Continuous Well-Composedness Implies Digital Well-Composedness in n-D

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Abstract
In this paper, we prove that when a n-D cubical set is continuously well-composed (CWC), that is, when the boundary of its continuous analog is a topological $(n-1)$-manifold, then it is digitally well-composed (DWC), which means that it does not contain any critical configuration. We prove this result thanks to local homology. This paper is the sequel of a previous paper where we proved that DWCness does not imply CWCness in 4D.

Keywords Well-composed · Topological manifolds · Critical configurations · Digital topology · Local homology

1 Introduction

Digital well-composedness (DWCness) is a strong property in digital topology, because it implies the equivalence of $2n$- and (3^n-1)-connectivities in a set and in its complement. A well-known application of this flavor of WCness is the tree of shapes [11,12], a powerful hierarchical representation of the objects in a gray-level [20] or color image [10]. On the other side, continuously well-composed (CWC) images are known as "counterparts" of n-dimensional manifolds (or in short, n-manifolds) in the sense that they do not have singularities (no "pinches") in their boundary. The consequence is that some geometric differential operators can be directly computed on the discrete sets, which can simplify or fasten specific algorithms.

DWCness and CWCness are known to be equivalent in 2D and in 3D [4,16]. As the sequel of [7] where we prove thanks to a counter-example that DWCness does not imply CWCness in 4D, we prove in this paper that CWCness implies DWCness in n-D.

Some other flavors of well-composednesses exist like well-composedness in the Alexandrov sense [2,8,9,20], well-composedness on arbitrary grids [2,23], weak well-composedness [5] or Euler well-composedness [6], but we will not go further into details here.

The plan is the following: Sect. 2 presents an intuitive explanation of the proof presented in this paper, Sect. 3 recalls the material necessary to our proof in matter of discrete topology; Sect. 4 contains the proof of the main result of this paper; Sect. 5 concludes the paper.

2 Intuitive Proof of our Main Theorem

Let us assume that we start from a finite set X of points of $\mathbb{Z}^n$. We want to show that when we dilate X by a unitary centered cube of radius $\frac{1}{2}$ in $\mathbb{R}^n$, then the topological properties of the resulting $\mathrm{CA}(X) \subset \mathbb{R}^n$, called the continuous analog of X, are related to the properties of the initial set X. More exactly, we want to prove Theorem 5, which asserts that when $\mathrm{CA}(X)$ is *regular* in the sense that its boundary is a topological manifold, then it means at the same time that the initial set X is *regular* in the discrete manner. Being regular in a discrete manner, in the context of discrete topology, means that a set does not contain critical configurations, well known to lead to topological issues. Using the technical terms, continuous well-composedness implies digital well-composedness.

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To prove that a regular CA(X) implies a regular X, we will proceed by counterposition, that is, we will prove that as soon as X contains (at least) one critical configuration, then the continuous counterpart CA(X) contains a *pinch* at the center $m \in \left(\frac{\mathbb{Z}}{2}\right)^n$ of this critical configuration (Sect. 4.3 is devoted to prove this fact). From a technical point of view, we will use local homology to compute local topological properties of CA(X) at m to show that it is not a homology manifold, and thus it is not a topological manifold either (we recall that topology manifoldness implies homology manifoldness).

The methodology is then straightforward: by assuming that X is not regular, we choose any of its critical configurations, we deduce its center m; since m belongs to the boundary of CA(X) according to Lemma 1, we can study the behavior of the boundary of CA(X) from a topological point of view around m thanks to local homology. These characteristics depend only on the configuration (we do a case-by-case study) as stated by Theorem 4. We will obtain that some homological issue appeared at m (since the local homology group of dimension $(n-1)$ will not be $\mathbb{Z}$ as stated in Property 4) and then we will conclude that the boundary of CA(X) is not a topological manifold.

Intuitively, this is the way the main proof of this paper will be done.

3 Discrete Topology

As usual in discrete topology, we will only work with *digital sets*, that is, non-empty strict subsets of $\mathbb{Z}^n$ which are finite or whose complementary in $\mathbb{Z}^n$ is finite.

3.1 Digital Topology and Digital well-composedness

Let $n \geq 2$ be a (finite) integer called the *dimension*. Now, let $\mathbb{B} = \{e^1, \dots, e^n\}$ be the (orthonormal) canonical basis of $\mathbb{Z}^n$. We use the notation p_i, where i belongs to $[\![1, n]\!]$, to determine the i^{th} coordinate of the point $p \in \mathbb{Z}^n$. We recall that the L^1-*norm* of a point $p \in \mathbb{Z}^n$ (seen as a vector) is denoted by $\|.\|_1$ and is equal to $\sum_{i \in [\![1,n]\!]} |p_i|$ where $|.|$ is the *absolute value*. Also, the L^∞-*norm* is denoted by $\|.\|_\infty$ and is equal to $\max_{i \in [\![1,n]\!]} |p_i|$.

For a given point $p \in \mathbb{Z}^n$, the $2n$-*neighborhood* in $\mathbb{Z}^n$ is denoted by $\mathcal{N}_{2n}(p)$ and is equal to $\{p' \in \mathbb{Z}^n \ ; \ \|p - p'\|_1 \leq 1\}$. In other words,

$$\mathcal{N}_{2n}(p) = \left\{ p + \lambda_i e^i \ ; \ \lambda_i \in \{-1, 0, 1\}, i \in [\![1, n]\!] \right\}.$$

For a given point $p \in \mathbb{Z}^n$, the $(3^n - 1)$-*neighborhood* in $\mathbb{Z}^n$ is denoted by $\mathcal{N}_{3^n-1}(p)$ and is equal to $\{p' \in \mathbb{Z}^n \ ; \ \|p - p'\|_\infty \leq 1\}$. In other words, $\mathcal{N}_{3^n-1}(p)$ equals:

$$\left\{ p + \sum_{i \in [\![1,n]\!]} \lambda_i e^i \ ; \ \lambda_i \in \{-1, 0, 1\}, \ i \in [\![1, n]\!] \right\}.$$

From now on, let ζ be a value in $\{2n, 3^n - 1\}$. The *starred ζ-neighborhood* of $p \in \mathbb{Z}^n$ is denoted by $\mathcal{N}^*_\zeta(p)$ and is equal to $\mathcal{N}_\zeta(p) \setminus \{p\}$. An element of the starred ζ-neighborhood of $p \in \mathbb{Z}^n$ is called a *ζ-neighbor* of p in $\mathbb{Z}^n$. Two points $p, p' \in \mathbb{Z}^n$ such that $p \in \mathcal{N}^*_\zeta(p')$ or equivalently $p' \in \mathcal{N}^*_\zeta(p)$ are said to be *ζ-adjacent*.

Let X be a subset of $\mathbb{Z}^n$. A finite sequence $\pi = (p^0, \dots, p^k)$ of points of X is called a *ζ-path* joining p^0 and p^k when p^0 is ζ-adjacent only to p^1 in π, p^k is ζ-adjacent only to p^{k-1} in π, and if for all $i \in [\![1, k-1]\!]$, p^i is ζ-adjacent only to p^{i-1} and to p^{i+1} in π. Such a path is said to be of *length* k.

A digital set $X \subset \mathbb{Z}^n$ is said to be *ζ-connected* when there exists a ζ-path into X joining any pair of points of X. A subset C of X which is ζ-connected and *maximal in the inclusion sense* (that is, there is no subset of X greater than C and ζ-connected) is said to be a *ζ-component* of X.

For any $q \in \mathbb{Z}^n$ and any $\mathcal{F} = (f^1, \dots, f^k) \subseteq \mathbb{B}$ ($\mathcal{F}$ can be an empty set), we denote by $S(q, \mathcal{F})$ the set:

$$\left\{ q + \sum_{i \in [\![1,k]\!]} \lambda_i f^i \ \middle| \ \lambda_i \in \{0, 1\}, \forall i \in [\![1, k]\!] \right\}.$$

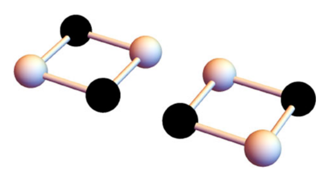

Fig. 1 The two connected sets depicted here represent 2D blocks. The two white points of the block depicted on the left side are 2-antagonists in this block, and draw a primary 2D critical configuration. In a same manner, the two white points antagonists in the block depicted on the right side draw a secondary critical configuration. Indeed, in a 2D space, all critical configurations are at the same time primary and secondary

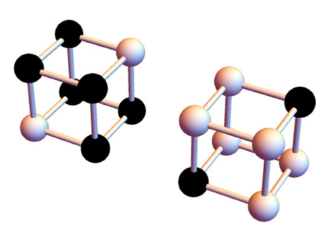

Fig. 2 The two connected sets depicted here represent 3D blocks. The white points of the block depicted on the left side are 3-antagonists in this block, and draw a primary 3D critical configuration. The set of six white points in the block depicted on the right side draws a secondary 3D critical configuration

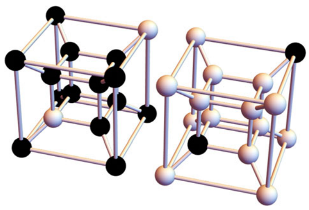

Fig. 3 The two connected sets depicted here represent 4D blocks. The white points of the block depicted on the left side are 4-antagonists in this block, and draw a primary 4D critical configuration. The set of fourteen white points in the block depicted on the right side draws a secondary 4D critical configuration

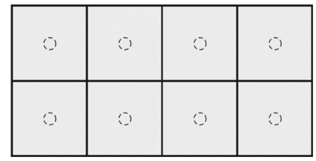

Fig. 4 The continuous analog of the set $\{0, 1\} \times \{0, 1, 2, 3\}$

We call this set the *block* associated with the pair $(q, \mathcal{F})$; its *center* is $q + \sum_{i \in [\![1,k]\!]} \frac{f^i}{2}$, and its *dimension*, denoted by $\dim(S)$, is equal to k. More generally, a set $S \subset \mathbb{Z}^n$ is said to be a *block* when there exists a pair $(q, \mathcal{F}) \in \mathbb{Z}^n \times \mathcal{P}(\mathbb{B})$ such that $S = S(q, \mathcal{F})$.

Then, we say that two points $p, p' \in \mathbb{Z}^n$ belonging to a block S are *antagonists* in S when the distance between them equals the maximal distance using the L^1 norm between two points in S; in this case we write $p' = \text{antag}_S(p)$. Note that the antagonist of a point p' in a block S containing p exists and is unique. Two points that are antagonists in a block of dimension $k \geq 0$ are said to be *k-antagonists*; k is then called the *order of antagonism* between these two points.

Note that in the particular case where p and p' are 0-antagonists, $p = p'$, the center of the block is equal to p, and the corresponding family of vectors is $\mathcal{F} = \emptyset$.

We say that a digital subset X of $\mathbb{Z}^n$ contains a *critical configuration* in a block S of dimension $k \in [\![2, n]\!]$ when there exist two points $\{p, p'\} \in \mathbb{Z}^n$ that are antagonists in S s.t. $X \cap S = \{p, p'\}$ (*primary case*) or s.t. $S \setminus X = \{p, p'\}$ (*secondary case*). Figures 1, 2 and 3 depict examples of critical configurations.

Then, a digital set $X \subset \mathbb{Z}^n$ is said to be *digitally well-composed (DWC)* [3] when it does not contain any critical configuration.

3.2 Basics in Topology and Continuous well-composedness

Definition 1 (Topological spaces [1,14]) Let T be a set, and let $\mathcal{U}$ be a set of subsets of T such that:

- T and $\emptyset$ are in $\mathcal{U}$,
- Any union of elements of $\mathcal{U}$ is in $\mathcal{U}$,
- Any finite intersection of elements of $\mathcal{U}$ is in $\mathcal{U}$.

Then, $\mathcal{U}$ is said to be a *topology*, and the pair $(T, \mathcal{U})$ is called a *topological space*. The elements of T are called the *points* of $(T, \mathcal{U})$, and the elements of $\mathcal{U}$ are called the *open sets* of $(T, \mathcal{U})$. We will abusively say that T is a topological space, assuming it is supplied with its topology $\mathcal{U}$.

An open set which contains a point of T is said to be a *neighborhood* of this point. For any subset Y of T, we denote by Y^c its *complement* in T; that is, $Y^c = T \setminus Y$. Let T be a topological space. A set $Y \subseteq T$ is said to be *closed* when it is the complement of an open set in T.

Definition 2 ([18]) A topological space M is said to be *locally Euclidean* of dimension $n \geq 0$ at $x \in M$ if x has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 3 A *second countable space* is a topological space X whose topology has a countable basis, that is, there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open sets of X such that any open subset of X can be written as a union of elements of some subfamily of $\mathcal{U}$.

Definition 4 A *Hausdorff space* is a topological space where distinct points have disjoint neighborhoods.

Definition 5 ([18]) A *topological n-manifold M with boundary* with $n \geq 0$ is a second countable Hausdorff space that is locally Euclidean of dimension n at each $x \in M$, and such that there exists for any $x \in M$ an open set U containing x and a homeomorphism $\phi_U : U \to \mathbb{R}^n$ or a homeomorphism $\phi_U : U \to \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.

Let us recall what is *continuous well-composedness* for n-D sets according to Latecki [16,17]. The *continuous analog* $\text{CA}(p)$ of a point $p \in \mathbb{Z}^n$ is the closed unit cube centered at this point with faces parallel to the coordinate planes:

$$\text{CA}(p) = \{p' \in \mathbb{R}^n \; ; \; \|p - p'\|_\infty \leq 1/2\}.$$

Note that for any $p \in \mathbb{Z}^n$, the topological space $\text{CA}(p)$ is an example of (connected and compact) topological manifold with boundary, and the set $\mathbb{R}^n$ is a topological manifold without boundary.

