Journal of Physics A: Mathematical and Theoretical (2023): 07 December 2023

DOI: https://doi.org/10.1088/1751-8121/ad0dc7

Journal of Physics A:
Mathematical and Theoretical

PAPER

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To cite this article: Emmanuel Briand *et al* 2024 *J. Phys. A: Math. Theor.* **57** 015205

IOP Publishing　　Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **57** (2024) 015205 (30pp)　　https://doi.org/10.1088/1751-8121/ad0dc7

All linear symmetries of the SU(3) tensor multiplicities

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Received 7 June 2023; revised 1 November 2023
Accepted for publication 17 November 2023
Published 7 December 2023

Abstract

The $SU(3)$ tensor multiplicities are piecewise polynomial of degree 1 in their labels. The pieces are the chambers of a complex of cones. We describe in detail this chamber complex and determine the group of all linear symmetries (of order 144) for these tensor multiplicities. We represent the cells by diagrams showing clearly the inclusions as well as the actions of the group of symmetries and of its remarkable subgroups.

Keywords: tensor multiplicities, Lie groups, SU(3), chamber complex

1. Introduction

The tensor multiplicities for $SU(k)$ appear in several different contexts in nuclear and particle physics, as well as quantum interferometry, [1–6]. For instance, in the classification of orbital states of particles in the nuclear shell model, the $SU(k)$ tensor multiplicities represent total angular momentum [1]. In quantum interferometry, the $SU(k)$ tensor multiplicities appear in the decomposition of input states into a direct sum of input states in distinct, finite irreducible representations [7]. They also appear in elementary particle physics in *the Eightfold way* pioneered by Gell–mann and Ne'eman [2, 3].

The $SU(k)$ tensor multiplicities also govern the decomposition into irreducibles of some restrictions of representations of the symmetric group, describe the structure constants for the multiplication in the cohomology ring of the Grassmannian, give a basis for the Schubert cycles, and are the structure constants for the multiplication of symmetric functions under the Schur basis, see [8].

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The $SU(k)$ tensor multiplicities afford a number of symmetries. One of these symmetries is obvious: the multiplicity of V_n in $V_\ell \otimes V_m$ is the same as in $V_m \otimes V_\ell$. Here V_ℓ, V_m … are the irreducible representations of $SU(k)$. The $SU(k)$ tensor multiplicities define a function on the lattice of the triples (ℓ, m, n) of weights for $SU(k)$. The aforementioned symmetry, $\ell \leftrightarrow m$, is *linear* since it is given by a linear function on this lattice. There is actually a well-known group of 12 linear symmetries for the $SU(k)$ tensor multiplicities (comprising $\ell \leftrightarrow m$), for any k.

In this paper, we focus on the case $k = 3$. Our main result is the determination, in an elementary way, of the full group of all linear symmetries for the $SU(3)$ tensor multiplicities. We find it has order 144, surprisingly bigger than the group of 12 *general symmetries* that hold for $SU(k)$ for any k. We determine the structure of this group: it is isomorphic to $\mathfrak{S}_2 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$ where $\mathfrak{S}_n$ are the symmetric groups.

It is known that, for any fixed k, the $SU(k)$ tensor multiplicities are given by piecewise polynomial formulas. The pieces (domains of validity of the polynomial formulas) are the chambers (maximal cells) of a complex of polyhedral cones (the *chamber complex*). In the case $k = 3$, we get a full, explicit description of the chamber complex from the combinatorial description (BZ triangles) given by Berenstein and Zelevinsky of these tensor multiplicities [9]. We get as well the action of the linear symmetries on the chamber complex. We associate to each cell of the chamber complex a diagram, in such a way that the action of the linear symmetries on the cells can be read easily from their diagrams.

Following Berenstein and Zelevinsky, we actually work with the *triple multiplicities of* $SU(3)$ rather than with the $SU(3)$ tensor multiplicities themselves. The triple multiplicities are an avatar of the tensor multiplicities, defined as the dimensions of the subspaces of invariants $(V_\ell \otimes V_m \otimes V_n)^{SU(3)}$. The advantage of this model is the straightforward description of the group of the 12 general symmetries: is generated by the 6 permutations of the factors V_ℓ, V_m, V_n, and the duality involution. Also the combinatorial description given by Berenstein and Zelevinsky is given in the setting of the triple multiplicities.

The tensor multiplicities for $SU(k)$ are essentially the same as the tensor multiplicities for GL_k. We translate our results to the GL_3-setting, obtaining for the corresponding Littlewood–Richardson coefficients the full group of linear symmetries (it has order 288) and the description of the chamber complex.

1.1. Relation with other works

Computing tensor multiplicities is a classical problem that has been widely considered. Some formulas, due to Steinberg [10], Racah [11], Klimyk [12, 13] apply for the very general case of tensor multiplicities for any Lie Group. In the case of $SU(k)$, there are combinatorial rules (Littlewood–Richardson rules and its avatars). In the specific case of $SU(3)$, extremely explicit formulas can be given. See [4, 14, 15] for early attempts. A very concise presentation is as the minimum of 18 linear forms [16]: see section 9.4. Rassart [17] studied the properties of the formulas for the $SU(k)$ tensor multiplicities and worked out explicitly the case of $SU(3)$ by means of some computer calculations. The present paper derives again Rassart's description for $SU(3)$, but avoiding the computer calculation. In addition, it unveils an extra structure (group action) on these formulas.

The full group of linear symmetries for the $SU(3)$ tensor multiplicities was first derived in [18], which is an earlier version of the present work. There, some computer calculations were involved. The present paper obtains the same results without computer calculations. Simple, clear proofs are used instead. The recent paper [19] derives again, by other means, the existence of 144 symmetries, and provides alternative descriptions of them, that are of interest.

1.2. Structure of the paper

Section 2 provides a presentation of the general framework. Section 3 introduces the BZ triangles for $SU(3)$. Section 4 contains the main result of this work: the determination of all linear symmetries of the triple multiplicities of $SU(3)$. In section 5, the full description of the chamber complex for the $SU(3)$ tensor multiplicities is derived, as well as the action of the linear symmetries on it. Section 6 is devoted to translating the previous results (symmetries and chamber complex) into the GL_3 tensor multiplicities setting. Section 7 presents a determinantal formula for the $SU(3)$ tensor multiplicities. Section 8 uses the description of the $SU(3)$ chamber complex to illustrate concretely known general stability properties of the Littlewood–Richardson coefficients. Finally, section 9 concludes with a discussion of some topics not contemplated in the main bulk of the paper.

2. Preliminaries

2.1. Representations of $SU(3)$ and their labels

The irreducible Lie group representations of $SU(3)$ are naturally labeled by their highest weights. These are vectors in an abstract vector space (the dual of the complexified Lie algebra of $SU(3)$). We will use as *numeric* labels the vectors of coordinates (ℓ_1,ℓ_2) of the highest weights in the basis of fundamental weights (*Dynkin labels*). We will denote with V_ℓ the irreducible representation of $SU(3)$ with Dynkin label $\ell=(\ell_1,\ell_2)$. The Dynkin labels of the irreducible representations of $SU(3)$ are the pairs of nonnegative integers.

The Dynkin label of the dual of the representation of V_ℓ with Dynkin label $\ell=(\ell_1,\ell_2)$ is ℓ read backwards, i.e. (ℓ_2,ℓ_1). We denote this by ℓ^*.

2.2. Triple multiplicities and tensor multiplicities

Consider Dynkin labels $\ell=(\ell_1,\ell_2)$, $m=(m_1,m_2)$ and $n=(n_1,n_2)$. After [9], the multiplicity of V_n in the tensor product $V_\ell\otimes V_m$ is equal to the dimension of the space of invariants

$$\left(V_\ell\otimes V_m\otimes V_n^*\right)^{SU(3)}.$$

We set

$$c\left(\ell;m;n\right)=\dim\left(V_\ell\otimes V_m\otimes V_n\right)^{SU(3)}. \tag{1}$$

These integers are called *triple multiplicities for* $SU(3)$. The multiplicity of V_n in $V_\ell\otimes V_m$ is thus $c(\ell;m;n^*)$.

The support of the triple multiplicities (the set of all triples (ℓ, m, n) of Dynkin labels such that $c(\ell; m; n) \neq 0$) generate a sublattice Λ_{TM} of $\mathbb{Z}^6$, which is defined by the condition:

$$\ell_1 + m_1 + n_1 \equiv \ell_2 + m_2 + n_2 \mod 3. \tag{2}$$

2.3. Linear symmetries

A *linear symmetry of the triple multiplicities* is a linear automorphism θ of the lattice Λ_{TM} that leaves the triple multiplicities unaffected. That is, such that the identity $c(\theta(\ell, m, n)) = c(\ell; m; n)$ holds.

The definition (1) of the triple multiplicities makes obvious that the six permutations of the three labels ℓ, m, n are linear symmetries for the triple multiplicities. Another linear symmetry is the 'duality' symmetry, corresponding to changing each irreducible representation with its dual:

$$(\ell, m, n) \leftrightarrow (\ell^*, m^*, n^*) .$$

The duality symmetry and the six label permutations generate a group of 12 linear symmetries isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_3$. We denote this group G_{g} and call it the *group of general linear symmetries* for $SU(3)$, because this group appears also as group of linear symmetries for the triple multiplicities of $SU(k)$ for any k.

2.4. Polyhedral description of the triple multiplicities

In a finite–dimensional real vector space $\mathcal{L}$ endowed with a full–rank lattice Λ, (a subgroup generated by a basis of $\mathcal{L}$), a *convex rational polyhedral cone* is the set of all linear combinations, with nonnegative real coefficients, of some fixed finite set of vectors. These cones can also be described as the solution sets of systems of finitely many linear equations $f_i(x) \geqslant 0$, where the f_i take integer values on the lattice points. In the sequel, *cone* means *convex rational polyhedral cone*.

A *pointed* cone C in $\mathcal{L}$ (cone not containing any line) and a linear projection $pr : \mathcal{L} \to \mathcal{L}'$, sending the lattice Λ onto a lattice Λ' and the cone C to a cone C', defines a *fiber-counting function* on Λ': its value at t is the number of lattice points in C with image t. See figure 1.

It follows from [9] that the function that associates to three irreducible representations of $SU(3)$ (and, more generally, of $SU(k)$) the corresponding triple multiplicity (or tensor multiplicity) is such a fiber-counting function. In this case, the lattice points in C are combinatorial objects called *Berenstein–Zelevinsky triangles*.

The analogous statement holds for the GL_3 tensor multiplicities.

A cone $C \subset \mathcal{L}$ and a projection $pr : \mathcal{L} \to \mathcal{L}'$, mapping C to C', also define a complex of cones subdividing C' (a collection of cones, with union C', such that the intersection of any two of them is a face of each; and the faces of any cone in this collection also belong to the collection.). This complex, called the *chamber complex*, has as open cells the sets of points belonging to the projections of exactly the same faces of C. Its maximal closed cells are called the *chamber* of the chamber complex. The fiber-counting function associated to C and pr is piecewise quasipolynomial, with the chambers as the domains of validity of the quasipolynomial formulas.

In the cases of the $SU(k)$ triple multiplicities, or of the $SU(k)$ or GL_k tensor multiplicities, the formulas are actually polynomials (see [17]).

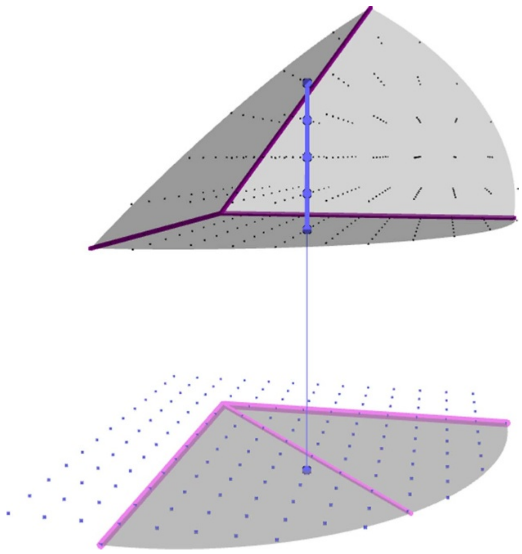


Figure 1. A cone projection and its fiber-counting function. Here a 3-dimensional cone (top) is projected to a 2.dimensional space, with two chambers. The associated fiber-counting function, defined on the lattice points of the 2-dimensional space, counts the lattice points in the corresponding fiber. One such fiber is represented (vertical segment in the top cone). The 'triple multiplicities' function for $SU(3)$ is such a fiber-counting function, where the lattice points in the top cone (7-dimensional) are the BZ triangles, and the projection is to a 6-dimensional space (the space of the $(\ell_1,\ell_2;m_1,m_2;n_1,n_2)$).

3. Berenstein–Zelevinsky triangles for $SU(3)$

The Littlewood–Richardson coefficients and the $SU(k)$ triple multiplicities can be calculated by means of the Littlewood–Richardson rule and its many avatars, see [20]. One of these avatars, introduced in [9], describes the $SU(k)$ triple multiplicities as counting combinatorial objects called *Berenstein-Zelevinsky triangles* ('BZ triangles' in the sequel). These are our main tools in this paper.

In this section, we review the definition of the BZ triangles for the $SU(3)$ triple multiplicities. (For the general description of the BZ triangles for $SU(k)$, for any k, see [9] or [20]). These are points in a 7-dimensional subspace $\mathcal{L}_{\text{BZ}}$ of a 9-dimensional space. We calculate a parameterization for the subspace $\mathcal{L}_{\text{BZ}}$ that will be used in the calculations of the next sections.

