Depósito de Investigación
Universidad de Sevilla

Depósito de investigación de la Universidad de Sevilla

https://idus.us.es/

Esta es la versión aceptada del artículo publicado en:

This is an accepted manuscript of a paper published in:

Discrete Applied Mathematics (Vol. 162): 10 January 2014

DOI: https://doi.org/10.1016/j.dam.2013.08.010

Copyright:

El acceso a la versión publicada del artículo puede requerir la suscripción de la revista.

Access to the published version may require subscription.

"This is an Accepted Manuscript of an article published by Elsevier in Discrete Applied Mathematics on 10 January 2014, available at: https://doi.org/10.1016/j.dam.2013.08.010."

On the connectivity and restricted edge-connectivity of 3-arc graphs

Camino Balbuena and Luis Pedro Montejano
Universitat Politècnica de Catalunya
Barcelona

Pedro García-Vázquez
Universidad de Sevilla
Sevilla

Abstract

A $3-arc$ of a graph G is a 4-tuple (y, a, b, x) of vertices such that both (y, a, b) and (a, b, x) are paths of length two in G. Let $\overleftrightarrow{G}$ denote the symmetric digraph of a graph G. The *3-arc graph* $X(G)$ of a given graph G is defined to have vertices the arcs of $\overleftrightarrow{G}$. Two vertices (ay), (bx) are adjacent in $X(G)$ if and only if (y, a, b, x) is a 3-arc of G. The purpose of this work is to study the edge-connectivity and restricted edge-connectivity of 3-arc graphs. We prove that the 3-arc graph $X(G)$ of every connected graph G of minimum degree $\delta(G) \geq 3$ has edge-connectivity $\lambda(X(G)) \geq (\delta(G) - 1)^2$; and restricted edge- connectivity $\lambda_{(2)}(X(G)) \geq 2(\delta(G) - 1)^2 - 2$ if $\kappa(G) \geq 2$. We also provide examples showing that all these bounds are sharp.

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [10] for terminology and definitions.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v, that is, the set of all vertices adjacent to v. The *degree* of a vertex v is $d(v) = |N_G(v)|$ and the minimum degree $\delta = \delta(G)$ of the graph G is the minimum degree over all vertices of G.

A graph G is called *connected* if every pair of vertices is joined by a path. If $S \subset V(G)$ and $G - S$ is not connected, then S is said to be a *cutset*. A *component* of a graph G is a maximal connected subgraph of G. A (noncomplete) connected graph is called *k-connected* if every cutset has cardinality at least k. The *connectivity* $\kappa(G)$ of a (noncomplete) connected graph G is defined as the maximum integer k such that G is k-connected. The *minimum* cutsets are those having cardinality $\kappa(G)$. The *connectivity* of a complete graph $K_{\delta+1}$ on $\delta + 1$ vertices is defined as $\kappa(K_{\delta+1}) = \delta$. Analogously, for edge connectivity an *edge-cut* in a graph G is a set W of edges of G such that $G - W$ is nonconnected. If W is a minimum edge-cut of a connected graph G, then $G - W$ contains exactly two components. Every connected graph on at least two vertices has an edge-cut. The *edge-connectivity* $\lambda(G)$ of a graph G is the minimum cardinality of an edge-cut of G. A classic result due to Whitney is that for every graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$. A graph is *maximally connected* if $\kappa(G) = \delta(G)$, and *maximally edge-connected* if $\lambda(G) = \delta(G)$.

Though the parameters κ, λ of connectivities give the minimum cost to disrupt the network, they do not take into account what remains after deletion. Even two graphs with the same connectivity κ, λ may be considered to have different reliabilities, since the number of minimum cutsets or edge-cuts is different. Superconnectivity is a stronger measure of connectivity, introduced by Boesch and Tindell in [8], whose study has deserved some attention in the last years, see for instance, [1, 6, 7, 19, 20]. A maximally connected [edge-connected] graph is called *super-κ* [*super-λ*] if for every cutset [edge-cut] W of cardinality $\delta(G)$ there exists a component C of $G - W$ of cardinality $|V(C)| = 1$. The study of super-κ [super-λ] graphs has a particular significance in the design of reliable networks, mainly due to the fact that attaining superconnectivity implies minimizing the number of minimum cutsets [edge-cuts] (see [7, 20]).

