Solutions for the Surface Quasigeostrophic equation

Trabajo fin de grado

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In hindsight, these five years studying mathematics and physics have been a beautiful endeavour. I would like to dedicate this project to everyone that has been close to me and helped me throughout. You know who you are. Thank you for nudging me towards a life of purpose and intention. Especially, I would like to dedicate this to my grandmother Mercedes, my staunchest protector.

Forever grateful,

Tomás

Contents

Abstract

The main aim of this project is to understand the surface quasi-geostrophic model (SQG) and prove the existence of global weak solutions under certain assumptions. Not only do we aim to understand the physics behind this model, but we also want to study it from a mathematical point of view.

The SQG system, which is derived in the situation of small Rossby and Ekman numbers and constant potential vorticity, is physically important as a model to explain atmospheric and oceanic dynamics. Moreover, it has also been studied mathematically on the grounds of having similarities with three-dimensional Euler equations.

In the first chapter, the main mathematical concepts that are necessary to tackle the SQG model are introduced. Among these, we find the definition of the Fourier transform on the n-dimensional torus as well as some important concepts of functional analysis, which are the cornerstone of the mathematical study that will be carried out in the forthcoming chapters.

In contrast, the second chapter provides some key concepts with regard to fluids, such as vorticity and geostrophic flow. Since the SQG model stems from physical considerations, it is, therefore, necessary to possess this background information. At the end of the chapter, the SQG system is shown (not derived yet) and compared to the three-dimensional Euler equations.

Next, in the third chapter, we derive the mathematical formulation of the SQG system. In order to do so, all the physical approximations are established previously, including the geostrophic and hydrostatic ones. In addition, the conservation of quasi-geostrophic potential vorticity, on which the SQG model is based, will be proved in detail.

In the fourth chapter, once the quasi-geostrophic setting has been introduced, we are finally ready to prove the important theorem of this project, which states the existence of global weak solutions for the model under study. Nevertheless, uniqueness is still an open problem.

Keywords: Fourier transform; potential vorticity; buoyancy; geostrophic flow; quasi-geostrophic; weak solution

Resumen

El objetivo principal de este trabajo es comprender el modelo ”surface quasi-geostrophic” (SQG) y probar la existencia de soluciones débiles globales bajo ciertas hipótesis. No solo pretendemos comprender la física detrás de este modelo, sino que también queremos estudiarlo desde un punto de vista matemático.

El sistema SQG, que se obtiene cuando las constantes de Rossby y Ekman son pequeños y la vorticidad potencial constante, es un modelo de importancia a nivel físico, pues permite explicar la dinámica atmosférica y oceánica. Además, también ha sido estudiado matemáticamente por tener similitudes con las ecuaciones de Euler tridimensionales.

En el primer capítulo se introducen los principales conceptos matemáticos necesarios para trabajar con el modelo SQG. Entre ellos encontramos la definición de la transformada de Fourier sobre el toro n-dimensional, así como algunos conceptos importantes del análisis funcional, que son de vital importancia para el estudio matemático que se llevará a cabo en los próximos capítulos.

Por el contrario, el segundo capítulo proporciona algunas definiciones importantes para el estudio de los fluidos, como la vorticidad y el flujo geostrófico. Dado que el modelo SQG se obtiene a partir de consideraciones físicas, es necesario conocer estos conceptos previamente. Al final del capítulo, se muestra el sistema SQG (todavía no deducido) y se compara con las ecuaciones de Euler tridimensionales. La unicidad sigue siendo un problema abierto.

A continuación, en el tercer capítulo, deducimos la formulación matemática del sistema SQG. Para ello se introducen previamente todas las aproximaciones físicas que son necesarias, incluidas las geostróficas e hidrostáticas. Además, se probará en detalle la conservación de la vorticidad potencial cuasi-geostrófica, en la que se basa el modelo SQG.

Una vez que se han explicado las raíces físicas del modelo SQG, en el cuarto y último capítulo probamos el teorema importante de este trabajo, que establece la existencia de soluciones débiles globales. Sin embargo, la unicidad sigue siendo un problema abierto.

Palabras clave: Transformada de Fourier; vorticidad potencial; empuje; flujo geostrófico; cuasi-geostrófico; solución débil

Chapter 1

Mathematical preliminaries

The aim of this section is to provide the reader with the necessary mathematical tools that we shall use in the forthcoming chapters. The first topic that we will tackle is Hilbert and Banach spaces, for the important role that they play as far as partial differential equations analysis is concerned. Whereas spaces like C^k or C^∞ have limitations when dealing with the solutions of partial differential equations of mathematical physics, functions belonging to Sobolev spaces, which are Banach spaces, represent a good compromise as they have some smoothness properties [1].

Moving on to the second subsection, we will define the Fourier transform in the torus and prove some important results, such as Plancherel's identity or Parseval's relation, which are rather powerful mathematical tools as we will see.

Thirdly, in the subsection Schwartz class and distributions, we will show the main definitions and prove some results as well. For further information, it is advisable to see an analysis book, for instance, [2]. The salient result of this part is the fact that we will be able to extend the concept of Fourier transform to a type of distribution.

Eventually, we will introduce the Riesz transform. In hindsight, this concept is fundamental with regard to the SQG equations, to be introduced later.

1.1 Hilbert and Banach spaces

Beginning with, let $e_j : 1 \leq j \leq n$ be the canonical basis in $\mathbb{R}^n$. We say that a function $f : \mathbb{R}^n \to \mathbb{C}$ is 2π-periodic in each variable if

$$f(x + 2\pi e_j) = f(x), \quad \forall x \in \mathbb{R}^n, 1 \leq j \leq n$$

The n-torus is defined as $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. That is, the n-dimensional torus is the quotient of the Euclidean space $\mathbb{R}^n$ by the set $(2\pi\mathbb{Z})^n$, so we identify equivalence classes with their canonical representatives in $[0, 2\pi)$. In other words, for $x, y \in \mathbb{R}^n$, we say that

$$x \equiv y$$

if $x - y \in (2\pi\mathbb{Z})^n$. In the case $n = 1$, this set can be geometrically viewed as a circle by bending the segment $[0, 2\pi]$ so that its endpoints are brought together. Whereas in the case $n = 2$, every point in $\mathbb{R}^2$ is identified with a point in the square $[0, 2\pi]^2$ and, in addition, the identification brings together the left and right sides of the square and also the top and bottom ones. It is well-known that the resulting figure in the two-dimensional case is a surface in $\mathbb{R}^3$ that looks like a donut. See Figure 1.1 below.

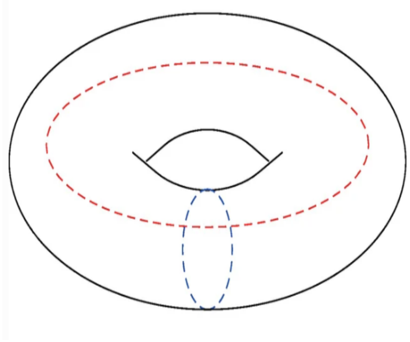

Figure 1.1: Graph of the two-dimensional torus $\mathbb{T}^2$. Image has been taken from [3].

Functions on $\mathbb{T}^n$ are functions on $\mathbb{R}^n$ that satisfy $f(x + m) = f(x)$ for all $x \in \mathbb{R}^n$ and $m \in (2\pi\mathbb{Z})^n$, so they are clearly 2π-periodic in each variable. Although the spaces that will be introduced later are typically defined on $\mathbb{R}^n$, in this project, they will be defined on the torus $\mathbb{T}^n$, for reasons of convenience that we will see in the forthcoming chapters. For further knowledge on this topic, I suggest consulting the book [4] and the article [5].

Definition 1.1 (Multi-index notation)**.** *Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$ be an n-tuple of non-negative integers. Then we define:*

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}, \quad \text{for } x \in \mathbb{R}^n,$$

$$|\alpha| := \sum_{j=1}^{n} \alpha_j,$$

$$\partial^\alpha := \prod_{j=1}^{n} \left(\frac{\partial}{\partial x_j} \right)^{\alpha_j} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

Definition 1.2 ($L^2(\mathbb{T}^n)$ space)**.** *$L^2(\mathbb{T}^n)$ is the space of the square-integrable measurable functions on $\mathbb{T}^n$:*

$$L^2(\mathbb{T}^n) = \left\{ f(x) \text{ Lebesgue measurable in } \mathbb{T}^n : \int_{\mathbb{T}^n} |f(x)|^2 dx < \infty \right\}.$$

Clearly, $L^2(\mathbb{T}^n)$ is a linear space, since if $f, g \in L^2$, then $\alpha f + \beta g \in L^2$ for any $\alpha, \beta \in \mathbb{R}$.

Definition 1.3 (Inner product)**.** *Let V be a vector space over $\mathbb{R}$. An inner product $\langle \cdot, \cdot \rangle$ is a function $V \times V \to \mathbb{C}$ with the following properties:*

1. *$\forall u \in V, \langle u, u \rangle \geq 0,$ and $\langle u, u \rangle = 0 \Leftrightarrow u = 0.$*
2. *$\forall u, v \in V,$ holds $\langle u, v \rangle = \langle v, u \rangle$.*
3. *$\forall u, v, w \in V,$ and $\forall \alpha, \beta \in \mathbb{R}$ holds $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$.*

Definition 1.4 (Inner product on $L^2(\mathbb{T}^n)$). *Let $f, g \in L^2(\mathbb{T}^n)$. Then we define the inner product:*

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{T}^n} f(x)\overline{g(x)}dx. \tag{1.1}$$

Furthermore, the norm on $L^2(\mathbb{T}^n)$ is defined by this inner product:

$$\|f\|_{L^2} = \sqrt{\langle f, f \rangle_{L^2}} = \left(\int_{\mathbb{T}^n} |f(x)|^2 dx \right)^{1/2}. \tag{1.2}$$

To see that (1.1) is indeed an inner product see pages 386 and 387 in the book [6].

Definition 1.5 (Banach space). *A Banach space is a linear space that is normed and complete.*

Definition 1.6 (Hilbert space). *A Hilbert space is a Banach space whose norm is given by an inner product.*

For instance, $L^2(\mathbb{T}^n)$ is a Hilbert space. We refer to [6] to see the proof. Other examples of Hilbert and Banach spaces are the following:

1. Sobolev spaces $H^k(\mathbb{T}^n)$ (Hilbert spaces based on L^2 norms):

$$H^k(\mathbb{T}^n) = \left\{ f(x) : \sum_{0 \le |\alpha| \le k} \left(\int_{\mathbb{T}^n} |\partial^\alpha f(x)|^2 dx \right)^{1/2} < \infty \right\}. \tag{1.3}$$

   Moreover, on the grounds of (1.3) it holds that $H^0(\mathbb{T}^n) = L^2(\mathbb{T}^n)$ and $H^k \subseteq H^{k-1} \subseteq ... \subseteq L^2$. The inner product in H^k is defined in terms of the L^2 inner product:

$$\langle f, g \rangle_{H^k} = \sum_{i=0}^{s} \left\langle \partial^i f, \partial^i g \right\rangle_{L^2}. \tag{1.4}$$

2. L^p spaces (Banach spaces):

$$L^p(\mathbb{T}^n) = \left\{ f(x) : \left(\int_{\mathbb{T}^n} |f(x)|^p dx \right)^{1/p} < \infty \right\}, \quad 1 \le p < \infty. \tag{1.5}$$

   For $1 \le p < \infty$, the norm in L^p is given by

$$\|f\|_{L^p} = \left(\int_{\mathbb{T}^n} |f(x)|^p dx \right)^{1/p}. \tag{1.6}$$

   For $p = \infty$, L^∞ denotes the space of bounded measurable functions on T^n and its norm is

$$\|f\|_{L^\infty} = \sup_{x \in \mathbb{T}^n} |f(x)|. \tag{1.7}$$

The many relationships of inclusion among these spaces, and their associated inequalities of norms, are important information for the analysis of PDE (see, for instance, [1] and [7]).

We finish this section with some definitions that will be used later.

Definition 1.7 (Linear functional)**.** *Let V be a vector space over a scalar field k. We say that $u : V \to k$ is a linear functional on V if it satisfies*

1. $u(v_1 + v_2) = u(v_1) + u(v_2), \forall v_1, v_2 \in V$,
2. $u(\alpha v) = \alpha u(v), \forall \alpha \in k, v \in V$.

Definition 1.8 (Dual space)**.** *Let X be a normed space. The dual of X is the vectorial space X' whose elements are continuous linear functionals on X.*

If $(X, \|\cdot\|_X)$ is a normed space and $f \in X'$, then

$$\|f\|_{X'} = \sup \{|f(x)| : x \in X, \|x\|_X \leq 1\}$$

defines a norm in X'. For further reading see [8].

Definition 1.9 (Weak convergence)**.** *Let X be a normed space. A sequence $\{x'_n\} \subset X'$ is weak* convergent to $x' \in X'$ and we write $x'_n \overset{*}{\rightharpoonup} x'$ if*

$$\langle x'_n, x \rangle \to \langle x', x \rangle, \forall x \in X.$$

1.2 Fourier transform on the torus

In this section, we will define the Fourier transform and prove a couple of important results of functional analysis that follow from it, for instance, Plancherel's identity and Parseval's relation.

Definition 1.10 (Fourier transform on the torus $\mathbb{T}^n$)**.** *Let $f \in L^1(\mathbb{T}^n)$ be a complex-valued function. The Fourier series of f on the torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ is*

$$\sum_{m \in \mathbb{Z}^n} \widehat{f}(m) e^{im \cdot x}, \tag{1.8}$$

where the Fourier coefficients are given by

$$\widehat{f}(m) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(x) e^{-im \cdot x} dx. \tag{1.9}$$

Proposition 1.1. *Let $f, g \in L^1(\mathbb{T}^n)$. Then for all $m, k \in \mathbb{Z}^n$, and $\lambda \in \mathbb{C}$ we have*

1. $\widehat{f + g}(m) = \widehat{f}(m) + \widehat{g}(m)$.
2. $\widehat{\lambda f}(m) = \lambda \widehat{f}(m)$.
3. $\widehat{\overline{f}}(m) = \overline{\widehat{f}(-m)}$.
4. $(e^{ik(\cdot)} f)^\wedge(m) = \widehat{f}(m - k)$.

Proof. 1. $\widehat{f+g}(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}(f(x)+g(x))e^{-im\cdot x}dx$

$$= \frac{1}{(2\pi)^n}\left(\int_{\mathbb{T}^n} f(x)e^{-im\cdot x}dx + \int_{\mathbb{T}^n} g(x)e^{-im\cdot x}dx\right)$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(x)e^{-im\cdot x}dx + \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} g(x)e^{-im\cdot x}dx = \widehat{f}(m)+\widehat{g}(m).$$

2. $\widehat{\lambda f}(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}(\lambda f(x))e^{-im\cdot x}dx = \lambda\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(x)e^{-im\cdot x}dx = \lambda\widehat{f}(m).$

3. $\widehat{\overline{f}}(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\overline{f(x)}e^{-im\cdot x}dx = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\overline{f(x)e^{im\cdot x}}dx = \frac{1}{(2\pi)^n}\overline{\int_{\mathbb{T}^n} f(x)e^{im\cdot x}dx}$
$= \overline{\widehat{f}(-m)}.$

4. $(e^{ik(\cdot)}f)^\wedge(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{ik\cdot x}f(x)e^{-im\cdot x}dx = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(x)e^{-i(m-k)\cdot x}dx$
$= \widehat{f}(m-k).$

□

We refer to [4] for the proof of the following proposition.

Proposition 1.2 (Fourier inversion)**.** *Suppose that $f \in L^1(\mathbb{T}^n)$ and that*

$$\sum_{m\in\mathbb{Z}^n}|\widehat{f}(m)| < \infty.$$

Then

$$f(x) = \sum_{m\in\mathbb{Z}^n}\widehat{f}(m)e^{im\cdot x} \quad a.e.,$$

and therefore f is almost everywhere equal to a continuous function.

Once the Fourier transform has been introduced, it is possible to give an alternative characterisation of the space H^s than (1.3), as stated in [9]. In fact, a function $f \in L^2(\mathbb{T}^n)$ belongs to $H^s(\mathbb{T}^n)$ if and only if

$$\sum_{m\in\mathbb{Z}^n}\left(1+|m|^2\right)^s|\widehat{f}(m)|^2 < \infty, \tag{1.10}$$

and the H^s norm is given by

$$\|f\|_{H^s} = \left(\sum_{\mathbb{Z}^n}\left(1+|m|^2\right)^s|\widehat{f}(m)|^2\right)^{1/2}.$$

What is more, the dual of $H^s(\mathbb{T}^n)$ is the space $H^{-s}(\mathbb{T}^n)$, whose norm is defined as

$$\|f\|_{H^{-s}} = \left(\sum_{\mathbb{Z}^n}\frac{1}{(1+|m|^2)^s}|\widehat{f}(m)|^2\right)^{1/2}.$$

Definition 1.11 (Complete orthonormal system)**.** *Let H be a separable Hilbert space with complex inner product $\langle \cdot, \cdot \rangle$. A subset $E \subset H$ is called orthonormal if $\langle f, g \rangle = 0$ for all f, g in E with $f \neq g$, while $\langle f, f \rangle = 1$ for all f in E. A complete orthonormal system is a subset of H with the additional property that the only vector orthogonal to all of its elements is the zero vector.*

Proposition 1.3. *Let H be a separable Hilbert space and let $\{\varphi_k\}_{k\in\mathbb{Z}}$ be an orthonormal system in H. Then the following are equivalent:*

1. *$\{\varphi_k\}_{k\in\mathbb{Z}}$ is a complete orthonormal system.*
2. *For every $f \in H$ we have*
$$\|f\|_H^2 = \sum_{k\in\mathbb{Z}} | \langle f, \varphi_k \rangle |^2.$$

The proof of Proposition 1.3 can be found in Rudin [10].

Let us now recall that $L^2(\mathbb{T}^n)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{T}^n} f(x)\overline{g(x)}dx.$$

Theorem 1.1 (Plancherel's identity)**.** *Let $f \in L^2(\mathbb{T}^n)$. It holds that*

$$\|f\|_{L^2}^2 = (2\pi)^n \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)|^2 \tag{1.11}$$

Proof. Let us compute the following integral in one dimension:

$$\int_0^{2\pi} e^{imx}\overline{e^{ikx}}dx = \int_0^{2\pi} e^{i(m-k)x}dx = \int_0^{2\pi} \cos((m-k)x)dx + i\int_0^{2\pi} \sin((m-k)x)dx. \tag{1.12}$$

Thus, on the one hand, it is clear that if $m = k$, then (1.12) is equal to 2π. On the other hand, if $m \neq k$ then

$$\int \cos((m-k)x)dx = \frac{\sin((m-k)x)}{m-k} \Rightarrow \int_0^{2\pi} \cos((m-k)x)dx = 0,$$

$$\int \sin((m-k)x)dx = -\frac{\cos((m-k)x)}{m-k} \Rightarrow \int_0^{2\pi} \sin((m-k)x)dx = 0,$$

and the integral (1.12) vanishes. Hence, if we define $\{\varphi_m\}$ as the sequence of functions $\xi \longmapsto \frac{1}{\sqrt{(2\pi)^n}} e^{im\cdot\xi}$ indexed by $m \in \mathbb{Z}^n$ we have

$$\int_{[0,2\pi]^n} \varphi_m(x)\overline{\varphi_k(x)}dx = \begin{cases} 1 & \text{if } m = k, \\ 0 & \text{if } m \neq k. \end{cases}$$

So the sequence $\{\varphi_m\}$ is an orthonormal set of functions. Now we are left to show the completeness. It holds that

$$\langle f, \varphi_m \rangle = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{T}^n} f(x)e^{-im\cdot x}dx = \sqrt{(2\pi)^n}\widehat{f}(m), \forall f \in L^2(\mathbb{T}^n).$$

Thus, if $\langle f, \varphi_m \rangle = 0$ for all $m \in \mathbb{Z}^n$ then $\widehat{f}(m) = 0$ for all $m \in \mathbb{Z}^n$, and by Proposition 1.2 we have that $f = 0$ a.e. Hence, $\{\varphi_m\}$ is a complete orthonormal set and we can apply Proposition 1.3 obtaining

$$\|f\|_{L^2}^2 = \sum_{m\in\mathbb{Z}^n} (2\pi)^n |\widehat{f}(m)|^2 = (2\pi)^n \sum_{k\in\mathbb{Z}^n} |\widehat{f}(m)|^2.$$

□

Theorem 1.2 (Parseval's relation)**.** *Let $f, g \in L^2(\mathbb{T}^n)$. It holds that*

$$\int_{\mathbb{T}^n} f\overline{g}dx = (2\pi)^n \sum_{m\in\mathbb{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)}. \tag{1.13}$$

Proof. First, we replace $f + g$ in (1.11) and it gives

$$\|f+g\|_{L^2}^2 = (2\pi)^n \sum_{m\in\mathbb{Z}^n} |\widehat{(f+g)}(m)|^2.$$

Now we expand the squares. On the one hand,

$$\begin{aligned} \|f+g\|_{L^2}^2 &= \int_{\mathbb{T}^2} |f(x)+g(x)|^2 dx = \int_{\mathbb{T}^2} (f(x)+g(x)) \cdot (\overline{f(x)+g(x)}) dx \\ &= \int_{\mathbb{T}^2} (f(x)+g(x)) \cdot (\overline{f(x)} + \overline{g(x)}) dx = \int_{\mathbb{T}^2} |f(x)|^2 dx + \int_{\mathbb{T}^2} |g(x)|^2 dx \\ &+ \int_{\mathbb{T}^2} \left(f(x)\overline{g(x)} + g(x)\overline{f(x)} \right) dx. \end{aligned} \tag{1.14}$$

On the other hand,

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n} |\widehat{(f+g)}(m)|^2 &= \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m) + \widehat{g}(m)|^2 = \sum_{m\in\mathbb{Z}^n} (\widehat{f}(m) + \widehat{g}(m)) \cdot (\overline{\widehat{f}(m)} + \overline{\widehat{g}(m)}) \\ &= \sum_{m\in\mathbb{Z}^n} \left(|\widehat{f}(m)|^2 + |\widehat{g}(m)|^2 + \widehat{f}(m)\overline{\widehat{g}(m)} + \widehat{g}(m)\overline{\widehat{f}(m)} \right). \end{aligned} \tag{1.15}$$

Recalling that according to (1.11) we have

$$\int_{\mathbb{T}^2} |f(x)|^2 dx = (2\pi)^n \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)|^2 < \infty,$$

$$\int_{\mathbb{T}^2} |g(x)|^2 dx = (2\pi)^n \sum_{m\in\mathbb{Z}^n} |\widehat{g}(m)|^2 < \infty.$$

We can easily compare equations (1.14) and (1.15) and derive the following identity:

$$\int_{\mathbb{T}^2} \left(f(x)\overline{g(x)} + g(x)\overline{f(x)} \right) dx = (2\pi)^n \sum_{m\in\mathbb{Z}^n} \left(\widehat{f}(m)\overline{\widehat{g}(m)} + \widehat{g}(m)\overline{\widehat{f}(m)} \right). \tag{1.16}$$