3.1. BZ triangles for $SU(3)$

Consider the *BZ graph* Γ, shown in figure 2: its vertices are the 9 points (i,j,k) with non-negative integer coordinates fulfilling $i+j+k=3$, different from $(1,1,1)$. There is an edge between any two vertices with difference $(1,-1,0)$, $(1,0,-1)$ or $(0,1,-1)$. The 9 vertices of the BZ graph are the 3 vertices of an equilateral triangle and the 6 vertices of a regular hexagon inscribed in the triangle. We will refer to the vertices of the BZ graph as Y_1,Y_2,Y_3 and $Z_1,\dots,Z_6$ as shown in figure 2.

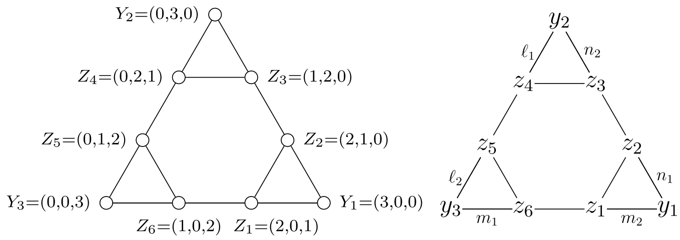


Figure 2. The BZ Graph Γ (left) and the coordinates of the BZ labelings (right).

Table 1. The objects in polyhedral geometric description of the $SU(3)$ Triple Multiplicities.

$\mathcal{L}_{\text{TM}}$	6D vector space of the $(\ell_1,\ell_2;m_1,m_2;n_1,n_2)$.
Λ_{TM}	Lattice of the integer points in $\mathcal{L}_{\text{TM}}$ fulfilling $\ell_1+m_1+n_1 \equiv \ell_2+m_2+n_2 \mod 3$.
TM	Cone in $\mathcal{L}_{\text{TM}}$.
lat(TM)	Lattice points in TM. Support of the triple multiplicities.
$\mathcal{K}_{\text{TM}}$	Chamber complex for the triple multiplicities. Subdivides TM.
$\mathcal{L}_{\text{BZ}}$	7D vector space of all BZ Graph labelings fulfilling (3). Endowed with a projection onto $\mathcal{L}_{\text{TM}}$.
Λ_{BZ}	Lattice of all integer labelings in $\mathcal{L}_{\text{BZ}}$. Projects onto Λ_{TM}.
BZ	Cone of all nonnegative labelings in $\mathcal{L}_{\text{BZ}}$. Projects onto TM.
lat(BZ)	Set of all Berenstein–Zelevinsky Triangles. (All lattice points in BZ).

For any labeling of the BZ graph, we will denote y_1,y_2,y_3 and $z_1,\ldots,z_6$ the labels of the vertices Y_1,Y_2,Y_3 and $Z_1,\ldots,Z_6$ (see figure 2). In the 9–dimensional space $\mathbb{R}^\Gamma$ of all BZ graph labelings, let $\mathcal{L}_{\text{BZ}}$ be the 7–dimensional subspace defined by the equations:

$$z_1 - z_4 = z_5 - z_2 = z_3 - z_6. \tag{3}$$

(See table 1 for a summary of the notations introduced in this section). The points of $\mathcal{L}_{\text{BZ}}$ are the BZ graph labelings such that any side of the hexagon sums as much as the opposite side. Let BZ be the cone of all points in $\mathcal{L}_{\text{BZ}}$ with *nonnegative* labels. Let lat(BZ) be the set of all integer points in the cone BZ. The *Berenstein–Zelevinsky triangle* (*BZ triangle* in the sequel) are the elements of lat(BZ); otherwise said, the BZ triangles are the labelings of the BZ graphs with nonnegative integer labels, fulfilling (3).

Let us introduce also the linear map $pr : \mathcal{L}_{\text{BZ}} \to \mathbb{R}^6$ that sends a BZ graph labeling in $\mathcal{L}_{\text{BZ}}$ to the tuple $(\ell_1,\ell_2;m_1,m_2;n_1,n_2)$ defined by

$$\begin{aligned} \ell_1 &= y_2+z_4, \quad & m_1 &= y_3+z_6, \quad & n_1 &= y_1+z_2, \\ \ell_2 &= y_3+z_5, \quad & m_2 &= y_1+z_1, \quad & n_2 &= y_2+z_3. \end{aligned} \tag{4}$$

These are the sums of the labels at some vertex Y_i and at one of the two neighboring vertices (see again figure 2). The linear map pr maps the integer points in $\mathcal{L}_{\text{BZ}}$ onto the lattice $\Lambda_{\text{TM}} \subset \mathbb{Z}^6$ of all points $(\ell_1,\ell_2,m_1,m_2,n_1,n_2)$ fulfilling (2). Then the triple multiplicity for the weights $\ell = (\ell_1, \ell_2)$, $m = (m_1, m_2)$ and $n = (n_1, n_2)$ counts the BZ triangles in the fiber over $t = (\ell_1, \ell_2; m_1, m_2; n_1, n_2)$ of the projection pr:

$$c(\ell; m; n) = \#\left(pr^{-1}(\ell; m; n) \cap \mathrm{lat}(\mathsf{BZ})\right).$$

3.2. A parameterization of the BZ triangles

We now parameterize the space $\mathcal{L}_{\mathrm{BZ}}$. We use as parameters the values of ℓ_1, ℓ_2, m_1, m_2, n_1, n_2 and the label $x = -y_1$ of vertex Y_1.

From (4) we get

$$\begin{aligned} z_4 &= \ell_1 - y_2, \quad z_6 = m_1 - y_3, \quad z_2 = n_1 - y_1, \\ z_5 &= \ell_2 - y_3, \quad z_1 = m_2 - y_1, \quad z_3 = n_2 - y_2. \end{aligned} \tag{5}$$

We set $\omega = z_4 - z_1$. After (3), there is also $\omega = z_6 - z_3 = z_2 - z_5$. Averaging these three expressions for ω gives

$$\omega = \frac{1}{3}(z_2 + z_4 + z_6 - z_1 - z_3 - z_5). \tag{6}$$

Replacing in (6) each z_i with its expression in terms of $t = (\ell_1, \ell_2; m_1, m_2; n_1, n_2)$ and the y_j from (5) yields

$$\omega = \frac{1}{3}(\ell_1 + m_1 + n_1 - \ell_2 - m_2 - n_2)$$

(the y_i cancel out).

Replacing in (3) each z_i with its expression in (5) yields three relations $y_i = y_1 + f_i(t)$, with the linear forms $f_i(t)$ displayed in table 2, where

$$\omega(t) = \frac{1}{3}(\ell_1 + m_1 + n_1 - \ell_2 - m_2 - n_2).$$

Replacing y_i with $y_1 + f_i(t)$ in (5) yields six relations $z_j = -y_1 - g_j(t)$, with the forms $g_j(t)$ also shown in table 2.

We finally set $x = -y_1$. For any $t = (\ell_1, \ell_2; m_1, m_2; n_1, n_2)$ and x, let $BZ(t, x)$ be the labeled BZ graph shown in figure 3. Then the linear map $(t, x) \mapsto BZ(t, x)$ establishes an isomorphism from $\mathbb{R}^6 \times \mathbb{R}$ to the linear span $\mathcal{L}_{\mathrm{BZ}}$ of the BZ triangles. This isomorphism of real vector spaces restricts to an isomorphism of lattices from $\Lambda \times \mathbb{Z}$ to the lattice Λ_{BZ} of the integer points of $\mathcal{L}_{\mathrm{BZ}}$.

Since $y_i = y_1 + f_i(t)$ and $x = -y_1$, the inequality $y_i \geqslant 0$ is equivalent to $f_i(t) \geqslant x$. Similarly, since $z_j = -y_1 - g_j(t)$, the inequality $z_j \geqslant 0$ is equivalent to $g_j(t) \leqslant x$. Therefore, under the parameterization considered here, the cone BZ is the set of solutions of

$$\begin{cases} \forall i, x \leqslant f_i(t), \\ \forall j, x \geqslant g_j(t). \end{cases}$$

4. Linear symmetries

In this section, we determine all linear symmetries for the triple multiplicities. As a first step, in 4.1, we find the group G_{BZ} of all linear symmetries for the set $\mathrm{lat}(\mathsf{BZ})$ of all BZ triangles. These are all automorphisms of Λ_{BZ} that stabilize the set $\mathrm{lat}(\mathsf{BZ})$. Then, in 4.2, we check that each element θ of G_{BZ} induces a linear symmetry θ' of the triple multiplicities. In such a

Table 2. The linear forms f_i and g_i.

$f_1(t) = 0,$	$f_2(t) = \ell_1 - m_2 - \omega(t)\,,$	$f_3(t) = \ell_2 - n_1 + \omega(t)\,,$
$g_1(t) = -m_2,$	$g_3(t) = \ell_1 - m_2 - n_2 - \omega(t)\,,$	$g_5(t) = -n_1 + \omega(t)\,,$
$g_2(t) = -n_1,$	$g_4(t) = -m_2 - \omega(t)\,,$	$g_6(t) = \ell_2 - m_1 - n_1 + \omega(t)$
	with $\omega(t) = \frac{1}{3}(\ell_1 + m_1 + n_1 - \ell_2 - m_2 - n_2)\,.$	

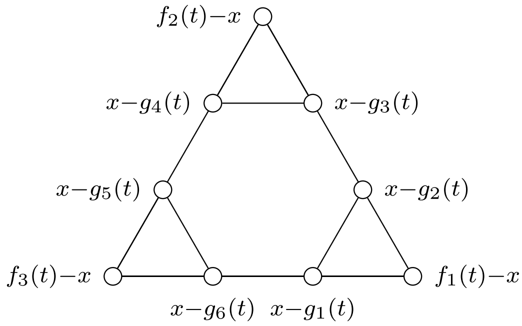


Figure 3. The labeled BZ graph $BZ(t,x)$. The linear forms $f_i(t)$ and $g_j(t)$ are those defined in table 2.

situation, we say that θ' *lifts to* θ, or that θ' *admits* θ *as a lifting*. We obtain this way a group of symmetries G_{l} of the triple multiplicities, isomorphic to G_{BZ}, of order 72: the group of symmetries 'that admit a lifting'. But not all symmetries of the triple multiplicities admit a lifting: the duality symmetry does not. The group of all linear symmetries, G, is thus bigger than G_{l}. In 4.3, we embed G into a permutation group (the group of permutations of all rays of the cone TM) that is found to have order 144. This is enough to conclude that G has exactly 144 elements, and is generated by G_{l} and the duality symmetry. In the course of the proof, we determine all rays of the cones BZ and TM.

As subgroup of interest, we also consider, besides G_{g} (group of general symmetries, see 2.3) the intersection $G_{\mathrm{lg}} = G_{\mathrm{g}} \cap G_{\mathrm{l}}$. The embedding of G into a group of permutations allows to elucidate the structure of the groups. The results are detailed in 4.4 and summarized in figure 4.

4.1. Symmetries for the BZ triangles of SU(3)

4.1.1. General symmetries for the BZ triangles. The BZ triangles have been introduced in [9] to make obvious some of the general symmetries of the $SU(3)$ tensor multiplicities. Namely, the 6 linear symmetries of the triangle $Y_1Y_2Y_3$ preserve the whole BZ graph, and permute the coordinates y_i and z_j of the BZ graph labelings, in such a way that the relations (3) are preserved. These six symmetries induce therefore symmetries of lat(BZ). They also induce symmetries of the triple multiplicities (since these are obtained as sums of the labels y_i and some neighboring z_j) forming a group G_{lg}. The group of the six linear symmetries of the triangle is generated by the reflections s_1 and s_2 with respect to the bisectors through Y_1 and Y_3. The effect on these generators on the BZ triangles and on the triple multiplicities is easily read on figure 5. The groups of the symmetries of the triangle, of the induced symmetries of the BZ triangles, and G_{lg} (induced symmetries of the triple multiplicities) are all isomorphic, and isomorphic to the symmetric group $\mathfrak{S}_3$.

Remember that the 12 general linear symmetries of the triple multiplicities for $SU(3)$ form a group G_{g} isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_3$, where the factor $\mathfrak{S}_2$ is generated by the duality involution

$$\begin{tikzcd}[column sep=small] & G & \\ G_{\mathrm{g}} \arrow[ur, phantom, sloped, "\subset"] & & G_{\mathrm{l}} \cong G_{\mathrm{BZ}} \arrow[ul, phantom, sloped, "\supset"] \\ & G_{\mathrm{lg}} \arrow[ul, phantom, sloped, "\subset"] \arrow[ur, phantom, sloped, "\supset"] & \end{tikzcd}$$

Group	Order	Structure
G (all linear symmetries)	144	$\mathfrak{S}_2 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$
G_{g} (general symmetries)	12	$\mathfrak{S}_3 \times \mathfrak{S}_2$
G_{l} (symmetries that lift)	72	$\mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_3$
G_{lg} $(= G_{\mathrm{l}} \cap G_{\mathrm{g}})$	6	$\mathfrak{S}_3$

Figure 4. Groups of linear symmetries for the triple multiplicities.

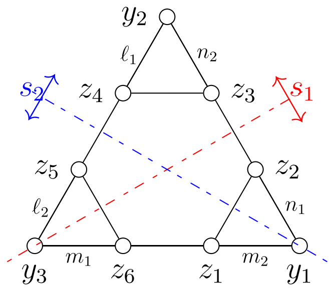


Figure 5. Action of the simple transpositions s_1 and s_2 on the BZ triangles and on the triple multiplicities.