In order to measure the super edge-connectivity we use the following parameter introduced by Esfahanian and Hakimi [11]. The *restricted edge-connectivity* $\lambda_{(2)} = \lambda_{(2)}(G)$ is the minimum cardinality over all *restricted edge-cuts* W, i.e., those such that there are no isolated vertices in $G - W$. A restricted edge-cut W is called a *$\lambda_{(2)}$-cut* if $|W| = \lambda_{(2)}$. Obviously for any $\lambda_{(2)}$-cut W, the graph $G - W$ consists of exactly two components

$C, \overline{C}$ and clearly $|V(C)| \geq 2, |V(\overline{C})| \geq 2$. A connected graph G is called $\lambda_{(2)}$-*connected* if $\lambda_{(2)}$ exists. Esfahanian and Hakimi [11] showed that each connected graph G of order $n(G) \geq 4$ except a star, is $\lambda_{(2)}$-connected and satisfies $\lambda_{(2)} \leq \xi$, where $\xi = \xi(G)$ denotes the *minimum edge-degree* of G defined as $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$. Furthermore, a $\lambda_{(2)}$-connected graph is said to be $\lambda_{(2)}$-*optimal* if $\lambda_{(2)} = \xi$. Recent results on this property are obtained in [2, 5, 12, 13, 18, 21, 23]. Notice that if $\lambda_{(2)} \leq \delta$, then $\lambda_{(2)} = \lambda$. When $\lambda_{(2)} > \delta$ (that is to say, when every edge cut of order δ isolates a vertex) the graph must be super-λ. Therefore, by means of this parameter we can say that a graph G is super-λ if and only if $\lambda_{(2)} > \delta$. Thus, we can measure the super edge-connectivity of the graph as the value of the restricted edge-connectivity $\lambda_{(2)}$.

Let $\overleftrightarrow{G}$ denote the symmetric digraph of a graph G. For adjacent vertices u, v of $V(G)$ we use (u, v) to denote the arc from u to v, and $(v, u)(\neq (u, v))$ to denote the arc from v to u. A *3-arc* is a 4-tuple (y, a, b, x) of vertices such that both (y, a, b) and (a, b, x) are paths of length two in G. The *3-arc graph* $X(G)$ of a given graph G is defined to have vertices the arcs of $\overleftrightarrow{G}$ and they are denoted as (uv). Two vertices $(ay), (bx)$ are adjacent in $X(G)$ if and only if (y, a, b, x) is a 3-arc of G, see [17, 22]. Equivalently, two vertices $(ax), (by)$ are adjacent in $X(G)$ if and only if $d_G(a, b) = 1$; that is, the tails a, b of the arcs $(a, x), (b, y) \in A(\overleftrightarrow{G})$ are at distance one in G. Thus the number of edges of $X(G)$ is $\sum_{uv \in E(G)} (d(u) - 1)(d(v) - 1)$ so that the minimum degree of $X(G)$ is $(\delta(G) - 1)^2$. There is a bijection between the edges of $X(G)$ and those of the 2-path graph $P_2(G)$, which is defined to have vertices the paths of length two in G such that two vertices are adjacent if and only if the union of the corresponding paths is a path or a cycle of length three, see [9]. Since $P_2(G)$ is a spanning subgraph of the second iterated line graph $L_2(G) = L(L(G))$ (see e.g. [14]), we have a relation between 3-arc graphs and line graphs. Some results on the connectivity of P_2-path graphs are studied e.g. in [3, 4, 15].

The purpose of this paper is to study the edge-connectivity, the restricted edge-connectivity and vertex-connectivity of the 3-arc graph $X(G)$ of a given graph G. The following theorem gather together the results on connectivity of 3-arc-graph $X(G)$ obtained by Knor and Zhou [16].

Theorem 1 [16] *Let G be a graph with minimum degree $\delta(G)$.*

(i) *$X(G)$ is connected if G is connected and $\delta(G) \geq 3$.*

(ii) $\kappa(X(G)) \geq (\kappa(G)-1)^2$ *if* $\kappa(G) \geq 3$.

