Chapter

FEATURES OF THE SPECIALISED KNOWLEDGE IN GEOMETRY OF PROSPECTIVE PRIMARY TEACHERS AT THE BEGINNING OF THEIR TRAINING

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ABSTRACT

As researchers and educators, we believe that if prospective primary teachers (henceforth PPTs) are to get the best from their future pupils, they should, as Hill, Rowan and Ball (2005) and Llinares and Krainer (2006) advocate, have a profound knowledge of the mathematics they ultimately intend to teach. This assertion correlates with our position that the first steps of a teacher's professional development are taken in their initial training, and hence at university (Carrillo et al., in press). In order to be able to design training courses that meet the needs of PPTs, in terms of both subject matter and related pedagogical issues, it is essential to gain an understanding of their current mathematical knowledge.

This study aims to explore the situation at three Andalusian[1] universities regarding the knowledge of PPTs in the area of metric geometry. None of the 737 students participating in the study had received any prior training in this area in any of their university courses.

The theoretical perspective from which we define the subdomains of teacher knowledge is intrinsic to mathematics, such that specialisation is distributed holistically across subdomains, rather than located within a single one. This position derives from our view of knowledge as inherent to the profession (Scheiner et al., 2017, p.10), and is consistent with the model Mathematics Teachers' Specialised Knowledge MTSK, Carrillo, Climent, Montes et al., 2018) which underpins our theoretical analysis. The mathematics picked up by the PPTs in their schooling prior to starting university represents a baseline from which their specialised knowledge for teaching needs to be developed during their initial training.

Working within an interpretative paradigm (Bassey 1999), we took the methodological design of a *survey* (Bryman, 2013) for testing trainees' knowledge of key elements of geometry. These had been selected as important in themselves and as essential entry-level items and had been the subject of previous studies into the knowledge of PPTs. Our findings identify a number of strengths in the participants' specialised knowledge, but above all, significant areas of weakness which could be rooted in obstacles caused during their own passage through primary and secondary education. Nevertheless, the pinpointing of these weaknesses could be regarded as an opportunity to reflect on how the training we offer can be improved, and by better understanding where trainees have difficulties, target specific areas of their knowledge which need to be developed. Finally, the results of this study provide support for something

[1] Andalusia is one of the 17 autonomous regions making up Spain and is located in the south of the Iberian Peninsula.

that, as teacher-trainers, we have been aware of for some time, namely the need to institute a selective test for entry to the degree in teaching, in order to guarantee a minimum level of basic mathematical knowledge. This would then allow initial training courses to dedicate more time to the task for which they are intended, that is, the construction of specialised knowledge of mathematical subject matter.

INTRODUCTION

The importance of researching the mathematics training of primary teachers has been endorsed by the TEDS-M study by the International Association for the Evaluation of Educational Achievement (IEA) (Tatto, Sharon, Senk, Ingvarson and Rowley, 2012), a project aimed at rigorously analysing the training received by primary and secondary teachers in 17 countries around the world. The current study forms part of a larger research project within our geographical context, which aims to understand the nature of mathematics teachers' knowledge and to identify the processes by which this is constructed in trainee primary teachers, including their specialised knowledge in both arithmetic (Montes et al., 2015) and geometry.

It is important that mathematics teachers in general, and those involved in primary education in particular, have a profound knowledge of the mathematics they teach in order to get the best from their pupils (Hill, Rowan and Ball, 2005; Llinares and Krainer, 2006), which requires and implies fully understanding the subject matter they teach (Ball, Thames, Phelps, 2008). In our view, professional development begins with teachers' initial training, that is, at university (Carrillo et al., in press), and this view is therefore most applicable to PPTs.

Numerous studies, both national and international in scope, have found an incomplete knowledge of geometry among PPTs, with the textbooks used often being identified as a principal source of confusion (López et al., 2015; Jaime, Chapa and Gutiérrez, 1992). A major limitation in many textbooks is the lack of variety in the examples they employ (Zazkis and Leikin, 2008), given that they are designed around a mode of teaching based on prototypical representations of geometrical shapes (Gualdrón and Gutiérrez, 2007), in which classification is given primacy over the principles of categorization into classes (Liñán-García, 2017; Barrantes and Zapata, 2008). In this regard, Barrantes and Blanco (2004) show how many PPTs are aware of Euclidian geometry, but not content such as spatial geometry or plane movements (isometry and homotethy). Zazkis and Leikin (2008), Fujita and Jones (2007) and Fujita (2012) have found that PPTs have limited knowledge of how to classify quadrilaterals, and often confuse the relationships between elements of plane shapes (Fujita and Jones, 2007; Guillén, 2000; Jaime and Gutiérrez, 1994). Thus, in considering parallelograms, for example, they might mistake certain kinds of regularity for evidence of symmetry. Elsewhere, researchers have found a widespread tendency to categorise geometric shapes according to how closely they resemble prototypes (Clements and Sarama, 2011), and to rely more on their mental image of a shape than on its definition (Vinner, 2011), to the point where they are able to draw a square, but are unable to provide a definition of one (Fujita and Jones, 2006).

With respect to triangles, PPTs are often found to use the terms *altitude* and *base* as if each triangle had only one of each, on the basis of the assumption that the foot of the altitude of a triangle or pyramid is always to be found in the interior of the shape (Blanco and Contreras, 2012; Barrantes and Blanco, 2006). In terms of the relationship between the area and the perimeter of flat shapes, they are easily led to believe that if two shapes have the same area, they then have the same perimeter (D’amore and Fadiño, 2007; Bosch et al., 2001), a misconception which stems from their procedural knowledge of the concepts (Zazkis, Sinclair and Liljedahl, 2013). They also have problems with visualising representations of flat shapes and unfolded geometric solids due to the static treatment the topic of geometry generally receives (Guillén, 2010; Barrantes and Zapata, 2008). Further points of difficulty include arithmetic systems for calculating the area of flat shapes (Zacharos, 2006; Zazkis, Sinclair and Liljedahl, 2013) and their knowledge of solids (Guillén, 2000), in particular with regard to specifying the characteristics of a cylinder (Tsamir et al., 2015).

With this theoretical overview in mind, the study aims to identify the baseline knowledge of geometry displayed by 737 prospective primary teachers at three Andalusian universities. We hence restrict the study to those knowledge subdomains most closely linked to the nature of this prior knowledge, as detailed in the following section. We hope our findings draw the attention of the relevant authorities to the need to improve the situation, and in the long term, it is intended that the strengths and weaknesses detected in this study contribute to improving the training provided by our institutions and others

THEORETICAL PERSPECTIVES

Mathematics Teachers' Specialised Knowledge as a Framework

Teachers' knowledge has been under scrutiny since Shulman (1986), the key feature of which work was the proposal of two knowledge domains: subject matter knowledge (SMK) and pedagogical content knowledge (PCK), consisting of the "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, 9). Chief among the multiple models that have since been proposed is that of Ball, Thames and Phelps (2008), Mathematical Knowledge for Teaching (MKT), notable for its suggestion of the subdomain denominated *specialised content knowledge*. The significance of this suggestion is that it recognised that teachers are required to understand their discipline in a way which is distinct from other professions. For all its merits, however, the model displayed certain limitations for research: the demarcation between subdomains was not clearly drawn, subdomains were defined in terms of teacher actions rather than elements of knowledge, the model did not accept external references in its characterisation of specialised knowledge, and the notion of specialisation was not limited to only one subdomain. It was these perceived limitations which led us to develop the analytical model *Mathematics Teachers' Specialised Knowledge* (MTSK) (Carrillo, Climent, Montes et al., 2018), which we will describe below.

The MTSK model (Carrillo et al., 2017; Carrillo et al., 2018) characterises the MK subdomains from an intrinsically mathematical perspective and takes the view that the notion of the teacher's specialised knowledge derives from their profession (Scheiner et al., 2017), in that it is concerned specifically with teaching mathematics. Hence it divides into the two recognisable broad domains, *Mathematical Knowledge* (MK) and *Pedagogical Content Knowledge* (PCK), but unlike other models it includes an affective domain which recognises teachers' conceptions of mathematics and of the teaching-learning process as elements which influence and define the other subdomains (see Figure 1).

