Bifurcations from a center at infinity in 3D piecewise linear systems with two zones

Abstract

We consider continuous piecewise linear systems in $\mathbb{R}^3$ with two zones under the assumption of having a linear center in the invariant manifold of the point at infinity. A specific conic projection is introduced, so that it is possible to analyze in a convenient way the dynamics near such a center at infinity in two qualitative different situations.

In the semi-homogeneous case, such a center is associated to the existence of a continuum of invariant semi-cones sharing the vertex at the origin; perturbing the configuration it is possible to detect the bifurcation of a limit cycle at infinity leading to the bifurcation of isolated invariant semi-cones.

For the non-homogeneous case, the cycles at infinity does not imply invariant semi-cones. However, the non-generic case when the center at infinity is associated to the existence of invariant cylinders for one of the involved vector fields, becomes rather interesting. It is possible then, by perturbing the other vector field, to get the bifurcation of a big limit cycle from infinity without destroying the center.

Keywords: Piecewise linear systems, bifurcation, Limit cycles

1. Introduction and setting of the problem

The class of piecewise linear systems has a distinguished, long pedigree. To give a couple of examples, we can quote the seminal book of Andronov et al. [3], where there appears a plenty of mechanical, electrical and control applications whose nonlinearities are adequately modeled as piecewise linear functions. As another remarkable milestone, the work by Levinson [20] on the forced VanderPol equation with a piecewise constant nonlinearity motivated Steve Smale to discover the horseshoe paradigm [27].

In fact, piecewise linear systems constitute the natural entry point for the analysis of piecewise smooth systems, an area of research where more and more attention is being payed to the specific case of discontinuous vector fields or Filippov systems [10], which are required in a growing number of modern engineering applications, as in power electronics, see [28]. It is clear however that without a perfect comprehension of the continuous case one cannot try to capture the subtleties of discontinuous instances.

While for planar, continuous piecewise linear systems the situation is rather satisfactory [11, 12], this is not the case for three-dimensional continuous piecewise linear systems. In fact, there is a lack of explicit characterizations even for seemingly elementary problems, as the stability issue for the equilibrium point at the origin of semi-homogeneous 3D systems, see [5]. Very few results about bifurcation of limit cycles are available, both in presence of symmetry [13, 24], and for the non-symmetric case [6, 14].

On the other hand, we want to emphasize that in order to get a dynamical global view of a differential system, it is very important to know the dynamical properties of the system at the invariant manifold associated to the point at infinity along with their possible bifurcations at or from infinity. To the best of our knowledge, there are not many works dealing with bifurcations at infinity for 3D differential systems. A general result on bifurcations from infinity under certain hypotheses well suited to control systems with only one nonlinearity being asymptotically linear at infinity appeared in [9], see also [18]. For polynomial vector fields under the additional assumption of reversibility, see [4] and [22]. In fact, the interest for the study of dynamics at infinity has undergone a certain upsurge, see [17], [21], and [29]. However, the quoted works do not tackle bifurcation phenomena at infinity; they mainly study bifurcation of limit cycles on the finite phase space and the existence of algebraic invariant manifolds for selected choices of parameters, sometimes working on the sphere $\mathbf{S}^2$ coming from Poincaré compactifications. Such Poincaré compactifications do not help adequately in the analysis of the kind of problems we are interested in, namely 3D piecewise linear systems with two zones and no symmetry, so that we need to develop specific techniques, as shown later.

Thus, we explore here some issues relative to the dynamics at infinity for a family of continuous three-dimensional piecewise linear systems with two zones, separated by a plane, focussing our attention on the study of bifurcations from a center at infinity and their repercussions in the finite dynamics. Entering already into the setting of our problem, the global vector field, which is nonlinear, comes from the matching of a *left* differential system of the form

$$
\begin{aligned}
\dot{x} &= (\lambda + 2\sigma)x - y, \\
\dot{y} &= (2\lambda\sigma + \sigma^2 + \omega^2)x - z, \\
\dot{z} &= \lambda(\sigma^2 + \omega^2)x + \mu,
\end{aligned}
\tag{1}
$$

for $x < 0$, and the *right* system

$$
\begin{aligned}
\dot{x} &= tx - y, \\
\dot{y} &= mx - z, \\
\dot{z} &= dx + \mu,
\end{aligned}
\tag{2}
$$

for $x > 0$. Here, we assume for the *left* system the spectrum

$$\{\lambda, \sigma \pm i\omega\}, \quad \omega > 0,$$

while for the *right* system we do not assume any specific eigenvalue configuration: the parameters t, m and d, representing respectively the trace, the sum of principal minor of order two and the determinant of the right matrix, are arbitrary. The *modal* parameter $\mu \in \{0, 1\}$ allows to distinguish two different configurations: the semi-homogeneous case $\mu = 0$ and the non-homogeneous case $\mu = 1$. Note that we adopt the so-called generalized Liénard form, as it was done in [6].

Our interest is to determine the possible bifurcations associated to the existence of a center at infinity, in a similar way to what has been done in previous works for ordinary centers, see [6, 13, 14]. The approach is rather different from the case of Hopf bifurcation at infinity with symmetry, see [2]. As shown later, we will need specific techniques, suitable for working near the point at infinity.

We remark that when $\mu = 0$ system (1)-(2) is semi-homogeneous, having the property that if $(x(\tau), y(\tau), z(\tau))$ is any solution then $(kx(\tau), ky(\tau), kz(\tau))$ is also a solution for every $k > 0$. Consequently, the existence of a periodic orbit implies the existence of an invariant semi-cone foliated by periodic orbits. Furthermore, if there exists a periodic orbit at infinity then there also exists an invariant semi-cone, this time not necessarily containing any periodic orbit. Reciprocally, any invariant semi-cone will determine a periodic orbit at infinity, so that we can establish in this semi-homogeneous case a correspondence between invariant semi-cones and periodic orbits at infinity. The study of existence of such invariant semi-cones has been tackled in several papers, see in particular [7] and [8]. Therefore, by assuming a center at infinity —it suffices the condition $\lambda = \sigma$ in (1)— we can conclude the existence of a continuum of invariant semi-cones, from which an isolated semi-cone could bifurcate, being associated to the existence of a limit cycle at infinity that bifurcates from the outermost periodic orbit of the original center. Our study completes the quoted works, by detecting a new bifurcation of semi-cones which had not been previously characterized.

When $\mu = 1$, no semi-cones are possible, so that limit cycle bifurcations from the center at infinity have not a counterpart for the global dynamics, in principle. However, under the stronger assumption $\lambda = \sigma = 0$, leading this time to the existence of a family of invariant cylinders for system (1), it turns out that the outermost periodic orbit of the center at infinity has then a higher degree of non-hyperbolicity. Thus, it is possible the bifurcation of a 'big' limit cycle from infinity without destroying the center. We obtain a new result in this direction, giving a complete characterization of such a bifurcation.

Our computations take advantage of a conic projection onto an adequate plane, namely a parallel plane to the left focal plane. This is the subject of Section 2, where we obtain for system (1)-(2) equivalent vector fields allowing the analysis of its dynamics near a selected chart at infinity. In Section 3, we derive a new bifurcation result for piecewise smooth planar systems with a linear center, see Proposition 2; such a result is crucial for the statement of Theorem 1 on bifurcation of invariant semi-cones in the semi-homogeneous case $\mu = 0$, which appears in Section 4. Finally, in Section 5 we deal with the non-homogeneous case $\mu = 1$, getting the characterization of a bifurcation of a limit cycle from infinity without symmetry, see Theorem 2. The proof of Theorem 2, which involves long computations, appears in Section 6.

2. Studying the dynamics near infinity via conic projections

Given a non-null vector $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, we consider a projection plane not passing through the origin

$$\Pi = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3, \mathbf{a}^\top \mathbf{x} = \delta\},$$

where $\delta = \pm 1$, as appropriate. Next, for any point $\mathbf{x}$ with $\mathbf{a}^\top \mathbf{x} \neq 0$, we take the straight line that joins the origin and the point $\mathbf{x}$, and we denote by $\mathbf{X} = (X, Y, Z)$ the intersection point of such a line with the plane Π, see Figure 1. Clearly, we can define a unique value $W = W(\mathbf{x}) \in \mathbb{R}$ satisfying

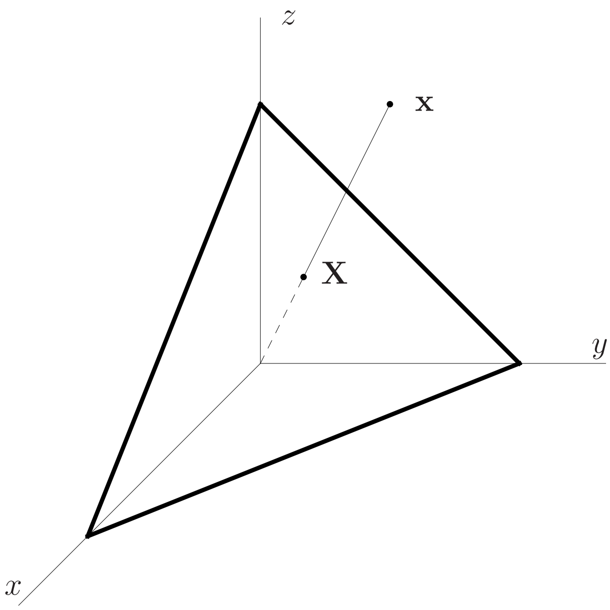

Figure 1: The conic projection on the plane Π.

the equalities

$$\mathbf{X} = W\mathbf{x} \iff \delta = \mathbf{a}^\top \mathbf{X} = W\mathbf{a}^\top \mathbf{x} \iff W = \frac{\delta}{\mathbf{a}^\top \mathbf{x}}. \tag{3}$$

Note that W gives, up to a constant factor, the inverse of the distance of the point $\mathbf{x}$ to the plane Π, so that the condition $W = 0$ corresponds with a certain chart at infinity, while the points on the plane Π satisfy $W = 1$. We will take this value W as a new variable, and after projecting the dynamics onto the plane Π, we select also as new variables two of the components of the projection $\mathbf{X}$, discarding for instance the last component Z, since we have from (3) the relationship $\mathbf{a}^\top \mathbf{X} = \delta$.

