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The *cg*-position value for games on fuzzy communication structures

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Abstract

A cooperative game for a set of agents establishes a fair allocation of the profit obtained for their cooperation. The best known of these allocations is the Shapley value. A communication structure defines the feasible bilateral communication relationships among the agents in a cooperative situation. Some solutions incorporating this information have been defined from the Shapley value: the Myerson value, the position value, etc. Later fuzzy communication structures were introduced. In a fuzzy communication structure the membership of the players and the relations among them are leveled. Several ways of defining the Myerson value for games on fuzzy communication structure were proposed, one of them is the Choquet by graphs (*cg*) version. Now we study in this work the *cg*-position value and its calculation. The *cg*-position value is defined as a solution for games with fuzzy communication structure which considers the bilateral communications as players. So, the Shapley value is applied for a new game (the link game) over the fuzzy sets of links in the fuzzy communication structure and the profit obtained for each link is allocated between both players in the link. As we see in our examples and results the *cg*-position value is more concerned with the graphical position of the players and their communications than the other *cg*-values. In this paper we also introduce a procedure to compute exactly the position value, avoiding to calculate the characteristic function of the link game for all coalitions. This procedure is used to determine the *cg*-position value. Finally we compare the new value with other *cg*-values in an applied example about the power of the groups in the European Parliament.

Keywords: game theory, fuzzy graphs, position value, Harsany's dividends, power indices, European Parliament

1. Introduction

A cooperative game with transferable utility over a finite set of players is defined as a function establishing the worth of each coalition (subset of players). The outcome of a game is a payoff vector, namely it is a vector in which each component represents the payment for each player because of their cooperation possibilities. The Shapley value [1] is the best known of these outcomes. The payoffs of the players in a game for the Shapley value are the expected worths of their marginal contributions to the coalitions containing them, i.e. the differences between the worth of the coalition and the worth of this coalition without each of

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them (see formula (1) in Section 2). The usual payoff vectors, also the Shapley value, suppose that all the communications are feasible and then all the cooperation possibilities. Moreover, they use the worths of all the coalitions in their formulations. Therefore, the payoff vector takes into account how the players cooperate among them and how the coalitions are formed. But usually this fact is not realistic and not all the coalitions are feasible. Several different models of games with partial cooperation have been studied: Aumann and Dreze [2], Owen [3], Myerson [4], etc.

Myerson [4] considered that the communication among the players is not always complete. He described the communication situation by a graph where the vertices are the players and the links are the feasible bilateral communications among them. This graph is named the communication structure of the game. Hence we will use both, graph or communication structure, alike. A communication value assigns a payoff vector to each game with a specific communication structure. The Myerson model supposes that the feasible coalitions are the connected sets in the graph and so only the worths of these coalitions should be used to elaborate outcomes. There exist several communication values defined from the Shapley value: the Myerson value [4], the position value [5], the average tree value [6], etc. The position value focuses the allocation of the profits on the links. A new cooperative game over the communications is defined by calculating the worth of a set of links as the sum of worths of the connected components in the subgraph generated by them. The Shapley value of this new game is a payoff vector for the links and the payoff of each link is allocated between both of he players in the link. Borm *et al.* [7] gave a characterization of the position value only for communication structures without cycles (trees). Later, Slikker [8] got a characterization for all the situations. Other characterizations or similar solutions can be found in van den Nouweland and Slikker [9], Ghintran *et al.* [10] and Ghintran [11]. The position value is directly related to the situation of the players in the graph as we can see in [7] where the authors proved that if the game only depends on the cardinality of the coalitions then the payoff of a player is proportional to his degree.

The problem of computing the Shapley value is NP-complete although it is P-complete for several particular cases, for instance the family of weighted majority games (see [12]). The Myerson value and the Shapley value were provided with several exact algorithms (for instance [13], [14], [15] and [16]). Particularly these algorithms were used to analyze the power in the European Union in [17] and [18]. References about computational aspects of cooperative game theory can be found in [19], but there is no analysis about algorithms to determine the position value.

Aubin [20] supposed uncertainty about the membership of the players in the coalitions studying games with fuzzy coalitions. To calculate the worth of a fuzzy coalition in a game it is necessary to consider a specific partition by levels of this fuzzy set. One of these partitions was defined by Tsurumi *et al.* [21] using the Choquet integral. Following this way, the uncertainty about the existence of the communications among the players can be extended. Recently, Jiménez-Losada *et al.* [22] introduced fuzzy graphs to analyze communication among players. Fuzzy graphs allow leveling the links between being feasible or not, and they also allow considering membership levels for the players. The idea of partition by levels was extended to fuzzy communication structures in [23], proposing different extensions of the Myerson value for fuzzy situations. In one of them, the Choquet by graphs (cg) option, players look for the biggest communication structure at the same level at each moment. Gallego *et al.* [24] studied the cg-Banzhaf value for fuzzy communication structures and the complexity of its calculation. In these models the Choquet integral determines the worth of a coalition with fuzzy links by intervals of levels of communication. The analysis of fuzzy communication structures can also allow to study games on communication networks with infinite range scaling by a sigmoid curve (the logistic function, for instance). This tool is usual in fuzzy networks as the reader can see in [25].

The main goal of this paper is to study the position value for games with fuzzy communication structure in the Choquet by graphs version. This paper is a logical continuation of our previous works about the Myerson value. We define the new solution using the Choquet integral, we get axioms for the cg-position value and we are also concerned with the computational aspects of the value. The cg-position value is also related with the fuzzy situation of the players, particularly with the fuzzy degree, as we will see later. The solution and the proposed algorithm are showed in an applied example, and it is compared to the other similar solutions. The organization of the paper is the following. Section 2 presents in short the background about cooperative games and communication structures which allows the reader to follow the paper. Section 3 is dedicated to games with fuzzy communication structure and the *cg*-partition. We define also the *cg*-position value in this section. We obtain in section 4 an axiomatization of the value. Section 5 analyzes the computation of the *cg*-position value and the time complexity of the algorithm. Finally in Section 6 we apply our solution to determine the power of the political groups in the European Parliament.

2. Preliminaries

2.1. Cooperative games

A *cooperative game with transferable utility* is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of $N = \{1, 2, ..., n\}$ are called *players*, the subsets $S \subseteq N$ *coalitions* and $v(S)$ is the *worth* of S. Let (N, v) be a game. A *null player* $i \in N$ satisfies $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$ with $i \in S$. The game (N, v) is 0-*normalized* if $v(\{i\}) = 0$ for all $i \in N$. The 0-*normalization* of (N, v) is a new game (N, v_0) with

$$v_0(S) = v(S) - \sum_{i \in S} v(\{i\}) \tag{1}$$

for each $S \subseteq N$. The *unanimity game* for coalition $T \subseteq N$, $T \neq \emptyset$, is (N, u_T) with $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. The characteristic function of every game (N, v) is a linear combination of unanimity games in N, that is

$$v = \sum_{\{T \subseteq N : T \neq \emptyset\}} \Delta_T^v u_T, \text{ with } \Delta_T^v = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S).$$

The coefficients of the above combination, Δ_T^v for all non-empty coalition $T \subseteq N$, are named *Harsanyi dividends* [26] of the game.

A *payoff vector* for the game (N, v) is any $x \in \mathbb{R}^N$ where, for each player $i \in N$, the number x_i represents the payment of i owing to his cooperation possibilities. A *value* for cooperative games assigns to each game (N, v) a payoff vector in $\mathbb{R}^N$. The *Shapley value* [1] of a game (N, v) is defined for any player $i \in N$ as

$$\phi_i(N, v) = \sum_{\{S \subseteq N : i \in S\}} \frac{(n - |S|)!(|S| - 1)!}{n!} [v(S) - v(S \setminus \{i\})] = \sum_{\{S \subseteq N : i \in S\}} \frac{\Delta_S^v}{|S|}. \tag{2}$$

This value satisfies the following axioms.

(S1) It is a linear function with respect to the characteristic function of the game, that is if $a_1, a_2 \in \mathbb{R}$ then for all games $(N, v_1), (N, v_2)$ we have $\phi(N, a_1v_1 + a_2v_2) = a_1\phi(N, v_1) + a_2\phi(N, v_2)$.

(S2) It satisfies the null player axiom, that is if $i \in N$ is a null player for the game (N, v) then $\phi_i(N, v) = 0$.

2.2. Communication structures

Let (N, v) be a cooperative game. Myerson [4] considered that the bilateral communications among players can modify the solution of a game. He represented the players' communications by a graph. An undirected *graph* $g = (S, A)$ is defined by a finite set S and a set A of unordered pairs of different members of S. The elements of S are named *vertices* and the elements of A are called *links*. Let $L = \{ij : i \neq j; i, j \in N\}$ denote the set of bilateral relations among the players in our game. Myerson defined a *communication structure* over N as a graph $g = (S, A)$ where $S \subseteq N$ is the subset of players who are genuinely active in the game (originally he only considered spanning graphs for the set of players, that is $S = N$) and $A \subseteq L$ is the set of feasible communications among them. Therefore we will use throughout the paper graph or communication structure alike. The set of all the communication structures over N is denoted by CS^N. Particularly $gN = (N, L)$ is the complete graph representing the total cooperation among all the players, and $\emptyset = (\emptyset, \emptyset)$ represents the total block situation where no player is active. If $i \in N$ then the *degree* of i is $d_i(g) = |\{ij \in A\}|$.