Next, we replace $g + if$ in 1.11 and expand the squares, as we did previously.

$$\|g+if\|_{L^2}^2 = (2\pi)^n \sum_{m\in\mathbb{Z}^n} |\widehat{(g+if)}(m)|^2.$$

The left-hand side term expands as

$$\begin{aligned}\|if+g\|_{L^2}^2 &= \int_{\mathbb{T}^2} |if(x)+g(x)|^2 dx = \int_{\mathbb{T}^2} (if(x)+g(x))\cdot(\overline{if(x)+g(x)})dx \\ &= \int_{\mathbb{T}^2} (if(x)+g(x))\cdot(\overline{g(x)} - i\overline{f(x)})dx = \int_{\mathbb{T}^2} |f(x)|^2 dx + \int_{\mathbb{T}^2} |g(x)|^2 dx \\ &+ \int_{\mathbb{T}^2} \left(if(x)\overline{g(x)} - ig(x)\overline{f(x)}\right) dx.\end{aligned} \tag{1.17}$$

Whereas for the right-hand side term, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |(\widehat{if+g})(m)|^2 &= \sum_{m\in\mathbb{Z}^n} |i\widehat{f}(m)+\widehat{g}(m)|^2 = \sum_{m\in\mathbb{Z}^n} (i\widehat{f}(m)+\widehat{g}(m))\cdot(\overline{i\widehat{f}(m)}+\overline{\widehat{g}(m)}) \\ &= \sum_{m\in\mathbb{Z}^n} \left(|\widehat{f}(m)|^2 + |\widehat{g}(m)|^2 + i\widehat{f}(m)\overline{\widehat{g}(m)} - i\widehat{g}(m)\overline{\widehat{f}(m)}\right).\end{aligned} \tag{1.18}$$

Comparing 1.17 and 1.18 yields the following identity:

$$\int_{\mathbb{T}^2} \left(f(x)\overline{g(x)} - g(x)\overline{f(x)}\right) dx = (2\pi)^n \sum_{k\in\mathbb{Z}^n} \left(\widehat{f}(k)\overline{\widehat{g}(k)} - \widehat{g}(k)\overline{\widehat{f}(k)}\right). \tag{1.19}$$

Eventually, we add equations 1.16 and 1.19 to get:

$$\int_{\mathbb{T}^n} f(x)\overline{g}(x)dx = (2\pi)^n \sum_{m\in\mathbb{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)},$$

which is Parseval's relation. □

Corollary 1.1. *For all $f, g \in L^2(\mathbb{T}^n)$ and $m \in \mathbb{Z}^n$ we have*

$$\widehat{fg}(m) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)\widehat{g}(m-k) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(m-k)\widehat{g}(k). \tag{1.20}$$

Proof. First of all, one has

$$\widehat{fg}(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(x)g(x)e^{-im\cdot x}dx = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(x)\overline{\overline{g(x)}e^{im\cdot x}}dx.$$

Next, Parseval's relation 1.13 yields

$$\int_{\mathbb{T}^n} f(x)\overline{\overline{g(x)}e^{im\cdot x}}dx = (2\pi)^n \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)\overline{\widehat{\overline{g(x)}e^{im\cdot x}}(k)}$$

. Using Proposition 1.1 (3) and (4) we eventually obtain

$$\widehat{fg}(m) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)\overline{\left(\overline{g(x)}e^{im\cdot x}\right)^{\wedge}(k)} = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)\overline{\widehat{\overline{g}}(k-m)} = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)\widehat{g}(m-k).$$

□

1.3 Schwartz class and distributions

Definition 1.12 (Schwartz class). *The Schwartz class, represented by $\mathcal{S}$, is the function space of all C^∞ functions that are 2π-periodic in each variable. That is to say*

$$\mathcal{S} = \mathcal{S}(\mathbb{T}^n, \mathbb{C}) := \{f : f \in C^\infty(\mathbb{T}^n, \mathbb{C})\}.$$

Remark 1.1. *It follows directly from definition 1.12 that*

$$\mathcal{S} = C^\infty(\mathbb{T}^n, \mathbb{C}) = C_c^\infty(\mathbb{T}^n, \mathbb{C}).$$

Proposition 1.4. *Suppose $f \in \mathcal{S}$. The following identity holds:*

$$(\partial^\beta f)^\wedge = (im)^\beta \widehat{f}.$$

Proof.

$$\begin{aligned}(\partial^\beta f)^\wedge(m) &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{-im\cdot x}\partial^\beta f(x)dx = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{-im\cdot x}\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}...\partial x_n^{\beta_n}}f(x)dx \\ &= \frac{(im)^\beta}{(2\pi)^n}\int_{\mathbb{T}^n} e^{-im\cdot x}f(x)dx = (im)^\beta \widehat{f}(m),\end{aligned}$$

where we have integrated by parts β times taking into account that the boundary terms vanish since f and its derivatives are periodic. ☐

The following result states an important property related to $\mathcal{S}$ and $L^p(\mathbb{T}^n)$ spaces, however, we will not provide its proof since it is not the core of this section. A proof can be found for instance on page 12 of [2].

Theorem 1.3. *$\mathcal{S}$ is dense in $L^p(\mathbb{T}^n)$ for $1 \le p < \infty$.*

Let $u : \mathcal{S} \to \mathbb{C}$ be a linear functional on the space $\mathcal{S}$ and let $\phi \in \mathcal{S}$ be a 2π-periodic function. We denote by $\langle u, \phi\rangle$ the number obtained by applying u to ϕ. Let us also consider in the Schwartz class $\mathcal{S}$ the topology defined by the norm

$$\|\phi\|_\alpha = \|\partial^\alpha \phi\|_{L^\infty}, \forall \alpha \in \mathbb{N}^n.$$

Hence

$$\phi_j \to \phi \text{ in } \mathcal{S} \Leftrightarrow \sup_{x\in\mathbb{T}^n} |\partial^\alpha(\phi(x) - \phi_j(x))| \to 0, \forall \alpha \in \mathbb{N}^n.$$

Definition 1.13 (Tempered distribution). *A tempered distribution is a continuous linear functional on $\mathcal{S}$. We denote by $\mathcal{S}'$ the space of tempered distributions.*

Given $u, v \in \mathcal{S}'$, we say that u=v on $\mathbb{T}^n$ if $\langle u, \phi\rangle = \langle v, \phi\rangle$ for all $\phi \in \mathcal{S}$. For further reading on this topic see [2].

Remark 1.2. *Every integrable function u on $\mathbb{T}^n$ can be regarded as a distribution if we define*

$$\langle u, \phi\rangle = \int_{\mathbb{T}^n} u(x)\phi(x)dx.$$

In this case, continuity stems from the Lebesgue dominated convergence theorem.

Now that the concept of distribution has been introduced, we move on to explain how we can extend operations from functions to distributions. We shall need a couple of definitions before.

Definition 1.14 (Linear operator)**.** *Let U and V be two vector spaces over a field k. We say that the map $T : U \to V$ is a linear operator if it satisfies*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall x, y \in U, \alpha, \beta \in k.$$

Although definition 1.14 provides the general definition, here we will only consider the case in which U and V are the same space and we will say that T is a linear operator on U.

Definition 1.15 (Dual of a linear operator)**.** *Let T be a linear operator on $\mathcal{S}$ that is continuous in the sense that if $\phi_j \to \phi$ in $\mathcal{S}$ then $T\phi_j \to T\phi$ in $\mathcal{S}$. Suppose there is another operator T' verifying*

$$\int_{\mathbb{T}^n} (T\phi)(x)\psi(x)dx = \int_{\mathbb{T}^n} \phi(x)(T'\psi)(x)dx, \forall \phi, \psi \in \mathcal{S}.$$

Then, we say that T' is the dual or transpose of T.

Taking into consideration definition 1.15, it is possible to extend T to act on distributions as

$$\langle Tu, \phi \rangle = \langle u, T'\phi \rangle . \tag{1.21}$$

Note that Tu on $\mathcal{S}$ is continuous since T' is assumed to be continuous.

The following proposition allows us to differentiate any distribution as many times as we want, obtaining other distributions.

Proposition 1.5. *Let $u \in \mathcal{S}'$ and $\phi \in \mathcal{S}$. Then*

$$\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle .$$

Proof. Let us define the operator $T = \partial^\alpha$ and let $\psi_1, \psi_2 \in \mathcal{S}$ be periodic functions. Integrating by parts $|\alpha|$ times we obtain

$$\int (T\psi_1)\psi_2 = \int (\partial^\alpha \psi_1)\psi_2 = (-1)^{|\alpha|} \int \psi_1(\partial^\alpha \psi_2).$$

Therefore, $T' = (-1)^{|\alpha|}\partial^\alpha$ and we conclude by (1.21). □

Definition 1.16 (Fourier transform of a tempered distribution)**.** *Given $u \in \mathcal{S}'$ we define $\widehat{u}$ via the formula*

$$\widehat{u}(m) = \frac{1}{(2\pi)^n} \left\langle u, e^{-im\cdot x} \right\rangle, \forall m \in \mathbb{Z}^n. \tag{1.22}$$

1.4 Riesz transform

Definition 1.17 (Riesz transform). *For any function $f \in L^2(\mathbb{T}^n)$, the Fourier transform of the j-th Riesz transform of f is given by*

$$\widehat{R_j f}(m) = \begin{cases} -i\dfrac{m_j}{|m|}\widehat{f}(m), & \text{if } |m| \neq 0, \\ 0, & \text{if } m = 0. \end{cases}$$

Proposition 1.6. *Let $f \in L^2(\mathbb{T}^n)$, then $R_j f \in L^2(\mathbb{T}^n)$.*

Proof. First of all, let us observe that

$$f \in L^2(\mathbb{T}^n) \Leftrightarrow \|f\|_{L^2}^2 < \infty \Leftrightarrow (2\pi)^n \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)|^2 < \infty \Leftrightarrow \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)|^2 < \infty.$$

Therefore

$$\sum_{m\in\mathbb{Z}^n} |\widehat{R_j f}(m)|^2 \leq \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)|^2 < \infty,$$

and it suffices to apply the same reasoning backward

$$\sum_{m\in\mathbb{Z}^n} |\widehat{R_j f}(m)|^2 < \infty \Leftrightarrow (2\pi)^n \sum_{m\in\mathbb{Z}^n} |\widehat{R_j f}(m)|^2 < \infty \Leftrightarrow \|R_j f\|_{L^2}^2 < \infty \Leftrightarrow R_j f \in L^2(\mathbb{T}^n).$$

□

Definition 1.17 endows the Riesz transform with usefulness regarding partial differential equations. In fact, suppose that $f \in \mathcal{S}$ is given and let u be a distribution such that $\nabla^2 u = f$, where ∇^2 denotes the Laplacian. Computing the Fourier transform in Laplace's equation gives

$$-|m|^2\widehat{u}(m) = \widehat{f}(m).$$

Hence, for all $1 \leq j, k \leq n$ one has

$$\widehat{\partial_j\partial_k u} = (im_j)(im_k)\widehat{u}(m) = -(im_j)(im_k)\frac{\widehat{f}(m)}{|m|^2} = -\widehat{R_j R_k(f)}. \tag{1.23}$$

In view of (1.23), we are capable of expressing the second-order derivatives of u in terms of the Riesz transforms of f. For further reading see [4].

Last, let us introduce a useful notation:

1. The perpendicular of a vector $v = (v_1, v_2)$ is $v^\perp = (-v_2, v_1)$.
2. $R^\perp f = (-R_2 f, R_1 f)$, where R_j is the j-th periodic Riesz transform (See Chapter 2 in [11]). Thus
$$\widehat{R^\perp f}(m) = -i\frac{m^\perp}{|m|}\widehat{f}(m).$$

Chapter 2

General insights about fluids and the SQG model

The Navier-Stokes equations are a set of second-order partial differential equations relating first and second derivatives of the vector field fluid velocity $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$, $u(x_1, ..., x_n, t) = (u_i(x_1, ..., x_n, t))_{1 \le i \le n}$ and first derivatives of the scalar field pressure $p : \mathbb{R}^n \times [0, \infty) \to (0, \infty)$, $p = p(x_1, ..., x_n, t)$. We will consider both to be C^∞ functions. Moreover, we take $n = 2$ or 3 for physical situations.

These equations have a wide range of applications in all areas of science as far as continuum phenomena are concerned. A fluid is said to be incompressible when its velocity field is divergence-free, i.e., $\nabla \cdot u = 0$, and in the case of an incompressible Newtonian fluid, the Navier-Stokes equations read

$$\rho \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \nu \nabla^2 u + \rho f, \tag{2.1a}$$

$$\nabla \cdot u = 0. \tag{2.1b}$$

Here, $f = (f_i(x_1, ..., x_n, t))_{1 \le i \le n}$ is the force term acting on every single fluid particle, ν is the viscosity of the fluid, which tells the ease with which the fluid flows when body forces are exerted on it, and $\rho = \rho(x, t)$ is the density of the fluid. For a derivation of (2.1a) see [12] or [13].

Regarding the motion of a fluid, the velocity field depends on both, position and time. Besides, the same applies to other physical properties of the fluid. Hence, it is useful to define a derivative that takes into consideration both, the rate of change and the change in position in the velocity field u. On the grounds of this reasoning, we define the material derivative as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla, \tag{2.2}$$

which is typically used in fluid dynamics.

Notice that using the material derivative, it is straightforward to rewrite 2.1a as

$$\rho \frac{Du}{Dt} = -\nabla p + \nu \nabla^2 u + \rho f.$$

In addition, on page 11 in [14] the well-known mass conservation equation is obtained

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{2.3}$$

or equivalently

$$\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0. \tag{2.4}$$

On the one hand, if ρ is constant, then 2.3 yields the incompressibility condition $\nabla \cdot u = 0$. On the other hand, according to 2.4, if our fluid is incompressible then its density ρ is conserved along the flow. That is to say

$$\nabla \cdot u = 0 \Rightarrow \frac{D\rho}{Dt} = 0.$$

Hence, the fact that density is constant in any fluid parcel, that is, any infinitesimal volume of fluid moving in the flow, is equivalent to incompressibility. In many situations, when dealing with incompressible fluids, it is common to take $\rho = 1$ for the sake of simplicity. Putting equations (2.1a) together with the incompressibility condition we obtain the following system of equations:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \displaystyle\sum_{1 \le j \le n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \sum_{1 \le j \le n} \frac{\partial^2 u_i}{\partial x_j^2} + f_i, \; i = 1, .., n. \\ \displaystyle\sum_{1 \le i \le n} \frac{\partial u_i}{\partial x_i} = 0. \end{cases} \tag{2.5}$$

If $\nu = 0$, system (2.5) is called Euler equations and it models ideal fluids, i.e. nonviscous and incompressible. On the contrary, if $\nu \neq 0$ the system is known as the Navier-Stokes equations, and it models the flow of incompressible fluids. For further knowledge on this topic [13] is recommended.

In the following, we will work with three-dimensional fluids, so we take $n = 3$ in (2.5). The velocity components will be denoted by

$$u_x = u_1, \quad u_y = u_2, \quad u_z = u_3$$

and the spatial ones by

$$x = x_1, \quad y = x_2, \quad z = x_3.$$

2.1 Vorticity and streamlines

Definition 2.1 (Vorticity). *We define the vorticity of a fluid as* $\omega = \nabla \times u$.

Vorticity ω plays a central role in geophysics. For further reading on vorticity see [14]. Our goal now is to express the Navier-Stokes equations in terms of the vorticity.

Proposition 2.1. *The following identity holds*

$$\frac{1}{2}\nabla(u \cdot u) = (u \cdot \nabla)u + u \times (\nabla \times u).$$

Proof. Let $\{e_x, e_y, e_z\}$ be the canonical basis in $\mathbb{R}^3$ such that $u = u_x e_x + u_y e_y + u_z e_z$. Plus, we use the notation $\partial_x, \partial_y, \partial_z$ for the partial derivatives with respect to x, y, and z, respectively. Then

$$u \times (\nabla \times u) = \begin{vmatrix} e_x & e_y & e_z \\ u_x & u_y & u_z \\ \partial_y u_z - \partial_z u_y & \partial_z u_x - \partial_x u_z & \partial_x u_y - \partial_y u_x \end{vmatrix}$$

$$= [u_y(\partial_x u_y - \partial_y u_x) - u_z(\partial_z u_x - \partial_x u_z)]e_x + [u_z(\partial_y u_z - \partial_z u_y) - u_x(\partial_x u_y - \partial_y u_x)]e_y + [u_x(\partial_z u_x - \partial_x u_z) - u_y(\partial_y u_z - \partial_z u_y)]e_z, \quad (2.6)$$

$$(u \cdot \nabla u)u = (u \cdot \nabla u)u_x e_x + (u \cdot \nabla u)u_y e_y + (u \cdot \nabla u)u_z e_z = (u_x \partial_x u_x + u_y \partial_y u_x + u_z \partial_z u_x)e_x + (u_x \partial_x u_y + u_y \partial_y u_y + u_z \partial_z u_y)e_y + (u_x \partial_x u_z + u_y \partial_y u_z + u_z \partial_z u_z)e_z, \quad (2.7)$$

$$\nabla(u \cdot u) = \nabla(u^2) = \partial_x(u^2)e_x + \partial_y(u^2)e_y + \partial_z(u^2)e_z. \quad (2.8)$$

On the one hand, let us now consider only the x-component in equations (2.6), (2.7) and add them:

$$u_y(\partial_x u_y - \partial_y u_x) - u_z(\partial_z u_x - \partial_x u_z) + u_x \partial_x u_x + u_y \partial_y u_x + u_z \partial_z u_x = u_x \partial_x u_x + u_y \partial_y u_y + u_z \partial_z u_z.$$

On the other hand, considering the x-component in equation (2.8) we have

$$\partial_x(u_x^2) + \partial_y(u_y^2) + \partial_z(u_z^2) = 2(u_x \partial_x u_x + u_y \partial_y u_y + u_z \partial_z u_z).$$

For the y and z components, the computations are analogous, so the proposition is proved. □

Proposition 2.2. *The following identity holds*

$$\nabla \times (u \times \omega) = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega + u(\nabla \cdot \omega) - \omega(\nabla \cdot u)$$

Proof. As we did in the proof of Proposition 2.1, we will only prove the identity for the x-component, since the computations are analogous regarding the other ones. Having said that, let us begin by expanding the x-component of the term $\nabla \times (u \times \omega)$, which yields:

$$\nabla \times (u \times \omega) \cdot e_x = \partial_y[u_x \omega_y - u_y \omega - x] - \partial_z[u_z \omega_x - u_x \omega_z]$$
$$= \omega_y \partial_y u_x + u_x \partial_y \omega_y - \omega_x \partial_y u_y - u_y \partial_y \omega_x - \omega_x \partial_z u_z - u_z \partial_z \omega_x + \omega_z \partial_z u_x + u_x \partial_z \omega_z. \quad (2.9)$$

Whereas for the terms on the right-hand side, we have

$$(\omega \cdot \nabla)u \cdot e_x = \omega_x \partial_x u_x + \omega_y \partial_y u_x + \omega_z \partial_z u_x,$$
$$(u \cdot \nabla)w \cdot e_x = u_x \partial_x \omega_x + u_y \partial_y \omega_x + u_z \partial_z \omega_x,$$
$$u(\nabla \cdot \omega) \cdot e_x = u_x(\nabla \cdot \omega) = u_x \partial_x \omega_x + u_x \partial_y \omega_y + u_x \partial_z \omega_z,$$
$$\omega(\nabla \cdot u) \cdot e_x = \omega_x \partial_x u_x + \omega_x \partial_y u_y + \omega_x \partial_z u_z.$$

All in all,

$$[(\omega \cdot \nabla)u - (u \cdot \nabla)\omega + u(\nabla \cdot \omega) - \omega(\nabla \cdot u)] \cdot e_x$$
$$= \omega_y \partial_y u_x + \omega_z \partial_z u_x - u_y \partial_y \omega_x - u_z \partial_z \omega_x + u_x \partial_y \omega_y + u_x \partial_z \omega_z - \omega_x \partial_y u_y - \omega_x \partial_z u_z. \quad (2.10)$$

It is now easy to check that (2.9) and (2.10) are the same and this completes the proof. □

Proposition 2.3. *If the body forces acting on the fluid are conservative, then equation (2.1a) is written in terms of the vorticity ω as:*

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \omega \cdot \nabla u + \nu \nabla^2 \omega. \tag{2.11}$$

Equation (2.11) is the so-called vorticity transport equation.

Proof. We will need the following results in the forthcoming calculations:

$$(u \cdot \nabla) u = \frac{1}{2} \nabla (u \cdot u) - u \times (\nabla \times u), \tag{2.12}$$

$$\nabla \times \nabla \phi = 0 \text{ for any scalar field } \phi, \tag{2.13}$$

$$\nabla \times (u \times \omega) = \omega \cdot \nabla u - u \cdot \nabla \omega + u \nabla \cdot \omega - \omega \nabla \cdot u. \tag{2.14}$$

To begin with, equation (2.13) is straightforward to prove. It suffices to assume that ϕ is regular enough and flip derivatives according to Schwarz's theorem. Equations (2.12) and (2.14) have been proved in propositions 2.1 and 2.2, respectively. Moreover, since $\nabla \cdot \omega = \nabla \cdot (\nabla \times u) = 0$ and $\nabla \cdot u = 0$, equation (2.14) can be simplified as

$$\nabla \times (u \times \omega) = \omega \cdot \nabla u - u \cdot \nabla \omega. \tag{2.15}$$

Substituting (2.12) in (2.1a) gives

$$\frac{\partial u}{\partial t} + \frac{1}{2} \nabla (u \cdot u) - u \times \omega = -\nabla p + \nu \Delta u + f. \tag{2.16}$$

Now, we calculate the curl of the left-hand side of (2.16), which yields

$$\frac{\partial \omega}{\partial t} + \frac{1}{2} \nabla \times \nabla (u \cdot u) - \nabla \times (u \times \omega). \tag{2.17}$$

By equation (2.13), $\nabla \times \nabla (u \cdot u) = 0$, and taking into consideration (2.15) as well, we conclude that the curl of the left-hand side is

$$\frac{\partial \omega}{\partial t} - \omega \cdot \nabla u + u \cdot \nabla \omega. \tag{2.18}$$

Taking the curl of the right-hand side in equation (2.16), we have

$$-\nabla \times (\nabla p) + \nabla \times f + \nu \nabla^2 \omega. \tag{2.19}$$

Again, equation (2.13) yields $\nabla \times (\nabla p) = 0$. Furthermore, by hypothesis, the body forces are conservative, so the curl of f equals zero. Hence, equation (2.19) simplifies to

$$\nu \nabla^2 \omega. \tag{2.20}$$

Eventually, equations (2.18) and (2.20) must be equal, so

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \omega \cdot \nabla u + \nu \nabla^2 \omega.$$

□

Before moving on to the next topic, let us mention that we will consider the two-dimensional case $n = 2$. In other words, we will neglect the variable z and focus only on the x and y components instead. Notice that since u is divergence-free, it is possible to introduce a streamfunction ψ such that

$$u = \nabla^{\perp}\psi = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right). \tag{2.21}$$

For further reading on the streamfunction [15] and [16] are recommended.