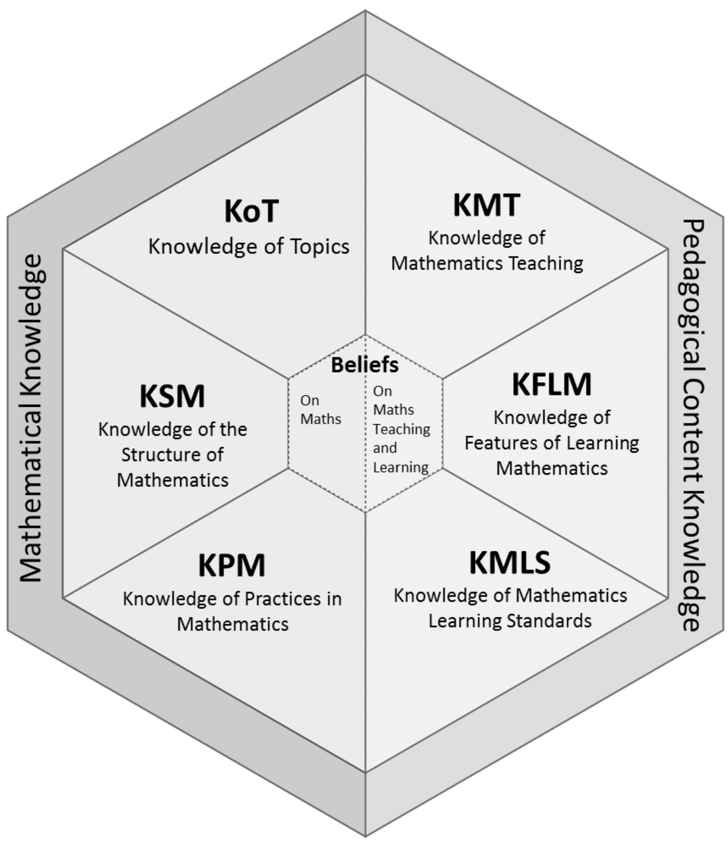

Figure 1. Mathematics Teachers' Specialised Knowledge (Carrillo, Climent et al., 2018).

As mentioned in the previous section, and due to the specific character of the matter under analysis, we focus solely on the domain of Mathematical Knowledge. This consists of three subdomains: *Knowledge of Topics* (KoT), K*nowledge of the Structure of Mathematics* (KSM) and *Knowledge of Practices in Mathematics* (KPM).

Knowledge of Topics (KoT) consists of the essential knowledge of the discipline, including all facets relating to a topic in mathematics. There are four categories of KoT in the model: *definitions, properties and their foundations; procedures; registers of representation;* and *phenomenology*. The *definitions, properties and their foundations* pertaining to a particular topic can be seen in, for example, the knowledge of the properties of lines and their relative positions on a plane or in space. The category of *procedures* considers the knowledge of *how the procedure is carried out, when it can be done*, *why it is done a particular way* and the *features of the outcome*. Thus, for example, knowledge of the metric system and its structure is distributed across all the descriptors above in relation to procedures for changing from one unit of measurement to another and the implications for measures of length, area and volume (we consider the magnitudes of length, area and volume as measurable characteristics of geometric constructions considered in terms of their composition and decomposition, on the basis of which we include them in knowledge of geometry).

As the name suggests, *registers of representation* constitute a teacher's knowledge of the different ways to represent a topic, from the symbolic to the graphical, among others, and includes the knowledge which enables them to switch from one representation to another, including within the same register. Hence, for example, a polygon can be described verbally in terms of its properties, or expressed graphically with the GeoGebra programme, or physically manipulated using teaching aids such as Geo Strips, to the effect that its attributes can be visualised irrespective of its position. The aspects of specialised knowledge which we refer to in the category of registers of representation imply an understanding of the equivalence of different registers and the mathematical treatment associated with each. This understanding would be evident, for example, in a verbal description of the procedure carried out in GeoGebra for constructing a parallel line, through one particular point to another given line, and its corresponding perpendicular, demonstrating how this is done on a plane with the GeoGebra and in space with Geo Strips.

Finally, *phenomenology* considers the teacher's knowledge of the applications of a particular topic and connects this to its meaning. An example of such knowledge would be when a teacher uses the Pythagorean Theorem to precisely measure irrational distances with a right triangle, or drawing on a Pythagorean triple, make a tool to construct and to measure right angles.

The following subdomain, *Knowledge of the Structure of Mathematics* (KSM), refers to the knowledge of interconceptual links between the different kinds of contents involved in teaching mathematics. Four such kinds of interconnections are recognised: *connections of increased complexity*, *connections of simplification*, and *transverse* and *auxiliary* connections. *Connections of increased complexity* enable the teacher to approach a more basic level of mathematics from a more advanced perspective, while those of *simplification* connect an element currently in focus with a previous, more basic, treatment of the same. *Transverse connections* make links across different fields of knowledge, for example, those that connect knowledge between equivalence relations and invariants from different geometric approaches (topological, projective or metric). Finally, *auxiliary connections* include knowledge used for other ends, involving the use of one element of knowledge when focusing on another. An example of this would be the connection employed in using two parallel lines to find all the quadrilaterals with at least two parallel sides.

Finally, Knowledge of Practices in Mathematics (KPM) concerns knowledge of doing and generating mathematics, and includes modes of validation and proof, the role of symbols and use of formal language, the processes involved in solving problems as a means of producing mathematics, the hierarchical organization and planning in relation to solving mathematical problems, particular procedures for mathematical work (for example, modelling), and the necessary and sufficient conditions for generating definitions. We can take as an example a teacher who finds him or herself in a situation-problem where it is necessary to calculate the number of diagonals in any polygon. The teacher would need to draw on a knowledge (KPM) of a series of steps, including the mathematicsization of the problem (correctly translating the problem statement into mathematical language and checking its validity). It would involve modelling (as a mathematical practice) and the use of definitions, scrutinising their mathematical validity. In addition, it would require knowing that, based on the observation of regularities in calculating the number of diagonals in polygons of an increasing number of sides, an inductive hypothesis relating diagonals to sides could be postulated, which would then need to be deductively demonstrated to ensure it worked for any polygon.

Specialised Knowledge in Geometry

As mentioned above, we consider any mathematical knowledge acquired by the PPTs prior to enrolling at university the foundation on which they will construct the specialised knowledge they will need to develop over the course of their studies, and hence a subset of this.

Taking the MTSK model as an organisational framework for this knowledge, we offer below a selection from the set of elements of geometric knowledge that we consider should be familiar to any student wishing to train as a primary teacher before they embark on the corresponding degree course. In making our selection, we have consulted the literature published on the topic and given careful consideration to the syllabus they will follow on their course. This knowledge will provide the theoretical foundations for the design of the survey we present in the following section.

On the basis of this premise, we focus chiefly on the subject matter constituting the conceptual bedrock of KoT, with the addition of certain aspects of KPM and KSM. The fact that the bulk of this knowledge pertains to KoT, with the inclusion of only a few aspects of KPM and KSM, reflects this premise, that is, our interest lies in the knowledge displayed by PPTs before they embark on their specialist training, in which the subdomains of KPM and KSM will predominate as the most pertinent to the task of teaching.

In the KoT subdomain, we highlight knowledge of the definitions, denominations (Puig, 1986) and meanings of the following main geometric items, along with the basic transformations applicable to them, their properties and the mathematical foundations for the properties and the connections between them: polygons, polyhedra and their elements, associated categorisations and classifications according to different criteria, plane and spatial motion (isometries and homothecias), the Pythagorean Theorem, perimeter, area and volume (including exploring the relationship between them), geometric proportionality (and its relation to numerical proportionality) and the metric system of measurement (and its relationship to the decimal system). We also include aspects concerning the registers of representation of the different topics, including the prevalence of the image of a concept over its definition as a result of the register of representation chosen or the shift from three to two dimensions when unfolding geometric solids. Also included is the knowledge for carrying out procedures for measuring and calculating areas, perimeters and volumes, as well as how to do elementary transformations and the expected results when applied to certain shapes and geometric solids. To round off our consideration of this subdomain, we include knowledge of the relationships between different classifications and categorizations, in respect of which we regard such relationships as a phenomenon of the classifications and categories themselves, thus providing us with information about the respondents' KoT phenomenonology.

Our selection also includes aspects of knowledge about *ways of doing mathematics*, pertaining to the subdomain Knowledge of Practices in Mathematics (KPM). We include knowledge of how to go about problem-solving, the definition or demonstration of a process, the role of examples and counter-examples in validating results and in generating definitions, and how to make a definition, differentiating between those attributes which are relevant, irrelevant, critical or incorrect. In this regard we bear in mind the notion of argumentation (Planas and Morera 2012, in turn citing Toulmin 1958) in knowledge of the processes associated with problem-solving as a means of producing mathematics, and knowledge of ways of validating and demonstrating, principally how to demonstrate results, and how this differs fundamentally from the demonstration of one or two cases in which an affirmation is fulfilled.

Finally, we have included in our inventory of essential knowledge certain elements of *Knowledge of the Structure of Mathematics* (KSM), in the form of transverse connections between geometrical and numerical knowledge. For example, the knowledge of the relationship between numerical and geometrical proportionality through the measure of magnitudes and as a result of the application of transformations to geometric constructs.