Let $g = (S, A) \in CS^N$ be a communication structure for the players of our game. A *subgraph* $g' = (S', A')$ of g is another graph which satisfies that $S' \subseteq S$ and $A' \subseteq A$. If $T \subseteq N$ is a coalition then we denote g_T as the subgraph of g using only the vertices in $T \cap S$ and the links in A among them. If $B \subseteq L$ then g_B is the subgraph of g using only the links in $B \cap A$ with the vertices joined by them. A path in g is defined by a sequence of vertices $(i_k)_{k=1}^{k=m}$ satisfying that $\{i_k, i_{k+1}\} \in A$ is a different link in g for each $k = 1, ..., m - 1$. The graph g is *connected* if for every pair of vertices there is a path in g containing them. Coalition T is *feasible* for g iff $T \subseteq S$ and g_T is connected. The family of all the feasible coalitions in g is denoted as $\mathcal{F}^g$. The *connected components* of g are the maximal subgraphs of

g which are connected. Hence the connected components of the graph determine a partition of the set of active players S in maximal feasible coalitions, and this partition is denoted as $N/g = \{H \subseteq S : g_H \text{ is a connected component of } g\}$. A vertex i is *isolated* in the graph g iff $\{i\} \in N/g$. For each link $ij \in A$ in g we will use the following graph $g_{-ij} = (S, A \setminus ij)$.

A *communication value* ψ assigns to every game (N, v) a function $\psi(N, v)$ determining a payoff vector for each communication structure $g \in CS^N$ over the players, $\psi(N, v)(g) \in \mathbb{R}^N$. In order to define communication values, Myerson [4] proposed to get a new game incorporating the information of every communication structure to the studied game. He introduced the following measure of the total profit for communication structures over the players of a game (N, v),

$$r^{(N,v)}(g) = \sum_{H \in N/g} v(H), \tag{3}$$

and $r^{(N,v)}(\emptyset) = 0$. Two communication values have been obtained from this measure using the Shapley value. Let (N, v) be a game. The *vertex game* for $g \in CS^N$, see [4], is the cooperative game (N, v^{Ng}) with $v^{Ng}(T) = r^{(N,v)}(g_T)$ for all $T \subseteq N$. So, the *Myerson value* is a communication value defined as $\mu(N, v)(g) = \phi(N, v^{Ng})$. In [24] a recurrence formula to obtain the dividends of the vertex game is provided,

$$\Delta_T^{v^{Ng}} = v^{Ng}(T) - \sum_{\{R \in \mathcal{F}^g : R \subset T\}} \Delta_R^{v^{Ng}} \tag{4}$$

for each $T \in \mathcal{F}^g$, and $\Delta_T^{v^{Ng}} = 0$ if $T \notin \mathcal{F}^g$. Particularly if $T \in \mathcal{F}^g$ then $v^{Ng}(T) = v(T)$.

On the other hand, the *link game*, see [7], is a game (L, v^{Lg}) defined over the links where for each graph $g \in CS^N$

$$v^{Lg}(B) = r^{(N,v)}(g_B) \quad \forall B \subseteq L. \tag{5}$$

Observe that the link game does not use the isolated vertices and therefore it loses information compared to the previous situation (the worths of the individual coalitions of the active isolated players). That is why they proposed the link game only for 0-normalized games. Hence v^{Lg} represents the link game of the 0-normalization of v from now on. Following [8] the *position value* is a communication value defined for a game (N, v) and any graph

$g = (S, A) \in CS^N$ as

$$\pi_i(N, v)(g) = v(\{i\}) + \frac{1}{2} \sum_{j \in N \setminus i} \phi_{ij}(L, v^{Lg}) \quad \forall i \in S, \tag{6}$$

and $\pi_i(N, v)(g) = 0$ otherwise.

2.3. Fuzzy sets and the Choquet integral

In this paper we use the operators $\wedge, \vee$ as the minimum and the maximum respectively. Let K be a finite set. A *fuzzy set* [27] in K is a function $\tau : K \to [0, 1]$. The family of fuzzy sets in K is denoted as $[0, 1]^K$. Each subset $Q \subseteq K$ is associated to the fuzzy set $e^Q \in [0, 1]^K$ with $e^Q(i) = 1$ if $i \in Q$ and $e^Q(i) = 0$ otherwise. Specifically, we denote $e^{\emptyset} = 0$. Let $\tau \in [0, 1]^K$. The *support* of τ is $\operatorname{supp}(\tau) = \{i \in K : \tau(i) \neq 0\}$ and the *image* of τ is the set $\operatorname{im}(\tau) = \{\lambda \in \mathbb{R} : \exists i \in K \text{ with } \tau(i) = \lambda\}$. If $t \in [0, 1]$ then the t-*cut* of τ is $[\tau]_t = \{i \in K : \tau(i) \geq t\}$. Another fuzzy set τ' is *comonotone* with τ if $[\tau(i) - \tau(j)][\tau'(i) - \tau'(j)] \geq 0$ for all $i, j \in K$. The *distance* between τ and another $\tau' \in [0, 1]^K$ is

$$D(\tau, \tau') = \bigvee_{i \in K} |\tau(i) - \tau'(i)|.$$

The *Choquet integral* [30] of a fuzzy set τ in K with respect to a set function over K, $f : 2^K \to \mathbb{R}$, is defined as

$$\int \tau \, df = \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) f([\tau]_{\lambda_k}),$$

with $im(\tau) = \{\lambda_1 < \cdots < \lambda_m\}$ and $\lambda_0 = 0$. First it was introduced only for non-negative functions (capacities) and later (see [31]) for all of them. Four interesting properties of the Choquet integral are the following:

(C1) $\int \tau \, d(af + a'f) = a \int \tau \, f + a' \int \tau \, f$, with $a, a' \in \mathbb{R}$.

(C2) If τ, τ' are comonotone and $\tau(i) + \tau'(i) \leq 1$ for every $i \in K$ then $\int (\tau + \tau') \, df = \int \tau \, df + \int \tau' \, df$.

(C3) If $t \in [0, 1]$ then $\int t\tau \, df = t \int \tau \, df$.

(C4) For each set function the Choquet integral is a continuous operator with respect to the fuzzy set using the distance D.

3. Fuzzy communication structures and the *cg*-position value

Jiménez-Losada *et al.* [22] introduced fuzzy communication structures for games as fuzzy graphs. Let (N, v) be a cooperative game and L the set of all the bilateral communications among the players in N.

Definition 1. *A fuzzy communication structure over N is an undirected fuzzy graph over N, namely a pair $\gamma = (\tau, \rho)$ with $\tau \in [0,1]^N$ the fuzzy set of vertices and $\rho \in [0,1]^L$ the fuzzy set of links satisfying $\rho(ij) \leq \tau(i) \wedge \tau(j)$ for all $ij \in L$. The set of fuzzy communication structures over N is denoted by FCS^N.*

Hence we will use fuzzy graph or fuzzy communication structure alike. We denote as $\gamma = 0$ the null fuzzy graph where $\tau = 0$ and $\rho = 0$. Every communication structure $g = (S, A) \in CS^N$ is identified with the fuzzy graph $g = (\tau, \rho)$ where $\tau = e^S$ and $\rho = e^A$. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure. The number $\tau(i)$ is interpreted as the real level of involvement of player $i \in N$ in the game v and the number $\rho(ij)$ represents the maximal level at which the link ij can be used. The set of vertices in γ is $V(\gamma) = \mathrm{supp}(\tau)$ and the set of links is $L(\gamma) = \mathrm{supp}(\rho)$. So, the *crisp version* of γ is the graph $g^\gamma = (V(\gamma), L(\gamma))$. We use the notation $N/\gamma = N/g^\gamma$ and $\mathcal{F}^\gamma = \mathcal{F}^{g^\gamma}$. The *minimal level* in γ is

$$\wedge\gamma = \left(\bigwedge_{i \in V(\gamma)} \tau(i)\right) \wedge \left(\bigwedge_{ij \in L(\gamma)} \rho(ij)\right). \tag{7}$$

A player i is *isolated* in γ if he is isolated in g^γ. The *fuzzy degree* of a vertex $i \in N$ in γ is

$$\delta_i(\gamma) = \sum_{j \in N \setminus i} \rho(ij). \tag{8}$$

The fuzzy degree can be seen as a ranking method of the different fuzzy graphs for each vertex (see [28] and [29] for more information about ranking methods). Another fuzzy graph $\gamma' = (\tau', \rho')$ over N is a *subgraph* of γ iff $\tau' \leq \tau$ and $\rho' \leq \rho$. We use in that case $\gamma' \leq \gamma$. We defined three binary operations for fuzzy graphs in [22]. Let $\gamma = (\tau, \rho), \gamma' = (\tau', \rho') \in FCS^N$ be two fuzzy graphs over N:

1) If $\tau(i)+\tau'(i)\leq 1$ for all $i\in N$ then $\gamma+\gamma'=(\tau+\tau',\rho+\rho')$.

2) If $\gamma'\leq\gamma$ and $\rho(ij)-\rho'(ij)\leq[\tau(i)-\tau'(i)]\wedge[\tau(j)-\tau'(j)]$ for all $i,j\in N$ then $\gamma-\gamma'=(\tau-\tau',\rho-\rho')$.

3) If $t\in[0,1]$ then $t\gamma=(t\tau,t\rho)$.

The reader can see that the subtraction of fuzzy graphs is not always feasible as the opposite operation of the sum, because we need to impose a new condition to obtain a new fuzzy graph (or changing the definition, see [22]). More information about fuzzy graphs in [32].

Aubin [20] introduced partitions by levels to determine the worth of a fuzzy coalition in a given cooperative game. Jiménez-Losada *et al.* [23], following Aubin and the Myerson model, defined a way to get the total profit in a fuzzy communication structure for a game. They introduced the concept of partition by levels of a fuzzy graph as a finite sequence $(g_k,s_k)_{k=1}^m$ of pairs (graph+level) which uses up the information of the fuzzy communication structure. In this paper we focus on one of these partitions based in the Choquet integral [30]. This partition for each fuzzy graph can be obtained by applying the following algorithm. Let $\gamma=(\tau,\rho)\in FCS^N$ be a fuzzy communication structure.

