully distributed, and no amount is left of an issue undistributed, if it is possible to do so. However, it does not require that all available amounts of the issues must be fully distributed. On the other hand, a feasible allocation that satisfies the condition in Axiom 1 is called Pareto efficient.

The second property states that equal claimants should receive the same in the final allocation. This is a basic requirement of fairness and non-arbitrariness. This is defined in the following axiom.

Axiom 2 (ETE). Given a rule R, it satisfies *equal treatment of equals*, if for every problem $(I, N, E, c, \alpha) \in \mathcal{MAC}$ and every pair of equal claimants $j, k \in N$, $R_j(I, N, E, c, \alpha) = R_k(I, N, E, c, \alpha)$.

The third property states the minimum that should be guaranteed to each claimant. In our case, these minimum amounts are determined from the analysis of the problems associated with each issue independently. In particular, the property states that a claimant should not receive less than what she would have received in the worst case if the rule had been applied to each problem separately to each of the issues. This is established in the following property.

Axiom 3 (GMA)**.** Given a rule R, it satisfies *guaranteed minimum award*, if for every problem $(I, N, E, c, \alpha) \in \mathcal{MAC}$,

$$R_j(I, N, E, c, \alpha) \geq \min\left\{R_j\left(\{i\}, N_i, e_i, c|_{N_i}\right) : i \in \alpha(j)\right\}, \forall j \in N,$$

where $N_i = \{k \in N : i \in \alpha(k)\}$, and $c|_{N_i}$ is the vector whose coordinates correspond to the claimants in N_i.

The fourth property is a requirement of robustness when some agents leave the problem with their allocations (see Thomson, 2011; Thomson, 2018). In particular, when a subset of claimants leave the problem respecting the allocations to those who remain, then it seems reasonable that claimants who leave will receive the same in the new problem as they did in the original. This is formally given in the following axiom.

Axiom 4 (CONS)**.** Given a rule R, it satisfies *consistency*, if for every problem $(I, N, E, c, \alpha) \in \mathcal{MAC}$, and $N' \subset N$, it holds that

$$R_j(I, N, E, c, \alpha) = R_j(I', N', E'^R, c|_{N'}, \alpha), \text{ for all } j \in N',$$

where $(I', N', E', c|_{N'}, \alpha) \in \mathcal{MAC}$, called the *reduced problem associated with* N', $I' = \{i \in I :$ there exists $k \in N'$such that $i \in \alpha(k)\}$, $E'^R = (e_1'^R, \ldots, e_m'^R)$ so that $e_i'^R = e_i - \sum_{j \in N \setminus N' : i \in \alpha(j)} R_j(I, N, E, c, \alpha)$, for all $i \in I'$, and $c|_{N'}$ is the vector whose coordinates correspond to the claimants in N'.

The last two properties are related to claimants' ability to manipulate the final allocation by splitting their claims among several new claimants or merging their claims into a single claimant. It seems sensible that if the claimants do this, they will not benefit and receive the same as they did in the original problem. These two possibilities are established in the following axioms.

Axiom 5 (NMS)**.** Given a rule R, it satisfies *non-manipulability by splitting*, if for every pair of problems $(I, N, E, c, \alpha), (I, N', E, c', \alpha') \in \mathcal{MAC}$, such that:

1. $N \subset N',\ S = \{i_1, \ldots, i_k\}$, such that $N' = (N \backslash S) \cup S_{i_1} \cup \ldots \cup S_{i_m}$, where S_{i_k} is the set of agents into which agent i_k has been divided including itself.
2. $c'_j = c_j, \forall j \in N \backslash S$ and $\sum_{k \in S_{i_h}} c'_k = c_{i_h}, h = 1, \ldots, m$,
3. $\alpha'(j) = \alpha(j), \forall j \in N \backslash S$ and $\alpha'(j) = \alpha(i_h), \forall j \in S_{i_h}, h = 1, \ldots, m$,

it holds

$$\sum_{j \in S_{i_h}} R_j(I, N', E, c', \alpha') = R_{i_h}(I, N, E, c, \alpha), h = 1, \ldots, m.$$

Axiom 6 (NMRM)**.** Given a rule R, it satisfies *non-manipulability by restricted merging*, if for every pair of problems $(I, N, E, c, \alpha), (I, N', E, c', \alpha') \in \mathcal{MAC}$, such that:

1. $N \subset N'$,
2. $c_j = c'_j, \forall j \in N \backslash \{j_0\}$ and $c_{j_0} = \sum_{k \in (N' \backslash N) \cup \{j_0\}} c'_k$,
3. $\alpha(j) = \alpha'(j), \forall j \in N \backslash \{j_0\}$ and $\alpha(j) = \alpha'(j_0), \forall j \in (N' \backslash N) \cup \{j_0\}$,

it holds

$$R_{j_0}(I, N, E, c, \alpha) = \sum_{j \in (N' \backslash N) \cup \{j_0\}} R_j(I, N', E, c', \alpha').$$