The *continuous analog $CA(X)$ of a digital set $X \subset \mathbb{Z}^n$* (see Fig. 4) is the union of the continuous analogs of the points belonging to the set X:

$$\text{CA}(X) = \bigcup_{p \in X} \text{CA}(p).$$

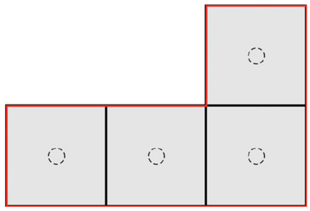

Fig. 5 Boundary of the continuous analog of a set: in dashed circles, the elements p of X, in gray the squares corresponding to $\mathrm{CA}(p)$ centered at the points p, and in red the boundary of the continuous analog of X all around X [This picture is better viewed in color.] (Color figure online)

However, contrary to $\mathrm{CA}(p)$, a topological space $\mathrm{CA}(X)$ (with X some digital subset of $\mathbb{Z}^n$) is not necessarily a topological manifold, as depicted later (see Fig. 11).

Then, we will denote by $\mathrm{bdCA}(X)$ the topological boundary (see Fig. 5) of $\mathrm{CA}(X)$:

$$\mathrm{bdCA}(X) = \mathrm{CA}(X) \setminus \mathrm{Int}(\mathrm{CA}(X)),$$

where $\mathrm{Int}(.)$ is the *(topological) interior operator*. That is, $\mathrm{Int}(\mathrm{CA}(X))$ is a subset of $\mathrm{CA}(X)$ which is open and maximal in the inclusion sense.

Let X be a subset of $\mathbb{Z}^n$. We say that X is a *continuous well-composed set (CWC)* when the boundary of its continuous analog $\mathrm{bdCA}(X)$ is a $(n-1)$-manifold, that is, if for any point $p \in X$, the (open) neighborhood of p in $\mathrm{bdCA}(X)$ is homeomorphic[1] to $\mathbb{R}^{n-1}$.

Note that it is well known that the boundary of the continuous analog is *self-dual*:

Proposition 1 *Let X be a digital subset of $\mathbb{Z}^n$, then:*

$$\mathrm{bdCA}(X) = \mathrm{bdCA}(X^c).$$

Thus, any digital set X subset of $\mathbb{Z}^n$ is CWC iff its complement X^c is CWC.

3.3 Local Homology

Since it will be useful in the sequel, let us recall that for A and B two sets, the *Cartesian product* of A and B is denoted by $A \times B$ and is equal to $\{(a, b)\ ;\ a \in A, b \in B\}$.

3.3.1 Cubical Sets

Definition 6 (Definition 2.1 p. 40 of [13]) An *elementary interval* is a closed interval $I \subset \mathbb{R}$ of the form

$$I = [l, l+1], \text{ or } I = \{l\},$$

for some $l \in \mathbb{Z}$. Elementary intervals that consist of a single point are said to be *degenerate*, while those of length 1 are said to be *nondegenerate*.

Definition 7 (Definition 2.3 p. 40 of [13]) An *elementary cube* Q in $\mathbb{R}^n$ is a finite product of elementary intervals, that is,

$$Q = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$$

where each I_i is an elementary interval. The set of elementary cubes in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$.

Note: It is important not to confuse the n-dimensional cubes $\mathrm{CA}(p)$ for $p \in \mathbb{Z}^n$, used to build the continuous analogs of discrete sets, with k-cubes (with $k \in [\![0, n]\!]$) used in cubical homology, which represent the faces of these n-dimensional cubes seen as cubical complexes and allow us to compute homology groups. Remark also that a translation by half coordinates is needed to convert $\mathrm{CA}(p)$ or its faces into a k-cube (and conversely). For example, in 1D, the 1-cube $[0, 1]$ is centered at $x = \frac{1}{2}$ when $\mathrm{CA}(0) = \left[-\frac{1}{2}, \frac{1}{2}\right]$ is centered at $x = 0$ and then we use a translation of $-\frac{1}{2}$ to convert the 1-cube into $\mathrm{CA}(0)$. However, these translations can be ignored in this paper since topological properties are preserved by translations in $\mathbb{R}^n$.

Definition 8 (Definition 2.4 p. 41 of [13]) Let $Q = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ be an elementary cube. The interval I_i is referred to as the ith *component* of Q and is written as $I_i(Q)$. The *dimension* of Q is defined to be the number of nondegenerate components in Q and is denoted by $\dim(Q)$. Also, we define

$$\mathcal{K}_k := \{Q \in \mathcal{K}\ ;\ \dim(Q) = k\}$$

and

$$\mathcal{K}_k^n := \mathcal{K}_k \cap \mathcal{K}^n.$$

Definition 9 (Definition 2.9 p. 43 of [13]) A set $\mathfrak{X} \subset \mathbb{R}^n$ is *cubical* if $\mathfrak{X}$ can be written as a finite union of elementary cubes. If it is a cubical set, we adopt the following notation:

$$\mathcal{K}(\mathfrak{X}) := \{Q \in \mathcal{K}\ ;\ Q \subseteq \mathfrak{X}\}$$

and

$$\mathcal{K}_k(\mathfrak{X}) := \{Q \in \mathcal{K}(\mathfrak{X})\ ;\ \dim(Q) = k\}.$$

Definition 10 (p. 47 of [13]) With each elementary k-cube $Q \in \mathcal{K}_k^n$, we associate an algebraic object $\widehat{Q}$ called an *elementary k-chain* of $\mathbb{R}^n$. The set of all elementary k-chains of $\mathbb{R}^n$ is denoted by

[1] We call *homeomorphism* a bicontinuous bijection. When there exists some homeomorphism $f : A \to B$ such that $B = f(A)$, we say that these spaces are *homeomorphic*.

$$\widehat{\mathcal{K}_k^n} := \{\widehat{Q}\ ;\ Q \in \mathcal{K}_k^n\},$$

and the set of all *elementary chains* of $\mathbb{R}^n$ is given by

$$\bigcup_{k=0}^{n} \widehat{\mathcal{K}_k^n}.$$

Given any finite collection $\{\widehat{Q_1}, \ldots, \widehat{Q_m}\}$, we are allowed to consider sums of the form

$$\alpha_1 \widehat{Q_1} + \cdots + \alpha_m \widehat{Q_m}$$

where $\alpha_1, \ldots, \alpha_m$ are arbitrary integers. In particular, for each $Q \in \mathcal{K}_k^n$, define $\widehat{Q} : \mathcal{K}_k^n \to \mathbb{Z}$ by

$$\widehat{Q}(P) := \begin{cases} 1 \text{ if } P = Q, \\ 0 \text{ otherwise,} \end{cases}$$

and let $0 : \mathcal{K}_k^n \to \mathbb{Z}$ be the *zero function*, namely, $0(Q) = 0$ for all $Q \in \mathcal{K}_k^n$ Then, $\widehat{Q}$ is the *elementary chain dual* to the elementary cube Q.

Definition 11 (Definition 2.16 p. 48 of [13]) The group C_k^n of k-dimensional chains of $\mathbb{R}^n$ (k-chains for short) is the free Abelian group generated by the elementary chains of $\widehat{\mathcal{K}}_k^n$. In particular, $\widehat{K}_k^n$ is the basis of C_k^n.

Definition 12 (Definition 2.23 p. 51 of [13]) Given two elementary cubes $P \in \mathcal{K}_k^n$ and $Q \in \mathcal{K}_{k'}^{n'}$, we define the *cubical product* of $\widehat{P}$ and $\widehat{Q}$ such as

$$\widehat{P} \diamond \widehat{Q} := \widehat{P \times Q}.$$

3.3.2 Chain Complexes and Boundary Operator

Definition 13 (Definition 2.27 p. 53 of [13]) Let $\mathfrak{X} \subset \mathbb{R}^n$ be a cubical set. Let $\widehat{\mathcal{K}}(\mathfrak{X}) := \{\widehat{Q}\ ;\ Q \in \mathcal{K}(\mathfrak{X})\}$. Then, $C_k(\mathfrak{X})$ is the subgroup of C_k^n generated by the elements of $\widehat{\mathcal{K}}_k(\mathfrak{X})$ and is referred to as the set of *k-chains of* $\mathfrak{X}$. Since we know that $\mathfrak{X} \subset \mathbb{R}^n$, it is not necessary to write a superscript n in $\widehat{K}_k(\mathfrak{X})$ and $C_k(\mathfrak{X})$.

Note that given any $c \in C_k(\mathfrak{X})$, we have the decomposition

$$c = \sum_{Q_i \in \mathcal{K}(X)} \alpha_i \widehat{Q}_i$$

where $\alpha_i \in \mathbb{Z}$.

Definition 14 (Definition 2.31 p. 54 of [13]) Given $k \in \mathbb{Z}$, the *cubical boundary operator* $\partial_k : C_k^n \to C_{k-1}^n$ is a homomorphism of free Abelian groups, which is defined for an elementary chain $\widehat{Q} \in \widehat{\mathcal{K}}_k^n$ by induction on the embedding number as follows. Consider first the case $n = 1$. Then, Q is an elementary interval and hence $Q = \{l\} \in \mathcal{K}_0^1$ or $Q = [l, l+1] \in \mathcal{K}_1^1$ for some $l \in \mathbb{Z}$. Define

$$\partial_k \widehat{Q} := \begin{cases} 0 \text{ if } Q = \{l\}, \\ \widehat{\{l+1\}} - \widehat{\{l\}} \text{ if } Q = [l, l+1]. \end{cases}$$

Note that k can take here two different values, $k = 0$ if $Q = \{l\}$ and $k = 1$ if $Q = \{l, l+1\}$.

Now assume that $n > 1$. Let $I = I_1(Q)$ and $P = I_2(Q) \times \cdots \times I_n(Q)$, then we can write that

$$\widehat{Q} = \widehat{I} \diamond \widehat{P}.$$

Define

$$\partial_k \widehat{Q} := \partial_{k_1} \widehat{I} \diamond \widehat{P} + (-1)^{k_1} \widehat{I} \diamond \partial_{k_2} \widehat{P},$$

where $k_1 = \dim(I)$ and $k_2 = \dim(P)$. Finally, we extend the definition to all chains by linearity; that is, if $c = \alpha_1 \widehat{Q}_1 + \cdots + \alpha_m \widehat{Q}_m$, then

$$\partial_k c := \alpha_1 \partial_k \widehat{Q}_1 + \cdots + \alpha_m \partial_k \widehat{Q}_m.$$

Proposition 2 *Let $Q = [0, 1]^k \subset \mathbb{R}^n$ be a k-elementary cube with $k \geq 1$. Then, the boundary of $\widehat{Q}$ equals*

$$\partial_k \widehat{Q} := \sum_{i=0}^{k-1} (-1)^i \operatorname{Alg}\left([0, 1]^i \times \{1\} \times [0, 1]^{k-1-i}\right) - \sum_{i=0}^{k-1} (-1)^i \operatorname{Alg}\left([0, 1]^i \times \{0\} \times [0, 1]^{k-1-i}\right),$$

where $\operatorname{Alg}(P)$ is just a notation representing $\widehat{P}$.

Proof The proof follows from Definitions 12 and 14. □

Proposition 3 (Proposition 2.39 p. 280 of [13]) *Let $\mathfrak{X} \subset \mathbb{R}^n$ be a cubical set. Then,*

$$\partial_k(C_k(\mathfrak{X})) \subseteq C_{k-1}(\mathfrak{X}).$$

Definition 15 (Definition 2.40 p. 59 of [13]) The boundary operator for the cubical set $\mathfrak{X}$ is defined to be

$$\partial_k^{\mathfrak{X}} : C_k(\mathfrak{X}) \to C_{k-1}(\mathfrak{X})$$

obtained by restricting $\partial_k : C_k^n \to C_{k-1}^n$ to $C_k(\mathfrak{X})$.

Definition 16 (Definition 2.41 p. 59 of [13]) The *cubical chain complex* for the cubical set $\mathfrak{X} \subset \mathbb{R}^n$ is

$$\mathcal{C}(\mathfrak{X}) := \{C_k(\mathfrak{X}), \partial_k^{\mathfrak{X}}\}_{k \in \mathbb{Z}},$$

where $C_k(\mathfrak{X})$ are the groups of cubical k-chains generated by $\mathcal{K}(\mathfrak{X})$ and $\partial_k^{\mathfrak{X}}$ is the cubical boundary operator restricted to $\mathfrak{X}$.

3.3.3 Homology Groups

Definition 17 (p. 60 of [13]) Let $\mathfrak{X} \subseteq \mathbb{R}^n$ be a cubical set. A k-chain $c \in C_k(\mathfrak{X})$ is called a *cycle* in $\mathfrak{X}$ if $\partial_k c = 0$. The set of all k-cycles in $\mathfrak{X}$, which is denoted by $Z_k(\mathfrak{X})$, is ker $\partial_k^{\mathfrak{X}}$ and forms a subgroup of $C_k(\mathfrak{X})$. Explicitly,

$$Z_k(\mathfrak{X}) := \ker \partial_k^{\mathfrak{X}} = C_k(\mathfrak{X}) \cap \ker \partial_k \subseteq C_k(\mathfrak{X}).$$

A k-chain $c' \in C_k(\mathfrak{X})$ is called a *boundary* in $\mathfrak{X}$ if there exists $c \in C_{k+1}(\mathfrak{X})$ such that $\partial_{k+1} c = c'$. Thus, the set of boundary elements in $C_k(\mathfrak{X})$, which is denoted by $B_k(\mathfrak{X})$, consists of the image of $\partial_{k+1}^{\mathfrak{X}}$. Since $\partial_{k+1}^{\mathfrak{X}}$ is a homomorphism, $B_K(\mathfrak{X})$ is a subgroup of $C_k(\mathfrak{X})$. Explicitly,

$$B_k(\mathfrak{X}) := \operatorname{im} \partial_{k+1}^{\mathfrak{X}} = \partial_{k+1}(C_{k+1}(\mathfrak{X})) \subseteq C_k(\mathfrak{X}).$$

Recall that since $\partial_k \partial_{k+1} = 0$ (Proposition 2.37, pp.58 of [13]), every boundary is a cycle and thus $B_k(\mathfrak{X})$ is a subgroup of $Z_k(\mathfrak{X})$.