Algorithm 1. *cg-partition*(γ)

$k\leftarrow 0$, $cg\leftarrow\emptyset$

while $\gamma\neq 0$ **do**

  $k\leftarrow k+1$

  $s_k\leftarrow\wedge\gamma$

  $g_k\leftarrow g^{\gamma}$

  $cg\leftarrow cg\cup\{(g_k,s_k)\}$

  $\gamma\leftarrow\gamma-s_kg_k$

end

Definition 2. *Let $\gamma\in FCS^N$. The Choquet by graphs (cg) partition by levels of γ is the family $cg=(g_k,s_k)_{k=1}^m$ obtained by the above algorithm, where $g_k\in CS^N$ and $s_k\in(0,1]$ for all k. It is denoted as $cg(\gamma)$.*

The *cg*-partition implies that players try to get first the biggest graph and second the top level to connect it.

Example 1. We consider the fuzzy graph (also used in [24]) $\gamma = (\tau, \rho)$ over the set $N = \{1,2,3,4\}$ with $\rho(1,3) = \rho(2,4) = 0.4$, $\tau(1) = \tau(4) = \rho(1,2) = \rho(1,4) = 0.5$, $\tau(2) = \rho(2,3) = 0.7$, $\tau(3) = 1$ and $\rho(3,4) = 0$. Figure 1 represents this fuzzy graph and its crisp version.

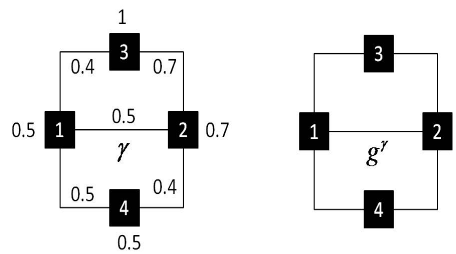

Figure 1. Fuzzy graph and crisp version.

The minimal level is $\wedge\gamma = 0.4$ and, for instance, $\delta_2(\gamma) = 1.6$. We can see in Figure 2 the cg-partition by levels of the above fuzzy graph.

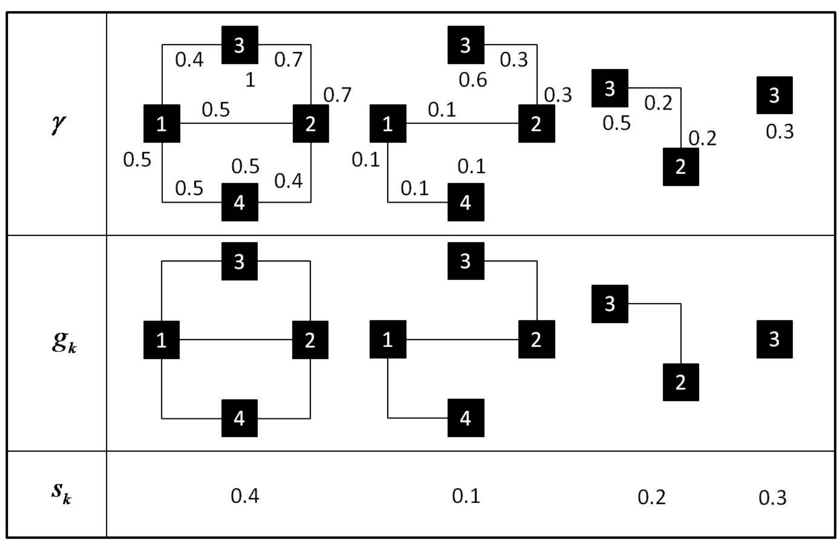

Figure 2. cg-partition.

We see several properties of the cg-partition that we will use later.

Proposition 1. *Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure. Algorithm 1 is well defined and the partition $cg(\gamma) = (g_k, s_k)_{k=1}^m$ obtained satisfies the following properties.*

1) $\gamma = \sum_{k=1}^m s_k g_k$.

2) $\sum_{\{k:i\in V(g_k)\}} s_k = \tau(i)$ *for all* $i \in N$.

3) $\sum_{ij\in L(\gamma)} \sum_{\{k\in\{1,\ldots,m\}:ij\in L(g_k)\}} s_k = \delta_i(\gamma)$ *for all* $i \in N$.

Proof. We show that the algorithm works well even though the result of a difference of fuzzy graphs is not always a new fuzzy graph. In the first step we take $s_1 = \wedge\gamma$ and $g_1 = g^{\gamma}$ then $s_1 g_1 \leq \gamma$ and for all $ij \in L(\gamma)$

$$\rho(ij) - s_1 \leq [\tau(i) - s_1] \wedge [\tau(j) - s_1]$$

because γ is a fuzzy graph. Obviously if $ij \notin L(\gamma)$ then the inequality is true because $\rho(ij) = 0$ and also the level of the link in $s_1 g_1$. Hence $\gamma - s_1 g_1$ is a new fuzzy graph. Suppose now following the algorithm that $\gamma = \gamma - \sum_{l=1}^{k-1} s_l g_l$ is a new subgraph of γ before the step k. We repeat the reasoning with s_k and the new γ. As we can continue applying the algorithm at any time we must finish with 1)

$$\gamma = \sum_{k=1}^{m} s_k g_k.$$

Condition 2) follows directly from the last equality. Also for each $i \in N$ we have 3) as

$$\delta_i(\gamma) = \sum_{j\in N\setminus i} \rho(ij) = \sum_{ij\in L(\gamma)} \sum_{\{k\in\{1,\ldots,m\}:ij\in L(g_k)\}} s_k. \qquad \square$$

Definition 3. *Given a game* (N, v)*, the cg-measure of the total profit in a fuzzy graph* γ *is*

$$\epsilon_{cg}^{(N,v)}(\gamma) = \sum_{k=1}^{m} s_k r^{(N,v)}(g_k)$$

where $cg(\gamma) = (g_k, s_k)_{k=1}^{m}$.

Now we introduce the *cg*-position value for games with fuzzy communication structure.

Definition 4. *A fuzzy communication value* F *assigns to each cooperative game* (N, v) *a payoff vector* $F(N, v)(\gamma) \in \mathbb{R}^N$ *for each fuzzy communication structure* $\gamma \in FCS^N$ *over its players.*

In [23] and [24] the cg-Myerson value and the cg-Banzhaf value for games with fuzzy communication structure were studied using the cg-partition. Now we define a fuzzy version of the link game (4) following Definition 3 in order to introduce the cg-position value. Let (N, v) be a cooperative game and $\gamma = (\tau, \rho) \in FCS^N$. If $A \subseteq L$ then $\gamma_A = (\tau_A, \rho_A) \in FCS^N$ is a new fuzzy graph given by

$$\tau_A(i) = \begin{cases} \tau(i), & \text{if } \exists j \text{ with } ij \in A \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \rho_A(ij) = \begin{cases} \rho(ij), & \text{if } ij \in A \\ 0, & \text{otherwise.} \end{cases}$$

Definition 5. *Let (N, v) be a game. For each fuzzy graph $\gamma \in FCS^N$ the cg-link game $\left(L, v_{cg}^{L\gamma}\right)$ is defined taking the links as players, for any $A \subseteq L$*

$$v_{cg}^{L\gamma}(A) = \epsilon_{cg}^{(N,v)}(\gamma_A).$$

where $cg(\gamma) = (g_k, s_k)_{k=1}^m$.

For the same reason of the crisp version, the cg-link game loses information from the worths of the individual coalitions and therefore it is only useful for 0-normalized games, thus we take $v_{cg}^{L\gamma}$ as the link game of the 0-normalization of v. So, the position value for the cg-partition is defined following (5).

Definition 6. *The cg-position value is the fuzzy communication value defined for each game (N, v) with fuzzy communication structure $\gamma = (\tau, \rho) \in FCS^N$ and every player $i \in N$*

$$P_i(N, v)(\gamma) = \tau(i)v(\{i\}) + \frac{1}{2} \sum_{j \in N \setminus i} \phi_{ij}\left(L, v_{cg}^{L\gamma}\right).$$

4. Axioms for the *cg*-position value

First we relate the cg-position value with the position value (4). Next lemma provides a formula in Choquet form to calculate our value by a linear combination of position values.

Lemma 2. *Let (N, v) be a game and $\gamma \in FCS^N$ a fuzzy communication structure. If the cg-partition of γ is $cg(\gamma) = (g_k, s_k)_{k=1}^m$ then*

1) $v_{cg}^{L\gamma} = \sum_{k=1}^m s_k v^{Lg_k}$, *2)* $P(N,v)(\gamma) = \sum_{k=1}^{m} s_k \pi(N,v)(g_k)$.

Proof. 1) Algorithm 1 determines $cg(\gamma) = (g_k, s_k)_{k=1}^{m}$ for a given fuzzy graph $\gamma \in FCS^N$.

Let $A \subseteq L$ be a set of links. It is possible to get the cg-partition of γ_A from $cg(\gamma)$. If $cg(\gamma_A) = \left(g'_p, s'_p\right)_{p=1}^{q}$ then there are indices $(k_p)_{p=1}^{q}$ with $\left(g_{k_p+1}\right)_A \neq \left(g_{k_p}\right)_A$ ($k_q = m$ if $(g_m)_A \neq 0$) such that

$$g'_p = (g_k)_A \quad \text{for all } k_p \leq k < k_{p+1} \text{ and } s'_p = \sum_{k=k_p}^{k_{p+1}-1} s_k.$$

So, we obtain for the game (N,v), from Definition 5 and Definition 3

$$v_{cg}^{L\gamma}(A) = \epsilon_{cg}^{(N,v)}(\gamma_A) = \sum_{p=1}^{q} s'_p v^{Lg'_p}(A) = \sum_{p=1}^{q} \sum_{k=k_p}^{k_{p+1}-1} s_k v^{Lg_k}(A) = \sum_{k=1}^{k_q} s_k v^{Lg_k}(A).$$

Observe that for all $k > k_q$ it holds $A \cap L\left((g_k)_A\right) = \emptyset$ and then $v^{Lg_k}(A) = 0$, thus

$$v_{cg}^{L\gamma} = \sum_{k=1}^{m} s_k v^{Lg_k}.$$

2) Since the linearity (S1) of the Shapley value and 1) we get for v,

$$\phi\left(N, v_{cg}^{L\gamma}\right) = \sum_{k=1}^{m} s_k \phi\left(N, v^{Lg_k}\right).$$