Definition 2.2 (Streamlines). *Streamlines are defined as lines that are everywhere tangential to the velocity field, that is to say, $u \cdot n = 0$, where n is the unit normal vector to the streamline.*

Proposition 2.4. *A line $l : (a, b) \subseteq \mathbb{R} \to \mathbb{R}^2$ is a streamline at a time $t \in \mathbb{R}_+$ if and only if the streamfunction ψ is constant along $l(s)$.*

Proof. $\Rightarrow$ Let $l(s)$ be a streamline. Then $u \parallel \partial_s l(s) \Rightarrow \nabla^{\perp}\psi \parallel \partial_s l(s) \Rightarrow \nabla\psi \perp \partial_s lx(s)$. Hence $\nabla\psi(l(s), t) \cdot \partial_s l(s) = 0 \Rightarrow \psi(l(s), t)$ is constant.

$\Leftarrow$ Let us assume that ψ is constant along the line $l(s)$, i.e., $\psi(l(s), t)$ is constant. We can now derive with respect to s and apply the previous reasoning backward, concluding that $l(s)$ is a streamline. □

Proposition 2.5. *Let ψ be the streamfunction defined in (2.21). For $n = 2$, the modulus of the vorticity is $\omega = \nabla^2\psi$.*

Proof. Since $n = 2$, we can write $u(x, y, t) = u_x(x, y, t)e_x + u_y(x, y, t)e_y$. Hence

$$\omega = |\nabla \times u| = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}.$$

But we also have that $u = \nabla^{\perp}\psi$, thus $u_x = -\frac{\partial \psi}{\partial y}$, $u_y = \frac{\partial \psi}{\partial x}$ and

$$\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2\psi.$$

□

Remark 2.1. *Note that in Proposition 2.5 an abuse of notation has been made, since we have used the variable ω for both, the vorticity vector and its modulus. The reader should be aware of this and distinguish whether ω is a vector or not depending on the context.*

Remark 2.2. *While n=2, the vector $\omega = \nabla \times u = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) e_z$ points the z direction, whereas ∇u lies within the xy plane. In conclusion: $\omega \cdot \nabla u = 0$. Moreover, if the flow is nonviscous, i.e. $\nu = 0$, we obtain the following system:*

$$\left.\begin{aligned} \frac{\partial \omega}{\partial t} + u \cdot \nabla\omega = 0, \quad (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+, \\ \omega = \nabla^2\psi, \quad u = \nabla^{\perp}\psi. \end{aligned}\right\} \tag{2.22}$$

The system (2.22) is an equivalent formulation for the Euler equation in the 2D case. It consists of two PDEs for ω and ψ rather than the initial formulation with three variables u_x, u_y, and p.

Theorem 2.1. *Let vorticity ω satisfy (2.22) and let fluid velocity u be divergence-free. Under the assumptions that $\omega(x,y,0) \in L^\infty$ and $\lim_{x,y\to\infty} \omega(x,y,t) = 0$, the L^p norms ($1 \leq p \leq \infty$) of ω are conserved for any $t \in \mathbb{R}_+$.*

Proof. Let $p \geq 0$ be an integer. Then

$$\begin{aligned}\frac{d}{dt}\frac{1}{1+p}\int_{\mathbb{R}^2} \omega(x,y,t)^{p+1}dx &= \int_{\mathbb{R}^2} \omega(x,y,t)^p \partial_t \omega(x,y,t)dx \\ &= -\int_{\mathbb{R}^2} u(x,y,t)\cdot\nabla\omega(x,y,t)\omega(x,y,t)^p dx \\ &= -\frac{1}{1+p}\int_{\mathbb{R}^2} u(x,y,t)\cdot\nabla\omega(x,y,t)^{p+1}dx \\ &= \frac{1}{1+p}\int_{\mathbb{R}^2} \nabla\cdot u(x,y,t)\omega(x,y,t)^{p+1}dx = 0.\end{aligned}$$

The last integral vanishes due to the divergence-free condition $\nabla\cdot u = 0$. Hence

$$\frac{1}{1+p}\frac{d}{dt}\left\|\omega(t)\right\|_{L^{p+1}}^{p+1} = 0 \Rightarrow \frac{d}{dt}\left\|\omega(t)\right\|_{L^{p+1}}^{p+1} = 0.$$

Integrating in time we get

$$\left\|\omega(t)\right\|_{L^{p+1}}^{p+1} = \left\|\omega(0)\right\|_{L^{p+1}}^{p+1}.$$

This proves that $\left\|\omega(t)\right\|_{L^{p+1}} = \left\|\omega(0)\right\|_{L^{p+1}}$ for any $t \in \mathbb{R}_+$ and for any $1 \leq p < +\infty$. We are now left to show that the L^∞ norm of the vorticity is also conserved in time. First of all, let us recall a result of functional analysis (proof can be found in [17]):

If there exists $q < +\infty$ such that $f \in L^\infty(S) \cap L^q(S)$, then: $\left\|f\right\|_{L^\infty} = \lim_{p\to\infty}\left\|f\right\|_{L^p}$.

By hypothesis we have that $\omega(x,0) \in L^\infty$, thus

$$\left\|\omega(0)\right\|_{L^\infty} = \lim_{p\to\infty}\left\|\omega(0)\right\|_{L^p} = \lim_{p\to\infty}\left\|\omega(t)\right\|_{L^p}. \tag{2.23}$$

This implies that the function $p \mapsto \left\|\omega(t)\right\|_{L^p}$ is bounded and therefore $\omega(x,t) \in L^\infty$. Eventually, we have that

$$\left\|\omega(t)\right\|_{L^\infty} = \lim_{p\to\infty}\left\|\omega(t)\right\|_{L^p} = \left\|\omega(0)\right\|_{L^\infty} \text{ by equation (2.23).}$$

□

2.2 Buoyancy and Brunt-Väisälä frequency

Let us imagine that there is a fluid parcel with a different density than that of its environment, for instance, an air parcel with density ρ' submerged in the atmosphere, with density ρ (see Figure 2.1). In addition, let us also consider a non-inertial reference frame that spins with the Earth, such that the x-axis points eastwards, the y-axis points northwards, and the z-axis points upwards. This topic will be explained further way in the next section. In this way, the acceleration due to gravity is given by $-ge_z$, where e_z is the unitary vector in the z-direction.

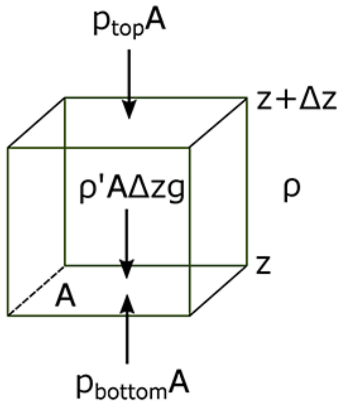

Figure 2.1: Forces acting on an air parcel with density ρ' in an environment with density ρ. Image has been taken from [18].

Let us now assume, for the sake of simplicity, that the fluid is ideal and it has no motion, i.e. $u = 0$. Then, equation (2.1a) yields

$$\nabla p = -\rho g e_z \Rightarrow \frac{\partial p}{\partial z} = -\rho g \Rightarrow \Delta p = -\rho g \Delta z,$$

which is known as the hydrostatic approximation. As we will see in Chapter 3, this is indeed a rather useful and valid approximation concerning atmospheric and oceanic dynamics. In addition, the partial derivatives of p with respect to x and y are zero, so we can safely neglect the movement along the x and y axes, and focus only on the dynamics in the z-direction.

According to Newton's second law, the z component of the acceleration experienced by the air parcel is

$$a_z = \frac{\sum F_z}{\rho' V}, \tag{2.24}$$

where $\sum F_z$ is the sum of all the forces acting on the air parcel along the z-direction, ρ' its density, and V its volume. Using the hydrostatic approximation we have

$$\begin{aligned} p_{bottom} - p_{top} = \rho g \Delta z \Rightarrow \sum F_z &= p_{bottom} A - p_{top} A - \rho' g A \Delta z \\ &= \rho g A \Delta z - \rho' g A \Delta z = -(\rho' - \rho) g A \Delta z. \end{aligned} \tag{2.25}$$

Taking into account that $V = A\Delta z$ and substituting (2.25) into (2.24), we eventually get

$$a_z = -\frac{(\rho' - \rho) g}{\rho'}. \tag{2.26}$$

According to (2.26), objects that are less dense than the fluid, float on its surface (a_z is positive). On the contrary, objects that are denser than the fluid, drown (a_z is negative). The force that makes objects float is called the buoyancy force and it is due to the increase of pressure with depth in a fluid.

Proposition 2.6 (Poisson equation). *If the flow of the fluid is an isentropic process, that is* $\frac{Ds}{Dt} = 0$, *then the quantity* $\frac{T}{p^{R/c_p}}$ *is conserved along the flow.*

Proof. Since we are working under the assumption that the air behaves as an ideal gas, the first law of thermodynamics can be expressed as

$$c_p \frac{DT}{Dt} = Q + \alpha \frac{Dp}{Dt}, \tag{2.27}$$

where c_p is the specific heat of the air at constant pressure, $\alpha = 1/\rho$ is the specific volume and Q is the net heating rate per unit mass. See pages 46 and 47 on [19]. In addition, we assume that c_p is constant and take advantage of the ideal gas assumption $p\alpha = R_d T$, rewriting equation (2.27) as

$$c_p \frac{D(\ln T)}{Dt} - R_d \frac{D(\ln p)}{Dt} = \frac{Q}{T},$$

where the quotient Q/T is the rate of change of entropy per unit mass, that is,

$$\frac{Ds}{Dt} = \frac{Q}{T}.$$

Therefore

$$c_p \frac{D(\ln T)}{Dt} - R_d \frac{D(\ln p)}{Dt} = \frac{Ds}{Dt}. \tag{2.28}$$

An isentropic process is an idealized thermodynamic process that is both adiabatic and reversible, hence, there is no exchange of heat between the system and its environment. Thus, equation (2.28) yields

$$c_p \frac{D(\ln T)}{Dt} - R_d \frac{D(\ln p)}{Dt} = 0 \Rightarrow \frac{D}{Dt}\left(\ln \frac{T}{p^{R/c_p}}\right) = 0 \Rightarrow \frac{T}{p^{R_d/c_p}} = \text{ constant.} \tag{2.29}$$

□

Equation (2.29) is known as the Poisson equation and leads to our next definition: potential temperature.

Definition 2.3 (Potential temperature)**.** *Let us suppose that there is a fluid element with temperature T and pressure p. The potential temperature θ of the fluid element is defined as follows:*

$$\theta = T\left(\frac{p_\theta}{p}\right)^{\frac{R}{c_p}}, \tag{2.30}$$

where p_θ is a reference pressure (usually 10^3 hPa) and c_p is the specific heat capacity at constant pressure.

The potential temperature θ of an air parcel represents the temperature that the parcel would have if it were expanded or compressed adiabatically from its initial state of pressure and temperature to a standard pressure p_θ. Since we know that the Poisson equation (2.29) holds for adiabatic flows, as a consequence, the potential temperature is a conserved quantity through adiabatic processes. To know more about potential temperature [14] and [20] can be useful.

Furthermore, assuming that the pressure is constant, the ideal gas law $p = \rho R_d T$, where R_d is the gas constant for dry air (see [19]), allows us to write the acceleration of the air parcel (2.26) in terms of the temperature and therefore in terms of the potential temperature:

$$a_z = g\frac{(T' - T)}{T} = g\frac{(\theta' - \theta)}{\theta}.$$

In light of the reasoning above, the two following definitions arise.

Definition 2.4 (Ocean Buoyancy). *In the ocean, buoyancy is defined as:*

$$b = -\frac{\rho g}{\rho_0}, \tag{2.31}$$

where ρ is the density, ρ_0 is a reference density and g is the gravity acceleration.

Definition 2.5 (Atmospheric buoyancy). *In the atmosphere, buoyancy is related to potential temperature through:*

$$b = \frac{g\theta}{\theta_0}, \tag{2.32}$$

where θ_0 is a reference potential temperature.

Let us consider the air parcel to be very small and that it experiences a small displacement δz along the z axis, so that θ_0 is the potential temperature at $z = 0$ and θ the potential temperature at δz. We can expand the potential temperature over $z = \delta z$ as:

$$\theta(\delta z) \simeq \theta_0 + \left.\frac{\partial \theta}{\partial z}\right|_{z=0} \delta z,$$

where $\theta_0 = \theta(0)$. Hence, the acceleration in the z direction of our small air parcel will be

$$a_z = \frac{d^2 \delta z}{dt^2} = g\frac{\theta_0 - \theta(\delta z)}{\theta(\delta z)} = -\frac{g}{\theta}\left.\frac{\partial \theta}{\partial z}\right|_{z=0} \delta z = -g\left.\frac{\partial \ln \theta}{\partial z}\right|_{z=0} \delta z$$

$$\Rightarrow \frac{d^2 \delta z}{dt^2} = -g\left.\frac{\partial \ln \theta}{\partial z}\right|_{z=0} \delta z. \tag{2.33}$$

Equation (2.33) is called the Brunt-Väisälä equation. We now distinguish three cases:

1. If $\left.\frac{\partial \ln \theta}{\partial z}\right|_{z=0} > 0$, then equation (2.33) has the form of the well-known simple harmonic oscillator equation
$$\frac{d^2 \delta z}{dt^2} = -N^2 \delta z,$$
where $N = \left(g\left.\frac{\partial \ln \theta}{\partial z}\right|_{z=0}\right)^{1/2}$ is called the Brunt-Väisälä frequency.

2. If $\left.\frac{\partial \ln \theta}{\partial z}\right|_{z=0} < 0$, then equation (2.33) has the form
$$\frac{d^2 \delta z}{dt^2} = \omega^2 \delta z,$$
with $\omega^2 = -g\left.\frac{\partial \ln \theta}{\partial z}\right|_{z=0}$. In this case, the “air bubble” would continue to rise.

3. If $\left.\dfrac{\partial \ln \theta}{\partial z}\right|_{z=0} = 0$, there is no acceleration exerted on the tiny air parcel.

Lemma 2.1. *The Brunt-Väisälä frequency can be expressed in terms of pressure and density as*

$$N^2 = g\left\{\frac{1}{\gamma}\frac{\partial \ln p}{\partial z} - \frac{\partial \ln \rho}{\partial z}\right\}_{z=0}, \tag{2.34}$$

where $\gamma = c_p/c_v$ is the adiabatic index.

Proof. Starting from $\theta = T\left(\frac{p_\theta}{p}\right)^{\frac{R}{c_p}}$ we have that

$$\frac{\partial \theta}{\partial z} = \left(\frac{p_\theta}{p}\right)^{\frac{R}{c_p}}\frac{\partial T}{\partial z} - T\left(\frac{p_\theta}{p}\right)^{\frac{R}{c_p}}\frac{R}{pc_p}\frac{\partial p}{\partial z} = \frac{\theta}{T}\frac{\partial T}{\partial z} - \frac{\theta R}{c_p p}\frac{\partial p}{\partial z},$$

Therefore

$$N^2 = g\left.\frac{\partial \ln \theta}{\partial z}\right|_{z=0} = g\left\{\frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{pc_p}\frac{\partial p}{\partial z}\right\}_{z=0} = g\left\{\frac{1}{T}\frac{\partial T}{\partial z} - \frac{\gamma-1}{\gamma}\frac{1}{p}\frac{\partial p}{\partial z}\right\}_{z=0},$$

Eventually, the ideal gas law yields

$$N^2 = g\left\{\frac{1}{\gamma p}\frac{\partial p}{dz} - \frac{1}{\rho}\frac{\partial \rho}{dz}\right\}_{z=0} = g\left\{\frac{1}{\gamma}\frac{\partial lnp}{\partial z} - \frac{\partial \ln \rho}{\partial z}\right\}_{z=0}.$$

□

Although 2.34 is, in fact, the most complete expression for N^2, since it includes the variation of pressure with height, however, concerning fluids, it is common to assume that the pressure remains fixed along a small vertical displacement and therefore take into consideration only the variation of density when defining N^2. That is to say:

$$N^2 = -\left.\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}\right|_{z=0}, \tag{2.35}$$

which is the expression of the Brunt-Väisälä frequency that will be used later on. For instance, a derivation of 2.34 can be found in [21].

2.3 Vertical vorticity and geostrophic flow

We defined vorticity as the curl of the fluid velocity, thus it represents a measure of the rotation of the fluid. However, when a fluid rotates we need to take into consideration two rotating mechanisms: fluid movement and Earth's rotation. In fact, let u_r be the relative velocity field of a non-inertial reference frame that spins with the Earth as the one shown in Figure 2.2 (green axes), then the total velocity field of the fluid will be given by

$$u = u_r + \Omega \times r,$$

where Ω is the Earth's angular velocity and r is the position vector that goes from the centre of the inertial frame in Figure 2.2 (blue axes) to a particular point in space.

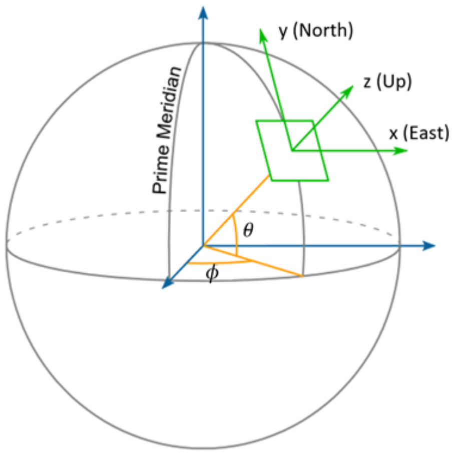

Figure 2.2: The green axes correspond to a non-inertial reference frame that spins with the Earth and it is typically used in meteorology. On the contrary, the blue axes correspond to a fixed, inertial reference frame that does not spin with the Earth.

Hence, the absolute vorticity ω is

$$\omega = \nabla \times u = \nabla \times u_r + \nabla \times (\Omega \times r). \tag{2.36}$$

Relative vorticity is defined as $\omega_r = \nabla \times u_r$. In addition, if we use the variables x', y', z' to denote the spatial coordinates in the reference frame that corresponds to the blue axes in Figure 2.2. and denote by $\{e_{x'}, e_{y'}, e_{z'}\}$ the correspondent basis of orthonormal vectors, then $\Omega = \Omega e_{z'}$ and $r = x'e_{x'} + y'e_{y'} + z'e_{z'}$. However, notice that, for simplicity purposes, we are making an abuse of the notation when using Ω for both, the vector and the scalar. All in all, we can compute

$$\Omega \times r = \begin{vmatrix} e_{x'} & e_{y'} & e_{z'} \\ 0 & 0 & \Omega \\ x' & y' & z' \end{vmatrix} = (\Omega x', \Omega y', 0) \Rightarrow \nabla \times (\Omega \times r) = 2\Omega,$$

and rewrite (2.36) as

$$\omega = \omega_r + 2\Omega. \tag{2.37}$$

Equation (2.37) states that absolute vorticity ω is equal to the relative vorticity ω_r plus the extra term 2Ω due to the Earth's rotation, which is called planetary vorticity. Defining a quantity that expresses the tendency to rotate along the horizontal direction is of particular interest. Let us denote by $\{e_x, e_y, e_z\}$ the basis of orthonormal vectors associated with the non-inertial reference frame in Figure 2.2 (green axes).

Definition 2.6 (Absolute vertical vorticity)**.** *The absolute vertical vorticity is defined as the z-component of the absolute vorticity, that is,*

$$\zeta_a = e_z \cdot \omega = e_z \cdot (\nabla \times u). \tag{2.38}$$

Definition 2.7 (Relative vertical vorticity)**.** *The relative vertical vorticity is defined as the z-component of the absolute vorticity, that is,*

$$\zeta = e_z \cdot \omega_r = e_z \cdot (\nabla \times u_r). \tag{2.39}$$

If we use the basis $\{e_x, e_y, e_z\}$ of orthonormal vectors of the non-inertial reference frame, Earth's angular rotation is given by $\Omega = \Omega \cos\theta e_y + \Omega \sin\theta e_z$.

Definition 2.8 (Coriolis term)**.** *The Coriolis term f is defined as the z-component of the planetary vorticity 2Ω:*

$$f = 2e_z \cdot \Omega = 2\Omega \sin\theta. \tag{2.40}$$

It is clear that f depends solely on the latitude θ, so that it is zero at the equator and maximum at the poles. It measures the spin of an object or fluid parcel due to the rotation of the Earth. Moreover, assuming that the change in latitude is small, we can expand f over a fixed latitude θ_0:

$$f = f_0 + \left.\frac{df}{d\theta}\right|_{\theta=\theta_0} \Delta\theta = f_0 + 2\Omega\cos\theta_0 \Delta\theta = f_0 + 2\Omega\cos\theta_0 \frac{\Delta y}{R}, \tag{2.41}$$

where $f_0 = 2\Omega\sin\theta_0$. Defining $\beta = \dfrac{2\Omega\cos\theta_0}{R}$, (2.41) can be written as

$$f = f_0 + \beta\Delta y. \tag{2.42}$$

Equation (2.42), which sets a linear variation of the Coriolis parameter, is known as the β-plane approximation. This approximation is useful concerning the theoretical analysis of many atmospheric or oceanic phenomena since it makes the equations much more tractable, but still considers the variation of the Coriolis parameter in space.

Remark 2.3. *From definitions (2.38),(2.39) and (2.40), it follows directly that ζ_a is the sum of the relative vertical vorticity and the Coriolis term:*

$$\zeta_a = e_z \cdot \omega = e_z \cdot \omega_r + 2e_z \cdot \Omega = \zeta + f. \tag{2.43}$$

Remark 2.4. *If u_x and u_y represent the velocity components along the x and y directions respectively, then the relative vertical vorticity reads*

$$\zeta = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}. \tag{2.44}$$

As a consequence, if we introduce a streamfunction ψ such that $u = \nabla^{\perp}\psi$, then

$$\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi. \tag{2.45}$$

Remark 2.5. *Carl Rossby proposed in 1940 (see [23]) that the local vertical component of the absolute vorticity ζ is the most important component for large-scale atmospheric flow, instead of the full three-dimensional vorticity vector. Therefore, in the following pages, when we mention absolute or relative vorticity, we are likely talking about the vertical component only. Analogously, although the Coriolis parameter is only the z-component of the planetary vorticity 2Ω, this component is the most interesting one concerning geophysics and, in many situations, it is common to call f by "planetary vorticity".*

Moving on to the topic of geostrophic flow, we need to introduce the Ekman transport. The wind only pushes the surface layer of water, and whenever this top layer is pushed away, water from the bottom is forced to rise to replace the water that has been moved

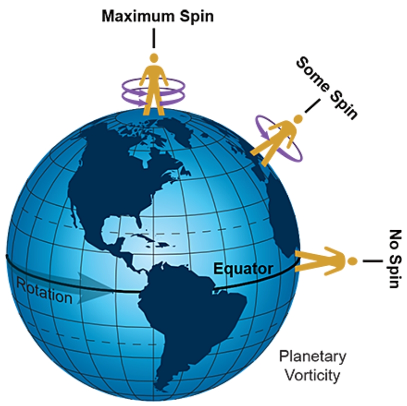

Figure 2.3: Planetary vorticity f is always positive in the Northern Hemisphere except at the equator where it is zero. Image has been taken from [22].

away. At the same time, the Coriolis effect causes the wind to shift in the direction that is pushing the water. This is the idea behind the Ekman transport system.