$(\ell;m;n) \leftrightarrow (\ell^*;m^*;n^*)$, and the elements of the factor $\mathfrak{S}_3$ are the permutations of the triple $(\ell;m;n)$. The elements of G_{lg} are the 3 even permutations of $(\ell;m;n)$, together with the 3 transpositions composed with the duality involution. Otherwise stated, G_{lg} is generated by $(\ell;m;n) \leftrightarrow (m^*;\ell^*;n^*)$ and by $(\ell;m;n) \leftrightarrow (\ell^*;n^*;m^*)$. This subgroup G_{lg} is isomorphic to $\mathfrak{S}_3$, but distinct from the group of the permutations of $(\ell;m;n)$.

Note that all of the above holds not only for the $SU(3)$ case considered here, but also for $SU(k)$ for any k (see [9, remark (a) p10]).

4.1.2. All symmetries of the BZ triangles of SU(3). As already mentioned, the BZ triangles generate a cone BZ in the subspace $\mathcal{L}_{\mathrm{BZ}}$ of $\mathbb{R}^\Gamma$ defined by the equations $z_1 - z_4 = z_3 - z_6 = z_5 - z_2$. We will determine its rays.

For any $T \in \Lambda$ whose triple multiplicity is 1, i.e. which is the projection of a unique BZ triangle. Denote this BZ triangle by Δ_T. We contend that the rays of BZ are generated by the eight BZ triangles $\Delta_{\overrightarrow{C_1}}, \Delta_{\overrightarrow{C_2}}, \Delta_{\overrightarrow{C_3}}, \Delta_{\overrightarrow{D_3}}, \Delta_{\overrightarrow{D_5}}, \Delta_{\overrightarrow{D_1}}, \Delta_{\overrightarrow{\triangleleft}}$ and $\Delta_{\overrightarrow{\triangleright}}$ shown in table 3. Following [16], we call them the *fundamental BZ triangles*.

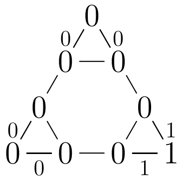
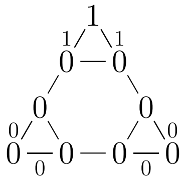
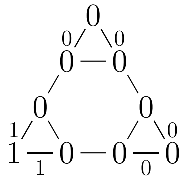
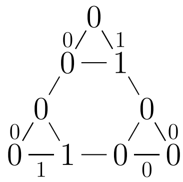
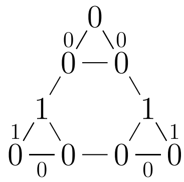
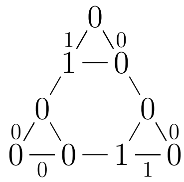
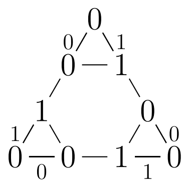
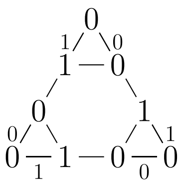
Indeed, given a point T in BZ, with labels y_i and z_j (as in figure 2), set (as in 3.2) $\omega = z_4 - z_1 = z_6 - z_3 = z_2 - z_5$. If $\omega \geqslant 0$, then T decomposes as

$$T = y_1 \cdot \Delta_{\overrightarrow{C_1}} + y_2 \cdot \Delta_{\overrightarrow{C_2}} + y_3 \cdot \Delta_{\overrightarrow{C_3}} + z_1 \cdot \Delta_{\overrightarrow{D_1}} + z_3 \cdot \Delta_{\overrightarrow{D_3}} + z_5 \cdot \Delta_{\overrightarrow{D_5}} + \omega \cdot \Delta_{\overrightarrow{\triangleright}} \tag{7}$$

with nonnegative coefficients; and if $\omega \leqslant 0$,

$$T = y_1 \cdot \Delta_{\overrightarrow{C_1}} + y_2 \cdot \Delta_{\overrightarrow{C_2}} + y_3 \cdot \Delta_{\overrightarrow{C_3}} + z_4 \cdot \Delta_{\overrightarrow{D_1}} + z_6 \cdot \Delta_{\overrightarrow{D_3}} + z_2 \cdot \Delta_{\overrightarrow{D_5}} + (-\omega) \cdot \Delta_{\overrightarrow{\triangleleft}}. \tag{8}$$

Table 3. The eight fundamental BZ triangles $\Delta_{\overrightarrow{C_i}}$, $\Delta_{\overrightarrow{D_i}}$, $\Delta_{\overrightarrow{\triangleleft}}$ and $\Delta_{\overrightarrow{\triangleright}}$ of the cone BZ, with their projections $\overrightarrow{C_i}$, $\overrightarrow{D_i}$, $\overrightarrow{\triangleleft}$ and $\overrightarrow{\triangleright}$.

 $\Delta_{\overrightarrow{C_1}}$ $\overrightarrow{C_1} = (00\|01\|10)$	 $\Delta_{\overrightarrow{C_2}}$ $\overrightarrow{C_2} = (10\|00\|01)$	 $\Delta_{\overrightarrow{C_3}}$ $\overrightarrow{C_3} = (01\|10\|00)$
 $\Delta_{\overrightarrow{D_3}}$ $\overrightarrow{D_3} = (00\|10\|01)$	 $\Delta_{\overrightarrow{D_5}}$ $\overrightarrow{D_5} = (01\|00\|10)$	 $\Delta_{\overrightarrow{D_1}}$ $\overrightarrow{D_1} = (10\|01\|00)$
 $\Delta_{\overrightarrow{\triangleleft}}$ $\overrightarrow{\triangleleft} = (01\|01\|01)$		 $\Delta_{\overrightarrow{\triangleright}}$ $\overrightarrow{\triangleright} = (10\|10\|10)$

This shows that the fundamental BZ triangles generate BZ. These are 8 vectors spanning the 7-dimensional space $\mathcal{L}_{\mathrm{BZ}}$. There is therefore a single linear relation between them, which is

$$\Delta_{\overrightarrow{D_1}} + \Delta_{\overrightarrow{D_3}} + \Delta_{\overrightarrow{D_5}} = \Delta_{\overrightarrow{\triangleleft}} + \Delta_{\overrightarrow{\triangleright}}. \tag{9}$$

Therefore, none of the fundamental BZ triangles is a positive linear combination of the other. This completes the checking of the fact that the fundamental triangles are generators of the rays of BZ.

Note that the fundamental BZ triangles do not generate only the vector space $\mathcal{L}_{\mathrm{BZ}}$, but the lattice of the integer points of $\mathcal{L}_{\mathrm{BZ}}$, since any point T in this lattice admits a decomposition (7) with integer coefficients.

The linear symmetries of lat(BZ) (the linear automorphisms of Λ_{BZ} that stabilize lat(BZ)) form a group G_{BZ}. Any element θ of G_{BZ} permutes the fundamental BZ triangles, since they are the minimal ray generators of the cone BZ. Taking into account the relation (9) between the fundamental BZ triangles, θ stabilizes the set $\{\Delta_{\overrightarrow{D_3}}, \Delta_{\overrightarrow{D_5}}, \Delta_{\overrightarrow{D_1}}\}$ (the only set of 3 fundamental BZ triangles whose sum is the sum of two other fundamental BZ triangles). Similarly, θ stabilizes $\{\Delta_{\overrightarrow{\triangleleft}}, \Delta_{\overrightarrow{\triangleright}}\}$ (the only set of two fundamental BZ triangles whose sum is the sum of three other fundamental BZ triangles). Finally, θ also stabilizes the set of the 3 remaining rays $\{\Delta_{\overrightarrow{C_1}}, \Delta_{\overrightarrow{C_2}}, \Delta_{\overrightarrow{C_3}}\}$. Reciprocally, any permutation of the 8 fundamental BZ triangles that stabilizes each of $\{\Delta_{\overrightarrow{C_1}},\Delta_{\overrightarrow{C_2}},\Delta_{\overrightarrow{C_3}}\}$, $\{\Delta_{\overrightarrow{D_3}},\Delta_{\overrightarrow{D_5}},\Delta_{\overrightarrow{D_1}}\}$ and $\{\Delta_{\overrightarrow{\triangleleft}},\Delta_{\overrightarrow{\triangleright}}\}$ lifts to a unique element of G_{BZ}, since it leaves the relation (9) unchanged. This shows that G_{BZ} is isomorphic to

$$\mathfrak{S}_{\left\{\Delta_{\overrightarrow{C_1}},\Delta_{\overrightarrow{C_2}},\Delta_{\overrightarrow{C_3}}\right\}}\times\mathfrak{S}_{\left\{\Delta_{\overrightarrow{D_3}},\Delta_{\overrightarrow{D_5}},\Delta_{\overrightarrow{D_1}}\right\}}\times\mathfrak{S}_{\{\Delta_{\overrightarrow{\triangleleft}},\Delta_{\overrightarrow{\triangleright}}\}}.$$

In particular, G_{BZ} has order $3!\times 3!\times 2! = 72$.

The elements of G_{BZ} can be described as linear symmetries of parts of the BZ graphs. The permutations of $\{\Delta_{\overrightarrow{C_1}},\Delta_{\overrightarrow{C_2}},\Delta_{\overrightarrow{C_3}}\}$ are obtained by applying the linear symmetries of the triangle $Y_1Y_2Y_3$, only to Y_1, Y_2, Y_3 (leaving unaffected the vertices Z_j). The elements of $\mathfrak{S}_{\{\Delta_{\overrightarrow{D_3}},\Delta_{\overrightarrow{D_5}},\Delta_{\overrightarrow{D_1}}\}}\times\mathfrak{S}_{\{\Delta_{\overrightarrow{\triangleleft}},\Delta_{\overrightarrow{\triangleright}}\}}$ are the linear symmetries of the hexagon $Z_1Z_2Z_3Z_4Z_5Z_6$ (dihedral group of order 12) applied to its vertices (leaving unaffected the vertices Y_i).

Alternatively, G_{BZ} is generated by the 6 linear symmetries of the BZ graph (the linear symmetries of the triangle $Y_1Y_2Y_3$, applied to all 9 vertices of the BZ graph) and the 12 linear symmetries of the hexagon $Z_1Z_2Z_3Z_4Z_5Z_6$ applied only to the vertices of this hexagon.

4.2. The linear symmetries of the triple multiplicities that lift to linear symmetries of the set of all BZ triangles

Any element θ of G_{BZ} is, *a priori*, of the form $BZ(t,x)\mapsto BZ(t',x')$. The symmetry $\theta\in G_{\mathrm{BZ}}$ induces a linear symmetry of the triple multiplicities if and only if t' is independent of x. Let us show that this is always the case. Let y_i and z_j be the coordinates of $BZ(t,x)$, and let y_i' and z_j' be the coordinates of $BZ(t',x')$. Then θ permutes the coordinates y_i, and also permutes the coordinates z_j. After the relations (4), the coordinates of t' are all of the form $y_i'+z_j'$. Each of them is thus y_p+z_q for some p and q. But $y_p+z_q=(f_p(t)-x)+(x-g_q(t))$, which is independent on x. This proves that t' is independent on x, as announced. As a conclusion, any element θ of G_{BZ} induces a linear symmetry of the triple multiplicities.

A direct calculation shows that the 8 fundamental BZ triangles have 8 distinct projections $\overrightarrow{C_i}$, $\overrightarrow{D_i}$, $\overrightarrow{\triangleleft}$ and $\overrightarrow{\triangleright}$ (given in table 3). As a consequence, the symmetries of the triple multiplicities induced by the 72 elements of G_{BZ} are all distinct. They form a subgroup G_{l} of the group of all symmetries of the triple multiplicities. The subgroup G_{l} is isomorphic to G_{BZ}, and thus to $\mathfrak{S}_2\times\mathfrak{S}_3\times\mathfrak{S}_3$, and has order 72. It can be characterized as the group of all linear automorphisms of the lattice Λ that stabilizes each of the sets $\{\overrightarrow{C_1},\overrightarrow{C_2},\overrightarrow{C_3}\}$, $\{\overrightarrow{D_1},\overrightarrow{D_3},\overrightarrow{D_5}\}$ and $\{\overrightarrow{\triangleleft},\overrightarrow{\triangleright}\}$.

The group G_{l} contains the group G_{lg} of the six 'general' symmetries that admit a lifting, which are easily recognized by their effect on the projections $\overrightarrow{C_i}$, $\overrightarrow{D_i}$, $\overrightarrow{\triangleleft}$ and $\overrightarrow{\triangleright}$. Namely, G_{lg} is the subgroup of the elements that permute the pairs $(\overrightarrow{C_1},\overrightarrow{D_5})$, $(\overrightarrow{C_2},\overrightarrow{D_3})$, $(\overrightarrow{C_3},\overrightarrow{D_1})$, and, only for the elements of order 2, swap $\overrightarrow{\triangleleft}$ with $\overrightarrow{\triangleright}$.

The duality involution $(\ell_1,\ell_2;m_1,m_2;n_1,n_2)\leftrightarrow(\ell_2,\ell_1;m_2,m_1;n_2,n_1)$, and the odd permutations of $(\ell;m;n)$ do not lift, since they do not stabilize $\{\overrightarrow{C_1},\overrightarrow{C_2},\overrightarrow{C_3}\}$. See the final remark 9.1 for a more straightforward argument for this fact.

