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Combinatorics on a family of reduced Kronecker coefficients ☆

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A R T I C L E   I N F O

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A B S T R A C T

The reduced Kronecker coefficients are particular instances of Kronecker coefficients, that nevertheless contain enough information to compute all Kronecker coefficients from them. In this note, we compute the generating function of a family of reduced Kronecker coefficients. We show that these reduced Kronecker coefficients count plane partitions. This allows us to check that these coefficients satisfy the saturation conjecture, and that they are weakly increasing. Thanks to its generating function, we can describe our family by a quasipolynomial, specifying its degree and period.

R É S U M É

Les coefficients de Kronecker réduits sont des coefficients de Kronecker particuliers, qui permettent néanmoins de recalculer tous les coefficients de Kronecker. Dans cette note, nous calculons la fonction génératrice d'une famille particulière de coefficients de Kronecker réduits. Nous exprimons sa relation avec les partitions planes, ce qui nous permet de vérifier que cette famille possède la propriété de saturation, ainsi que la propriété de monotonie. Grâce à cette fonction génératrice, nous pouvons décrire les coefficients considérés au moyen d'une formule quasi polynomiale, dont nous précisons le degré et la période.

1. Introduction

With the original aim of trying to understand the rate of growth experienced by the Kronecker coefficient as we increase the sizes of its rows, we investigate the family of reduced Kronecker coefficients $\overline{g}^{(k)}_{(k^a),(k^b)}$.

The Kronecker coefficient are the multiplicities appearing in the decomposition into irreducible of the tensor product of two irreducible representations of the symmetric group. They also appear naturally in the study of the general lineal group and the unitary group. The reduced Kronecker coefficients, on the other hand, are particular instances of Kronecker coefficients, believed to be simpler to understand than the Kronecker coefficients, but that contain enough information to recover them.

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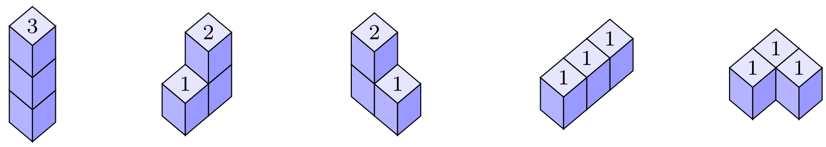
In Theorem 1 we compute the generating function $\mathcal{F}_{a,b} = \sum_k \overline{g}^{(k)}_{(k^a),(k^b)} x^k$. We achieve this by giving an explicit bijection between a family of colored partitions, and the Kronecker tableaux of Orellana and Ballantine, [1]. Since $\mathcal{F}_{a,b}$ turns out to be rational, these coefficients obey a lineal recurrence.

In Theorem 5 we give an striking connection to plane partitions: the reduced Kronecker coefficient $\overline{g}^{(k)}_{(k^a),(k^a)}$ counts the number of plane partitions of k fitting on a $2 \times a$ rectangle. We obtain this result by comparing the generating function obtained in Theorem 1 with MacMahon's classical formula. For instance, the reduced Kronecker coefficient for $a = 4$ and $k = 3$ is $\overline{g}^{(3)}_{(3,3,3,3),(3,3,3,3)} = 5$, and there are 5 plane partitions of 3 fitting inside an 2×4 rectangle:



These results imply that this family of coefficients satisfies the saturation conjecture for the reduced Kronecker coefficients of Kirillov and Klyachko, and that they are weakly increasing. Finally we go back to our original aim. In Theorem 7, we show that the family $\overline{g}^{(k)}_{(k^a),(k^a)}$ is described by a quasipolynomial of degree $2a - 1$ and period dividing the least common multiple of $1, 2, \ldots, a + 1$.

In a different context, plane partitions have appeared in the study of the Kronecker coefficients on the work of E. Vallejo, see [12].

2. Reduced Kronecker coefficients

In 1938 Murnaghan observed that the Kronecker coefficients $g^{\lambda}_{\mu,\nu}$ always stabilize when we increase the sizes of the first parts of the three labeling partitions, see [10] and [11]. For example, the sequence $g^{(k,2,2,1)}_{(k,3,2),(k-1,4,2)} = 18, 35, 40, 40, \ldots$, defined for $k \geq 5$, has 40 as its stable value.