We calculate the *cg*-position value for a player $i \in N$. We denote k_i the last level in the algorithm such that $i \in V(g_{k_i})$. If $k > k_i$ then for each $j \in N \setminus i$ the link ij is a null player for the link game v^{Lg_k} and using Proposition 1(2) and (S2),

$$\begin{aligned} P_i(N,v)(\gamma) &= \tau(i)v(\{i\}) + \frac{1}{2}\sum_{j\in N} \phi_{ij}\left(L, v_{cg}^{L\gamma}\right) \\ &= \sum_{k=1}^{k_i} s_k v(\{i\}) + \frac{1}{2}\sum_{j\in N\setminus i}\sum_{k=1}^{k_i} s_k \phi_{ij}\left(L, v^{Lg_k}\right) \\ &= \sum_{k=1}^{k_i} s_k \left[v(\{i\}) + \frac{1}{2}\sum_{j\in N\setminus i} \phi_{ij}\left(L, v^{Lg_k}\right)\right]. \end{aligned}$$

We obtain from Definition 6, $P_i(N,v)(\gamma) = \sum_{k=1}^{m} s_k \pi_i(N,v)(g_k)$. $\square$

Now we look for an axiomatization of the cg-position value following [8]. Consider the following axioms for a given fuzzy communication value F. Let (N,v) be a cooperative game and $\gamma \in FCS^N$ be a fuzzy communication structure over N. The payoffs obtained by the players are efficient for each connected component with respect to the measure of the fuzzy graph with this model. If $S \subseteq N$ is a coalition then $\gamma_S = (\tau_S, \rho_S) \in FCS^N$ is the subgraph of γ defined as

$$\tau_S(i) = \begin{cases} \tau(i), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \rho_S(i,j) = \begin{cases} \rho(i,j), & \text{if } i,j \in S \\ 0, & \text{otherwise.} \end{cases}$$

Fuzzy efficiency by components. For all $S \in N/\gamma$ it holds

$$\sum_{i \in S} F_i(N,v)(\gamma) = \epsilon_{cg}^{(N,v)}(\gamma_S).$$

The above axiom implies the efficiency for all the players from Definition 3, that is

$$\sum_{i \in N} F_i(N,v)(\gamma) = \epsilon_{cg}^{(N,v)}(\gamma).$$

The subgraph $\gamma_{-ij}^t = (\tau_{-ij}^t, \rho_{-ij}^t)$ represents the fuzzy graph γ modified by reducing to $t \in [0, \rho(ij)]$ the capacity of $ij \in L(\gamma)$, that is, $\tau_{-ij}^t = \tau$ and $\rho_{-ij}^t = \rho$ except for $\rho_{-ij}^t(ij) = t$. We denote as $\wedge_i \gamma = \wedge_{ik \in L(\gamma)} \rho(ik)$ the lowest level of communication for a non isolated player $i \in N$, i.e. the minimal degree of the player. For two non isolated players in the fuzzy communication structure we consider the notation $\wedge_{ij}\gamma = (\wedge_i \gamma) \wedge (\wedge_j \gamma) > 0$, that is the *minimal common degree* of the players.

Balanced total fuzzy threats. Let $i,j \in N$ be two different non isolated players and $t \in [0, \wedge_{ij}\gamma]$. Then

$$\sum_{ih \in L(\gamma)} \left[F_j(N,v,\gamma) - F_j\left(N,v,\gamma_{-ih}^t\right)\right] = \sum_{jh \in L(\gamma)} \left[F_i(N,v,\gamma) - F_i\left(N,v,\gamma_{-jh}^t\right)\right].$$

This axiom means that the total loss for a player because of the drop of the communications of another player is balanced, and then anyone can threaten another in these terms.

First we prove that the cg-position value satisfies these axioms.

Theorem 3. *The cg-position value satisfies fuzzy efficiency by components and balanced total fuzzy threats.*

Proof. Let (N, v) be a game. We take $\gamma \in FCS^N$ with cg-partition $cg(\gamma) = (g_k, s_k)_{k=1}^m$.
Fuzzy efficiency by components. Borm *et al.* [7] proved that π is efficient by components, that is for all $g \in CS^N$ and for all $S \in N/g$ we get $\sum_{i\in S} \pi_i(N, v)(g) = v(S)$. Let $S \in N/\gamma$ be a connected component of γ. Let $S \subseteq N$ be a coalition. It is possible to get the cg-partition of γ_S from $cg(\gamma)$. If $cg(\gamma_S) = \left(g'_p, s'_p\right)_{p=1}^q$ then there are indices $(k_p)_{p=1}^q$ with $\left(g_{k_p+1}\right)_S \neq \left(g_{k_p}\right)_S$ ($k_q = m$ if $(g_m)_S \neq 0$) such that

$$g'_p = (g_k)_S \quad \text{for all } k_p \leq k < k_{p+1} \text{ and } s'_p = \sum_{k=k_p}^{k_{p+1}-1} s_k.$$

It holds that $T \in N/(g_k)_S$ iff $T \in N/g_k$ with $T \subseteq S$. We have by the above lemma

$$\begin{aligned}
\sum_{i\in S} P_i(N, v)(\gamma) &= \sum_{i\in S}\sum_{k=1}^m s_k \pi_i(N, v)(g_k) = \sum_{k=1}^m \sum_{i\in S} s_k \pi_i(N, v)(g_k) \\
&= \sum_{k=1}^m s_k \sum_{T\in N/(g_k)_S} \sum_{i\in T} \pi_i(N, v)(g_k) = \sum_{p=1}^q \sum_{k=k_p}^{k_{p+1}-1} s_k \sum_{T\in N/(g_k)_S} v(T) \\
&= \sum_{p=1}^q \sum_{k=k_p}^{k_{p+1}-1} s_k r^{(N,v)}((g_k)_S) = \sum_{p=1}^q s'_p r^{(N,v)}(g'_p) = \epsilon_{cg}^{(N,v)}(\gamma_S).
\end{aligned}$$

Balanced total fuzzy threats. Slikker [8] proved that π satisfies balanced total threats, that is for all pairs of players $i, j \in N$ and $g \in CS^N$ it holds

$$\sum_{ih\in L(g)} \left[\pi_j(N, v)(g) - \pi_j(N, v)(g_{-ih})\right] = \sum_{jh\in L(g)} \left[\pi_i(N, v)(g) - \pi_i(N, v)(g_{-jh})\right].$$

Let $i, j \in N$ and $t \in [0, \wedge_{ij}\gamma]$. There exists $k_t \in \{1, ..., m\}$ such that $0 \leq t - \sum_{k=0}^{k_t-1} s_k < s_{k_t}$ supposing $s_0 = 0$. We consider the following partition by levels for γ equivalent to $cg(\gamma)$ (if we set the pairs with the same g_k then we obtain the cg-partition),

$$\left\{ (g_k, s_k)_{k=1}^{k_t-1}, \left(g_{k_t}, t - \sum_{k=1}^{k_t-1} s_k \right), (g_{k_t}, s_{k_t} - t), (g_k, s_k)_{k=k_t+1}^{k_{ih}}, (g_k, s_k)_{k=k_{ih}+1}^{m} \right\}.$$

For each $ih \in L(\gamma)$ (or jh) there is $k_{ih} \in \{k_t, ..., m\}$ with

$$\sum_{k=1}^{k_{ih}} s_k = \rho(ih).$$

We take for γ_{-ih}^t the partition

$$\left\{ (g_k, s_k)_{k=1}^{k_t-1}, \left(g_{k_t}, t - \sum_{k=1}^{k_t-1} s_k \right), ((g_{k_t})_{-ih}, s_{k_t} - t), ((g_k)_{-ih}, s_k)_{k=k_t+1}^{k_{ih}}, (g_k, s_k)_{k=k_{ih}+1}^{m} \right\}.$$

Observe that these partitions are equivalent to the cg-partitions of the corresponding fuzzy graphs. Hence using Lemma 2,

$$\begin{aligned}
\sum_{ih \in L(\gamma)} \left[P_j(N,v)(\gamma) - P_j(N,v)(\gamma_{-ih}^t) \right] &= (s_{k_t} - t) \sum_{ih \in L(\gamma)} [\pi_j(N,v)(g_{k_t}) - \pi_j(N,v)((g_{k_t})_{-ih})] \\
&+ \sum_{ih \in L(\gamma)} \sum_{k=k_t+1}^{k_{ih}} s_k [\pi_j(N,v)(g_k) - \pi_j(N,v)((g_k)_{-ih})] \\
&= (s_{k_t} - t) \sum_{ih \in L(\gamma)} [\pi_j(N,v)(g_{k_t}) - \pi_j(N,v)((g_{k_t})_{-ih})] \\
&+ \sum_{k=k_t+1}^{m} s_k \sum_{ih \in L(g_k)} [\pi_j(N,v)(g_k) - \pi_j(N,v)((g_k)_{-ih})] \\
&= (s_{k_t} - t) \sum_{jh \in L(\gamma)} [\pi_i(N,v)(g_{k_t}) - \pi_i(N,v)((g_{k_t})_{-jh})] \\
&+ \sum_{k=k_t+1}^{m} s_k \sum_{jh \in L(g_k)} [\pi_i(N,v)(g_k) - \pi_i(N,v)((g_k)_{-jh})] \\
&= \sum_{jh \in L(\gamma)} \left[P_i(N,v)(\gamma) - P_i(N,v)(\gamma_{-jh}^t) \right]. \qquad \square
\end{aligned}$$

Next theorem says that our fuzzy communication value is the only one satisfying these two axioms.