Due to the Coriolis effect, the surface water usually moves in a direction that forms about 45^o with the wind direction. Once this top layer has been displaced, the bottom ones will rise forming a different angle, since they are less affected by the wind. As you go deeper into the ocean, you have less motion and in a different direction, so that the overall result is an ever-turning spiraling rising column of water, as shown in Figure 2.4. Regarding fluid dynamics, gyres are an important topic to address. A gyre is a large-scale

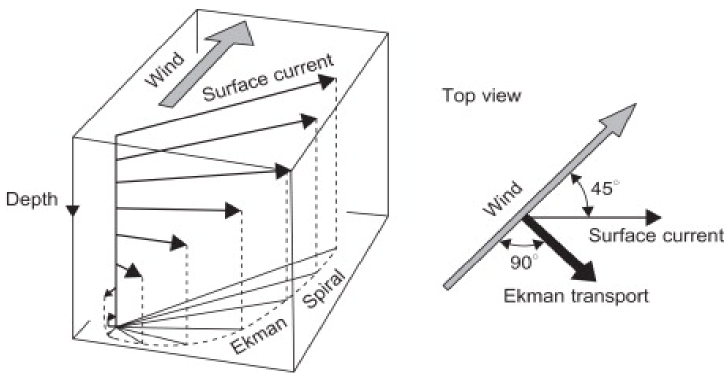

Figure 2.4: Ekman's transport illustration. Image has been taken from [24].

system of rotating ocean currents. Subtropical gyres are centered near 30 degrees latitude in the North and South Atlantic, the North and South Pacific, and the Indian Ocean. Concerning these gyres, Ekman transport will cause surface waters to move toward the central region of the gyre, producing a broad mound of water. Surface water will begin to flow downhill and, eventually, a balance will develop between the Coriolis force and the force arising from the water pressure gradient, so that water parcels flow around the gyre and parallel to contours of elevation of sea level. We call this current geostrophic flow.

In Chapter 3, we shall show that the velocity of the geostrophic flow is given by the formula:

$$u^g = \frac{1}{\rho f} e_z \times \nabla p.$$

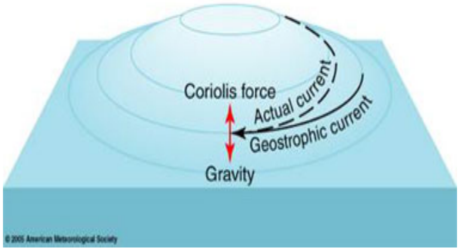

Figure 2.5: The horizontal movement of surface water arising from a balance between the pressure gradient force and the Coriolis force is known as geostrophic flow. Image has been taken from [25].

2.4 Potential vorticity and SQG model

Let us now think of an ice skater spinning. It is well-known that if the skater extends his arms, he will spin slower. On the contrary, he will spin faster if she contracts her arms. Nevertheless, the angular momentum remains always constant. A rotating column of air or water acts somehow in a similar way: it rotates faster if it is stretched into a narrower column and it rotates slower if it is squashed into a wider column. We want to define a quantity that plays a role similar to the angular momentum of an ice skater but for the rotating column. This quantity must remain constant while the column spins.

Definition 2.9 (Barotropic potential vorticity)**.** *The barotropic potential vorticity PV of a fluid column is defined as the quotient between absolute vorticity and the height of the column:*

$$PV = \frac{\zeta + f}{H}. \tag{2.46}$$

In [23], the Swedish-born American meteorologist Carl Rossby showed that for a one-layer shallow water system, the barotropic potential vorticity is conserved along an adiabatic and frictionless flow, that is,

$$\frac{D(PV)}{Dt} = \frac{D}{Dt}\left(\frac{\zeta + f}{H}\right) = 0. \tag{2.47}$$

This result is known as the Rossby theorem and is the key property of shallow water potential vorticity. An example of its applications is shown in Figure 2.6.

Definition 2.10 (Quasi-geostrophic potential vorticity)**.** *The quasi-geostrophic potential vorticity, also known as the pseudo-potential-vorticity, is defined as:*

$$q = f + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right). \tag{2.48}$$

where N is the so-called Brunt-Väisälä frequency and ψ is a streamfunction, i.e. $u = \nabla^{\perp}\psi$.

An analogous result to that of the barotropic potential vorticity PV can be proved for the quasi-geostrophic potential vorticity q. In fact, we will prove in Chapter 3 that in the absence of forcing and dissipation q is conserved along the geostrophic flow, that is,

$$\frac{Dq}{Dt} = 0.$$

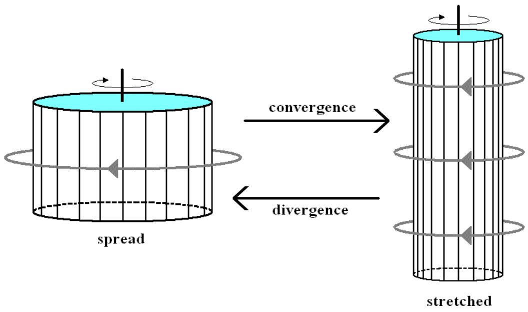

Figure 2.6: On the left-hand side column the height is lower in comparison with the column on the right-hand side. Since PV is constant, then the right-hand side column must have greater absolute vorticity. Image has been taken from [26].

The importance of this result is that given the potential vorticity q, one can invert equation (2.48) to obtain the streamfunction ψ, which allows us to know the geostrophic wind field and the potential temperature field. This is called the principle of potential vorticity inversion. For further information concerning this topic, it is worth seeing [23] and [27]. Nonetheless, solving equation (2.48) requires knowledge of boundary conditions, which will be discussed in Chapter 3.

The surface quasi-geostrophic model (SQG) is a mathematical model based on the quasi-geostrophic approximation which is used to explain the dynamics of the ocean and the atmosphere. It assumes that, besides potential vorticity conservation, surface buoyancy is also a conserved quantity. The 2D surface quasi-geostrophic equation reads

$$\left.\begin{aligned} &\frac{\partial\theta}{\partial t}+u\cdot\nabla\theta=0, \quad (x,y,t)\in\mathbb{R}^2\times\mathbb{R}_+, \\ &u=\nabla^\perp\psi, \quad \theta=-(-\nabla^2)^{1/2}\psi. \end{aligned}\right\} \quad \text{(SQG)} \tag{2.49}$$

Here, $\theta=\theta(x,y,t)$ is a non-dimensional variable related to the surface buoyancy, $u=u(x,y,t)$ is the fluid velocity, and $\psi=\psi(x,y,t)$ is the streamfunction. The operator $(-\nabla^2)^{1/2}$ is defined via the Fourier transform by $\widehat{(-\nabla^2)^{1/2}f}(m)=|m|\,\widehat{f}(m)$, where $\widehat{f}$ is the Fourier transform of the function f. This equation has a geophysical origin and will be studied in more detail in the next chapters.

In addition to this, we can consider a more general problem given by

$$\left.\begin{aligned} &\frac{\partial\theta}{\partial t}+u\cdot\nabla\theta=0, \quad (x,y,t)\in\mathbb{R}^2\times\mathbb{R}_+, \\ &u=\nabla^\perp\psi, \quad \theta=-(-\nabla^2)^{1-\alpha/2}\psi. \end{aligned}\right\} \tag{2.50}$$

We define the operator $\Lambda^\gamma=(-\nabla^2)^{\gamma/2}$ as $\widehat{\Lambda^\gamma f(m)}=|m|^\gamma\,\widehat{f}(m)$, with $\widehat{f}$ the Fourier transform of the function f.

Remark 2.6. *For $\alpha=0$, problem (2.50) has the same form as 2D Euler (2.22). Whereas for $\alpha=1$, problem (2.50) corresponds to the surface-quasigeostrophic equation (SQG).*

2.5 3D Euler and 2D SQG

Considering an inviscid fluid, that is $\nu = 0$, the vorticity transport equation (2.11) reads

$$\partial_t \omega + u \cdot \nabla \omega = (\nabla u)\omega, \tag{2.51}$$

that together with

$$\nabla \cdot \omega = \nabla \cdot u = 0 \tag{2.52}$$

corresponds to the 3D incompressible Euler equations for an inviscid fluid. Moreover, applying the perpendicular gradient $\nabla^\perp$ to the SQG equation $\partial_t \theta + u \cdot \theta = 0$ we obtain

$$\partial_t(\nabla^\perp \theta) + u \cdot \nabla(\nabla^\perp \theta) = (\nabla u)\nabla^\perp \theta. \tag{2.53}$$

Since the fluid that we are considering is incompressible then $\nabla \cdot u = 0$. Besides, it is clear that $\nabla \cdot (\nabla^\perp \theta) = 0$. Hence

$$\nabla \cdot (\nabla^\perp \theta) = \nabla \cdot u = 0. \tag{2.54}$$

All in all, comparing equations (2.51) and (2.52) to (2.53) and (2.54), we come to the conclusion that SQG is a 2D model of the 3D Euler equations, since $\nabla^\perp \theta = \left(-\frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial x}\right)$ is a 2D vector that plays the same role as the vorticity ω in 3D Euler.

In 1994, Constantin, Majda, and Tabak proved in [28] the local well-posedness in $H^s(s \geq 3)$ of the SQG model and studied its analogy with 3D incompressible Euler. Although the local existence of solutions in H^s has been proved, global well-posedness is still an open problem for either of them.

However, Beale, Kato, and Majda proved in [29] a result stating that if the solution for 3D Euler's equations fails to be regular past a certain time, then the vorticity $\omega = \nabla \times u$ must necessarily become unbounded. The local existence theorem for Euler's equations is stated as:

Theorem 2.2. *Suppose an initial velocity field u_0 is specified in $H^s, s \geq 3$, with $\|u_0\|_{H^3} \leq N_0$, for some $N_0 > 0$. Then, there exists $T_0 > 0$, depending only on N_0, so that the equations*

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0,$$
$$\nabla \cdot u = 0,$$

have a solution in the class: $u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ at least for $T = T_0(N_0)$.

Nonetheless, this result does not state anything as far as the regularity of the solution is concerned. The following theorem states a condition that must be fulfilled if the solution blows up:

Theorem 2.3 (Beale-Kato-Majda criterion)**.** *Let u be a solution of Euler's equations and suppose there is a time T_* such that the solution cannot be continued in the class $C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ to $T = T_*$. Assume that T_* is the first such time. Then*

$$\int_0^{T_*} \|\omega\|_{L^\infty} \, dt = \infty$$

and in particular

$$\limsup_{t \to T_*} \|\omega\|_{L^\infty} = \infty.$$

For a proof of the theorem see [29]. Equivalently, the Beale-Kato-Majda criterion states that $\int_0^{T_*} \|\omega\|_{L^\infty} \, dt < \infty$ implies that there is no blow-up at time T. Furthermore, in the SQG case, due to the analogy of the equations, we have that if the condition $\int_0^{T_*} \left\|\nabla^\perp \theta\right\|_{L^\infty} dt < \infty$ holds, then there is no blow-up at time T.

The following table makes it easier to compare the SQG and the 3D Euler models mentioned previously.

	3D EULER	2D SQG
Variable	$\omega = \nabla \times u$	$\nabla^\perp \theta = \left(-\frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial x}\right)$
Equations	$\partial_t w + u \cdot \nabla w = (\nabla u)\omega$	$\partial_t(\nabla^\perp \theta) + u \cdot \nabla(\nabla^\perp \theta) = (\nabla u)\nabla^\perp \theta$
	$\nabla \cdot \omega = \nabla \cdot u = 0$	$\nabla \cdot (\nabla^\perp \theta) = \nabla \cdot u = 0$
BKM	$\int_0^{T_*} \|\omega\|_{L^\infty} \, dt < \infty$	$\int_0^{T_*} \left\|\nabla^\perp \theta\right\|_{L^\infty} dt < \infty$

Table 2.1: Comparison between 3D Euler and 2D SQG. BKM stands for the Beale-Kato-Majda criterion.

Chapter 3

General formulation of Surface Quasi-Geostrophy (SQG)

3.1 Geostrophic and hydrostatic approximations

In order to describe the atmospheric dynamics, we shall use spherical coordinates. We will denote by θ the latitude and by ϕ the longitude (blue axes in Figure 2.2). The position vector in spherical coordinates (r, θ, ϕ) is given by

$$\vec{r} = (r\cos\theta\cos\phi, r\cos\theta\sin\phi, r\sin\theta) \tag{3.1}$$

Hence

$$\begin{aligned}
\frac{\partial\vec{r}}{\partial r} &= (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta) \Rightarrow \left|\frac{\partial\vec{r}}{\partial r}\right| = 1,\\
\frac{\partial\vec{r}}{\partial\theta} &= (-r\sin\theta\cos\phi, -r\sin\theta\sin\phi, r\cos\theta) \Rightarrow \left|\frac{\partial\vec{r}}{\partial\theta}\right| = r,\\
\frac{\partial\vec{r}}{\partial\phi} &= (-r\cos\theta\sin\phi, r\cos\theta\cos\phi, 0) \Rightarrow \left|\frac{\partial\vec{r}}{\partial\phi}\right| = r\cos\theta.
\end{aligned}$$

So the local orthogonal unit vectors in the directions of increasing r, θ, and ϕ are, respectively,

$$\begin{aligned}
e_r &= (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta),\\
e_\theta &= (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta),\\
e_\phi &= (-\sin\phi, \cos\phi, 0).
\end{aligned}$$

The velocity in spherical coordinates (r, θ, ϕ) is obtained by differentiating 3.1 with respect to time, and it is straightforward to see that it can be expressed as

$$u = \frac{d\vec{r}}{dt} = \dot{r}e_r + r\dot{\theta}e_\theta + r\dot{\phi}\cos\theta e_\phi.$$

Next, differentiating 3.1 with respect to time we obtain the acceleration:

$$\begin{aligned}
a =& \frac{du}{dt} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\cos^2\theta)e_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta)e_\theta\\
&+ (2\dot{r}\dot{\phi}\cos\theta - 2r\dot{\theta}\dot{\phi}\sin\theta + r\ddot{\phi}\cos\theta)e_\phi.
\end{aligned} \tag{3.2}$$

Note that we have used the notation $\vec{r}$ for the position vector, while we have opted not to use this "arrow" notation for the velocity and acceleration. This has been done in order to avoid confusion, since the variable r is used for the radial coordinate. Besides, the letter u has been chosen for the velocity so as to be consistent with the notation used in the previous chapter.

Let us now consider a non-inertial reference frame that spins with the Earth and such that the x-axis points towards the east, the y-axis towards the north, and the z-axis points outwards, as shown in Figure 2.2 (green axes). In this reference frame, the local unitary vectors in each direction are

$$\begin{cases} \text{Eastward: } e_x = e_\phi. \\ \text{Northward: } e_y = e_\theta. \\ \text{Upward: } e_z = e_r. \end{cases}$$

Moreover, we will use the following notation for the components of the velocity:

$$\begin{cases} u_x = r\dot{\phi}\cos\theta, \\ u_y = r\dot{\theta}, \\ u_z = \dot{r}. \end{cases} \tag{3.3}$$

In this way, $u = u_x e_x + u_y e_y + u_z e_z$, so it is clear that, concerning atmospheric and oceanic dynamics, u_x and u_y represent the horizontal components of the velocity, whereas u_z represents the vertical velocity of the fluid.

Proposition 3.1. *In terms of u_x,u_y, and u_z introduced in (3.3) and assuming that r is constant and equal to the Earth's radius, that is, $r = R$, the acceleration is given by*

$$\begin{aligned} a = &\left(\frac{Du_x}{Dt} + \frac{u_x u_z}{R} - \frac{u_x u_y \tan\theta}{R}\right) e_x + \left(\frac{Du_y}{Dt} + \frac{u_y u_z}{R} + \frac{u_x^2 \tan\theta}{R}\right) e_y \\ &+ \left(\frac{Du_z}{Dt} - \frac{u_y^2}{R} - \frac{u_x^2}{R}\right) e_z \end{aligned} \tag{3.4}$$

Proof. To start with,

$$\begin{aligned} \frac{Du_x}{Dt} + \frac{u_x u_z}{R} - \frac{u_x u_y \tan\theta}{R} &= \dot{r}\dot{\phi}\cos\theta + R\ddot{\phi}\cos\theta - R\dot{\phi}\dot{\theta}\sin\theta + \frac{R\dot{\phi}\cos\theta\dot{r}}{R} - \frac{R^2\dot{\theta}\dot{\phi}\cos\theta\tan\theta}{R} \\ &= 2\dot{r}\dot{\phi}\cos\theta - 2R\dot{\theta}\dot{\phi}\sin\theta + R\ddot{\phi}\cos\theta. \end{aligned}$$

In addition,

$$\frac{Du_y}{Dt} + \frac{u_y u_z}{R} + \frac{u_x^2 \tan\theta}{R} = \dot{r}\dot{\theta} + R\ddot{\theta} + \frac{R\dot{r}\dot{\theta}}{R} + \frac{R^2\dot{\phi}^2 \cos\theta^2 \tan\theta}{R} = 2\dot{r}\dot{\theta} + R\ddot{\theta} + R\dot{\phi}^2 \sin\theta\cos\theta.$$

Furthermore,

$$\frac{Du_z}{Dt} - \frac{u_y^2}{R} - \frac{u_x^2}{R} = \ddot{r} - \frac{R^2\dot{\theta}^2}{R} - \frac{R^2\dot{\phi}^2 \cos\theta^2}{R} = \ddot{r} - R\dot{\theta}^2 - R\dot{\phi}^2\cos^2\theta.$$

□

From now on we will assume r=R for the sake of simplicity, which is a very good approximation for the regions of the atmosphere with which meteorologists work. Moreover, in the case of the atmosphere, the viscosity of the air can be neglected when working at a height above 2 km. For further reading on these approximations see [30]. In light of these considerations and assuming our fluid to be non-viscous, Newton's second law reads

$$a = -\frac{\nabla p}{\rho} - g e_z - 2\Omega \times u, \tag{3.5}$$

where $-\nabla p/\rho$ is the acceleration resulting from the pressure-gradient force, $-g e_z$ is the standard acceleration due to gravity, and$-2\Omega \times u$ is the Coriolis acceleration term (see page 34 in [30]).

On the one hand, we can express the gradient in spherical coordinates, thus for the pressure-gradient term we have

$$\begin{aligned} -\frac{\nabla p}{\rho} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} e_r - \frac{1}{R\rho}\frac{\partial p}{\partial \theta} e_\theta - \frac{1}{R\rho\cos\theta}\frac{\partial p}{\partial \phi} e_\phi \\ &= -\frac{1}{R\rho\cos\theta}\frac{\partial p}{\partial \phi} e_x - \frac{1}{R\rho}\frac{\partial p}{\partial \theta} e_y - \frac{1}{\rho}\frac{\partial p}{\partial r} e_z. \end{aligned} \tag{3.6}$$

On the other hand, the Coriolis term gives

$$\begin{aligned} -2\Omega \times u &= -2 \begin{vmatrix} e_x & e_y & e_z \\ 0 & \Omega\cos\theta & \Omega\sin\theta \\ u_x & u_y & u_z \end{vmatrix} \\ &= -2(\Omega u_z \cos\theta - \Omega u_y \sin\theta) e_x - 2\Omega u_x \sin\theta e_y + 2\Omega u_x \cos\theta e_z. \end{aligned} \tag{3.7}$$

Eventually, by substituting (3.6) and (3.7) into equation (3.5) and equating this to (3.4), we obtain the following identities for each component:

$$\frac{Du_x}{Dt} + \frac{u_x u_z}{R} - \frac{u_x u_y \tan\theta}{R} = -\frac{1}{R\rho\cos\theta}\frac{\partial p}{\partial \phi} + 2\Omega u_y \sin\theta - 2\Omega u_z \cos\theta, \tag{3.8a}$$

$$\frac{Du_y}{Dt} + \frac{u_y u_z}{R} + \frac{u_x^2 \tan\theta}{R} = -\frac{1}{R\rho}\frac{\partial p}{\partial \theta} - 2\Omega u_x \sin\theta, \tag{3.8b}$$

$$\frac{Du_z}{Dt} - \frac{u_y^2 + u_x^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + 2\Omega u_x \cos\theta. \tag{3.8c}$$

It is conventional to define x and y as eastward and northward distances, respectively, such that $Dx = R\cos\theta D\phi$ and $Dy = RD\theta$. Since the spherical coordinate r is the distance from the centre of the Earth, it is related to the height z by $r = R + z$. Thus, the horizontal velocity components are $u_x = Dx/Dt$ and $u_y = Dy/Dt$, whereas for the vertical component, we have $u_z = Dz/Dt$, and it holds that

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial y}\frac{\partial y}{\partial \theta} = R\frac{\partial p}{\partial y}, \tag{3.9a}$$

$$\frac{\partial p}{\partial \phi} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial \phi} = R\cos\theta\frac{\partial p}{\partial x}, \tag{3.9b}$$

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial z}. \tag{3.9c}$$

In conclusion, using (3.9a), (3.9b), and (3.9c), the derived equations (3.8a), (3.8b), and (3.8c) can be rewritten as

$$\frac{Du_x}{Dt} + \frac{u_x u_z}{R} - \frac{u_x u_y \tan\theta}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega u_y \sin\theta - 2\Omega u_z \cos\theta, \tag{3.10a}$$

$$\frac{Du_y}{Dt} + \frac{u_y u_z}{R} + \frac{u_x^2 \tan\theta}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u_x \sin\theta, \tag{3.10b}$$

$$\frac{Du_z}{Dt} - \frac{u_x^2 + u_y^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u_x \cos\theta. \tag{3.10c}$$

This set of equations (3.10a)-(3.10c), which are, respectively, the eastward, northward, and vertical component momentum equations, governs the dynamics of the atmospheric flow. In the following, we shall perform a scale analysis of (3.10a)-(3.10c), which will simplify these equations by neglecting the terms with the lowest order of magnitude.

In meteorology, a synoptic scale is commonly used, which refers to a horizontal length scale of the order of 10^6 m or more. As far as we consider large spatial scales ($L \sim 10^6$ m horizontally and $H \sim 10^4$ m vertically), the horizontal velocities u_x and u_y are typically of the order of $U = 10\,\mathrm{m\,s^{-1}}$, while the vertical velocity u_z is about $W = 10^{-2}\,\mathrm{m\,s^{-1}}$. Hence, the typical time scale is $\Delta t = \dfrac{L}{U} = 10^5$ s. Furthermore, the following physical properties are well-known:

- Earth radius: $R = 6.378 \cdot 10^6\,\mathrm{m} \sim 10^7\,\mathrm{m}$.
- Frequency of rotation of the Earth: $\Omega = 7.272 \cdot 10^{-5}\,\mathrm{s^{-1}} \sim 10^{-4}\,\mathrm{s^{-1}}$.
- Gravitational acceleration: $g = 9.81\,\mathrm{m\,s^{-2}} \sim 10\,\mathrm{m\,s^{-2}}$.
- Density of air in the atmosphere: $\rho \sim 1\,\mathrm{kg\,m^{-3}}$.

In addition to the magnitudes above, we need to consider:

- Horizontal pressure fluctuation scale: $\delta p/\rho \sim 10^3\,\mathrm{m^2\,s^{-2}}$.
- Coriolis term: $f_0 = 2\Omega \sin\theta_0 \sim \Omega \sim 10^{-4}\,\mathrm{s^{-1}}$.

Horizontal pressure fluctuation is normalised by the density for the scale estimate to be valid at all heights in the troposphere, despite the fact that both δp and ρ decrease exponentially with height. Furthermore, in the Coriolis term estimation, we have considered a centred latitude $\theta_0 = 45^\circ$. For further knowledge on the characteristic scales mentioned above, I suggest seeing pages 38 and 39 in [30]. At this point, we are finally ready to carry out the scale analysis of (3.10a)-(3.10c). For instance, let us first determine the approximate order of magnitude of the individual terms in equation (3.10a):

$$\frac{Du_x}{Dt} = \frac{\partial u_x}{\partial t} + u_x \cdot \nabla u_x \sim \frac{U^2}{L} \sim 10^{-4}\,\mathrm{m\,s^{-2}}.$$

$$\frac{u_x u_z}{R} \sim \frac{UW}{R} \sim 10^{-8}\,\mathrm{m\,s^{-2}}.$$

$$\frac{u_x u_y \tan\theta_0}{R} \sim \frac{U^2}{R} \sim 10^{-5}\,\mathrm{m\,s^{-2}}.$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} \sim \frac{\delta p}{\rho L} \sim 10^{-3}\,\mathrm{m\,s^{-2}}.$$

$$2f_0 u_y \sin\theta_0 \sim \Omega U \sim 10^{-3}\,\mathrm{m\,s^{-2}}.$$

$$2f_0 u_z \cos\theta_0 \sim \Omega W \sim 10^{-6}\,\mathrm{m\,s^{-2}}.$$

Hence, the terms with the highest order of magnitude in equation (3.10a) are $\frac{1}{\rho}\frac{\partial p}{\partial x}$ and $2\Omega u_y \sin\theta$. According to our scale analysis, the other terms are negligibly small and therefore can be neglected. The same reasoning can be performed in equation (3.10b), coming to the conclusion that the pressure-gradient force term and the Coriolis force term are the most relevant ones. Table 3.1 contains the most relevant data with regard to what has just been said.

x-Equation	$\frac{Du_x}{Dt}$	$\frac{u_x u_z}{R}$	$\frac{-u_x u_y \tan\theta}{R}$	$\frac{-1}{\rho}\frac{\partial p}{\partial x}$	$2\Omega u_y \sin\theta$	$-2\Omega u_z \cos\theta$
y-Equation	$\frac{Du_y}{Dt}$	$\frac{u_y u_z}{R}$	$\frac{u_x^2 \tan\theta}{R}$	$\frac{-1}{\rho}\frac{\partial p}{\partial y}$	$-2\Omega u_x \sin\theta$	
Scales	U^2/L	UW/R	U^2/R	$\delta p/\rho L$	$f_0 U$	$f_0 W$
Order ($\mathrm{m\,s^{-2}}$)	10^{-4}	10^{-8}	10^{-5}	10^{-3}	10^{-3}	10^{-6}

Table 3.1: Scale analysis of the horizontal momentum equations (3.10a),(3.10b).

This is indeed the core of the geostrophic approximation, which neglects every term in equations (3.10a) and (3.10b) but the Coriolis term due to the horizontal velocities and the pressure-gradient one. This approximation is generally true in the deep ocean over large spatial and long temporal scales ($L > 100\,\mathrm{km}, \Delta t > 2\,\mathrm{days}$). Thus, the geostrophic approximation reads

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega u_y \sin\theta = 0,$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u_x \sin\theta = 0,$$

or equivalently

$$\begin{cases} \frac{1}{\rho}\frac{\partial p}{\partial x} = f u_y, \\ \frac{1}{\rho}\frac{\partial p}{\partial y} = -f u_x. \end{cases} \tag{3.11}$$

Alternatively, if we carry out the previous scale analysis in equation (3.10c), we find out that the terms with the highest order of magnitude are the gravitational and pressure-gradient force ones. However, in this case, we need to take into account that pressure decreases by about an order of magnitude from the ground to the tropopause, so the vertical pressure gradient may be scaled by P_0/H, where $P_0 \sim 10^5\,\mathrm{Pa}$ is the surface pressure. Neglecting the other terms and only considering the ones mentioned above, we get the hydrostatic approximation:

$$\frac{\partial p}{\partial z} = -\rho g \tag{3.12}$$

This approximation is valid when vertical accelerations are small in comparison to the gravitational one. Again, table 3.2 contains the results of the scale analysis of equation (3.10c).

z-Equation	$\frac{Du_z}{Dt}$	$-\frac{u_x^2 + u_y^2 w}{R}$	$\frac{-1}{\rho}\frac{\partial p}{\partial z}$	$-g$	$2\Omega u_x \cos\theta$
Scales	UW/L	U^2/R	$P_0/\rho H$	g	$f_0 U$
Order ($\mathrm{m\,s^{-2}}$)	10^{-7}	10^{-5}	10	10	10^{-3}

Table 3.2: Scale analysis of the vertical momentum equation (3.10c).

Remark 3.1. *On the one hand, equations (3.11) represent a steady-state balance between the pressure gradient and Coriolis forces. On the other hand, (3.12) means that the pressure at any point in the ocean/atmosphere is due to the weight of the water/air above it.*

3.2 Potential vorticity conservation

Definition 3.1 (Geostrophic wind). *The geostrophic wind is the theoretical wind that results from an exact balance between the Coriolis force and the pressure-gradient force. The velocity components of the geostrophic wind are*

$$\begin{cases} u_x^g = -\dfrac{1}{\rho f}\dfrac{\partial p}{\partial y}, \\ u_y^g = \dfrac{1}{\rho f}\dfrac{\partial p}{\partial x}. \end{cases} \tag{3.13}$$

Remark 3.2. *The geostrophic wind is a horizontal flow and its velocity in vectorial form is given by*

$$u^g = \frac{1}{\rho f} e_z \times \nabla p. \tag{3.14}$$

Definition 3.2 (Rossby number (Ro)). *The Rossby number is a dimensionless number defined as the ratio of inertial force to Coriolis force. Explicitly:*

$$Ro = \frac{U}{Lf}, \tag{3.15}$$

where U is the horizontal velocity scale, f is the Coriolis parameter, and L is the horizontal length scale.

Bearing in mind that

$$\frac{Du_x}{Dt} \sim \frac{U^2}{L}, \quad fu_x \sim fU,$$

we have

$$\frac{Du_x/Dt}{fu_x} \sim \frac{U^2/L}{fU} = \frac{U}{Lf} = Ro,$$

so it is clear that when $Ro \ll 1$, the Coriolis force plays a key role concerning the dynamics of the fluid under study. In order for the geostrophic approximation to be acceptable, it is necessary that the Rossby number be negligibly small.

Remark 3.3. *Another dimensionless number widely used in geophysics is the Ekman number (Ek), defined as the ratio of viscous forces to Coriolis forces and it can be expressed as*

$$Ek = \frac{\nu}{2\Omega L^2 \sin\theta},$$

where ν is the kinetic viscosity of the fluid. In our case, since we are considering non-viscous fluids, then $\nu = 0$ and $Ek = 0$. Nevertheless, one should notice that not only do we need a small Rossby number for the geostrophic approximation to be valid, but we also require the Ekman number to be small.

Going on with the mathematical computations, it is helpful to partition the pressure and density into parts that represent the fields in the absence of motion and perturbations to those fields due to the motion. In absence of motion, i.e. $u_x = u_y = u_z = 0$, equations (3.10a)-(3.10c) yield

$$\frac{\partial p_s}{\partial x} = \frac{\partial p_s}{\partial x} = 0, \tag{3.16}$$

$$\frac{\partial p_s}{\partial z} = -\rho_s g, \tag{3.17}$$

where $p_s(z)$ is a standard pressure that corresponds to the horizontally averaged pressure at each height, whereas $\rho_s(z)$ is the corresponding standard density. According to equation (3.17), both of them are in exact hydrostatic equilibrium. For further reading on this topic see pages 41 and 42 in [30]. We may then write the total pressure and density fields as

$$\begin{cases} p = p_s(z) + p'(x,y,z,t), \\ \rho = \rho_s(z) + \rho'(x,y,z,t), \end{cases} \tag{3.18}$$

where the fields p' and ρ' are the deviations from the standard values of pressure and density, p_s and ρ_s, respectively.

The Boussinesq approximation replaces the density by a constant mean value, ρ_s, everywhere except in the buoyancy term in the vertical momentum equation (3.10c). This approximation stems from the fact that the standard density varies across the lowest kilometer of the atmosphere by only about 10%, and the fluctuating component of density deviates from the basic state (standard) by only a few percentage points. However, this is not always the case, and density fluctuations cannot be neglected in order to represent the buoyancy force. For further information see page 117 in [30].

On the one hand, under the Boussinesq approximation, combining the geostrophic approximation (3.11) with (3.18), we have

$$\begin{cases} \dfrac{\partial p}{\partial x} = \dfrac{\partial p'}{\partial x} = \rho_s f u_y, \\ \dfrac{\partial p}{\partial y} = \dfrac{\partial p'}{\partial y} = -\rho_s f u_x. \end{cases} \tag{3.19}$$

On the other hand, the hydrostatic approximation (3.12) combined with (3.18) yields

$$\frac{\partial p'}{\partial z} = -\rho' g. \tag{3.20}$$

Therefore, under our approximations, the pressure perturbation field is in hydrostatic equilibrium with the density perturbation field.

Proposition 3.2. *If the Rossby number Ro is small, the density perturbation ρ' is much smaller than the static density ρ_s, that is $\rho' \ll \rho_s$.*

Proof. Let us first notice that (3.19) implies $p' \sim \rho_s f U L$. Similarly, from equation (3.20) we have that $\rho' \sim \dfrac{p'}{gH}$. Hence

$$\rho' \sim \frac{\rho_s f U L}{gH} \Rightarrow \frac{\rho'}{\rho_s} \sim Ro \frac{f^2 L^2}{gH}.$$

By hypothesis, $Ro \ll 1$. Besides, considering the typical scales in meteorology: $\dfrac{f^2 L^2}{gH} \sim \dfrac{10^{-8} \cdot 10^{12}}{10 \cdot 10^4} \sim 0.1$, and we conclude that $\rho' \ll \rho_s$. □

Proposition 3.3. *Under the assumptions that $Ro \ll 1$ and also that the horizontal spatial scale is much lower than the Earth radius, that is, $L \ll R$, then the function defined as $\psi = \dfrac{p}{\rho_s f}$ is a streamfunction.*

Proof. Let us start by evaluating the partial derivative of p/f with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{p}{f}\right) = \frac{1}{f}\frac{\partial p}{\partial y} - \frac{p}{f^2}\frac{\partial f}{\partial y}. \tag{3.21}$$

Bearing in mind that

$$\frac{\partial f}{\partial y} = \frac{1}{R}\frac{\partial f}{\partial \theta} = \frac{2\Omega \cos\theta}{R} = \beta, \tag{3.22}$$

we can compare the two terms in (3.21):

$$\frac{\dfrac{p}{f^2}\dfrac{\partial f}{\partial y}}{\dfrac{1}{f}\dfrac{\partial p}{\partial y}} = \frac{\dfrac{1}{f}\dfrac{\partial f}{\partial y}}{\dfrac{1}{p}\dfrac{\partial p}{\partial y}} \sim \frac{\beta/f}{1/L} \sim \frac{L}{R}\cot\theta. \tag{3.23}$$

Therefore, as long as the ratio L/R is negligibly small, the second term in equation (3.21) can be safely neglected. This is actually equivalent to considering that the Coriolis parameter is constant along any horizontal direction, i.e., $f \simeq f_0 = 2\Omega \sin\theta_0$. In other words, the condition $L \ll R$ means that we are working over a fixed latitude θ_0.

Moreover, since $Ro \ll 1$, we can use the geostrophic approximation (3.11). Thus, on the one hand

$$\frac{\partial \psi}{\partial y} = \frac{1}{f_0\rho_s}\frac{\partial p}{\partial y} = -\frac{\rho_s f_0 u}{f_0 \rho_s} = -u_x,$$

but on the other hand

$$\frac{\partial \psi}{\partial x} = \frac{1}{f_0\rho_s}\frac{\partial p}{\partial x} = \frac{\rho_s f_0 v}{f_0 \rho_s} = u_y.$$

In conclusion, $(u_x, u_y) = \nabla^\perp \psi = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right)$ and ψ is a streamfunction. □

From this point onwards, we will assume that $L \ll R$, and therefore we can consider a constant Coriolisis parameter: $f_0 = 2\Omega \sin\theta_0$.

Proposition 3.4. *Under the geostrophic and hydrostatic approximations, buoyancy b is related to the streamfunction $\psi = \frac{p}{\rho_s f_0}$ as*

$$b = f_0 \frac{\partial \psi}{\partial z}. \tag{3.24}$$

Proof. The derivative of the streamfunction ψ with respect to z yields

$$\frac{\partial \psi}{\partial z} = \frac{1}{f_0\rho_s}\frac{\partial p}{\partial z} - \frac{p}{f_0\rho_s^2}\frac{\partial \rho_s}{\partial z}.$$

Next, we will show that the second term in the previous equation is smaller than the first one. To do so, we use the relation between the Brunt-Väisälä frequency and the derivative of ρ_s with respect to z.

$$N^2 = -\frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z} \Rightarrow \frac{\partial \rho_s}{\partial z} = \frac{-\rho_s N^2}{g}.$$

Therefore

$$\frac{\partial \psi}{\partial z} = \frac{1}{f_0\rho_s}\frac{\partial p}{\partial z} + \frac{pN^2}{f_0\rho_s g}.$$

The typical values of the Brunt-Väisälä frequency in the ocean and the atmosphere are $N \sim 10^{-2}\,\mathrm{s}^{-1}$ (see [31]). Hence

$$\frac{pN^2}{f_0\rho_s g} \sim \frac{10^5 \cdot 10^{-4}}{10^{-4}\cdot 10} = 10^4\,\mathrm{m\,s^{-1}},$$

whereas

$$\frac{1}{f_0\rho_s}\frac{\partial p}{\partial z} \sim \frac{10}{10^{-4}} = 10^5\,\mathrm{m\,s^{-1}}.$$

In conclusion, $\frac{1}{f_0\rho_s}\frac{\partial p}{\partial z} \gg \frac{pN^2}{f_0\rho_s g}$ and the derivative of ψ with respect to z can be approximated by

$$\frac{\partial \psi}{\partial z} = \frac{1}{f_0\rho_s}\frac{\partial p}{\partial z} = -\frac{\rho g}{f_0\rho_s} = \frac{b}{f_0} \Rightarrow b = f_0\frac{\partial \psi}{\partial z}.$$

□

Let us recall that the quasi-geostrophic potential vorticity was defined in (2.48) as

$$q = f + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) \simeq f_0 + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right). \tag{3.25}$$

Definition 3.3 (Geopotential). *We define the geopotential as the potential function* $\Phi(x, y, z, t)$ *such that*

$$\frac{\partial\Phi}{\partial z} = g. \tag{3.26}$$

Note that the hydrostatic relation (3.12) establishes a relationship between pressure and height in each vertical column of the atmosphere. Thus, it is possible to use pressure p as the independent vertical coordinate, instead of height z, which would be the dependent one. These considerations lead to describing the thermodynamic state of the atmosphere through the fields $\Phi(x, y, p, t)$ and $T(x, y, p, t)$. For example, assuming an ideal gas behaviour, i.e. $p = \rho R_d T$, we have

$$g = \frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial p}\frac{\partial p}{\partial z} = -\rho g\frac{\partial\Phi}{\partial p} = -\frac{pg}{R_d T} \Rightarrow \frac{\partial\Phi}{\partial p} = -\frac{R_d T}{p},$$

which is an equivalent formulation of the hydrostatic approximation (3.12) but in terms of the geopotential and pressure.

Furthermore, as shown on pages 21 and 22 in [30], the following identities hold

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = \left(\frac{\partial\Phi}{\partial x}\right)_p, \tag{3.27}$$

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = \left(\frac{\partial\Phi}{\partial y}\right)_p, \tag{3.28}$$

where the subindexes p, z indicate that the derivatives are evaluated holding p and z constant, respectively. Therefore, the isobaric system has a salient advantage: the density no longer appears in the pressure gradient force. In light of equations (3.27) and (3.28), the components of the geostrophic wind can be rewritten as

$$\begin{cases} u_x^g = -\dfrac{1}{f}\dfrac{\partial\Phi}{\partial y}. \\ u_y^g = \dfrac{1}{f}\dfrac{\partial\Phi}{\partial x}. \end{cases} \tag{3.29}$$

In addition, because p is the independent vertical coordinate, the material derivative expands as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{Dx}{Dt}\frac{\partial}{\partial x} + \frac{Dy}{Dt}\frac{\partial}{\partial y} + \frac{Dp}{Dt}\frac{\partial}{\partial p} = \frac{\partial}{\partial t} + u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

where $\omega = Dp/Dt$ is called the "omega" vertical motion and it plays the same role in the isobaric coordinate system that $u_z = Dz/Dt$ plays in height coordinates. Nevertheless, the horizontal velocity can be decomposed into geostrophic and ageostrophic components

$$u_x = u_x^g + u_x^a, \quad u_y = u_y^g + u_y^a,$$

and assuming that $u_x^a \ll u_x^g, u_y^a \ll u_y^g$, then we can suppose that the main mechanism by which advection occurs in middle latitudes is given by the geostrophic wind. Hence, the material derivative is approximated by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p} \simeq \frac{\partial}{\partial t} + u_x^g \frac{\partial}{\partial x} + u_y^g \frac{\partial}{\partial y}.$$

In other words,

$$\frac{D}{Dt} \simeq \frac{\partial}{\partial t} + u^g \cdot \nabla,$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ is the horizontal gradient operator, also known as gradient at constant pressure. From now on, ∇ will be used to denote this operator. What is more, since we are working under the assumption $u \simeq u^g$, the z-component of the velocity will be neglected.

Definition 3.4 (Geopotential tendency)**.** *The geopotential tendency χ of a flow is the Eulerian derivative of the geopotential with respect to time:*

$$\chi = \frac{\partial \Phi}{\partial t}.$$

Proposition 3.5. *The geopotential tendency satisfies the following equation:*

$$\frac{1}{f_0} \nabla^2 \chi = -u \cdot \nabla \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) + f_0 \frac{\partial \omega}{\partial p},$$

where the only unknowns are Φ and χ.

Proof. Let us start from the QG vorticity equation (see pages 151 and 152 in [30] for a derivation):

$$\frac{D\zeta}{Dt} = f_0 \frac{\partial \omega}{\partial p} - \frac{\partial f}{\partial y} u_y = f_0 \frac{\partial \omega}{\partial p} - \beta u_y.$$

Assuming a geostrophic flow, one has

$$\zeta = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = \frac{1}{f_0} \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) = \frac{1}{f_0} \nabla^2 \Phi.$$

Hence

$$\frac{D}{Dt} \left(\frac{1}{f_0} \nabla^2 \Phi \right) = f_0 \frac{\partial \omega}{\partial p} - \frac{\partial f}{\partial y} u_y = f_0 \frac{\partial \omega}{\partial p} - \beta u_y$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\frac{1}{f_0} \nabla^2 \Phi \right) + u \cdot \nabla \left(\frac{1}{f_0} \nabla^2 \Phi \right) = f_0 \frac{\partial \omega}{\partial p} - \beta u_y,$$

which is equivalent to

$$\frac{1}{f_0} \nabla^2 \chi = -u \cdot \nabla \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) + f_0 \frac{\partial \omega}{\partial p}.$$

□

Proposition 3.6 (QG geopotential tendency equation in its adiabatic form)**.** *In the absence of forcing and dissipation, the following equation holds:*

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0 u \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}u \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right],$$

where the only unknown is the geopotential Φ.

Proof. According to (3.29), the geostrophic wind in terms of the geopotential Φ is given by

$$u \simeq u^g = \frac{1}{f}k \times \nabla\Phi \simeq \frac{1}{f_0}k \times \nabla\Phi. \tag{3.30}$$

Substituting (3.30) in the equation given by Proposition 3.5, we obtain

$$\frac{1}{f_0}\nabla^2\chi = -\left(\frac{1}{f_0}k \times \nabla\Phi\right) \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial \omega}{\partial p}. \tag{3.31}$$

Our goal now is to get rid of the term $f_0\dfrac{\partial \omega}{\partial p}$ in the previous equation (3.31). In order to do so, we shall need the QG thermodynamic equation (see [32] for its derivation):

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p} \tag{3.32}$$

, where $\sigma = -\dfrac{1}{\rho}\dfrac{\partial \ln\theta}{\partial p}$ is the static stability parameter, $\kappa = R_d/c_p$ and J is the rate of heating per unit mass due to radiation, conduction, and latent heat release.