4.3. All linear symmetries of the $SU(3)$ triple multiplicities

Remember that G is the group of all linear symmetries of the triple multiplicities. Any element of G stabilizes the cone TM; therefore it permutes the rays of TM, and also their minimal ray generators. These minimal ray generators are among the elements of

$$R=\left\{\overrightarrow{C_1},\overrightarrow{C_2},\overrightarrow{C_3},\overrightarrow{D_1},\overrightarrow{D_3},\overrightarrow{D_5},\overrightarrow{\triangleleft},\overrightarrow{\triangleright}\right\}.$$

These vectors span not only the vector space $\mathcal{L}_{\mathsf{TM}} = \mathbb{R}^6$, but also the lattice Λ_{TM}, since they are the projections of the fundamental BZ triangles that span the lattice Λ_{BZ} of the integer points in $\mathcal{L}_{\mathrm{BZ}}$.

The following relations between the vectors in R hold:

$$\overrightarrow{C_1} + \overrightarrow{C_2} + \overrightarrow{C_3} = \overrightarrow{D_1} + \overrightarrow{D_3} + \overrightarrow{D_5} = \overrightarrow{\triangleleft} + \overrightarrow{\triangleright}. \tag{10}$$

Both can be checked straightforwardly. (Actually the second relation is obtained from (9) by projecting. For the fact that the other relation is *not* obtained by such a projection, see section 9.1). Since R is a set of 8 vectors spanning $\mathbb{R}^6$, there exist two independent relations between the elements of R, which are just (10). In particular, there cannot be any relation expressing one vector from R as a positive linear relation of others. This shows that R is exactly the set of all minimal ray generators of TM.

We can thus embed G into the group of permutations of R. Moreover, any element of G stabilizes the subset $\{\overrightarrow{\triangleleft}, \overrightarrow{\triangleright}\}$ (the only pair of vectors of R whose sum is sum of three other vectors of R). Similarly, any element of G stabilizes each of $\{\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}\}$ and $\{\overrightarrow{D_1}, \overrightarrow{D_3}, \overrightarrow{D_5}\}$, or swaps them. Indeed, these two sets are the only sets of three vectors of R whose sums are also sums of two vectors of R.

As a conclusion, the embedding of G into the group of permutations of R takes its value in the subgroup H of the group of all permutations of R, that stabilizes $\{\overrightarrow{\triangleleft}, \overrightarrow{\triangleright}\}$ and $\{\{\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}\}, \{\overrightarrow{D_1}, \overrightarrow{D_3}, \overrightarrow{D_5}\}\}$. This subgroup is isomorphic to $\mathfrak{S}_2 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$, where $\wr$ denotes the wreath product of groups (see for example [21, p 187]). The subgroup H thus has order $2 \times (2 \times (3!)^2)$, which is 144. On the other hand, G contains the subgroup G_{l} of order 72 consisting of the symmetries that admit a lifting. Therefore, G has order some multiple of 72. The group G also contains some elements not in G_{l}, such as the duality automorphism. Therefore G has order at least 144. This proves that G has order 144 exactly, and is isomorphic to H.

4.4. The subgroups of symmetries as permutation groups

Let us summarize and complete the results in this section. Firstly, the group G of all linear symmetries of the triple multiplicities of $SU(3)$ has order 144 and is isomorphic to $\mathfrak{S}_2 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$. It embeds into the group $\mathfrak{S}_R$ of the permutations of R as the stabilizer of $\{\{\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}\}, \{\overrightarrow{D_1}, \overrightarrow{D_3}, \overrightarrow{D_5}\}\}$.

The subgroup G_{l} of all symmetries that lift to linear symmetries of $\mathrm{lat}(\mathsf{BZ})$ has order 72 and embeds into $\mathfrak{S}_R$ as

$$\mathfrak{S}_{\left\{\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}\right\}} \times \mathfrak{S}_{\left\{\overrightarrow{D_1}, \overrightarrow{D_3}, \overrightarrow{D_5}\right\}} \times \mathfrak{S}_{\left\{\overrightarrow{\triangleleft}, \overrightarrow{\triangleright}\right\}}.$$

Each element of G_{l} lifts uniquely to a linear symmetry of $\mathrm{lat}(\mathsf{BZ})$, and therefore $G_{\mathrm{l}} \cong G_{\mathrm{BZ}}$.

Let us consider the group G_{g} of the 12 'general symmetries', generated by the permutations of $(\ell; m; n)$ and the duality involution $(\ell; m; n) \leftrightarrow (\ell^*; m^*; n^*)$. The permutations of $(\ell; m; n)$ permute the pairs (C_1, D_3), (C_2, D_5) and (C_3, D_1), while the duality involution swaps the terms of each of these pairs (i.e. swaps (C_1, C_2, C_3) with (D_3, D_5, D_1)), and additionally swaps $\overrightarrow{\triangleright}$ with $\overrightarrow{\triangleleft}$.

Finally, the subgroup $G_{\mathrm{lg}} = G_{\mathrm{l}} \cap G_{\mathrm{g}}$, made of the 6 'general symmetries with a lifting', is generated by the two involutions s_1 swapping $(\ell; m; n)$ with $(m^*; \ell^*; n^*)$, and s_2 swapping $(\ell; m; n)$ with $(\ell^*; n^*; m^*)$. The involution s_1 swaps $\overrightarrow{D_3}$ with $\overrightarrow{D_5}$, $\overrightarrow{C_1}$ with $\overrightarrow{C_2}$ and $\overrightarrow{\triangleright}$ with $\overrightarrow{\triangleleft}$. The other generator s_2 of G_{lg} swaps $\overrightarrow{D_5}$ with $\overrightarrow{D_1}$, $\overrightarrow{C_2}$ with $\overrightarrow{C_3}$ and $\overrightarrow{\triangleright}$ with $\overrightarrow{\triangleleft}$

5. The chamber complex for the triple multiplicities for $SU(3)$

In this section. we show that the support of the triple multiplicities is the set of all lattice points of a cone TM, covered by 18 smaller cones $C(i,j)$, that are the domains for linear formulas for the triple multiplicities. The cones $C(i,j)$ generate a chamber complex $\mathcal{K}_{\mathrm{TM}}$ subdividing TM. We check that the $C(i,j)$ are full-dimensional, and thus that they are the chambers of $\mathcal{K}_{\mathrm{TM}}$. We also show that all $C(i,j)$ are simplicial (i.e. have exactly as many facets as the dimension of the ambient space, which is 6). Then we encode combinatorially all cells of the chamber complex. As an aside, we are able to count the cells of each dimension. Finally, we associate to each cell a diagram that makes clear the action of the group G of linear symmetries of the triple multiplicities, and its subgroups, on the cells.

5.1. The chamber complex $\mathcal{K}_{\mathrm{TM}}$

We have seen at the end of section 3.2 that the cone BZ is defined by a system of 18 inequalities:

$$\begin{cases} \forall i \in \{1,2,3\}, & x \leqslant f_i(t), \\ \forall j \in \{1,2,3,4,5,6\}, & x \geqslant g_j(t), \end{cases}$$

with the 3 forms f_i and the 6 forms g_j defined in table 2. This system can be summarized by the condition:

$$\max_q g_q(t) \leqslant x \leqslant \min_p f_p(t). \tag{11}$$

As a consequence, the labels $t = (\ell;m;n)$ of the non-zero triple multiplicities all belong to the subset TM of $\mathcal{L}_{\mathrm{TM}}$ defined by

$$\max_q g_q(t) \leqslant \min_p f_p(t).$$

The above inequality summarizes the system of 18 linear inequalities:

$$\forall i \in \{1,2,3\},\ \forall j \in \{1,2,3,4,5,6\},\ g_j(t) \leqslant f_i(t).$$

This is why TM is a cone.

For any fixed $t = (\ell;m;n) \in \mathrm{lat}(\mathsf{TM})$, the triple multiplicity $c(t)$ counts the integers x that fulfill (11). Therefore

$$c(t) = 1 + \max\left(0, \min_p f_p(t) - \max_q g_q(t)\right).$$

This shows that the triple multiplicity function is piecewise polynomial of degree 1 (i.e. piecewise linear with constant term). For each $i \in \{1,2,3\}$ and each $j \in \{1,2,\ldots,6\}$, the linear formula

$$c(t) = 1 + f_i(t) - g_j(t)$$

holds for all lattices points t in the set $C(i,j)$ defined by

$$\min_p f_p(t) = f_i(t) \geqslant g_j(t) = \max_q g_q(t).$$

The $C(i,j)$ are 18 cones covering TM, and, clearly, the intersection of any two of them is a face of each. Therefore the cones $C(i,j)$, together with all their faces, are the cells of a complex of (convex rational polyhedral) cones subdividing TM. This complex is the chamber complex of the $SU(3)$ triple multiplicities (see section 2.4). We denote this complex $\mathcal{K}_{\mathrm{TM}}$.

5.2. All 18 cones $C(i,j)$ are full-dimensional

We prove this now.

For each of the cones $C(i,j)$, let $\widehat{C}(i,j)$ be its inverse image under the projection $p : \mathsf{BZ} \to \mathbb{R}^6$ that sends $BZ(t,x)$ (the BZ triangle defined in figure 3) to t. Then $\widehat{C}(i,j)$ is the set of points $BZ(t,x)$ fulfilling: '$f_i(t) = \min_p f_p(t)$; $g_j(t) = \max_q g_q(t)$; and $f_i(t) \geqslant g_j(t)$'. Rewrite the condition '$f_i(t) = \min_p f_p(t)$' as '$f_i(t) - x = \min_p (f_p(t) - x)$'. In terms of the coordinates y_i and z_j of the space $\mathcal{L}_{\mathrm{BZ}}$, this is '$y_i = \min_p y_p$'. Likewise, rewrite '$g_j(t) = \max_q g_q(t)$' as '$x - g_j(t) = \min_q (x - g_q(t))$'; this is '$z_j = \min_q z_q$'. Finally, rewrite '$f_i(t) \geqslant g_j(t)$' as '$f_i(t) - x \geqslant -(x - g_j(t))$', which is '$y_i \geqslant -z_j$', a condition that always holds in BZ. As a conclusion, $\widehat{C}(i,j)$ is the set of points of BZ that fulfill: '$y_i = \min_p y_p$ and $z_j = \min_q z_q$'. Now, the group G_{BZ}, that permutes the coordinates of the BZ triangles, acts transitively on the pairs of coordinates (y_i, z_j). As a consequence, G_{BZ} permutes transitively the cones $\widehat{C}(i,j)$. Accordingly, G_{I} permutes transitively the cones $C(i,j)$. The cones $C(i,j)$ are thus either all full-dimensional, or all degenerated. But these cones cannot be all degenerated, because their union TM is full-dimensional. We conclude that the cones $C(i,j)$ are all full-dimensional.

5.3. All cells of the chamber complex are simplicial

This is what we prove now. Since any cell of the chamber complex is a face of some chamber, and any face of any simplicial cone is simplicial, it is enough to prove that all chambers are simplicial.

Consider a chamber $C(i,j)$. It is defined by the conditions: $f_i = \min_p f_p$; $g_j = \min_q g_q$; $f_i \geqslant g_j$. These conditions translate into the system of inequalities:

$$\begin{cases} \forall p \neq i, \quad f_p \geqslant f_i \quad \text{(2 inequalities)} \\ \forall q \neq j, \quad g_p \leqslant g_j \quad \text{(5 inequalities)} \\ f_i \geqslant g_q \quad \text{(1 inequality)} \end{cases}.$$

Two of these inequalities are actually consequences of the other six. Indeed, from the equations $z_1 - z_4 = z_3 - z_6 = z_5 - z_2$ that hold on the linear span of the BZ triangles, and the equations $z_i = x - g_i(t)$, follow the relations:

$$g_1(t) - g_4(t) = g_5(t) - g_2(t) = g_3(t) - g_6(t). \tag{12}$$

Using these relations, one gets that, for all j,

$$g_{j+1} - g_j = g_{j+1} - g_{j+4} + g_{j+4} - g_j = g_{j+3} - g_j + g_{j+4} - g_j,$$

Table 4. The conditions whose conjunctions define the cells of the chamber complex $\mathcal{K}_{\text{TM}}$.

C_1	$\min_p f_p(t) = f_1(t)$
C_2	$\min_p f_p(t) = f_2(t)$
C_3	$\min_p f_p(t) = f_3(t)$
D_1	$\max_q g_q(t) = \max(g_1(t), g_4(t))$
D_3	$\max_q g_q(t) = \max(g_3(t), g_6(t))$
D_5	$\max_q g_q(t) = \max(g_5(t), g_2(t))$
$\triangleleft$	$\max_q g_q(t) = \max(g_1(t), g_3(t), g_5(t))$
$\triangleright$	$\max_q g_q(t) = \max(g_2(t), g_4(t), g_6(t))$
$\star$	$\min_p f_p(t) = \max_q g_q(t)$

where the indices are considered modulo 6 (e.g. if $j = 3$ then $j + 4 = 1$). Therefore $g_{j+1} \leqslant g_j$ follows from $g_{j+3} \leqslant g_j$ and $g_{j+4} \leqslant g_j$. Similarly, $g_{j-1} \leqslant g_j$ follows from $g_{j+3} \leqslant g_j$ and $g_{j+2} \leqslant g_j$. As a consequence, $C(i,j)$ is defined by the smaller system:

$$\begin{cases} f_{i+1} \geqslant f_i \\ f_{i+2} \geqslant f_i \\ g_{j+2} \leqslant g_j \\ g_{j+3} \leqslant g_j \\ g_{j+4} \leqslant g_j \\ f_i \geqslant g_j \end{cases} . \tag{13}$$

Since $C(i,j)$ is a 6-dimensional pointed cone, defined by a system of only 6 linear inequalities, it is simplicial and all 6 inequalities are essential, i.e. the corresponding equations define the facets of $C(i,j)$.