DATA, METHODS AND DATA ANALYSIS STRATEGY

The aim of this study was to explore PPTs' understanding of basic geometry as they embark on their degree course. It adopted an interpretative paradigm (Bassey 1999) and took the methodological design of a *survey* (Bryman, 2013) in order to provide a representative sample from which to examine the current state of affairs. Data-gathering was carried out using a multiple-choice test (with each question offering 4 options, only one of which was correct), specially designed for this purpose in accordance with the theoretical foundations described above. Each question left space for a brief explanation for the choice of response.

In total there were 737 respondents, all PPTs from the first year Primary Education degree at the University of Seville, the CEU Cardenal Spínola Centre for University Studies (affiliated to the University of Seville), and from the first and second years of the degree in Primary Education at the University of Huelva[2]. In each case, the survey was given to the students before they had received any specialised training in

[2] The variation in the composition of the groups from the universities of Seville and Huelva are due to differences in the syllabuses in force at each institution, which results in differences in the scheduling of certain subjects.

geometry, both in terms of subject matter and teaching approaches. All respondents were fully informed of the different stages and objectives of the study they were participating in and gave their explicit agreement.

The survey was completed within a ninety-minutes class slot, such that the PPTs had plenty of time to consider their responses and explain their reasoning in the spaces provided. Use of calculators was not allowed as none of the questions required any kind of calculation, but a blank sheet of paper was given to each respondent for any notes they wished to make. This sheet did not form part of the study. The PPTs completed the test by circling the correct options; if they wished to change their mind, they simply crossed out their original answer and chose another. All tests which conformed to these instructions were considered valid and included in the survey.

Data analysis was carried out over two consecutive stages, taking a dual perspective. The first of these was a quantitative analysis of all the respondents' answers, which allowed us to see the relative frequencies of the options for each question. Given that each incorrect option, represented a potential error that the PPTs might fall into, the quantitative analysis was followed up in the second stage by an interpretative analysis aiming to uncover the root cause of both the errors and correct answers, and to characterise them using the categories and indicators of the MTSK model (Carrillo et al., 2018), along with our own theoretical sensibility (Strauss and Corbin, 1994).

The Test Survey as an Instrument for Collecting Data: Design Process and Structure

The initial pilot multiple-choice test was developed through an investigator triangulation (Flick, 2007). It originally consisted of 20 items and was trailled with a group of students who did not form part of the study. A number of other questions were held in reserve should it become necessary to substitute any items found to be unusable in the piloting stage. This was effectively the case with some items. After analysing the results of the pilot test and drawing on Bryman's (2013) theoretical suggestions, the research team substituted some of the questions and rewrote others so as to avoid misinterpretations. By this means we could be sure that errors were not the result of factors unconnected with the specialised knowledge of the PPTs participating in the survey. The end result was a test of 17 multiple-choice questions, each with four options, only one of which was correct. The other three options were not arbitrary; as mentioned above, they were selected to reflect the kinds of errors and misunderstandings that the PPTs could be expected to make, based on the results of the theoretical review, and thus enabled the qualitative analysis of the next stage. The discussion below of the final version of the test considers what we felt a priori each option might suggest about the PPTs' knowledge of geometry.

1. Consider the following shapes:

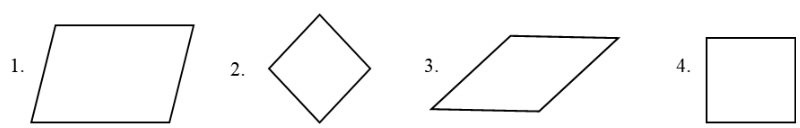

Figure 2. Shapes for question 1.

Which is a rhombus?

a. Only shape 2
b. Shapes 2 and 4
c. Shapes 2, 3 and 4
d. Only shape 1

This question is aimed at obtaining information about the PPTs' knowledge of the definition of a rhombus (KoT, d*efinitions, properties and their foundations*). It also considers the role of their mental image in identifying the shape, with position potentially being a significant factor (KoT, r*egisters of representation*) with respect to categorising shapes according to their properties (KoT, *procedures: how a procedure is carried out, when it can be carried out*). Hence, the aim of the question is to set a context which will provide clues to the PPTs' knowledge of the equivalence of shapes when they undergo rigid motions on a plane (KoT, *procedures: characteristics of the result*).

Option *a* focuses on the image of a rhombus in what can be considered its prototypical position; *b* identifies rhombi with squares, either as a consequence of the prototypical position in Figure 2, or on the basis that figures 2 and 4 have the same properties; option *c* presupposes that position is not among the properties for categorising rhombi, and that squares can be considered special cases of rhombi (categorisation of classes); finally *d* indicates a confusion between the notion of rhombus and rhomboid.

2. The Phythagorean Theorem can be applied to:
    a. isosceles triangles
    b. equilateral triangles
    c. obtuse triangles
    d. none of the above triangles

This question seeks to gather information about the respondents' KoT (*definitions, properties and their foundations*) with regard to the conditions in which the Pythagorean Theorem can be applied, and consequently their KPM (*necessary and sufficient conditions for generating definitions*), as it is this knowledge which enables them to determine which of the options fail to meet the conditions necessary to apply the theorem. By the same token, the question also provides information about respondents' knowledge of the classification of triangles in terms of the measures of their angles and of their sides (KoT, *definitions, properties and their foundations*), and the connection we can establish between the two, to the effect that it can be seen as the same phenomenon (KoT, *phenomenology*). The choice of options *b* and *c* reveals a lack of awareness of the incompatibility between right triangles and equilateral or obtuse triangles, and the kind of triangle (in terms of angles) which fulfil the conditions of the theorem; *d* could indicate that the respondent does not consider it possible for an isosceles triangle to also be a right triangle.

3. An eight-sided polygon is drawn on a sheet of 1cm squared paper. What is the length of the perimeter?
   a. 8 cm
   b. Approximately 9.6 cm
   c. Approximately 8.4 cm
   d. Approximately 8.8 cm

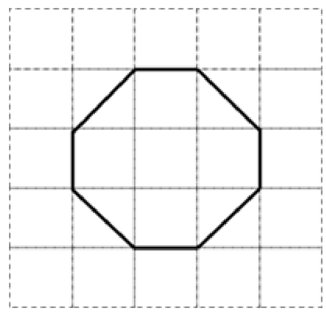

Figure 3. Shape for question 3.

This question, drawing on Hernández et al. (2002), provides information about the PPTs' KoT (*definitions, properties and their foundations*) with respect to identifying right triangles from their representation on the grid, which means interpreting the diagonal of the squares forming the grid as the hypotenuse of a right triangle, as well as recognising the irregularity of the octagon in virtue of the characteristics of the grid on which it is drawn. Linked to this, the respondents' KoT (*register of representation*) also comes into play in that the register of representation chosen for the octagon might be influential in the recognition of the right triangles and might provide information as to whether their image of the concept takes prevalence over its definition (potentially confusing certain regularities of the octagon in the diagram with the definition of a regular octagon). The question is also directed at obtaining information about the respondents' KoT (*procedures*), as it requires them to be aware of how to calculate the length of the perimeter and how to arrive at an approximation of the square roots so as to discriminate from among the three latter options (*how the procedure is* *carried out*, *when and why it is done* – the calculation of the length itself, involving Pythagoras' Theorem to work out the length of four sides of the octagon – and also *features of the result* – awareness that the result cannot be a natural number, so any such options can be ruled out). Option *a* indicates a predominance of the prototypical image (Vinner, 2011) of a regular octagon, while *b* and *c* suggest recognition of the fact that the answer cannot be a natural number, but at the same time point to a flaw in approximating the calculation for failing to realise that four times the square root of two must necessarily be greater than 5.

4. What is the area of a rectangle measuring 0.2 metres wide by 0.25 decametres long?
   a. 50 square decimetres
   b. 0.5 metres
   c. 0.5 square decametres
   d. 50 decimetres

This question is aimed at obtaining information about respondents' KoT (*definitions, properties and their foundations*) with respect to the metric system (MS) by offering a range of measurement scales for determining the area of a rectangle, from which the respondents need to identify the one appropriate to the task. Further potential information includes KoT (*procedures*), regarding calculating the area, (*how the procedure is carried out*), and the implicit requisite for all the measurements to use the same scale (*when and why the procedure can be done*). Finally, the question also provides information on *features of the result*, as there must be units assigned and these must be multidimensional given the characteristics of the initial data. Knowledge of the procedure for calculating the area of a rectangle is linked to an auxiliary connection in KSM in that it requires the respondent to know the procedure for multiplying non-integer numbers to calculate the size of the area, and a transverse connection in KSM in establishing a connection between the decimal system (DS) and the MS. The incorrect options point up, on the one hand, a lack of awareness that the result should be expressed as a unit of area, even if the numerical calculation in invitingly correct (*b* and *d*), and on the other, problems in using the MS (*c*).