We say that two cycles $c_1, c_2 \in Z_k(\mathfrak{X})$ are *homologous* and we write $c_1 \sim c_2$ if $c_1 - c_2$ is a boundary in $C_k(\mathfrak{X})$, that is, $c_1 - c_2 \in B_k(\mathfrak{X})$. The *equivalence classes* are then the elements of the quotient group $Z_k(\mathfrak{X})/B_k(\mathfrak{X})$.

Definition 18 (Definition 2.42 p. 60 of [13]) The *k-th homology group* is the quotient group

$$\mathbb{H}_k(\mathfrak{X}) := Z_k(\mathfrak{X})/B_k(\mathfrak{X}).$$

The homology of $\mathfrak{X}$ is the collection of all homology groups of $\mathfrak{X}$. The shorthand notation for this is

$$\mathbb{H}(\mathfrak{X}) := \{H_k(\mathfrak{X})\}_{k \in \mathbb{Z}}.$$

Definition 19 (Definition 2.43 p. 60 of [13]) Given $c \in Z_k(\mathfrak{X})$, $[c] \in H_k(\mathfrak{X})$ is the homology class of c in $\mathfrak{X}$.

Definition 20 (Definition 2.50 p. 67 of [13]) A sequence of vertices $V_0, \ldots, V_n \in \mathcal{K}_0(X)$ is an *edge path* in X if there exists edges $E_1, \ldots, E_n \in \mathcal{K}_1(X)$ such that V_{i-1}, V_i are the two faces of E_i for $i = 1, \ldots, n$. For $V, V' \in \mathcal{K}_0(X)$, we write $V \sim_X V'$ if there exists an edge path $V_0, \ldots, V_n \in \mathcal{K}_0(X)$ in X such that $V = V_0$ and $V' = V_n$. We say that X is *edge-connected* if $V \sim_X V'$ for any $V, V' \in \mathcal{K}_0(X)$.

3.3.4 Relative Homology

Now, we recall some background in matter of *relative homology*.

Definition 21 (Definition 9.1 p. 280 of [13]) A pair of cubical sets $\mathfrak{X}$ and A with the property that $A \subseteq \mathfrak{X}$ is called *cubical pair* and is denoted by $(\mathfrak{X}, A)$.

Relative homology is used to compute how two spaces $A, \mathfrak{X}$ such that $A \subseteq \mathfrak{X}$ differ from each other. Intuitively, we want to compute the homology of $\mathfrak{X}$ *modulo* A: we want to ignore the set A and everything connected to it. In other words, we want to work with chains belonging to $C(\mathfrak{X})/C(A)$, which leads to the following definition:

Definition 22 (Definition 9.3 p. 280 of [13]) Let $(\mathfrak{X}, A)$ be a cubical pair. The *relative chains of* $\mathfrak{X}$ *modulo* A are the elements of the quotient groups

$$C_k(\mathfrak{X}, A) := C_k(\mathfrak{X})/C_k(A).$$

The equivalence class of a chain $c \in \mathcal{C}(\mathfrak{X})$ relative to $\mathcal{C}(A)$ is denoted by $[c]_A$. Note that for each k, $C_k(\mathfrak{X}, A)$ is a free Abelian group. The *relative chain complex of* $\mathfrak{X}$ *modulo* A is given by

$$\{C_k(\mathfrak{X}, A), \partial_k^{(\mathfrak{X},A)}\}$$

where $\partial_k^{(\mathfrak{X},A)} : C_k(\mathfrak{X}, A) \to C_{k-1}(\mathfrak{X}, A)$ is defined by

$$\partial_k^{(\mathfrak{X},A)}[c]_A := [\partial^{\mathfrak{X}} c]_A.$$

Obviously, this map satisfies $\partial_{k-1}^{(\mathfrak{X},A)} \partial_k^{(\mathfrak{X},A)} = 0$. The relative chain complex gives rise to the *relative k-cycles*:

$$Z_k(\mathfrak{X}, A) := \ker \partial_k^{(\mathfrak{X},A)},$$

the *relative k-boundaries*

$$B_k(\mathfrak{X}, A) := \operatorname{im} \partial_{k+1}^{(\mathfrak{X},A)},$$

and finally the *relative homology groups*:

$$\mathbb{H}_k(\mathfrak{X}, A) := Z_k(\mathfrak{X}, A)/B_k(\mathfrak{X}, A).$$

Note that for $c \in C_k(\mathfrak{X})$, we can write $[c]_A = c + C_k(A)$ using the coset notation since $[c]_A$ represents the equivalence class whose representative is c.

Proposition 4 (Proposition 9.4 p. 281 of [13]) *Let* $\mathfrak{X}$ *be an (edge-)connected cubical set and let* A *be a non-empty cubical subset of* $\mathfrak{X}$. *Then,*

$$\mathbb{H}_0(\mathfrak{X}, A) = 0.$$

3.3.5 Exact Sequences

Definition 23 (Definition 9.15 p. 289 of [13]) A sequence of groups and homomorphisms

$$\cdots \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to \ldots$$

is said to be *exact* at G_2 when

$$\text{im } \psi_3 = \ker \psi_2.$$

It is an *exact sequence* if it is exact at every group.

Corollary 1 (The exact homology sequence of a pair (Corollary 9.26 p. 297 of [13])) *Let* $(\mathfrak{X}, A)$ *be a cubical pair. Then, there is a long exact sequence:*

$$\cdots \to \mathbb{H}_{k+1}(A) \xrightarrow{\iota_*} \mathbb{H}_{k+1}(\mathfrak{X}) \xrightarrow{\pi_*} \mathbb{H}_{k+1}(\mathfrak{X}, A) \xrightarrow{\partial_*} \mathbb{H}_k(A) \to \ldots$$

where $\iota : \mathcal{C}(A) \hookrightarrow \mathcal{C}(\mathfrak{X})$ *is the inclusion map and* $\pi : \mathcal{C}(\mathfrak{X}) \to \mathcal{C}(\mathfrak{X}, A)$ *is the quotient map.*

3.3.6 The First Isomorphism Theorem

Let us briefly recall the first isomorphism theorem, critical to compute homology groups in the diagrams depicted at the end of the paper.

Theorem 1 *The first isomorphism theorem states that for two groups G and H, with* ϕ *a homomorphism from G to H, then* $G/\ker(\phi) \simeq \text{im }(\phi)$.

3.3.7 Mayer–Vietoris Sequence of a Pair

Theorem 2 (p. 142 of [19]) *A* cubical subset $\mathfrak{X}_0$ *of a cubical set* $\mathfrak{X}$ *is a cubical set which is a subset of* $\mathfrak{X}$*. Let* $\mathfrak{X}$ *be a cubical set; let* $\mathfrak{X}_0, \mathfrak{X}_1$ *be two cubical subsets. of* $\mathfrak{X}$ *such that* $\mathfrak{X} = \mathfrak{X}_0 \cup \mathfrak{X}_1$*. Let* $L = \mathfrak{X}_0 \cap \mathfrak{X}_1$*. Then, there exists an exact sequence:*

$$\cdots \to \mathbb{H}_k(L) \overset{\phi_k}{\to} \mathbb{H}_k(\mathfrak{X}_0) \oplus \mathbb{H}_k(\mathfrak{X}_1) \overset{\psi_k}{\to} \mathbb{H}_k(\mathfrak{X}) \overset{\partial_k}{\to} \mathbb{H}_{k-1}(L) \to \ldots$$

called the Mayer–Vietoris sequence of $(\mathfrak{X}_0, \mathfrak{X}_1)$.

The interested reader can refer to the proof of this theorem in [19] (pp. 142) to get the details about which homomorphisms were used to obtain such a remarkable result.

3.3.8 Manifolds and Local Homology

Definition 24 ([21]) A cubical set $\mathfrak{X}$ is said to be locally a *homological* n-manifold at $x \in \mathfrak{X}$ if the homology groups $\{\mathbb{H}_i(\mathfrak{X}, \mathfrak{X} \setminus \{x\})\}_{i \in \mathbb{Z}}$ satisfy:

$$\mathbb{H}_i(\mathfrak{X}, \mathfrak{X} \setminus \{x\}) = \begin{cases} \mathbb{Z} \text{ when } i = n, \\ 0 \quad \text{otherwise.} \end{cases}$$

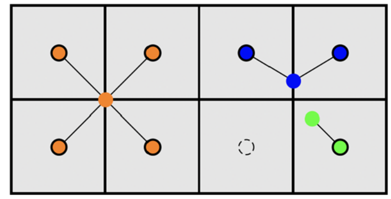

Fig. 6 How to compute $\xi(z)$ (the encircled disks) from a given point z (the not-encircled disks of the same color) (Color figure online)

Then, $\mathfrak{X}$ is said to be a *n-dimensional homological manifold* if it is locally an n-dimensional homological manifold at each point $x \in \mathfrak{X}$.

Theorem 3 ([21]) *A topological manifold is a homological manifold.*

More details about local homology can be found in [15, 22].

4 The Proof that CWCness Implies DWCness in *n*-D

To prove that CWCness implies DWCness in n-D, we proceed by counterposition: we prove that when a digital set contains a primary or secondary critical configuration, then the boundary of its continuous analog is not a homological $(n-1)$-manifold, and then not a topological $(n-1)$-manifold. In the sequel, we will use the notations described in Table 1 and progressively detailed along this section.

4.1 Properties of the Continuous Analog Operator

We define the *round operator* round$(\cdot)$ for any value $v \in \mathbb{R} \setminus \left(\frac{\mathbb{Z}}{2} \setminus \mathbb{Z}\right)$ as $\text{round}(v) = w$ where w is the integer such that $v \in]w - \frac{1}{2}, w + \frac{1}{2}[$.

Notations 1 *From now on, we will write for* $z \in \mathbb{R}^n$ *and for* $\varepsilon > 0$*:*

$$B_\infty(z, \varepsilon) := \{x \in \mathbb{R}^n \; ; \; \|x - z\|_\infty < \varepsilon\}.$$

Notations 2 *Let z be an element of* $\mathbb{R}^n$*. We define (see Fig. 6):*

$$\xi(z) := \left\{q \in \mathbb{Z}^n \; ; \; z \in \text{CA}(q)\right\}$$

Remarkably, $\xi(z)$ *is also the intersection of the closed ball* $B_\infty(z, 1/2)$ *with* $\mathbb{Z}^n$.

For a given $z \in \mathbb{R}^n$, the following notation is an alternative way to determine which points of $\mathbb{Z}^n$ are the centers of the continuous analogs which contain z. Due to its definition based on the Cartesian product, it will be easier to manage it in our n-dimensional proofs.

Table 1 Summary of the main notations of Sect. 4.1

$\mathfrak{X}^c$	$\mathbb{R}^n \setminus \mathfrak{X}$		$\mathfrak{X} \subset \mathbb{R}^n$
X^c	$\mathbb{Z}^n \setminus X$		$X \subset \mathbb{Z}^n$
$\mathrm{CA}(p)$	The continuous analog of p		$p \in \mathbb{Z}^n$
$\mathrm{CA}(X)$	The continuous analog of the set X		$X \subset \mathbb{Z}^n$
$\mathrm{bdCA}(X)$	Boundary of the continuous analog of the set X		$X \subset \mathbb{Z}^n$
$B_\infty(z, \varepsilon)$	$\{x \in \mathbb{R}^n \ ; \ \|x - z\|_\infty < \varepsilon\}$	Notation 1	$z \in \mathbb{R}^n, \varepsilon > 0$
$\xi(z)$	$\{q \in \mathbb{Z}^n \ ; \ z \in \mathrm{CA}(q)\}$	Notation 2	$z \in \mathbb{R}^n$
$\xi^{\mathrm{alt}}(z)$	$\times_{i \in [\![1,n]\!]} \xi_i^{\mathrm{alt}}$	Notation 3	$z \in \mathbb{R}^n$
$\epsilon(v)$	ϵ-operator	Notation 4	$v \in \mathbb{R}$
$\mathrm{CA}_{1D}(v)$	$[v - \frac{1}{2}, v + \frac{1}{2}]$	Notation 5	$v \in \mathbb{R}$
$\mathrm{CA}_{1D}(T)$	$\cup_{v \in T} \mathrm{CA}_{1D}(v)$	Notation 5	$T \subset \mathbb{Z}$

Notations 3 *Let z be an element of $\mathbb{R}^n$. We define:*

$$\xi^{\mathrm{alt}}(z) := \times_{i \in [\![1,n]\!]} \xi_i^{\mathrm{alt}},$$

where for any $i \in [\![1, n]\!]$,

$$\xi_i^{\mathrm{alt}} := \begin{cases} \left\{z_i - \frac{1}{2}, z_i + \frac{1}{2}\right\} & \text{when } z_i \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}, \\ \{\mathrm{round}(z_i)\} & \text{otherwise.} \end{cases}$$