Theorem 4. *There is only one fuzzy communication value satisfying fuzzy efficiency by components and balanced total fuzzy threats.*

Proof. As our fuzzy communication value P satisfies both of the axioms then it is only necessary to prove the uniqueness. Consider F another fuzzy communication value satisfying these axioms. The proof of the uniqueness is by recurrence in $K_\gamma = |L(\gamma)|$. If $K_\gamma = 0$ then all the players are isolated in γ and the fuzzy efficiency by components implies $F_i(N,v)(\gamma) = \tau(i)v(\{i\})$ for all $i \in N$. We suppose true the uniqueness, namely $F = P$, for all $\gamma \in FCS^N$ with $K_\gamma < p$. Now, let $\gamma \in FCS^N$ with $K_\gamma = p$. We will find a unique feasible payoff for the players in each connected component. If $S \in N/\gamma$ with $S = \{i\}$ then by fuzzy efficiency by components we get $P_i(N,v)(\gamma) = \tau(i)v(\{i\})$. Suppose $|S| > 1$. We take $i \in S$ and the other players in the component $S \setminus i = \{j_1, ..., j_q\}$. We look for $F_i(N,v)(\gamma), F_{j_1}(N,v)(\gamma), ..., F_{j_q}(N,v)(\gamma)$. Applying the balanced total fuzzy threats axiom with $t = 0$ to every pair of players i, j_k with $k = 1, ..., q$ we obtain

$$\begin{aligned}
\sum_{ih \in L(\gamma)} \left[F_{j_1}(N,v)(\gamma) - F_{j_1}(N,v)(\gamma^0_{-ih})\right] &= \sum_{j_1 h \in L(\gamma)} \left[F_i(N,v)(\gamma) - F_i(N,v)(\gamma^0_{-j_1 h})\right] \\
&\vdots \\
&\vdots \\
\sum_{ih \in L(\gamma)} \left[F_{j_q}(N,v)(\gamma) - F_{j_q}(N,v)(\gamma^0_{-ih})\right] &= \sum_{j_q h \in L(\gamma)} \Big[F_i(N,v)(\gamma) - F_i(N,v)(\gamma^0_{-j_q h})\Big].
\end{aligned}$$

We denote $Q_i = |\{ih \in L(\gamma)\}|$ and Q_{j_k} in the same way. These numbers $Q_i, Q_{j_k} \neq 0$ because these vertices are in the same connected component. Each link $j_k h$ (or ih) verifies that $K_{\gamma^0_{-j_k h}} < p$, thus $F = P$ over them. Adding the equations for S by the fuzzy efficiency by components we get the following linear system,

$$\begin{aligned}
Q_i F_{j_1}(N,v)(\gamma) - Q_{j_1} F_i(N,v)(\gamma) &= \sum_{ih\in L(\gamma)} P_{j_1}(N,v)(\gamma^0_{-ih}) - \sum_{j_1 h\in L(\gamma)} P_i(N,v)(\gamma^0_{-j_1 h}) \\
&\vdots \\
&\vdots \\
Q_i F_{j_q}(N,v)(\gamma) - Q_{j_q} F_i(N,v)(\gamma) &= \sum_{ih\in L(\gamma)} P_{j_q}(N,v)(\gamma^0_{-ih}) - \sum_{j_q h\in L(\gamma)} P_i(N,v)(\gamma^0_{-j_q h}) \\
F_{j_1}(N,v)(\gamma) + \cdots + F_{j_q}(N,v)(\gamma) &+ F_i(N,v)(\gamma) = \epsilon^{(N.v)}_{cg}(\gamma_S).
\end{aligned}$$

This is a determined compatible system and then $F = P$. $\square$

Borm *et al.* [7] showed that the position value is proportional to the centrality measure of the players, the degree of the vertex in this case, if the link game only depends on the number of links. Our version too.

Theorem 5. *Let (N,v) be a 0-normalized game and let γ be a fuzzy communication structure such that $v^{Lg}(A) = |L(g_A)|$ for all $g \in CS^N$ with $g \le g^\gamma$. It holds that there exists $K > 0$ with $P_i(N,v)(\gamma') = K\delta_i(\gamma')$ for all $\gamma' \le \gamma$.*

Proof. Suppose a game (N,v) and a fuzzy graph γ satisfying the condition of the statement. Let $\gamma' \le \gamma$. Following [7] there is $K > 0$ with $\pi_i(N,v)(g) = Kd_i(g)$ for all $g \in CS^N$ with $g \le g^{\gamma'}$. By Lemma 2 and Proposition 1 considering the cg-partition of γ'

$$\begin{aligned}
P_i(N,v)(\gamma') &= \sum_{k=1}^m s_k \pi_i(N,v)(g_k) = \sum_{k=1}^m s_k K d_i(g_k) \\
&= K \sum_{ij\in L(\gamma)} \sum_{\{k\in\{1,\ldots,m\}: ij\in L(g_k)\}} s_k = K\delta_i(\gamma). \qquad \square
\end{aligned}$$

We study now properties of our fuzzy communication value derived from its Choquet form. For all $A \subseteq L$ we denote $g^A = (N,A) \in CS^N$. Given a game (N,v) we define for every $i \in N$ the set function over L,

$$f_i^v(A) = \frac{1}{2} \sum_{ij\in L} \phi_{ij}(L, v^{Lg^A}) \quad \forall A \subseteq L.$$

Lemma 6. *Let (N, v) be a game. For all $\gamma = (\tau, \rho) \in FCS^N$ and $i \in N$,*

$$P_i(N, v)(\gamma) = \tau(i)v(\{i\}) + \int \rho \, df_i^v.$$

Proof. Let $\gamma = (\tau, \rho) \in FCS^N$. From Definition 6 it is enough to see that $\int \rho \, df_i^v = \frac{1}{2} \sum_{ij \in L} \phi_{ij}(L, v_{cg}^{L\gamma})$. So, if $im(\rho) = \{\lambda_1 < \cdots < \lambda_m\}$ and $\lambda_0 = 0$ then

$$\int \rho \, df_i^v = \sum_{p=1}^{q} (\lambda_p - \lambda_{p-1}) f_i^v([\rho]_{\lambda_p}) = \frac{1}{2} \sum_{ij \in L} \sum_{p=1}^{q} (\lambda_p - \lambda_{p-1}) \phi_{ij}(L, v^{Lg^{\rho_{\lambda_p}}}).$$

For each k observe that $v^{Lg^{\rho_{\lambda_k}}} = v^{Lg_{\rho_{\lambda_k}}}$. Suppose now $\{t_1 < \cdots < t_m\} = im(\tau) \cup im(\rho)$ and $t_0 = 0$. If $cg(\gamma) = (s_k, g_k)_{k=1}^m$ then Algorithm 1 implies that $s_k = t_k - t_{k-1}$. But for all t_k such that $t_k \notin im(\rho)$ it holds $v^{Lg_k} = v^{Lg_{k-1}}$. Thus we get

$$s_{k-1}\phi_{ij}(L, v^{Lg_{k-1}}) + s_k \phi_{ij}(L, v^{Lg_k}) = (t_k - t_{k-2})\phi_{ij}(L, v^{Lg_k}).$$

Going on this reasoning we can use only the numbers in $im(\rho)$ and we have for each link ij,

$$\sum_{p=1}^{q} (\lambda_p - \lambda_{p-1}) \phi_{ij}(L, v^{Lg_{\rho_{\lambda_p}}}) = \sum_{k=1}^{m} s_k \phi_{ij}(L, v^{Lg_k}). \qquad \square$$

Next theorem shows interesting properties of the *cg*-position value. Let $\gamma = (\tau, \rho), \gamma' = (\tau', \rho') \in FCS^N$ two fuzzy graphs. We say that they are *link-comonotone* if ρ, ρ' are comonotone as fuzzy sets over L. The *link-distance* between γ and γ' is defined as $D(\gamma, \gamma') = D(\rho, \rho')$.

Theorem 7. *The cg-position value P satisfies the following properties.*

1) P is a linear function for the game.

2) P is a comonotone function by links. Let (N, v) be a game, for all two link-comonotone fuzzy communication structures $\gamma, \gamma' \in FCS^N$ and $t \in [0, 1]$ it holds

$$P(N, v)(t\gamma + (1 - t)\gamma') = tP(N, v)(\gamma) + (1 - t)P(N, v)(\gamma').$$

3) P is a continuous function.

Proof. 1) Let $(N,v),(N,v')$ be two games, a,a' two real numbers and $\gamma=(\tau,\rho)\in FCS^N$ a fuzzy communication structure. Property (C1) and Lemma 6 imply that for every $i\in N$

$$\begin{aligned} P_i(N,av+a'v')(\gamma) &= \tau(i)[av(\{i\})+a'v'(\{i\})]+\int \rho\, df_i^{av+a'v'} \\ &= a\tau(i)v(\{i\})+a\int \rho\, df_i^{v}+a'\tau(i)v'(\{i\})+a'\int \rho\, df_i^{v'} \\ &= aP_i(N,v)(\gamma)+a'P_i(N,v')(\gamma). \end{aligned}$$

2) Let $\gamma=(\tau,\rho),\gamma'=(\tau',\rho')\in FCS^N$ be two link-comonotone fuzzy communication structures. Obviously, for each $t\in[0,1]$, the convex combination $t\gamma+(1-t)\gamma'$ is a new fuzzy graph because for all $i\in N$ we have $t\tau(i)+(1-t)\tau'(i)\leq 1$. Moreover $t\rho$ and $(1-t)\rho'$ are comonotone fuzzy sets in L. Properties (C2), (C3) and Lemma 6 say that for each $i\in N$,

$$\begin{aligned} P_i(N,v)(t\gamma+(1-t)\gamma') &= [t\tau(i)+(1-t)\tau'(i)]v(\{i\})+\int (t\rho+(1-t)\rho')\, df_i^{v} \\ &= t\tau(i)v(\{i\})+t\int \rho\, df_i^{v}+(1-t)\tau'(i)v(\{i\})+(1-t)\int \rho'\, df_i^{v} \\ &= tP_i(N,v)(\gamma)+(1-t)P_i(N,v)(\gamma'). \end{aligned}$$

3) Lemma 6 shows that $P_i(N,v)$ is a sum of a linear operator for τ and a Choquet integral for ρ. As both of them are continuous then $P_i(N,v)$ is continuous. $\square$

Example 2. Suppose the fuzzy communication structure γ in Figure 3 for $N=\{1,2,3\}$ and the game with $v(S)=|S|-1$ if $S\neq\emptyset$ and $v(\emptyset)=0$. It is a 0-normalized game but the link game does not satisfy the other condition in Theorem 5, $v^{Lg}(L)=v(N)=2\neq 3$. In order to obtain the payoffs of our value we use Theorem 7 (2). We take the decomposition in Figure 3 of γ with γ_1,γ_2 in the order of the figure and $t=1/3$. Observe that γ_1 and γ_2 are link-comonotone.