5. How many axes of symmetry are there in a rhomboid?
   a. Two, coinciding with its diagonals
   b. Four, its two diagonals and the line segments intersecting the opposite sides at the mid-point
   c. There are no lines of symmetry
   d. Two, the line segments intersecting the opposite sides at the mid-point

Drawing on Hill, Schilling and Ball (2004), this question is designed to obtain information about KoT (*definitions, properties and their foundations*) relating to the key features of a rhomboid and the relationship between them (lines of symmetry and diagonals), in which a significant influence is likely to be the prototypical image of a rhomboid (Vinner, 2011). The question also touches on the mathematical practice of making generalisations from the particular – inappropriately in this instance (the square and the rhombus), as it is common to confuse certain regularities in parallelograms with the concept of symmetry in the shape. Within the same category, it also gathers information on the properties defining a rhomboid and the characteristics of an axis of symmetry in a shape, distinguishing halves which are equal in area with those which are symmetrical, along with the elements involved.

Options *a*, *b* and *d* reflect the confusion between symmetry and the diagonal or mediatrix or misidentification of a rhomboid. In this respect, we should make it clear that we have opted for a perspective of exclusivity in defining the shapes, hence employing classification (an intraconceptual focus) as opposed to categorization of classes (an interconceptual focus) (Liñán-García, 2017), as students tend to arrive at university with an exclusive perspective in respect to defining constructs according to categories.

6. Which of the expressions below gives the area of the shape? (the lengths of the sides are given in the same unit):
   a. $a^2 + 5a$
   b. $(a + 5)^2$
   c. $a^2 + 5$
   d. $a^2 + 25$

Figure 4. Shape for question 6.

This question, based on Hill *et al.* (2004), is designed to obtain information on KoT (*procedures*); here, the knowledge of the properties of the given plane shapes underpins the recognition that the diagram can be interpreted as a single figure or as a composite of two. This in turn allows the area to be calculated as the sum of the areas of the two shapes, or as the area of a single shape whose sides are the sum of the lengths of the two together. This knowledge – (*how the procedure is carried out, when and why it is done*) – also recognises the equivalence of the two procedures. In addition, the question challenges respondents' understanding of the registers of representation employed, as they are expected to identify the correct algebraic expression from a corresponding graphical register. Finally, we can note an auxiliary connection (KSM) in the form of using an algebraic expression to arrive at the conclusion that the area of the quadrilateral is a^2+5a.

Options *b*, *c* and *d* involve three types of error in the calculation of the area of a parallelogram or the associated algebraic expression, which might be due to the shift from a graphic to an algebraic system.

7. A box is able to contain exactly 24 identical cubes, which fill it completely. What is the maximum number of cubes that could be placed in a box whose dimensions (width, height and length) are double those of the first?
   a. Double the number of cubes, that is, 48
   b. 96 cubes
   c. 8 times more cubes
   d. 72 cubes

Based on Alsina, Burgués and Fortuny (1991), this question seeks to gather data on the PPTs' KoT (*procedures: how and when a procedure is carried out*) regarding the procedures for calculating volume, specifically the volume of a prism based on the dimensions of its edges, the *number of cubes which fit* along the length of each side so as to arrive at the total number of cubes in the prism, and the implications for the multiplication by which the answer is arrived at (*why a procedure is done this way*). Linked to this later we can also mention KoT (*definitions, properties and their foundations*) in terms of identifying the use of a one-dimensional unit of measurement for finding the volume, and the characterisation of the scale between the two boxes (a scale factor of 2), which implicates the knowledge that the target volume requires the consecutive multiplications of the three dimensions measured in cubes.

Option *a* demonstrates that respondents have applied the length scale to the ratio of the two volumes, and *b* that they have applied the ratio 2^2. Option *d* suggests they have applied a non-proportional model, as this simply triples the original number of cubes.

8. The altitude of a triangle ...
   a. … never coincides with a side
   b. … depends on the side taken as reference
   c. … always falls on the midpoint of a side
   d. … always intersects one of the sides

This question is designed to probe the PPTs' KoT regarding the definition, properties and foundations of altitudes of triangles, a topic in which prototypical images often play a crucial role. It might also provide information about the respondents' awareness of the value of counterexamples for ruling out certain options (KPM, *forms of demonstration*). The task draws on Blanco and Contreras (2012) and tackles the difference between the image and definition of a concept (Vinner, 2011), here the altitude of a triangle, and whether they contradict each other.

The type of triangle chosen by the PPTs to justify selecting or rejecting the options is significant in this question in terms of the classification of angles. Hence option *a* might be selected by those respondents who are unaware of the case of a right triangle, whilst option *c* might be selected by those limiting their consideration to isosceles and equilateral triangles (confusing altitude for median), and *d* rules out obtuse triangles.

9. A pizzeria offers two sizes of round pizzas of the same thickness. The smaller size is 30 cm in diameter and costs 3€, while the larger one is 40 cm in diameter and costs 4€. Which is the best value for money?
    a. The large one
    b. The small one
    c. Both are equal value
    d. It depends on the number of people who are going to share it

This question, taken from Contreras et al. (2012), is designed to explore the KoT which leads respondents to fix on either the area, circumference or diameter in order to establish the basis of the proportional comparison and solve the problem. It checks their awareness of the need for the same unit of measurement to be applied to each magnitude (KoT, *definitions, properties and their foundations*). We can also observe their KoT (*procedures: how and when a procedure is carried out*) as the calculation of the area of a circle is necessary. And in the same category we can see whether they were able to apply the functional type of proportional reasoning, relating each area to its price.

We indirectly obtain information about respondents' knowledge of how to solve problems (KPM, *processes associated with problem-solving as a means of producing Mathematics*), as it is necessary to make the restriction implicit in the problem statement explicit (Planas and Morera 2012, in turn citing Toulmin 1958): the fact that the size of the diameter is included implies that both *round* pizzas (which indicates being of a circular shape or similar) are circles.

Were they to solve the problem by comparing like figures, they should be aware of the need to work with areas and not diameters, since placing the focus on the latter or on the perimeter would lead to option *c*. Option *d* was included as a distractor.

10. Which of the definitions below describes solely a square?
    a. A parallelogram with two equal diagonals
    b. A quadrilateral whose diagonals bisect its angles
    c. A quadrilateral with four equal angles
    d. A parallelogram with two equal and perpendicular diagonals

Inspired by the work of Zazkis and Leikin (2008), this question seeks to obtain information about the PPTs' (KoT, *definitions, properties and their foundations*) regarding the definitions and properties of squares, parallelograms, quadrilaterals, diagonals and bisection, the equivalence between the given properties and how several properties together create a subset within a classification (systems for categorising classes, Liñán-García 2017), and the concepts of the size of an angle and the length of diagonals. The question also provides information about the respondents' KPM (*Ways of validating and demonstrating*) since in order to check whether the given properties define a square it is necessary to come up with a counterexample to ensure that the set of such properties does actually exclude the shapes that cannot be considered a square.

Options *a* and *c* are true for all rectangles, and hence not only for squares, while *b* is true for all rhombi, and hence not only for squares. The correct answer requires an understanding of how parallelograms are categorised into classes.

11. Figure B has been made by cutting Figure A into the three parts shown and redistributing them. Indicate the correct answer:
    a. A and B have the same perimeter
    b. B has a longer perimeter than A, but a smaller area than A
    c. B has a longer perimeter than A and the same area
    d. B has a longer perimeter than A and a larger area than A

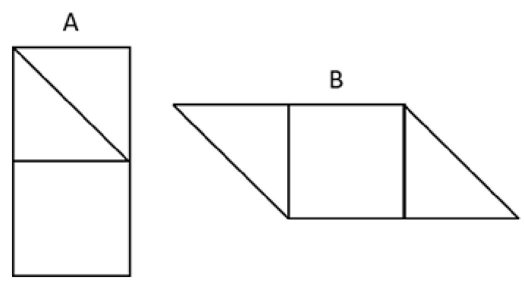

Figure 5. Shapes for question 11.

This question aims to explore the PPTs' KoT regarding the relationship between area and perimeter, and requires knowledge of the respective definitions, properties and the basis for these, as well as how each is calculated, which involves KoT (*procedures: how and when a procedure is carried out*).

Understanding the relationship between area and perimeter in the shift from one shape to the other might involve the recognition of the role of hypotenuse in the two right triangles in Figure B, either by analogy with the sides and diagonal of a square, or simply by visual comparison of the corresponding lengths. In either case, the respondent should arrive at the correct answer, *c*. By contrast, option *a* not only implies a linear relationship between area and perimeter, but also requires discarding the

Pythagorean Theorem and the recognition that the diagonal of a square is always longer than any of its sides. Option *d* indicates another misconception of the relationship between area and perimeter, as it is predicated on the erroneous idea that *the larger the perimeter, the larger the area*.