Proposition 5 *For any $z \in \mathbb{R}$, we have the following property:*

$$\xi(z) = \xi^{\mathrm{alt}}(z).$$

Proof Let z be an element of $\mathbb{R}^n$. Let us remark that $q \in \xi(z)$ is equivalent to say that $q \in \mathbb{Z}^n$ such that $z \in \mathrm{CA}(q)$, that is, $\|z - q\|_\infty \leq \frac{1}{2}$.
Now let q be an element of $\xi^{\mathrm{alt}}(z)$. Then, for any $i \in [\![1, n]\!]$, $q_i \in \xi_i^{\mathrm{alt}}$, which implies that we have 3 possible cases:

- when $z_i \notin \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}$, $q_i = \mathrm{round}(z_i) \in \mathbb{Z}$ and then $|z_i - q_i| \leq \frac{1}{2}$,
- when $z_i \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}$ and $q_i = z_i - \frac{1}{2}$, $q_i \in \mathbb{Z}$ and $|q_i - z_i| \leq \frac{1}{2}$,
- when $z_i \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}$ and $q_i = z_i + \frac{1}{2}$, $q_i \in \mathbb{Z}$ and $|q_i - z_i| \leq \frac{1}{2}$,

then $\|q - z\|_\infty \leq \frac{1}{2}$ and $q \in \mathbb{Z}^n$, then $q \in \xi(z)$.
Now let q be an element of $\xi(z)$. Then, $q \in \mathbb{Z}^n$ such that $\|z - q\|_\infty \leq \frac{1}{2}$. Then, for any $i \in [\![1, n]\!]$, $|z_i - q_i| \leq \frac{1}{2}$. The consequence is that for any $i \in [\![1, n]\!]$, $-\frac{1}{2} \leq q_i - z_i \leq \frac{1}{2}$, that is:

$$z_i - \frac{1}{2} \leq q_i \leq z_i + \frac{1}{2}. \tag{1}$$

When $z_i \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}$, we obtain that $q_i \in [\![z_i - \frac{1}{2}, z_i + \frac{1}{2}]\!]$ since $q_i \in \mathbb{Z}$, then $q_i \in \{z_i - \frac{1}{2}, z_i + \frac{1}{2}\}$. When $z_i \notin \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}$, we obtain that there exists a unique q_i that satisfies (1), and this value is $\mathrm{round}(z_i)$, then $q_i \in \{\mathrm{round}(z_i)\}$. The proof is done. □

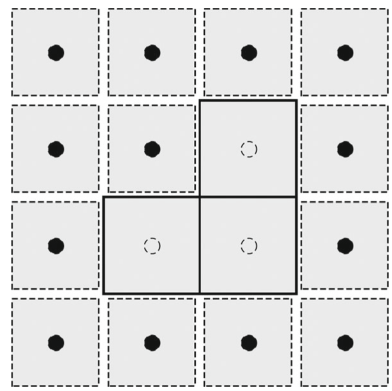

Fig. 7 When the interior of the continuous analog of p intersects the continuous analog of some set X, then p belongs to X

The goal of this section is to prove Theorem 4 (see page 15). As the reader will understand easily, before proving such a theorem, we need to understand how the continuous analog relates the points of $\mathbb{Z}^n$ and the unitary cubes centered at points of $\mathbb{Z}^n$.

Proposition 6 *Let X be a subset of $\mathbb{Z}^n$ and let p be an element of $\mathbb{Z}^n$. Then,*

$$\{\mathrm{Int}(\mathrm{CA}(p)) \cap \mathrm{CA}(X) \neq \emptyset\} \Rightarrow \{p \in X\}.$$

Proof This proposition is depicted in Fig. 7. Let us assume that $z \in \mathrm{Int}(\mathrm{CA}(p)) \cap \mathrm{CA}(X)$. Since $z \in \mathrm{Int}(\mathrm{CA}(p))$, then $\|z - p\|_\infty < \frac{1}{2}$. In addition, since $z \in \mathrm{CA}(X)$, there exists some $q \in X$ such that $\|z - q\|_\infty \leq \frac{1}{2}$. Since $\|q - p\|_\infty = \|q - z + z - p\|_\infty \leq \|q - z\|_\infty + \|z - p\|_\infty < 1$, then $q = p$, and then $p \in X$. □

As we can see in the next proposition, the continuous analog is also strongly related to ξ.

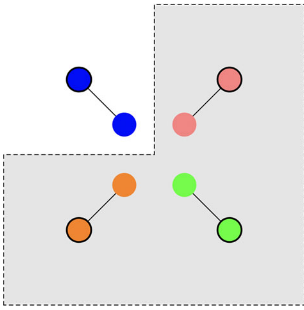

Fig. 8 When a point z (each non-encircled colored disk) belongs to the interior of the continuous analog of some set X (see the gray dashed component), then $\xi(z)$ (depicted by the encircled disks of the same color) is included in X (Color figure online)

Proposition 7 *Let X be a subset of $\mathbb{Z}^n$, and let z be an element of $\mathbb{R}^n$. Then,*

$$z \in \text{Int}(\text{CA}(X)) \Rightarrow \xi(z) \subseteq X.$$

Proof This proposition is depicted in Fig. 8. Let us assume that z belongs to $\text{Int}(\text{CA}(X))$, then there exists some neighborhood V_z of z such that $V_z \subseteq \text{CA}(X)$. Then, there exists some small value $\varepsilon > 0$ such that $B_\infty(z, \varepsilon) \subseteq V_z \subseteq \text{CA}(X)$. Now, two cases are possible:

- either $z \in \mathbb{Z}^n$, then $\xi(z) = \{z\}$, and $z \in \text{Int}(\text{CA}(z))$, thus $z \in \text{Int}(\text{CA}(z)) \cap \text{CA}(X)$, which implies by Proposition 6 that $z \in X$, then $\xi(z) \subseteq X$.
- or $z \notin \mathbb{Z}^n$, then for every $q \in \xi(z)$, there exists a point q_ε defined such as:

  $$q_\varepsilon := z + \frac{\varepsilon}{2}(q - z),$$

  which belongs to $B_\infty(z, \varepsilon) \subseteq \text{CA}(X)$. Also, we can reformulate:

  $$q_\varepsilon := \left(1 - \frac{\varepsilon}{2}\right) z + \frac{\varepsilon}{2} q,$$

  which leads easily to $q_\varepsilon \in \text{Int}(\text{CA}(q))$, thus it satisfies:

  $$q_\varepsilon \in \text{Int}(\text{CA}(q)) \cap \text{CA}(X),$$

  and then $q \in X$ by Proposition 6. We can conclude with $\xi(z) \subseteq X$.

This concludes the proof. □

Since in Theorem 4, we will use the boundary operator used on the continuous analog, we can assume that we will need the following proposition relating the continuous analog of a set and the one of its complementary.

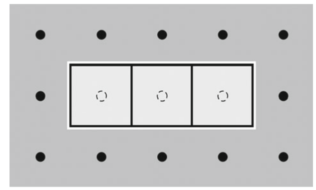

Fig. 9 Let X be the set of three points of $\mathbb{Z}^2$ pictured as dashed circles. The interior of the continuous analog of a set X (in light gray) does not intersect the continuous analog of the complementary of X (in dark gray)

Proposition 8 *Let X be a subset of $\mathbb{Z}^n$. Then,*

$$\text{Int}(\text{CA}(X)) \cap \text{CA}(X^c) = \emptyset.$$

Proof This proposition is depicted in Fig. 9. Let us assume that there exists some $z \in \text{Int}(\text{CA}(X)) \cap \text{CA}(X^c)$. Because $z \in \text{Int}(\text{CA}(X))$, by Proposition 7, the set $\xi(z)$ satisfies $\xi(z) \subseteq X$. Let us denote by $\#(.)$ the cardinality operator. Then,

- either $\#(\xi(z)) = 1$, and we are in the case where there exists a unique $p \in X$ such that $\|p - z\|_\infty \leq \frac{1}{2}$, then for all $q \in \mathbb{Z}^n \setminus \{p\}$ (containing X^c), $\|q - z\|_\infty > \frac{1}{2}$, and then $z \notin \text{CA}(X^c)$: we obtain a contradiction.
- or $\#(\xi(z)) \geq 2$, then for all $p \in \xi(z)$, $\|p - z\|_\infty = \frac{1}{2}$, when for every $q \in \mathbb{Z}^n \setminus \xi(z)$, $\|q - z\|_\infty > \frac{1}{2}$. Because $\xi(z) \subseteq X$, $q \subseteq \mathbb{Z}^n \setminus \xi(z)$, and then for any $q \in X^c$, $\|q - z\|_\infty > \frac{1}{2}$. This way, $z \notin \text{CA}(X^c)$; one more time, we obtain a contradiction.

The proof is done. □

Now let us recall and prove an elementary property of the continuous analog relative to the continuous analog of the complementary, it will be used in the next proposition.

Proposition 9 *Let X be a subset of $\mathbb{Z}^n$, then:*

$$\text{Int}(\text{CA}(X^c)) = (\text{CA}(X))^c.$$

Proof Let z be an element of $\text{Int}(\text{CA}(X))$. Then, there exists some neighborhood V_z of z which is included in $\text{CA}(X)$. Then, $V_z \cap (\text{CA}(X))^c = \emptyset$. Let us assume that:

$$z \in \text{CA}(X^c) \tag{2}$$

then there exists $y \in X^c$ such that $\|z - y\|_\infty \leq \frac{1}{2}$. Because $\text{CA}(y)$ is closed, then $V_z \cap \text{CA}(y) \neq \emptyset$. However by (2),

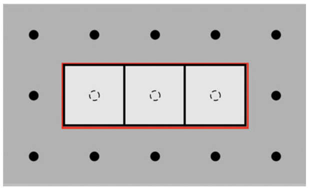

Fig. 10 The boundary of the continuous analog of a digital set X (see the red closed curve between the set of the dark points and the set of the dashed points) can be computed as the intersection of the continuous analog of X and the one of its complementary in $\mathbb{Z}^n$. [This picture is better viewed in color.] (Color figure online)

$V_z \subset \mathrm{CA}(X)$, which is equivalent to $V_z \subseteq \mathrm{Int}(CA(X))$, and by Proposition 8,

$$\mathrm{Int}(\mathrm{CA}(X)) \cap \mathrm{CA}(X^c) = \emptyset,$$

then $V_z \cap \mathrm{CA}(X^c) = \emptyset$. We obtain a contradiction. Then, (2) is false, that is, $z \in (\mathrm{CA}(X^c))^c$.
Let us prove the converse inclusion. Let z be an element of $(\mathrm{CA}(X))^c$. Since $\mathrm{CA}(X)$ is closed, $(\mathrm{CA}(X))^c$ is open, and then there exists an open neighborhood V_z of z such that $V_z \subseteq (\mathrm{CA}(X))^c$. It means that V_z is included into $\mathrm{CA}(X^c)$ since $\mathrm{CA}(X) \cup \mathrm{CA}(X^c) = \mathbb{R}^n$. However, the fact that V_z is included in $\mathrm{CA}(X^c)$ means that z belong to $\mathrm{Int}(\mathrm{CA}(X^c))$. The proof is done. □

As said before, we need properties relative to the boundary of the continuous, analog; here we are going to show that we can reformulate its topological boundary as an intersection of two continuous analogs, which will make the proofs easier in the sequel.

Proposition 10 *Let us define the non-empty digital strict subset X of $\mathbb{Z}^n$, then, the topological boundary* $\mathrm{bdCA}(X) := \mathrm{CA}(X) \setminus \mathrm{Int}(CA(X))$ *of the continuous analog of X is equal to:*

$$\mathrm{CA}(X) \cap \mathrm{CA}(X^c).$$

Proof This proposition is depicted in Fig. 10.
Let us assume that $X \neq \emptyset \neq X^c$. Let us prove the double inclusion. Let z be an element of $\mathrm{bdCA}(X)$. Then, $z \in \mathrm{CA}(X)$, and $z \notin \mathrm{Int}(CA(X))$. This last property means that for any neighborhood V_z of z, $V_z \cap \mathrm{CA}(X)^c \neq \emptyset$. However, $\mathrm{CA}(X) \cup \mathrm{CA}(X^c) = \mathbb{R}^n$, then $\mathrm{CA}(X)^c \subseteq \mathrm{CA}(X^c)$, and then $V_z \cap \mathrm{CA}(X^c) \neq \emptyset$. Since $\mathrm{CA}(X^c)$ is closed in $\mathbb{R}^n$ and since any neighborhood of z intersects $CA(X^c)$, z belongs to $\mathrm{CA}(X^c)$. Then, we have proven that $\mathrm{bdCA}(X) \subseteq \mathrm{CA}(X) \cap \mathrm{CA}(X^c)$.
Let us now prove the converse inclusion. Let z be an element of $\mathrm{CA}(X) \cap \mathrm{CA}(X^c)$. By hypothesis, $z \in \mathrm{CA}(X)$. Since $z \in \mathrm{CA}(X^c)$, then $z \notin \mathrm{Int}(\mathrm{CA}(X))$ by Proposition 9. The proof is done. □

In the sequel, we are going to show that the center of a critical configuration belongs to the boundary of the continuous analog of this critical configuration, but before we have to prove this elementary property.