Flipping derivatives in (3.32) yields

$$-\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right) - u \cdot \nabla\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{\kappa J}{p} = 0 \Rightarrow -\frac{\partial \chi}{\partial p} - u \cdot \nabla\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{\kappa J}{p} = 0. \tag{3.33}$$

The salient idea of the proof is to differentiate equation (3.33) with respect to the pressure and multiply by $\dfrac{f_0}{\sigma}$. Hence

$$\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right) - u \cdot \nabla\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{\kappa J}{p}\right] = 0. \tag{3.34}$$

Furthermore, assuming that σ is constant with respect to p, equation (3.34) can be rewritten as

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}u \cdot \nabla\left(\frac{\partial \Phi}{\partial p}\right)\right] - f_0\frac{\partial \omega}{\partial p} - f_0\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right). \tag{3.35}$$

Notice that if we add equations (3.31) and (3.35), the term $f_0\dfrac{\partial \omega}{\partial p}$ will disappear and, multiplying by f_0 as well, we obtain the identity

$$\begin{aligned}\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = &- f_0 u \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) \\ &- \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}u \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] - f_0^2\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right).\end{aligned} \tag{3.36}$$

Equation (3.36) is called the QG geopotential tendency equation. Eventually, under the assumption of an adiabatic flow, it holds that $J = 0$ and we finally obtain the so-called QG geopotential tendency equation in the adiabatic form. Thus, the proposition is proved. □

Proposition 3.7. *Quasi-geostrophic potential vorticity q can be expressed in terms of the geopotential Φ as*

$$q = f_0 + \frac{1}{f_0}\nabla^2\Phi + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right).$$

Proof. First of all, combining 3.29 and 2.44, we have

$$\zeta = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2\Phi,$$

and by equation 2.45 we obtain

$$\nabla^2\psi = \frac{1}{f_0}\nabla^2\Phi. \tag{3.37}$$

Secondly, notice that it is possible to define a geostrophic streamfunction as

$$\psi = \frac{\Phi}{f_0}, \tag{3.38}$$

since the following identities hold

$$\frac{\partial\psi}{\partial x} = \frac{1}{f_0}\frac{\partial\Phi}{\partial x} = u_y, \qquad \frac{\partial\psi}{\partial y} = \frac{1}{f_0}\frac{\partial\Phi}{\partial y} = -u_x.$$

Hence

$$\frac{\partial\Phi}{\partial p} = f_0\frac{\partial\psi}{\partial p} = f_0\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial p} = -\frac{f_0}{\rho g}\frac{\partial\psi}{\partial z},$$

and as a consequence

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right) = \frac{\partial}{\partial p}\left(-\frac{f_0^2}{\sigma}\frac{1}{\rho g}\frac{\partial\Phi}{\partial z}\right) = \frac{1}{\rho g}\frac{\partial}{\partial z}\left(\frac{f_0^2}{\sigma\rho g}\frac{\partial\psi}{\partial z}\right). \tag{3.39}$$

Next, we aim to establish a relationship between the static stability parameter σ and the Brunt-Väisälä frequency N^2. By definition, one has

$$\sigma = -\frac{RT_0}{p}\frac{\partial ln\theta_0}{\partial p} = \frac{RT_0}{p\rho g}\frac{\partial ln\theta_0}{\partial z} = \frac{1}{g\rho^2}\frac{\partial ln\theta_0}{\partial z} = \frac{N^2}{\rho^2 g^2} \Leftrightarrow N^2 = \rho^2 g^2\sigma.$$

Therefore, assuming that N^2 varies slightly along the vertical direction, equation 3.39 reads

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right) = \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right). \tag{3.40}$$

Eventually, using equations 3.37 and 3.40 we obtain

$$q = f_0 + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) = f_0 + \frac{1}{f_0}\nabla^2\Phi + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right).$$

□

Theorem 3.1. *In the absence of forcing and dissipation, the quasi-geostrophic potential vorticity q is conserved along the geostrophic flow, that is,*

$$\frac{\partial q}{\partial t} + u\cdot\nabla q = 0,$$

where ∇ is the horizontal gradient.

Proof. To show the statement, we start from the QG geopotential tendency equation in the adiabatic form, given by Proposition 3.6:

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0 u \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}u \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right].$$

By the product rule, the last term on the right-hand side expands as

$$-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}u \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] = \frac{f_0^2}{\sigma}\frac{\partial u}{\partial p} \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right) - u \cdot \nabla \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial \Phi}{\partial p}\right), \tag{3.41}$$

and since we are assuming a geostrophic flow, we also have

$$f_0 \frac{\partial u}{\partial p} = k \times \nabla\left(\frac{\partial \Phi}{\partial p}\right) \Rightarrow \frac{\partial u}{\partial p} \perp \nabla\left(\frac{\partial \Phi}{\partial p}\right).$$

Thus, the first term on the right-hand side in equation (3.41) vanishes, that is,

$$-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}u \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] = -u \cdot \nabla \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial \Phi}{\partial p}\right). \tag{3.42}$$

Next, we substitute (3.42) in the QG geopotential tendency equation (adiabatic) and simplify:

$$\begin{aligned}\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi &= -f_0 u \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - u \cdot \nabla \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial \Phi}{\partial p}\right) \\ &= -f_0 u \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right).\end{aligned}$$

Hence

$$\frac{\partial}{\partial t}\left[\frac{1}{f_0}\nabla^2\Phi + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] + u \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right) = 0. \tag{3.43}$$

But $\dfrac{\partial f}{\partial t} = 0$, so equation (3.43) is equivalent to

$$\frac{\partial}{\partial t}\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] + u \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right) = 0.$$

In other words

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + u \cdot \nabla q = 0.$$

□

3.3 Formulation of the SQG model

Since we are only interested in studying the QGPV (quasi-geostrophic potential vorticity) anomalies and the Coriolis term f_0 is constant, we can omit this term in the expression of the potential vorticity (3.25). The cornerstone of the Quasi-Geostrophic (QG) theory is the conservation of the QGPV along the geostrophic flow trajectories. However, as we will see later, in addition to QGPV conservation, it is necessary to introduce the concept of surface buoyancy.

Definition 3.5 (Surface buoyancy). *Surface buoyancy is defined as:*

$$b_s = b(x, y, z = 0) = f_0 \left.\frac{\partial \psi}{\partial z}\right|_{z=0}. \tag{3.44}$$

Taking into consideration what has been mentioned above, we will write the quasi-geostrophic potential vorticity as

$$q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right). \tag{3.45}$$

According to (3.45), it would be possible to determine the streamfunction field ψ if the three-dimensional distribution of PV were known at a given time. As long as $N^2 > 0$, (3.45) is an elliptic equation and hence, boundary conditions are to be defined in order to solve this equation. Let us set aside the lateral conditions for now and focus on the vertical ones. For instance, in the case of the atmosphere, we can think of a semi-infinite vertical domain, so the vertical boundary condition is to be imposed at the surface $z = 0$. Nevertheless, equation (3.44) already gives us the boundary condition at $z = 0$ that we are seeking.

Cutting a long story short, both QGPV in the fluid interior and surface buoyancy are needed in order to determine the three-dimensional streamfunction field:

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} + \dfrac{\partial}{\partial z}\left(\dfrac{f_0^2}{N^2}\dfrac{\partial \psi}{\partial z}\right) = q, \\ \left.\dfrac{\partial \psi}{\partial z}\right|_{z=0} = \dfrac{b_s}{f_0}. \end{cases} \tag{3.46}$$

This reasoning mentioned above is known as the principle of potential vorticity inversion. We should bear in mind that knowing the streamfunction field ψ is important on the grounds of the fact that it allows us to know other quantities such as velocity, buoyancy, pressure fields, etc. It stands to reason that different values of the surface buoyancy b_s will yield different solutions of (3.46).

Let us distinguish two cases now:

1. Interior-induced dynamics:

   This case stems from removing the surface buoyancy ($b_s = 0$) and allowing for QGPV anomalies in the fluid interior ($q \neq 0$). Hence, the interior QGPV is conserved along the geostrophic flow and we have the following equations:

$$\textbf{(QG)} \begin{cases} q = \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} + \dfrac{\partial}{\partial z}\left(\dfrac{f_0^2}{N^2}\dfrac{\partial \psi}{\partial z}\right), \\ f_0 \left.\dfrac{\partial \psi}{\partial z}\right|_{z=0} = 0, \\ \dfrac{\partial q}{\partial t} + u \cdot \nabla q = 0. \end{cases} \tag{3.47}$$

   The set of equations (3.47) is the classical model of the quasi-geostrophic theory (QG) and its solution is represented by the interior streamfunction ψ_{int}.

2. Surface-induced dynamics:

   On the contrary, this situation assumes no interior QGPV anomalies ($q = 0$). Besides, conservation of the surface buoyancy along the surface geostrophic flow is assumed. Hence

$$\textbf{(SQG)} \begin{cases} q = \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} + \dfrac{\partial}{\partial z}\left(\dfrac{f_0^2}{N^2}\dfrac{\partial \psi}{\partial z}\right) = 0, \\ b_s = f_0 \left.\dfrac{\partial \psi}{\partial z}\right|_{z=0}, \\ \dfrac{\partial b_s}{\partial t} + u_s \cdot \nabla b_s = 0, \end{cases} \tag{3.48}$$

   where $u_s = u(x, y, z = 0)$. This set of equations (3.48) comes from the surface quasi-geostrophic theory and its solution, the SQG flow, is represented by ψ_{sqg}.

As a matter of fact, due to the linearity of (3.46), its general solution ψ will account for both, the QG and SQG contributions: $\psi = \psi_{int} + \psi_{sqg}$ (see [33]). In the following, we will study the SQG system (3.48), which is the core of this project.

Let us now define a non-dimensional variable $\theta = \dfrac{b_s}{f_0}$. Thus

$$b_s = f_0 \left.\frac{\partial \psi}{\partial z}\right|_{z=0} \Rightarrow \theta = \left.\frac{\partial \psi}{\partial z}\right|_{z=0}$$

and

$$\frac{\partial b_s}{\partial t} + u_s \cdot \nabla b_s = 0 \Rightarrow \frac{\partial \theta}{\partial t} + u_s \cdot \nabla \theta = 0.$$

Note that θ only depends on the spatial coordinates x,y but not on z since the surface buoyancy b_s does not depend on z. Therefore: $\theta = \theta(x, y, t), \forall (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+$.

Moreover, let us also assume that the Brunt-Väisälä frequency N^2 is constant, so that the vertical coordinate z can be replaced by zN/f_0, which will be used as the new vertical coordinate. Under these assumptions, it is straightforward to see that

$$q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \nabla^2 \psi.$$

Hence, the non-dimensional form of the surface quasi-geostrophic equations (SQG) reads

$$\nabla^2 \psi = 0, \quad z > 0, \tag{3.49a}$$

$$\theta = \left.\frac{\partial \psi}{\partial z}\right|_{z=0}, \tag{3.49b}$$

$$\lim_{z \to +\infty} \frac{\partial \psi}{\partial z} = 0, \tag{3.49c}$$

$$u_s = \left(- \left.\frac{\partial \psi}{\partial y}\right|_{z=0}, \left.\frac{\partial \psi}{\partial x}\right|_{z=0} \right), \tag{3.49d}$$

$$\frac{\partial \theta}{\partial t} + u_s \cdot \nabla \theta = 0. \tag{3.49e}$$

Remark 3.4. *Note that the set of equations (3.49a)-(3.49e) corresponds to the atmospheric case, since we are considering a semi-infinite domain $z > 0$ in the vertical direction. In fact, equation (3.49c) is a boundary condition which means that when the height z is big enough, then nothing happens.*

Remark 3.5. *In order to tackle the oceanic case, it would be enough to consider a finite domain in the vertical direction and replace the boundary condition (3.49c) by*

$$\left.\frac{\partial \psi}{\partial z}\right|_{z=-H} = 0, \tag{3.50}$$

where H would be the depth of the bottom of the ocean.

Let us consider now equations (3.49a) and (3.49b):

$$\begin{cases} \nabla^2 \psi = 0, z > 0, \\ \left.\dfrac{\partial \psi}{\partial z}\right|_{z=0} = \theta. \end{cases}$$

In the following, we will assume ψ to be 2π-periodic so that it belongs to $L^1(\mathbb{T}^n)$ and we can compute its Fourier transform. In fact, in the horizontal Fourier domain equation (3.49a) reads

$$-|m|^2 \widehat{\psi}(m,z) + \frac{\partial^2 \widehat{\psi}}{\partial z^2}(m,z) = 0, \quad z > 0. \tag{3.51}$$

Equation (3.51) is a second-order differential equation in the variable z, whose solution is given by

$$\widehat{\psi}(m,z) = A(m)e^{|m|z} + B(m)e^{-|m|z}, \quad z > 0, \tag{3.52}$$

and differentiating with respect to z yields

$$\frac{\partial \widehat{\psi}}{\partial z}(m,z) = |m|A(m)e^{|m|z} - |m|B(m)e^{-|m|z}.$$

So as to determine the coefficients $A(m)$ and $B(m)$ in (3.52), we shall impose the boundary conditions (3.49b),(3.49c) on the solution (3.52). In the horizontal Fourier domain, (3.49b) and (3.49c) read, respectively,

$$\widehat{\theta}(m) = \left.\widehat{\frac{\partial \psi}{\partial z}}(m,z)\right|_{z=0} = \left.\frac{\partial \widehat{\psi}}{\partial z}(m,z)\right|_{z=0}, \tag{3.53}$$

$$\lim_{z\to+\infty} \widehat{\frac{\partial \psi}{\partial z}}(m,z) = \lim_{z\to+\infty} \frac{\partial \widehat{\psi}}{\partial z}(m,z) = 0. \tag{3.54}$$

Proposition 3.8. *From equations (3.49a)-(3.49c), the following identity is obtained in the horizontal Fourier domain:*

$$\widehat{\psi}(m,z) = -\frac{\widehat{\theta}(m)}{|m|}e^{-|m|z}, \quad z > 0, \tag{3.55}$$

where $\widehat{\psi}(m,z)$ is the two-dimensional Fourier transform of ψ at the altitude z and $\widehat{\theta}(m)$ is the Fourier transform of θ.

Proof. On the hand one, the boundary condition (3.53) gives

$$\widehat{\theta}(m) = |m|A(m) - |m|B(m). \tag{3.56}$$

On the other hand, the boundary condition (3.54) gives

$$\lim_{z\to+\infty}\left(|m|A(m)e^{|m|z} - |m|B(m)e^{-|m|z}\right) = \lim_{z\to+\infty}\left(|m|A(m)e^{|m|z}\right) = 0.$$

Thus, it necessarily holds that

$$A(m) = 0. \tag{3.57}$$

Combining (3.56) and (3.57) we get

$$\widehat{\theta}(m) = |m|B(m). \tag{3.58}$$

Eventually, we substitute $A(m)$ and $B(m)$ obtained from equations (3.57) and (3.58) in equation (3.52), obtaining

$$\widehat{\psi}(m,z) = -\frac{\widehat{\theta}(m)}{|m|}e^{|m|z}, \quad z > 0.$$

□

Note that taking $z = 0$ in equation (3.55) gives the following expression relating buoyancy to the streamfunction:

$$\widehat{\theta} = -|m|\widehat{\psi_s}. \tag{3.59}$$

Moreover, let us comment on the fact that the identity (3.59) is equivalent to the condition $\theta = -(-\nabla^2)^{1/2}\psi_s$, by definition of the operator $(-\nabla^2)^{1/2}$. In conclusion, the 2D SQG model can be expressed as follows:

$$\left.\begin{aligned} &\frac{\partial\theta}{\partial t} + u_s\cdot\nabla\theta = 0, \quad (x,y,t)\in\mathbb{R}^2\times\mathbb{R}_+,\\ &u_s = \nabla^\perp\psi_s, \quad \theta = -(-\nabla^2)^{1/2}\psi_s, \end{aligned}\right\} \quad \text{(SQG)} \tag{3.60}$$

which is indeed the same formulation as (2.49). The only difference would be the fact that in (2.49) the subindex "s" for the velocity have been omitted for the sake of simplicity. Nonetheless, although we shall use this simplified notation later, one ought to bear in mind that u and ψ stand for the velocity and the streamfunction at the surface.

Furthermore, we can rewrite the conditions $u = \nabla^\perp\psi$ and $\theta = -(-\nabla^2)^{1/2}\psi$ in (2.49) in terms of the Riesz transform. The reasoning is the following:

$$u = \nabla^\perp\psi \Rightarrow \widehat{u}(m) = im^\perp\widehat{\psi}(m) = -\frac{im^\perp}{|m|}\widehat{\theta}(m), \tag{3.61}$$

where $m^\perp$ is the perpendicular vector of m, that is, if $m = (m_1, m_2)$, then $m^\perp = (-m_2, m_1)$. In light of definition 1.17, we conclude that (3.61) is equivalent to $u = R^\perp\theta$. Thus, 2D SQG can be equivalently stated as

$$\left.\begin{aligned} &\frac{\partial\theta}{\partial t} + u_s\cdot\nabla\theta = 0, \quad (x,y,t)\in\mathbb{R}^2\times\mathbb{R}_+,\\ &u_s = R^\perp\theta. \end{aligned}\right\} \quad \text{(SQG)} \tag{3.62}$$

This formulation is widely used in the scientific literature, take, for instance, [11].

Chapter 4

Existence of Global Weak Solutions

4.1 Weak solutions

Let us consider the 2D SQG system on the torus:

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad (x,t)\in\mathbb{T}^2\times\mathbb{R}_+, \\ u = R^\perp\theta. \end{cases} \tag{4.1}$$

with $\theta = \theta(x,t)$ a non-dimensional scalar field, $u = u(x,t)$ the fluid velocity and $\psi = \psi(x,t)$ the corresponding streamfunction. For the sake of simplicity, we have introduced the notation ∂_t for the partial derivative with respect to time. Besides, in contrast with the previous chapters, we will use the variable $x \in \mathbb{T}^2$ for the spatial coordinates, on the grounds of the fact that it simplifies the notation a lot. Nonetheless, the reader should always bear in mind that x will denote a two-dimensional variable.

Let $\varphi \in C^\infty(\mathbb{T}^2\times[0,T])$ be a test function such that φ is 2π-periodic in space and $\varphi(x,T) = 0\ \forall x\in\mathbb{T}^2$. If a function θ satisfies (4.1), then we can multiply by φ and integrate over $\mathbb{T}^2\times[0,T]$ obtaining

$$\int_0^T\int_{\mathbb{T}^2}(\partial_t\theta + u\cdot\nabla\theta)\varphi dxdt = 0. \tag{4.2}$$

On the one hand, integration by parts in time yields

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2}\partial_t\theta\varphi dxdt &= -\int_0^T\int_{\mathbb{T}^2}\theta\partial_t\varphi dxdt + \int_{\mathbb{T}^2}[\varphi(x,T)\theta(x,T) - \varphi(x,0)\theta(x,0)]dx \\ &= -\int_0^T\int_{\mathbb{T}^2}\theta\partial_t\varphi dxdt - \int_{\mathbb{T}^2}\varphi(x,0)\theta(x,0)dx.\end{aligned} \tag{4.3}$$

On the other hand, integrating by parts in space and taking into account periodicity gives

$$\int_{\mathbb{T}^2}(u\cdot\nabla\theta)\varphi dx = -\int_{\mathbb{T}^2}\theta\varphi\nabla\cdot u dx - \int_{\mathbb{T}^2}\theta u\cdot\nabla\varphi dx.$$

However, since $u = \nabla^\perp\psi \Rightarrow \nabla\cdot u = 0$, so

$$\int_{\mathbb{T}^2}(u\cdot\nabla\theta)\varphi dx = -\int_{\mathbb{T}^2}\theta u\cdot\nabla\varphi dx.$$

Therefore

$$\int_0^T \int_{\mathbb{T}^2} (u \cdot \nabla\theta)\varphi dxdt = -\int_0^T \int_{\mathbb{T}^2} (u \cdot \nabla\varphi)\theta dxdt. \tag{4.4}$$

Eventually, we replace (4.3) and (4.4) in equation (4.2) and we obtain

$$\int_0^T \int_{\mathbb{T}^2} \theta(x,t)(\partial_t\varphi(x,t) + u(x,t)\cdot\nabla\varphi(x,t))dxdt + \int_{\mathbb{T}^2} \varphi(x,0)\theta(x,0)dx = 0. \tag{4.5}$$

In light of the previous equation (4.5), the definition of weak solution arises.

Definition 4.1 (Weak solution). *Let $\theta_0 \in L^2(\mathbb{T}^2)$. A function $\theta \in L^\infty([0,T]; L^2(\mathbb{T}^2))$ is a weak solution of equation (4.1) if for every test function $\varphi \in C^\infty(\mathbb{T}^2 \times [0,T])$ such that $\varphi(x,T) = 0\ \forall x \in \mathbb{T}^2$ it holds that*

$$\int_0^T \int_{\mathbb{T}^2} \theta(x,t)(\partial_t\varphi(x,t) + u(x,t)\cdot\nabla\varphi(x,t))dxdt + \int_{\mathbb{T}^2} \varphi(x,0)\theta_0(x)dx = 0, \tag{4.6}$$

where $u = R^\perp\theta$, for almost every $t \in [0,T]$.

4.2 Global existence theorem

It is possible to prove the existence of weak solutions for problem (4.1) using integration by parts and a commutator estimate. In addition, we shall also prove the conservation of the Hamiltonian and the upper bound for the kinetic energy.

First of all, let us recall some results of functional analysis that we shall need later on.

Theorem 4.1 (Arzelà-Ascoli). *Let X be a compact metric space and let $C(X)$ denote the space of all continuous functions on X with values in $\mathbb{C}$, endowed with the metric*

$$dist(f,g) = max\{|f(x) - g(x)| : x \in X\}.$$

If a sequence $\{f_n\} \subset C(X)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.

We do not provide the proof of this theorem here, it can be found in [34] or [35].

Theorem 4.2. *Let X be a separable normed space and $\{x'_n\} \subset X'$ a bounded sequence. Then, there exists a subsequence of $\{x'_n\}$ that is weak* convergent.*

For a proof, we refer to [36],[35].

Proposition 4.1. *Let $(X, \|\cdot\|)$ be a normed space and let $\{x_n\} \subset X$ be a sequence that is weak* convergent, that is, there exists $x \in X$ such that $x_n \overset{*}{\rightharpoonup} x$. Then, the sequence is bounded and*

$$\|x\|_X \leq \liminf_{n\to\infty} \|x_n\|_X.$$

The proof can be found in [35].

Theorem 4.3. *Let $(X\,\|\cdot\|_X)$ be a normed space and $(X', \|\cdot\|_{X'})$ its dual. For any $x \in X$ we have*

$$\|x\|_X = \sup\{|f(x)| : f \in X', \|f\|_{X'} \le 1\} = \max\{|f(x)| : f \in X', \|f\|_{X'} \le 1\}.$$

Theorem 4.3 is a consequence of Hahn-Banach's theorem; we refer to [8] or [37] to see the proof.

Lemma 4.1 (Generalised Minkowski's inequality for sums)**.** *The inequality*

$$\left(\sum_{i=1}^{\infty}\left|\sum_{k=1}^{\infty} Q_{i,k}\right|^p\right)^{1/p} \le \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} |Q_{i,k}|^p\right)^{1/p}$$

is valid for $p > 1$.

The proof of this lemma can be found in [38], for instance.

Definition 4.2. *Let P be an operator and f, u sufficient regular functions. We define the commutator $[P, f]u = P(fu) - f(Pu)$.*

Let us recall that the operator Λ is defined via the Fourier transform as

$$\widehat{\Lambda f}(m) = ((-\nabla^2)^{1/2} f)^\wedge(m) = |m|\widehat{f}(m),$$

for any $f \in L^1(\mathbb{T}^2)$.