12. The least number of faces a polyhedron can have is a. Three b. Four c. Five d. Six

The question is designed to obtain information about the PPTs' KoT (*definitions, properties and their foundations*) regarding polyhedra and the characteristics governing their construction. Also influential in this respect is the respondents' knowledge of the graphical register (KoT, *registers of representation*).

Taking the cube as a starting point, and basing their argument on a graphical representation or using a physical resource, the PPTs should arrive at the conclusion that *c* and *d* are not optimal in terms of minimising faces, while *a* is impossible, although a degree of confusion between a polygon and polyhedron might intrude here.

13. Which of the following are correct definitions of a square? a. It is a parallelogram with equal sides b. It is two horizontal lines perpendicular to two vertical lines c. It is a rhombus with a 90º angle d. It is a shape with equal angles

This question draws on the same source (Zazkis and Leikin, 2008) as question 10 and complements it with information about the PPTs' KoT regarding the definition and properties of a square in terms of its sides and angles. It also explores their knowledge of the intraconceptual relationships of this knowledge in considering the attributes of a square and how the relationships between these contribute to the process of categorization (Liñán-García 2017); this in turn provides insights into the respondents' KoT (*procedures: how and when a procedure is carried out*).It also sheds light on KPM with respect to PPTs' ability to differentiate between conditions that are necessary but not sufficient, and those that are necessary and sufficient, as well as the requirement for a definition to characterise all and only the set of items it refers to (*necessary and sufficient conditions for generating definitions*), and the use of counterexamples to rule out invalid options (*ways of validating and demonstrating*).

For many PPTs, the notion of a square as a special case of a rhombus defies their expectations, especially without regard for its position, and the reference to just one of its angles instead of the four makes them resist accepting option *c* as a valid option. Nevertheless, where respondents recognise that this observation implies that the other three options must also be rhombi, they demonstrate a knowledge of how the properties of parallelograms and quadrilaterals can be expressed, and this should lead them to identify the set made up solely of the square, and hence choose *c* as the correct answer. As well as suggesting rectangles, option *b* is imprecise in its use of language, and establishes horizontal-vertical orientation as a necessary characteristic. The same can be said of *d*, which permits the inclusion of shapes with more than four sides, while *a* is valid for all rhombi.

14. How many different angles are there in the diagram below?
    a. 16
    b. 4
    c. 8
    d. 12

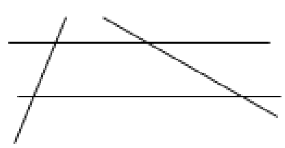

Figure 6. Shape for question 14.

We decided to pose this question about the angles formed by four-line segments, two of which are parallel, taking into account only convex angles[3]. There are four intersections around which the angles are represented. Respondents need to mobilise their knowledge of the definition of an angle, of the relationship between opposite angles and between angles when parallel lines are cut by transversal lines (KoT, *definitions, properties and their foundations*); their interpretation and mathematical reasoning in the graphical register are also brought into play (KoT, *registers of representation*). Option *a* gives the total number of angles represented in the diagram irrespective of size; *c* indicates the application of the properties of parallel lines, but not the property of complementary angles, while *d* is a distractor.

15. The template for a cylinder is formed from a 4 cm by 6 cm rectangle. What is the radius of the base?
    a. It is impossible to form a circular base with these dimensions.
    b. $3/\pi$ cm
    c. cm
    d. cm

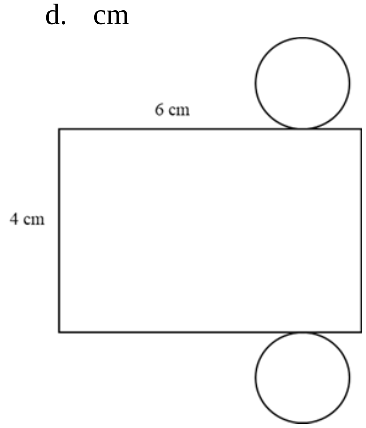

Figure 7. Shape for question 15.

[3] Compulsory education in Spain typically only covers this type of angle.

This question aims to find out the PPTs' KoT (*definitions, properties and their foundations*) of the definitions and properties of a circumference and a cylinder, as it is these that enable them to identify the size of the circumference of the base with the length of one of the sides of the rectangle forming the lateral surface of the cylinder. Respondents also need to be aware of the role of π in expressing the ratio between the circumference and its diameter. The question also gathers data about KoT (*registers of representation*) with respect to switching between representations of the cylinder (from two to three dimensions), and between the problem statement and the accompanying diagram. Finally, information about respondents' KoT (*procedures: how and when a procedure is carried out*) is also gathered in this question regarding their knowledge of how to calculate the radius, given the size of the corresponding circumference.

Options *c* and *d* indicate that the respondent has failed to take into account the definition of as the proportion between the diameter and the size of the circumference, while *a* is a distractor.

16. The foot of the altitude of a pyramid falls...
    a. … always at the centre of the base of the pyramid.
    b. … at some point within the base.
    c. … on the plane containning the base.
    d. … never at a vertex.

This question, taking its cue from Blanco and Contreras (2012), gathers information about the PPTs' knowledge in terms of the *definitions, properties and their foundations* regarding the possible locations of the foot of the altitude of a pyramid.

The definition the respondents fall back on to tackle this question, along with the degree to which their mental image of the pyramid is generalised or not (Vinner, 2011), have a significant influence on their choice of answer: *a* a square-based right pyramid; *b* any kind of oblique pyramid whose height falls within the base; *d* excludes pyramids with three perpendicular edges. These latter two cases underline the value of counter-examples for validating mathematical arguments (KPM).

17. If the two larger squares are the same size, which of the following has the larger area?
    a. The four circles inscribed in the smaller squares.
    b. The circle inscribed in the larger square.
    c. The large circle has the same area as the four smaller circles.
    d. It is impossible to determine.

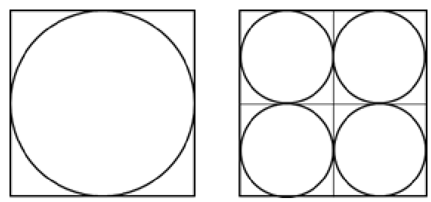

Figure 8. Shape for question 17.

This question, implicitly requiring the calculation of each of the given shapes, captures data on KoT (*procedures: how and when a procedure is carried out, why it is carried out in this way and the features of the result*), as, given that the large circle is inscribed in a square whose side is double the length of the sides of the squares in which the other circles are inscribed, the solution to the problem implies establishing proportional relationships between radii, and hence, between areas, which can be arrived at by using the diagram to establish the relationship. Thus the question provides the opportunity to gather information on respondents' KoT *registers of representation*. More specifically, the need to establish the relationship between the shapes gives access to information about their KSM (transverse connections) when it comes to making a connection between numerical and geometric proportionality.

Options *a* and *b* suggest an image of the concept of area somewhat removed from its definition, while *d* is a distractor.

Below is a summary of the information, according to the MTSK model, that the test form is designed to provide. Questions 1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14 and 16 consider KoT, *definitions, properties and their foundations and phenomenology*; questions 1, 3, 4, 6, 9, 11, 13 and 17 consider *procedures*; and questions 1, 6, 14 and 15 consider *registers of representation* and their interconnectedness. For its part, KPM is explored through various indicators in questions 2, 8, 9, 10, 13 and 16. Finally, KSM is considered in terms of transverse connections in questions 4 and 17, and in terms of auxiliary connections in questions 4 and 6.

RESULTS

This section presents the results according to the dual focus – quantitative and qualitative – of data analysis. First, we describe the results obtained in terms of frequencies (expressed as percentages) and then, from a qualitative perspective, we give a detailed analysis of our interpretation of the PPTs' understanding of geometry, highlighting their strengths and weaknesses.

Figure 9 shows that 54.7% of the PPTs answered at least 14 questions, while 78.5% had attempted at least 10.

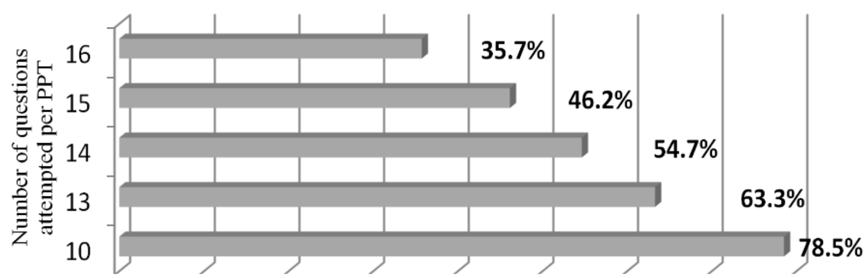

Figure 9. Percentage of PPTs attempting stated minimum number of questions.