Proposition 11 *The center m of a block S in $\mathbb{Z}^n$ satisfies the following relation:*

$$\forall p \in S,\ m \in \mathrm{CA}(p).$$

Proof Let S be a block which can be written $S(q, \mathcal{F})$ with $\mathcal{F} = \cup_{i \in \mathcal{I}} \{e^i\}$ and with $\mathcal{I} \subseteq [\![1, n]\!]$. Then, by definition, any $p \in S$ can be written as:

$$p := q + \sum_{i \in \mathcal{I}} \lambda_i\, e^i,$$

with $\lambda_i \in \{0, 1\}$. Then, the value $\|p - m\|_\infty$ is equal to $\max_{i \in [\![1,n]\!]} |p_i - m_i|$. When $i \in [\![1, n]\!]$ does not belong to $\mathcal{I}$, then $m_i = p_i$. Then, $\|p - m\|_\infty$ is equal to $\max_{i \in \mathcal{I}} |p_i - m_i|$. When i belongs to $\mathcal{I}$, we have two possible cases: either $\lambda_i = 0$ and $|p_i - m_i| = |q_i - (q_i + \frac{1}{2})| = \frac{1}{2}$, or $\lambda_i = 1$ and $|p_i - m_i| = |(q_i + 1) - (q_i + \frac{1}{2})| = \frac{1}{2}$. The conclusion is that when $\mathcal{I}$ is empty, that is, when S is a block of one point, we have that $p \in S$ is equal to m and then $m \in \mathrm{CA}(p)$, and that, when $\dim(S) \geq 1$, for any $p \in S$, $\|p - m\|_\infty = \frac{1}{2}$ and then we obtain one more time that $m \in \mathrm{CA}(p)$. □

Now, we can assert and prove the property that the center of a critical configuration belongs to the boundary of the continuous analog of this critical configuration; the aim being to show in the sequel that we can easily find elements of the boundary of the continuous analogs where the continuous analog of a not DWC set is not a homological manifold.

Lemma 1 *Let X be a digital subset of $\mathbb{Z}^n$. When X contains a critical configuration in the block S, then the center m of S belongs to* $\mathrm{bdCA}(X)$.

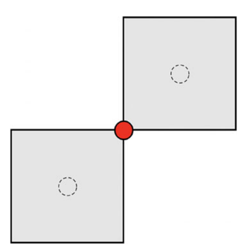

Fig. 11 A digital set X containing a critical configuration in some block S (see the dashed circles in the squares), then the center m (in red at the center of the figure) belongs to the boundary of the continuous analog of X. [This picture is better viewed in color.] (Color figure online)

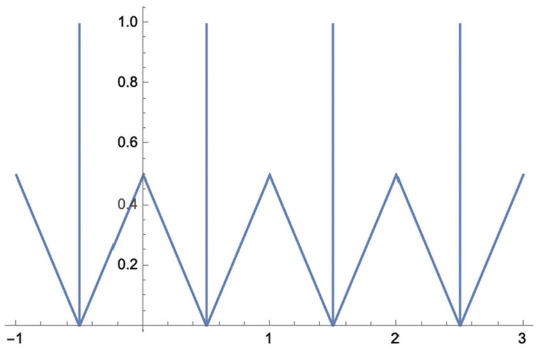

Fig. 12 The graph of the mapping $v \to \epsilon(v)$

Proof This lemma is depicted in Fig. 11. When X contains a critical configuration, there exists some block S of dimension $k \geq 2$ such that $X \cap S = \{p, p'\}$ (or such that $S \setminus X = \{p, p'\}$) with $p' = \text{antag}_S(p)$. Let $q \in S$ be a $2n$-neighbor of p (then $q \neq p'$ since they are $(k-1)$-antagonists). Now let m be the center of S. Two cases are possible: in the primary case, $p \in X$, and then $q \in X^c$. By Proposition 11, m belongs then to $\text{CA}(p) \cap \text{CA}(q) \subseteq \text{CA}(X) \cap \text{CA}(X^c)$, and then m belongs to $\text{bdCA}(X)$ by Proposition 10. The secondary case follows a similar reasoning. □

The following notation represents the maximal radius of the open ball which fits in $\text{Int}(\text{CA}(\xi(z)))$ (1D and n-D cases).

Notations 4 *Let us define the operator* $\epsilon : \mathbb{R} \to [0, 1]$ *such that for any* $v \in \mathbb{R}$*:*

$$\epsilon(v) := \begin{cases} 1 & \text{when } v \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}, & \text{(I)} \\ \frac{1}{2} & \text{when } v \in \mathbb{Z}, & \text{(II)} \\ v - \left(\lceil v \rceil - \frac{1}{2}\right) & \text{when } \lceil v \rceil - v < v - \lfloor v \rfloor, & \text{(III)} \\ \lfloor v \rfloor + \frac{1}{2} - v & \text{when } \lceil v \rceil - v > v - \lfloor v \rfloor, & \text{(IV)} \end{cases}$$

(see Fig. 12). When we are in cases (II), (III) *or* (IV), *we obtain:*

$$\emptyset \neq \left]v - \epsilon(v), v + \epsilon(v)\right[\subseteq \left]\text{round}(v) - \frac{1}{2}, \text{round}(v) + \frac{1}{2}\right[.$$

Based on ϵ *defined for real values, we define by extension:*

$$\forall z \in \mathbb{R}^n, \; \epsilon(z) := \min_{i \in [\![1,n]\!]} \epsilon(z_i).$$

To be able to switch between the 1D and the n-D continuous analogs (needed in Proposition 16 seen at page 13), we introduce a simplified definition of the 1D version here.

Notations 5 *For* $v \in \mathbb{R}$, *let us denote by* $\text{CA}_{1D}(v) := [v - \frac{1}{2}, v + \frac{1}{2}]$. *For any* $p \in \mathbb{Z}^n$, *we have that*

$$\times_{i \in [\![1,n]\!]} \text{CA}_{1D}(p_i) = \text{CA}(p).$$

Now, for $R \subseteq \mathbb{Z}$, *let us denote:*

$$\text{CA}_{1D}(R) := \cup_{v \in R} \text{CA}_{1D}(v).$$

Property 1 *For any family* $\{E_i\}_{i \in [\![1,n]\!]}$ *of subsets of* $\mathbb{Z}$, *we have the following property:*

$$\times_{i \in [\![1,n]\!]} \text{CA}_{1D}(E_i) = \text{CA}(\times_{i \in [\![1,n]\!]} E_i).$$

Proof Let us prove the case $n = 2$:

$$\begin{aligned} &\text{CA}_{1D}(E_1) \times \text{CA}_{1D}(E_2) \\ &= \left(\bigcup_{p_1 \in E_1} \text{CA}_{1D}(p_1)\right) \times \left(\bigcup_{p_2 \in E_2} \text{CA}_{1D}(p_2)\right), \\ &= \bigcup_{p_1 \in E_1} \bigcup_{p_2 \in E_2} \text{CA}_{1D}(p_1) \times \text{CA}_{1D}(p_2), \\ &= \bigcup_{p_1 \in E_1} \bigcup_{p_2 \in E_2} \text{CA}((p_1, p_2)), \\ &= \bigcup_{p \in E_1 \times E_2} \text{CA}(p), \\ &= \text{CA}(E_1 \times E_2). \end{aligned}$$

The case $n \geq 2$, n finite, follows the same reasoning. □

In the following proposition, we show that a little open ball centered at a given $z \in \mathbb{R}^n$ is included in the continuous analog of $\xi(z)$, which shows how $\epsilon(z)$ and $\xi(z)$ are related.

Proposition 12 *For any* $z \in \mathbb{R}^n$, *we have the following property:*

$$B_\infty(z, \epsilon(z)) \subseteq \text{Int}\left(\text{CA}(\xi(z))\right).$$

Proof The intuition of this proof is depicted in Fig. 13. Let us define $\mathcal{I}_{\frac{1}{2}}(z) := \{i \in [\![1,n]\!] \; z_i \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}\}$. Now let us observe that:

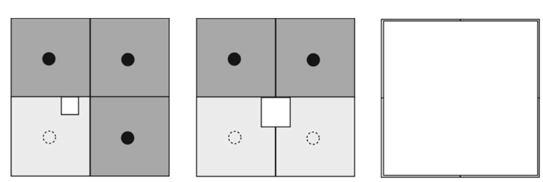

Fig. 13 The set $B_\infty(z, \epsilon(z))$ (see each white square) is always included in $\text{Int}\left(\text{CA}(\xi(z))\right)$ (see the union of the squares in light gray centered at the elements of $\xi(z)$). Furthermore, the intersection of $\text{CA}(\mathbb{Z}^n \setminus \xi(z))$ and $B(z, \epsilon(z))$ is equal to the empty set

$$B_\infty(z,\epsilon(z)) = \times_{i\in[\![1,n]\!]} B_\infty(z_i,\epsilon(z)),$$
$$\subseteq \times_{i\in[\![1,n]\!]} B_\infty(z_i,\epsilon(z_i)),$$
$$\subseteq \times_{i\in[\![1,n]\!]} \,]z_i-\epsilon(z_i), z_i+\epsilon(z_i)[\,.$$

Now, let i be an element of $\mathcal{I}_{\frac{1}{2}}(z)$, then $\xi(z_i)=\{z_i-\frac{1}{2}, z_i+\frac{1}{2}\}$, which implies that $\text{Int}(\text{CA}_{1D}(\xi(z_i))=]z_i-1, z_i+1[$, and then:

$$]z_i-\epsilon(z_i), z_i+\epsilon(z_i)[\subseteq \text{Int}(\text{CA}_{1D}(\xi(z_i)).$$

Besides, when i is an element of $[\![1,n]\!]\setminus\mathcal{I}_{\frac{1}{2}}(z)$, $\xi(z_i)=\{\text{round}(z_i)\}$, then

$$\text{Int}(\text{CA}_{1D}(\xi(z_i)))=]\text{round}(z_i)-\frac{1}{2}, \text{round}(z_i)+\frac{1}{2}[,$$

and since we are in cases (II), (III) or (IV), we obtain:

$$]z_i-\epsilon(z_i), z_i+\epsilon(z_i)[\subseteq \text{Int}(\text{CA}_{1D}(\xi(z_i))).$$

Finally,

$$B_\infty(z,\epsilon(z)) \subseteq \times_{i\in[\![1,n]\!]} \,]z_i-\epsilon(z_i), z_i+\epsilon(z_i)[\,,$$
$$\subseteq \times_{i\in[\![1,n]\!]} \text{Int}(\text{CA}_{1D}(\xi(z_i))),$$
$$\subseteq \text{Int}(\times_{i\in[\![1,n]\!]} \text{CA}_{1D}(\xi(z_i))),$$
$$\subseteq \text{Int}(\text{CA}(\times_{i\in[\![1,n]\!]} \xi(z_i))),$$
$$\subseteq \text{Int}(\text{CA}(\xi(z))).$$

This concludes the proof. □

In the following proposition, we show the complementary of the previous proposition.

Proposition 13 *For any $z\in\mathbb{R}^n$,*

$$\text{CA}(\mathbb{Z}^n\setminus\xi(z))\cap B_\infty(z,\epsilon(z))=\emptyset.$$

Proof The intuition of this proposition is depicted in Fig. 13. By Proposition 12,

$$B_\infty(z,\epsilon(z))\subseteq \text{Int}(\text{CA}(\xi(z))),$$

then $B_\infty(z,\epsilon(z))\cap(\text{Int}(\text{CA}(\xi(z))))^c=\emptyset$, so by Proposition 9, $B_\infty(z,\epsilon(z))\cap\text{CA}(\mathbb{Z}^n\setminus\xi(z))=\emptyset$. □

The following proposition is very important, since it is the first to show that we can restrict the set X to $X\cap\xi(z)$ when we compute its intersection with the neighborhood of $B_\infty(z,\epsilon(z))$. This is one of the keys of Theorem 4.

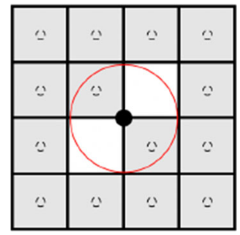

Fig. 14 At z (depicted by a small black disk), only the topology of $\text{Int}(\text{CA}(X\cap\xi(z)))\cap B_\infty(z,\epsilon(z))$ matters when we look at $\text{Int}(\text{CA}(X))$. The same reasoning applies for the continuous analog and for its boundary. [This picture is better viewed in color.] (Color figure online)

Proposition 14 *For any $z\in\mathbb{R}^n$ and for any $X\subset\mathbb{Z}^n$,*

$$\text{Int}(\text{CA}(X))\cap B_\infty(z,\epsilon(z))$$
$$=\text{Int}(\text{CA}(X\cap\xi(z)))\cap B_\infty(z,\epsilon(z)).$$

Proof The intuition of this proof is depicted in Fig. 14. The converse inclusion is immediate. Now, for the direct inclusion, let us assume that x belongs to $\text{Int}(\text{CA}(X))\cap B_\infty(z,\epsilon(z))$. This is equivalent to say that there exists some neighborhood V_x of x which is included in $\text{CA}(X)$ and in $B_\infty(z,\epsilon(z))$. However, $V_x\subseteq B_\infty(z,\epsilon(z))$. Since $V_x\subseteq \text{CA}(X)$,

$$V_x\subseteq\text{CA}(X)\cap B_\infty(z,\epsilon(z)),$$

which is included in:

$$\text{CA}(X\cap\xi(z))\cap B_\infty(z,\epsilon(z))$$
$$\bigcup \text{CA}(X\setminus\xi(z))\cap B_\infty(z,\epsilon(z)),$$

where the second term is included in $\text{CA}(\mathbb{Z}^n\setminus\xi(z))\cap B_\infty(z,\epsilon(z))$ which is equal to the empty set by Proposition 13. Then, $V_x\subseteq\text{CA}(X\cap\xi(z))\cap B_\infty(z,\epsilon(z))$, which means that $x\in\text{Int}(\text{CA}(X\cap\xi(z))\cap B_\infty(z,\epsilon(z)))$, which is equal to $\text{Int}(\text{CA}(X\cap\xi(z)))\cap B_\infty(z,\epsilon(z))$. This concludes the proof. □

The following proposition is the complementary part of the previous proposition, since it concerns the continuous analog and not its interior.