Let us see how this change behaves when dealing with a non homogeneous linear system of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}. \tag{4}$$

We remark that a particular instance of the above projection was introduced for a homogeneous case in [7]. We note first that from (3) we have

$$\dot{W} = -\frac{\delta \mathbf{a}^\top (A\mathbf{x} + \mathbf{b})}{(\mathbf{a}^\top \mathbf{x})^2} = -\delta W^2 \mathbf{a}^\top \left(A\frac{\mathbf{X}}{W} + \mathbf{b} \right) = -\delta W \mathbf{a}^\top (A\mathbf{X} + W\mathbf{b}), \tag{5}$$

so that, as expected, the plane $W = 0$ is invariant. Furthermore, from (3) and (5), we also have

$$\dot{\mathbf{X}} = \dot{W}\frac{\mathbf{X}}{W} + W\left(A\frac{\mathbf{X}}{W} + \mathbf{b} \right) = -\delta \mathbf{a}^\top (A\mathbf{X} + W\mathbf{b})\mathbf{X} + A\mathbf{X} + W\mathbf{b}. \tag{6}$$

In principle, the above system is not linear any longer, but when one focusses on the the dynamics at infinity, that is the dynamics on the plane $W = 0$, it is possible to get there a linear system by choosing adequately the vector $\mathbf{a}$. As shown next, it suffices to take for such vector a left eigenvector associated to a real eigenvalue of the matrix A.

Proposition 1. *If the vector* $\mathbf{a}$ *is a left eigenvector associated to a real eigenvalue* λ *of the matrix* A*, then system* (6) *becomes linear when it is restricted to the invariant plane* $W = 0$*.*

PROOF. We see that

$$\mathbf{a}^\top A\mathbf{X} = \lambda \mathbf{a}^\top \mathbf{X} = \lambda\delta,$$

so that (5) reduces to

$$\dot{W} = -\delta^2 \lambda W - \delta \mathbf{a}^\top \mathbf{b} W^2 = -\lambda W - \delta \mathbf{a}^\top \mathbf{b} W^2, \tag{7}$$

and system (6) becomes

$$\dot{\mathbf{X}} = (A - \lambda I)\mathbf{X} - \delta \mathbf{a}^\top \mathbf{b} W\mathbf{X} + W\mathbf{b}, \tag{8}$$

where we have used that $\delta^2 = 1$ and I stands for the identity matrix. Note that system (8) reduces for $W = 0$ to the linear homogeneous system

$$\dot{\mathbf{X}} = (A - \lambda I)\mathbf{X},$$

with a zero eigenvalue, but after eliminating one of the components of $\mathbf{X}$, say Z, by using the linear relation $\mathbf{a}^\top \mathbf{X} = \delta$, the reduced planar system becomes also linear but non-homogeneous and without such a zero eigenvalue. The proof is complete.

Let us consider the above projection for system (1)-(2). Denoting with A_L, A_R, the matrices giving the linear parts in each half-space, we apply Proposition 1 by taking as director vector of the projection plane the left-eigenvector $\mathbf{a} = (\lambda^2, -\lambda, 1)^\top$ of A_L, which is associated to its real eigenvalue λ. We have $\mathbf{b} = (0, 0, \mu)^\top$, so that $\mathbf{a}^\top \mathbf{b} = \mu$, and considering first system (1), from (7) we obtain for $X < 0$

$$\dot{W} = -\lambda W - \delta\mu W^2,$$

and from (8)

$$\dot{\mathbf{X}} = \begin{pmatrix} 2\sigma & -1 & 0 \\ 2\lambda\sigma + \sigma^2 + \omega^2 & -\lambda & -1 \\ \lambda(\sigma^2+\omega^2) & 0 & -\lambda \end{pmatrix} \mathbf{X} - \begin{pmatrix} \delta\mu X W \\ \delta\mu Y W \\ \delta\mu Z W - \mu W \end{pmatrix}, \tag{9}$$

where we must take into account the condition

$$\lambda^2 X - \lambda Y + Z = \delta, \tag{10}$$

which defines the projection plane Π. After eliminating Z by substituting $Z = \delta - \lambda^2 X + \lambda Y$ into the two first components of (9), we get the planar system

$$\begin{array}{rcl} \dot{X} & = & 2\sigma X - Y - \delta\mu X W, \\ \dot{Y} & = & -\delta + [(\lambda+\sigma)^2 + \omega^2]X - 2\lambda Y - \delta\mu Y W, \end{array} \tag{11}$$

for $X < 0$. Before proceeding further, we note that at infinity (for $W = 0$) this planar system has an equilibrium point at $(\overline{X}, \overline{Y})$ with $\overline{Y} = 2\sigma\overline{X}$ and

$$\overline{X} = \frac{\delta}{(\lambda-\sigma)^2 + \omega^2},$$

so that this point is a real equilibrium point if $\delta = -1$, and furthermore it becomes a linear center when $\lambda = \sigma$. Thus, as we know that this last case is the configuration capable of generating limit cycles by perturbation, we will assume in the sequel $\delta = -1$.

We need to apply the same projection for points in the half-space $x > 0$. From (5) with $\delta = -1$ and (10), we now obtain for $X > 0$

$$\dot{W} = W\left(\lambda^2, -\lambda, 1\right)\begin{pmatrix} tX - Y \\ mX - Z \\ dX + \mu W \end{pmatrix} = [\mathrm{CP}_R(\lambda)X - \lambda]\, W + \mu W^2,$$

where $\mathrm{CP}_R(\lambda) = -\lambda^3 + t\lambda^2 - m\lambda + d$ is the characteristic polynomial of the linear part A_R in the region $x > 0$ evaluated in λ. Also, from (6) we have

$$\dot{\mathbf{X}} = (A_R - \lambda I)\mathbf{X} + [\mathrm{CP}_R(\lambda)X + \mu W]\,\mathbf{X} + \mu W\mathbf{e}_3, \tag{12}$$

and after eliminating Z as before, we get the planar system

$$\begin{array}{rcl} \dot{X} & = & (t-\lambda)X - Y - \mathrm{CP}_R(\lambda)X^2 + \mu XW, \\ \dot{Y} & = & 1 + (m+\lambda^2)X - 2\lambda Y - \mathrm{CP}_R(\lambda)XY + \mu YW. \end{array} \tag{13}$$

From expressions (11) and (13), it is possible to study the possible bifurcation of limit cycles at infinity, by considering the dynamics on the plane $W = 0$. In next section, we derive a bifurcation result for a rather more general situation, which includes the cases we are interested in.

3. A limit cycle bifurcation from a center in planar PWS systems.

In this section, we obtain an auxiliary result for planar piecewise smooth systems to be later applied to our study of the dynamics at infinity for 3D PWL systems with two zones. We study a planar system composed by two different vector fields: a linear system with focus dynamics in the half-plane $x \leq 0$ and a polynomial system in the half-plane $x \geq 0$, with the specific property of making possible the concatenation of orbits near the origin. More precisely, at the separation straight line $x = 0$ and for small $|y| > 0$ the dynamics is *crossing* and the global vector field becomes continuous yet non-smooth at the origin with a tangency of the orbits there at such a line (*visible* from the left and *invisible* from the right).

Thus, we consider the piecewise smooth system

$$\begin{array}{rcl} \dot{x} & = & 2\gamma x - y, \\ \dot{y} & = & 1 + (1+\gamma^2)x, \end{array} \tag{14}$$

for $x < 0$, and

$$\begin{array}{rcl} \dot{x} & = & -y + a_1x + b_1y^2 + a_2xy + b_2y^3 + a_3x^2 + b_3xy^2 + c_3y^4, \\ \dot{y} & = & 1 + A_2x + B_2y^2 + A_3xy + B_3y^3, \end{array} \tag{15}$$

for $x > 0$. Here, the subscripts of coefficients indicate the order of quasi-homogeneity of different monomials, see [1]. The quasi-homogeneity comes from a scaling of variables and time of the form $x = \varepsilon^2 X$, $y = \varepsilon Y$, $t = \varepsilon\tau$, leading to a type $(2,1)$ with the zero-order vector field being $(-y,1)^\top$, which is the natural choice for our case of invisible tangency at the origin from the right. Note that in the second equation of (15) the original term $A_1 y$ has been removed by a single linear change of variables, in a similar way to what is done in Proposition 3.1 of [15].

Clearly, the *left* system is a linear system with a focus at

$$(\bar{x},\bar{y}) = \left(-\frac{1}{1+\gamma^2}, -\frac{2\gamma}{1+\gamma^2}\right),$$

having an orbit with *visible tangency* to the y-axis at the origin. Note also that for $\gamma=0$ we have a linear center at the point $(-1,0)$. Since the eigenvalues of the system matrix are $\gamma \pm i$, the focus at $(\bar{x},\bar{y})$ is stable for $\gamma<0$, being unstable for $\gamma>0$. If we consider an orbit starting at the point $(0,y_0)$ with $y_0>0$, then from the first equation of (14) we obtain that $\dot{x}<0$, so that the orbit will enter the half-plane $x<0$ to surround the focus; under certain conditions, we can assume that after a time τ_L near 2π such an orbit eventually arrives to a point $(0,y_1)$ with $y_1<0$, using only the half-plane $x\leq 0$. Then, we must have

$$e^{\gamma\tau_L}\begin{pmatrix} \cos\tau_L+\gamma\sin\tau_L & -\sin\tau_L \\ (\gamma^2+1)\sin\tau_L & \cos\tau_L-\gamma\sin\tau_L \end{pmatrix}\begin{pmatrix} -\bar{x} \\ y_0-\bar{y} \end{pmatrix} = \begin{pmatrix} -\bar{x} \\ y_1-\bar{y} \end{pmatrix} \tag{16}$$

where we have written explicitly the matrix exponential corresponding to system (14).