Ten of the 17 questions making up the test (that is, nearly 60%) the most common error was the most frequent answer, according to the review carried out in the section *Explanation and background* (labelled in Figure

10 as error 1). A further two questions were answered most frequently with another error. This means that the wrong answers outnumbered the right in 12 out of 17 cases. Furthermore, when the aggregate errors (all three incorrect options added together) are compared with the correct response in each case (Figure 11), in only two questions do more respondents answer correctly than incorrectly, specifically question 6, with 63.1%, and question 17, with 64.9%. These initial results offer an overall preliminary impression of the weaknesses of the PPTs' knowledge of geometry, which we will analyse in greater depth below.

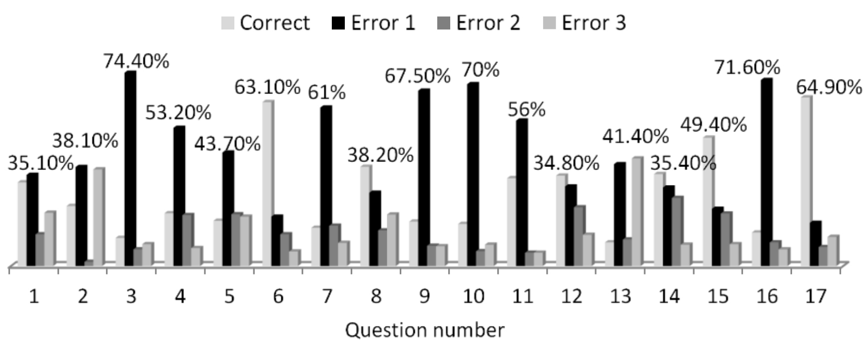

Figure 10. Relative frequencies of the four options in each question according to Table 1.

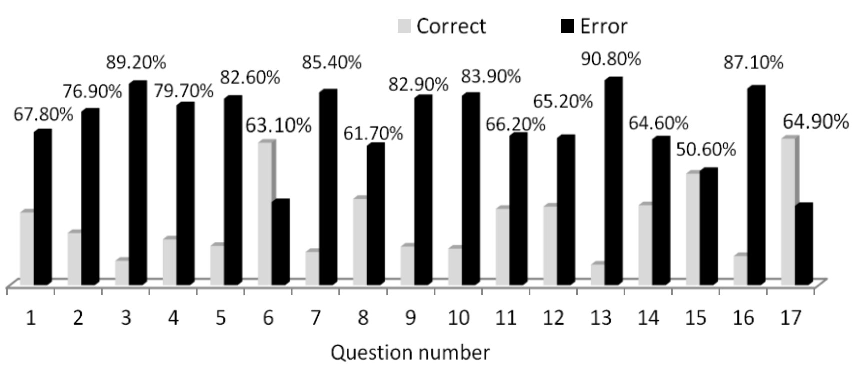

Figure 11. Proportion of correct answers to aggregated incorrect answers.

In the description of the results of the analysis below, the PPTs' choices of response to the questions have been grouped according to the categories within the subdomain *Knowledge of Topics* (KoT) of the MTSK model. These categories consist of *definitions*, *properties and their foundations*, *procedures* (including the indicators *how and when and why a procedure is carried out, and characteristics of the result*), and *registers of representation*. Following discussion of these categories, we consider elements of the subdomains *Knowledge of Practices in Mathematics* (KPM) and *Knowledge of the Structure of Mathematics* (KSM).

Table 1 shows the percentage of respondents for each option, forming the basis for the analysis of each kind of knowledge which follows, taking into account the observations of the section above.

Table 1. Relative frequencies of the options (shading indicates correct answer)

	Options			
Question	A	B	C	D
1	35.10%	12.20%	32.20%	20.50%
2	23.10%	38.10%	1.60%	37.20%
3	74.40%	10.80%	6.40%	8.40%
4	20.30%	53.20%	19.60%	6.90%
5	43.70%	19.90%	17.40%	19.00%
6	63.10%	12.20%	19.00%	5.60%
7	61.00%	15.50%	14.70%	8.90%
8	13.70%	38.20%	19.90%	28.20%
9	17.10%	7.80%	67.50%	7.60%
10	5.70%	8.20%	70.00%	16.20%
11	56.00%	5.10%	33.80%	5.10%
12	30.60%	34.80%	22.60%	12.00%
13	39.20%	10.30%	9.10%	41.40%
14	26.20%	35.40%	30.20%	8.20%
15	20.20%	49.40%	22.00%	8.40%
16	71.60%	9.10%	12.90%	6.40%
17	7.30%	16.60%	64.90%	11.20%

Knowledge of Definitions, Properties and Their Foundations, and Phenomenology (KoT)

1. Definition and Image of Polygons

Within knowledge of registers of representation, and definitions and their properties (KoT), one positive point that can be noted is that the orientation of the shape in question did not greatly influence the choice of answer (question 13) regarding the definition of polygon. Only 10.2% of respondents chose the option which featured this aspect (*a square is two horizontal lines perpendicular to two vertical lines*). This point is reinforced by the 44.4% of aggregated answers in question 1 (options *b* and *c*), which show an understanding of the equivalence of shapes which have undergone a rigid motion of the plane, in this case a rotation. However, it would seem the PPTs do rely on a prototypical image, in which orientation appears to be the chief defining characteristic, for a rhombus (with option *a* in question 1 attracting 35.1% of respondents, the highest frequency) and for an octagon (with 74.4% of respondents identifying the shape in question 3 as regular by choosing option *a*). We can conclude that, by choosing the graphic register of representation, the characteristics of the prototypical images in this register influenced the PPT's choice of response, as a result of which we consider that the register of representation selected did exert an influence on their answers.

A rhomboid was identified by 36.4% (options *c* and *d* in question 5), whereas greater difficulties were encountered with the idea of a polyhedron, with 30.6% of respondents potentially confusing polyhedrons and polygons (option *a* in question 12) and 12% potentially associating polyhedra exclusively with cubes or prisms (by stating that the minimum number of faces on a polyhedron is 6). Again, we can surmise a misguided correspondence between the image of a concept and its definition.

The PPTs display a degree of knowledge about the relationships between properties (16.2% in question 10 and 10% in question 13 choose the correct answer). In the written definitions of a square (questions 10 and 13) the definitions which attract by far the most respondents are those couched in conventional terms, with such options selected by 70% in question 10 (where option *c* refers to angles) and more than 80% in question 13 (where options *a* and *d* refer to equal sides and angles).

2. Definition and Image of the Altitude of a Triangle and of a Pyramid

Knowledge relating to the definition and properties of the altitudes of a pyramid and a triangle would also seem to be related to the prototypical image of geometrical shapes, albeit this would appear to be more marked in the former (right pyramids with regular bases) than in the latter (no consideration of altitudes of non-acute angles). In question 16, the option stating that the foot of the altitude of a pyramid falls within its base was chosen by 71.6% of respondents, while the other incorrect options (*b* and *d*) involving characteristics only applicable to certain constructions together accounted for 15.5%. The correct answer, which admitted non-right pyramids, edges perpendicular to the base (as well as those which are not), and both regular and non-regular bases, was chosen by 12.9% of the PPTs. Regarding the altitude of a triangle (question 8), 38.2% correctly considered it a property of any one of the sides over the full range of triangle types, while 61.8% excluded the altitudes of obtuse triangles and non-equilateral isosceles triangles, assuming we adopt a categorisation of triangles according to their sides.

3. Definition and Image of Axes of Symmetry of a Polygon

Question 5 (axes of symmetry of a rhomboid) tests the PPTs' knowledge of the classification of quadrilaterals, in which a rhomboid is considered a quadrilateral with two pairs of equal and parallel sides, but the pairs of unequal length with respect to each other. The options take into account how the definitions of the shapes are dealt with in the school syllabus. The majority appear to identify the axis of symmetry with diagonals (43.7% identifying only diagonals, and 19.9% identifying diagonals and the line segments intersecting the opposite sides at the mid-point). Respondents seemed to consider that, given certain regularities, the rhomboid ought to be symmetrical.

5. Integration of claSsifications of Triangles According to Sides and Angles. The Pythagorean Theorem

In question 2, on the application of the Pythagorean theorem, 23.1% of the PPTs demonstrate a phenomenological knowledge of how to classify triangles according to angles and sides (Liñán-García, 2017), correctly identifying that the theorem can be applied to an isosceles triangle (option *a*), thus implicitly acknowledging the relationship between this class and that of right triangles. By contrast, 37.2% considered that none of the first three options was valid (option *d*), presumably unaware or forgetful of the fact that it is possible for triangles to be both right-angled and isosceles. Another 38.1% considered that it is possible to apply the theorem to equilateral triangles, which could be indicative of a limitation in their previous work on the topic, which neglected to contrast the sizes of the angles in equilateral and right triangles, and how each type meets the requirement for the sum of interior angles to be 180°. On a positive note, nearly all the respondents showed themselves to be aware that the theorem cannot be applied to obtuse triangles (with only 1.6% choosing the corresponding option, *d*). That said, however, the majority (74.4%) failed to recognise that question 3 was premised on the application of the theorem (to approximate the sides of the octagon lying on the hypotenuse of the right triangles formed by the squares) and opted to assume instead that the eight sides were of equal length.