Proposition 15 *Let X be a digital subset of $\mathbb{Z}^n$ and let z be an element of $\mathbb{R}^n$. Then,*

$$\text{CA}(X)\cap B_\infty(z,\epsilon(z))=\text{CA}(X\cap\xi(z))\cap B_\infty(z,\epsilon(z)).$$

Proof The intuition of this proof is depicted in Fig. 14. For any digital set $X\subset\mathbb{Z}^n$ and for any $z\in\mathbb{R}^n$, we have:

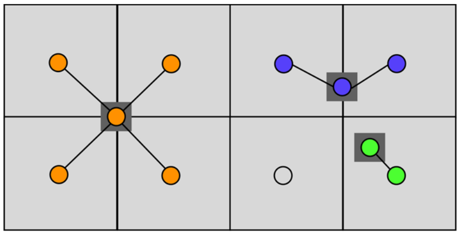

Fig. 15 Points z are depicted by colored disks surrounded by a dark gray solid rectangle corresponding to the ball $B_\infty(z, \varepsilon)$. The balls intersect always the interior of the continuous analog $CA(p)$ of each point p belonging to $\xi(z)$ (depicted by encircled disks of the same color as z) (Color figure online)

$$\begin{aligned}\mathrm{CA}(X) \cap B_\infty(z, \epsilon(z)) &= \bigcup_{p \in X} \mathrm{CA}(p) \cap B_\infty(z, \epsilon(z)), \\ &= \left(\bigcup_{p \in X \cap \xi(z)} \mathrm{CA}(p) \cap B_\infty(z, \epsilon(z)) \right) \\ &\bigcup \left(\bigcup_{p' \in X \setminus \xi(z)} \mathrm{CA}(p') \cap B_\infty(z, \epsilon(z)) \right).\end{aligned}$$

However we can remark that the second term in the union is included in $\bigcup_{p' \in \mathbb{Z}^n \setminus \xi(z)} \mathrm{CA}(p') \cap B_\infty(z, \epsilon(z))$ by Proposition 13, which is equal to the empty set. This concludes the proof. □

Grouping together the two previous propositions, we can assert in the following lemma that shows that even for the boundary of the continuous analog, only the set $X \cap \xi(z)$ counts.

Lemma 2 *Let X be a digital subset of $\mathbb{Z}^n$ and let z be an element of $\mathbb{R}^n$. Then,*

$$\mathrm{bdCA}(X) \cap B_\infty(z, \epsilon(z)) = \mathrm{bdCA}(X \cap \xi(z)) \cap B_\infty(z, \epsilon(z)).$$

In other words, the boundary of X in the neighborhood of z depends only on $X \cap \xi(z)$.

Proof It follows directly from Propositions 14 and 15. The intuition of this proof is depicted in Fig. 14. □

4.2 Properties of antagonists and blocks

There comes an additional property of the neighborhood of $z \in \mathbb{R}^n$ relatively to the interior of the continuous analog of $\xi(z)$. This proposition will be used in Lemma 3 at page 14 to show that we have some remarkable properties when we use the continuous analog on antagonists.

Proposition 16 *For any $z \in \mathbb{R}^n$, any $\varepsilon > 0$, and any $p \in \xi(z)$,*

$$B_\infty(z, \varepsilon) \cap \mathrm{Int}(\mathrm{CA}(p)) \neq \emptyset. \tag{3}$$

Proof The intuition of this proposition is depicted in Fig. 15. Let ε be a real value greater than $\frac{1}{2}$. In this case, $B_\infty(z, \varepsilon)$ contains $\xi(z)$ thus for any $p \in \xi(z)$, (3) is true.
Now, let us assume $\varepsilon \in \left]0, \frac{1}{2}\right]$. Then, for any coordinate $i \in [\![1, n]\!]$, we have two possibilities. When z_i belongs to $(\left(\frac{\mathbb{Z}}{2}\right)^n \setminus \mathbb{Z}^n)$: $\xi(z_i) = \{z_i - \frac{1}{2}, z_i + \frac{1}{2}\}$; otherwise, $\xi(z_i) = \mathrm{round}(z_i)$. In both cases, for any $p_i \in \xi(z_i)$:

$$B_\infty(z_i, \varepsilon) \cap \mathrm{Int}(\mathrm{CA}_{1D}(p_i)) \neq \emptyset.$$

Using this property, we obtain that for any $p \in \xi(z)$ and for any $i \in [\![1, n]\!]$, $p_i \in \xi(z_i)$, thus:

$$]z_i - \varepsilon, z_i + \varepsilon[\;\cap\; \left]p_i - \frac{1}{2}, p_i + \frac{1}{2}\right[\neq \emptyset,$$

which means that by using the n-D Cartesian product, we obtain that (3) is true. This concludes the proof. □

The intuition of the following proposition is the following: a block is defined using its lexicographically lowest vertex (with all coordinates minimal), when ξ is defined with respect to its center (using the alternative definition of $\xi(z)$ as the intersection of $\mathbb{Z}^n$ with the ball centered at z and of radius $\frac{1}{2}$). One translation transforms the block to ξ. It is the same translation that transforms elementary cubes to closed unit cubes.

Proposition 17 *Let m be an element of $\left(\frac{\mathbb{Z}}{2}\right)^n$, and let S be the block centered at m. Then,*

$$S = \xi(m).$$

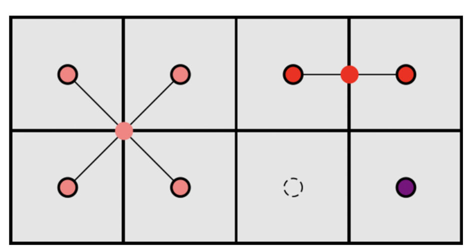

Fig. 16 For m a center of some block S, we can compute this same block just by applying the operator ξ to m: the cardinality of $\xi(m)$ is equal to 4 (on the left side), 2 (on the right top side) and 1 (on the right down side) in the pink, red and purple cases, respectively. [This picture is better viewed in color.] (Color figure online)

Proof This proposition is depicted in Fig. 16. Let m be an element of $\left(\frac{\mathbb{Z}}{2}\right)^n$. Let $(q, \mathcal{F}) \in \mathbb{Z}^n \times \mathbb{B}$ such that $S = S(q, \mathcal{F})$, we can write $\mathcal{F} = \{f^i\}_{i \in [\![1,k]\!]}$ where $k := \dim(S)$. By definition of m, we have $m = q + \sum_{i \in [\![1,k]\!]} \frac{f^i}{2}$.
For any $p \in S$, there exist $(\lambda_i)_{i \in [\![1,k]\!]} \in \{0, 1\}^k$ such that $p = q + \sum_{i \in [\![1,k]\!]} \lambda_i f^i$. Then,

$$\|p - m\|_\infty = \left\| \sum_{i \in [\![1,k]\!]} \lambda'_i f^i \right\|_\infty$$

where $(\lambda'_i)_{i \in [\![1,k]\!]} \in \{-\frac{1}{2}, \frac{1}{2}\}^k$, which implies that $\|p - m\|_\infty \leq \frac{1}{2}$, and then $m \in \mathrm{CA}(p)$, leading to $p \in \xi(m)$.
Conversely, for any $p \in \xi(m)$, $p \in \mathbb{Z}^n$ and $\|p - m\|_\infty \leq \frac{1}{2}$, which means that for any $i \in [\![1, n]\!]$, $|p_i - m_i| \leq \frac{1}{2}$, or equivalently:

$$m_i - \frac{1}{2} \leq p_i \leq m_i + \frac{1}{2}. \tag{4}$$

When $m_i \in \mathbb{Z}$, (4) is equivalent to $p_i = m_i$ since $p_i \in \mathbb{Z}$, and when $m_i \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}$, (4) is equivalent to $p_i \in \{m_i - \frac{1}{2}, p_i + \frac{1}{2}\}$; we call this property (R_1). Let us define $\mathcal{I} := \{i \in [\![1, n]\!]\ ;\ m_i \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}\}$, then $\mathcal{F} = \{e^i\}_{i \in \mathcal{I}}$. Also we can remark that for any $i \in [\![1, n]\!]$,

$$q_i = \begin{cases} m_i & \text{if } m_i \in \mathbb{Z}, \\ m_i - \frac{1}{2} & \text{otherwise.} \end{cases}$$

Then, we can rewrite S in the following manner:

$$\begin{aligned} S &= S(q, \mathcal{F}), \\ &= \{q + \textstyle\sum_{i \in \mathcal{I}} \lambda_i e^i\ ;\ \lambda_i \in \{0, 1\}, \forall i \in \mathcal{I}\}, \\ &= \{m + \textstyle\sum_{i \in \mathcal{I}} \lambda'_i e^i\ ;\ \lambda'_i \in \{-\frac{1}{2}, \frac{1}{2}\}, \forall i \in \mathcal{I}\}. \end{aligned}$$

Besides, by (R_1), p can be rewritten as $m + \sum_{i \in \mathcal{I}} \lambda'_i e^i$ with $\lambda'_i \in \{-\frac{1}{2}, \frac{1}{2}\}$ for each $i \in \mathcal{I}$, then $p \in S$. □

The next lemma shows that when we use k-antagonists, with $k \geq 2$, the interior and the union operators commute.

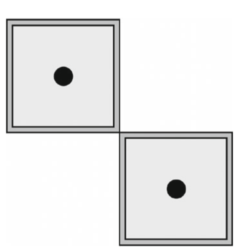

Fig. 17 When p and p' are 2-antagonists (see the black disks), the interior of the union of the continuous analog of $\{p, p'\}$ is equal to the union of the interiors of the continuous analogs of p and p'. Furthermore, the intersection of the two interiors is equal to the empty set

This assertion will be used in Lemma 4, showing that bdCA and the union operators also commute in this same configuration.

Lemma 3 *Let p, p' be two k-antagonists in a block S, with $k \geq 2$. Then, we have the following relation:*

$$\mathrm{Int}(\mathrm{CA}(p)) \cup \mathrm{Int}(\mathrm{CA}(p')) = \mathrm{Int}(\mathrm{CA}(\{p, p'\})).$$

Proof The intuition of this lemma is depicted in Fig. 17. The fact that

$$\mathrm{Int}(\mathrm{CA}(p)) \cup \mathrm{Int}(\mathrm{CA}(p')) \subseteq \mathrm{Int}(\mathrm{CA}(\{p, p'\}))$$

is obvious since for any two subsets A, B of a topological space, $\mathrm{Int}(A) \cup \mathrm{Int}(B) \subseteq \mathrm{Int}(A \cup B)$.
Now let us prove that if p does not belong to $\mathrm{Int}(\mathrm{CA}(p)) \cup \mathrm{Int}(\mathrm{CA}(p'))$, then p does not belong to $\mathrm{Int}(\mathrm{CA}(\{p, p'\}))$. Obviously, when p does not belong to $\mathrm{CA}(p) \cup \mathrm{CA}(p')$, then p cannot belong to $\mathrm{Int}(\mathrm{CA}(\{p, p'\}))$. Then, let us prove that if p belongs to:

$$\begin{aligned} &\mathrm{CA}(p) \cup \mathrm{CA}(p') \setminus (\mathrm{Int}(\mathrm{CA}(p)) \cup \mathrm{Int}(\mathrm{CA}(p'))) \\ &\quad \subseteq \mathrm{bdCA}(p) \cup \mathrm{bdCA}(p'), \end{aligned}$$

then it does not belong to $\mathrm{Int}(\mathrm{CA}(\{p, p'\}))$.
Let m be the center of S, then by Proposition 17, $S = \xi(m)$. Then,

$$\mathrm{CA}(p) \cup \mathrm{CA}(p') = \mathrm{CA}(\{p, p'\}) \subseteq \mathrm{CA}(S) = \mathrm{CA}(\xi(m)).$$

It means that two cases are possible when $z \in \mathrm{CA}(\xi(m))$:

- Either $z \in \mathrm{Int}(\mathrm{CA}(\xi(m)))$, then by Proposition 16, for any $\varepsilon > 0$, and for any $q \in \xi(z)$, $B_\infty(z, \varepsilon) \cap \mathrm{CA}(q) \neq \emptyset$. Then, the smallest set $E \subseteq \mathbb{Z}^n$ verifying that $B_\infty(z, \varepsilon) \subseteq \mathrm{CA}(E)$ contains $\xi(z)$. In other words, if a set $F \subseteq \mathbb{Z}^n$ does not contain $\xi(z)$, then $B_\infty(z, \varepsilon) \nsubseteq \mathrm{CA}(F)$. Two subcases are then possible:
  - If $z = m$, then by Proposition 17, $\xi(z) = \xi(m) = S$. Then, for $F := \{p, p'\} \subset \mathbb{Z}^n$, $F \not\supseteq \xi(z) = S$ because $\dim(S) \geq 2$, and then $B_\infty(z, \varepsilon) \nsubseteq \mathrm{CA}(F)$, and finally $z \notin \mathrm{Int}(\mathrm{CA}(\{p, p'\}))$.
  - If $z \neq m$, then $\xi(z)$ is a ℓ-D block with $\ell \in [\![1, k-1]\!]$. This way, $F := \{p, p'\} \subset \mathbb{Z}^n$ does not contain $\xi(z)$. Indeed, if F contains $\xi(z)$, then $F = \xi(z)$ (since $\xi(z)$ contains at least two points and F contains exactly two points), which implies that $\xi(z)$ is a 1D block made of two $2n$-neighbors. It would imply that p and p' are $2n$-neighbors, which is impossible since $k \geq 2$. Then, F does not contain $\xi(z)$, and

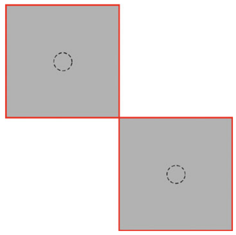

Fig. 18 When p and p' are 2-antagonists, the boundary (in red all around the two squares) of the continuous analog of $\{p, p'\}$ is equal to the union of the boundaries of the continuous analogs of p and p'. [This picture is better viewed in color.] (Color figure online)

then $B_\infty(z, \varepsilon) \nsubseteq \mathrm{CA}(F) = \mathrm{CA}(\{p, p'\})$ and then $z \notin \mathrm{Int}(\mathrm{CA}(\{p, p'\}))$.