Note that in *NMS* we move from the allocation problem with a set of claimants N to the problem with a set of claimants N', i.e., one of the claimants is split into several new claimants, one of whom has the same name as in N. However, in *NMRM*, we move from the problem in N' to the problem in N, i.e., several claimants merge into one claimant who has the same name as in N', but all merged claimants are homologous. Thus, we are only considering the merging of homologous claimants. For this reason, we call this axiom non-manipulability by "restricted" merging. Nevertheless, it seems reasonable from the perspective of symmetry of both properties, because when one claimant is split into several new claimants, these are homologous in the new problem.

CPA satisfies all the properties mentioned above. We establish this in the following theorem.

Theorem 1. *CPA for multi-issue bankruptcy problems with crossed claims satisfies PEFF, ETE, GMA, CONS, NMS, and NMRM.*

Proof. We go axiom by axiom.

- CPA satisfies *PEFF* and *GMA* by definition.
- If two claimants are equal, then CPA allocates both the same, since the procedure for calculating the rule treats, in each step, in the same way to all active equal claimants, so if two claimants are equal, they stop receiving at the same step. Therefore, CPA satisfies *ETE*.
- Given $(I, N, E, c, \alpha) \in \mathcal{MAC}$ and $(I', N', E'^{CPA}, c|_{N'}, \alpha) \in \mathcal{MAC}$ the reduced problem associated with $N' \subset N$. Let us consider the following sets obtained from the application of CPA to (I, N, E, c, α):

  $$\mathcal{A}^s = \mathcal{N}^s \setminus \mathcal{N}^{s+1}, s = 1, \ldots, r, \text{ and } \mathcal{B}^s = \mathcal{I}^s \setminus \mathcal{I}^{s+1}, s = 1, \ldots, r.$$

  We now consider the following sets: $N' \cap \mathcal{A}^s, s = 1, \ldots, r$. Taking into account the definitions of E'^{CPA}, $\mathcal{N}^s$, and $\mathcal{I}^s$, it is evident that claimants in $N' \cap \mathcal{A}^s$ cannot receive more than ρ^s times their claims, because, otherwise, the available amounts of issues $e_i'^{CPA}$, $i \in \mathcal{B}^s$, would be exceeded. Thus, from the definition of CPA, claimants in $N' \cap \mathcal{A}^s$ have to receive exactly ρ^s times their claims. Therefore, the claimants in $(I', N', E'^{CPA}, c|_{N'}, \alpha)$ receive the same as in (I, N, E, c, α). Hence, CPA satisfies consistency.
- Note that when one claimant splits into several new claimants, CPA for the new problem will have the same number of iterations as in the original one, since the claim for each issue will be obviously the same in each step. Therefore, all split claimants will receive the same proportion of their claims which coincides with the proportion obtained by the split claimant in the original problem. Thus, the aggregate allocation of the split claimants in the new problem coincides with the allocation of the split claimant in the original problem.
- When two homologous claimants merge into a new claimant, we can make completely analogous reasoning as in the case of a claimant splitting into several new claimants. Therefore, CPA also satisfies NMRM. □

6. Characterization

In this section, the aim is to achieve a better knowledge of the CPA rule for $\mathcal{MAC}$ by describing it in a unique way as a combination of some reasonable axioms. We characterize the *CPA* rule by means of *PEFF*, *ETE*, *GMA*, *CONS*, and *NMS*. Therefore, the CPA rule can be considered as a desirable way to distribute a set of issues among their claimants. Before giving the characterization of CPA, we need the following lemmas.

Lemma 1. *Let $(I, N, E, c, \alpha) \in \mathcal{MAC}$, such that $|I| = 1$. If a rule R satisfies PEFF, ETE, and NMS, then*

$$R_i(I, N, E, c, \alpha) = \frac{c_1}{\sum_{j\in N} c_j} e, \text{ for all } i \in N.$$

Proof. First note that in this case the function α is irrelevant. Let $R_1, R_2, \dots, R_n$ be the allocations for claimants in N, respectively. By *PEFF*, and the definition of rule, we know that there are $\beta_i \in [0, 1]$, $i \in N$, such that $R_i = \beta_i c_i$, $i \in N$, and $\sum_{i\in N} \beta_i c_i = e$.