From (16) we obtain the two equations

$$\begin{array}{rcl} -y_0(1+\gamma^2)\sin\tau_L+\cos\tau_L-\gamma\sin\tau_L &=& e^{-\gamma\tau_L}, \\ y_1(1+\gamma^2)\sin\tau_L+\cos\tau_L+\gamma\sin\tau_L &=& e^{\gamma\tau_L}, \end{array} \tag{17}$$

and rescaling the variables y_0 and y_1 by $(1+\gamma^2)$, that is, taking $\tilde{y}_0 = y_0(1+\gamma^2)$ and $\tilde{y}_1 = y_1(1+\gamma^2)$, we can write the equations in the more compact form

$$\begin{array}{rcl} \cos\tau_L-(\tilde{y}_0+\gamma)\sin\tau_L &=& e^{-\gamma\tau_L}, \\ \cos\tau_L+(\tilde{y}_1+\gamma)\sin\tau_L &=& e^{\gamma\tau_L}. \end{array} \tag{18}$$

From these equations, we proceed in what follows by looking for only one equation relating the three variables $\tilde{y}_0$, $\tilde{y}_0$ and γ, that is, we aim to eliminate the time τ_L. Multiplying both equations in (18) we obtain

$$\cos^2\tau_L+(\tilde{y}_1-\tilde{y}_0)\cos\tau_L\sin\tau_L-(\tilde{y}_0+\gamma)(\tilde{y}_1+\gamma)\sin^2\tau_L = 1,$$

and reordering terms we see that

$$(\tilde{y}_1 - \tilde{y}_0)\cos\tau_L \sin\tau_L = [1 + (\tilde{y}_1 + \gamma)(\tilde{y}_0 + \gamma)]\sin^2\tau_L,$$

that is

$$\tan\tau_L = \frac{\tilde{y}_1 - \tilde{y}_0}{1 + (\tilde{y}_1 + \gamma)(\tilde{y}_0 + \gamma)}, \tag{19}$$

where we have discarded a spurious solution, corresponding to $\sin\tau_L = 0$. As we want to study the bifurcation at $\gamma = 0$, and then $\tau_L = 2\pi$, we do the change $\tau_L = 2\pi + s$, and using the trigonometric formula for the tangent of the sum of two angles, we get the equality

$$s = \arctan(\tilde{y}_1 + \gamma) - \arctan(\tilde{y}_0 + \gamma).$$

Thus, for small values around the point $(s, \gamma, \tilde{y}_0, \tilde{y}_1) = (0, 0, 0, 0)$ we obtain the expansion of s in terms of $(\gamma, \tilde{y}_0, \tilde{y}_1)$

$$\begin{aligned} s &= \textstyle\sum_{k=0}^{\infty}(-1)^k \dfrac{(\tilde{y}_1 + \gamma)^{2k+1} - (\tilde{y}_0 + \gamma)^{2k+1}}{2k+1} = \\ &= \tilde{y}_1 - \tilde{y}_0 - \tfrac{1}{3}(\tilde{y}_1^3 - \tilde{y}_0^3) - \gamma(\tilde{y}_1^2 - \tilde{y}_0^2) - \gamma^2(\tilde{y}_1 - \tilde{y}_0) + \cdots \end{aligned} \tag{20}$$

Now, subtracting both equations in (18) and replacing τ_L by $2\pi + s$, it holds

$$(\tilde{y}_0 + \tilde{y}_1 + 2\gamma)\sin s = 2\sinh(\gamma(2\pi + s)) \tag{21}$$

Computing the series expansion of (21) near the point $(s, \gamma, \tilde{y}_0, \tilde{y}_1) = (0, 0, 0, 0)$, and replacing s by expression (20), we obtain an expression depending only on $(\gamma, \tilde{y}_0, \tilde{y}_1)$. From such expression, we can deduce the following expansion of γ in terms of the original values y_0 and y_1,

$$\gamma = \frac{y_1^2 - y_0^2}{4\pi} - \frac{y_1^4 - y_0^4}{8\pi} - \frac{y_1^5 - y_0^2 y_1^3 - y_0^3 y_1^2 + y_0^5}{12\pi^2} + O(6). \tag{22}$$

Note that, as it can be deduced from geometrical considerations, the above expression $\gamma := \gamma(y_0, y_1)$ satisfies the condition $\gamma(-y_1, -y_0) = -\gamma(y_0, y_1)$, and also that each term of any fixed degree in (y_0, y_1) must contain the factor $(y_0 + y_1)$. Thus, in the above expansion $\gamma = \sum c_{ij} y_0^i y_1^j$, we must have $c_{ij} - c_{ji} = 0$ for $i + j$ odd, $c_{ij} + c_{ji} = 0$ for $i + j$ even, and $c_{ii} = 0$.

Now, we compute the expansion for the inverse of the right Poincaré map P_R in terms of y_0. To do that, we compute the successive time derivatives for the right system (15), to get a sufficiently high order approximation of $(x(\tau), y(\tau))$ such $(x(0), y(0)) = (0, y_0)$. Next, by assuming a negative value for the time $\tau_R < 0$ and imposing that $x(\tau_R) = 0$, we get an expansion of the right flight time $\tau_R < 0$, in terms of y_0. Thus, after substituting such a time expansion in the expression for $y_1 = y(\tau_R)$, we get

$$y_1 = -y_0 + \frac{2}{3}\alpha y_0^2 - \frac{4}{9}\alpha^2 y_0^3 + \frac{2}{135}\beta y_0^4 - \frac{2}{405}\delta y_0^5 + \cdots \tag{23}$$

where $\alpha = a_1 + b_1$,

$$\beta = 22a_1^3 + 9a_1a_2 - 9a_1A_2 + 18a_3 + 9A_3 + 66a_1^2b_1 + 9a_2b_1 + 75a_1b_1^2 +$$
$$40b_1^3 + 18a_1b_2 + 45b_1b_2 + 9a_1B_2 + 18b_1B_2 + 9b_3 + 27B_3 + 27c_3,$$

and

$$\delta = 52a_1^4 + 208a_1^3b_1 + 54a_1^2a_2 - 180a_1^2A_2 + 366a_1^2b_1^2 +$$
$$108a_1^2b_2 - 126a_1^2B_2 + 108a_1a_2b_1 - 252a_1A_2b_1 + 108a_1a_3 +$$
$$370a_1b_1^3 + 378a_1b_1b_2 - 198a_1b_1B_2 + 54a_1b_3 + 27a_1B_3 +$$
$$162a_1c_3 + 27a_2A_2 + 54a_2b_1^2 + 81A_2^2 - 45A_2b_1^2 +$$
$$81A_2b_2 + 135A_2B_2 + 108a_3b_1 + 27A_3b_1 + 160b_1^4 +$$
$$270b_1^2b_2 - 72b_1^2B_2 + 54b_1b_3 + 27b_1B_3 + 162b_1c_3 + 81B_2^2.$$

Note that the right Poincaré map in (23) is an involution, which is related to the invisible character of the tangency at the origin. Replacing (23) in (22), we obtain the existence of a periodic orbit passing for the point $(0, y_0)$ with small $y_0 > 0$, whenever

$$\gamma = -\frac{\alpha}{3\pi}y_0^3 + \frac{\rho}{3\pi}y_0^5 + \frac{\alpha^2}{3\pi}y_0^4 + \frac{\nu}{3\pi}y_0^6 + \cdots \tag{24}$$

where

$$\rho = \alpha - \frac{4\alpha^3}{9} - \frac{\beta}{45},$$

and

$$\nu = \frac{2\beta\alpha}{135} - \frac{2\alpha}{3\pi} - \frac{5\alpha^2}{3} + \frac{4\alpha^4}{27} + \frac{2\delta}{135}.$$

We remark that for $\alpha = 0$, that is $b_1 = -a_1$, we have

$$\beta_0 := \beta|_{\alpha=0} = -9(a_1^3 + a_1A_2 - 2a_3 - A_3 + 3a_1b_2 + a_1B_2 - b_3 - 3B_3 - 3c_3), \tag{25}$$

and

$$\delta_0 := \delta|_{\alpha=0} = 27\left(a_1^2 A_2 - a_1 A_3 + a_2 A_2 + 3A_2^2 + 3A_2 b_2 + 5A_2 B_2 + 3B_2^2\right), \tag{26}$$

so that

$$\rho_0 := \rho|_{\alpha=0} = -\frac{\beta_0}{45}, \quad \nu_0 := \nu|_{\alpha=0} = \frac{2\delta_0}{135}. \tag{27}$$

To determine the number of positive solutions y_0 of (24) for (α, γ) in a neighborhood of $(0,0)$, we could do the change of variables

$$\gamma = \frac{\epsilon}{3\pi} y_0^2, \tag{28}$$

where the new variable ϵ is to be introduced instead of γ. Note that $\mathrm{sign}(\epsilon) = \mathrm{sign}(\gamma)$. We would obtain, after dividing by y_0^2, that equation (24) simplifies to

$$\epsilon = -\alpha y_0 + \rho(\alpha) y_0^3 + \alpha^2 y_0^2 + \nu(\alpha) y_0^4 + \cdots, \tag{29}$$

so that we would recognize the cusp catastrophe scenario for the triplet (ϵ, α, y_0), restricted to $y_0 \geq 0$, see [19]. Thus, provided that $\rho_0 = \rho(0) \neq 0$, the equation corresponding to lower order terms in (α, y_0), namely

$$\epsilon + \alpha y_0 - \rho_0 y_0^3 = 0, \tag{30}$$

determines the bifurcation behavior. We deduce that in the original parameter plane (α, γ) there must exist, for equation (24), a curve corresponding to saddle-node bifurcation points. Working directly with equation (24), the following result follows. The assertions concerning the stability of limit cycles come easily from the local properties of half-Poincaré maps, see Proposition 3 in [16] for a full study of the half-return map P_L and expression (23) for the half-return map P_R^{-1}.