- Or $z \in \mathrm{bdCA}(\xi(m))$. Then, let us assume that z belongs to $\mathrm{Int}(\mathrm{CA}(\{p, p'\}))$. Then, there exists a neighborhood V_z of z such that $V_z \subseteq \mathrm{CA}(\{p, p'\})$. However, $\xi(m) = S \supseteq \{p, p'\}$ and then $\mathrm{CA}(\{p, p'\}) \subseteq \mathrm{CA}(\xi(m))$, then $V_z \subseteq \mathrm{CA}(\xi(m))$, then z belongs to $\mathrm{Int}(\mathrm{CA}(\xi(m)))$, which leads to a contradiction. Then, $z \notin \mathrm{Int}(\mathrm{CA}(\{p, p'\}))$.

The proof is done. □

The following results show one property of the continuous analog when we use antagonists.

Proposition 18 *Let p and p' be two k-antagonists in a block of $\mathbb{Z}^n$ with $k \geq 1$. Then,*

$$\mathrm{Int}(\mathrm{CA}(p)) \cap \mathrm{CA}(p') = \emptyset = \mathrm{Int}(\mathrm{CA}(p')) \cap \mathrm{CA}(p).$$

Proof The intuition of this proof is depicted in Fig. 17. Let us prove that $\mathrm{Int}(\mathrm{CA}(p)) \cap \mathrm{CA}(p') = \emptyset$: $\mathrm{Int}(\mathrm{CA}(p)) = \{z \in \mathbb{R}^n \ ; \ \|z - p\|_\infty < \frac{1}{2}\}$, and $\mathrm{CA}(p') = \{z \in \mathbb{R}^n \ ; \ \|z - p'\|_\infty \leq \frac{1}{2}\}$. If the intersection $\mathrm{Int}(\mathrm{CA}(p)) \cap \mathrm{CA}(p')$ is not empty, there exists an element $z \in \mathrm{Int}(\mathrm{CA}(p)) \cap \mathrm{CA}(p')$ and then:

$$\|p - p'\|_\infty \leq \|p - z\|_\infty + \|z - p'\|_\infty < 1,$$

which is impossible because $k \geq 1$. □

The following lemma is the second key of Theorem 4 (operators bdCA and union commute when we use k-antagonists with $k \geq 2$).

Lemma 4 *Let p and p' be two k-antagonists in a block of $\mathbb{Z}^n$ with $k \geq 2$. Then, we have:*

$$\mathrm{bdCA}(\{p, p'\}) = \mathrm{bdCA}(p) \cup \mathrm{bdCA}(p').$$

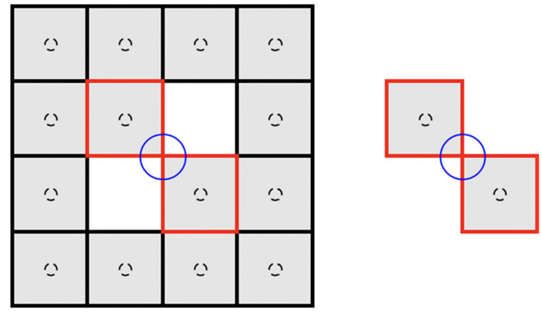

Fig. 19 When X contains a critical configuration $\{p, p'\}$ of center m, the boundary of $\mathrm{CA}(X)$ behaves like the union of the boundaries of the continuous analogs of p and p' in the neighborhood of m (see the part of the red self-crossing curve included in the blue circle). [This picture is better viewed in color.] (Color figure online)

Proof The intuition of this proof is depicted in Fig. 18. The term $\mathrm{bdCA}(p) \cup \mathrm{bdCA}(p')$ is equal to:

$$\mathrm{CA}(p) \setminus \mathrm{Int}(\mathrm{CA}(p)) \cup \mathrm{CA}(p') \setminus \mathrm{Int}(\mathrm{CA}(p')),$$

since $\mathrm{CA}(p)$ and $\mathrm{CA}(p')$ are closed sets. By Proposition 18,

$$\mathrm{Int}(\mathrm{CA}(p)) \cap \mathrm{CA}(p') = \emptyset = \mathrm{Int}(\mathrm{CA}(p')) \cap \mathrm{CA}(p),$$

then $\mathrm{bdCA}(p) \cup \mathrm{bdCA}(p')$ is equal to:

$$(\mathrm{CA}(p) \cup \mathrm{CA}(p')) \setminus \mathrm{Int}(\mathrm{CA}(p)) \setminus \mathrm{Int}(\mathrm{CA}(p')).$$

This term is equal by Lemma 3 to:

$$(\mathrm{CA}(p) \cup \mathrm{CA}(p')) \setminus \mathrm{Int}(\mathrm{CA}(p) \cup \mathrm{CA}(p')),$$

which is in fact $\mathrm{bdCA}(\{p, p'\})$. □

Theorem 4 *Let X be a digital subset of $\mathbb{Z}^n$. When X contains a critical configuration (of order $k \in [\![2, n]\!]$) at some block S of center m, then for all $\varepsilon \in]0, \epsilon(m)]$:*

$$\begin{aligned}\mathrm{bdCA}(X) \cap B_\infty(m, \varepsilon) \\ = (\mathrm{bdCA}(p) \cup \mathrm{bdCA}(p')) \cap B_\infty(m, \varepsilon).\end{aligned}$$

In other words, the boundary of $\mathrm{CA}(X)$ behaves *like the union of the boundaries of the continuous analogs of p and p' in the neighborhood of m.*

Proof This theorem is depicted in Fig. 19. Let us treat first the primary case: $X \cap S = \{p, p'\}$. Then, by Lemma 2, and by choosing $z := m$, we obtain that:

$$\begin{aligned}\mathrm{bdCA}(X) \cap B_\infty(m, \epsilon(m)) \\ = \mathrm{bdCA}(X \cap \xi(m)) \cap B_\infty(m, \epsilon(m)),\end{aligned}$$

and since $m \in \left(\frac{\mathbb{Z}}{2}\right)^n$, by Proposition 17, then $\xi(m) = S$, then:

$$\begin{aligned} &\mathrm{bdCA}(X) \cap B_\infty(m, \epsilon(m)) \\ &\quad = \mathrm{bdCA}(\{p, p'\}) \cap B_\infty(m, \epsilon(m)). \end{aligned}$$

Since p and p' are k-antagonists with $k \geq 2$, by Lemma 4, we obtain:

$$\begin{aligned} &\mathrm{bdCA}(X) \cap B_\infty(m, \epsilon(m)) \\ &\quad = (\mathrm{bdCA}(p) \cup \mathrm{bdCA}(p')) \cap B_\infty(m, \epsilon(m)). \end{aligned}$$

Now let us treat the secondary case: $S \setminus X = \{p, p'\}$. Then, the fact that X contains a secondary critical configuration is equivalent to say that X^c contains a primary critical configuration: $X^c \cap S = \{p, p'\}$. Then, by Proposition 1, $\mathrm{bdCA}(X) = \mathrm{bdCA}(X^c)$, and then by following the same reasoning as for the primary case:

$$\begin{aligned} &\mathrm{bdCA}(X) \cap B_\infty(m, \epsilon(m)) \\ &\quad = \mathrm{bdCA}(X^c) \cap B_\infty(m, \epsilon(m)), \\ &\quad = \mathrm{bdCA}(X^c \cap \xi(m)) \cap B_\infty(m, \epsilon(m)), \\ &\quad = \mathrm{bdCA}(\{p, p'\}) \cap B_\infty(m, \epsilon(m)), \\ &\quad = (\mathrm{bdCA}(p) \cup \mathrm{bdCA}(p')) \cap B_\infty(m, \varepsilon). \end{aligned}$$

This concludes the proof. □

Corollary 2 *Let us assume that a digital set $X \subset \mathbb{Z}^n$ contains a critical configuration in some block S of center m such that $X \cap S = \{p, p'\}$ or $S \setminus X = \{p, p'\}$. If $\mathrm{bdCA}(p) \cup \mathrm{bdCA}(p')$ is not locally Euclidean of dimension $(n-1)$, then $\mathrm{bdCA}(X)$ is not locally Euclidean of dimension $(n-1)$ neither. In other words, it is sufficient to show that the set $\{p, p'\}$ of X is not CWC to show that X is not CWC.*

4.3 The *n*-D Proof

From now on, in this subsection, we assume that we have a digital set $X \subset \mathbb{Z}^n$ which contains some primary critical configuration at the block S of center $m \in \left(\frac{\mathbb{Z}}{2}\right)^n$ and such that $X \cap S = \{p, p'\}$. In addition, we define:

$$\mathfrak{X}_{p,p'} := \mathrm{bdCA}(p) \cup \mathrm{bdCA}(p').$$

The notations of Sect. 4.3 are summarized in Table 2. Thanks to Lemma 1, we know that m belongs to $\mathfrak{X}_{p,p'}$, and thanks to Corollary 2, we know that if $\mathfrak{X}_{p,p'}$ is not locally homeomorphic to $]0,1[^{n-1}$ at m, then $\{p, p'\}$ is not CWC, and then X is not CWC neither.

To prove that $\{p, p'\}$ is not CWC, we are going to use homology. Indeed, if we can prove that:

$$\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, \mathfrak{X}_{p,p'} \setminus \{m\}) \neq \mathbb{Z},$$

then $\mathfrak{X}_{p,p'}$ is not a homological manifold at m, and then it is not a topological manifold. For this aim, we will use the first isomorphism theorem.

Since $\mathfrak{X}_{p,p'}$ is the union of two $(n-1)$-spheres sharing a $(n-k)$-cube, we can deduce its homology groups:

Property 2 *The homology groups of $\mathfrak{X}_{p,p'}$ are the following:*

$$\begin{cases} \mathbb{H}_0(\mathfrak{X}_{p,p'}) = \mathbb{Z}, \\ \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}) = \mathbb{Z} \oplus \mathbb{Z}, \\ \mathbb{H}_{k \in \mathbb{Z} \setminus \{0, n-1\}}(\mathfrak{X}_{p,p'}) = 0. \end{cases}$$

Now let us define:

$$A = \mathfrak{X}_{p,p'} \setminus \{m\},$$

then we obtain the following values of the homology groups of A (they will be used to prove next that the homology group $\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A)$ is not equal to $\mathbb{Z}$).

Property 3 *Let $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Let $A = \mathfrak{X}_{p,p'} \setminus \{m\}$, then:*

- *When $k = n = 2$, we have:*

$$\begin{cases} \mathbb{H}_0(A) = \mathbb{Z}^2, \\ \mathbb{H}_{k \in \mathbb{Z}^*}(A) = 0, \end{cases}$$

- *When $k = 2$ and $n = 3$, we have:*

$$\begin{cases} \mathbb{H}_0(A) = \mathbb{Z}, \\ \mathbb{H}_1(A) = \mathbb{Z}, \\ \mathbb{H}_{k \in \mathbb{Z} \setminus \{0,1\}}(A) = 0, \end{cases}$$

- *When $k = 2$ and $n \geq 4$, we have:*

$$\begin{cases} \mathbb{H}_{n-1}(A) = 0, \\ \mathbb{H}_{n-2}(A) = \mathbb{Z}, \end{cases}$$

- *When $k = n \geq 3$, we have:*

$$\begin{cases} \mathbb{H}_0(A) = \mathbb{Z}^2, \\ \mathbb{H}_{n \in \mathbb{Z}^*}(A) = 0, \end{cases}$$

- *When $k = n - 1$ and $n \geq 4$, we have:*

$$\begin{cases} \mathbb{H}_n(A) = 0, \\ \mathbb{H}_{n-1}(A) = 0, \\ \mathbb{H}_{n-2}(A) = 0, \end{cases}$$

- *When $k \in [\![3, n-2]\!]$ and $n \geq 5$, we have:*

$$\begin{cases} \mathbb{H}_n(A) = 0, \\ \mathbb{H}_{n-1}(A) = 0, \\ \mathbb{H}_{n-2}(A) = 0, \end{cases}$$

Table 2 Summary of the main notations of Sect. 4.3

$n \geq 2$	The dimension of the ambient space
X	A digital subset of $\mathbb{Z}^n$ which is not DWC
S	One of the blocks where a critical configuration occurs in X
$k \geq 2$	The antagonism order of S relatively to X
p, p'	The two k-antagonists in S
$X \cap S = \{p, p'\}$	The studied primary critical configuration of X
$m = \frac{p+p'}{2}$	The center of S
$\mathfrak{X}_{p,p'}$	$\text{bdCA}(p) \cup \text{bdCA}(p')$

Proof Let us decompose A this way for the sequel:

$$\begin{cases} K_0 = \text{bdCA}(p) \setminus \{m\}, \\ K_1 = \text{bdCA}(p') \setminus \{m\}, \\ \mathfrak{I} = K_0 \cap K_1, \\ A = K_0 \cup K_1. \end{cases}$$

Now let us treat each case separately.