5.4. Combinatorial encoding of the cells

Let

$$\Omega = \{C_1, C_2, C_3, D_1, D_3, D_5, \triangleleft, \triangleright, \star\}$$

be the set of nine conditions defined in table 4.

As a subset of TM (i.e. assuming that all inequalities $f_p(t) \geqslant g_q(t)$ are fulfilled), the chamber $C(i,j)$ is defined by

'$\min_p f_p(t) = f_i(t)$ and $\max_q g_q(t) = g_j(t)$'.

The first of the two conditions in this conjunction is exactly C_i. The second one is easily seen to be equivalent to '$D_{j'}$ and ω_j', where $j' = j$ and $\omega_j = \triangleright$ if j is odd, and $j' = j + 3$ and $\omega_j = \triangleleft$ if j is even.

Therefore, the chamber $C(i,j)$ is defined in TM by 'C_i and $D_{j'}$ and ω_j'.

Because $C(i,j)$ is simplicial, there is a one-to-one correspondence between the faces σ of $C(i,j)$ and the sets X of equations obtained from the defining inequalities (13), which are:

$$f_{i+1} = f_i,\ f_{i+2} = f_i,\ g_{j+2} = g_j,\ g_{j+3} = g_j,\ g_{j+4} = g_j, f_i = g_j. \tag{14}$$

In this correspondence, σ corresponds to X when σ is defined, as a subset of $C(i,j)$, by the system of all equations in X; and then X is the set of all equations from the list that hold everywhere on σ.

The equations in (14) can be restated in terms of conditions in $\Omega \setminus \{C_i, D_{j'}, \omega_j\}$. Indeed, the first two equations in (14) are C_{i+1} and C_{i+2} (taking into account the assumption $f_i = \min_p f_p$). The last one is $\star$. One checks easily that the other three conditions ($g_{j+2} = g_j$, $g_{j+3} = g_j$, $g_{j+4} = g_j$) are equivalent to $D_{j'+2}$, ω_j' and $D_{j'+4}$ respectively, where ω_j' is the element of $\{\triangleleft, \triangleright\}$ that is not ω_j. The above one-to-one correspondence becomes a one-to-one correspondence between the faces σ of $C(i,j)$ and the subsets X of $\Omega \setminus \{C_i, D_{j'}, \omega_j\}$.

Note that, when σ corresponds to X, then σ is defined *in TM* by $X \cup \{C_i, D_{j'}, \omega_j\}$; and $X \cup \{C_i, D_{j'}, \omega_j\}$ is the set of all conditions in Ω that hold on σ. We define two maps Ψ and *Sol*: for any cell σ of the chamber complex, Ψ_σ is the set of all conditions in Ω that hold everywhere in σ; for any subset $X \subset \Omega$, *SOLX* is the set of points t in TM that fulfill all conditions in X. From what precedes, we get that for any chamber $C(i,j)$, Ψ and *Sol* induce bijections, inverse of each other, between the set of all faces of $C(i,j)$ and the subsets X of Ω containing each of C_i, $D_{j'}$ and $\omega_{j'}$.

Together with the fact that each cell of $\mathcal{K}_{\mathrm{TM}}$ is the face of some chamber, this is enough to deduce that Ψ and *Sol* induce bijections, inverse of each other, between the set of all faces of the chamber complex, and the set S of all subsets of Ω meeting each of $\{C_1, C_2, C_3\}$, $\{D_1, D_3, D_5\}$ and $\{\triangleleft, \triangleright\}$.

Rather than dealing with Ψ, which reverses inclusions, let us introduce, for any cell σ, the complement $\overline{\Psi}(\sigma)$ of $\Psi(\sigma)$ in Ω (the set of all conditions that fail to hold everywhere on σ). Then $\overline{\Psi}$ is an inclusion–preserving bijection from the set of all cells of $\mathcal{K}_{\mathrm{TM}}$, to the set $\overline{S}$ of all parts of Ω that contain none of $\{C_1, C_2, C_3\}$, $\{D_1, D_3, D_5\}$ and $\{\triangleleft, \triangleright\}$. The minimum element of $\overline{S}$ is the empty set, that corresponds to the $\{0\}$ cell. The minimal non-empty elements of $\overline{S}$ are the one-element subsets from Ω, naturally in bijection with Ω; they correspond to the rays of the chamber complex. Since each cell is simplicial, its dimension is the number of rays it contains. Therefore, the dimension of any cell σ is the cardinality of $\overline{\Psi}(\sigma)$.

Given any set E, denote by *Subsets* E (resp. *PSubsets* E) the set of all subsets E (resp. of all proper subsets of E, i.e. all subsets of E distinct from E). The ranked poset of the cells of the chamber complex (ordered by inclusion) is isomorphic to the poset of the elements of $\overline{S}$, which is itself isomorphic to

$$\begin{aligned} &\mathit{PSubsets}\{C_1, C_2, C_3\} \times \mathit{PSubsets}\{D_1, D_3, D_5\} \\ &\qquad \times \mathit{PSubsets}\{\triangleleft, \triangleright\} \times \mathit{Subsets}\{\star\}. \end{aligned}$$

The generating series of a ranked poset is the polynomial $\sum_i a_i q^i$ where q is a variable and a_i is the number of elements of rank i. When N is the cardinality of E, the generating series of *Subsets* E is $(1+q)^N$, and the generating series for *PSubsets* E is $(1+q)^N - q^N$, which gives $1 + 3q + 3q^2$ for $N = 3$, and $(1+2q)$ when $N = 2$.

Therefore, $\overline{S}$ is a ranked poset with rank generating series

$$\left(1 + 3q + 3q^2\right)^2 (1+2q)(1+q).$$

This expands as

$$1 + 9q + 35q^2 + 75q^3 + 93q^4 + 63q^5 + 18q^6.$$

The coefficients in this expansion are thus the numbers of faces of each dimension in the chamber complex of $SU(3)$ ('f-vector'). In particular, we recover that the chamber complex has 18 chambers, and observe that it has nine rays. Eight of them have already been obtained, as the eight rays of the cone TM. The group G of linear symmetries of the chamber complex

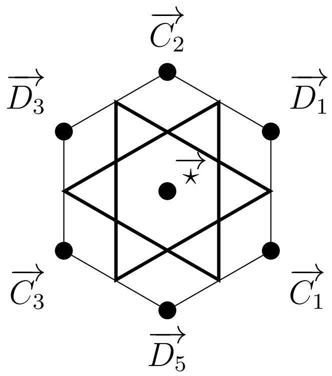


Figure 6. The items associated to the rays of the chamber complex.

permutes the nine rays of the chamber complex. On the other hand, we know that G stabilizes the eight rays of the cone TM. Therefore, G fixes the ninth ray. In particular, this ray must be fixed by the six permutations of $(\ell;m;n)$ and by the duality involution. It follows that this ninth ray is generated by $(11|11|11)$. We denote the generator by $\overrightarrow{\star}$ and refer to its ray as the *internal ray*, since it is the only one not on the border of TM.

Each ray generator must fulfill all conditions from Ω but one. One checks (using the expressions in coordinates in table 3) that, for each $X \in \Omega$, the condition not fulfilled by the ray generator $\overrightarrow{X}$ is X. This justifies *a posteriori* the coincidence of notations.

Let us observe finally that the nine ray generators obtained here are related by

$$\overrightarrow{C_1} + \overrightarrow{C_2} + \overrightarrow{C_3} = \overrightarrow{D_1} + \overrightarrow{D_3} + \overrightarrow{D_5} = \overrightarrow{\triangleleft} + \overrightarrow{\triangleright} = \overrightarrow{\star}. \tag{15}$$

Indeed, these relations are those in (10), except for $\overrightarrow{\triangleleft} + \overrightarrow{\triangleright} = \overrightarrow{\star}$, which is, in coordinates,

$$(01|01|01) + (10|10|10) = (11|11|11)\,.$$

Since these are 9 vectors spanning a 6–dimensional vector space, and since (15) already gives 3 independent relations, there are no more independent relations.

5.5. *Cell diagrams and actions of the groups on the cells.*

There is a convenient pictorial way to describe the action of G and its subgroups on the cells. Draw a regular hexagon with vertices labeled with $\overrightarrow{C_1}$, $\overrightarrow{D_1}$, $\overrightarrow{C_2}$, $\overrightarrow{D_3}$, $\overrightarrow{C_3}$ and $\overrightarrow{D_5}$, in this order. Draw also the triangle whose vertices are the midpoints of the sides $\overrightarrow{C_2}\overrightarrow{D_1}$, $\overrightarrow{C_3}\overrightarrow{D_3}$ and $\overrightarrow{D_5}\overrightarrow{C_1}$ of the hexagon ('left–pointing triangle') and the triangle whose vertices are the midpoints of the three remaining sides ('right–pointing triangle'). Label accordingly these two triangles with $\overrightarrow{\triangleleft}$ and $\overrightarrow{\triangleright}$.

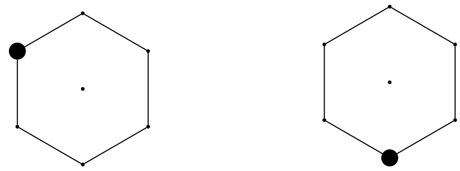
Label finally the center of this hexagon with $\overrightarrow{\star}$,

We call the six vertices of the hexagon, its center and the two triangles the *items of the cell diagrams*. See figure 6.

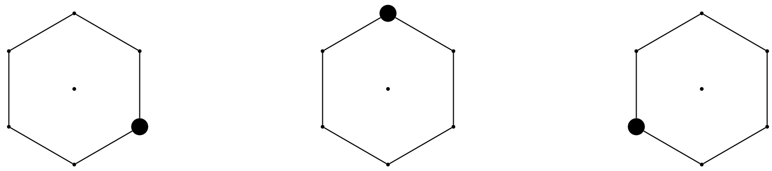
Given any cell of the chamber complex, its *diagram* is obtained by superposing the items of the ray it contains. Table 5 displays the diagrams of the nine rays, and figure 7 shows the diagrams of some other cells.

The missing items in the diagram of a cell σ provide the conditions defining σ, i.e. the set $\Psi(\sigma)$. For instance, the diagram of the chamber $C(1,1)$ has all items except those of C_1, D_1

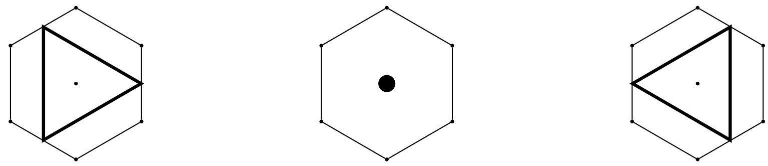
Table 5. The diagrams of the nine rays of the chamber complex, with the coordinates $(\ell_1, \ell_2|m_1, m_2|n_1, n_2)$ of their minimal generator.



$\overrightarrow{D_3} = (00\|10\|01)$	$\overrightarrow{D_5} = (01\|00\|10)$	$\overrightarrow{D_1} = (10\|01\|00)$



$\overrightarrow{C_1} = (00\|01\|10)$	$\overrightarrow{C_2} = (10\|00\|01)$	$\overrightarrow{C_3} = (01\|10\|00)$



$\overrightarrow{\triangleright} = (01\|01\|01)$	$\overrightarrow{\star} = (11\|11\|11)$	$\overrightarrow{\triangleleft} = (10\|10\|10)$

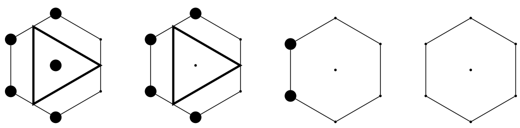


Figure 7. Diagrams of some cells of the chamber complex. From left to right: the diagram of the chamber $C(1,1)$, defined by the conditions (C_1), (D_1) and $\triangleleft$; the diagram of the external facet of $C(1,1)$, defined in $C(1,1)$ by $f_1 = g_1$; the diagram of the 2–dimensional face of $C(1,1)$ defined by the equations $f_1 = f_2 = g_1 = g_4 = g_5$; the diagram of the $\{0\}$ cell.

and $\triangleleft$ (see figure 7). This corresponds to the fact that $C(1,1)$ is defined by the conditions (C_1), (D_1) and $(\triangleleft)$.

The action of the group G and its subgroups on the cells is easily read from the diagrams.

Consider first the subgroup G_{lg}: it acts on the items as the group of symmetries of the triangle $\overrightarrow{C_1}\overrightarrow{C_2}\overrightarrow{C_3}$ (since s_1 acts as the reflection with respect to the axis $\overrightarrow{D_1}\overrightarrow{C_3}$, and s_2 as the reflection with respect to the axis $\overrightarrow{D_3}\overrightarrow{C_1}$).

The duality involution acts on the items as the central symmetry (with respect to the center of the hexagon). Therefore, G_{g} (being generated by G_{lg} and the duality involution) acts on the items as the group of the symmetries of the hexagon.