The main results contained in this paper are the following:
Let G be a connected graph with minimum degree $\delta(G) \geq 3$.

(i) $\lambda(X(G)) \geq (\delta(G)-1)^2$.

(ii) $\lambda_{(2)}(X(G)) \geq 2(\delta(G)-1)^2 - 2$ *if* $\kappa(G) \geq 2$.

(iii) $\kappa(X(G)) \geq \min\{\kappa(G)(\delta(G)-1), (\delta(G)-1)^2\}$.

(iv) $X(G)$ *is super-*κ *if* $\kappa(G) = \delta(G)$ *and* $\delta(X(G)) = (\delta(G)-1)^2$.

2 Results on the edge-connectivity and restricted edge-connectivity of 3-arc graphs

Let $X(G)$ be the 3-arc graph of a graph G. If (ay) and (bx) are adjacent in $X(G)$ then the edge $(ay)(bx)$ will be called an *ab-edge* (or *ba*-edge). Observe that $(ay)(bx) = (bx)(ay)$ but $(ay) \neq (ya)$ and $(bx) \neq (xb)$. For any edge $ab \in E(G)$ let $\mathcal{V}^a_{ab} = \{(ay) \in V(X(G)) : y \in N_G(a) - b\}$. Observe that the induced subgraph of $X(G)$ by the set $\mathcal{V}^a_{ab} \cup \mathcal{V}^b_{ba}$ is the complete bipartite graph $K_{|\mathcal{V}^a_{ab}|,|\mathcal{V}^b_{ba}|} = K_{d(a)-1,d(b)-1}$.

If W is a minimal edge cut of a connected graph G, then, $G - W$ necessarily contains exactly two components C and $\overline{C}$, so it is usual to denote an edge cut W as $[C, \overline{C}]$ where $[C, \overline{C}]$ denotes the set of edges between C and its complement $\overline{C}$.

Lemma 2 *Let G be a graph and* $[C, \overline{C}]$ *an edge-cut of* $X(G)$. *Let* $ab \in E(G)$, *if* $[C, \overline{C}]$ *contains ab-edges, then it contains at least* $\min\{d(a)-1, d(b)-1\}$ *ab-edges.*

Proof: Suppose that $(ay)(bx)$ is an edge of $[C, \overline{C}]$ such that $(ay) \in V(C)$ and $(bx) \in V(\overline{C})$. Then $\mathcal{V}^a_{ab} \cap V(C) \neq \emptyset$ and $\mathcal{V}^b_{ba} \cap V(\overline{C}) \neq \emptyset$. Let denote by $|\mathcal{V}^a_{ab} \cap V(C)| = r_a \geq 1$, $|\mathcal{V}^b_{ba} \cap V(C)| = r_b \geq 0$, $|\mathcal{V}^a_{ab} \cap V(\overline{C})| = \overline{r}_a \geq 0$ and $|\mathcal{V}^b_{ba} \cap V(\overline{C})| = \overline{r}_b \geq 1$. Moreover, these numbers must satisfy $r_a + \overline{r}_a = d(a) - 1$ and $r_b + \overline{r}_b = d(b) - 1$. Furthermore, the number of *ab*-edges contained in $[C, \overline{C}]$ is $r_a\overline{r}_b + r_b\overline{r}_a$, that is,

$$|[C, \overline{C}]| \geq r_a\overline{r}_b + r_b\overline{r}_a. \quad (1)$$

If $r_b = 0$, then $\overline{r}_b = d(b) - 1$. As $r_a \geq 1$, (1) implies $|[C, \overline{C}]| \geq d(b) - 1$ and the lemma follows. Similarly, if $\overline{r}_a = 0$, the result is also true. Therefore, we can assume that $r_a, r_b, \overline{r}_a, \overline{r}_b \geq 1$. In this case $r_a\overline{r}_b + r_b\overline{r}_a \geq r_a + \overline{r}_a = d(a) - 1$, and $r_a\overline{r}_b + r_b\overline{r}_a \geq r_b + \overline{r}_b = d(b) - 1$, and the result holds. □

Suppose that $[C, \overline{C}]$ is an edge-cut of $X(G)$. Let denote by $\omega(\alpha) = \{e \in E(G) : e = \alpha\beta\}$ and define $\mathcal{A} = \{\alpha\beta \in E(G) : (\alpha y)(\beta x) \in [C, \overline{C}]\}$. Then, as a consequence of the above lemma, we have $|[C, \overline{C}]| \geq |\mathcal{A}|(\delta(G) - 1)$. Next we prove that $|[C, \overline{C}]| \geq (\delta(G) - 1)^2$.