6. Opposite Angles at a Vertex and Angles Formed When Two Parallel Lines Are Cut by a Transversal

Slightly more than a third of the PPTs identified the equality of opposite angles by the vertex and angles formed when parallel lines are cut by a transversal (that is, 35.4% chose the correct answer to question 14), while 30.2% failed to recognise the equality in one of the two cases, and 26.2% failed to do so in both cases.

None of the PPTs made any comment regarding concave angles, which were not taken into account in the responses, as mentioned above in the description of question 14.

7. The Relationship between Area and Perimeter, and the Metric System

In question 11, around 61% of respondents chose the option predicated on a linear relationship between perimeter and area (of which, 56% maintain that the perimeter remains the same in the two diagrams, while a further 5.1% state that if the perimeter grows, so too does the area). At the same time, 60% confuse a unit of length with a unit of area (thus in question 4, 53.2% chose the metre, and 6.9% the decimetre, as valid responses to a question about area), while the 19.6% who selected dam^2 might have problems with the relationship between units of area.

8. The Relationship between Scale factor, Volume and Area

A weakness appears relating to scalar reasoning in question 7, concerning the new volume of a box when each of its three dimensions are doubled in length, given that 61% of PPTs recognised that a linear scale factor was involved, but chose the option which *doubled* the original volume. Only 14% realised that the scale should be cubed, while 15.5% considered that it should be raised to the power of two. In question 17, 16.6 % chose the single large circle as having a larger area than the four smaller ones, possibly as a result of purely their visual impression, without considering the ratio between the radii of the circles and their areas.

Knowledge of Procedures (KoT)

1. Categorization of Classes

In question 1, 32.2% of respondents recognised that a square can be considered a special case of a rhombus (a pictorial representation), although in question 13 (a verbal definition) only 9.1% seemed to accept this (a square is a rhombus with a 90° angle). It would seem that the PPTs expect the characteristics typically used in definitions (e.g., all the angles are the same) to be explicitly present in the verbal definition for this to be correct (Blanco and Contreras, 2012). On the Van Hiele scale of geometric understanding (1986), such a way of thinking is somewhat lower than might be desired in a future primary teacher. At the same time, the primacy of the visual representation over the definition of a concept is again brought to the fore (Vinner, 2011), a consequence of which is the tendency to categorise geometric shapes on the basis of their similarity to prototypes (Clements and Sarama, 2011).

2. Calculating the Perimeter of a Polygon

Turning to question 3 again, 10.8% seem to understand the *how*, *when* and *why* of the procedure for calculating the perimeter of the polygon, as they made allowance for the difference in length between the sides which were vertical or horizontal and those which were oblique (which they were alerted to not only by the drawing, but also by the properties of the shape given in the problem statement). Nevertheless, the majority of respondents chose the first option (*the perimeter of the octagon measures 8 cm*), without taking account of the shape's irregularity, even though the diagram was drawn on a 1 cm squared grid where it was a straightforward procedure to apply the Pythagorean Theorem to show that the oblique sides were longer than the horizontally and vertically oriented lines. The prevalence of the visual representation over the definition (Vinner, 2011) can once again be seen at work here, with the regularity of the polygon preventing them from taking into account the *why* and the *characteristics of the result* when applying the procedure.

3. Calculating the Area of a Plane Shape

20.3% of PPTs chose the correct answer to question 4, indicating that they were aware of the discrepancy in the units of measure involved (two being units of length, one being a unit of area), and of the procedure for calculating the area of a rectangle, which suggests that they also have the requisite knowledge concerning the *when* and the *why* of the procedure. There might also be indications of knowledge of the characteristics of the result.

However, the fact that 53.2% of PPTs chose an option which gave a unit of length as the answer, is indicative of their difficulties in distinguishing between variegated units of measure, and how one unit of measure (in this case length) relates to another (area). It is clear that their knowledge of *how a procedure is done* fails to take account of the differences in the units of measure, which leads into the question of *when a procedure* can be done (the units of measure at the start of the procedure must necessarily be the same). Had they taken into account *the characteristics of the result* which they chose (0.5 m), they might have realised that, the question being about an area, the unit of measure could not be linear, and that a multiplication of two decimals, one in tenths and the other in hundredths, can never give a result solely in tenths.

In question 6, calculating the area of a rectangle by decomposition into smaller shapes, the majority of respondents answered correctly, although 19% fail to identify the dimensions of the rectangle, or have problems calculating the area by means of decomposition.

The perception of area and perimeter are not independent, and the fact that two shapes have the same area fosters the belief that they have the same perimeter (D'Amore and Fadiño, 2007; Bosch et al., 2001). It would seem that knowledge of *how* to calculate area and perimeter, without knowing the *when* and the *why*, generates confusion between the two (Zazkis, Sinclair and Liljedahl, 2013). This is highlighted in the most chosen answer to question 11 (56%), although it should also be pointed out that 89.8% of the PPTs did recognise that the area is not affected by the transformation proposed by the question, demonstrating their knowledge of calculating areas by composition.

In question 9 (the area of a circle), 17.1% chose the correct answer, which required an understanding of both how to calculate the area of each pizza (on the assumption that they are circular) and how to establish the functional ratio between area and cost in each case (involving the use of a composite unit, specifically an amount of money per square centimetre). The large majority of PPTs (67.5%) chose the option in which the functional ratio was the same, thus concurring with other studies which have identified ratios as an area of difficulty (Pérez-Bueno, Liñán-García and Barrera-Castarnado, 2018).

In question 17, the correct answer was chosen by 64.9% of PPTs, suggesting that these respondents know how to calculate the area of a circle, although it is also possible that they arrived at the right answer by reasoning proportionality (the square with the largest area is divided into four smaller ones of equal size, all of which are inscribed with a circle at the same scale). The difference between the percentages of correct answers in these two questions (9 and 17) could lie in the respondents taking the diameter as the point of comparison in question 9, instead of the area of the circle, rather than confusion about how to calculate the area of a circle.

Knowledge of Registers of Representation (KoT)

The fact that 35.4% of the PPTs correctly identified the number of different-sized angles in an diagram made up of four line segments, two of which were parallel (question 14), indicates that those PPTs correctly interpreted the geometric drawing and were able to extract the implicit information from it, albeit, as mentioned above, without making any comment on, for example, the concave angles, which were equally defined in the diagram.

Regarding the template for a cylinder (question 15), 49.4% were able to correctly interpret the relationship between the dimensions of the rectangle forming the sides of the cylinder and the circles forming its bases, and to successfully merge the information from the diagram with that given in the problem statement. In this respect they demonstrate their knowledge of the relationship between the lengths of the diameter and the circumference (the scalar ratio π) so as to be able to establish the length of the radius of the base. 30.4% seemed unable to visualise the cylinder from the template. In question 17, which also required respondents to use information provided in the graphical register, 11.2% considered it impossible to determine the answer, while 64.9% chose the correct option, demonstrating their knowledge of extracting implicit data from a visual representation.

We can also note difficulties in interpreting the graphical representation of shapes like a rhombus and a rhomboid. In question 1, 20,5% of PPTs chose the fourth option, showing that what they considered to be a rhombus was in fact the drawing of a rhomboid not rhombus. Finally, we can note that in question 6 more than 60% correctly identified the algebraic expression for the area of a rectangle decomposed into two. This would seem to be the result of the familiarity with simple algebraic expressions which the PPTs acquire over the course of their schooling.

Knowledge of Practices in Mathematics (KPM)

With regard to the definition of a square (question 10), 70% affirmed that *a quadrilateral with four equal angles* defined solely a square. If, from the other options, we add further properties which are not exclusive to a square, this figure rises to over 80%, indicating that the majority of PPTs do not discriminate between *necessary* conditions and *necessary and sufficient* conditions. This interpretation is consistent with the responses to question 13, again dealing with the definition of a square. In this instance, over 80% chose one of the options which were limited to necessary conditions, but were not sufficient to constitute a definition. Nevertheless, the 9.1% who gave the right answer could have arrived at it by using counterexamples to eliminate invalid options.

On the Pythagorean Theorem (question 2), 37.2% stated that the Theorem can be applied to *none of the above triangles* (isosceles, equilateral and obtuse). This again suggests that they failed to distinguish a necessary and sufficient condition (that of being a right triangle) from other characteristics which give rise to a crossed categorisation (such as being isosceles and right-angled), or from the categorisation of classes.