Consider the following chain of problems:

$$(I, N, E, c, \alpha) \longrightarrow (I, N(q), E, c(q), \alpha)$$

where the first problem is the original, the second is the problem in which each claimant i is split into a number of identical claimants k_i, $k_i \in \mathbb{N}_+$, with claims exactly equal to $q \in \mathbb{R}_+$. We now distinguish two cases:

1. $c_1, c_2, \dots, c_n \in \mathbb{Q}_+$. In this case, there exists $q \in \mathbb{Q}_+$ such that $c_i = k_i q, k_i \in \mathbb{N}_+, i \in N$. Now, by *PEFF* and *ETE*, we have that

   $$R_j(I, N(q), E, c(q), \alpha) = \beta q, \; j \in N(q).$$

   On the other hand, by *NMS*, it holds for every $i \in N$ that

   $$\beta_i c_i = k_i \beta q = \beta c_i \Rightarrow \beta_i = \beta$$

2. $c_1, c_2, \dots, c_n \in \mathbb{R}_+$. In this case, for each $\varepsilon > 0$, there exists $q \in \mathbb{R}_+$ such that $c_i = k_i(q)q + \varepsilon_i(q), k_i(q) \in \mathbb{N}_+$, and $\varepsilon_i(q) < \frac{\varepsilon}{n}$, for all $i \in N$.
   Now, by *ETE*, we have the following equality for the second problem:

   $$\left(\sum_{i\in N} k_i(q)\right)\beta(q)q + \sum_{i\in N} \delta_i(q)\varepsilon_i(q) = e,$$

   where $\beta(q) \in [0, 1]$, and $\delta_i(q) \in [0, 1]$ for all $i \in N$. This equality can be written as follows:

   $$\beta(q)\sum_{i\in N}(c_i - \varepsilon_i(q)) + \sum_{i\in N} \delta_i(q)\varepsilon_i(q) = e,$$

   or equivalently,

   $$\frac{e}{\sum_{i\in N} c_i} - \beta(q) = \frac{\sum_{i\in N}\{(\delta_i(q) - \beta(q))\varepsilon_i(q)\}}{\sum_{i\in N} c_i},$$

   taking limits on both sides when q goes to zero, we obtain that $\lim_{q\to 0^+} \beta(q) = \frac{e}{\sum_{i\in N} c_i}$.
   On the other hand, by *NMS*, for each q and for each $i \in N$,

$$\beta_i c_i = k_i(q)\beta(q)q + \delta_i(q)\varepsilon_i(q) = \beta(q)c_i + (\delta_i(q) - \beta(q))\varepsilon_i(q).$$

   Since $\lim_{q\to 0^+} \beta(q) = \frac{e}{\sum_{i\in N} c_i}$, $\beta_i = \frac{e}{\sum_{i\in N} c_i}$, for each $i \in N$. □

Lemma 2. *For each problem $(I, N, E, c, \alpha) \in \mathcal{MAC}$, and each rule R that satisfies PEFF, ETE, GMA and NMS, if for each $N' \subset N$ with $|N'| = |N| - 1$, we have $R_i(I, N, E, c, \alpha) = CPA_i(I', N', E'^R, c|_{N'}, \alpha)$ for all $i \in N'$, then $R(I, N, E, c, \alpha) = CPA(I, N, E, c, \alpha)$.*

Proof. We first prove that if there is $R_i = R_i(I, N, E, c, \alpha) = CPA_i(I, N, E, c, \alpha)$, then the result holds. Indeed, let us consider R in the conditions of the statement, and $R_i = CPA_i(I, N, E, c, \alpha)$. We now consider $N' = N\backslash\{i\}$, since $R_i = CPA_i(I, N, E, c, \alpha)$,

$$(I', N', E'^R, c|_{N'}, \alpha) = (I', N', E'^{CPA}, c|_{N'}, \alpha).$$

By hypothesis, we have that for all $k \in N'$,

$$R_k(I, N, E, c, \alpha) = CPA_k(I', N', E'^R, c|_{N'}, \alpha) = CPA_k(I', N', E'^{CPA}, c|_{N'}, \alpha).$$

Moreover, since CPA satisfies consistency,

$$CPA_k(I, N, E, c, \alpha) = CPA_k(I', N', E'^{CPA}, c|_{N'}, \alpha) \text{ for all } k \in N'.$$

Therefore, $CPA_k(I, N, E, c, \alpha) = R_k(I, N, E, c, \alpha)$ for all $k \in N'$.

Let us consider R in the conditions of the statement and we assume without loss of generality that $\beta_1 = \frac{R_1}{c_1} \le \beta_2 = \frac{R_2}{c_2} \le \dots \le \beta_{|N|} = \frac{R_{|N|}}{c_{|N|}}$, where for the sake of simplicity, we denote $R_k(I, N, E, c, \alpha)$ by R_k for each $k \in N$.