Lemma 3 *Let G be a graph and $[C, \overline{C}]$ an edge-cut of $X(G)$. Let $ab \in E(G)$ and suppose that $ab \in \mathcal{A}$. Then $|(\omega(a) \cup \omega(b)) \cap \mathcal{A}| \geq (\delta - 1)^2$.*

Proof: Suppose that for all $y \in N(a) - b$, $ay \in \mathcal{A}$. Then there are at least δ different ay-edges in $[C, \overline{C}]$, and by Lemma 2 the number of ay-edges in $[C, \overline{C}]$ is at least $\delta(\delta - 1) > (\delta - 1)^2$. The same occurs if for every $x \in N(b) - a$, $bx \in \mathcal{A}$. Therefore we may assume that there exists $y_0 \in N_G(a) - b$ such that $ay_0 \notin \mathcal{A}$ and there exists $x_0 \in N_G(b) - a$ such that $bx_0 \notin \mathcal{A}$.

As $ab \in \mathcal{A}$, $(ay')(bx') \in [C, \overline{C}]$ for some $y' \in N(a) - b$ and $x' \in N(b) - a$, and without loss of generality we may suppose that $(ay') \in V(C)$, $(bx') \in V(\overline{C})$. Suppose that $(ay_0)(bx_0) \notin [C, \overline{C}]$. Without loss of generality we may assume that $(ay_0), (bx_0) \in V(\overline{C})$ in which case $(ay')(bx_0) \in [C, \overline{C}]$ because $(ay') \in V(C)$. Then we can continue the proof assuming that there is an edge $(ay)(bx) \in [C, \overline{C}]$ such that $bx \notin \mathcal{A}$, i.e., there are no bx-edges in $[C, \overline{C}]$.

First suppose that $\mathcal{V}^x_{xb} \cap V(C) \neq \emptyset$. Let $B = \{x' \in N_G(b) \setminus \{x, a\} : (x'z) \in V(C)\}$ and $\overline{B} = \{x' \in N_G(b) \setminus \{x, a\} : (x'z) \in V(\overline{C})\}$. Observe that for all $x' \in B \cup \overline{B}$, $(x'z)$ is adjacent to $(bx) \in V(\overline{C})$, and $(x'z)$ is adjacent to (ba). Hence the edge-cut $[C, \overline{C}]$ must contain $|B|$ different bx'-edges. Moreover, since (ba) is adjacent to every $(xb') \in \mathcal{V}^x_{xb}$ and $bx \notin \mathcal{A}$, then $(ba) \in V(C)$ because our assumption $\mathcal{V}^x_{xb} \cap V(C) \neq \emptyset$. Hence $[C, \overline{C}]$ also contains $|\overline{B}|$ different bx'-edges yielding that $[C, \overline{C}]$ contains at least $|B| + |\overline{B}| + |\{ab\}| = d(b) - 1 \geq \delta - 1$ different bv-edges with $v \in N(b)$ and by Lemma 2, the result holds.

Second suppose that $\mathcal{V}^x_{xb} \subset V(\overline{C})$. Hence $\mathcal{V}^b_{ba} \subset V(\overline{C})$ because every $(bx') \in \mathcal{V}^b_{ba}$ is adjacent to every $(xb') \in \mathcal{V}^x_{xb}$ and $[C, \overline{C}]$ does not contain bx-edges. If $ay \notin \mathcal{A}$, reasoning for ay in the same way as for bx we get that $\mathcal{V}^a_{ab} \subset V(C)$. Thus as $\mathcal{V}^b_{ba} \subset V(\overline{C})$ it follows that $[C, \overline{C}]$ contains at least

$(d(a)-1)(d(b)-1) \geq (\delta-1)^2$ ab-edges and the lemma holds. Therefore, suppose that $ay \in \mathcal{A}$.