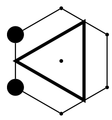
In what regards G_{l}, recall that it permutes the rays as

$$\mathfrak{S}_{\left\{\overrightarrow{C_1},\overrightarrow{C_2},\overrightarrow{C_3}\right\}} \times \mathfrak{S}_{\left\{\overrightarrow{D_1},\overrightarrow{D_3},\overrightarrow{D_5}\right\}} \times \mathfrak{S}_{\left\{\overrightarrow{\triangleleft},\overrightarrow{\triangleright}\right\}}.$$

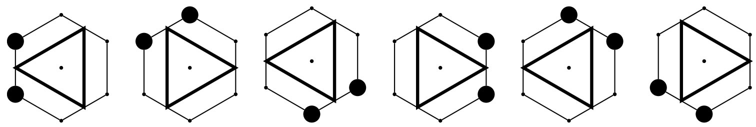
Each factor of this decomposition acts as the group of symmetries of the triangle $C_1C_2C_3$ acting only on part of the diagram (leaving unaffected the other parts): $\mathfrak{S}_{\{\overrightarrow{C_1},\overrightarrow{C_2},\overrightarrow{C_3}\}}$ on the triangle $\overrightarrow{C_1}\overrightarrow{C_2}\overrightarrow{C_3}$; $\mathfrak{S}_{\{\overrightarrow{D_1},\overrightarrow{D_3},\overrightarrow{D_5}\}}$ on the triangle $\overrightarrow{D_1}\overrightarrow{D_3}\overrightarrow{D_5}$; and $\mathfrak{S}_{\{\overrightarrow{\triangleleft},\overrightarrow{\triangleright}\}}$ on the left and right pointing triangles.

This describes the action of all elements of the group G, since G is generated by G_l and the duality involution.

Example. We consider the 3-dimensional cell represented by the following diagram and its orbits under the groups of symmetries:

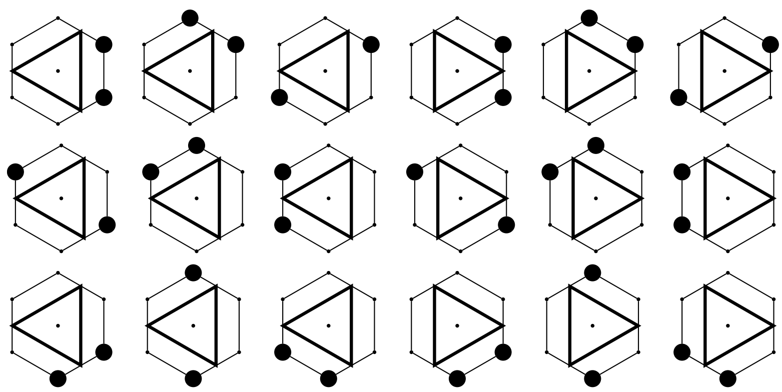


The elements of its orbit under G_{lg}, obtained by applying alternatively s_1 and s_2, are the cells with the following diagrams: This set of diagrams is stabilized by the central symmetry



(duality involution for the cells). Therefore, the orbit under G_g coincide with this orbit under G_{lg}.

The orbit under G_l and the orbit under G also coincide, and correspond to the following set of diagrams, obtained by applying the hexagon symmetries only to the C-vertex, only to the D-vertex, or only to the triangles:



6. Linear symmetries and chamber complex for the Littlewood–Richardson coefficients of GL_3

In this section, we translate the results obtained for the $SU(3)$ triple multiplicities into the setting of the Littlewood–Richardson coefficients (tensor multiplicities for the general linear groups GL_k, here for GL_3).

6.1. From $SU(3)$ to GL_3

For any weakly decreasing sequence of integers $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, let W_λ be the irreducible representation of GL_3 whose highest weight sends the diagonal matrix with entries x_1, x_2, x_3 to $\lambda_1 x_2 + \lambda_2 x_2 + \lambda_3 x_3$. All irreducible finite-dimensional representations of GL_3 are of this form. The representation W_λ is *polynomial* when $\lambda_3 \geqslant 0$, and then the label λ is an integer partition.

The irreducible representations V_ℓ of $SU(3)$ are exactly the restrictions of the irreducible representations of W_λ of GL_3. Precisely, for $\ell = (\ell_1, \ell_2)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, V_ℓ is the restriction of W_λ when

$$\begin{cases} \ell_1 = \lambda_1 - \lambda_2, \\ \ell_2 = \lambda_2 - \lambda_3. \end{cases} \tag{16}$$

Note that all GL_3 irreducible representations $W_{\lambda_1+k, \lambda_2+k, \lambda_3+k}$, for k integer (positive or negative), restrict to the same representation of $SU(3)$.

A necessary condition for the GL_3 tensor multiplicity $mult\,(W_\nu; W_\lambda \otimes W_\mu)$ (multiplicity of W_λ in the tensor product $W_\mu \otimes W_\nu$) to be non-zero is that $|\lambda| + |\mu| = |\nu|$ (where $|\lambda|$ stands for the sum of the coordinates of λ). If λ, μ and ν are partitions (i.e. all three representations are polynomial), this tensor multiplicity is called a *Littlewood–Richardson coefficient* and denoted $c^{\nu}_{\lambda;\mu}$.

The relation

$$mult\,(W_\nu; W_\lambda \otimes W_\nu) = mult\,(V_{n^*}; V_\ell \otimes V_m) = c\,(\ell; m; n)$$

holds when V_ℓ, V_m and V_{n^*} are the restrictions of W_λ, W_μ and W_ν respectively, that is when:

$$\begin{cases} \ell_1 = \lambda_1 - \lambda_2, \quad m_1 = \mu_1 - \mu_2, \quad n_2 = \nu_1 - \nu_2, \\ \ell_2 = \lambda_2 - \lambda_3, \quad m_2 = \mu_2 - \mu_3, \quad n_1 = \nu_2 - \nu_3. \end{cases} \tag{17}$$

(We prefer to denote as V_{n^*} the restriction of W_ν, rather than as V_n, to get indices as simple as possible for the triple multiplicities).

6.2. The chamber complex for the Littlewood–Richardson coefficients

We now describe the Littlewood–Richardson coefficients and, more generally, the GL_3 tensor multiplicities, in the geometric language of cones.

The support of the GL_3 tensor multiplicities (the set of triples of labels $(\lambda; \mu; \nu)$ such that the corresponding tensor multiplicity is nonzero) is a set of integer points in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. This support is contained in the 8-dimensional subspace $\mathcal{L}_{\text{LRC}}$ defined by the linear equation $|\lambda| + |\mu| = |\nu|$. Let $\Lambda_{\text{LRC}} = \mathbb{Z}^9 \cap \mathcal{L}_{\text{LRC}}$ be the lattice of all integer points in $\mathcal{L}_{\text{LRC}}$.

Formulas (17) define a projection $\pi : \mathcal{L}_{\text{LRC}} \to \mathcal{L}_{\text{TM}}$. The space $\mathcal{L}_{\text{LRC}}$ decomposes as

$$\mathcal{L}_{\text{LRC}} = \mathcal{L}^0_{\text{LRC}} \oplus \ker \pi \tag{18}$$

where $\mathcal{L}^0_{\text{LRC}}$ is the 6-dimensional subspace defined by the equations $\lambda_3 = \mu_3 = 0$, and $\ker\pi$ is the kernel of the projection π. This kernel is a 2-dimensional subspace, with basis $(\overrightarrow{\lambda}, \overrightarrow{\mu})$ where

$$\overrightarrow{\lambda} = (1,1,1;0,0,0;1,1,1), \quad \overrightarrow{\mu} = (0,0,0;1,1,1;1,1,1).$$

The projection π maps isomorphically $\mathcal{L}^0_{\text{LRC}}$ onto $\mathcal{L}_{\text{TM}}$; its inverse defines an embedding $\iota : \mathcal{L}_{\text{TM}} \hookrightarrow \mathcal{L}_{\text{LRC}}$ with image $\mathcal{L}^0_{\text{LRC}}$. The decomposition (18) together with the isomorphism $\mathcal{L}^0_{\text{LRC}} \cong \mathcal{L}_{\text{TM}}$, define an isomorphism

$$\mathcal{L}_{\mathsf{TM}} \times \mathbb{R}^2 \cong \mathcal{L}_{\text{LRC}}. \tag{19}$$

Under this isomorphism: $\Lambda_{\text{TM}} \times \mathbb{Z}^2$ corresponds to Λ_{LRC}; the cone $\mathsf{TM} \times \mathbb{R}^2$ corresponds to a cone H (known as the *Horn cone*); and $\text{lat}(\mathsf{TM}) \times \mathbb{Z}^2$ corresponds to $\text{lat}(\mathsf{H})$ (the set of lattice points of the Horn cone), and this is the support of the GL_3 tensor multiplicities. The Horn cone is better known as the set of spectra of triples of Hermitian matrices (A,B,C) that fulfill $A+B=C$, see [22, 23].

Under the isomorphism (19) we have also that: the cone $\mathsf{TM} \times (\mathbb{R}^+)^2$ corresponds to a cone H^+ (known as the *positive Horn cone*), intersection of H with the subset defined by $\lambda_3 \geqslant 0$ and $\mu_3 \geqslant 0$; and the set $\text{lat}(\mathsf{TM}) \times (\mathbb{Z}^+)^2$ corresponds to $\text{lat}(\mathsf{H}^+)$ (lattice points of the positive Horn cone). This is the support of the Littlewood–Richardson coefficients.

The polynomial formula $1 + L_{i,j}$, that holds for the triple multiplicities in a chamber $C(i,j)$ of the chamber complex $\mathcal{K}_{\text{TM}}$, gives (by plugging the expressions (17) of ℓ, m, n in terms of λ, μ, ν) a formula for the GL_3 tensor multiplicities that holds in the cone that corresponds to $C(i,j) \times \mathbb{R}^2$. These cones are the chambers of a chamber complex subdividing H. Each of its chamber is isomorphic to some $C(i,j) \times \mathbb{R}^2$.

The same formulas apply, of course, to the Littlewood–Richardson coefficients. These are defined only for λ_3, μ_3, ν_3 nonnegative. The domains for these formulas correspond thus to the products $C(i,j) \times (\mathbb{R}^+)^2$ and are the chambers of a chamber complex $\mathcal{K}_{\text{LRC}}$ subdividing H^+. We will denote with $C^\bullet(i,j)$ the chamber obtained from $C(i,j)$. Precisely, it is the cone generated by $\iota(C(i,j))$ and the vectors $\overrightarrow{\lambda}$, $\overrightarrow{\mu}$.

The rays of $\mathcal{K}_{\text{LRC}}$ are generated by the vectors $\overrightarrow{\lambda}$, $\overrightarrow{\mu}$ and the images under the embedding ι of the ray generators for $\mathcal{K}_{\text{TM}}$. These are given in table 6.

The chambers of $\mathcal{K}_{\text{LRC}}$ are 8–dimensional and generated by 8 rays (the embeddings of the 6 rays of the corresponding chamber from $\mathcal{K}_{\text{TM}}$, plus $\overrightarrow{\lambda}$ and $\overrightarrow{\mu}$). As a consequence, they are all simplicial. The cells of $\mathcal{K}_{\text{LRC}}$ are in order–preserving bijection with the Cartesian product of the set of all chambers of $\mathcal{K}_{\text{TM}}$ with $\mathit{Subsets}\{\overrightarrow{\lambda}, \overrightarrow{\mu}\}$. Its rank generating function is thus the rank generating function of $\mathcal{K}_{\text{TM}}$, multiplied with $(1+q)^2$, which gives

$$\left(1+3q+3q^2\right)^2 (1+2q)(1+q)^3.$$

6.3. Linear symmetries of the Littlewood–Richardson coefficients.

Recall the decomposition (18): $\mathcal{L}_{\text{LRC}} = \mathcal{L}^0_{\text{LRC}} \oplus \ker\pi$. Among the 11 ray generators of the chamber complex $\mathcal{K}_{\text{LRC}}$ (table 6), the vectors $\overrightarrow{\lambda}$ and $\overrightarrow{\mu}$ form a basis of $\ker\pi$, while the other nine generators belong to $\mathcal{L}^0_{\text{LRC}}$. Therefore, $\overrightarrow{\lambda}$ and $\overrightarrow{\mu}$ are not involved in any linear relation between these 11 vectors. As what regards the other nine generators, they are related by

$$\overrightarrow{C_1^\bullet} + \overrightarrow{C_2^\bullet} + \overrightarrow{C_3^\bullet} = \overrightarrow{D_1^\bullet} + \overrightarrow{D_3^\bullet} + \overrightarrow{D_5^\bullet} = \overrightarrow{\triangleleft^\bullet} + \overrightarrow{\triangleright^\bullet} = \overrightarrow{\star^\bullet}.$$

Table 6. The ray generators $u=(\ell_1\ell_2|m_1m_2|n_1n_2)$ for $\mathcal{K}_{\mathrm{TM}}$, and the corresponding ray generators $v=(\lambda_1\lambda_2\lambda_3|\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3)$ for $\mathcal{K}_{\mathrm{LRC}}$. They are determined from each other by the conditions: $u=\pi(v)$ with $v\in\mathcal{L}^0_{\mathrm{LRC}}$; and $v=\iota(u)$.

ray generator u of $\mathcal{K}_{\mathrm{TM}}$	ray generator v of $\mathcal{K}_{\mathrm{LRC}}$	Name of v in [17]
$\overrightarrow{D}_1=(10\|01\|00)$	$\overrightarrow{D}^{\bullet}_1=(100\|110\|111)$	d_2
$\overrightarrow{D}_3=(00\|10\|01)$	$\overrightarrow{D}^{\bullet}_3=(000\|100\|100)$	g_2
$\overrightarrow{D}_5=(01\|00\|10)$	$\overrightarrow{D}^{\bullet}_5=(110\|000\|110)$	e_1
$\overrightarrow{C}_1=(00\|01\|10)$	$\overrightarrow{C}^{\bullet}_1=(000\|110\|110)$	e_2
$\overrightarrow{C}_2=(10\|00\|01)$	$\overrightarrow{C}^{\bullet}_2=(100\|000\|100)$	g_1
$\overrightarrow{C}_3=(01\|10\|00)$	$\overrightarrow{C}^{\bullet}_3=(110\|100\|111)$	d_1
$\overrightarrow{\triangleright}=(01\|01\|01)$	$\overrightarrow{\triangleright}^{\bullet}=(110\|110\|211)$	c
$\overrightarrow{\star}=(11\|11\|11)$	$\overrightarrow{\star}^{\bullet}=(210\|210\|321)$	b
$\overrightarrow{\triangleleft}=(10\|10\|10)$	$\overrightarrow{\triangleleft}^{\bullet}=(100\|100\|110)$	f
	$\overrightarrow{\lambda}=(111\|000\|111)$	a_1
	$\overrightarrow{\mu}=(000\|111\|111)$	a_2

that come from (15) by applying the embedding of $\mathcal{L}_{\mathrm{TM}}$ into $\mathcal{L}_{\mathrm{LRC}}$.