Proposition 2. *Consider system* (14)-(15) *and define the parameters* $\alpha = a_1 + b_1$, β_0 *as in* (25), *and assume* $\beta_0 \neq 0$. *The following statements hold.*

- **(a)** *If* $\gamma = 0$, *then the outermost orbit of the center, which is tangent to the* y*-axis, is unstable from outside when* $\beta_0 \alpha > 0$, *being stable from outside and surrounded by one unstable limit cycle when* $\beta_0 \alpha < 0$.
- **(b)** *If* $\alpha = 0$ *the system has no periodic orbits if* $\beta_0 \gamma > 0$, *having one limit cycle surrounding the focus for small* $\beta_0 \gamma < 0$.

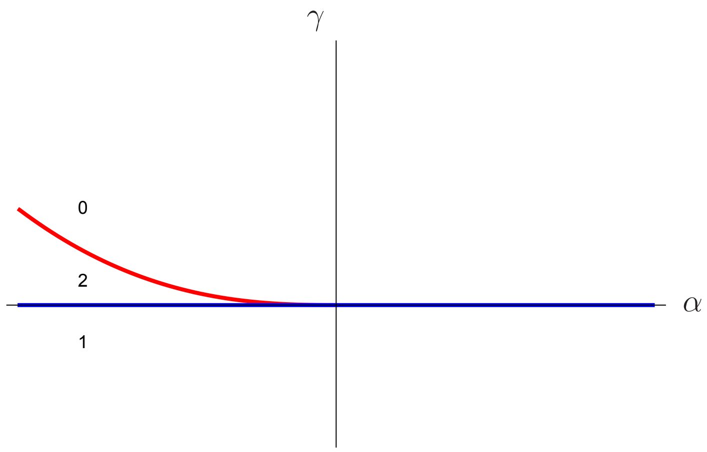

Figure 2: The number of positive roots for (24) when $\beta_0 > 0$ near the origin of the parameter plane (α, γ).

(c) *If $\beta_0\alpha > 0$, then there are no periodic orbits for small $\alpha\gamma > 0$, and one periodic solution for small $\alpha\gamma < 0$.*

(d) *If $\beta_0\alpha < 0$, then there is one periodic orbit for small $\alpha\gamma > 0$.*

(e) *When $\beta_0\alpha < 0$ and $\alpha\gamma < 0$ is small, there exists a curve of the form*

$$\mathrm{SN}(\alpha, \gamma) = \gamma + \frac{2\alpha}{5\pi}\left(-\frac{3\alpha}{\beta_0}\right)^{\frac{3}{2}} + O(\alpha^2) = 0, \tag{31}$$

whose graph separates, in the plane (α, γ), a region with two limit cycles from a region with no limit cycles, so determining saddle-node bifurcation points for periodic orbits. More precisely, the following cases arise.

(i) *If $\mathrm{SN}(\alpha, \gamma)\gamma < 0$, then the focus is surrounded by two limit cycles.*

(ii) *If $\mathrm{SN}(\alpha, \gamma) = 0$, then the focus is surrounded by just one semi-stable limit cycle.*

(iii) *If $\mathrm{SN}(\alpha, \gamma)\gamma > 0$, then the focus is surrounded by no limit cycles.*

Obviously, when $\gamma \neq 0$, its sign determines the stability of the focus, and the stability of limit cycles, if any, can be deduced accordingly (alternating stabilities, when needed).

According to Proposition 2, in Figure 2 we show the number of positive solutions of (24) for different regions of the parameter plane (α, γ) under the assumption $\beta_0 > 0$, so that there appears a small region with two positive solutions in the second quadrant of such parameter plane. The curve indicating the change between two and none roots can be approximated by using the expression of $\mathrm{SN}(\alpha, \epsilon)$, and corresponds to a saddle-node bifurcation of periodic orbits.

Remark 1. *Proposition 2 greatly extends the focus-center-limit cycle bifurcation studied in [26] for the piecewise-linear case.*

3.1. Example: the simplest continuous non-piecewise linear example with two limit cycles

To illustrate the above analysis, it is interesting to consider the simplest case where system (14)-(15) becomes continuous but not being a purely piecewise linear case. It is known that continuous piecewise linear systems with two zones separated by a straight line can have at most one limit cycle, see [11, 25]; starting from the piecewise linear case, by adding only one quadratic term that preserves the continuity of the vector field, we will see that two limit cycles are possible.

Effectively, if we take the 'left' system as in (14) and the 'right' system as follows,

$$\begin{aligned} \dot{x} &= -y + a_1 x + a_3 x^2, \\ \dot{y} &= 1, \end{aligned} \tag{32}$$

we see that all the coefficients in (15) are zero excepting a_1 and a_3, so that $\alpha = a_1$ and $\beta_0|_{a_1=0} = 18a_3$. Looking for the sector in the plane (α, γ) with two limit cycles, we simulated system (14)-(32) for $a_1 = -1$, $a_3 = 1$ and $\gamma = 0.025$, obtaining the two limit cycles of Figure 3.

4. Semi-homogeneous piecewise linear systems: bifurcation of limit cycles at infinity and invariant semi-cones.

In this section, we will study the homogeneous version of system (1)-(2) corresponding to $\mu = 0$, namely

$$\begin{aligned} \dot{x} &= (\lambda + 2\sigma)x - y, \\ \dot{y} &= (2\lambda\sigma + \sigma^2 + \omega^2)x - z, \\ \dot{z} &= \lambda(\sigma^2 + \omega^2)x, \end{aligned} \tag{33}$$

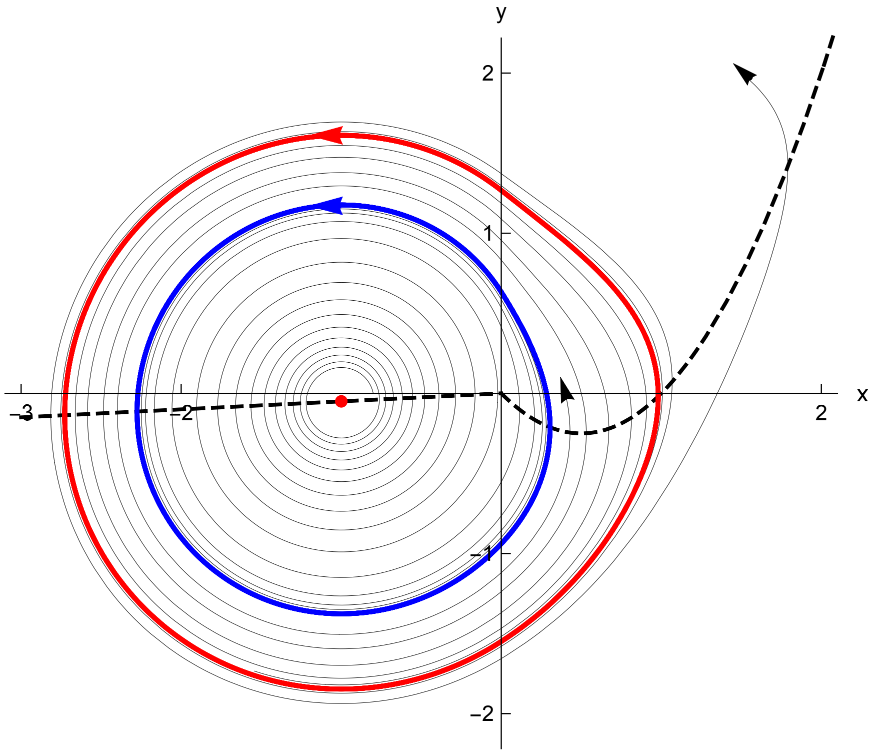

Figure 3: The two limit cycles for system (14)-(32) for $\alpha = a_1 = -1$, $a_3 = 1$ and $\gamma = 0.025$. We show the vertical nullcline with a dashed line. The unstable focus is surrounded by a stable limit cycle (in blue) and an unstable limit cycle (in red).

for $x < 0$ with $\omega > 0$, and

$$\begin{array}{rcl} \dot{x} & = & tx - y, \\ \dot{y} & = & mx - z, \\ \dot{z} & = & dx, \end{array} \tag{34}$$

for $x > 0$.

From the analysis in Section 3, we get our first main result. Recall that $\mathrm{CP}_R(\lambda) = -\lambda^3 + t\lambda^2 - m\lambda + d$ is the characteristic polynomial of the linear part A_R in the region $x > 0$ evaluated in λ.

Theorem 1. *Consider system* (33)-(34)*, assuming that*

$$\Gamma = \mathrm{CP}_R(\lambda)\big|_{t=3\lambda} = d - m\lambda + 2\lambda^3 \neq 0, \tag{35}$$

and define the bifurcation parameters $\varepsilon = t - 3\lambda$ and $\eta = \sigma - \lambda$. The following statements hold.