We know that there exists $v \in N_G(a) - y$ such that $av \notin \mathcal{A}$. As (va') is adjacent to (ay) for all $(va') \in \mathcal{V}^v_{va}$ it follows that $\mathcal{V}^v_{va} \subset V(C)$ (because $(ay) \in V(C)$ and $av \notin \mathcal{A}$). Hence $\mathcal{V}^a_{av} \subset V(C)$ because every $(ay') \in \mathcal{V}^a_{av}$ is adjacent to $(va') \in \mathcal{V}^v_{va}$. As $\mathcal{V}^b_{ba} \subset V(\overline{C})$ it follows that $[C, \overline{C}]$ contains at least $(d(a)-2)(d(b)-1)$ ab-edges. Further, as $ay \in \mathcal{A}$, by Lemma 2, $[C, \overline{C}]$ also contains at least $\delta - 1$ ay-edges, yielding that the number of au-edges contained $|[C, \overline{C}]|$ is at least $(\delta-2)(\delta-1) + (\delta-1) = (\delta-1)^2$, and the lemma holds. □

Theorem 4 *Let G be a connected graph with minimum degree $\delta \geq 3$. Then*

$$\lambda(X(G)) \geq (\delta-1)^2.$$

Proof: Let $[C, \overline{C}]$ be a minimum edge-cut of $X(G)$ and $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. As G is connected and $\delta \geq 3$, then $X(G)$ is connected yielding that $|\mathcal{A}| \geq 1$. So considering $ab \in \mathcal{A}$, and using Lemma 3 we get $|[C, \overline{C}]| \geq (\delta-1)^2$, following the theorem. □

The following corollary is an immediate consequence from Theorem 4, and from the fact that if G is a graph of minimum degree δ having an edge xy such that $d(x) = \delta$ and $d(y') = \delta$ for all $y' \in N_G(x) - y$, then the minimum degree of $X(G)$ is $\delta(X(G)) = (\delta-1)^2$.

Corollary 5 *Let G be a connected graph of minimum degree $\delta \geq 3$ having an edge xy such that $d(x) = \delta$ and $d(y') = \delta$ for all $y' \in N_G(x) - y$. Then the 3-arc graph $X(G)$ of G is maximally edge-connected.*

Figure 1 shows a 3-regular graph G with $\lambda(G) = 1$ and its 3-arc graph $X(G)$ which has $\lambda(X(G)) = 4 = \delta(X(G))$. However $X(G)$ is not super-λ and hence is not $\lambda_{(2)}$-optimal. And Figure 2 shows a 3-regular graph G with $\lambda(G) = \kappa(G) = 2$, and its 3-arc graph $X(G)$ which has $\lambda(X(G)) = 4$ and $\lambda_{(2)}(X(G)) = 6 = \xi(X(G))$, i.e., this graph is $\lambda_{(2)}$-optimal. In what follows we give a lower bound on the restricted edge-connectivity $\lambda_{(2)}(X(G))$ where G is a graph having connectivity $\kappa(G) \geq 2$.

Two edges which are incident with a common vertex are *adjacent*.

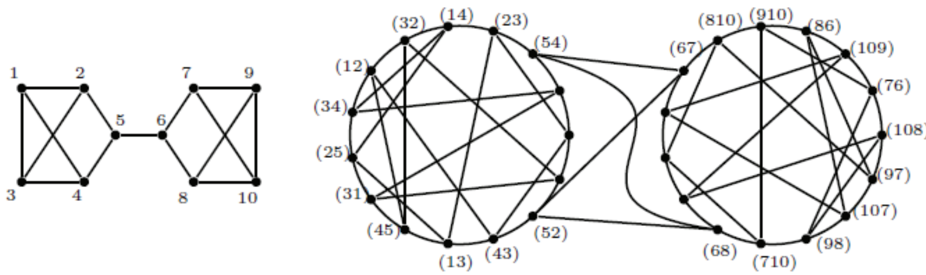

Figure 1: A 3-regular graph with $\lambda = 1$ and its 3-arc graph.