Denote by $\alpha[n]$ the sequence of integers defined by prepending a first part of size $n - |\alpha|$ to α, then the reduced Kronecker coefficient $\overline{g}^{\gamma}_{\alpha\beta}$ is defined to be the stable limit of the sequence $g^{\gamma[n]}_{\alpha[n]\beta[n]}$.

The reduced Kronecker coefficients are interesting objects of their own right. Littlewood observed that when $|\alpha| + |\beta| = |\gamma|$, they coincide with the Littlewood–Richardson coefficient. They have be shown to contain enough information to recover from them the Kronecker coefficients, [3]. Recently, the reduced Kronecker coefficients have been used to investigate the rate of growth of the Kronecker coefficients, [4].

Murnaghan noticed that not only each particular sequence $g^{\gamma[n]}_{\alpha[n]\beta[n]}$ stabilizes, but that the Kronecker product $s_{\alpha[n]} * s_{\beta[n]}$, written in the Schur basis, is also stable. Moreover, in [3, Theorem 1.2], it is proved that the Kronecker product $s_{\alpha[n]} * s_{\beta[n]}$ stabilizes at $stab(\alpha, \beta) = |\alpha| + |\beta| + \alpha_1 + \beta_1$. Therefore, if $n \geq stab(\alpha, \beta, \gamma)$, with $stab(\alpha, \beta, \gamma) = \min\{stab(\alpha, \beta), stab(\alpha, \gamma), stab(\beta, \gamma)\}$, we have that $\overline{g}^{\gamma}_{\alpha\beta} = g^{\gamma[n]}_{\alpha[n]\beta[n]}$.

3. The generating function of a family of reduced Kronecker coefficient

We compute the generating function for a special family of reduced Kronecker coefficients.

Theorem 1. *Fix integers $a \geq b \geq 0$. Consider the sequence of reduced Kronecker coefficients $\left\{\overline{g}^{(k)}_{(k^a),(k^b)}\right\}_{k \geq 0}$.*

1. *If $a = b$, the generating function for the reduced Kronecker coefficients $\overline{g}^{(k)}_{(k^a),(k^a)}$ is*

$$\mathcal{F}_{a,a} = \frac{1}{(1-x)\cdot(1-x^2)^2\cdots(1-x^a)^2\cdot(1-x^{a+1})}$$

2. *If $a = b + 1$, then $\overline{g}^{(k)}_{(k^a),(k^b)} = 1$ for all $k \geq 0$. That is $\mathcal{F}_{b+1,b} = \frac{1}{1-x}$.*
3. *If $a > b + 1$, $\overline{g}^{(k)}_{(k^a),(k^b)} = 0$, except for $k = 0$ that it is 1.*

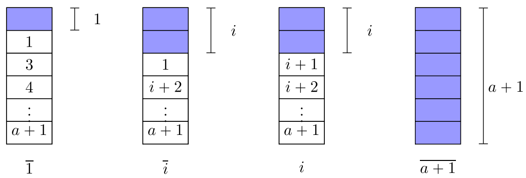
A SSYT of shape λ/α and type ν/α such that it reverse reading word is an α-lattice permutation is called a *Kronecker tableau* of shape λ/α and type ν/α if either $\alpha_1 = \alpha_2$, or $\alpha_1 > \alpha_2$ and any one of the following two conditions is satisfied: the number of 1's in the second row of λ/α is exactly $\alpha_1 - \alpha_2$ or the number of 2's in the first row of λ/α is exactly $\alpha_1 - \alpha_2$. Let $k^{\lambda}_{\alpha\nu}$ denote total number of Kronecker tableaux of shape λ/α and type ν/α.

Lemma 2. *(See Thm. 3.2(a) [1].) Let n and p be positive integers such that* $n \geq 2p$*. Let* $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ *and* ν *be partitions of n. If* $\lambda_1 \geq 2p-1$*, then* $g_{\lambda,\nu}^{(n-p,p)} = \sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda \cap \nu}} k_{\alpha\nu}^{\lambda}$.