(a) *For $\eta = 0$ there exists a continuum of invariant semi-cones totally contained in the region $x \le 0$. The system undergoes for $\varepsilon = 0$ a bifurcation from the most external semi-cone of the above continuum, so that for $\varepsilon\Gamma < 0$ and small, there appears one invariant isolated semi-cone that crosses the plane $x = 0$, containing the above semi-cones.*

(b) *For $\varepsilon = 0$ the system undergoes a bifurcation when $\eta = 0$, so that when $\Gamma\eta > 0$ and small there are no invariant semi-cones, and when $\Gamma\eta < 0$ and small there appears an invariant isolated semi-cone that crosses the plane $x = 0$ bifurcating from the continuum of semi-cones that exists for $\eta = 0$.*

(c) *For small values of $\Gamma\eta > 0$ and $\varepsilon\eta > 0$, there are no isolated invariant semi-cones.*

(d) *For small $\Gamma\eta < 0$, there exists a unique isolated invariant semi-cone.*

(e) *For small $\Gamma\eta > 0$ and $\varepsilon\eta < 0$, there exists a function with the expansion*

$$S(\varepsilon) := -\frac{18\varepsilon}{5\pi}\left(-\frac{3\varepsilon}{\Gamma}\right)^{\frac{3}{2}} + O(\varepsilon^3) \tag{36}$$

such that

(i) *if $(\eta - S(\varepsilon))\sigma < 0$, there exists two isolated invariant semi-cones crossing the plane $x = 0$.*

(ii) *if $\eta = S(\varepsilon)$, then there exists an invariant isolated semi-cone that crosses the plane $x = 0$.*

(iii) *if $(\eta - S(\varepsilon))\sigma > 0$, there are no invariant isolated crossing semi-cones.*

Proof. As mentioned in the introduction, we can determine the number of invariant semi-cones in system (33)-(34) by studying their periodic orbits at infinity. After making the conic projection of Section 2, we obtain the dynamics at the plane $W = 0$ that represents the invariant manifold of the point at infinity, which is ruled by the differential system

$$\begin{aligned} \dot{X} &= 2\sigma X - Y \\ \dot{Y} &= 1 + [(\lambda+\sigma)^2 + \omega^2]X - 2\lambda Y, \end{aligned} \tag{37}$$

for $X < 0$, and

$$\begin{aligned} \dot{X} &= (t-\lambda)X - Y - \mathrm{CP}_R(\lambda)X^2, \\ \dot{Y} &= 1 + (m+\lambda^2)X - 2\lambda Y - \mathrm{CP}_R(\lambda)XY. \end{aligned} \tag{38}$$

for $X > 0$.

To facilitate the analysis, taking into account the results of Section 3, we first make a single change of variables to write the left planar systems in Liénard's form, namely

$$\begin{aligned} u &= X, \\ v &= -2\lambda X + Y, \end{aligned} \tag{39}$$

After this change the system (37)-(38) becomes

$$\begin{aligned} \dot{u} &= 2(\sigma-\lambda)u - v \\ \dot{v} &= 1 + [(\sigma-\lambda)^2 + \omega^2]u, \end{aligned} \tag{40}$$

for $u < 0$, and

$$\begin{aligned} \dot{u} &= (t-3\lambda)u - v - \mathrm{CP}_R(\lambda)u^2, \\ \dot{v} &= 1 + (3\lambda^2 - 2t\lambda + m)u - \mathrm{CP}_R(\lambda)uv, \end{aligned} \tag{41}$$

for $u > 0$.

We consider a new change of variables in order to write system (40)-(41) in the form of the system (14)-(15), namely

$$x = \omega^2 u, \quad y = \omega v, \quad \hat{\tau} = \omega\tau. \tag{42}$$

After this change of variables, the system (40)-(41) becomes

$$\begin{aligned} \dot{x} &= 2(\hat{\sigma}-\hat{\lambda})x - y, \\ \dot{y} &= 1 + [(\hat{\sigma}-\hat{\lambda})^2 + 1]x, \end{aligned} \tag{43}$$

for $x < 0$, and

$$\begin{aligned} \dot{x} &= (\hat{t}-3\hat{\lambda})x - y - \widehat{\mathrm{CP}}_R(\hat{\lambda})x^2, \\ \dot{y} &= 1 + (3\hat{\lambda}^2 - 2\hat{t}\hat{\lambda} + \hat{m})x - \widehat{\mathrm{CP}}_R(\hat{\lambda})xy, \end{aligned} \tag{44}$$

for $x > 0$, where

$$\hat{\lambda} = \frac{\lambda}{\omega}, \quad \hat{\sigma} = \frac{\sigma}{\omega}, \quad \hat{t} = \frac{t}{\omega}, \quad \hat{m} = \frac{m}{\omega^2}, \quad \hat{d} = \frac{d}{\omega^3},$$

and $\widehat{\mathrm{CP}}_R(\hat{\lambda}) = -\hat{\lambda}^3 + \hat{t}\hat{\lambda}^2 - \hat{m}\hat{\lambda} + \hat{d}$. Comparing (43)-(44) with (14)-(15) we obtain the parameter values

$$\gamma = \hat{\sigma} - \hat{\lambda}, \quad a_1 = \hat{t} - 3\hat{\lambda}, \quad A_2 = 3\hat{\lambda}^2 - 2\hat{t}\hat{\lambda} + \hat{m}, \quad A_3 = a_3 = -\widehat{\mathrm{CP}}_R(\hat{\lambda}),$$

and that the remaining parameters in (15) vanish. Since $b_1 = 0$, the parameter $\alpha = a_1$ in (24) , so that

$$\beta_0 = -27\, \widehat{\mathrm{CP}}_R(\hat{\lambda})\Big|_{\hat{t}=3\hat{\lambda}}.$$

Note that $\omega\gamma = \eta$, and $\omega a_1 = \varepsilon$. Also, in (35) we get

$$\Gamma = \omega^3\, \widehat{\mathrm{CP}}_R(\hat{\lambda})\Big|_{\hat{t}=3\hat{\lambda}},$$

so that $\mathrm{sign}(\Gamma) = -\,\mathrm{sign}(\beta_0)$.

To prove the theorem, it suffices now to apply Proposition 2, by identifying the limit cycles there with isolated invariant crossing semi-cones. Naturally, the stability character of such limit cycles is inherited by the corresponding semi-cones, regarding the attractiveness of such invariant manifolds; we did not give explicitly such stability in the different statements for brevity.

The study of invariant cones in PWL homogeneous systems began in [7], where the existence of a saddle-node bifurcation of invariant cones was conjectured. In [8] such a bifurcation is analyzed in a particular situation: they consider the rather degenerate situation when the global system is purely linear under just one-parameter perturbation within the PWL context. Thus, Theorem 1 completes the quoted works, by describing the codimension-two bifurcation of invariant cones in the general case, which includes a codimension-one curve of saddle-node bifurcation of invariant cones.

5. Bifurcation of a limit cycle from a center at infinity in PWL systems with two zones

Here, we deal with the analysis near infinity of the non-homogeneous case of system (1)-(2), that is $\mu = 1$. We know that at the invariant manifold at infinity ($W = 0$) the dynamics is the same as before, so that the previous results on bifurcation of limit cycles at infinity apply, but now periodic orbits at infinity do not translate into invariant semi-cones for the global dynamics. However, in the special case $\lambda = \sigma = 0$, it is possible to control the bifurcation of a limit cycle from infinity that appears by moving the parameters of the right system.

Thus, we will consider a specific instance of system (1)-(2), namely

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= \omega^2 x - z, \\ \dot{z} &= 1, \end{aligned} \tag{45}$$

for $x < 0$, and

$$\begin{aligned} \dot{x} &= tx - y, \\ \dot{y} &= mx - z, \\ \dot{z} &= dx + 1, \end{aligned} \tag{46}$$

for $x > 0$. A first observation concerns the parameter $\omega > 0$, which is not relevant, as stated in the following preliminary result. Since the proof is direct, it is omitted.

Lemma 1. *In system* (1)-(2) *we can assume* $\omega = 1$ *without loss of generality, since it suffices to make the change of variables*

$$\hat{\tau} = \omega\tau, \quad \hat{x} = \omega^3 x, \quad \hat{y} = \omega^2 y, \quad \hat{z} = \omega z, \tag{47}$$

and to use then as new parameters

$$\hat{t} = \frac{t}{\omega} \quad \hat{m} = \frac{m}{\omega^2}, \quad \hat{d} = \frac{d}{\omega^3}. \tag{48}$$

In the sequel, we suppose $\omega = 1$, discarding the hats in variables and parameters to alleviate notation. However, to come back to the original system, we must take into account Lemma 1.

It is also remarkable that system (45), when considered in the whole $\mathbb{R}^3$, does not admit invariant semi-cones, but it possesses the first integral

$$H(x, y) = (x - z)^2 + (y + 1)^2,$$

so that the center at infinity is now associated to the family of invariant cylinders $H(x, y) = k$ for $k > 0$ arbitrary, which share as their common axis the straight line $x = z, y = -1$.

Another observation concerning the existence of periodic orbits for system (45)-(46) comes from the following straightforward result.

Lemma 2. *If $d \geq 0$, then system* (45)-(46) *has no periodic orbits.*

PROOF. It suffices to observe that under hypothesis $d \geq 0$, the derivative $\dot{z}$ is positive everywhere and so we cannot have closed orbits.

Our final main result, which takes into account the above necessary condition, is the following.