First, for $\alpha(1)$, for every $i \in \alpha(1)$, we take $\gamma_i > 0$ such that $\gamma_i \sum_{j\in N: i\in\alpha(j)} c_j = e_i$. We now define $\gamma_1 = \min\{\gamma_i : i \in \alpha(1)\}$, and we assume that γ_1 is without loss of generality obtained for issue 1.

Second, $\beta_1 \le \gamma_1$, otherwise, we would have that

$$\sum_{j:1\in\alpha(j)} \beta_j c_j \ge \beta_1 \sum_{j:1\in\alpha(j)} c_j > \gamma_1 \sum_{j:1\in\alpha(j)} c_j = e_1,$$

which is a contradiction.

Third, by Lemma 1 and *GMA*, $R_1 \ge \min\{\frac{c_1}{\sum_{j:i\in\alpha(j)} c_j} e_i : i \in \alpha(1)\} = \gamma_1 c_1$. Therefore, $\beta_1 = \gamma_1$. Now, by *PEFF*, $\beta_j = \gamma_1$ for all $j \in N$ such that $1 \in \alpha(j)$.

Fourth, for each $N' \subset N$ with $1 \in N'$ and $|N'| = |N| - 1$, by hypothesis and the definition of CPA, we have that β_1 coincides with the λ^1's of the iterative procedures for calculating each $CPA(I', N', E'^R, c|_{N'}, \alpha)$. This implies that $\beta_1 = \lambda^1 = \min\{\lambda_i^1 : i \in \mathcal{I}'^1\}$, for each $N' = N\backslash\{k\}, k \in N\backslash\{1\}$, where

$$\lambda_i^1 \sum_{j\in\mathcal{N}'^1: i\in\alpha(j)} c_j^1 = e_i^1 - \delta(i, k)\beta_k c_k, \forall i \in I',$$

where $\delta(i, k) = 1$ if $i \in \alpha(k)$, and 0 otherwise. Since $R_1 = \beta_1 c_1 = CPA_i(I', N', E'^R, c|_{N'}, \alpha)$, $\arg\min\{\lambda_i^1 : i \in \mathcal{I}'^1\} \in \alpha(1)$, otherwise, claimant 1 would obtain more than $\beta_1 c_1$ which is a contradiction. In particular, the minimum will be attained, although possibly among others, at issue 1, because $\beta_j = \gamma_1$ for all $j \in N$ such that $1 \in \alpha(j)$.

On the other hand, by definition of *CPA*, we have that $\lambda^1 = \min\{\lambda_i^1 : i \in \mathcal{I}^1\}$, where

$$\lambda_i^1 \sum_{j\in\mathcal{N}^1: i\in\alpha(j)} c_j^1 = e_i^1, \forall i \in \mathcal{I}.$$

This λ^1 can be also obtained by solving the following simple linear program:

$$\begin{aligned} \lambda^1 = \max \quad & \lambda \\ \text{subject to } & \lambda \sum_{j\in\mathcal{N}^1: i\in\alpha(j)} c_j^1 \le e_i^1, \forall i \in \mathcal{I} \\ & \lambda \ge 0. \end{aligned}$$

It is obvious that $\lambda^1 \le \gamma_1$, because of the definition of γ_1. Now, since R satisfies *PEFF*, γ_1 is a feasible solution for the linear program above and $\lambda^1 \le \gamma_1$, γ_1 is an optimal solution of the problem. Therefore, the inequality associated with issue 1 is saturated in the optimal solution and by definition of *CPA* claimant 1 will obtain $\gamma_1 c_1 = \beta_1 c_1 = R_1$, i.e., $R_1(I, N, E, c, \alpha) = CPA_1(I, N, E, c, \alpha)$. □

Theorem 2. *CPA is the only rule that satisfies PEFF, ETE, GMA, CONS, and NMS.*

Proof. We distinguish three cases, depending on the number of claimants in the problem.

1. $|N| = 1$. In this case, all rules that satisfy *PEFF* coincide with CPA.
2. $|N| = 2$. We distinguish two cases:
   (a) $\alpha(1) \cap \alpha(2) = \varnothing$. In this situation, since the rule satisfies *PEFF* we can consider two separate problems of only one claimant each. Now by applying the case $|N| = 1$, all rules that satisfy *PEFF* coincide with CPA.
   (b) $\alpha(1) \cap \alpha(2) \ne \varnothing$. We consider other two cases:

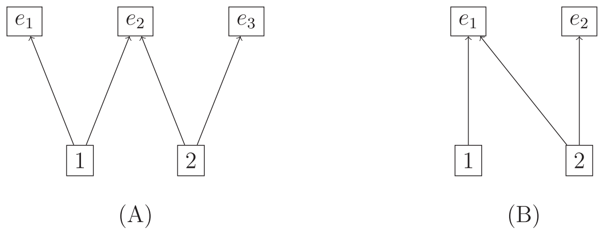

Fig. 2. Basic 2-claimants situations when $\alpha(1)\cap\alpha(2)\neq\varnothing$ and $\alpha(1)\neq\alpha(2)$.

i. $\alpha(1)=\alpha(2)$. By *GMA*, Lemma 1, and the definition of rule,

$$R_1=\frac{c_1}{c_1+c_2}e,\ \text{and } R_2=\frac{c_2}{c_1+c_2}e,$$

where e is the minimum of the available amounts of the issues.

ii. $\alpha(1)\neq\alpha(2)$. First note that by Lemma 1, we know that for every single issue we obtain the proportional distribution of the available amount among the corresponding claimants. Therefore, in order to apply *GMA*, we can consider without loss of generality the two situations shown in Fig. 2.

We next analyze the two situations in Fig. 2:

A. By *GMA*, $c_1\geq e_1$, $c_2\geq e_3$, and $c_1+c_2\geq e_2$,

$$R_1\geq\min\left\{e_1,\frac{c_1}{c_1+c_2}e_2\right\},\ \text{and } R_2\geq\min\left\{\frac{c_2}{c_1+c_2}e_2,e_3\right\}$$

If $\min\{e_1,\frac{c_1}{c_1+c_2}e_2\}=e_1$, then $R_1=e_1$, and by *PEFF* $R_2=\min\{c_2,e_2-e_1,e_3\}$. If $\min\{e_1,\frac{c_1}{c_1+c_2}e_2\}=\frac{c_1}{c_1+c_2}e_2$, then we have two possibilities:

- $\min\{\frac{c_2}{c_1+c_2}e_2,e_3\}=e_3$, then $R_2=e_3$, and by *PEFF* $R_1=\min\{c_1,e_2-e_3,e_1\}$.
- $\min\{\frac{c_2}{c_1+c_2}e_2,e_3\}=\frac{c_2}{c_1+c_2}e_2$, then $R_1=\frac{c_1}{c_1+c_2}e_2$ and $R_2=\frac{c_2}{c_1+c_2}e_2$.

B. By *GMA*, $c_1\geq e_1$, $c_2\geq e_3$, and $c_1+c_2\geq e_2$,

$$R_1\geq\frac{c_1}{c_1+c_2}e_1,\ \text{and } R_2\geq\min\left\{\frac{c_2}{c_1+c_2}e_1,e_2\right\}$$

Now reasoning as in the previous case,

$$R_1=\frac{c_1}{c_1+c_2}e_1,\ \text{and } R_2=\frac{c_2}{c_1+c_2}e_1$$

or

$$R_1=\min\{c_1,e_1-e_2\},\ \text{and } R_2=e_2$$

3. $|N|=3$. Let R be a rule that satisfies *PEFF*, *ETE*, *GMA*, *CONS*, and *NMS*, and let $(I,N,E,c,\alpha)\in\mathcal{MAC}$, then we have that

$$R(I,N,E,c,\alpha)=CPA(I,N,E,c,\alpha).$$

Indeed, for each $N'=\{i_1,i_2\}\subset N$ such that $|N'|=2$, since R satisfies CONS,

$$R_{i_k}(I',N',E'^R,c|_{N'},\alpha)=R_{i_k}(I,N,E,c,\alpha),k=1,2,$$

and since $|N'|=2$, we have that

$$R_{i_k}(I',N',E'^R,c|_{N'},\alpha)=CPA_{i_k}(I',N',E'^R,c|_{N'},\alpha),k=1,2.$$

Since we can take all possible $N'=\{i_1,i_2\}\subset N$, by Lemma 2

$$R(I,N,E,c,\alpha)=CPA(I,N,E,c,\alpha).$$

4. $|N|\leq k$. Let us suppose that for each (I,N,E,c,α) with $|N|\leq k$, $R(I,N,E,c,\alpha)=CPA(I,N,E,c,\alpha)$.
5. $|N|=k+1$. For each $N'\subset N$ such that $|N'|=k$, since R satisfies CONS,

$$R_i(I',N',E'^R,c|_{N'},\alpha)=R_i(I,N,E,c,\alpha),i\in N'.$$

and since $|N'|\leq k$, we have that

$$R_i(I',N',E'^R,c|_{N'},\alpha)=CPA_i(I',N',E'^R,c|_{N'},\alpha),i\in N'.$$

Finally, since we can take all possible $N'\subset N$ with $|N'|=k$, by Lemma 2,

$$R(I,N,E,c,\alpha)=CPA(I,N,E,c,\alpha).\qquad\square$$

Proposition 2. *Properties in Theorem 2 are logically independent.*

Proof. We consider the five possibilities:

- (No *PEFF*) The null rule satisfies all properties but *PEFF*.
- (No *ETE*) Consider an order on the set of claimants and a rule which reimburses each claimant all that can be, in that order, until it is not possible to do so. If we assume that when a claimant splits into several new claimants or some claimants leave, the order in which the claims are attended is preserved, and this rule satisfies *PEFF*, *GMA*, *CONS*, and *NMS*, but not *ETE*.
- (No *GMA*) Consider a rule that has two phases. In the first phase, each issue is distributed proportionally, but only among those claimants that only demand the corresponding issue. In the second phase, the amounts of each issue are updated accordingly and distributed among the rest of the claimants applying CPA. This rule satisfies *PEFF*, *ETE*, *CONS*, and *NMS*, but not *GMA*.
- (No *CONS*) For every problem $(I,N,E,c,\alpha)\in\mathcal{MAC}$, consider the following rule defined in two steps:
  1. First, we allocate to each claimant j, $\min\left\{PROP_j\left(\{i\},N_i,e_i,c|_{N_i}\right):i\in\alpha(j)\right\}$.
  2. Next, we revise down the available amounts of issues and the claims, and we assume without loss of generality that $e'_1\leq e'_2\leq\ldots\leq e'_m$. Then we begin to distribute each state proportionally among the claimants, starting from the smallest to the largest quantity available. It is not until one state has been fully distributed or the claimants fully satisfied that we move on to the next update of the claims. We continue until all the states have been distributed as much as possible.

  The allocation to each claimant is the sum of everything that she has obtained in each of the steps of the procedure described.

  This rule satisfies *GMA*, *PEFF*, and *ETE*. Moreover, using arguments like those used in Theorem 1, it can be shown that this rule satisfies *NMS*. However, it does not satisfy *CONS* since this rule does not coincide with CPA, and if we consider reduced problems with $|N'|=2$, by *PEFF*, *ETE*, *GMA*, and *NMS*, we obtain the allocations prescribed by CPA.
- (No *NMS*) The CEA rule for MAC satisfies all properties but *NMS* (Acosta et al., 2022). $\square$

7. Numerical example based on real-world data

In this section, we consider the applied limits corresponding to a representative local normative of spills to the sewage network established by the Honorary Granada City Council (2000, BOP N° 129, 30/05/2000). These emission limits of some pollutants are shown in Table 1 and correspond to initial emission limits for pollutant categories of 5.6 ppm for benzenoids, 1400 ppm for oxygen-demanding wastes and 250 ppm for nitrogen eutrophic nutrients.

Now suppose that the city council wants to impose more stringent limitations on the concentrations (ppm) of each of the three categories of pollutants mentioned in Section 1 (benzenoids, oxygen-demanding wastes, and eutrophic nutrients), independent of the emission limits for each of the pollutants. In this sense, what the city council intends is to control emissions more by categories

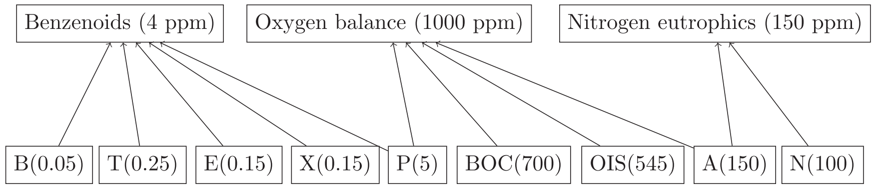

Fig. 3. Relationship between families of pollutants and substances.

Table 1
Maximum allowed values (ppm) for some pollutants.

Pollutants	Maximum value
Benzene	0.05
Toluene	0.25
Ethylbenzene	0.15
Xylenes	0.15
Phenols	5
BOC	700
OIS	545
Ammoniacal compounds	150
Nitrate compounds	100

Table 2
Maximum allowed values (ppm) for some pollutants after limiting the emissions of the three groups of pollutants.

Pollutants	Original maximum value	New maximum value
Benzene	0.05	0.036
Toluene	0.25	0.179
Ethylbenzene	0.15	0.107
Xylenes	0.15	0.107
Phenols	5	3.571
BOC	700	509.639
OIS	545	396.79
Ammoniacal compounds	150	90
Nitrate compounds	100	60

of pollutants than by each of the pollutants themselves, respecting, at the same time, the limitations on the emissions of each substance. Suppose the city council sets the following limits for each of the groups of pollutants: 4 ppm for benzenoids, 1000 ppm for oxygen-demanding wastes, and 150 ppm for nitrogen eutrophic nutrients. The emission limit scenario is shown in Fig. 3.

Associated with the situation described above we consider the following multi-issue allocation problem with crossed claims $MAC = (I, N, E, c, \alpha)$ with

- $I = \{1, 2, 3\}$;
- $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$;
- $E = (4, 1000, 150)$;
- $c = (0.05, 0.25, 0.15, 0.15, 5, 700, 545, 150, 100)$; and
- $\alpha(1) = \{1\}$, $\alpha(2) = \{1\}$, $\alpha(3) = \{1\}$, $\alpha(4) = \{1\}$, $\alpha(5) = \{1, 2\}$, $\alpha(6) = \{2\}$, $\alpha(7) = \{2\}$, $\alpha(8) = \{2, 3\}$, and $\alpha(9) = \{3\}$.

It is obvious that the emission limits for each of the three pollutant categories are not sufficient to guarantee the emission limits for each of the substances; therefore, their limits must be recalculated. To do this, we can now apply the procedure described above to calculate the constrained proportional awards rule for MAC problems.

Step 1. $\mathcal{N}^1 = N$, $\mathcal{I}^1 = I$,
$E^1 = (4, 1000, 150)$, $c^1 = (0.05, 0.25, 0.15, 0.15, 5, 700, 545, 150, 100)$
– $\lambda_1^1 = 0.714$
– $\lambda_2^1 = 0.714$
– $\lambda_3^1 = 0.600$
$\lambda^1 = \min\{0.714, 0.714, 0.600\} = 0.600$

$$a_1^1 = 0.03, a_2^1 = 0.15, a_3^1 = 0.09, a_4^1 = 0.09, a_5^1 = 3, a_6^1 = 420, a_7^1 = 327, a_8^1 = 90, a_9^1 = 60$$

Step 2. $\mathcal{N}^2 = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{I}^2 = \{1, 2\}$,
$E^2 = (0.64, 160, 0)$, $c^2 = (0.02, 0.1, 0.06, 0.06, 2, 280, 218, 60, 40)$
– $\lambda_1^2 = 0.286$
– $\lambda_2^2 = 0.32$
$\lambda^2 = \min\{0.286, 0.32\} = 0.286$

$$a_1^2 = 0.01, a_2^2 = 0.03, a_3^2 = 0.02, a_4^2 = 0.02, a_5^2 = 0.57, a_6^2 = 80, a_7^2 = 62.29, a_8^2 = 0, a_9^2 = 0$$

Step 3. $\mathcal{N}^3 = \{6, 7\}$, $\mathcal{I}^2 = \{2\}$,
$E^3 = (0, 17.143, 0)$, $c^3 = (0.014, 0.071, 0.043, 0.043, 1.429, 200, 155.714, 60, 40)$
– $\lambda_2^3 = 0.048$
$\lambda^3 = \min\{0.048\} = 0.048$

$$a_1^2 = 0, a_2^2 = 0, a_3^2 = 0, a_4^2 = 0, a_5^2 = 0, a_6^2 = 9.64, a_7^2 = 7.50, a_8^2 = 0, a_9^2 = 0$$

Step 4. $\mathcal{N}^4 = \varnothing$, $\mathcal{I}^2 = \varnothing$.

$$CPA(\mathcal{I}, N, E, c, \alpha) = (0.036, 0.179, 0.107, 0.107, 3.571, 509.639, 396.79, 90, 60).$$

Therefore, if the local government wants to limit the emissions of the three categories of pollutants below 4 ppm, 1000 ppm, and 150 ppm but keeps a fixed limit of emissions for each of the pollutant substances, an alternative would be to limit the emissions of the pollutant substances to the new values in the third column of Table 2.

Thus, we can observe how the constrained proportional awards rule for multi-issue problems with crossed claims can help authorities to design new water quality policies, in particular, how the limits of emissions of different substances can be established considering the categories of water pollutants that have different effects on the quality of water.

As already mentioned in the introduction, water quality control management usually falls on local authorities, as they are the institutions closest to the problem. Normally, local authorities tend to set limits on discharges of pollutants to maintain adequate levels of water quality for use. These limits are usually modified from time to time, however, conditions can change in short periods of time, so having management tools that allow you to quickly adapt to changes is relevant. As shown in the numerical example, the proposed management methodology responds quite well to this more dynamic type of management that allows the pollutant emission limits to be adapted in a timely manner. In addition, the use of the pollutant categories and the constrained proportional awards rule means that the adaptations are staggered considering the different levels of abatement in the categories, as seen in Table 2. Moreover, if a category of pollutants did not experience reductions in its discharges, the limits on pollutants that only belonged to that category would not change.

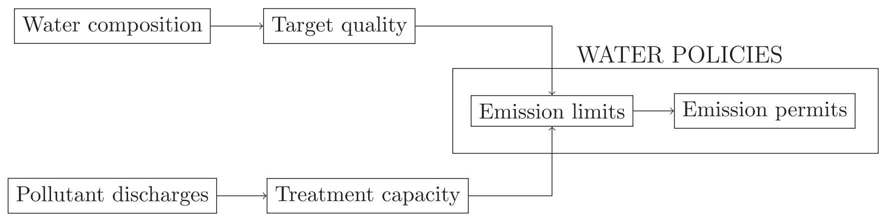

Fig. 4. Some relationships between different elements for the design of water quality policies.