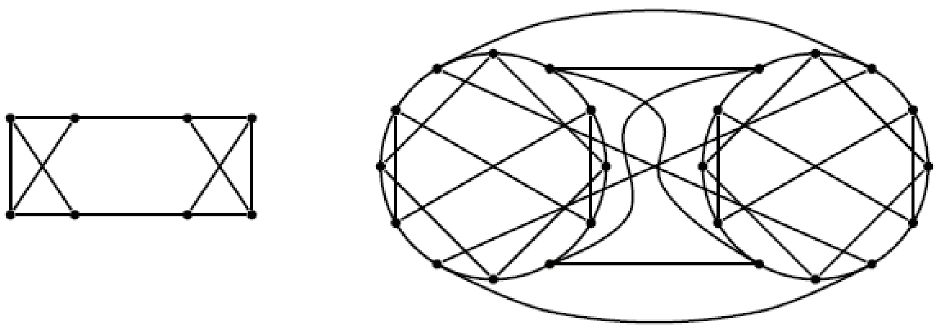

Figure 2: A 3-regular graph with $\lambda = 2$ ($\kappa = 2$) and its 3-arc graph.

Lemma 6 *Let G be a graph with minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$. Let $[C, \overline{C}]$ be a restricted edge-cut of $X(G)$ and consider the set $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. Then there are at least two nonadjacent edges in $\mathcal{A}$.*

Proof: Clearly $\mathcal{A} \neq \emptyset$, because $X(G)$ is connected. Thus let $(ay) \in V(C)$ and $(bx) \in V(\overline{C})$ be two adjacent vertices in $X(G)$, which implies that $ab \in \mathcal{A}$. Since $[C, \overline{C}]$ is a restricted edge-cut, then there exist $(uy') \in V(C)$ and $(wx') \in V(\overline{C})$ adjacent to (ay) and (bx) in $X(G)$, respectively. Observe that we may assume that $u \neq w$ because $\delta \geq 3$. Since G is 2-connected we can find a path $R : u = r_0, r_1, \ldots, r_k = w$ from u to w in $G - a$. As $\delta \geq 3$, there exists $v_i \in N(r_i) \setminus \{r_{i-1}, r_{i+1}\}$ for each $i = 1, \ldots, k-1$. Moreover we may choose $v_0 = y'$ and $v_k = x'$. Then the path R induces in $X(G)$ the path $R^* : (uy'), (r_1v_1), \ldots, (r_{k-1}v_{k-1}), (wx')$ (observe that if $k = 1$ then $R^* : (uy'), (wx')$). Since $(uy') \in V(C)$ and $(wx') \in V(\overline{C})$, it follows that $[C, \overline{C}] \cap E(R^*) \neq \emptyset$, hence $r_ir_{i+1} \in \mathcal{A}$ for some $i \in \{0, \ldots, k\}$. Since $a \notin V(R)$ then $a \notin \{r_i, r_{i+1}\}$.

Now reasoning analogously, we can find a path $S : u = s_0, s_1, \ldots, s_\ell = w$ from u to w in $G - b$ that induces a path S^* from $(uy') \in V(C)$ to $(wx') \in V(\overline{C})$. This implies that $[C, \overline{C}] \cap E(S^*) \neq \emptyset$, hence $s_js_{j+1} \in \mathcal{A}$ for some $j \in \{0, \ldots, \ell\}$. Since $b \notin V(S)$ then $b \notin \{s_j, s_{j+1}\}$.

As $ab, r_ir_{i+1}, s_js_{j+1} \in \mathcal{A}$, $a \notin \{r_i, r_{i+1}\}$ and $b \notin \{s_j, s_{j+1}\}$, it follow that al least two of the edges of $\{ab, r_ir_{i+1}, s_js_{j+1}\}$ are nonadjacent. □

Theorem 7 *Let G be a graph with minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$. Then $X(G)$ has restricted edge-connectivity $\lambda_{(2)}(X(G)) \geq 2(\delta - 1)^2 - 2$.*