Theorem 2. *System* (45)-(46) *with $0 < m \neq \omega^2$ and $d < 0$ undergoes a limit cycle bifurcation from a center at infinity for $d - \omega^2 t = 0$, so that one limit cycle of great amplitude appears for $(m - \omega^2)(d - \omega^2 t) < 0$ and $|d - \omega^2 t|$ sufficiently small. In particular, if $m - \omega^2 < 0$ then one limit cycle bifurcates for $d - \omega^2 t > 0$ and it is orbitally asymptotically stable, while when $m - \omega^2 > 0$ the limit cycle bifurcates for $d - \omega^2 t < 0$, being then unstable.*

The proof of Theorem 2 appears in Section 6. We remark that, up to the best of our knowledge, it constitutes the first bifurcation result from infinity in 3D systems without any symmetry; see [4] and [22] for cases with some reversibility and [2] for symmetric PWL systems.

We note that Theorem 2, when considering t as the only bifurcation parameter, leads to the critical value $t_{c1} = d/\omega^2$, where the bifurcation from infinity takes place. Besides, we know from Theorem 1 in [6] that for the critical value $t_{c2} = d/m$ there appears a focus-center-limit cycle bifurcation; more precisely, the following analogous result applies, see the quoted paper for a proof.

Proposition 3. *System* (45)-(46) *with $0 < m \neq \omega^2$ and $d < 0$ undergoes a limit cycle bifurcation from a bounded center at the right zone for $d - mt = 0$, so that one limit cycle of finite amplitude appears for $(m - \omega^2)(d - mt) > 0$ and $|d - mt|$ sufficiently small. In particular, if $m - \omega^2 < 0$ then one limit cycle bifurcates for $d - mt < 0$ and it is orbitally asymptotically stable, while when $m - \omega^2 > 0$ the limit cycle bifurcates for $d - mt > 0$, being then unstable.*

Under the common hypotheses of Theorem 2 and Proposition 3, if we consider for instance the case $m - \omega^2 < 0$, then

$$t_{c2} = \frac{d}{m} < \frac{d}{\omega^2} = t_{c1} < 0,$$

so that there exists $\varepsilon > 0$ such that one stable limit cycle appears for $t_{c2} < t < t_{c2} + \varepsilon$ and also for $t_{c1} - \varepsilon < t < t_{c1}$, this time being of great amplitude. The situation is the opposite when $m - \omega^2 > 0$, for then

$$t_{c1} = \frac{d}{\omega^2} < \frac{d}{m} = t_{c2} < 0,$$

and so, for some $\varepsilon > 0$ one unstable limit cycle of great amplitude appears for $t_{c1} < t < t_{c1} + \varepsilon$ and also for $t_{c2} - \varepsilon < t < t_{c2}$, this time being of finite amplitude. It is natural to state the following conjecture, which deserves to be investigated, surely requiring different techniques to the ones developed in this paper.

Conjecture 3. *System (45)-(46) with $0 < m \neq \omega^2$ and $d < 0$ has at least one limit cycle for all the values of t between t_{c1} and t_{c2}. In particular, if $m < \omega^2$ then for $t_{c2} < t < t_{c1}$ one limit cycle exists and it is orbitally asymptotically stable, while one unstable limit cycle exists for $t_{c1} < t < t_{c2}$ when $m > \omega^2$.*

Proving or disproving the above conjecture will help to characterize the appearance of additional limit cycles in the 3D piecewise linear version of the Hopf-pitchfork bifurcation, see [24]. Such study needs additional research efforts and should be the scope of a future work.

6. Proof of Theorem 2

After applying Lemma 1, we can assume $\omega = 1$ without loss of generality. To work near the most interesting chart at infinity, we resort again to the conic projection of Section 1, taking into account that here

$$A_L = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

so that $\mathbf{a}^\top = (0, 0, 1)$, corresponding to a left eigenvector of the zero eigenvalue. Thus, the change of variables reduces to

$$x = \frac{X}{W}, \quad y = \frac{Y}{W}, \quad z = -\frac{1}{W}, \tag{49}$$

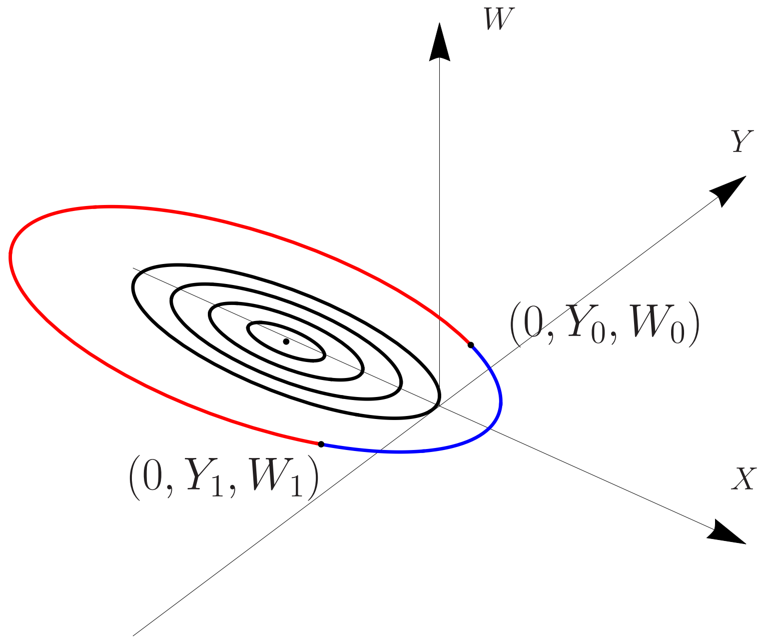

Figure 4: The limit cycle (in red, the part with $X < 0$, in blue for $X > 0$) bifurcates from the persistent center at the plane $W = 0$, which is located in the region $X \leq 0$.

where we assume $W > 0$. Accordingly, we arrive to the differential system

$$\begin{aligned} \dot{X} &= -Y + XW, \\ \dot{Y} &= 1 + X + YW, \\ \dot{W} &= W^2, \end{aligned} \tag{50}$$

for $X < 0$, and

$$\begin{aligned} \dot{X} &= tX - Y + dX^2 + XW, \\ \dot{Y} &= 1 + mX + dXY + YW, \\ \dot{W} &= dXW + W^2, \end{aligned} \tag{51}$$

for $X > 0$. In the invariant plane $W = 0$ corresponding to the manifold at infinity, it is easy to see now that for system (50) we have indeed a linear center at $(X, Y, W) = (-1, 0, 0)$, which is responsible for a period annulus tangent to the straight-line $X = W = 0$ at the origin. This center is clearly a consequence of the invariant cylinders of system (45), see Figure 4.

To study the possible limit cycle bifurcation from the center at infinity, let us recall the closing equations technique for periodic orbits starting with system (50)-(51). Assume that after integrating the orbit starting at $(X, Y, W) = (0, Y_0, W_0)$ with $Y_0 > 0$ and using the left vector field, we come back to the plane $X = 0$ at a certain point $(X, Y, W) = (0, Y_1, W_1)$ with $Y_1 < 0$. Thus, we can define the left return map $(Y_1, W_1) = P_L(Y_0, W_0)$. Clearly, if we integrate the orbit starting at the point $(0, Y_1, W_1)$ using now the right vector field, then, in order to get a periodic orbit, we must return to the plane $X = 0$ exactly at the point $(X, Y, W) = (0, Y_0, W_0)$.

Solving the differential system (50) by taking as initial condition any point $(0, Y_0, W_0)$ in the quadrant of the plane $X = 0$ with $Y_0 > 0$ and $W_0 > 0$, we get

$$\begin{aligned} X(\tau) &= \frac{\cos\tau - (Y_0 + W_0)\sin\tau}{1 - \tau W_0} - 1, \\ Y(\tau) &= \frac{\sin\tau + (Y_0 + W_0)\cos\tau - W_0}{1 - \tau W_0}, \\ W(\tau) &= \frac{W_0}{1 - \tau W_0}, \end{aligned} \tag{52}$$

where such expressions are valid for us while $X(\tau) \leq 0$ and, in particular, we must require $\tau < 1/W_0$. For instance, assume $W_0 < 1/(2\pi)$; if we take $\tau = 2\pi < 1/W_0$, we see that

$$X(2\pi) = \frac{1}{1 - 2\pi W_0} - 1 > 0,$$

so that the solution has already become invalid. By continuity, there exists $\tau_L = \tau_L(Y_0, W_0) < 2\pi$ such that $X(\tau) < 0$ for $0 < \tau < \tau_L$ and $X(\tau_L) = 0$. Imposing the last equality and defining the deviation time with respect to a complete tour $s = 2\pi - \tau_L$, we solve the first expression in (52) for W_0 to obtain

$$W_0 = \frac{1 - \cos s - Y_0 \sin s}{2\pi - s + \sin s}. \tag{53}$$

Clearly, if we define $Y_1 = Y(\tau_L)$ and $W_1 = w(\tau_L)$, we see that the orbit with initial condition $(0, Y_0, W_0)$ eventually arrives to the point $(0, Y_1, W_1)$ using only the half-space $X \leq 0$. Substituting the W_0 value (53) into the last two expressions in (52) and using s instead of τ_L, we get

$$Y_1 = \frac{(2\pi - s)(Y_0 \cos s - \sin s) + 2(\cos s - 1) + Y_0 \sin s}{(2\pi - s)(\cos s + Y_0 \sin s) + \sin s}, \tag{54}$$

and

$$W_1 = \frac{\cos s + Y_0 \sin s - 1}{(2\pi - s)(\cos s + Y_0 \sin s) + \sin s}, \tag{55}$$

which give us, along with (53) a parametric representation of the 'left' half-return map in terms of (Y_0, s), namely

$$(Y_1, W_1) = \widetilde{P}_L(Y_0, s) = P_L\left(Y_0, W_0(Y_0, s)\right). \tag{56}$$

Consider now the right system (51). Here, instead of looking for solutions in closed form, we compute recursively the successive time derivatives for X, Y and W, in order to get higher order expansions $X(\tau)$, $Y(\tau)$ and $W(\tau)$ satisfying the initial conditions $X(0) = 0$, $Y(0) = Y_0$ and $W(0) = W_0$. Taking now the expansion for $X(\tau)$ and imposing the return condition $X(\tau) = 0$ backwards in time, we can obtain an expansion of the (negative) flight time τ_R in a power series in Y_0 with coefficients depending on (t, m, d, W_0), namely

$$\begin{aligned} \tau_R &= -2Y_0 + \frac{2}{3}(t + 2W_0)Y_0^2 + \frac{2}{9}(3m - 2t^2 - 5tW_0 - 8W_0^2)Y_0^3 + \\ &+ \frac{2}{135}(27d - 54mt + 22t^3 - 108mW_0 + 78t^2W_0 + 150tW_0^2 + 200W_0^3)Y_0^4 \\ &+ \cdots \end{aligned} \tag{57}$$

Such an expression and other similar expansions in the sequel have been obtained thanks to the software Mathematica [23].