Regarding the question of validating conjectures (in this instance geometric), 22% used counterexamples to check the affirmations about the altitude of a pyramid (question 16), which enabled them to rule out that the foot always falls somewhere within the base (including the centre). They also seemed to be aware of the significance of the quantifiers *never* and *always*. Something similar can be observed in their knowledge of the altitude of a triangle (question 8), in which 38.2% seemed to use counterexamples to rule out the three incorrect options, and again would seem to have understood the use of *never* and *always*.

Question 9 could be indirectly highlighting a strength in their knowledge of how to solve problems (KPM, *processes associated with problem-solving as a means of producing mathematics*). Here, the aggregated majority (92.4%) choose an answer relating to the calculation of the area of the stated surfaces, suggesting that, given the problem statement talks of the lengths of the diameters of the pizzas, they applied the supposition that the pizzas were necessarily circular.

Knowledge of Structures in Mathematics (KSM)

As stated above, 20.3% of the PPTs who answered question 4 took into account both the units of measure involved (two of which were measures of length and one of area), and the procedure for calculating the area of a rectangle. We can note here weaknesses in their knowledge of the relationship between the metric system of measurement and the decimal system (KSM, transverse connections), and in the relationship between product and the calculation of area (KSM, auxiliary connections). In the same vein, we can see that 69.9% of the PPTs in question 7 perceive an equivalence between numerical and geometric proportionality (KSM, transverse connections). On the other hand, we can note a strength in question 6, where 63.1% correctly identify the algebraic expression giving the area of the rectangle which has been decomposed (KSM, auxiliary connections).

CONCLUSION

An overview of the errors committed in the test showed that many were typical of the early years of school, suggesting that the PPTs participating in the study had not had occasion to extend their knowledge of geometry since compulsory education. The meticulous identification of errors achieved in this study provides us with a better understanding of which elements of geometry should be included in degree courses for primary teachers, and how particular aspects should be approached. At the same time, before taking any specific decisions about applying measures to improve the initial training of these professionals, we are aware that there are two factors we should reflect on. First, the errors might be resistant to correction (Brodie, 2005), or the treatment of the related aspects in previous stages was inadequate to overcome them. Second, it is essential to precisely define the basic mathematical knowledge that can be expected of entrants to primary education degrees (Castro et al., 2014) and hence establish the necessary tests for accrediting this knowledge for new entrants to training courses. It would be inappropriate at university to go back over areas which should have been learnt earlier. Moreover, it is fundamental that research in this field should be directed towards defining the specialised knowledge of mathematics which PPTs should have acquired on finishing their studies.

Our findings also identified certain strengths, chiefly in terms of the elements of procedural learning more related to *how* than to *when*, *why* and the *characteristics of the result*. The elements in need of review arising from the study include certain transverse aspects of KPM and KSM. In terms of KPM, the understanding of problem statements based on formal logic can be highlighted, with special attention given to understanding the characteristics of a good definition, being aware of the conditions (necessary and/or sufficient) in which a mathematical result can be applied and having a degree of familiarity with problem-solving. In terms of KSM, the areas of greatest importance would seem to be the connections between geometry and numbers, such as the relationship between proportionality and geometry, and the parallels between the decimal system and the metric system.

With respect to plane shapes, the elements comprising them and the relationship between the two, we concur with Climent and Carrillo (2002) in noting a tendency for PPTs to identify the base of the triangle uniquely with the horizontal axis and likewise the altitude with the vertical. In this study, we also noted a tendency for the images to prevail over definitions.

This was the case of the definition of a square and rectangle, where the PPTs did not consider a square a special case of a rectangle, and in the properties of concepts such as regularity, whereby certain regularities, like having equal angles, were confused with the quality of being a regular shape or solid. In very much the same way, the findings concerning the definition and properties of the altitude of any pyramid, saw very little use of the meaning of the essential characteristics of the shapes (Tsamir *et al*., 2015), which is also true of the elements and relationships of geometric solids (Barrantes and Zapata, 2008). We can also underline the lack of understanding of the relationship between area and perimeter, and the difficulty many PPTs found in recognising the conservation of area after shape transformations. Nevertheless, it should be noted the graphical register of representation enabled the identification of some geometric characteristics and the relationship between them.

The use of algebra in geometry, which other studies have found to be a cause of error (Zazkis, Sinclair and Liljedahl, 2013; Zacharos, 2006), represented in ours an area of strength, and in conjunction with the graphical register of representation, it enabled PPTs to solve the corresponding problem. This might be accounted for by a cultural factor, namely the tendency, within the context of Spanish compulsory education, to overindulge the symbolic (numeric and algebraic). Notwithstanding their association of geometry with the measurement of lengths, areas and volumes, the PPTs found difficulties with the metric system and its structural relationship with the decimal system, which could be indicative of a weakness in their number sense (NCTM, 2000). The way to calculate volume seems poorly grasped, and there is little evidence that PPTs are aware of how to conserve proportionality across dimensions. If to this we add the confusion arising in comparing shapes, then not only is their ability to recognise similarities called into question, but also their difficulties in identifying, analysing and applying proportionality to everyday situations (Contreras et al., 2012). The PPTs' failure to grasp the numerical structure of proportionality is a significant weakness in their geometric knowledge. To this we can add, following Pérez-Bueno, Liñán-García and Barrera-Castarnado (2018), the need to underline the kind of proportionality in play, whether scalar or functional, and in the case of the latter, the importance of using the appropriate units of measurement in the problem.

With respect to the PPTs' Knowledge of Practices in Mathematics, we can again identify a strength, albeit indirectly, in terms of their understanding of how to solve problems, in that in order to find a solution the problem in question, the PPTs were required to hypothesise the restrictions on the shape which some figures might take.

Finally, the concept of *symmetrical shape* was confused with certain regularities or geometrical patterns, as in the case of a parallelogram with unequal sides and angles. One of the PPTs' major difficulties was an understanding of the concept of symmetry, in that they tended to identify half of a shape in terms of size (in this case, area) with half of a shape in terms of symmetry.

These findings confirm our original position: the geometrical knowledge selected for this study needs to be problematized and reconstructed on the basis of what a future educational professional will require. Many of the educational obstacles that appear from the nursery onwards still persist, enshrined in traditional teaching modes and an exclusive use of the textbook, which together frequently "*create mental schemes unsuitable for developing open and diverse thinking in the pupil*" (Barrantes and Zapata, 2008, 56), with the errors rooted in the primary and secondary stages prevailing above all.

It is our belief that a teacher's professional development starts with their initial training (Carrillo et al., in press), and hence we consider the findings of Ma (1999) on practising teachers' knowledge highly relevant. This researcher argues that teacher training is a period in which changes can and should be made, given that inadequate mathematical training coupled with sketchy knowledge among primary teachers results in a vicious circle producing poor quality understanding among their pupils. Our study identified certain strengths among the PPTs, but a large number of weaknesses. If the circle is to be broken, then it would appear that it can only take place through proper training, based on the existing knowledge of school mathematics of candidates for PPT, so that trainees in due course provide a better mathematical grounding for their future charges. The knowledge which must be achieved relates to the four properties of primary teachers' profound understanding of fundamental mathematics: basic ideas, connectedness, multiple perspectives and longitudinal coherence (Ma, op cit., 122).

In our study, the first of these, basic ideas, can be seen in KoT (*definitions, procedures, registers of representation and phenomenology*), the definitions, properties, elements and image of both polygons and geometrical solids, the procedures for calculating perimeter and area, the implicit relationships between a definition and its utility, and between the distinct classifications and categorisations of a single construct (for example, the interpretation of the relationship between the classifications or categorisations of triangles from the measurement of their angles or the measurement of their sides as a phenomenon of the same). The second, connections between these ideas, reflected in KoT, in the case of intraconceptual connections, and in KSM in the case of interconceptual connections (viz., connections of complexity and of simplification, and transverse and auxiliary connections), can be seen in the relationships, for example, between numerical and geometric proportionality. The third, multiple perspectives of the same phenomenon, can be seen in KoT regarding phenomenology, and in KPM regarding the various ways of approaching a problem, recognising their equivalence, such as might be the use of a counterexample and the differentiation between any conditions and those which are necessary and sufficient for a definition. Finally, longitudinal coherence, can be seen in the connections of complexity and simplification, as well as the adaptation to both the curriculum and expert proposals such as the NCTM (2000) and TEDS-M (Tatto et al., 2012). These four properties which Ma (1999) discusses would clearly be supported by Knowledge of Practices in Mathematics (KPM), that is the *how*, the *when* and the *why* we do mathematics.

We conclude by reiterating that whatever training takes place, it needs to take into account the deficiencies detected in the PPTs. In this way we can break the tendency to repeat the cycles observed in the educational system and ensure that the necessary mathematical foundations of their specialised knowledge are as sufficiently well-founded that they can put it to use as soon as they start to exercise their profession. Likewise, following Montes et al. (in press), we consider the MTSK model a useful organiser for teacher training, based in this particular case, on the weaknesses and strengths in the PPTs' specialised knowledge of mathematics.

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