Finally, the reduced Kronecker coefficient $\overline{g}_{(k^a),(k^a)}^{(k)}$ is equal to the Kronecker coefficient $g_{(k^a)[N],(k^a)[N]}^{(k)[N]}$, when $N \geq stab(\lambda,\mu,\nu)$. Therefore, $\overline{g}_{(k^a),(k^a)}^{(k)}$ is equal to the number of Kronecker tableaux of shape $(3k,k^a)/\alpha$ and type $(3k,k^a)/\alpha$, where α is a partition of k.

Since $\mathcal{F}_{a,a}$ is the generating function for colored partitions with parts in $\mathcal{A} = \left\{\bar{1}, 2, \bar{2}, \ldots, a, \bar{a}, \overline{a+1}\right\}$ (i.e. weakly decreasing sequences of integers, ordered by saying that $\bar{i} < i$, and such that both entries i and $\bar{i}$ have weight i), it suffices to define a bijection between colored partitions with parts in $\mathcal{A}$, and Kronecker tableaux counted by $\overline{g}_{(k^a),(k^a)}^{(k)}$.

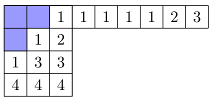
The following algorithm defines a bijection between colored partitions of k with parts in $\mathcal{A}$, and Kronecker tableaux of shape $(3k,k^a)/\alpha$ and type $(3k,k^a)/\alpha$, where α is a partition of k. With a colored partition β of k with parts in $\mathcal{A}$, we associate a Kronecker tableau $T(\beta)$ as follows. First, we identify each element of $\mathcal{A}$ to a column of height $a+1$: if $i \in \{2,3,\ldots,a-1,a\}$ then



Note that it is always possible to order the columns corresponding to the parts of β in such a way that we obtain a SSYT. If we write β as $\left(\bar{1}^{m_{\bar{1}}}\bar{2}^{m_{\bar{2}}}2^{m_2}\ldots\overline{a+1}^{m_{\overline{a+1}}}\right)$, then m_i will denote the number of times that the column i appears in the tableau that we are building. We read the partition α from our SSYT by counting the number of blue boxes in each row: $\alpha_{a+1} = m_{\overline{a+1}}$, $\alpha_i = \alpha_{i+1} + m_i + m_{\bar{i}}$ for $i = 2,\ldots,a$, and $\alpha_1 = \alpha_2 + m_{\bar{1}}$.

This SSYT defines the first columns of $T(\beta)$. We build the rest of the Kronecker tableau of shape $(3k,k^a)/\alpha$ as follows: complete the ith row with entry i, for $i = 2,\ldots,a+1$, and the first row with the remaining numbers of the type $(3k,k^a)/\alpha$ in weakly increasing order from left to right.

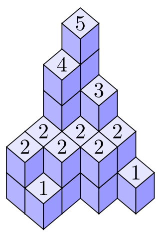
For instance, for $a = 3$, and $\beta = (\bar{2},\bar{1})$, $T(\beta)$ equals



a Kronecker tableau with $\lambda = \nu = (9,3,3,3)$ and $\alpha = (2,1)$. This map is well defined and bijective. To conclude the proof, we show that in the other two cases, there is at most one Kronecker tableau that satisfies all requirements.

4. Plane partitions and reduced Kronecker coefficients

In this section we establish a link between the family of reduced Kronecker coefficients $\overline{g}_{(k^a),(k^a)}^{(k)}$ and plane partitions. A *plane partition* is a finite subset $\mathcal{P}$ of positive integer lattice points, $\{(i,j,k)\} \subset \mathbb{N}^3$, such that if (r,s,t) lies in $\mathcal{P}$ and if (i,j,k) satisfies that $1 \leq i \leq r$, $1 \leq j \leq s$ and $1 \leq k \leq t$, then (i,j,k) also lies in $\mathcal{P}$. Let $\mathcal{B}(r,s,t)$ be the set of plane partitions fitting in an $r \times s$ rectangle and with biggest part equal to t. That is, $\mathcal{B}(r,s,t) = \{(i,j,k) \mid 1 \leq i \leq r,\ 1 \leq j \leq s,\ 1 \leq k \leq t\}$. As an illustration we present a plane partition in $\mathcal{B}(4,4,5)$,



We show that the generating function for the reduced Kronecker coefficients, obtained in Theorem 1, coincides with the classical generating functions for plane partitions.