After substituting the expression for τ_R into the expansions for $Y(\tau)$ and $W(\tau)$, we get expansions for the inverse of the right Poincaré map $(Y_1, W_1) = P_R^{-1}(Y_0, W_0)$, in terms of (Y_0, W_0) with coefficients depending on (t, m, d), namely

$$\begin{aligned} Y_1 &= -Y_0 + \frac{2}{3}(t + 2W_0)Y_0^2 - \frac{4}{9}(t^2 + 4tW_0 + 7W_0^2)Y_0^3 + \\ &+ \frac{2}{135}(27d - 9mt + 22t^3 - 18mW_0 + 123t^2W_0 + 375tW_0^2 + 560W_0^3)Y_0^4 + \\ &+ \cdots, \end{aligned} \tag{58}$$

and

$$
\begin{aligned}
W_1 &= W_0 - 2W_0^2 Y_0 + \frac{2}{3}(tW_0^2 + 8W_0^3)Y_0^2 - \\
&- \frac{2}{9}(3dW_0 - 3mW_0^2 + 2t^2W_0^2 + 17tW_0^3 + 68W_0^4)Y_0^3 + \cdots .
\end{aligned}
\tag{59}
$$

Note that the invisible tangency along the W-axis assures here the property $P_R^{-1}(0, W_0) = (0, W_0)$, while the invariant character of the plane $W = 0$ implies that $P_R^{-1}(Y_0, 0) = (Y_1, 0)$.

Using (53), we can get the parametric representation of P_R^{-1}, namely

$$
(Y_1, W_1) = \widetilde{P}_R^{-1}(Y_0, s) = P_R^{-1}\left(Y_0, W_0(Y_0, s)\right). \tag{60}
$$

From (56) and (60), we can write the closing equations in terms of Y_0 and s, that is

$$
\widetilde{P}_L(Y_0, s) = \widetilde{P}_R^{-1}(Y_0, s); \tag{61}
$$

their solutions with $Y_0 > 0$, $s > 0$ correspond with periodic orbits of the whole system.

We start by considering the first equation in (61), and by implicit function theory it is possible to solve for s, getting the expansion

$$
s = 2Y_0 - \frac{2t}{3}Y_0^2 - \frac{2}{9}(3-2t^2)Y_0^3 + \frac{2}{135}(45t - 22t^3 + 9mt - 27d)Y_0^4 + O(Y_0^5). \tag{62}
$$

Proceeding as prescribed by the Lyapunov-Schmidt procedure, we substitute such expansion in the second equation of (61), to obtain, after a surprising cancellation of all the terms up to Y_0^5, the bifurcation equation

$$
0 = t(d-t)Y_0^6 - 2t(d-t)Y_0^7 + \frac{(d-t)\varphi + 6d^2(1-m)}{15}Y_0^8 + O(Y_0^9), \tag{63}
$$

with $\varphi = 3d + 6md - 30t - 6mt + 43t^3$. Here, we select the trace t as the main bifurcation parameter. By direct inspection of the coefficients in (63), we detect two critical values for t, namely $t_c = 0$, and $t_c = d$. In both cases, the first two terms vanish and the third one is different from zero, provided that $d \neq 0$ and $m \neq 1$.

To detect possible bifurcating periodic orbits, we should consider first the critical value $t_c = 0$. However, in such a case the bifurcating periodic orbit belongs to the plane $W = 0$. We do not consider this bifurcation any more because the new limit cycle remains at infinity, so that it cannot be observed in practice.

Regarding the possible bifurcation of periodic orbits in the case of the critical value $t_c = d$, we solve (63) in a neighborhood of such critical value, obtaining

$$\begin{aligned} t &= d - \frac{2d(m-1)}{5} Y_0^2 + \frac{d(m-1)(4\pi d + 5)}{15\pi} Y_0^3 - \\ &- \frac{d(m-1)(175d + 104\pi d^2 - 138\pi m)}{525\pi} Y_0^4 - \\ &- \frac{d(m-1)(490d^2 - 525m + 420d\pi + 248d^3\pi - 804dm\pi)}{1575\pi} Y_0^5 - \\ &- \frac{d(m-1)C_1(t,m,d)}{70875\pi^2} Y_0^6 + O(Y_0^7), \end{aligned} \tag{64}$$

where

$$\begin{aligned} C_1(t,m,d) &= 7875 - 7875m + 42525d\pi + 20300d^3\pi - 59850dm\pi + \\ &+ 36552d^2\pi^2 + 9256d^4\pi^2 - 50856d^2m\pi^2 + 14094m^2\pi^2. \end{aligned}$$

Substituting this expression in (62) we get

$$\begin{aligned} s &= 2Y_0 - \frac{2d}{3} Y_0^2 - \frac{2(3-2d^2)}{9} Y_0^3 + \frac{2d(27m - 22d^2)}{135} Y_0^4 + \\ &+ \frac{2(135d - 45md + 81\pi + 90\pi d^2 + 52\pi d^4 - 162\pi md^2)}{405\pi} Y_0^5 - \\ &- \frac{2dC_2(t,m,d)}{14175\pi} Y_0^6 + O(Y_0^7) \end{aligned} \tag{65}$$

where

$$C_2(t,m,d) = 9450d - 3675md + 5828\pi d^2 + 1500\pi d^4 - 8028\pi md^2 + 2025\pi m^2.$$

Now, if we translate this expression to (53) and (55) we get

$$\begin{aligned} W_0 &= -\frac{d}{3\pi} Y_0^3 + \frac{d^2}{3\pi} Y_0^4 + \frac{d(9m - 14d^2)}{45\pi} Y_0^5 + \\ &+ \frac{d(45 - 45m + 90\pi d - 216\pi md + 116\pi d^3)}{405\pi^2} Y_0^6 + O(Y_0^7), \end{aligned} \tag{66}$$

and a very similar expression for W_1 (in fact, the first different term appears in the sixth degree coefficient)

$$\begin{aligned} W_1 &= -\frac{d}{3\pi}Y_0^3 + \frac{d^2}{3\pi}Y_0^4 + \frac{d(9m-14d^2)}{45\pi}Y_0^5 + \\ &+ \frac{d(45-45m+180\pi d-216\pi md+116\pi d^3)}{405\pi^2}Y_0^6 + O(Y_0^7). \end{aligned} \tag{67}$$

Thus, recalling that $d<0$, these positive values for W_0 and W_1 indicate that a periodic orbit, not contained in the plane $W=0$, bifurcates at the critical value $t_c = d$. From (64) we see that the bifurcation takes place when $\mathrm{sign}(t-d) = \mathrm{sign}(m-1)$

To check the stability of this periodic orbit, we resort to the following lemma, useful for PWL systems with two zones separated by the plane $x=0$.

Lemma 3. *Assume that τ_L, τ_R denote the flight times for a periodic orbit of system* (1)-(2)*, with $(0, y_0, z_0)$ and $(0, y_1, z_1)$ as their intersection points with the plane $x=0$, so that $P_L(y_0, z_0) = (y_1, z_1)$ and $P_R(y_1, z_1) = (y_0, z_0)$. The following statement holds.*

The derivative DP of the return map $P = P_R \circ P_L$, when evaluated at the fixed point (y_0, z_0), shares its two eigenvalues with the product of matrix exponentials $B = \exp(A_L\tau_L)\cdot\exp(A_R\tau_R)$, for which the third eigenvalue is always the unity.