- When $k = n = 2$, A is homotopy equivalent to a 0-sphere since it is a set of two empty 2-cubes minus their intersection.
- When $k = 2$ and $n = 3$:
  - A is made of two 3-cubes sharing a 1-cube minus its center, then it is connected and $\mathbb{H}_0(A) = \mathbb{Z}$.
  - $\mathfrak{I}$ is homotopy equivalent to a 0-sphere and then:

  $$\begin{cases} \mathbb{H}_0(\mathfrak{I}) = \mathbb{Z}^2, \\ \mathbb{H}_{i \in \mathbb{Z}^*}(\mathfrak{I}) = 0, \end{cases}$$

  we obtain then the Mayer–Vietoris sequence depicted below:

  $$\begin{array}{ccccc} \mathbb{H}_3(\mathfrak{I}) = 0 & \xrightarrow{\iota_3} & \mathbb{H}_3(K_0) \oplus \mathbb{H}_3(K_1) = 0 & \xrightarrow{\pi_3} & \mathbb{H}_3(A) = 0 \\ & \xleftarrow{\partial_3} & & & \\ \mathbb{H}_2(\mathfrak{I}) = 0 & \xrightarrow{\iota_2} & \mathbb{H}_2(K_0) \oplus \mathbb{H}_2(K_1) = 0 & \xrightarrow{\pi_2} & \mathbb{H}_2(A) = 0 \\ & \xleftarrow{\partial_2} & & & \\ \mathbb{H}_1(\mathfrak{I}) = 0 & \xrightarrow{\iota_1} & \mathbb{H}_1(K_0) \oplus \mathbb{H}_1(K_1) = 0 & \xrightarrow{\pi_1} & \mathbb{H}_1(A) = \mathbb{Z} \\ & \xleftarrow{\partial_1} & & & \\ \mathbb{H}_0(\mathfrak{I}) = \mathbb{Z}^2 & \xrightarrow{\iota_0} & \mathbb{H}_0(K_0) \oplus \mathbb{H}_0(K_1) = \mathbb{Z}^2 & \xrightarrow{\pi_0} & \mathbb{H}_0(A) = \mathbb{Z} \\ & \xleftarrow{\partial_0} & & & \\ \mathbb{H}_{-1}(\mathfrak{I}) = 0 & & & & \end{array}$$

  thus $\mathbb{H}_1(A) = \mathbb{Z}$.
- When $k = 2$ and $n \geq 4$, $\mathfrak{I}$ is a $(n-k-1)$-sphere with $(n-k-1) = (n-3) \geq 1$ and then $\mathbb{H}_0(\mathfrak{I}) = \mathbb{Z}$, $\mathbb{H}_{n-3}(\mathfrak{I}) = \mathbb{Z}$, and $\mathbb{H}_{i \in \mathbb{Z} \setminus \{0, n-3\}} = 0$. At the same time, K_0 and K_1 are contractile and then $\mathbb{H}_0(K_0) = \mathbb{H}_0(K_1) = \mathbb{Z}$ and $\mathbb{H}_{i \in \mathbb{Z}^*}(K_0) = \mathbb{H}_{i \in \mathbb{Z}^*}(K_1) = 0$. Then, we obtain the Mayer–Vietoris sequence depicted below:

  $$\begin{array}{ccccc} & & \mathbb{H}_{n-2}(K_0) \oplus \mathbb{H}_{n-2}(K_1) = 0 & \xrightarrow{\psi_{n-2}} & \mathbb{H}_{n-2}(A) = \mathbb{Z} \\ & \xleftarrow{\partial_{n-2}} & & & \\ \mathbb{H}_{n-3}(\mathfrak{I}) = \mathbb{Z} & \xrightarrow{\phi_{n-3}} & \mathbb{H}_{n-3}(K_0) \oplus \mathbb{H}_{n-3}(K_1) = 0 & & \end{array}$$

  thus $\mathbb{H}_{n-2}(A) = \mathbb{Z}$.
- When $k = n \geq 3$, A is a set of two empty n-cubes minus their intersection (a vertex), and then it is homotopy equivalent to a 0-sphere.
- When $k = n-1$ and $n \geq 4$, then $\mathfrak{I}$ is homotopy equivalent to a 0-sphere and K_0 and K_1 are contractile, then we have:

  $$\begin{cases} \mathbb{H}_0(\mathfrak{I}) = \mathbb{Z}^2, \\ \mathbb{H}_{i \in \mathbb{Z}^*}(\mathfrak{I}) = 0, \end{cases}$$

  $$\begin{cases} \mathbb{H}_0(K_0) = \mathbb{Z}, \\ \mathbb{H}_{i \in \mathbb{Z}^*}(K_0) = 0, \end{cases}$$

  and:

  $$\begin{cases} \mathbb{H}_0(K_1) = \mathbb{Z}, \\ \mathbb{H}_{i \in \mathbb{Z}^*}(K_1) = 0, \end{cases}$$

which leads to the results depicted below:

$$\mathbb{H}_{n-1}(K_0) \oplus \mathbb{H}_{n-1}(K_1) = 0 \xrightarrow{\psi_{n-1}} \mathbb{H}_{n-1}(A) = 0 \xrightarrow{\partial_{n-1}} \mathbb{H}_{n-2}(\mathfrak{I}) = 0 \xrightarrow{\phi_{n-2}} \mathbb{H}_{n-2}(K_0) \oplus \mathbb{H}_{n-2}(K_1) = 0 \xrightarrow{\psi_{n-2}} \mathbb{H}_{n-2}(A) = 0 \xrightarrow{\partial_{n-2}} \mathbb{H}_{n-3}(\mathfrak{I}) = 0$$

then $\mathbb{H}_{n-1}(A) = \mathbb{H}_{n-2}(A) = 0$.

– When $k \in [\![3, n-2]\!]$ and $n \geq 5$, then $\mathfrak{I}$ is homotopy equivalent to a $(n-k-1)$-sphere since it is equal to a $(n-k)$-ball minus its center, and because $(n-k-1) \geq 1$, we have:

$$\begin{cases} \mathbb{H}_0(\mathfrak{I}) = \mathbb{Z}, \\ \mathbb{H}_{n-k-1}(\mathfrak{I}) = \mathbb{Z}, \\ \mathbb{H}_{i \in \mathbb{Z} \setminus \{0, n-k-1\}}(\mathfrak{I}) = 0, \end{cases}$$

Also, K_0 and K_1 are contractile, and then:

$$\begin{cases} \mathbb{H}_0(K_0) = \mathbb{Z}, \\ \mathbb{H}_{i \in \mathbb{Z}^*}(K_0) = 0, \end{cases}$$

and:

$$\begin{cases} \mathbb{H}_0(K_1) = \mathbb{Z}, \\ \mathbb{H}_{i \in \mathbb{Z}^*}(K_1) = 0, \end{cases}$$

We obtain then the results depicted below:

$$\mathbb{H}_{n-1}(K_0) \oplus \mathbb{H}_{n-1}(K_1) = 0 \xrightarrow{\psi_{n-1}} \mathbb{H}_{n-1}(A) = 0 \xrightarrow{\partial_{n-1}} \mathbb{H}_{n-2}(\mathfrak{I}) = 0 \xrightarrow{\phi_{n-2}} \mathbb{H}_{n-2}(K_0) \oplus \mathbb{H}_{n-2}(K_1) = 0 \xrightarrow{\psi_{n-2}} \mathbb{H}_{n-2}(A) = 0 \xrightarrow{\partial_{n-2}} \mathbb{H}_{n-3}(\mathfrak{I}) = 0$$

then $\mathbb{H}_{n-1}(A) = \mathbb{H}_{n-2}(A) = 0$.

This concludes the proof. □

Now that we know the important values of the homology groups of A, let us prove the following property induced by Property 3.

Property 4 *When we have $n \geq 2$ and $k = 2$, then:*

$$\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A) = \mathbb{Z}^3,$$

and when we have $n \geq 3$ and $k \in [\![3, n]\!]$, then:

$$\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A) = \mathbb{Z}^2.$$

In other words, for any $n \geq 2$ and any $k \in [\![2, n]\!]$, we have $\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A) \neq \mathbb{Z}$.

Proof These results follow from the six following computations:

Step 1 : $\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A)$ when $k = n = 2$

$$\mathbb{H}_1(A) = 0 \xrightarrow{\iota_1} \mathbb{H}_1(\mathfrak{X}_{p,p'}) = \mathbb{Z}^2 \xrightarrow{\pi_1} \mathbb{H}_1(\mathfrak{X}_{p,p'}, A) = \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{H}_0(A) = \mathbb{Z}^2 \xrightarrow{\iota_0} \mathbb{H}_0(\mathfrak{X}_{p,p'}) = \mathbb{Z} \xrightarrow{\pi_0} \mathbb{H}_0(\mathfrak{X}_{p,p'}, A) = 0$$

Step 2: $\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A)$ when $k = 2$ and $n = 3$

$$\mathbb{H}_2(A) = 0 \xrightarrow{\iota_2} \mathbb{H}_2(\mathfrak{X}_{p,p'}) = \mathbb{Z}^2 \xrightarrow{\pi_2} \mathbb{H}_2(\mathfrak{X}_{p,p'}, A) = \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{H}_1(A) = \mathbb{Z} \xrightarrow{\iota_1} \mathbb{H}_1(\mathfrak{X}_{p,p'}) = 0$$

Step 3: $\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A)$ when $k = 2$ and $n \geq 4$

$$\mathbb{H}_{n-1}(A) = 0 \xrightarrow{\iota_{n-1}} \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}) = \mathbb{Z}^2 \xrightarrow{\pi_{n-1}} \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A) = \mathbb{Z}^3 \xrightarrow{\partial_{n-1}} \mathbb{H}_{n-2}(A) = \mathbb{Z} \xrightarrow{\iota_{n-2}} \mathbb{H}_{n-2}(\mathfrak{X}_{p,p'}) = 0$$

Step 4: $\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A)$ when $k = n \geq 3$

$$\mathbb{H}_{n-1}(A) = 0 \xrightarrow{\iota_{n-1}} \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}) = \mathbb{Z}^2 \xrightarrow{\pi_{n-1}} \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A) = \mathbb{Z}^2 \xrightarrow{\partial_{n-1}} \mathbb{H}_{n-2}(A) = 0$$

Step 5: $\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A)$ when $k = n-1$ and $n \geq 4$

$$\mathbb{H}_{n-1}(A) = 0 \xrightarrow{\iota_{n-1}} \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}) = \mathbb{Z}^2 \xrightarrow{\pi_{n-1}} \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A) = \mathbb{Z}^2 \xrightarrow{\partial_{n-1}} \mathbb{H}_{n-2}(A) = 0$$

Step 6: $\mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A)$ when $n \geq 5$ and $k \in [\![3, n-2]\!]$

$$\mathbb{H}_{n-1}(A) = 0 \xrightarrow{\iota_{n-1}} \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}) = \mathbb{Z}^2 \xrightarrow{\pi_{n-1}} \mathbb{H}_{n-1}(\mathfrak{X}_{p,p'}, A) = \mathbb{Z}^2 \xrightarrow{\partial_{n-1}} \mathbb{H}_{n-2}(A) = 0$$

The proof is done. □

Based on Property 4, it follows that $\mathfrak{X}_{p,p'}$ is not locally a homological manifold at m, and then $\mathfrak{X}_{p,p'}$ is not locally Euclidean of dimension $(n-1)$ at m. From this, we can conclude that $\mathfrak{X}_{p,p'}$ is not locally a topological $(n-1)$-manifold at m. Since $\mathfrak{X}_{p,p'}$ behaves like $\text{bdCA}(X)$ in the neighborhood of m by Theorem 4, then X is not CWC. When X contains a secondary critical configuration, the reasoning is the same, as explained in Corollary 2.

Theorem 5 *For any digital set $X \subset \mathbb{Z}^n$, $n \geq 2$, X is DWC when X is CWC. In other words, CWCness implies DWCness in n-D, $n \geq 2$.*

5 Conclusion

We have shown in this paper that CWCness implies DWCness in n-D, which can be summarized by saying that when we do not have any topological issue in the boundary of the continuous analog of a digital subset of $\mathbb{Z}^n$, then this last set does not contain any critical configuration, which implies that its connectivities are equivalent.

By gathering the properties relative to well-composedness coming from [7] and from the current paper, we can see that we obtain:

$$CWC \Rightarrow HWC \Rightarrow DWC,$$

where we call homology-well-composedness (HWCness) the property of a cubical set to have a homology manifold as boundary. Conversely, we know that:

$$DWC \not\Rightarrow HWC,$$

but we do not know if HWCness implies CWCness. We propose to study this last point in future works:

$$HWC \overset{?}{\Rightarrow} CWC.$$

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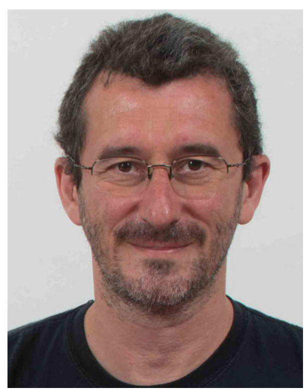

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