Theorem 3. *(See P. MacMahon, [9].) The generating function for plane partitions in $\mathcal{B}(r, s, t)$ is*

$$pp_t(x; r, s) = \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1 - x^{i+j+t-1}}{1 - x^{i+j-1}}$$

Since $\frac{1-x^{l+k}}{1-x^l} = 1 + x^l + \cdots + x^{\lfloor \frac{k}{l} \rfloor l} + \mathcal{O}(k+1)$, for all $l \geq 1$, this generating function for plane partitions can be rewritten as a generating function computed resembling the one appearing in Theorem 1.

Lemma 4. *Let $r = \min(a, l)$ and $s = \max(a, l)$. Then, the generating function for the plane partitions fitting inside an $l \times a$ rectangle is*

$$\prod_{j=r}^{s} \left(\frac{1}{1 - x^j} \right)^r \cdot \prod_{i=1}^{r-1} \left(\frac{1}{1 - x^i} \right)^i \left(\frac{1}{1 - x^{s+i}} \right)^{r-i}$$

From this we conclude that the reduced Kronecker coefficients $\bar{g}^{(k)}_{(k^a),(k^a)}$ count plane partitions.

Theorem 5. *The reduced Kronecker coefficient $\bar{g}^{(k)}_{(k^a),(k^a)}$ counts the number of plane partitions of k fitting inside a $2 \times a$ rectangle.*

5. Consequences

5.1. Saturation hypothesis

Let denote by $\{C(\alpha^1, \ldots, \alpha^n)\}$ any family of coefficients depending on the partitions $\alpha^1, \ldots, \alpha^n$. The family $\{C(\alpha^1, \ldots, \alpha^n)\}$ satisfies the *saturation hypothesis* if the conditions $C(\alpha^1, \ldots, \alpha^n) > 0$ and $C(s \cdot \alpha^1, \ldots, s \cdot \alpha^n) > 0$ for all $s > 1$ are equivalent, where $s \cdot \alpha = (s\alpha_1, s\alpha_2, \ldots)$.

The Littlewood–Richardson coefficients satisfy the saturation hypothesis, as Knutson and Tao show in [8]. On the other hand, the Kronecker coefficients are known not to satisfy it. For example $g^{(n,n)}_{(n,n),(n,n)}$ is equal to 1 if n is even, and to 0 otherwise, see [2].

In [7] and [6], Kirillov and Klyachko have conjectured that the reduced Kronecker coefficients satisfy the saturation hypothesis. From the combinatorial interpretation for the reduced Kronecker coefficients $\bar{g}^{(k)}_{(k^a),(k^a)}$ in terms of plane partitions we verify their conjecture for our family of coefficients.

Corollary 6. *The saturation hypothesis holds for the coefficients $\bar{g}^{(k)}_{(k^a),(k^a)}$. In fact, $\bar{g}^{(sk)}_{((sk)^a),((sk)^a)} > 0$ for all $s \geq 1$. Moreover, the sequences of coefficients obtained by, either fixing k or a, and then letting the other parameter grow, are weakly increasing.*

5.2. Quasipolynomiality

In Theorem 1, we computed the generating function $\mathcal{F}_{a,b}$ for the reduced Kronecker coefficients. In this section, we study the implications of this calculation. We concentrate on the non-trivial case, $a = b$.

Theorem 7. *Let $\mathcal{F}_a = \mathcal{F}_{a,a}$ be the generating function for the reduced Kronecker coefficients $\bar{g}^{(k)}_{(k^a),(k^a)}$.*

Let ℓ be the least common multiple of $1, 2, \ldots, a, a+1$.

(i) *The generating function $\mathcal{F}_a$ can be rewritten as*

$$\mathcal{F}_a = \frac{P_a(x)}{(1 - x^\ell)^{2a}}$$

where $P_a(x)$ is a product of cyclotomic polynomials, and $\deg(P_a(x)) = 2\ell a - (a+2)a < 2a\ell - 1$.