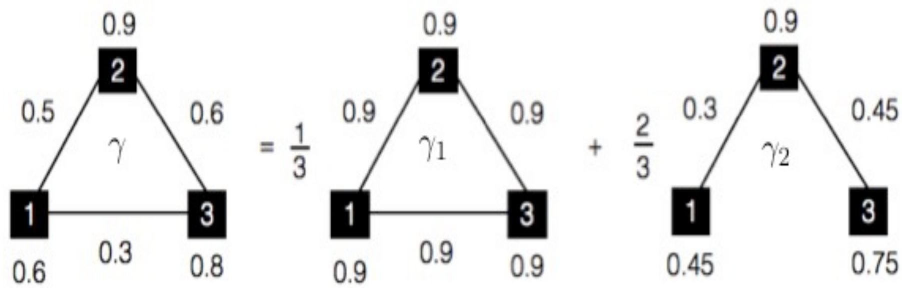

Figure 3. Fuzzy graph γ and decomposition.

Now we get the value for both of the fuzzy graph in the decomposition. The first one has only one level 0.9, and then the cg-position value coincides with the position value. The game and the structure are symmetric and therefore the payoffs are using $v(N) = 2$

$$P(N,v)(\gamma_1) = 0.9 \cdot (2/3, 2/3, 2/3) = \left(\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right).$$

The fuzzy graph γ_2 satisfies both of the conditions in Theorem 5 for our game. Then the payoffs are proportional to the fuzzy degrees of the vertices. As $\epsilon_{cg}^{(N,v)}(\gamma) = 0.75$ (the graph is connected but the efficiency depends on the levels) and the fuzzy degrees $(0.3, 0.75, 0.45)$ then

$$P(N,v)(\gamma_2) = \left(\frac{3}{20}, \frac{3}{8}, \frac{9}{40}\right).$$

So, the cg-position value is

$$P(N,v)(\gamma) = \frac{1}{3}P(N,v)(\gamma_1) + \frac{2}{3}P(N,v)(\gamma_2) = \left(\frac{3}{10}, \frac{9}{20}, \frac{7}{20}\right).$$

We consider now players 2 and 3. We test the balanced total fuzzy threats axiom for them. The minimal levels are $\wedge_2(\gamma) = 0.5, \wedge_3(\gamma) = 0.3$ and then $\wedge_{23}(\gamma) = 0.3$. We take a threat of $t = 0.2 \in [0, 0.3]$. Following the same method than before we get

$$P_3(N,v)(\gamma) - P_3(N,v)(\gamma_{-12}^{0.2}) + P_3(N,v)(\gamma) - P_3(N,v)(\gamma_{-23}^{0.2}) = \left(\frac{7}{20} - \frac{23}{60}\right) + \left(\frac{7}{20} - \frac{11}{60}\right) = \frac{2}{15}.$$

Also in the same way,

$$P_2(N,v)(\gamma)-P_2(N,v)(\gamma^{0.2}_{-13})+P_2(N,v)(\gamma)-P_2(N,v)(\gamma^{0.2}_{-23}) = \left(\frac{9}{20}-\frac{17}{60}\right)+\left(\frac{9}{20}-\frac{29}{60}\right) = \frac{2}{15}.$$

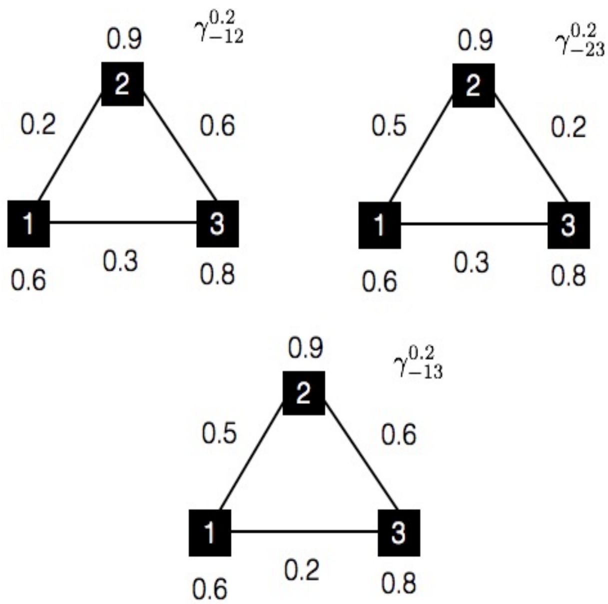

Figure 4. Communication reduction.

5. Computing the *cg*-position value

The goal of this section is to formulate an algorithm which determines the cg-position value and to study the complexity of it. Let A be an algorithm. The *time complexity* of A is measured by a function $f:\mathbb{Z}_+\to\mathbb{Z}_+$ where $f(n)$ is the maximal number of iterations in a universal Turing machine (before halting) in relation with the size of the input n. Let $f,g:\mathbb{Z}_+\to\mathbb{Z}_+$. Following $\mathcal{O}\Omega\Theta$-notation, proposed by Knuth [35], we use $f=\mathcal{O}(g)$ if there are $c,n_0\in\mathbb{Z}_+$ such that $f(n)\le cg(n)$ for all $n\ge n_0$. In that case we say f *is of the order of* g.

Let $\gamma=(\tau,\rho)\in FCS^N$ be a fuzzy communication structure. We store the fuzzy set of vertices and the fuzzy set of edges in an upper triangular matrix $\gamma=[\gamma(i,j)]_{N\times N}$ where

$\gamma(i,i) = \tau(i)$ for every $i \in N$ and $\gamma(i,j) = \rho(ij)$ when $i < j$. So,

$$\gamma = \begin{bmatrix} \gamma(1,1) & \gamma(1,2) & \cdots & \gamma(1,n) \\ 0 & \gamma(2,2) & \cdots & \gamma(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \gamma(n,n) \end{bmatrix}$$

We can also represent the crisp graph g^γ corresponding to the graph γ by a matrix g^γ such that $g^\gamma(i,j) = \lceil \gamma(i,j) \rceil$ for all $i, j \in N$.

We suppose a 0-normalized games from now on. In order to compute the cg-position value we provide the position value with a process to determine it. This procedure is called *PVDL procedure* (position value by dividends of the link game). Let (N, v) be a game with communication structure $g = (S, A)$. The dual graph of g is another graph g^* such that the links in g are the vertices in g^* and there is a link in g^* between each two adjacent links of g. Suppose known the family of feasible sets of links $\mathcal{F}^{g^*}$, we can obtain the dividends of the link game by a recurrence formula.

Lemma 8. *Let (N, v) be a 0-normalized game with communication structure $g = (S, A)$. The Harsanyi dividend of the link game (L, v^{Lg}) for each $E \in \mathcal{F}^{Lg}$ is*

$$\Delta_E^{v^{Lg}} = v\left(\bigcup_{ij \in E} \{i, j\}\right) - \sum_{\{B \in \mathcal{F}^{Lg} : B \subset E\}} \Delta_B^{v^{Lg}},$$

and $\Delta_E^{v^{Lg}} = 0$ otherwise.

Proof. We observe that the link game for the graph g coincides with the vertex game for the dual graph g^*. If $E \in \mathcal{F}^{Lg}$ then

$$v^{Lg}(E) = v\left(\bigcup_{ij \in E} \{i, j\}\right).$$

So, we use (3) and (2) to get the equality $\square$

Using Lemma 1 we apply next algorithm to determine the Shapley value of the link game (L, v^{Lg}). Let $g = (S, A)$, $l = |A|$ and $\{E_k^h : h = 1, ..., E(k)\}$ the set of elements in $\mathcal{F}^{Lg}$ of cardinality k with $k = 1, ..., l$.

Algorithm 2 *dividends_link* $\left(L, v^{L_g}\right)$

$\Delta_{\emptyset}^{v^{Lg}} \leftarrow 0$

for k **from** 1 **to** l

  for h **from** 1 **to** $E(k)$

  $\Delta_{E_k^h}^{v^{Lg}} \leftarrow v\left(\bigcup_{ij \in E_k^h} \{i,j\}\right) - \sum_{\left\{B \in \mathcal{F}^{Lg} : B \subset E_k^h\right\}} \Delta_B^{v^{Lg}}$

  end

end

Next result follows from Theorem 6 in [24].

Lemma 9. *Let (N, v) be a 0-normalized game with communication structure $g = (S, A)$. The time complexity of computing the dividends of the link game (L, v^{Lg}), when $\mathcal{F}^g$ is known, is $\mathcal{O}\left(3^l\right)$ where $l = |A|$.*

So, PVDL procedure works in three steps.

PVDL procedure $PVDL(N, v, g)$

1. We obtain the dual graph g^* of g and its set of feasible coalitions of links, $\mathcal{F}^{g^*}$.
2. We compute the Shapley value of the link game using Algorithm 2.
3. We calculate the position value $\pi(N, v)(g)$ by (5).