Proof: Let $[C, \overline{C}]$ be a restricted edge-cut of $X(G)$ and consider the set $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. From Lemma 6, $\mathcal{A}$ contains two nonadjacent edges ab and cd. By Lemma 3, the number of au-edges and bv-edges, $u, v \in N(a) \cup N(b)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$, and the number of cu-edges and dv-edges, $u, v \in N(c) \cup N(d)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$. If $|[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \leq 2$ then $|[C, \overline{C}]| \geq 2(\delta - 1)^2 - |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \geq 2(\delta - 1)^2 - 2$. If $3 \leq |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \leq 4$ then we may assume without loss of generality that $ac, bd \in \mathcal{A}$, hence, by applying Lemma 3, the number of au-edges and cv-edges, $u, v \in N(a) \cup N(c)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$, and the number of bu-edges and dv-edges, $u, v \in N(b) \cup N(d)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$. Thus,

$$\begin{aligned} |[C, \overline{C}]| &\geq 2(\delta - 1)^2 - |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| + 2(\delta - 1)^2 - |[\{a, c\}, \{b, d\}] \cap \mathcal{A}| \\ &\geq 4(\delta - 1)^2 - 8 \\ &\geq 2(\delta - 1)^2 - 2, \end{aligned}$$

since $\delta \geq 3$. Hence the theorem is valid. □

Figure 3 shows that $\lambda(G) \geq 2$ is not enough to guarantee that $\lambda_{(2)}(X(G)) \geq 2(\delta - 1)^2 - 2$. In this example G is a 4-regular graph with $\lambda = 2$ and $\kappa = 1$, but $\lambda_{(2)}(X(G)) = 12 < 16$.

The following corollary is an immediate consequence from Theorem 7, and from the fact that if G is graph of minimum degree δ having an edge

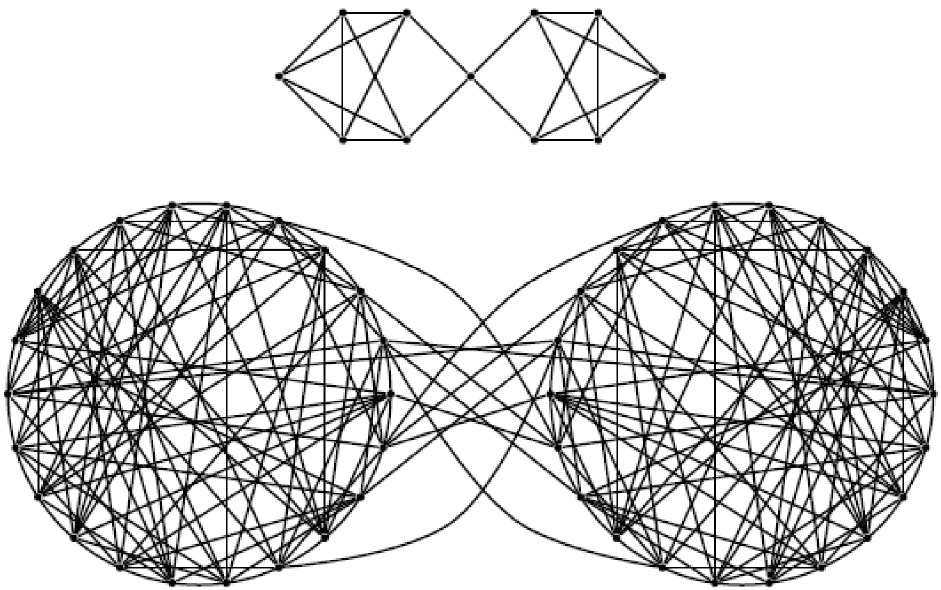

Figure 3: The 3-arc graph of a 4-regular graph with $\kappa = 1$ and $\lambda = 2$ with $\lambda_{(2)}(X(G)) = 12$.

xy such that $d(x) = \delta$, $d(y) = \delta$ and such that every $w \in (N_G(x) - y) \cup (N_G(y) - x)$ also has degree δ, then the minimum edge degree of $X(G)$ is $\xi(X(G)) = 2(\delta - 1)^2 - 2$.