(ii) *The polynomial P_a is the generating function for colored partitions with parts in $\mathcal{A} = \{\bar{1}, 2, \bar{2}, \ldots, a, \bar{a}, a+1\}$, where parts j and $\bar{j}$ appear with multiplicity less than ℓ/j times.*

(iii) *The coefficients of P_a are positive and palindrome, but in general are not a concave sequence.*

(iv) *The coefficients $\bar{g}^{(k)}_{(k^a),(k^a)}$ are described by a quasipolynomial of degree $2a - 1$ and period dividing ℓ. In fact, we have checked that the period is exactly ℓ for $a \leq 10$.*

(v) *The coefficients $\bar{g}^{(k)}_{(k^a),(k^a)}$ satisfy a formal reciprocity law: $x^{a(a+2)} \mathcal{F}_a(x) = \mathcal{F}_a(\frac{1}{x})$.*

Idea of Proof. Item (i) follows from the expression of $\mathcal{F}_a$ and $(1-x^l)^{2a}$ as a product of cyclotomic polynomials. For (ii), write $(1-x^l)^{2a}\mathcal{F}_a$ as a product of polynomials of the form $\frac{(1-x^l)}{(1-x^j)} = 1 + x^j + x^{2j} + \cdots + x^{\lfloor \frac{l}{j} \rfloor}$, and remark that each part appears with bounded multiplicity. Using (ii), P_a has positive coefficients and they are a palindrome because the first cyclotomic polynomial does not appear in P_a. For $a=2$, we have that $P_2(x) = x^{16} + x^{15} + 3x^{14} + 4x^{13} + 7x^{12} + 9x^{11} + 10x^{10} + 13x^9 + 12x^8 + 13x^7 + 10x^6 + 9x^5 + 7x^4 + 4x^3 + 3x^2 + x + 1$ is not a concave sequence. Finally, (iv) follows from Proposition 4.13 of [5] and (v) from the computation of $\mathcal{F}_a(\frac{1}{x})$.

As an illustration, we include the following example that shows that the coefficients $\bar{g}^{(k)}_{(k^2),(k^2)}$ are given by the following quasipolynomial of degree 3 and period 6:

$$\bar{g}^{(k)}_{(k^2),(k^2)} = \begin{cases} 1/72k^3 + 1/6k^2 + \ \ 2/3k + 1 & \text{if } k \equiv 0 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 5/18 & \text{if } k \equiv 1 \mod 6 \\ 1/72k^3 + 1/6k^2 + \ \ 2/3k + 8/9 & \text{if } k \equiv 2 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 1/2 & \text{if } k \equiv 3 \mod 6 \\ 1/72k^3 + 1/6k^2 + \ \ 2/3k + 7/9 & \text{if } k \equiv 4 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 7/18 & \text{if } k \equiv 5 \mod 6 \end{cases}$$

5.3. On the growth of the Kronecker coefficients

Murnaghan famously observed that the sequences obtained by adding cells to the first parts of the partitions indexing a Kronecker coefficients are eventually constant. Fix three arbitrary partitions. In [4], it is shown that the sequences obtained by adding cells to the first parts of the three partitions indexing a reduced Kronecker coefficients are described by a linear quasipolynomial of period 2. These sequences can be interpreted as adding cells to the second rows of the partitions indexing a Kronecker coefficient, while keeping their fist parts very long in comparison.

An interesting question is then to describe what happens when we add cells to arbitrary rows of the partitions indexing a Kronecker (and reduced Kronecker) coefficient. The results presented in this note can be seen as a contribution to this investigation. We have shown that for any a, the sequence $\bar{g}^{k}_{(k^a),(k^a)}$ is described by a quasipolynomial of degree $2a-1$ and period dividing ℓ.

On the other hand, in [4] it is also shown that when we fix three partitions, and start adding cells to their first columns, the sequences obtained are eventually constant. From our combinatorial interpretation for $\bar{g}^{k}_{(k^a),(k^a)}$ in terms of plane partitions of k fitting a $2 \times a$ rectangle, we conclude that, for k fixed, the sequences $\{\bar{g}^{k}_{(k^a),(k^a)}\}_{a \geq 1}$ always stabilize. Their limit value is precisely the number of plane partitions of k with at most two parts. These sequences stabilize at $a = k$.

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