Theorem 10. *Let (N, v) be a 0-normalized game with communication structure $g = (S, A)$. The time complexity of computing the position value using the PVDL procedure is $\mathcal{O}\left(3^l\right)$ where $l = |A|$.*

Proof. We follow the three steps of the PVDL procedure. For the graph g we determine the dual graph g^*. The edges of g are the vertices of g^*. There are at most l players in the maximal coalition induced by g^*. To obtain the edges of g^*, we need at most $l(l-1)$ comparisons between the edges since two vertices in g^* are connected if and only if the intersection between the corresponding edges in g is not empty. Therefore we need $\mathcal{O}(l^2)$ time to determine the dual graph g^*. In order to calculate the set $\mathcal{F}^{g^*}$ of feasible coalitions induced by g^*, we have to look at the subsets of vertices of g^* which generate a connected subgraph in g^*. In the worst case g^* is a complete graph, there are l vertices and $l(l-1)$ edges. It is known that we can determine if a graph is connected or not by a Depth First Search (DFS) which needs $\mathcal{O}(|\text{vertices}| + |\text{links}|)$ time; in this case $\mathcal{O}(l + l(l-1)) = \mathcal{O}(l^2)$. Therefore to obtain $\mathcal{F}^{g^*}$ we need $\mathcal{O}(l^2 2^l)$ time. Now, Lemma 9 says that we obtain the dividends of the link game in $\mathcal{O}(3^l)$. Finally, we obtain the position value for each player $i \in N$,

$$\pi_i(N,v)(g) = \frac{1}{2} \sum_{j \in N \setminus i} \left(\sum_{\{B \in \mathcal{F}^{L_g} : ij \in B\}} \frac{\Delta_B^{v^{L_g}}}{|B|} \right).$$

After storing the dividends of game (L, v^{L_g}), for each link player $ij \in L$, it is required in the worst case $\mathcal{O}\left(2^l\right)$ time to compute the Shapley value $\phi_{ij}(L, v^{L_g})$. The link player ij has degree $l-1$ at most, therefore to compute the position value for player i it is required $\mathcal{O}\left(l2^l\right)$ time. Then, for all players it is required $\mathcal{O}\left(nl2^l\right)$ time. We consider then $\max\{l^2 2^l, 3^l, nl2^l\}$ and so the PVDL procedure needs $\mathcal{O}\left(3^l\right)$ time. $\square$

The procedure to determine the cg-position value is the following.

cg-position procedure

1. Using Algorithm 1 we construct the partition $cg(\gamma) = (g_k, s_k)_{k=1}^m$.
2. We calculate $PVDL(N, v, g_k)$ for each k.
3. Using Lemma 2 we obtain the cg-position value.

Next result also follows from Theorem 6 in [24].

Lemma 11. *Let (N,v) be a 0-normalized game with fuzzy communication structure $\gamma = (\tau, \rho)$. The time complexity of computing its cg-partition $cg(\gamma)$ by Algorithm 1 is $\mathcal{O}\left(c(a+b)\right)$ where $a = |\text{im}(\tau)|$, $b = |\text{im}(\rho)|$ and $c = |\text{im}(\tau) \cup \text{im}(\rho)|$.*

Theorem 12. *Let (N,v) be a 0-normalized game with fuzzy communication structure $\gamma = (\tau, \rho)$ and $cg(\gamma) = (s_k, g_k)_{k=1}^m$. The time complexity of computing the cg-position value using the cg-position procedure is $\mathcal{O}\left(l3^l\right)$, where $l = |L(\gamma)|$.*

Proof. Using Lemma 11 we obtain the cg-partition in $\mathcal{O}\left(c(a+b)\right)$ where $a = |\text{im}(\tau)|$, $b =$

$|\mathrm{im}(\rho)|$ and $c = |\mathrm{im}(\tau) \cup \mathrm{im}(\rho)|$. Now, using Lemma 2

$$P(N,v)(\gamma) = \sum_{k=1}^{m} s_k \pi(N,v)(g_k).$$

Hence it is necessary to execute m times the *PVDL procedure* to obtain $\pi(N,v)(g_k)$, $k \in \{1,...,m\}$, but there are no links in L_{g_k}, for all $k \in \{m-l,...,l\}$, since $\rho(i,j) \leq \tau(i) \wedge \tau(j)$ in γ. Therefore, the only significative iterations are the first l, so it is necessary $\mathcal{O}\left(l3^l\right)$ time from Theorem 10. $\square$

Example 3. Suppose the game (N,v) be given by $N = \{1,2,3,4,5\}$, and

$$v(S) = \begin{cases} 1, & \text{if } S = \{2,3,4,5\} \text{ or if both } |S| \geq 2 \text{ and } 1 \in S \\ 0, & \text{otherwise,} \end{cases}$$

with the fuzzy communication structure γ represented in Figure 5. The crisp graph g^γ and the dual graph $(g^\gamma)^*$ can be seen in Figure 6 and 7.

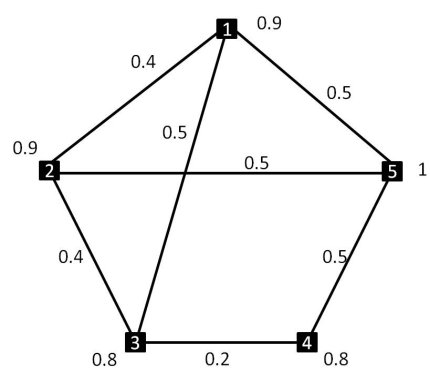

Figure 5: Fuzzy graph γ

$$\gamma = \begin{bmatrix} 0.9 & 0.4 & 0.5 & 0 & 0.5 \\ 0 & 0.9 & 0.4 & 0 & 0.5 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0.8 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

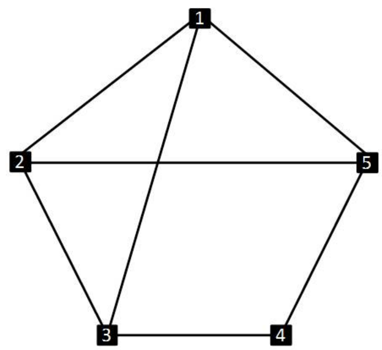

Figure 6: Graph g^γ

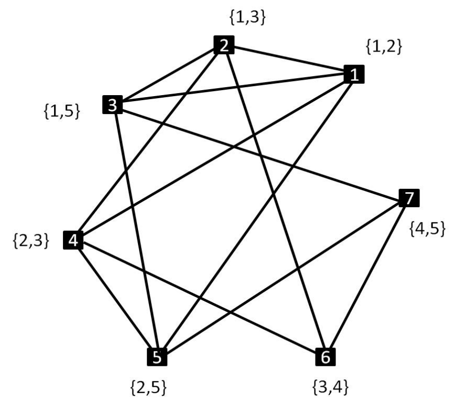

Figure 7: Dual graph $(g^\gamma)^*$

In the dual graph $(g^\gamma)^*$ the vertices are the links in the graph g^γ, we rename these vertices in sequential order, $\{1, 2, 3, 4, 5, 6, 7\}$. Table 1 includes the matrices γ_k corresponding to the algorithm *cg-partition* $cg(\gamma) = (g_k, s_k)_{k=1}^{k=6}$.

$\gamma_1 = \begin{bmatrix} 0.9 & 0.4 & 0.5 & 0 & 0.5 \\ 0 & 0.9 & 0.4 & 0 & 0.5 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0.8 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $s_1 = 0.2$	$\gamma_2 = \begin{bmatrix} 0.7 & 0.2 & 0.3 & 0 & 0.3 \\ 0 & 0.7 & 0.2 & 0 & 0.3 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.3 \\ 0 & 0 & 0 & 0 & 0.8 \end{bmatrix}$ $s_2 = 0.2$
$\gamma_3 = \begin{bmatrix} 0.5 & 0 & 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0 & 0 & 0.1 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0 & 0.6 \end{bmatrix}$ $s_3 = 0.1$	$\gamma_4 = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$ $s_4 = 0.3$
$\gamma_5 = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 \end{bmatrix}$ $s_5 = 0.1$	$\gamma_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}$ $s_6 = 0.1$

Table 1. The *cg*-partition of the fuzzy graph γ.

We calculate now the *cg*-position value using the position values of the communication structures in the cg-partition $cg(\gamma) = (g_k, s_k)_{k=1}^{k=6}$, Table 2 includes the position values $\pi\,(N, v)(g_k)$

for $k = 1, ..., 6$.

	$\pi(N,v)(g_1)$	$\pi(N,v)(g_2)$	$\pi(N,v)(g_3)$	$\pi(N,v)(g_4)$	$\pi(N,v)(g_5)$	$\pi(N,v)(g_6)$
1	0.442857	0.475	0.5	0.	0.	0
2	0.17619	0.175	0.	0.	0.	0.
3	0.17619	0.166667	0.25	0.	0.	0.
4	0.0285714	0.008333	0.	0.	0.	0.
5	0.176190	0.175	0.25	0.	0.	0.

Table 2. The position values for all crisp graphs.

Finally, Table 3 compares the classical Shapley value, $\phi(N,v)$, the cg-Myerson value, $M(N,v)(\gamma)$ (see [23]) and the cg-position value, $P(N,v)(\gamma)$.

Players	$\phi(N,v)$	$M(N,v)(\gamma)$	$P(N,v)(\gamma)$
1	0.6	0.286667	0.233571
2	0.1	0.053333	0.0702381
3	0.1	0.07	0.0935714
4	0.1	0.02	0.00738095
5	0.1	0.07	0.0952381

Table 3. Shapley, cg-Myerson and cg-position values.

6. Values in the European Parliament: *cg*-position versus *cg*-Myerson value

The Treaties of Maastricht (1992) and Lisbon (2009) regulate the functions of the European Parliament in a context of the co-decision procedure with the Council of the European Union. The European Parliament pretends to be the ideologic representation of the european citizens, but currently the channel of voting is the set of national political parties in each member state. Hence, the relations among these groups are partial because of the national interests. The European Parliament is organized in political groups depending on the ideologic feeling. The different political parties of the member countries present a list of candidates in their own countries and later they assume the membership to a specific group in the chamber. Therefore, the behavior of a group is not homogeneous because it is made conditional on the countries relationships. A group needs to verify two conditions: it must contain at least twenty five seats and it must represent at least one-quarter of the member countries. Those members of the chamber who do not belong to any political group are known as non-attached members.