8. Discussion and conclusion

As already stated in the Introduction, water resource management includes two major issues: water quantity (when there is scarcity) and water quality (when there is degradation). In both cases, allocation problems usually arise, and one way to address them is through game theory (see Dinar & Hogarth, 2015). The use of game theory to solve this type of problem seems adequate because the solutions are usually based on principles that seek the reasonableness and acceptability of the result for all the parties involved. This is in line with what was proposed in Liu et al. (2011). In the case of water quality, the most common allocation problems are the distribution of costs of water cleaning treatments and wastewater (or pollutant) discharge permits (see Section 2). In this paper, we address a different allocation problem in the design of water quality policies, the problem of establishing limits of emissions of pollutant substances when an authority wants to guarantee different parameters of water quality, which are affected by the main categories of water pollutants. Based on this approach, we propose a new methodology to manage pollutant emissions that is illustrated with a numerical example. Although linked to a real-world case, the allocation produced by the suggested method cannot be contrasted in a real-world decision-making context or discussed with relevant stakeholders, which is a limitation of this work.

In the design of water quality policies, it is necessary to consider, on the one hand, the physical-chemical composition of the water and the target of its composition in accordance with the use to be given to the water and, on the other hand, pollutant discharges and the capacity for water cleaning treatment of these substances (see Fig. 4). All these elements can change for different reasons over time, so the design of water quality management policies must be adaptable, flexible and coordinated (see, Beck, 1981; Helmer & Hespanhol, 1997). This means that it is interesting to have a methodology such as the one presented here to systematically recalculate the emission limits of pollutants when the conditions change, for example from a dry to a wet period.

In water quality policies, emission limits are usually set for each of the polluting substances (see, for example, Table 1 for the case presented in Section 7). However, these substances can have more than one negative effect on water quality, so they could belong to several categories of pollutants (see, for example, Fig. 3 in Section 7). This should lead to consideration of a more complex approach that goes from the emission limits singled out by substances to the limits of the categories of polluting substances according to their effects on water quality. This does not mean that limits are not set for each polluting substance, but that the category structure of pollutants should also be considered, which leads to the approach that has been presented in this work for the design of water quality policies. In addition, both Beck (1981) and Helmer & Hespanhol (1997) suggest the use of more complex models for better management of water quality control, given that water is an essential resource for the existence of life.

Moreover, as has been shown throughout this work, the introduction of greater complexity in the problem of setting emission limits for polluting substances into water has not meant an excessive practical difficulty of calculation, as shown in the numerical example in Section 7 in contrast, it does have greater theoretical complexity. Therefore, an applicable methodology in water quality management is proposed, which is supported by an adequate mathematical foundation.

On the other hand, Endres (1985) and Kuosmanen & Laukkanen (2011) propose designing environmental policies that consider pollutant interactions. In this work, this approach is followed in some way since the problem is approached from the perspective of the categories of pollutants and interactions among the different pollutants are considered. In this sense, the interactions considered in this work are additive and other types of interactions (non-linear relationships) would be interesting to study in the future.

Although it seems reasonable to use the well-known principle of proportionality as it is done in this work, to analyse the use of other solutions to bankruptcy problems based on other principles of fairness different from proportionality, such as the random arrival rule (O'Neill, 1982) or the Talmud rule (Aumann & Maschler, 1985), would be of interest. The first of these rules is related to the well-known Shapley value (Shapley, 1953), which satisfies desirable properties in line with principles of fairness (Algaba et al., 2019c), see also Algaba et al. (2019b) for an update on theoretical and applied aspects of this value, while the second is related to the nucleolus (Schmeidler, 1969). However, the first seems easier to study than the second in the context of multi-issue allocation problems with crossed claims.

Finally, we can find other real cases in which the proposed approach could be applied. For example, in Section 6 of Acosta et al. (2022) the similarity between multi-issue bankruptcy problems with crossed claims and set covering problems, using the results of Bergantiños et al. (2020), is established. Therefore, the methodology described here could be applied to allocation costs in set covering problems. Another type of real case where this approach could be applied is in material resource planning (MRP) systems. In situations where several raw materials are available and used by different production plants, the raw materials must be allocated to the production plants for their operation. Obviously, once a certain resource is assigned to a production plant, other resources have to be assigned to it so that production is possible, since several resources are needed to produce. The different material resources play the role of issues, and the production plants play the role of claimants. The structure of crossed demands would be derived from the material resources necessary for each production plant. In this way, the methodology proposed in this work could also be used for the allocation of material resources to different production plants.

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