Proposition 4.2. *Let $\varphi, \psi \in C^\infty(\mathbb{T}^2 \times [0, T])$. Then we have the following commutator estimate:*

$$\|[\Lambda, \nabla\varphi]\psi\|_{L^2} \le 2\pi\,\|\psi\|_{L^2} \sum_{l\in\mathbb{Z}^2} |l||\widehat{\nabla\varphi}(l)|.$$

Proof. By definition 4.2 and Plancherel's identity 1.1 we have

$$\begin{aligned} \|[\Lambda, \nabla\varphi]\psi\|_{L^2} &= \|\Lambda(\nabla\varphi\psi) - \nabla\varphi(\Lambda\psi)\|_{L^2} \\ &= 2\pi\left(\sum_{n\in\mathbb{Z}^2} |(\Lambda(\nabla\varphi\psi) - \nabla\varphi(\Lambda\psi))^\wedge(n)|^2\right)^{1/2}. \end{aligned} \tag{4.7}$$

Let us now compute the term within the sum in 4.7.

$$\begin{aligned} (\Lambda(\nabla\varphi\psi) - \nabla\varphi(\Lambda\psi))^\wedge(n) &= |n|\widehat{\nabla\varphi\psi}(n) - \widehat{\nabla\varphi\nabla\psi}(n) \\ &= |n|\sum_{l\in\mathbb{Z}^2}\widehat{\nabla\varphi}(l)\widehat{\psi}(n-l) - \sum_{l\in\mathbb{Z}^2}\widehat{\nabla\varphi}(l)\widehat{\Lambda\psi}(n-l) \\ &= |n|\sum_{l\in\mathbb{Z}^2}\widehat{\nabla\varphi}(l)\widehat{\psi}(n-l) - \sum_{l\in\mathbb{Z}^2}\widehat{\nabla\varphi}(l)|n-l|\widehat{\psi}(n-l) \\ &= \sum_{l\in\mathbb{Z}^2}(|n| - |n-l|)\widehat{\nabla\varphi}(l)\widehat{\psi}(n-l). \end{aligned} \tag{4.8}$$

Moreover, $|n| = |n-l+l| \leq |n-l| + |l| \Rightarrow |n| - |n-l| \leq |l|$. Hence, substituting equality 4.8 in equation 4.7 yields

$$\begin{aligned} \|[\Lambda, \nabla\varphi]\psi\|_{L^2} &= 2\pi \left(\sum_{n\in\mathbb{Z}^2} \left| \sum_{l\in\mathbb{Z}^2} (|n| - |n-l|) \widehat{\nabla\varphi}(l) \widehat{\psi}(n-l) \right|^2 \right)^{1/2} \\ &\leq 2\pi \left(\sum_{n\in\mathbb{Z}^2} \left| \sum_{l\in\mathbb{Z}^2} |l| \left|\widehat{\nabla\varphi}(l)\right| \left|\widehat{\psi}(n-l)\right| \right|^2 \right)^{1/2}, \end{aligned} \tag{4.9}$$

and using Lemma 4.1 in 4.9 we obtain

$$\begin{aligned} \|[\Lambda, \nabla\varphi]\psi\|_{L^2} &\leq 2\pi \sum_{l\in\mathbb{Z}^2} \left(\sum_{n\in\mathbb{Z}^2} \left(|l| \left|\widehat{\nabla\varphi}(l)\right| \left|\widehat{\psi}(n-l)\right| \right)^2 \right)^{1/2} \\ &= 2\pi \sum_{l\in\mathbb{Z}^2} |l| \left|\widehat{\nabla\varphi}(l)\right| \left(\sum_{n\in\mathbb{Z}^2} \left|\widehat{\psi}(n-l)\right|^2 \right)^{1/2}. \end{aligned} \tag{4.10}$$

Eventually, bearing in mind that

$$\sum_{n\in\mathbb{Z}^2} \left|\widehat{\psi}(n-l)\right|^2 = \sum_{n\in\mathbb{Z}^2} \left|\widehat{\psi}(n)\right|^2 = \|\psi\|_{L^2},$$

we can rewrite 4.10 as

$$\|[\Lambda, \nabla\varphi]\psi\|_{L^2} \leq 2\pi \sum_{l\in\mathbb{Z}^2} |l| \left|\widehat{\nabla\varphi}(l)\right| \|\psi\|_{L^2}.$$

□

Proposition 4.3 (Sobolev inequality)**.** *Let $\epsilon > 0$ be an integer and let $f \in H^{1+\epsilon}(\mathbb{T}^2)$. Then*

$$\|f\|_{L^\infty} \leq C_\epsilon \|f\|_{H^{1+\epsilon}},$$

where $C_\epsilon > 0$ is a constant that depends on ϵ.

Proof. By Proposition 1.2

$$f(x) = \sum_{m\in\mathbb{Z}^2} \widehat{f}(m) e^{im\cdot x},$$

and we also have the following estimates:

$$\begin{aligned} |f(x)| &= \left| \sum_{m\in\mathbb{Z}^2} \widehat{f}(m) e^{im\cdot x} \right| \leq \sum_{m\in\mathbb{Z}^2} |\widehat{f}(m)||e^{im\cdot x}| = \sum_{m\in\mathbb{Z}^2} \frac{(1+|m|^2)^{(1+\epsilon)/2}}{(1+|m|^2)^{(1+\epsilon)/2}} |\widehat{f}(m)| \\ &\leq \left(\sum_{m\in\mathbb{Z}^2} \frac{1}{(1+|m|^2)^{1+\epsilon}} \right)^{1/2} \left(\sum_{m\in\mathbb{Z}^2} (1+|m|^2)^{1+\epsilon} |\widehat{f}(m)|^2 \right)^{1/2}. \end{aligned}$$

Since the series

$$\sum_{m\in\mathbb{Z}^2} \frac{1}{(1+|m|^2)^{1+\epsilon}} < \infty,$$

is convergent, then we can denote its root square by C_ϵ and it holds that

$$|f(x)| \leq C_\epsilon \|f\|_{H^{1+\epsilon}}, \forall x \in \mathbb{T}^2 \Rightarrow \|f\|_{L^\infty} \leq C_\epsilon \|f\|_{H^{1+\epsilon}}.$$

□

Proposition 4.4. *Let $\epsilon > 0$ be an integer and let $f \in L^2(\mathbb{T}^2)$. Then, there exists $0 < \lambda < 1$ that depends on ϵ such that*

$$\|f\|_{H^{-1}} \leq \|f\|^{\lambda}_{H^{-(2+\epsilon)}} \|f\|^{1-\lambda}_{L^2}.$$

Proof. Let us observe that

$$\|f\|^2_{H^{-1}} = \sum_{m\in\mathbb{Z}^2} \frac{1}{1+|m|^2}|\widehat{f}(m)|^2 = \sum_{m\in\mathbb{Z}^2} \frac{1}{1+|m|^2}|\widehat{f}(m)|^{2\lambda}\widehat{f}(m)|^{2(1-\lambda)}. \tag{4.11}$$

Next, we use Hölder's inequality in (4.11) with $p = \dfrac{1}{\lambda}$ and $q = \dfrac{1}{1-\lambda}$. Thus

$$\|f\|^2_{H^{-1}} \leq \left(\sum_{m\in\mathbb{Z}^2} \frac{1}{(1+|m|^2)^{1/\lambda}}|\widehat{f}(m)|^2\right)^{\lambda} \left(\sum_{m\in\mathbb{Z}^2} |\widehat{f}(m)|^2\right)^{1-\lambda} \tag{4.12}$$

Eventually, we can take λ such that $\lambda = \dfrac{1}{2+\epsilon}$. In this way, it is clear that $0 < \lambda < 1$ and (4.12) reads

$$\|f\|^2_{H^{-1}} \leq \|f\|^{2\lambda}_{H^{-(2+\epsilon)}} \|f\|^{1-2\lambda}_{L^2}.$$

That is to say,

$$\|f\|_{H^{-1}} \leq \|f\|^{\lambda}_{H^{-(2+\epsilon)}} \|f\|^{1-\lambda}_{L^2}.$$

□

Remark 4.1. *Under the same assumptions, it can be shown in an analogous way that*

$$\|f\|_{H^{-1/2}} \leq \|f\|^{\lambda}_{H^{-(2+\epsilon)}} \|f\|^{1-\lambda}_{L^2}.$$

In this case, we need to take $\lambda = \dfrac{2}{2+\epsilon}$, so it still holds that $0 < \lambda < 1$.

For any $f \in L^1(\mathbb{T}^2)$, let us define the norm

$$\|f\|_s = \left(\sum_{m\in\mathbb{Z}^2} |m|^{2s}|\widehat{f}(m)|^2\right)^{1/2}.$$

Proposition 4.5. *Let θ be a solution of (4.1) and have zero average on the torus, that is,*

$$\widehat{\theta}(0,t) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \theta(x,t)dx = 0.$$

Then, this condition is preserved in time, and the norms $\|\theta\|_{H^s}$ and $\|\theta\|_s$ are equivalent.

Proof. First of all, let us show that $\widehat{\theta}(0,t) = \widehat{\theta}(0,0)$ for any $t \in [0,T]$.

$$\partial_t \widehat{\theta}(0,t) = \frac{1}{(2\pi^2)} \int_{\mathbb{T}^2} \partial_t \theta(x,t) dx = -\frac{1}{(2\pi^2)} \int_{\mathbb{T}^2} u \cdot \nabla \theta dx = \frac{1}{(2\pi^2)} \int_{\mathbb{T}^2} \nabla \cdot u \theta dx = 0,$$

where we have used integration by parts and the fact that $\nabla \cdot u = 0$. Now, we are ready to prove that the two norms are equivalent. On the one hand,

$$\|\theta\|_{H^s} = \sum_{m \in \mathbb{Z}^2} (1 + |m|^2)^s |\widehat{\theta}(m)|^2 \geq \sum_{m \in \mathbb{Z}^2} |m|^{2s} |\widehat{\theta}(m)|^2 = \|\theta\|_s .$$

On the other hand,

$$\begin{aligned} \|\theta\|_{H^s} &= \sum_{m \in \mathbb{Z}^2, m \neq 0} (1 + |m|^2)^s |\widehat{\theta}(m)|^2 \leq \sum_{m \in \mathbb{Z}^2, m \neq 0} (2|m|^2)^s |\widehat{\theta}(m)|^2 \\ &= 2^s \sum_{m \in \mathbb{Z}^2, m \neq 0} |m|^{2s} |\widehat{\theta}(m)|^2 = 2^s \|\theta\|_s . \end{aligned}$$

All in all,

$$\|\theta\|_s \leq \|\theta\|_{H^s} \leq 2^s \|\theta\|_s ,$$

and the norms are equivalent. □

In the following, we will assume that the scalar field θ has zero average on the torus and, therefore, $\|\theta\|_{H^s}$ and $\|\theta\|_s$ are equivalent norms.

In addition, since $u = R^\perp \theta$, using Plancherel's identity (1.11) it is straightforwrad to see that

$$\|u\|_{L^2}^2 = \|\theta\|_{L^2}^2 .$$

Theorem 4.4. *For each $T > 0$ and $\theta_0 \in L^2(\mathbb{T}^2)$ there exists a weak solution $\theta \in L^\infty([0,T]; L^2(\mathbb{T}^2))$ in the sense of definition 4.1 of the SQG equation (4.1). Moreover, this solution preserves the Hamiltonian*

$$-\frac{1}{2} \int_{\mathbb{T}^2} \psi(t) \theta(t) dx = -\frac{1}{2} \int_{\mathbb{T}^2} \psi_0 \theta_0 dx, \tag{4.13}$$

and satisfies the kinetic energy inequality

$$\frac{1}{2} \int_{\mathbb{T}^2} |u(t)|^2 dx \leq \frac{1}{2} \int_{\mathbb{T}^2} |u_0|^2 dx. \tag{4.14}$$

Proof. **Existence of global weak solution.** To begin with, we shall prove the convergence of the non-linear term using integration by parts. Let $\varphi \in C^\infty(\mathbb{T}^2 \times [0,T])$ be a test function such that $\varphi(x,T) = 0 \; \forall x \in \mathbb{T}^2$. Since $u = R^\perp \theta$, then $u = \nabla^\perp \psi$ and $\theta = -\Lambda \psi$. What is more,

$$\psi \in H^1 \Rightarrow \nabla^\perp \psi \in L^2 \Rightarrow \Lambda(\nabla^\perp \psi) \in H^{-1},$$

$$\varphi \in C^\infty \Rightarrow \nabla \varphi \in C^\infty,$$

so we have that

$$\begin{aligned}\int_{\mathbb{T}^2} \theta u \cdot \nabla\varphi dx &= -\int_{\mathbb{T}^2} \Lambda\psi\nabla^\perp\psi \cdot \nabla\varphi dx = -\left\langle \Lambda\psi, \nabla^\perp\psi \cdot \nabla\varphi \right\rangle_{L^2,L^2} \\ &= \left\langle \Lambda(\nabla^\perp\psi), \psi\nabla\varphi \right\rangle_{H^{-1},H^1} + \left\langle \Lambda\psi, \psi\nabla^\perp \cdot \nabla\varphi \right\rangle_{L^2,L^2} \\ &= \left\langle \Lambda(\nabla^\perp\psi), \psi\nabla\varphi \right\rangle_{H^{-1},H^1} = \left\langle \nabla^\perp\psi, \Lambda(\psi\nabla\varphi) \right\rangle_{L^2,L^2} \\ &= \int_{\mathbb{T}^2} \nabla^\perp\psi \cdot \Lambda(\psi\nabla\varphi) dx.\end{aligned} \tag{4.15}$$

According to definition 4.2, $[\Lambda, \nabla\varphi]\psi = \Lambda(\psi\nabla\varphi) - \nabla\varphi\Lambda\psi$ and it holds that

$$\begin{aligned}\int_{\mathbb{T}^2} u \cdot [\Lambda, \nabla\varphi]\psi dx &= \int_{\mathbb{T}^2} u \cdot \Lambda(\psi\nabla\varphi) dx - \int_{\mathbb{T}^2} \Lambda\psi u \cdot \nabla\varphi dx \\ &= \int_{\mathbb{T}^2} \nabla^\perp\psi \cdot \Lambda(\psi\nabla\varphi) dx + \int_{\mathbb{T}^2} \theta u \cdot \nabla\varphi dx \\ &= 2\int_{\mathbb{T}^2} \nabla^\perp\psi \cdot \Lambda(\psi\nabla\varphi) dx.\end{aligned}$$

Thus, equation (4.15) is equivalent to

$$\int_{\mathbb{T}^2} \theta u \cdot \nabla\varphi dx = \frac{1}{2}\int_{\mathbb{T}^2} u[\Lambda, \nabla\varphi]\psi dx. \tag{4.16}$$

Let us observe that for any test function $\varphi \in C^\infty$, we can apply Scharwz's inequality and Proposition 4.2 obtaining

$$\int_{\mathbb{T}^2} u[\Lambda, \nabla\varphi]\psi dx \leq \|u\|_{L^2} \|[\Lambda, \nabla\varphi]\psi\|_{L^2} \leq 2\pi \|u\|_{L^2} \|\psi\|_{L^2} \sum_{l\in\mathbb{Z}^2} |l||\widehat{\nabla\varphi}(l)|,$$

and we also have that

$$\begin{aligned}\sum_{l\in\mathbb{Z}^2} |l||\widehat{\nabla\varphi}(l)| = \sum_{l\in\mathbb{Z}^2} \frac{|l|^{2+\epsilon}}{|l|^{1+\epsilon}}|\widehat{\nabla\varphi}(l)| &\leq \left(\sum_{l\in\mathbb{Z}^2} \frac{1}{|l|^{2+2\epsilon}}\right)^{1/2} \left(\sum_{l\in\mathbb{Z}^2} |l|^{4+2\epsilon}|\widehat{\nabla\varphi}(l)|^2\right)^{1/2} \\ &\leq \left(\sum_{l\in\mathbb{Z}^2} \frac{1}{|l|^{2+2\epsilon}}\right)^{1/2} \left(\sum_{l\in\mathbb{Z}^2} (1+|l|^2)^{2+\epsilon}|\widehat{\nabla\varphi}(l)|^2\right)^{1/2} = C_\epsilon \|\nabla\varphi\|_{H^{2+\epsilon}},\end{aligned} \tag{4.17}$$

where $C_\epsilon = \left(\sum_{l\in\mathbb{Z}^2} \frac{1}{|l|^{2+2\epsilon}}\right)^{1/2} < \infty$.

Therefore, using Proposition 4.2 together with (4.17) we obtain the following commutator estimate:

$$\|[\Lambda, \nabla\varphi]\psi\|_{L^2} \leq C_\epsilon \|\nabla\varphi\|_{H^{2+\epsilon}} \|\psi\|_{L^2} \leq C_\epsilon \|\varphi\|_{H^{3+\epsilon}} \|\psi\|_{L^2}, \tag{4.18}$$

for any $\epsilon > 0$. Notice that we have omitted the constant 2π in 4.18 since it does not change anything and it would suffice to redefine the constant C_ϵ.

The method to prove the existence of a weak solution of (4.1) that we use here is based on Galerkin approximations. Let n be a positive integer. We consider the orthogonal projector P_n in $H^0 = L^2$ onto the span S_n of the Fourier modes $e^{im\cdot x}$ with $0 < |m| \leq n$, that is,

$$P_n f(x) = \sum_{|m|\leq n} \widehat{f}(x)e^{im\cdot x}, \forall f \in L^1(\mathbb{T}^2).$$

Our goal is to construct a weak solution using any appropriate sequence of approximations $\{\theta_n\}$, called Galerkin truncations. For further reading on this topic see page 64 in [39]. For $t = 0$ we have that $\theta(x, 0) = \theta_0(x)$, hence we take $\theta_n(x, 0) = P_n\theta_0(x)$. Whereas in the case $t > 0$ we have to solve $\theta_t(x, t) + u(x, t) \cdot \nabla\theta(x, t) = 0$ with $u(x, t) = R^\perp\theta(x, t)$, thus the Galerkin truncations are given by $\dot{\theta}_n(x, t) + P_n(u_n(x, t)\cdot \nabla\theta_n(x, t)) = 0$ with $u_n(x, t) = R^\perp\theta_n(x, t)$. All in all, the n-th Galerkin truncation is an ODE system in the finite-dimensional space S_n, defined by

$$\begin{cases} \dot{\theta}_n + P_n(u_n \cdot \nabla\theta_n) = 0, & t > 0, \\ u_n = R^\perp\theta_n, & t \geq 0, \\ \theta_n = P_n\theta_0, & t = 0. \end{cases} \tag{4.19}$$

Notice that since $u_n = R^\perp\theta_n$, Plancherel's identity (1.11) yields $\|u_n\|^2_{L^2} = \|\theta_n\|^2_{L^2}$. To see that 4.19 is indeed an ODE system we write down the Fourier series of the terms involved. For θ_n we have

$$\theta_n(x, t) = \sum_{|m|\leq n} c_m(t)e^{im\cdot x}, \tag{4.20}$$

and therefore $c_m(t)$ are the Fourier coefficients of θ_n. We omit the dependency with n in order to simplify the notation. The series for the term $\dot{\theta}_n$ follows directly from 4.20:

$$\dot{\theta}_n(x, t) = \sum_{|m|\leq n} \dot{c_m}(t)e^{im\cdot x}. \tag{4.21}$$

Next, we use Corollary 1.1 so as to compute the Fourier transform of $u_n \cdot \nabla\theta_n$:

$$\widehat{u_n \cdot \nabla\theta_n}(m) = \sum_{k\in\mathbb{Z}^2} \widehat{u_n}(m - k) \cdot \widehat{\nabla\theta_n}(k) = \sum_{|k|\leq n} \widehat{u_n}(m - k) \cdot \widehat{\nabla\theta_n}(k). \tag{4.22}$$

The Fourier coefficients $\widehat{u_n}(m - k)$ and $\widehat{\nabla\theta_n}(k)$ are easily computed:

$$\widehat{u_n}(m - k) = \widehat{R^\perp\theta_n}(m - k) = -i\frac{(m - k)^\perp}{|m - k|}\widehat{\theta_n}(m - k) = -i\frac{(m - k)^\perp}{|m - k|}c_{m-k}(t), \tag{4.23}$$

$$\widehat{\nabla\theta_n}(k) = ik\widehat{\theta_n}(k) = ikc_k(t), \tag{4.24}$$

and substituting (4.23) and (4.24) into (4.22) yields

$$\widehat{u_n \cdot \nabla\theta_n}(m) = \sum_{|k|\leq n} \frac{(m - k)^\perp}{|m - k|}c_{m-k}(t) \cdot kc_k(t).$$

Therefore, the series for the term $P_n(u_n \cdot \nabla\theta_n)$ is

$$\begin{aligned} P_n(u_n \cdot \nabla\theta_n) &= \sum_{|m|\leq n} \widehat{u_n \cdot \nabla\theta_n}(m)e^{im\cdot x} \\ &= \sum_{|m|\leq n} \left(\sum_{|k|\leq n} \frac{(m - k)^\perp}{|m - k|}c_{m-k}(t) \cdot kc_k(t) \right) e^{im\cdot x}. \end{aligned} \tag{4.25}$$

If we substitute equations (4.21) and (4.25) in equation $\dot{\theta}_n + P_n(u_n \cdot \nabla\theta_n) = 0$ of (4.19), we obtain

$$\sum_{|m|\leq n}\left(\dot{c}_m(t) + \sum_{|k|\leq n}\frac{(m-k)^\perp}{|m-k|}c_{m-k}(t)\cdot kc_k(t)\right)e^{im\cdot x} = 0,$$

but $e^{im\cdot x}$ is a complete orthogonal set, so

$$\dot{c}_m(t) + \sum_{|k|\leq n}\frac{(m-k)^\perp}{|m-k|}c_{m-k}(t)\cdot kc_k(t) = 0, \forall\, |m| \leq n \tag{4.26}$$

The initial condition in (4.19) reads $\theta_n(x,0) = P_n\theta_0(x)$, thus

$$\sum_{|m|\leq n} c_m(0)e^{im\cdot x} = \sum_{|m|\leq n}\widehat{\theta_0}(m)e^{im\cdot x} \Rightarrow \sum_{|m|\leq n}\left(c_m(0) - \widehat{\theta_0}(m)\right)e^{im\cdot x} = 0$$

$$\Rightarrow c_m(0) = \widehat{\theta_0}(m), \forall\, |m| \leq n. \tag{4.27}$$

After these computations, we finally conclude that the ODE system that stems from (4.19) is

$$\begin{cases} \dot{c}_m(t) + \sum_{|k|\leq n} c_{m-k}(t)c_k(t)\dfrac{(m-k)^\perp}{|m-k|}\cdot k = 0, \quad t > 0, \\ c_m(0) = \widehat{\theta_0}(m), \end{cases} \tag{4.28}$$

for every m such that $|m| \leq n$. It is easy to see that using Picard's theorem the local in time existence and uniqueness of solutions.