As a consequence, the group of linear automorphisms of the 'Littlewood–Richardson function' $(\lambda;\mu;\nu)\mapsto c^{\nu}_{\lambda;\mu}$ decomposes as the product $G_1\times G_2$, where G_1 is the subgroup of the linear automorphisms fixing $\mathcal{L}^0_{\mathrm{LRC}}$, and G_2 is the subgroup of the automorphisms fixing $\overrightarrow{\lambda}$ and $\overrightarrow{\mu}$. The subgroup G_2 is isomorphic to the group G of all symmetries of the triple multiplicities. The subgroup G_1 is not trivial: it contains the involution swapping $\overrightarrow{\lambda}$ and $\overrightarrow{\mu}$. It is thus isomorphic to $\mathfrak{S}_2$.

As a conclusion, the group of all linear symmetries of the function $(\lambda;\mu;\nu)\mapsto c^{\nu}_{\lambda;\mu}$ has order 288, and is isomorphic to $\mathfrak{S}_2\times G$, and thus to $\mathfrak{S}_2\times\mathfrak{S}_2\times(\mathfrak{S}_3\wr\mathfrak{S}_2)$.

The generator of G_1, swapping $\overrightarrow{\lambda}$ and $\overrightarrow{\mu}$, is easily interpreted, and holds for GL_k for any k: the representation $W_{(1^k)}$ of GL_k is the one dimensional representation where $g\in GL_k$ acts by multiplication by $\det(g)^k$; for any λ,

$$W_{\lambda+(1^k)}\cong W_\lambda\otimes W_{(1^k)}$$

and therefore, for any λ and μ,

$$W_{\lambda+(1^k)}\otimes W_\mu\cong W_\lambda\otimes W_{(1^k)}\otimes W_\mu\cong W_\lambda\otimes W_{\mu+(1^k)}$$

which gives that for any λ, μ and ν,

$$c^{\nu}_{\lambda+(1^k);\mu}=c^{\nu}_{\lambda;\mu+(1^k)}.$$

7. Triple multiplicities and Littlewood–Richardson coefficients as volumes

In this section, we observe that the linear part of the $SU(3)$ triple multiplicity $c(\ell;m;n)$ for can be interpreted as the volume of a parallelotope (higher–dimensional parallelepiped), or, equivalently, as a determinant.

Remember that any chamber $C(i,j)$ of $\mathcal{K}_{\mathrm{TM}}$ has 6 rays: one of them is interior (not on the border of TM), with generator $\overrightarrow{\star}$; the other 5 are exterior.

We contend that when $t=(\ell;m;n)$ lies in the chamber $C(i,j)$, we have

$$c(t)=1+\mathrm{Vol}_{\Lambda_{\mathrm{TM}}}\left(\Pi\left(t_1,\ldots,t_5,t\right)\right) \tag{20}$$

where t_1, …, t_5 are the minimal generators of the 5 exterior rays of $C(i,j)$, $\Pi(t_1,\ldots,t_5,t)$ is the parallelotope generated by them and t (the set of all combinations $x_1t_1+\ldots+x_5t_5+x_6t$ such that all x_i fulfill $0\leqslant x_i\leqslant 1$); and $\mathrm{Vol}_{\Lambda_{\mathrm{TM}}}$ is the volume with respect to the lattice Λ_{TM}, i.e. the volume normalized in such a way that the fundamental domains of the lattice have volume 1.

Let us check this. On $C(i,j)$, we have $c(t)=1+L_{i,j}(t)$, with $L_{i,j}$ linear. We should prove that $L_{i,j}$ coincides with the volume of the parallelotope.

On $C(i,j)$, both $L_{i,j}(t)$ and $\mathrm{Vol}_{\Lambda_{\mathrm{TM}}}\left(\Pi(t_1,\ldots,t_5,t)\right)$ are linear in t; both vanish on the external facet of $C(i,j)$ (generated by t_1, …, t_5). It is enough to check that both evaluate equally at $\overrightarrow{\star}$ to conclude that they are equal. We have $c(\overrightarrow{\star})=2$. Therefore $L_{i,j}(\overrightarrow{\star})=1$.

We will show that $\mathrm{Vol}_{\Lambda_{\mathrm{TM}}}\left(\Pi(t_1,\ldots,t_5,t)\right)=1$ as well, that is that $\Pi(t_1,\ldots,t_5,\star)$ is a fundamental domain for Λ. This amounts in showing that the set $B=\{t_1,\ldots,t_5,\overrightarrow{\star}\}$ is a basis for Λ.

The 9 minimal generators for the rays of $\mathcal{K}_{\mathrm{TM}}$ (table 5) span the lattice Λ_{TM} (section 4.3). Between them hold the relations (15).

The set B contains all minimal generators for the rays of $\mathcal{K}_{\mathrm{TM}}$, with the exception of: one of the $\overrightarrow{C_i}$, one of the $\overrightarrow{D_j}$, and one of $\overrightarrow{\triangleleft}$ and $\overrightarrow{\triangleright}$. This follows from the fact that the chambers are the maximal elements of the poset of cells of $\mathcal{K}_{\mathrm{TM}}$, and from the description of this poset given in section 5.4

But then, by (15), the set B also spans Λ_{TM}. Since B has 6 elements and Λ_{TM} has rank 6, we conclude that B is a basis of Λ_{TM}, which was what was to be demonstrated.

Formula (20) can be written using a determinant. Let $M(t_1,t_2,\ldots)$ be the matrix whose columns give the coordinates $(\ell_1,\ell_2;m_1,m_2;n_1,n_2)$ of the vectors $t_1,t_2,\ldots$ Then

$$|\det M\left(t_1,\ldots,t_5,t\right)|=\mathrm{Vol}_{\mathbb{Z}^6}\left(\Pi\left(t_1,\ldots,t_5,t\right)\right).$$

But because Λ_{TM} has index 3 in $\mathbb{Z}^6$ (since Λ_{TM} is defined by (2)), there is the relation $\mathrm{Vol}_{\Lambda_{\mathrm{TM}}}=3\ \mathrm{Vol}_{\mathbb{Z}^6}$. As a consequence, for $t=(\ell,m,n)\in C(i,j)$,

$$c(\ell;m;n)=1+\frac{1}{3}\left|\det M\left(t_1,\ldots,t_5,t\right)\right|. \tag{21}$$

For example: Chamber $C(1,1)$ is generated by $\overrightarrow{C_2}$, $\overrightarrow{C_3}$, $\overrightarrow{D_3}$, $\overrightarrow{D_5}$, $\overrightarrow{\triangleright}$ and $\overrightarrow{\star}$. Since

$$\det\left(\overrightarrow{C_2},\overrightarrow{C_3},\overrightarrow{D_3},\overrightarrow{D_5},\overrightarrow{\triangleright},\overrightarrow{\star}\right)>0,$$

we have that for any $t=(\ell_1,\ell_2;m_1,m_2;n_1,n_2)$ in chamber $C(1,1)$:

$$\det\left(\overrightarrow{C_2},\overrightarrow{C_3},\overrightarrow{D_3},\overrightarrow{D_5},\overrightarrow{\triangleright},t\right)\geqslant 0,$$

and

$$\begin{aligned} c\left(\ell_1,\ell_2;m_1,m_2;n_1,n_2\right)&=1+\frac{1}{3}\det\left(\overrightarrow{C_2},\overrightarrow{C_3},\overrightarrow{D_3},\overrightarrow{D_5},\overrightarrow{\triangleright},t\right),\\ &=1+\frac{1}{3}\det\begin{bmatrix}1&0&0&0&0&\ell_1\\0&1&0&1&1&\ell_2\\0&1&1&0&0&m_1\\0&0&0&0&1&m_2\\0&0&0&1&0&n_1\\1&0&1&0&1&n_2\end{bmatrix}.\end{aligned}$$

There one such formula for each chamber. If the chamber t belongs to is unknown, we can evaluate all 18 formulas. If all results are negative, then $c(t)=0$; otherwise, $c(t)$ is the minimum of the 18 number obtained (see 9.4).

Formulas analogous to (20), proved with similar arguments, hold for the Littlewood–Richardson coefficients of GL_3. Any chamber $C^\bullet(i,j)$ of $\mathcal{K}_{\text{LRC}}$ has as minimal generators for its rays $\overrightarrow{\star^\bullet}$, $\overrightarrow{\lambda}$, $\overrightarrow{\mu}$ and 5 other vectors v_1, …, v_5 from table 6. For any vectors $w_1, w_2, \ldots$ in $\mathcal{L}_{\text{LRC}}$ let $M^\bullet(w_1,w_2,\ldots)$ be the matrix whose columns give the coordinates $(\lambda_1,\lambda_2,\lambda_3;\mu_1,\mu_2,\mu_3;\nu_1,\nu_2)$ (the component ν_3 is dropped) of $w_1, w_2, \ldots$ Then, for any lattice point $w=(\lambda;\mu;\nu)\in C^\bullet(i,j)$, there is

$$\begin{aligned} c^{\nu}_{\lambda;\mu} &= 1+\mathrm{Vol}_{\Lambda_{\text{LRC}}}\left(\Pi\left(v_1,\ldots,v_5,\overrightarrow{\lambda},\overrightarrow{\mu},w\right)\right)\\ &= 1+\left|\det M^\bullet\left(v_1,\ldots,v_5,\overrightarrow{\lambda},\overrightarrow{\mu},w\right)\right|. \end{aligned}$$

Formulas (20) and (21) do not aim at improving implemented calculations of $SU(3)$ triple multiplicities (the more explicit formulas in 9.4 are better suited for this). Rather, they give qualitative information on the nature and properties of the triple multiplicities function. For instance, the interpretation of (20) (except for the constant 1) as the continuous volume of a parallelotope provides an interesting visual explanation for the following fact: if any of $\ell_1,\ell_2,m_1,m_2,n_1,n_2$ are 0, then the corresponding triple multiplicity is 1. Namely, whenever one of the Dynkin labels vanishes, we see that the parallelotope has volume 0 since the dimension drops. The case when $\ell_2=0$ is given by Pieri's rule.

8. Stability

Stability in representation theory refers to the following property exhibited by functions F defining structural constants in terms of their labels: for any fixed label u and any 'well-chosen' label v, the sequence with general term $F(u+kv)$ (depending on the integer k) is eventually constant. This property was observed first in some instances by Murnaghan [24] for Kronecker coefficients (tensor multiplicities for the symmetric groups), then in greater generality in [25, 26], and widely generalized for other families of representation–theoretic settings [26, 27]. For triple multiplicities and Littlewood–Richardson coefficients, the stability property is easily explained thanks to the combinatorial models.

The case under consideration in this paper, of the triples multiplicities for $SU(3)$, provides an even simpler 'toy example'.

Claim: Let $u\in\mathrm{lat}(\mathsf{TM})$ such that $c(u)=1$. Then, for any $t\in\mathrm{lat}(\mathsf{TM})$, there exists a positive integer k_0 such that

$$\forall k\geqslant k_0,\quad c(t+ku)=1+\min_{i,j}\{L_{i,j}(t):u\in C(i,j)\}$$

is independent of k.

Proof. Set $L(\ell,m,n)=c(\ell,m,n)-1$.

The key is to observe that there exists some chamber $C(i_0,j_0)$ and some index k_0 such that $t+ku\in C(i_0,j_0)$ for all $k\geqslant k_0$. Indeed, '$t+ku\in C(i_0,j_0)$ for k big enough' is equivalent to '$u+\varepsilon t\in C(i_0,j_0)$ for $\varepsilon>0$ small enough', since $C(i_0,j_0)$ is defined by linear inequalities (set

$\varepsilon = 1/k$). Then u lies in the same chamber $C(i_0,j_0)$. Since $c(u) = 1$, we have $L(u) = 0$. Since $u \in C(i_0,j_0)$, this yields $L_{i_0,j_0}(u) = 0$. On the other hand, for all $k \geqslant k_0$,

$$L(t+ku) = L_{i_0,j_0}(t+ku) = L_{i_0,j_0}(t) + kL_{i_0,j_0}(u) = L_{i_0,j_0}(t).$$

Further, we note that since $t+ku \in C(i_0,j_0)$, we have that $L_{i_0,j_0}(t+ku) = \min_{i,j} L_{i,j}(t+ku)$. Let $C(i,j)$ be any chamber. We have thus

$$L_{i_0,j_0}(t) = L_{i_0,j_0}(t+ku) \leqslant L_{i_0,j_0}(t+ku) = L_{i_0,j_0}(t) + kL_{i_0,j_0}(u).$$

Consider now any the chamber $C(i,j)$ containing u. Then $L_{i,j}(u) = L(u) = 0$. Therefore $L_{i_0,j_0}(t) \leqslant L_{i,j}(t)$.

This shows that $L_{i_0,j_0}(t)$ is the minimum of all $L_{i,j}(t)$ over all chambers $C(i,j)$ containing u. $\square$

The result not only tells us that the sequence stabilizes, but also yields its stable value: this is $1 + L_{i,j}(t)$, where $C(i,j)$ is any of the chambers containing $u + \varepsilon t$ for small positive ε.

9. Final remarks

9.1. Symmetries that cannot be lifted

We have seen in section 4.2 that, among the 12 general symmetries for the $SU(3)$ triple multiplicities, only 6 of them can be lifted to symmetries of the cone of the BZ triangles. This followed from the calculation of the full group of symmetries of lat(BZ). Let us give here instead a short argument of this fact. We will show that $\ell \leftrightarrow m$ does not lift to a symmetry of lat(BZ).

Consider the eight vectors $\overrightarrow{C_i}$, $\overrightarrow{D_j}$, $\overrightarrow{\triangleleft}$ and $\overrightarrow{\triangleright}$, all defined in table 3. Each is the projection of a unique BZ triangle $\Delta_{\overrightarrow{C_i}}$, $\Delta_{\overrightarrow{D_j}}$, $\Delta_{\overrightarrow{\triangleleft}}$ and $\Delta_{\overrightarrow{\triangleright}}$ (the 'fundamental' BZ triangles), also defined in table 3.

Remember the following relation:

$$\Delta_{\overrightarrow{D_1}} + \Delta_{\overrightarrow{D_3}} + \Delta_{\overrightarrow{D_5}} = \Delta_{\overrightarrow{\triangleleft}} + \Delta_{\overrightarrow{\triangleright}}. \tag{9}$$

On the other hand, the involution $\ell \leftrightarrow m$ fixes $\overrightarrow{\triangleleft}$ and $\overrightarrow{\triangleright}$, and swaps $\overrightarrow{D_1}$, $\overrightarrow{D_3}$ and $\overrightarrow{D_5}$ with $\overrightarrow{C_3}$, $\overrightarrow{C_2}$ and $\overrightarrow{C_1}$ respectively. But, last, as one can check:

$$\Delta_{\overrightarrow{C_3}} + \Delta_{\overrightarrow{C_2}} + \Delta_{\overrightarrow{C_1}} \neq \Delta_{\overrightarrow{\triangleleft}} + \Delta_{\overrightarrow{\triangleright}}. \tag{22}$$

A would–be lifting of $\ell \leftrightarrow m$ would fix $\Delta_{\overrightarrow{\triangleleft}}$ and $\Delta_{\overrightarrow{\triangleright}}$, and map $\Delta_{\overrightarrow{D_1}}$, $\Delta_{\overrightarrow{D_3}}$, $\Delta_{\overrightarrow{D_5}}$ to $\Delta_{\overrightarrow{C_3}}$, $\Delta_{\overrightarrow{C_2}}$ and $\Delta_{\overrightarrow{C_1}}$. Then applying this lifting to (9) would give in (22) an equality. Therefore (9) together with (22) provide a clear obstruction for the existence of a lifting of $\ell \leftrightarrow m$.

This argument adapts straightforwardly for the Littlewood–Richardson coefficients, using for instance the hive model (see [20]): replace in the above formulas each ray generator $\overrightarrow{X}$ of $\mathcal{K}_{\text{TM}}$ with the corresponding ray generator $\overrightarrow{X^\bullet}$ of $\mathcal{K}_{\text{LRC}}$ (see table 6), and the unique BZ triangle $\Delta_{\overrightarrow{X}}$ above $\overrightarrow{X}$, with the unique hive above $\overrightarrow{X^\bullet}$.

The impossibility of lifting $\lambda \leftrightarrow \mu$ in the GL_4 case was already pointed out in [20]. The above calculation shows this already happens for GL_3.

9.2. The $SU(2)$ case

In the $SU(2)$ case, each of ℓ, m and n has only one coordinates. In this case, it follows from the Pieri rule that $c(\ell,m,n)$ counts the integer solutions x of

$$2x = \ell + m + n \text{and } x \leqslant \ell \text{and } x \leqslant m \text{and } x \geqslant 0.$$

As a consequence, $c(\ell,m,n) = 1$ when

$$\ell + m + n \equiv 0 \mod 2 \text{ and } \ell \leqslant m + n \text{and } m \leqslant \ell + n \text{ and } n \leqslant m + n,$$

and otherwise $c(\ell,m,n) = 0$.

9.3. About the $SU(4)$ case

In contrast to the $SU(3)$ case where the chamber complex for the triple multiplicities has 18 chambers, in the $SU(4)$ case, we have calculated that there are 67769 chambers. In this case, the group of linear symmetries has order 12, i.e. there are only the 12 general linear symmetries. Indeed, any symmetry of the triple multiplicities is also a symmetry of the corresponding cone TM, but the cone TM for $SU(4)$ affords only the 12 general linear symmetries.

9.4. Formulas with minima and maxima

The two following formulas for the triple multiplicities hold on Λ_{TM}:

$$c(t) = \max\left(0, 1 + \min_i f_i(t) - \max_j g_j(t)\right) \tag{23}$$

and

$$c(t) = \max\left(0, 1 + \min_{i,j}\left(f_i(t) - g_j(t)\right)\right). \tag{24}$$

Both formulas appear in [16, formulas (16) and (17)]. Both interpret as counting lattice points in the fibers of a projection $(x,t) \in \mathbb{R} \times \mathbb{R}^6 \mapsto t$ restricted to a cone. The two formulas correspond to two different cones: for (23), the cone is defined by the inequalities $f_i(t) \geqslant x$ and $x \geqslant g_j(t)$ (as in our presentation); for (24), the cone is defined by the inequalities $0 \leqslant x$ and $x \leqslant f_i(t) - g_j(t)$, which is the region 'below the graph' of the piecewise linear concave function $\min_{i,j}(f_i(t) - g_j(t))$. It is interesting to observe that the transformation mapping the first cone to the second, corresponds to a simple 'justification' (to use the term from typography) of the fibers, i.e. pushing each fiber until its bottom is at level 0, see figure 8. The parameters α used in [16, section 3.7] are precisely the x–coordinates in the second description.

9.5. Concise tensor product rule for $SU(3)$

General procedures and formulas for tensor multiplicities have been widely considered, with powerful procedures or formulas holding for $SU(k)$ for all k, or for Lie groups in general [10–12]. But in the particular case of $SU(3)$, they cannot compete with the very explicit form of the ad hoc formula (24).

Using our study of the symmetries of the $SU(3)$ tensor multiplicities, we propose a slight improvement of the presentation of (24). Let us first recall that the group G of all linear symmetries acts transitively on the chambers. Under the subgroup G_{lg}, there are 3 orbits of 6

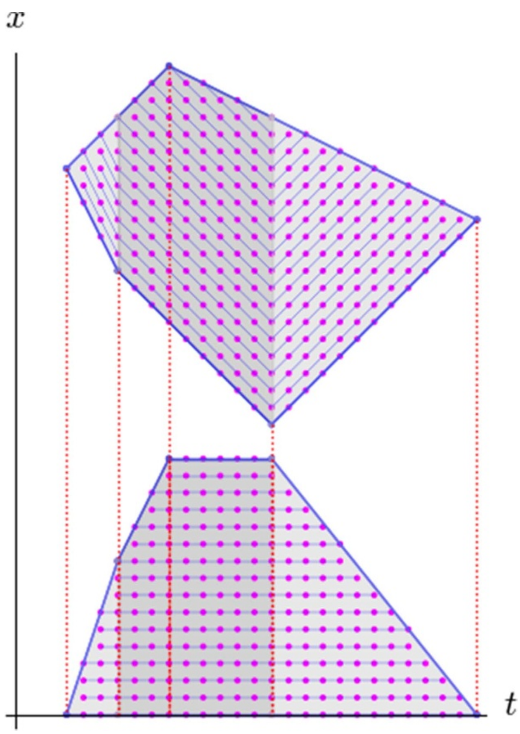
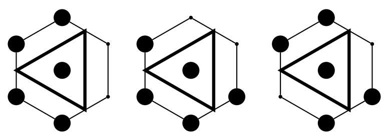


Figure 8. Top: a transversal section of a cone defined by a system of inequalities $g_j(t) \leqslant x \leqslant f_i(t)$. Bottom: the corresponding transversal version of its justification, defined by the inequalities $0 \leqslant x \leqslant f_i(t) - g_j(t)$.

chambers each. For each orbit, all chambers have the same diagram up to a symmetry of the hexagon. The classes of diagrams corresponding to the three G_{lg}-orbits are:



Let θ be the symmetry of order 3 that permutes cyclically $\overrightarrow{C_1}$, $\overrightarrow{C_2}$ and $\overrightarrow{C_3}$, while fixing each of the other 5 ray generators of TM. Then θ maps any chamber to a chamber in another G_{lg}-orbit. Therefore, the transformations $g \circ \theta^i$, for $i \in \{0, 1, 2\}$ and $g \in G_{\mathrm{lg}}$, permute transitively the 18 chambers. Each chamber contains a unique facet of the cone TM, and each facet of TM is contained in a unique chamber. Therefore the transformations $g \circ \theta^i$ permute transitively the facets of TM, as well their primitive inner normals. These inner normals are precisely the linear forms $f_i - g_j$. Note also that $f_1 - g_1$ is the coordinate m_2 (see table 2). After its description in 4.1.1, the group G_{lg} permutes transitively the coordinates $\ell_1, \ell_2, m_1, m_2, n_1$ and n_2. Therefore, the forms $f_i - g_j$ are the six coordinate functions ℓ_1, ℓ_2, m_1, m_2, n_1, n_2, and their compositions with θ and θ^2.

We obtain the following concise presentation of (24):

$$c(t) = \max(0, 1 + \min(\mathrm{mincoord}(t), \mathrm{mincoord}(\theta(t)), \mathrm{mincoord}(\theta^2(t)))$$

where mincoord means 'minimum coordinate'.

The transformation θ admits the following description:

$$\theta(\ell_1,\ell_2,m_1,m_2,n_1,n_2) = (\omega+\ell_1',\omega+\ell_2',\omega+m_1',\omega+m_2',\omega+n_1',\omega+n_2')$$

where $\omega = (\ell_1+m_1+n_1-\ell_2-m_2-n_2)/3$, and $(\ell_1',\ell_2',m_1',m_2',n_1',n_2')$ is the image of $(\ell_1,\ell_2,m_1,m_2,n_1,n_2)$ under the linear map with matrix

$$\begin{bmatrix} P & Q & P \\ P & P & Q \\ Q & P & P \end{bmatrix} \quad \text{where } P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In order to get the multiplicity of the $SU(3)$ irreducible representation V_{n_1,n_2} in $V_{\ell_1,\ell_2} \otimes V_{m_1,m_2}$, one should apply this rule with $t = (\ell_1,\ell_2,m_1,m_2,n_2,n_1)$ i.e. with n_1 and n_2 swapped.

10. Conclusion

We have determined the full group of linear symmetries of the $SU(3)$ tensor multiplicities. This amounts to two results. The first result is the discovery of additional symmetries, proper to the $SU(3)$ case, with respect to the known 12 symmetries that hold in general for the $SU(k)$ tensor multiplicities. The second result is that the list is complete, and thus there is no point in looking for more symmetries of the same kind. Using the action of these symmetries, we have obtained a more structured description for the piecewise linear formulas for the $SU(3)$ tensor multiplicities. We obtained that there is 'essentially one formula' since the group permutes transitively the domains of the formulas.

We believe that the knowledge of the $SU(3)$ tensor symmetries contributes to the understanding of finer analysis of $SU(3)$ tensor products and the unveiling of unexpected connections with other objects. For instance, the paper [19] solves the *missing label problem* for the tensor products of $SU(3)$ representations, by introducing operators decomposing univocally each isotypic component. There, the relevance of the choice of these operators is underlined by the fact that they fulfill the same 144 symmetries as the corresponding tensor multiplicities. A similarity with the 'magic square' symmetries of the $SU(2)$ Clebsch–Gordan coefficients is observed [19, section 2.3].

Our study does not cover *non-linear symmetries*, for instance piecewise linear symmetries such as the one discovered by Coquereaux and Zuber [16], and extended partially for $SU(k)$ for $k > 3$ in [28, 29]. It would be interesting to study systematically such symmetries.

Another possible generalization is for the fusion multiplicities for the affine algebra $\widehat{su}(3)_k$ at level k, that deform the $SU(3)$ tensor multiplicities [30]. It is natural to ask in what extent the results obtained here (symmetries and formulas) can be extended to these fusion coefficients, specially in view of the remarks in [16, section 5].

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

This work was partially supported by the grants PID2020-117843GB-I00, funded by MCIN/AEI/10.13039/501100011033, and P20_01056 (Junta de Andalucía/PAIDI 2020/FEDER).

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