In the seventh legislature there were seven political groups in the European Parliament plus the non-attached seats. So, we consider in our example the following groups corresponding to 2012, it was already proposed in [24] to compute the cg-Banzhaf value:

1. European People's Party (Christian Democrats), 265 members.
2. Progressive Alliance of Socialists and Democrats, 183 members.
3. Alliance of Liberals and Democrats for Europe, 84 members.
4. European Conservatives and Reformists, 55 members.
5. Greens/European Free Alliance, 55 members.
6. European United Left - Nordic Green Left, 35 members.
7. Europe of Freedom and Democracy, 29 members.
8. Non-attached Members, 29 members.

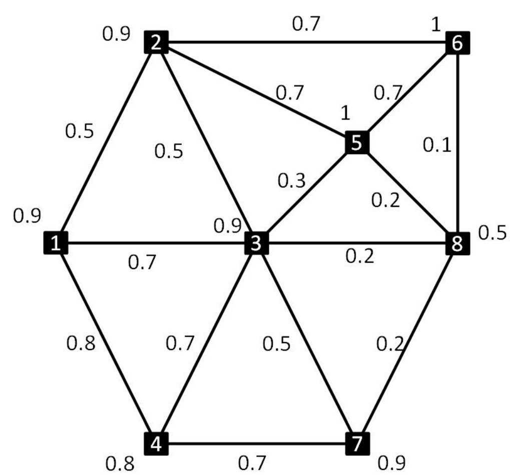

Figure 8: EP fuzzy graph γ

We consider the game of political representation groups of the European Parliament in 2012 with 735 seats and a quota of 368. The corresponding weighted voting game, called the

EP-game, is represented by $v(S) = 1$ if the sum of the number of seats of the groups in S is greater or equal to 368, and $v(S) = 0$ otherwise. We can summarize the bilateral relations among the different groups, and the degree of cohesion of every group, by a fuzzy graph $\gamma = (\tau, \rho)$ over $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, in this case $\tau(i)$ is interpreted as the membership capacity of the groups the voting day. Number $\rho(i, j)$ means the maximal level of agreement that groups i, j can reach in this voting. This fuzzy graph for the EP-game is represented in Figure 8. The matrix representation of the EP fuzzy graph γ is:

$$\gamma = \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0.5 & 0 & 0.7 & 0.7 & 0 & 0 \\ 0 & 0 & 0.9 & 0.7 & 0.3 & 0 & 0.5 & 0.2 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.7 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Next we compare the results. Table 4 shows the Shapley value (ϕ), the position value (π) and the *cg*-position value (P). This table allows us to compare the crisp option with the fuzzy version of the position value. So, we can see clearly how including new information in the game changes the payoffs, first considering only the graph and second considering the fuzzy graph. Table 5 compares the Shapley value, the Myerson value (μ) and the *cg*-Myerson value (M) in the same way. Finally, Table 6, Figure 9 and Figure 10 show a comparison among all the *cg*-values studied in the literature: the *cg*-Myerson value, the *cg*-position value and the *cg*-Banzhaf graph value (B and, we applied it to this example and we compared it with the crisp version in [24]). The sum of the *cg*-Banzhaf graph payoffs is different to the others and then we have to compare them related to their sums (B_N). So, we can see how the position value increases the power of the groups with more degree (according to their seats). We see in Figure 8 that ADLE is the vertex with more fuzzy degree and it is the group with the best increasing of payoff in the cg-position value compared to the cg-Myerson value.

Players	Groups	votes	$\phi\,(N,v)$	$\pi(N,v)(g^{\gamma})$	$P(N,v)(\gamma)$
1	PPE	265	0.4214290	0.276504	0.226201
2	S&D	183	0.1785710	0.224578	0.117805
3	ADLE	84	0.1309520	0.225688	0.181412
4	CRE	55	0.0738095	0.0826326	0.11746
5	Greens-ALE	55	0.0738095	0.0651779	0.0193003
6	GUE/NGL	35	0.0404762	0.0329532	0.00810578
7	EDF	29	0.0404762	0.0393828	0.0202077
8	NI	29	0.0404762	0.053083	0.00950702

Table 4 Power indices in the European Parliament (I)

Players	Groups	Votes	$\phi(N,v)$	$\mu(N,v)(g)$	$M(N,v)(\gamma)$
1	PPE	265	0.4214290	0.370238	0.253095
2	S&D	183	0.1785710	0.232143	0.125476
3	ADLE	84	0.1309520	0.175000	0.159048
4	CRE	55	0.0738095	0.0630952	0.0983333
5	Greens-ALE	55	0.0738095	0.0464286	0.0216667
6	GUE/NGL	35	0.0404762	0.0202381	0.0097619
7	EDF	29	0.0404762	0.0464286	0.0233333
8	NI	29	0.0404762	0.0464286	0.0092857

Table 5. Power indices in the European Parliament. (II)

Players	Groups	Votes	$P(N,v)(\gamma)$	$M(N,v)(\gamma)$	$B(N,v)(\gamma)$	$B(N,v)(\gamma)_N$
1	PPE	265	0.226201	0.2530950	0.357813	0.256891
2	S&D	183	0.117805	0.1254760	0.192188	0.137981
3	ADLE	84	0.181412	0.1590480	0.201563	0.144712
4	CRE	55	0.11746	0.0983333	0.107813	0.0774038
5	Greens-ALE	55	0.0193003	0.0216667	0.0390625	0.0280449
6	GUE/NGL	35	0.00810578	0.0097619	0.0171875	0.0123397
7	EDF	29	0.0202077	0.0233333	0.0421875	0.0302885
8	NI	29	0.00950702	0.0092857	0.0171875	0.0123397

Table 6. Power indices in the European Parliament. (III)

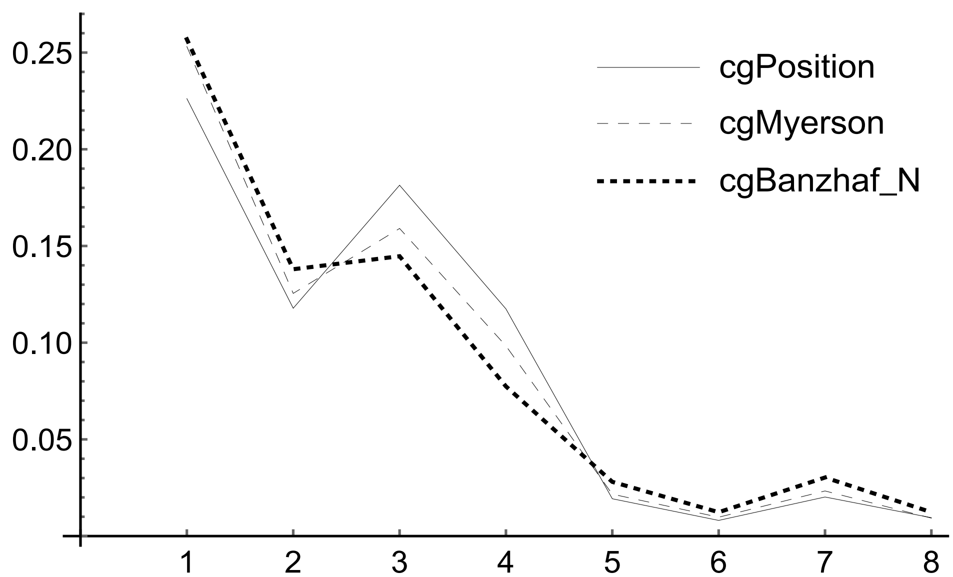

Figure 9: Comparative graph cg-values in EP game (I).

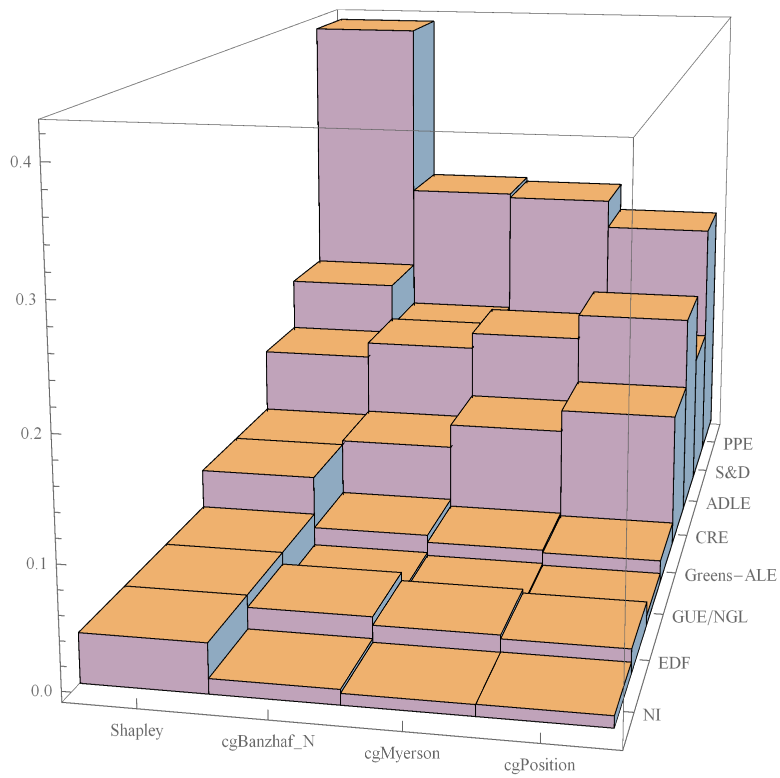

Figure 10: Comparative graph cg-values in EP game (II).

7. Conclusions

We have defined a version of the position value for games with fuzzy communication situations. The position value gives a different vision to that of the Myerson value or the graph Banzhaf value for communication structures, conceding greater importance to the situation of the players in the structure. Fuzzy graphs allow us to study situations in cooperative games where the communication among the agents should be shared. The cg-position value can be obtained by using the position values of a partition of the fuzzy graph. We have provided with an axiomatization of the new value and an algorithm to calculate it. We have shown an application of the value as a power index to determine the power of the groups in the European Parliament. We can see that the fuzziness of the cooperation among the groups implies a drop in powers of all groups.

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