Using Parseval's relation 1.2 it is straightforward to see that for any $f, g \in L^2(\mathbb{T}^2)$ the following identities hold:

$$\int_{\mathbb{T}^2} P_n f P_n\overline{g}dx = \int_{\mathbb{T}^2} P_n f\overline{g}dx = \int_{\mathbb{T}^2} fP_n\overline{g}dx = (2\pi)^2\sum_{|m|\leq n}\widehat{f}(m)\overline{\widehat{g}(m)}. \tag{4.29}$$

So it is clear that we can omit the orthogonal projector P_n when integrating as long as there is a multiplying function that is projected. Moreover, since $u_n = R^\perp\theta_n$, then $\nabla \cdot u_n = 0$ and we have

$$\begin{aligned}\int_{\mathbb{T}^2}\theta_n u_n\cdot\nabla\theta_n dx = -\int_{\mathbb{T}^2}\theta_n\nabla\cdot u_n\theta_n dx - \int_{\mathbb{T}^2}\nabla\theta_n\cdot u_n\theta_n dx \\ \Rightarrow \int_{\mathbb{T}^2}\theta_n u_n\cdot\nabla\theta_n dx = 0\end{aligned} \tag{4.30}$$

In light of (4.29) and (4.30), it holds that

$$\int_{\mathbb{T}^2}\theta_n P_n(u_n\cdot\nabla\theta_n)dx = \int_{\mathbb{T}^2}\theta_n u_n\cdot\nabla\theta_n dx = 0. \tag{4.31}$$

Furthermore

$$\int_{\mathbb{T}^2}\theta_n P_n(u_n\cdot\nabla\theta_n)dx = -\int_{\mathbb{T}^2}\theta_n\dot{\theta}_n dx = -\frac{1}{2}\int_{\mathbb{T}^2}\frac{d}{dt}|\theta_n|^2 dx. \tag{4.32}$$

Combining (4.31) and (4.32) gives

$$\frac{d}{dt}\left\|\theta_n(t)\right\|_{L^2}^2 = 0,$$

so

$$(2\pi)^2 \sum_{|m|\le n} |c_m(t)|^2 = \|\theta_n(t)\|_{L^2}^2 = \|\theta_n(0)\|_{L^2}^2 = \|P_n\theta_0\|_{L^2}^2 \le \|\theta_0\|_{L^2}^2 . \tag{4.33}$$

According to inequality (4.33), the trajectory of the system lies in a ball in S_n. Since S_n is a compact space, we have that the solution for the Galerkin ODE (4.19) exists for every $t > 0$.

With the benefit of hindsight, we know that the condition $u_n = R^\perp\theta_n$ can be also stated as $u_n = \nabla^\perp\psi_n, \theta_n = -\Lambda\psi_n$. Next, we proceed with ψ_n as we did with θ_n before.

$$\begin{aligned}\int_{\mathbb{T}^2} \psi_n P_n(u_n\cdot\nabla\theta_n)dx &= \int_{\mathbb{T}^2} \psi_n u_n\cdot\nabla\theta_n dx \\ &= -\int_{\mathbb{T}^2} \nabla\psi_n\cdot u_n\theta_n dx - \int_{\mathbb{T}^2} \psi_n\nabla\cdot u_n\theta_n dx.\end{aligned} \tag{4.34}$$

The first integral on the right-hand side in (4.34) vanishes since $\nabla\psi_n\cdot u_n = \nabla\psi_n\cdot\nabla^\perp\psi_n = 0$. Besides, since $u_n = \nabla^\perp\psi_n$ then $\nabla\cdot u_n = 0$ and the other integral on the right-hand side vanishes as well. Therefore

$$\int_{\mathbb{T}^2} \psi_n P_n(u_n\cdot\nabla\theta_n)dx = \int_{\mathbb{T}^2} \psi_n u_n\cdot\nabla\theta_n dx = 0. \tag{4.35}$$

In addition, we also have

$$\int_{\mathbb{T}^2} \psi_n P_n(u_n\cdot\nabla\theta_n)dx = -\int_{\mathbb{T}^2} \psi_n\dot{\theta}_n dx. \tag{4.36}$$

Since $\dot{\theta}_n(x,t)\in\mathbb{R}$ for any $(x,t)\in\mathbb{T}^2\times\mathbb{R}_+$, then $\dot{\theta}_n = \overline{\dot{\theta}_n}$ and Parseval's relation (1.13) yields

$$\begin{aligned}\int_{\mathbb{T}^2} \psi_n\dot{\theta}_n dx &= (2\pi)^2\sum_{|m|\le n}\widehat{\psi_n}(m)\overline{\widehat{\dot{\theta}}_n}(m) = -4\pi^2\sum_{|m|\le n}\frac{1}{|m|}\widehat{\theta_n}(m)\overline{\widehat{\dot{\theta}}_n}(m) \\ &= -\frac{4\pi^2}{2}\frac{d}{dt}\sum_{|m|\le n}\frac{1}{|m|}|\widehat{\theta_n}(m)|^2 = -2\pi^2\frac{d}{dt}\left\|\theta_n\right\|_{H^{-1/2}}^2 ,\end{aligned} \tag{4.37}$$

where we have used that $\theta_n = -\Lambda\psi_n \Rightarrow \widehat{\psi_n}(m) = -\frac{1}{|m|}\widehat{\theta_n}(m)$. Now, we substitute (4.37) into (4.36):

$$\int_{\mathbb{T}^2} \psi_n P_n(u_n\cdot\nabla\theta_n)dx = 2\pi^2\frac{d}{dt}\left\|\theta_n\right\|_{H^{-1/2}}^2 , \tag{4.38}$$

and combining (4.35) and (4.38) we obtain

$$\frac{d}{dt}\left\|\theta_n(t)\right\|_{H^{-1/2}}^2 = 0.$$

Thus

$$\|\theta_n(t)\|^2_{H^{-1/2}} = \|P_n\theta_0\|^2_{H^{-1/2}} . \tag{4.39}$$

Again, taking advantage of the fact that $\nabla \cdot u_n = 0$, we integrate by parts:

$$\begin{aligned}
\int_{\mathbb{T}^2} \varphi P_n(u_n \cdot \nabla\theta_n)dx &= \int_{\mathbb{T}^2} P_n\varphi u_n \cdot \nabla\theta_n dx \\
&= -\int_{\mathbb{T}^2} \nabla P_n\varphi \cdot u_n\theta_n dx - \int_{\mathbb{T}^2} P_n\varphi\nabla \cdot u_n\theta_n dx \\
&= -\int_{\mathbb{T}^2} \nabla P_n\varphi \cdot u_n\theta_n dx.
\end{aligned} \tag{4.40}$$

Moreover, using (4.40) and Proposition 4.3, for any test function $\varphi \in H^{2+\epsilon}$ holds

$$\begin{aligned}
\left|\int_{\mathbb{T}^2} \varphi\dot{\theta_n}(t)dx\right| = \left|\int_{\mathbb{T}^2} \varphi P_n(u_n \cdot \nabla\theta_n)dx\right| &= \left|\int_{\mathbb{T}^2} \nabla P_n\varphi \cdot u_n\theta_n dx\right| \\
&\leq \|\nabla P_n\varphi\|_{L^\infty} \|\theta_n\|_{L^2} \|u_n\|_{L^2} \leq C_\epsilon \|\nabla\varphi\|_{H^{1+\epsilon}} \|\theta_n(t)\|^2_{L^2} \\
&= C_\epsilon \|\varphi\|_{H^{2+\epsilon}} \|\theta_n(t)\|^2_{L^2} ,
\end{aligned}$$

with $\epsilon > 0$. Then, inequality (4.33) gives

$$\left|\int_{\mathbb{T}^2} \varphi\dot{\theta_n}(t)dx\right| \leq C_\epsilon \|\varphi\|_{H^{2+\epsilon}} \|\theta_0\|^2_{L^2} .$$

Since $H^{-(2+\epsilon)}$ is the dual space of $H^{2+\epsilon}$ (see [40]), we can apply Theorem 4.3 to conclude that

$$||\dot{\theta}_n(t)||_{H^{-(2+\epsilon)}} \leq C_\epsilon \|\theta_0\|^2_{L^2} , \tag{4.41}$$

uniformly on n. Let $T > 0$ be arbitrary but fixed and let $t_1, t_2 \in [0, T]$. Then,

$$\|\theta_n(t_2) - \theta_n(t_1)\|_{H^{-(2+\epsilon)}} = \left\|\int_{t_1}^{t_2} \dot{\theta}_n(t)dt\right\|_{H^{-(2+\epsilon)}} \leq \int_{t_1}^{t_2} ||\dot{\theta}_n(t)||_{H^{-(2+\epsilon)}} dt,$$

where the last inequality can be seen in [41]. Hence, using (4.41) we have

$$\|\theta_n(t_2) - \theta_n(t_1)\|_{H^{-(2+\epsilon)}} \leq C_\epsilon \|\theta_0\|^2_{L^2} (t_2 - t_1). \tag{4.42}$$

In conclusion, (4.33) means that the sequence of approximations $\{\theta_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded in $C([0, T]; H^{-(2+\epsilon)})$, and (4.42) means that the sequence is equicontinuous in $C([0, T]; H^{-(2+\epsilon)})$. Therefore, we can use Theorem 4.1 obtaining

$$\theta_n(t) \to \theta(t) \text{ in } C([0, T]; H^{-(2+\epsilon)}). \tag{4.43}$$

Above, one should be careful with this notation since we are using θ_n to denote the subsequence of $\{\theta_n(t)\}_{n\in\mathbb{N}}$. Moreover, since $L^\infty([0, T]; L^2(\mathbb{T}^2))$ is the dual space of $L^1([0, T]; L^2(\mathbb{T}^2))$ (see [41]), then, we can use Theorem 4.2 to conclude that there exists another subsequence θ_n such that

$$\theta_n(t) \overset{*}{\rightharpoonup} \theta(t) \text{ in } L^\infty([0, T]; L^2(\mathbb{T}^2)). \tag{4.44}$$

By proposition 4.4 we have

$$\|\theta_n(t) - \theta(t)\|_{H^{-1}} \leq \|\theta_n(t) - \theta(t)\|^\lambda_{H^{-(2+\epsilon)}} \|\theta_n(t) - \theta(t)\|^{1-\lambda}_{L^2} , \tag{4.45}$$

with $\lambda > 0$. On the one hand, $\|\theta_n(t) - \theta(t)\|^{\lambda}_{H^{-(2+\epsilon)}} \to 0$ due to (4.43). On the other hand,

$$\|\theta_n(t) - \theta(t)\|_{L^2} \leq \|\theta_n(t)\|_{L^2} + \|\theta(t)\|_{L^2} \leq 2\,\|\theta_0\|_{L^2}\,, \tag{4.46}$$

so the term $\|\theta_n(t) - \theta(t)\|^{1-\lambda}_{L^2}$ is uniformly bounded. Therefore, the right-hand side of (4.45) tend to zero, and so does the term on the left-hand side. In other words,

$$\theta_n(t) \to \theta(t) \text{ in } L^\infty([0,T]; H^{-1}(\mathbb{T}^2)). \tag{4.47}$$

The properties above (4.44) and (4.47) imply, respectively, that

$$u_n(t) \overset{*}{\rightharpoonup} u(t) \text{ in } L^\infty([0,T]; L^2(\mathbb{T}^2)), \tag{4.48}$$

$$\psi_n(t) \to \psi(t) \text{ in } L^\infty([0,T]; L^2(\mathbb{T}^2)), \tag{4.49}$$

with $\psi = -\Lambda^{-1}\theta$ and $u = \nabla^\perp\psi = R^\perp\theta$, for almost every $t \in [0,T]$.

For each $n \in \mathbb{N}$, $T > 0$ and $\varphi \in C^\infty$ we can integrate by parts as we did at the beginning of this chapter but with (4.19) instead, and it yields

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}^2} \theta_n(x,t)\partial_t\varphi(x,t)dxdt + \int_{\mathbb{T}^2} \theta_0(x)P_n\varphi(x,0)dx\\ &+ \int_0^T \int_{\mathbb{T}^2} \theta_n(x,t)u_n(x,t)\cdot\nabla P_n\varphi(x,t)dxdt = 0\end{aligned} \tag{4.50}$$

Using (4.44), we obtain that the first term in (4.50) weak* converges in $L^\infty([0,T]; L^2(\mathbb{T}^2))$ for almost every $t \in [0,T]$, because

$$\int_0^T \int_{\mathbb{T}^2} \theta_n(x,t)\partial_t\varphi(x,t)dxdt \to \int_0^T \int_{\mathbb{T}^2} \theta(x,t)\partial_t\varphi(x,t)dxdt$$

by definition of weak* convergence. Plus, by (4.16),

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}^2} \theta u\cdot\nabla\varphi dxdt - \int_0^T \int_{\mathbb{T}^2} \theta_n u_n\cdot\nabla P_n\varphi dxdt\\ &= \frac{1}{2}\int_0^T \int_{\mathbb{T}^2} u[\Lambda, \nabla\varphi]\psi dxdt - \frac{1}{2}\int_0^T \int_{\mathbb{T}^2} u_n[\Lambda, \nabla P_n\varphi]\psi_n dxdt,\end{aligned} \tag{4.51}$$

but we also have the identity

$$[\Lambda, \nabla P_n\varphi]\psi_n = [\Lambda, \nabla\varphi]\psi - [\Lambda, \nabla(\varphi - P_n\varphi)]\psi - [\Lambda, \nabla P_n\varphi](\psi - \psi_n). \tag{4.52}$$

We need some computations to prove (4.52). First, we expand the three terms on the right-hand side of (4.52):

$$[\Lambda, \nabla\varphi]\psi = \Lambda(\nabla\varphi\psi) - \nabla\varphi\Lambda\psi, \tag{4.53}$$

$$\begin{aligned}[\Lambda, \nabla(\varphi - P_n\varphi)]\psi =&\Lambda(\nabla(\varphi - P_n\varphi)\psi) - \nabla(\varphi - P_n\varphi)\Lambda\psi\\ =&\Lambda(\nabla\varphi\psi - \nabla P_n\varphi\psi) - \nabla\varphi\Lambda\psi + \nabla P_n\varphi\Lambda\psi\\ =&\Lambda(\nabla\varphi\psi) - \Lambda(\nabla P_n\varphi\psi) - \nabla\varphi\Lambda\psi + \nabla P_n\varphi\Lambda\psi,\end{aligned} \tag{4.54}$$

$$\begin{aligned}[\Lambda, \nabla P_n\varphi](\psi - \psi_n) =&\Lambda(\nabla P_n\varphi(\psi - \psi_n)) - \nabla P_n\varphi\Lambda(\psi - \psi_n)\\ =&\Lambda(\nabla P_n\varphi\psi - \nabla P_n\varphi\psi_n) - \nabla P_n\varphi\Lambda(\psi - \psi_n)\\ =&\Lambda(\nabla P_n\varphi\psi) - \Lambda(\nabla P_n\varphi\varphi_n) - \nabla P_n\varphi\Lambda\psi + \nabla P_n\varphi\Lambda\psi_n.\end{aligned} \tag{4.55}$$

At this point, subtracting (4.54) and (4.55) to (4.53) gives

$$\Lambda(\nabla P_n\varphi\psi_n) - \nabla P_n\varphi\Lambda\psi_n = [\Lambda, \nabla P_n\varphi]\psi_n,$$

and this proves (4.52). Therefore, the right-hand side in (4.51) is equal to

$$\frac{1}{2}\int_0^T\int_{\mathbb{T}^2}\left((u-u_n)[\Lambda,\nabla\varphi]\psi + u_n[\Lambda,\nabla(\varphi - P_n\varphi)]\psi + u_n[\Lambda,\nabla P_n\varphi](\psi-\psi_n)\right)dxdt. \tag{4.56}$$

Let us study each term in (4.56). According to (4.18),

$$\|[\Lambda,\nabla\varphi]\psi\|_{L^2} \leq C_\epsilon \|\varphi\|_{H^{3+\epsilon}} \|\psi\|_{L^2} \Rightarrow [\Lambda,\nabla\varphi]\psi \in L^2.$$

Moreover, $u_n(t) \overset{*}{\rightharpoonup} u(t)$ in $L^\infty([0,T];L^2(\mathbb{T}^2))$. Thus,

$$\int_0^T\int_{\mathbb{T}^2} u_n[\Lambda,\nabla\varphi]\psi dxdt \to \int_0^T\int_{\mathbb{T}^2} u[\Lambda,\nabla\varphi]\psi dxdt,$$

and for the first term in (4.56) we have

$$\int_0^T\int_{\mathbb{T}^2}(u-u_n)[\Lambda,\nabla\varphi]\psi dxdt \to 0.$$

Moving on to the second term, we have

$$\begin{aligned}\int_{\mathbb{T}^2} u_n[\Lambda,\nabla(\varphi-P_n\varphi)]\psi dx &\leq \|u_n\|_{L^2}\|[\Lambda,\nabla(\varphi-P_n\varphi)]\psi\|_{L^2}\\ &\leq C_\epsilon\|u_n\|_{L^2}\|\varphi-P_n\varphi\|_{H^{3+\epsilon}}\|\psi\|_{L^2},\end{aligned} \tag{4.57}$$

and since $\|u_n(t)\|^2_{L^2} = \|\theta_n(t)\|^2_{L^2} \leq \|\theta_0\|^2_{L^2}$, the following estimate for (4.57) holds:

$$\int_{\mathbb{T}^2} u_n[\Lambda,\nabla(\varphi-P_n\varphi)]\psi dx \leq C_\epsilon\|\theta_0\|_{L^2}\|\varphi-P_n\varphi\|_{H^{3+\epsilon}}\|\psi\|_{L^2}.$$

Bearing in mind that

$$\widehat{(\varphi - P_n\varphi)}(m) = \widehat{\varphi}(m) - \widehat{P_n\varphi}(m) = \begin{cases} 0, & \text{if } |m|\leq n,\\ \widehat{\varphi}(m), & \text{if } |m|>n,\end{cases}$$

we obtain

$$\|\varphi-P_n\varphi\|_{H^{3+\epsilon}} = \sum_{m\in\mathbb{Z}^2}|m|^{2(3+\epsilon)}|\widehat{(\varphi-P_n\varphi)}(m)|^2 = \sum_{|m|>n}|m|^{2(3+\epsilon)}|\widehat{\varphi}(m)|^2 \to 0.$$

Hence,

$$\int_{\mathbb{T}^2} u_n[\Lambda,\nabla(\varphi-P_n\varphi)]\psi dx \to 0.$$

Eventually, we apply the same reasoning concerning the last term in (4.56):

$$\begin{aligned}\int_{\mathbb{T}^2} u_n[\Lambda,\nabla P_n\varphi](\psi-\psi_n)dx &\leq \|u_n\|_{L^2}\|[\Lambda,\nabla P_n\varphi](\psi-\psi_n)\|_{L^2}\\ &\leq C_\epsilon\|u_n\|_{L^2}\|P_n\varphi\|_{H^{3+\epsilon}}\|\psi-\psi_n\|_{L^2}\\ &\leq C_\epsilon\|\theta_0\|_{L^2}\|\varphi\|_{H^{3+\epsilon}}\|\psi-\psi_n\|_{L^2},\end{aligned}$$

but according to (4.49), $\psi_n \to \psi$ in $L^\infty([0,T]; L^2(\mathbb{T}^2))$, so $\|\psi - \psi_n\|_{L^2} \to 0$ and

$$\int_{\mathbb{T}^2} u_n[\Lambda, \nabla P_n\varphi](\psi - \psi_n)dx \to 0.$$

Cutting a long story short, as $n \to \infty$, equation (4.56) goes to zero and, as a consequence, so does the left-hand side in equation (4.51). In other words,

$$\int_0^T \int_{\mathbb{T}^2} \theta_n u_n \cdot \nabla P_n \varphi dx dt \to \int_0^T \int_{\mathbb{T}^2} \theta u \cdot \nabla \varphi dx dt.$$

Hence, all the terms in (4.50) converge and we have

$$\int_0^T \int_{\mathbb{T}^2} \theta(x,t)(\partial_t \varphi(x,t) + u(x,t) \cdot \nabla\varphi(x,t))dxdt + \int_{\mathbb{T}^2} \theta_0(x)\varphi(x,0)dx = 0,$$

which proves that there exists a weak solution of (4.1).

Kinetic energy inequality. Since u and θ have the same L^2-norm, (4.33) implies that

$$\|u_n(t)\|_{L^2}^2 \leq \|u_n(0)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 .$$

Hence, we can apply Proposition 4.1 and conclude that

$$\|u(t)\|_{L^2}^2 \leq \liminf_{n\to\infty} \|u_n(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 ,$$

which is the sought kinetic energy inequality.

Conservation of the Hamiltonian. First of all, let us observe that

$$\begin{aligned}\|\theta(t)\|_{H^{-1/2}}^2 &= \sum_{m\in\mathbb{Z}^2, |m|\neq 0} \frac{1}{|m|}|\widehat{\theta}(m)|^2 = \sum_{m\in\mathbb{Z}^2, |m|\neq 0} \frac{1}{|m|}\widehat{\theta}(m)\overline{\widehat{\theta}}(m) \\ &= -\sum_{m\in\mathbb{Z}^2} \widehat{\psi}(m)\overline{\widehat{\theta}}(m) = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \psi(t)\theta(t)dx,\end{aligned}$$

so

$$\|\theta_0\|_{H^{-1/2}}^2 = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \psi_0\theta_0 dx.$$

Analogously, it can be shown that

$$\|\theta_n(t)\|_{H^{-1/2}}^2 = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \psi_n(t)\theta_n(t)dx.$$

Using Remark 4.1, we have that for any $\epsilon > 0$ there exists $0 < \lambda < 1$ such that

$$\|\theta_n(t) - \theta(t)\|_{H^{-1/2}} \leq \|\theta_n(t) - \theta(t)\|_{H^{-(2+\epsilon)}}^{\lambda} \|\theta_n(t) - \theta(t)\|_{L^2}^{1-\lambda} .$$

Hence, using (4.43) and (4.46), we conclude that

$$\theta_n \to \theta \text{ in } L^\infty([0,T]; H^{-1/2}(\mathbb{T}^2)), \tag{4.58}$$

and as a consequence

$$\int_{\mathbb{T}^2} \psi_n(t)\theta_n(t)dx \to \int_{\mathbb{T}^2} \psi(t)\theta(t)dx.$$

At this point, we are finally ready to conclude the proof of the conservation of the Hamiltonian. On the one hand,

$$\|\theta_n(t)\|^2_{H^{-1/2}} = \|P_n\theta_0\|^2_{H^{-1/2}} \to \|\theta_0\|^2_{H^{-1/2}} = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\psi_0\theta_0 dx,$$

where we have used (4.39). While on the other hand

$$\|\theta_n(t)\|^2_{H^{-1/2}} = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\psi_n(t)\theta_n(t)dx \to -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\psi(t)\theta(t)dx.$$

Hence

$$\int_{\mathbb{T}^2}\psi(t)\theta(t)dx = \int_{\mathbb{T}^2}\psi_0\theta_0 dx$$

and the Hamiltonian is preserved.

□

Remark 4.2. *In Theorem 4.4 it has been proved that for any $\theta_0 \in L^2(\mathbb{T}^2)$, there exists a global-in-time weak solution, as defined in 4.1, to equations (4.1). Nevertheless, the uniqueness of weak solutions is still an open problem. For less regular initial data, the non-uniqueness of the weak solution was proved in [42].*

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