Corollary 8 *Let G be a graph of minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$ having an edge xy such that $d(x) = \delta$, $d(y) = \delta$ and such that every $w \in (N_G(x) - y) \cup (N_G(y) - x)$ also has degree δ. Then the 3-arc graph $X(G)$ has restricted edge connectivity $\lambda_{(2)}(X(G)) = \xi(X(G)) = 2(\delta - 1)^2 - 2$.*

Acknowledgement

Research supported by the Ministry of Science and Innovation, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02/MTM; also by Catalonian goverment under proyect 2009 SGR 1298.

References

[1] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez and X. Marcote. On the restricted connectivity and superconnectivity in graphs with given girth. *Discrete Math.*, 307(6):659–667, 2007.

[2] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez and X. Marcote. Sufficient conditions for λ'-optimality of graphs with smaill conditional diameter. *Inf. Process. Lett.*, 95:429–434, 2005.

[3] C. Balbuena and D. Ferrero. Edge-connectivity and super edge-connectivity of P_2-path graphs. *Discrete Math.*, 269(1-3):13–20, 2003.

[4] C. Balbuena and P. García-Vázquez. A sufficient condition for P_k-path graphs being r-connected. *Discrete Appl. Math.*, 155:1745–1751, 2007.

[5] C. Balbuena, P. García-Vázquez and X. Marcote. Sufficient conditions for λ'-optimality in graphs with girth g. *J. Graph Theory*, 52(1):73–86, 2006.

[6] C. Balbuena, K. Marshall and L.P. Montejano. On the connectivity and superconnected graphs with small diameter. *Discrete Appl. Math.*, 158(5):397–403, 2010.

[7] F.T Boesch. Synthesis of reliable networks–A survey. *IEEE Trans. Reliability*, 35:240–246, 1986.

[8] F. Boesch and R. Tindell. Circulants and their connectivities. *J. Graph Theory*, 8:487–499, 1984.

[9] H.J. Broersma and C. Hoede. Path graphs. *J. Graph Theory*, 13:427–444, 1989.

[10] G. Chartrand and L. Lesniak. Graphs and Digraphs. Third edition. Chapman and Hall, London, UK, 1996.

[11] A.H. Esfahanian and S.L. Hakimi. On computing a conditional edge-connectivity of a graph. *Inf. Process. Lett.*, 27:195–199, 1988.

[12] A. Hellwig and L. Volkmann. Sufficient conditions for graphs to be λ'-optimal, super-edge- connected, and maximally edge-connected. *J. Graph Theory*, 48:228–246, 2005.

[13] A. Hellwig and L. Volkmann. Sufficient conditions for λ'-optimality in graphs of diameter 2. *Discrete Math.*, 283:113–120, 2004.

[14] M. Knor and L. Niepel. Connectivity of iterated line graphs. *Discrete Appl. Math.*, 125:255–266, 2003.

[15] M. Knor, L. Niepel and M. Malah. Connectivity of path graphs. *Australas. J. Combin.*, 25:175–184, 2002.

[16] M. Knor and S. Zhou. Diameter and connectivity of 3-arc graphs. *Discrete Math.*, 310:37–42, 2010.

[17] C.H. Li, C.E. Praeger and S. Zhou. A class of finite symmetric graphs with 2-arc transitive quotients. *Math. Proc. Cambridge Phil. Soc.*, 129:19–34, 2000.

[18] S. Lin and S. Wang. Super p-restricted edge connectivity of line graphs. *Information Sciences*, 179:3122–3126, 2009.

[19] J. Meng. Connectivity and super edge-connectivity of line graphs. *Graph Theory Notes of New York*, XL:12–14, 2001.

[20] T. Soneoka. Super edge-connectivity of dense digraphs and graphs. *Discrete Appl. Math.*, 37/38:511–523, 1992.

[21] Z. Zhang. Sufficient conditions for restricted-edge-connectivity to be optimal. *Discrete Math.*, 307(22):2891–2899, 2007.

[22] S. Zhou. Imprimitive symmetric graphs, 3-arc graphs and 1-designs. *Discrete Math.*, 244:521–537, 2002.

[23] Q. Zhu, J.M. Xu, X. Hou and M. Xu. On reliability of the folded hypercubes. *Information Sciences* 177:1782-1788, 2007.