Proof. See Proposition 3 in [6]

To apply the above lemma we first write the expression for τ_R, after substituting in (57) the expression (64) for t and (66) for W_0, taking into account that we need to use P_R instead of P_R^{-1} (that is, the time τ_R now stands for a positive forward time. Thus, we get

$$\begin{aligned} \tau_R &= 2Y_0 - \frac{2d}{3}Y_0^2 + \frac{2}{9}(2d^2-3m)Y_0^3 - \frac{2d(45+22d^2-72m)}{135}Y_0^4 + \\ &+ \frac{2(135d-45md+216\pi d^2+52\pi d^4-288\pi md^2+81\pi m^2)}{405\pi}Y_0^5 - \\ &- 2d\left(\frac{d(18-7m)}{27\pi} + \frac{20d^4}{189} + \frac{m(71m-56)}{105} + \frac{8d^2(1236-1511m)}{14175}\right)Y_0^6 + \\ &+ O(Y_0^7). \end{aligned} \tag{68}$$

To avoid the eigenvalue computations of Lemma 3, we just determine the expansions in powers of Y_0 for the trace and determinant of the involved product of matrix exponentials, where $\tau_L = 2\pi - s$, with s given in (65) and τ_R in (68). We obtain

$$
\begin{aligned}
\mathrm{trace}(B) &= 3 + 2dY_0 + \frac{4d^2}{3}Y_0^2 + \frac{2d(18+10d^2-33m)}{45}Y_0^3 + \\
&+ \frac{2d(-45+45m+9\pi d+8\pi d^3-72\pi dm)}{135\pi}Y_0^4 + \\
&+ 2d\left(\frac{d(m+1)}{9\pi} + m\frac{313m-104(2+d^2)}{525} + d^2\frac{140d^2-342}{14175}\right)Y_0^5 + \\
&+ \frac{2dC_{\mathrm{trace}(B)}}{14175\pi}Y_0^6 + O(Y_0^7),
\end{aligned}
\tag{69}
$$

with

$$
\begin{aligned}
C_{\mathrm{trace}(B)} &= -735d^2 + 6300m + 1260d^2m - 6300m^2 - 5922\pi d - 384\pi d^3 + \\
&+ 40\pi d^5 + 8964\pi dm - 776\pi md^3 + 2178\pi dm^2,
\end{aligned}
\tag{70}
$$

and

$$
\begin{aligned}
\det(B) &= 1 + 2dY_0 + \frac{4d^2}{3}Y_0^2 + + \frac{2d(18+10d^2-33m)}{45}Y_0^3 + \\
&+ \frac{2d(-45+45m+9\pi d+8\pi d^3-72\pi dm)}{135\pi}Y_0^4 + \\
&+ 2d\left(\frac{d(m+1)}{9\pi} + m\frac{313m-104(2+d^2)}{525} + d^2\frac{140d^2-342}{14175}\right)Y_0^5 + \\
&+ \frac{2dC_{\det(B)}}{14175\pi}Y_0^6 + O(Y_0^7),
\end{aligned}
\tag{71}
$$

with

$$
\begin{aligned}
C_{\det(B)} &= -735d^2 + 6300m + 1260d^2m - 6300m^2 - 3402\pi d - 384\pi d^3 + \\
&+ 40\pi d^5 + 6444\pi dm - 776\pi md^3 + 2178\pi dm^2.
\end{aligned}
\tag{72}
$$

Note that all the coeficients in (69) and (71) of degrees one to five coincide, but

$$C_{\text{trace}(B)} - C_{\det(B)} = 2510\pi d(m-1).$$

The two eigenvalues μ_1, μ_2 of $DP(y_0, z_0)$ satisfy the quadratic equation

$$\mu^2 - p\mu + q = 0,$$

where $p = \mu_1 + \mu_2 = trace(B) - 1$ and $q = \mu_1\mu_2 = det(B)$. The quadratic equation has roots within the unit circle of the complex plane if and only if $|q| < 1$ and $|p| < 1 + q$. As $d < 0$, from (71) we see that the first condition is fulfilled, and from (69) that $p = trace(B) - 1 \in (2, 3)$, for $Y_0 > 0$ and suffciently small. We conclude that the bifurcating periodic orbit will be stable if

$$p - q - 1 < 0.$$

In our case, we get

$$p - q - 1 = \text{trace}(B) - \det(B) - 2 = \frac{16d^2(m-1)}{45} Y_0^6 + O(Y_0^7), \tag{73}$$

so that the bifurcating periodic orbit will be stable if $m < 1$. From (64), then $t - d < 0$, and for $Y_0 > 0$ sufficiently small, we see that the stable case leads to $t - d < 0$. Otherwise, if $m > 1$, the unstable bifurcating periodic orbit appears for $t - d > 0$.

The theorem is completely shown, by resorting to the change of variables and parameters of Lemma 1.

Conflict of interest

The authors declare that they have no conflict of interest.

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References

[1] A. Algaba, E. Freire, E. Gamero and C. García, *Quasi-homogeneous normal forms*, J. Computational and Applied Mathematics **150** (2003), 193–216.

[2] A. Amador, E. Freire, E. Ponce, J. Ros and E. Vela, *Hopf Bifurcation at Infinity in 3D piecewise linear systems with symmetry*, Preprint (2019), submitted for publication.

[3] A. A. Andronov, A. A. Vitt, S. E. Khaikin, *Theory of oscillators*, Dover Publications, Inc., New York, 1987. Translated from the Russian by F. Immirzi, Reprint of the 1966 translation.

[4] C.A. Buzzi, J. Llibre, J.C.Medrado, *Periodic orbits for a class of reversible quadratic vector field on* $\mathbb{R}^3$. Journal of Mathematical Analysis and Applications **335** (2007), 1335–1346.

[5] V. Carmona, E. Freire, E. Ponce and F. Torres, *The continuous matching of two stable linear systems can be unstable*, Discrete and Continuous Dynamical Systems **16** (2006), 689–703.

[6] V. Carmona, E. Freire, E. Ponce, J. Ros and F. Torres, *Limit cycle bifurcation in 3D continuous piecewise linear systems with two zones. Application to Chua's circuit,* International Journal of Bifurcation and Chaos **15** (2005), 3153–3164.

[7] V. Carmona, E. Freire, E. Ponce, F. Torres, *Bifurcation of invariant cones in piecewise linear homogeneous systems,* International Journal of Bifurcation and Chaos **15** (2005) 2469–2484.

[8] V. Carmona, S. Fernández-García, E. Freire, *Saddle-node bifurcation of invariant cones in 3D piecewise linear systems,* Physica D **241** (2012) 623–635.

[9] P. Diamond, N. A. Kuznetsov, D. Rachinskii, *On the Hopf bifurcation in control systems with a bounded nonlinearity asymptotically homogeneous at infinity*, Journal of Differential Equations **175** (2001) 1–26.

[10] A. F. Filippov, *Differential Equations With Discontinuous Righthand Sides*, American Mathematical Society Translations, Series 2, (AMS), 1964.

[11] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones*, Int. J. Bifurcations and Chaos **8** (1998), 2073–2097.

[12] E. Freire, E. Ponce, F. Rodrigo, and F. Torres, *Bifurcation sets of continuous piecewise linear systems with three zones*, Int. J. Bifurcation and Chaos **12** (2002), 1675–1702.

[13] E. Freire, E. Ponce and J. Ros, *The Focus-Center-Limit Cycle Bifurcation in Symmetric 3D Piecewise Linear Systems*, SIAM J. Applied Mathematics. **65** (2005), 1933–1951.

[14] E. Freire, E. Ponce, and J. Ros, *A Biparametric Bifurcation In 3D Continuous Piecewise Linear Systems with Two Zones. Application to Chuas Circuit*, Int. J. Bifurcations and Chaos **17** (2007), 445–457.

[15] E. Freire, E. Ponce and F. Torres, *Canonical discontinuous planar piecewise linear systems*, SIAM J. Applied Dynamical Systems **11** (2012), 181–211.

[16] E. Freire, E. Ponce and F. Torres, *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dynamics **78** (2014), 251–263.

[17] B.R. de Freitas, J. Llibre, J. Medrado, *Limit cycles of continuous and discontinuous piecewise-linear differential systems in R3*, Journal of Computational and Applied Mathematics **338** (2018) 311–323 .

[18] J.N. Glover, *Hopf bifurcations from infinity*, Nonlinear Analysis, Theory, Methods & Applicattons, **13** (1989),1393–1398.

[19] M. Golubitsky and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, volume 51 of Applied Mathematical Sciences. Springer-Verlag, New York, NY, 1985.

[20] N. Levinson, *A second order differential equation with singular solutions*, Annals of Mathematics, Second Series, Vol. 50, No. 1 (Jan., 1949), 127–153.

[21] J. Llibre, A. Rodrigues, *On the dynamics of a system that bridges the gap between Lorenz and Chen systems*, International Journal of Bifurcation and Chaos **25** (2015),1550122 (9 pages).

[22] J. Llibre, M. Messias, *Large amplitude oscillations for a class of symmetric polynomial differential systems in* $\mathbb{R}^3$, Anais da Academia Brasileira de Ciências **79** (2007), 1–13.

[23] Wolfram Research, Inc.,*Mathematica*,Version 10.4, Champaign, IL (2016).

[24] E. Ponce, J. Ros and E. Vela, *Unfolding the fold-Hopf bifurcation in piecewise linear continuous differential systems with symmetry,* Physica D **250** (2013), 34–46.

[25] E. Ponce, J. Ros and E. Vela, *Limit cycle and boundary equilibrium bifurcations in continuous planar piecewise linear systems,* International Journal of Bifurcation and Chaos **25** (2015), 1530008 (18 pages).

[26] E. Ponce, J. Ros and E. Vela, *The focus-center-limit cycle bifurcation in discontinuous planar piecewise linear systems without sliding*, in Progress and Challenges in Dynamical Systems, Springer Proceedings in Mathematics & Statistics, vol. 54, Springer, 2013, pp. 335349.

[27] S. Smale, *Finding a horseshoe on the beaches of Rio*, The Mathematical Intelligencer **20** (1998) 39–44. https://doi.org/10.1007/BF03024399

[28] H. Sira-Ramirez and R. Silva-Ortigoza, *Control Design Techniques in Power Electronics Devices*, Springer, 2006.

[29] Z. Wei, I. Moroz, Z. Wang, J.C. Sprott and T. Kapitaniak, *Dynamics at Infinity, Degenerate Hopf and Zero-Hopf Bifurcation for Kingni-Jafari System with Hidden Attractors*, International Journal of Bifurcation and Chaos **26** (2016), 1650125 (16 pages).

Declaration of interests

 The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: