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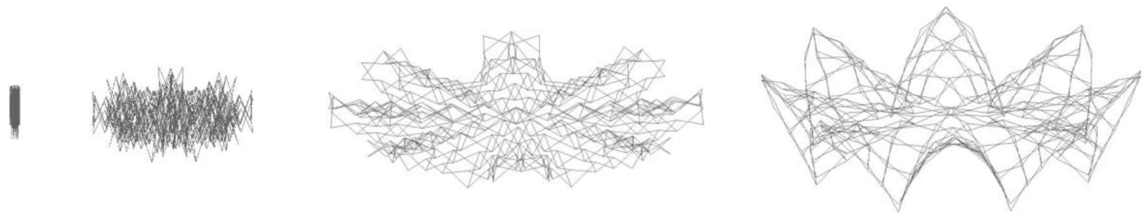
Geometric method to design bistable and non - bistable deployable structures of straight scissors based on the convergence surface

Abstract

Since Leonardo Da Vinci designed the first deployable structure with scissors some centuries ago, many authors have tried to improve its designs using new geometries. However, these strategies to make deployable structures have limits because they can only be applied in some geometries and the level of the resolution obtained is limited. In this paper a new geometric method to design bistable and non – bistable deployable structures with straight scissors is going to be developed. Using this method, the only limit in the design of deployable structures with straight scissors is the designer's imagination, for example, with this method the designer will be able to design as deployable the Sydney Opera House or a common building with pillars, etc… Consequently, this method is not based on load or forces that can be applied in the structure. An application example is the figure attached in this Abstract where the reader can see the Manantiales Restaurant in Xochimilco designed with straight scissors and without any geometric incompatibility.



1. Introduction

In 1985 Felix Escrig Pallares discovered that a deployable structure of straight scissors will be geometrically convergent (all rods will be a single rod in the folded position) if equation 1 is satisfied in Figure 1 [1] [2] [3] [4].

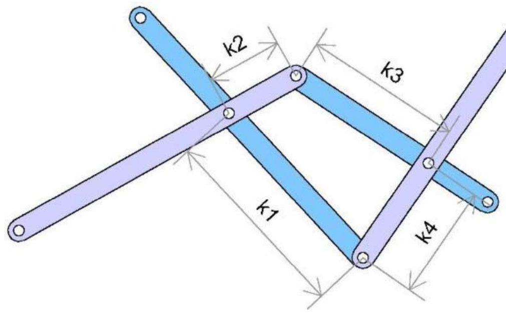


Fig. 1. Geometrically convergence condition in a deployable structure with straight scissors developed by Felix Escrig Pallares in 1985.

$$k_1 + k_2 = k_3 + k_4 \tag{1}$$

Luis Sánchez Cuenca [5] discovered some years later that Eq. (1) can be satisfied if the deployable structure is designed using the following ellipse property (Fig. 2): In any point of an ellipse, the addition of the lenghts of its vector – radios is a constant number and this number is equal to the longer axis, Eq. (2).

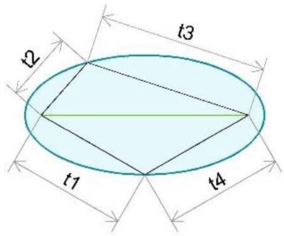


Fig. 2. Fundamental property of the ellipse.

$$t_1 + t_2 = t_3 + t_4 = \text{Longer axis} \tag{2}$$

The affirmation done by Luis Sanchez Cuenca was correct and this idea allowed the design of innovative deployable structures [6] (Fig. 3).

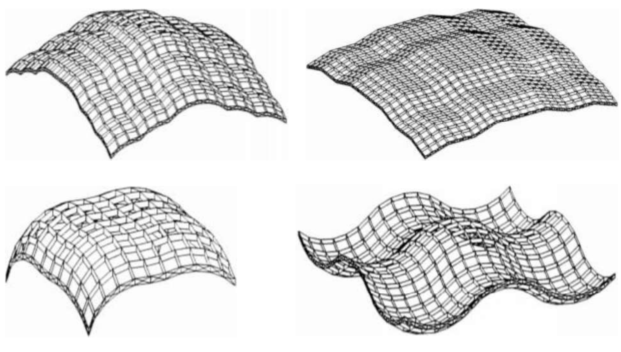


Fig. 3. Designs done by Luis Sanchez Cuenca that were published in the paper “Geometric Models for expandable structures”.

The main researcher who continued the trajectory of Luis Sanchez Cuenca was Niels De Temmerman. This researcher developed the first equations to design deployable structures with straight rods using ellipsoids. This work can be read in his PhD [7].

Later, this researcher and a PhD student (Kelvin Roovers) published two interesting papers where they develop particular strategies to design deployable structures in two situations: The axis of the scissors are parallel (Translational units) [8] [9] or the axis of the scissors are not parallel (Polar units) [10] [11].

Also, they design a deployable structure with a conic shape in the paper of the translational units. This structure is composed with triangles. Consequently, the rods of the 2nd. frequency are going to have a deformation (Blued rods in Fig. 4).

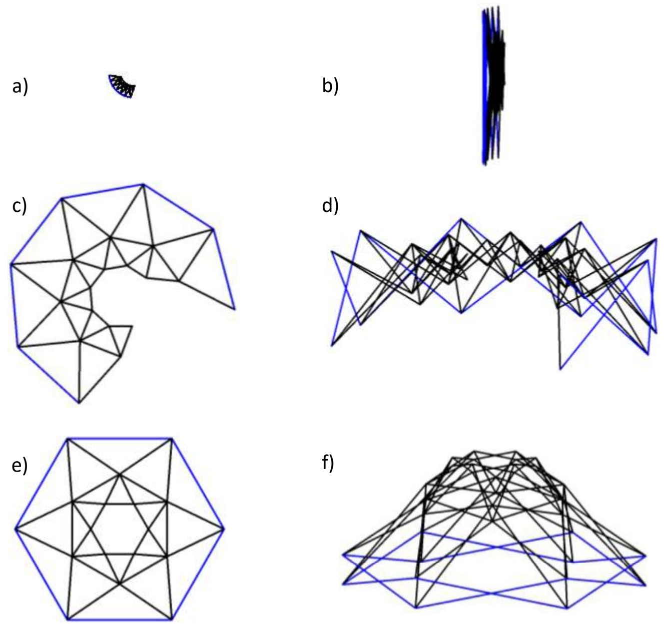


Fig. 4. (a) and (b) Deployment process at 0%; (c) and (d) Deployment process at 50%; (e) and (f) Deployment process at 100%.

Other authors who have also researched in the field of straight scissors are Félix Escrig Pallares, Jose Sánchez Sánchez and Juan Pérez Valcárcel [12] [13]. They have published several papers about transformable structures and their most important project is the roof for the olympic pool in the San Pablo sports center (Seville, 1997).

This roof is composed of two modular spheres. Consequently, one module can be connected with the next one changing the size of the space that is covered by the structure. In Figure 5 we can see 3 connected modules (Fig. 5).

To avoid instability problems, the designers put a rod in the diagonal of each module when the structure was in its final position. This rod allows increasing the rigidity of the structure.

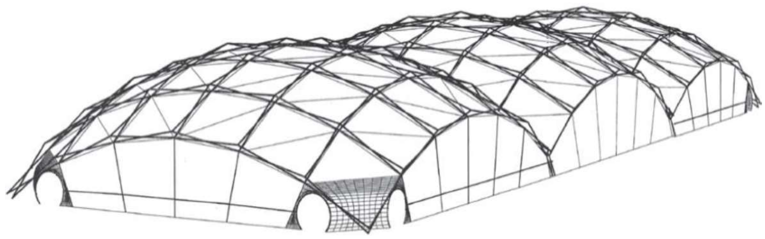


Fig. 5. Deployable roof that is composed of straight scissors, 3 spheres and rods in the diagonals to increase the stability.

Once each sphere had been built in a workshop, the spheres were transported to the sports center in the folded position. In the olympic pool the structure was deployed with a hoist.

Fig. 6. (Left figure) Test of the deployment process; (Right figure) Deployment process in the olympic pool.

When the deployment process was finished, the structure was fixed to the ground and the lighting was installed (Fig. 7).

Fig. 7. (Left figure) Interior space that is created by the structure; (Right figure) Exterior space that is created by the structure.

However, not all scissors that are currently used are composed of straight rods. Some years ago, Chuck Hoberman discovered angulated scissors, where the middle articulation does not belong to the line that joins the extreme articulations [14]. Likewise, the lengths from this middle point to the extreme points have to be the same. This situation means that the extreme articulations are going to be in the same line during the deployment process (Fig. 8).

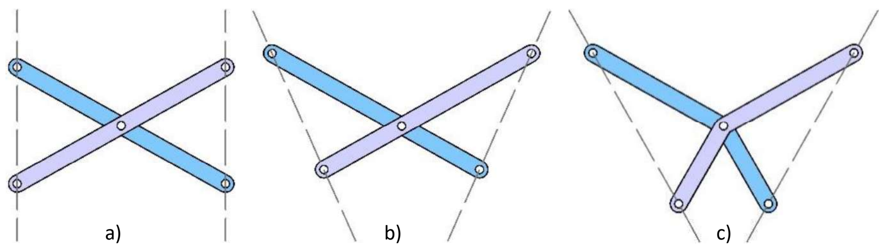


Fig. 8. (a) Straight rods with a parallel axis (Translational units); (b) Straight rods with no parallel axis (Polar units); Angulated scissors.

The Hoberman sphere is the most famous work of Chuck Hoberman in this field [15]. In this deployable structure, all scissors target the same point during the deployment process (the sphere's center) (Fig. 9).

a) b)

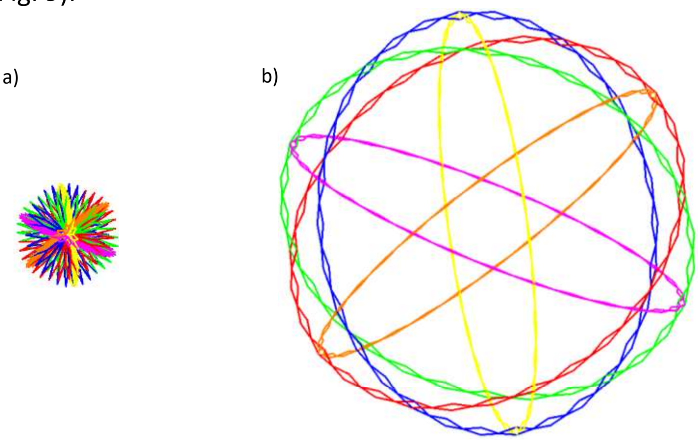


Fig. 9. (a) Hoberman sphere in folded position; (b) Hoberman sphere in unfolded position.

Later, Sergio Pellegrino developed the limit positions of the Hoberman scissors [16]. In this research, the maximum radius and the minimum radius of deployment are obtained (The Hoberman scissors have not a convergence in a line in the folded position). Also, the joints effect in the deployable structure was investigated.

Likewise, Pellegrino developed the design of cover elements in deployable structures of scissors. These elements allow a real application of this type of structures.

On the other hand, when the researchers of this field speak about Hoberman scissors we think normally about scissors that aren't straight. However, a modification of the Hoberman scissors is the curved scissors. Marios C. Phocas investigated this type of scissors to design a deployable structure with a cylindrical shape and with a cover element [17].

Another important author in this field was Charis Gantes. The most important deployable structure designed by this research was a deployable sphere where the main directions have only one degree of freedom. The rest of the space of the dome is covered with auxiliar scissors (Fig. 10). This structure is developed in the publications of this researcher [18].

a) b) c)

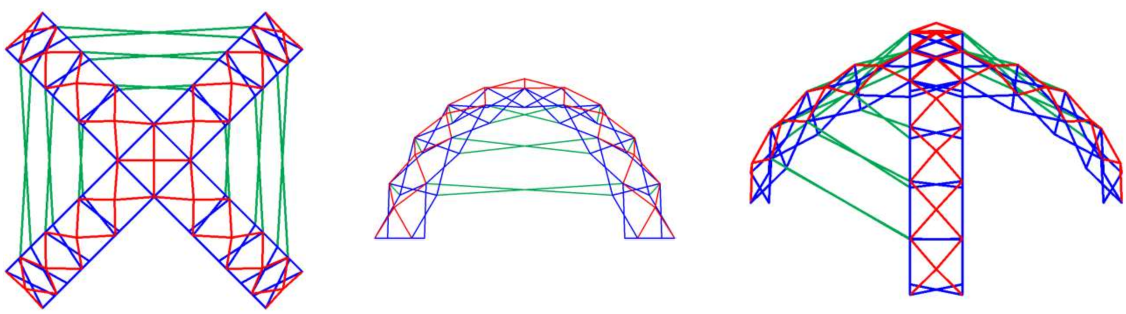


Fig. 10. Deployable structure designed by Charis Gantes. (a) Floor view, (b) Section view, (c) Perspective view. The blue lines are the scissors of the main directions. The red lines are the rods for the rigidity. The green lines are the scissors to close the space.

Yenal Akgün did an important work in the field of the geometric calculus of depoyable structure with straight scissors [19]. In this paper, this research develops different equations for the design of flat deployable structures. This work and the work of Niels de Temmerman and Kelvin Roovers are the origin of the research of this paper.

The main difference between these researches (Yenal Akgün and Niels de Temmerman - Kelvin Roovers) is that the first one develops the calculus using the geometry of the scissor and the second one uses an ellipse as a tool to obtain the final equations.

The research of this paper has a relationship with the work of Niels de Temmerman and Kelvin Roovers because the equations are also obtained using an ellipse in the plane or an ellipsoid in the space. However, these researches don't obtain the set of points of the space that satisfies the Escrig equation. The strategy of the previous works is based on iterative methods that give an approximate solution.

In this paper the iterative methods and the equations for an exact solution are going to be developed using a different mathematical process.

Also, Niels de Temmernan and Kelvin Roovers obtained the relationship between two ellipses if the axes are paralell (Translational units) [8] [10] (Proportionality ratio). However, the relationship between two ellipses if the axes are not paralell (Polar units) has been never developed. In this paper the relationship for translational units and for polar units is going to be developed using differential geometry.

On the other hand, an important aspect that has to be considered in the life of a deployable structure is the behaviour in the final position. In this state, the structure needs to support loads and new forces are going to have an important influence.

Juan Perez-Valcarcel is a research that has done an important work in this field.This autor develops different methods to calculate deployable structures [2] in function of:

a) Straight rods with the middle articulation in a fixed position.
b) Straight rods with the middle articulation in a variable position.
c) Curved rods.

This is an interesting field of research because allow us to go from the line model (Used during the design process) to the real mode.

2. Application of the method for deployable structures of straight scissors to a curve

2.1. Mathematical development

The goal of this method in a curve is to design as deployable any curve in the plane or in the space with the condition that all scissors have to be a line in the folded position. This situation means that the associated ellipse of each scissor has to be a tangent during the deployment process and the cross point between the rods has to be in this tangent (consequently the Felix Escrig equation is going to be satisfied in all scissors).

If this condition is satisfied, we will say that the deployable structure is "geometrically convergent". The figure where the mathematical development of the method in a curve is going to be developed is Fig.11

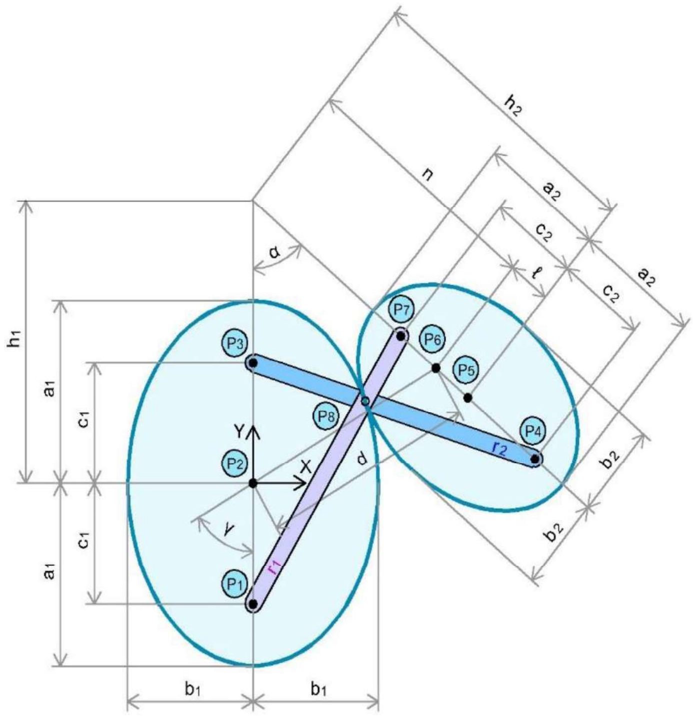


Fig. 11. Convergence between two ellipses in the plane.

Fig. 11 shows how there are three types of geometrical parameters where each one is going to have a physical meaning:

- The input parameters: These parameters are defined by the initial ellipse. There are 3:

* a_1 = This variable is the semimajor axis of the initial ellipse.

* b_1 = This variable is the semiminor axis of the initial ellipse.

* c_1 = This variable is the focal distance of the initial ellipse.

- The parameters that can be controlled by the designer or the control parameters: The designer can give a value to these parameters to obtain different results in the output parameters. There are 4:

* d = This variable is the distance between the center of the initial ellipse and the point that depends on the ℓ value.

* γ = This variable is the minor angle between the axis of the initial ellipse and the line that is given by the "d" variable.

* α = This variable is the angle between the axis of the initial ellipse and the axis of the final ellipse.

* ℓ = This is the most important parameter that can be controlled by the designer because this parameter allows modifying the relative position of the scissor with respect to the curve that we want to design as deployable. Its value can be from - ∞ to + ∞. However, there are 3 values that are the most important.

The first one is ℓ = -c_2. This value means that the inferior focus of the final ellipse will be in the curve that we want to design as deployable (the P_6 point will be in the P_4 position and both points will be in the curve).

The second one is ℓ = 0. This value means that the center of the final ellipse will be in the curve that we want to design as deployable (the P_6 point will be in the P_5 position and both points will be in the curve).

The third one is ℓ = +c_2. This value means that the superior focus of the final ellipse will be in the curve that we want to design as deployable (the P_6 point will be in the P_7 position and both points will be in the curve).

- The output parameters: These parameters are obtained once the input parameters and the control parameters are defined. The value of the output parameters gives us a scissor that satisfies the geometric convergence. There are 3:

* a_2 = This variable is the semimajor axis of the final ellipse.

* b_2 = This variable is the semiminor axis of the final ellipse.

* c_2 = This variable is the focal distance of the final ellipse.

In Fig. 11 we can also see that the "n" and the "h_2" parameters are defined. These parameters can be obtained as a function of γ, α and "d" and they will be important in the geometrical interpretation of the mathematical development.

Consequently, each point will have the next position:

$$P_1 = \left(P_{1x}, P_{1y}\right) = (0, -c_1) \tag{3}$$

$$P_2 = \left(P_{2x}, P_{2y}\right) = (0,0) \tag{4}$$

$$P_3 = \left(P_{3x}, P_{3y}\right) = (0, c_1) \tag{5}$$

$$P_4 = \left(P_{4x}, P_{4y}\right) = (d \cdot \sin\gamma + (\ell + c_2) \cdot \sin\alpha \,, d \cdot \cos\gamma - (\ell + c_2) \cdot \cos\alpha) \tag{6}$$

$$P_5 = \left(P_{5x}, P_{5y}\right) = (d \cdot \sin\gamma + \ell \cdot \sin\alpha \,, d \cdot \cos\gamma - \ell \cdot \cos\alpha) \tag{7}$$

$$P_6 = \left(P_{6x}, P_{6y}\right) = (d \cdot \sin\gamma \,, d \cdot \cos\gamma) \tag{8}$$

$$P_7 = \left(P_{7x}, P_{7y}\right) = (d \cdot \sin\gamma + (\ell - c_2) \cdot \sin\alpha \,, d \cdot \cos\gamma - (\ell - c_2) \cdot \cos\alpha) \tag{9}$$

$$P_8 = \left(P_{8x}, P_{8y}\right) = (x_c, y_c) \tag{10}$$

The goal is to obtain the cross point between the r_1 and r_2 lines and after that we will set that this point has to belong to the ellipse with a_1, b_1 and c_1 parameters. We haveto do 4 steps:

Step 1: The c_2 value will be obtained using the equations developed at the end of this section, Eq. (21), Eq. (22) and Eq. (23).

Step 2: The a_2 value will be obtained using the fundamental ellipse equation: $\overline{P_7P_8} + \overline{P_4P_8} = 2 \times a_2$

Step 3: Once the c_2 value and the a_2 value are calculated, the b_2 value can be obtained as: $a_2^2 = b_2^2 + c_2^2$

Step 4: Ellipse 2 will be ellipse 1 and we have to do the previous steps again to obtain the new ellipse.

Consequently, r_1 is defined by the fundamental equation of a line, Eq. (11).

$$r_1 : y = m \cdot x + n \tag{11}$$

If we replace the points positions in Eq. (11) we obtain:

$$m = \frac{P_{1y} - P_{7y}}{P_{1x} - P_{7x}} \tag{12}$$

$$n = \frac{P_{7y} \cdot P_{1x} - P_{7x} \cdot P_{1y}}{P_{1x} - P_{7x}} \tag{13}$$

Consequently:

$$r_1 : y = \frac{P_{1y} - P_{7y}}{P_{1x} - P_{7x}} \cdot x + \frac{P_{7y} \cdot P_{1x} - P_{7x} \cdot P_{1y}}{P_{1x} - P_{7x}} \tag{14}$$

If we do the same process with r_2 we obtain:

$$r_2 : y = \frac{P_{3y} - P_{4y}}{P_{3x} - P_{4x}} \cdot x + \frac{P_{4y} \cdot P_{3x} - P_{4x} \cdot P_{3y}}{P_{3x} - P_{4x}} \tag{15}$$

We know that $P_{1x} = 0$ and $P_{3x} = 0$. If we replace these conditions we obtain:

$$r_1 : y = \frac{P_{7y} - P_{1y}}{P_{7x}} \cdot x + P_{1y} \tag{16}$$

$$r_2 : y = \frac{P_{4y} - P_{3y}}{P_{4x}} \cdot x + P_{3y} \tag{17}$$

The cross point between r_1 and r_2 is x_c and y_c:

$$\frac{P_{7y} - P_{1y}}{P_{7x}} \cdot x_c + P_{1y} = \frac{P_{4y} - P_{3y}}{P_{4x}} \cdot x_c + P_{3y} \tag{18}$$

$$x_c = \frac{P_{3y} - P_{1y}}{\frac{P_{7y} - P_{1y}}{P_{7x}} - \frac{P_{4y} - P_{3y}}{P_{4x}}} \tag{19}$$

$$y_c = \frac{P_{4y} - P_{3y}}{P_{4x}} \cdot \frac{P_{3y} - P_{1y}}{\frac{P_{7y} - P_{1y}}{P_{7x}} - \frac{P_{4y} - P_{3y}}{P_{4x}}} + P_{3y} \tag{20}$$

Once the cross point between r_1 and r_2 has been obtained we replace the points value: Eq. (21) and Eq. (22).

$$x_c = \frac{c_1 \cdot [(d \cdot \sin\gamma)^2 + (\ell^2 - c_2^2) \cdot (\sin\alpha)^2 + 2 \cdot d \cdot \ell \cdot \sin\gamma \cdot \sin\alpha]}{c_1 \cdot (d \cdot \sin\gamma + \ell \cdot \sin\alpha) + c_2 \cdot d \cdot \sin(\alpha + \gamma)} \tag{21}$$

$$y_c = \frac{c_1 \cdot \begin{bmatrix} c_2^2 \cdot \sin(2 \cdot \alpha) + 2 \cdot c_1 \cdot c_2 \cdot \sin\alpha + d^2 \cdot \sin(2 \cdot \gamma) + \\ +2 \cdot d \cdot \ell \cdot \sin(\alpha - \gamma) - \ell^2 \cdot \sin(2 \cdot \alpha) \end{bmatrix}}{2 \cdot c_1 \cdot (d \cdot \sin\gamma + \ell \cdot \sin\alpha) + 2 \cdot c_2 \cdot d \cdot \sin(\alpha + \gamma)} \tag{22}$$

$$\forall (x_c, y_c) \in \left(\frac{x_c}{b_1}\right)^2 + \left(\frac{y_c}{a_1}\right)^2 = 1 \tag{23}$$

If $\alpha = 0$ (the axes of both ellipses are parallel), the previous equation can be simplified as:

$$c_2 = c_1 \cdot \left[\sqrt{\left(\frac{d \cdot \sin\gamma}{b_1}\right)^2 + \left(\frac{\ell - d \cdot \cos\gamma}{a_1}\right)^2} - 1\right] \tag{24}$$

To show the practical use of Eq. (23), some examples have been done. In these examples a random curve has be designed as deployable with straight scissors using different ℓ values (Fig. 12.). Also, the ℓ value can change during the design process; for example, we can design a scissor with $\ell = 0$ and the next scissor can be designed with $\ell = c_2$.

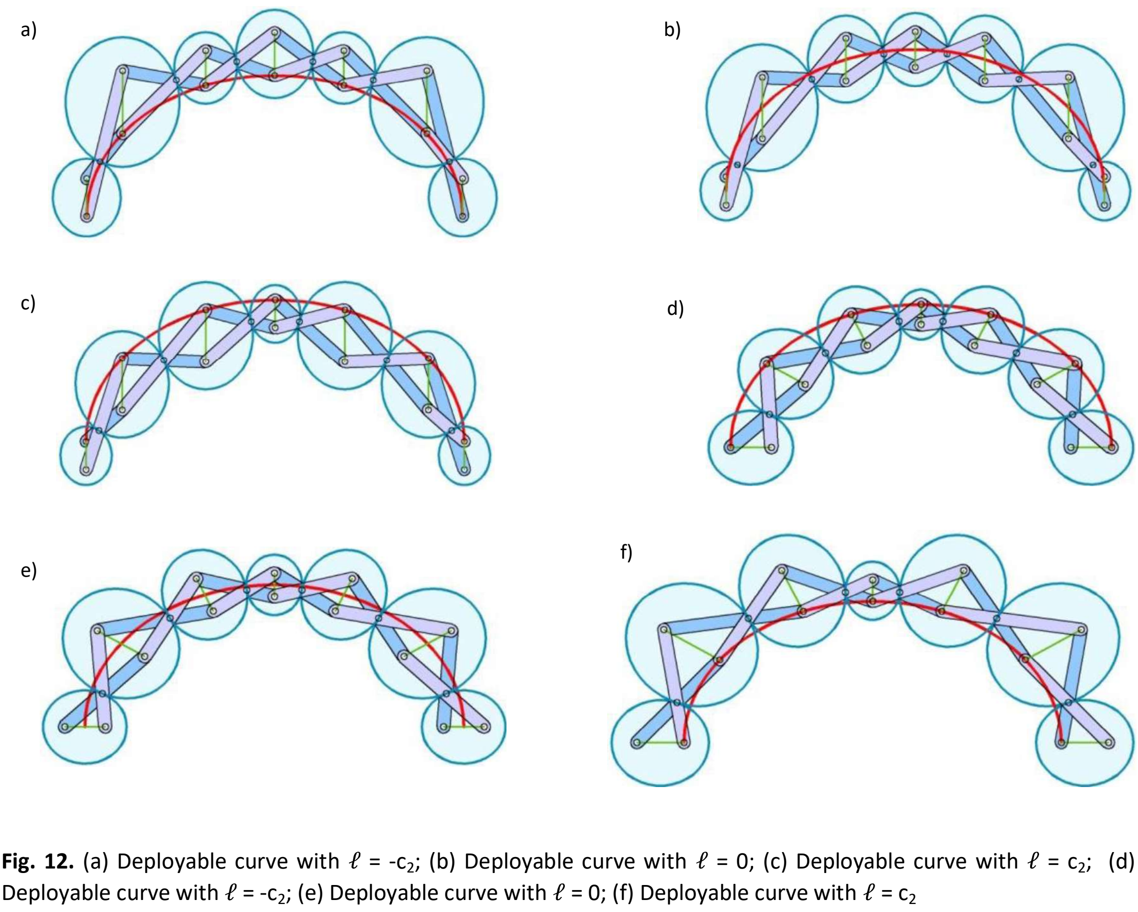


Fig. 12. (a) Deployable curve with $\ell = -c_2$; (b) Deployable curve with $\ell = 0$; (c) Deployable curve with $\ell = c_2$; (d) Deployable curve with $\ell = -c_2$; (e) Deployable curve with $\ell = 0$; (f) Deployable curve with $\ell = c_2$

2.2. Geometrical interpretation of the mathematical development

The equation that has been obtained in the previous section allows us to calculate the scissor that satisfies the convergence for a position and an orientation.

However, this strategy is not suitable if we want to design a surface as deployable with straight scissors because the equations are quite big, and we have to solve this using iteration methods.

To avoid this situation a geometrical extrapolation is going to be developed using the equation of the previous section. Consequently, we are going to obtain an equivalence between the mathematical tool and the geometrical tool to do the design process easier.

From Figure 11 we can obtain:

$$\frac{\sin\alpha}{d} = \frac{\sin\gamma}{n} = \frac{\sin(\alpha+\gamma)}{h_1} \tag{25}$$

The following step is to obtain the $\cos\alpha$ value in function of the input parameters. Using Eq. (25) we can derive:

$$\frac{\sin\alpha}{d} = \frac{\sin\alpha\cdot\cos\gamma + \cos\alpha\cdot\sin\gamma}{h_1} \tag{26}$$

$$\cos\alpha = \frac{\sin\alpha}{\sin\gamma}\cdot\left(\frac{h_1}{d} - \cos\gamma\right) \tag{27}$$

Also, from Eq. (25) can be derived:

$$\sin\alpha = \frac{d}{n}\cdot\sin\gamma \tag{28}$$

Replacing Eq. (28) in Eq. (27):

$$\cos\alpha = \frac{1}{n}\cdot(h_1 - d\cdot\cos\gamma) \tag{29}$$

If Eq. (28) is replaced in Eq. (21) we obtain Eq. (30)

$$x_c = \frac{c_1\cdot d\cdot\sin\gamma\cdot[(1+v)^2 - u^2]}{c_1\cdot(1+v) + u\cdot h_1} \tag{30}$$

And if Eq. (28) and Eq. (29) are replaced in Eq. (22) we obtain Eq. (31)

$$y_c = \frac{c_1\cdot\begin{bmatrix} u\cdot c_1 + d\cdot\cos\gamma + v\cdot d\cdot\cos\gamma + \\ +(h_1 - d\cdot\cos\gamma)\cdot(u^2 - v^2 - v)\end{bmatrix}}{c_1\cdot(1+v) + u\cdot h_1} \tag{31}$$

Where:

$$u = \frac{c_2}{n} \tag{32}$$

$$v = \frac{\ell}{n} \tag{33}$$

"u" and "v" are non-dimensional parameters where "u" will be the variable to iterate and "v" will be a parameter with the same function of the ℓ parameter. This situation means: If v = -u has the same meaning in comparison with $\ell = -c_2$ or if v = 0 has the same meaning in comparison with $\ell = +c_2$

Eq. (30) and Eq. (31) are in polar units. To translate these equations to Cartesian units we are going to use Eq. (34) and Eq. (35)

$$\cos\gamma = \frac{y}{d} \tag{34}$$

$$\sin\gamma = \frac{x}{d} \tag{35}$$

If Eq. (34) and Eq. (35) are replaced in Eq. (30) and in Eq. (31) we obtain:

$$x_c = \frac{c_1 \cdot x \cdot [(1+v)^2 - u^2]}{c_1 \cdot (1+v) + u \cdot h_1} \tag{36}$$

$$y_c = \frac{c_1 \cdot [u \cdot c_1 + y + v \cdot y + (h_1 - y) \cdot (u^2 - v^2 - v)]}{c_1 \cdot (1+v) + u \cdot h_1} \tag{37}$$

We cannot forget that x_c and y_c belong to Eq. (23). Consequently, if Eq. (36) and Eq. (37) are replaced in Eq. (23) and after some calculus we obtain Eq. (38)

$$\left(\frac{y - f_1}{f_2}\right)^2 + \left(\frac{x}{f_3}\right)^2 = 1 \tag{38}$$

Where:

If the axes of the ellipses are not parallel (Translational units):

$$f_1(\text{If } \alpha \neq 0) = -\frac{h_1 \cdot u^2 + c_1 \cdot u - h_1 \cdot v - h_1 \cdot v^2}{(1+v)^2 - u^2} \tag{39}$$

$$f_2(\text{If } \alpha \neq 0) = \left[\frac{a_1}{(1+v)^2 - u^2}\right] \cdot \left[1 + v + \frac{u \cdot h_1}{c_1}\right] \tag{40}$$

$$f_3(\text{If } \alpha \neq 0) = \left[\frac{b_1}{(1+v)^2 - u^2}\right] \cdot \left[1 + v + \frac{u \cdot h_1}{c_1}\right] \tag{41}$$

If the axes of the ellipses are parallel (Polar units):

$$f_1(\text{If } \alpha = 0) = \ell \tag{42}$$

$$f_2(\text{If } \alpha \neq 0) = a_1 \cdot \left(1 + \frac{c_2}{c_1}\right) \tag{43}$$

$$f_3(\text{If } \alpha \neq 0) = b_1 \cdot \left(1 + \frac{c_2}{c_1}\right) \tag{44}$$

Eq. (38) is the equation of a family of ellipses that are displaced in their axis direction in function of the “u” value. Consequently, if we give a value to the “u” parameter we will obtain an ellipse such that all of its points will satisfy the Felix Escrig equation from the initial ellipse, Eq. (1).

In Figure 13 the convergence ellipse for u = 0.2 and v = 0 has been represented. In this figure it can be observed that for any point of this ellipse (yellow ellipse) the cross point of each scissor belongs to the initial ellipse (blue ellipse).

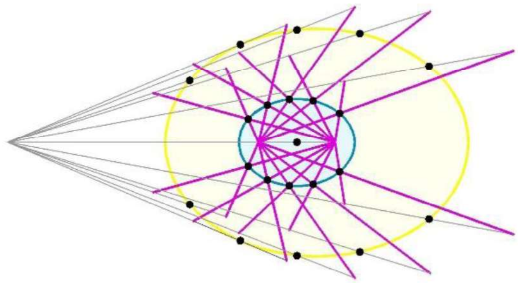


Fig. 13. Graphical representation of scissors with u = 0.2 and = 0

2.3. Step by step examples

The goal of these examples is to find the scissor in the final point of a curve (O point in Figure 14) that completes the structure of the example and that satifies the geometrical convergence.

Consequently, the values of the initial ellipse (a_1, b_1 and c_1) and the position parameters (d and γ) are going to be the same in all the examples.

The parameters that are going to change in each example are α (the angle between the ellipses axes) and ℓ (the relative position of the scissor with respect to the curve that is going to be designed as deployable).

The reader (Table 1) can see in these examples how the process is to use this method in a curve. Also, the numerical values are given for other persons to be able to reproduce the examples.

The initial structure of scissors is shown in Figure 14.

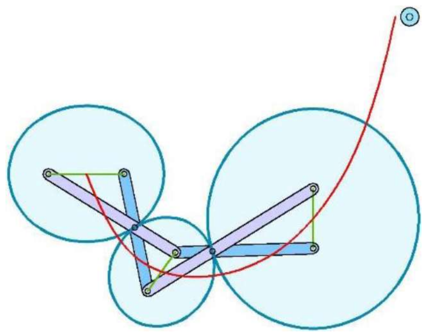


Fig. 14. Problem that has to be solved in each example: To find the scissors that finish in point O and that satisfy the geometric convergence for different values of α and ℓ.

The following values will be the same in all the examples:

a_1 = 289.93109809 mm
b_1 = 279.15357736 mm
c_1 = 78.31552776 mm
d = 573.75929178 mm
γ = 22.21730152°

Each example will be solved using at first the geometrical interpretation of the mathematical development and later using the equation of the mathematical development.

Also, the main advantage of the geometrical interpretation of the mathematical development is that it does not need any mathematical software because it only uses displaced ellipses in the axis direction. However, its main disadvantage is the imprecision that the designer is going to have when he/she wants to put a scissor in an exact point of the curve (he/she has to iterate the "u" value if si $\alpha \neq 0$ or the c_2 value if $\alpha = 0$).

On the other hand, the main advantage of the equation of the mathematical development, Eq. (21), Eq. (22) and Eq. (23), is that the designer has the exact solution without any iteration. However, its main disadvantage is that this equation has to be solved using a mathematical software if $\alpha \neq 0$ because the equation is implicit.

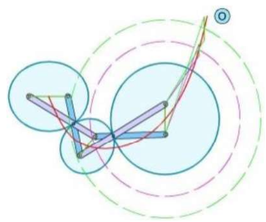
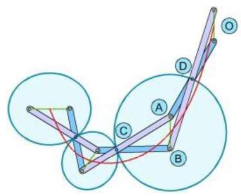
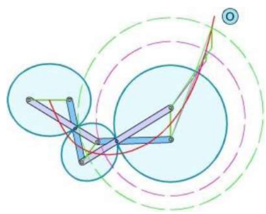
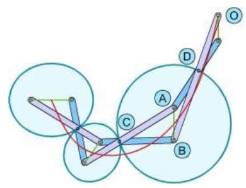
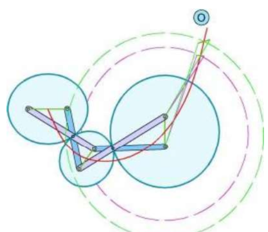
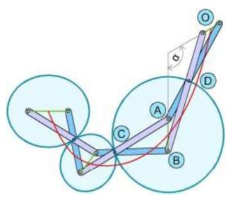
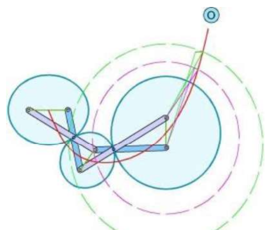
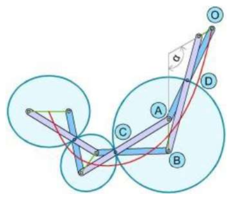
Step by step example 1 ($\alpha = 0$ and $\ell = 0$)		Step by step example 2 ($\alpha = 0$ and $\ell = +c_2$)	
Geometrical interpretation	Mathematical development	Geometrical interpretation	Mathematical development
			
1º Iteration (purple ellipse): c_2 = 30 mm	c_2 = 77.53659728 mm	1º Iteration (purple ellipse): c_2 = 25 mm	c_2 = 62.21825402 mm
2º Iteration (green ellipse): c_2 = 60 mm	With this c_2 value: $\overline{AC} + \overline{BC} = \overline{AD} + \overline{BD}$ Where: $\overline{AC} = 313.2541$ mm $\overline{AD} = 217.8345$ mm $\overline{BC} = 266.6080$ mm $\overline{BD} = 362.0276$ mm	2º Iteration (green ellipse): c_2 = 50 mm	With this c_2 value: $\overline{AC} + \overline{BC} = \overline{AD} + \overline{BD}$ Where: $\overline{AC} = 313.2541$ mm $\overline{AD} = 219.3415$ mm $\overline{BC} = 266.6080$ mm $\overline{BD} = 360.5206$ mm
Step by step example 3 ($\alpha = 120^0$ and $v = 0$)		Step by step example 4 ($\alpha = 120^0$ and $v = -u$)	
Geometrical interpretation	Mathematical development	Geometrical interpretation	Mathematical development
			
1º Iteration (purple ellipse): u = 0.1	u = 0.19531353 and c_2 = 48.92845405	1º Iteration (purple ellipse): u = 0.075	u = 0.14987326 and c_2 = 37.54510325 mm
2º Iteration (green ellipse): u = 0.15	With this c_2 value: $\overline{AC} + \overline{BC} = \overline{AD} + \overline{BD}$ Where: $\overline{AC} = 313.2541$ mm $\overline{AD} = 217.2196$ mm $\overline{BC} = 266.6080$ mm $\overline{BD} = 362.6425$ mm	2º Iteration (green ellipse): u = 0.1	With this c_2 value: $\overline{AC} + \overline{BC} = \overline{AD} + \overline{BD}$ Where: $\overline{AC} = 313.2541$ mm $\overline{AD} = 216.1271$ mm $\overline{BC} = 266.6080$ mm $\overline{BD} = 363.7350$ mm

Table 1. Step by step examples.

2.4. Application examples

The goal of this section is to use the method that is developed in this paper to design 2 curves as deployable: The Fermat spiral and the Lemniscata. In both examples the mathematical method is going to be used.

2.4.1. Application example 1: Fermat spiral

a) Definition: The Fermat spiral is a plane and transcendent curve. Also, this curve is unlimited, continuous and has a double mirror with respect to the geometric center. The discoverer of this curve was Menealo de Alejandria and the equation was developed by Pierre de Fermat. This spiral is considered as an evolution of the Archiimedes spiral.

b) Equation:

$$d = \pm a \cdot \sqrt{\beta} \tag{45}$$

c) Graphic representation (Fig. 15):

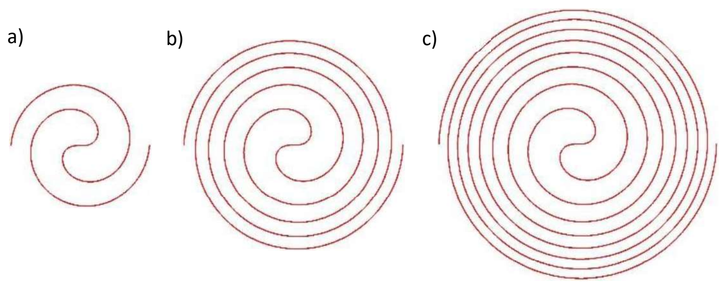


Fig. 15. (a) Graphic representation with $0 < \beta < 360^o$; (b) Graphic representation with $0 < \beta < 900^o$; (c) Graphic representation with $0 < \beta < 1440^o$.

d) Values of the deployment parameters:

- In the concave sections: $l = c_2$

- In the convex sections: $l = -c_2$

- In the center of the curve: $l = 0$

- In the complete curve: $\alpha = 0$

e) Deployable structure of a Fermat spiral with $0 < \beta < 360^{o}$ (Fig. 16):

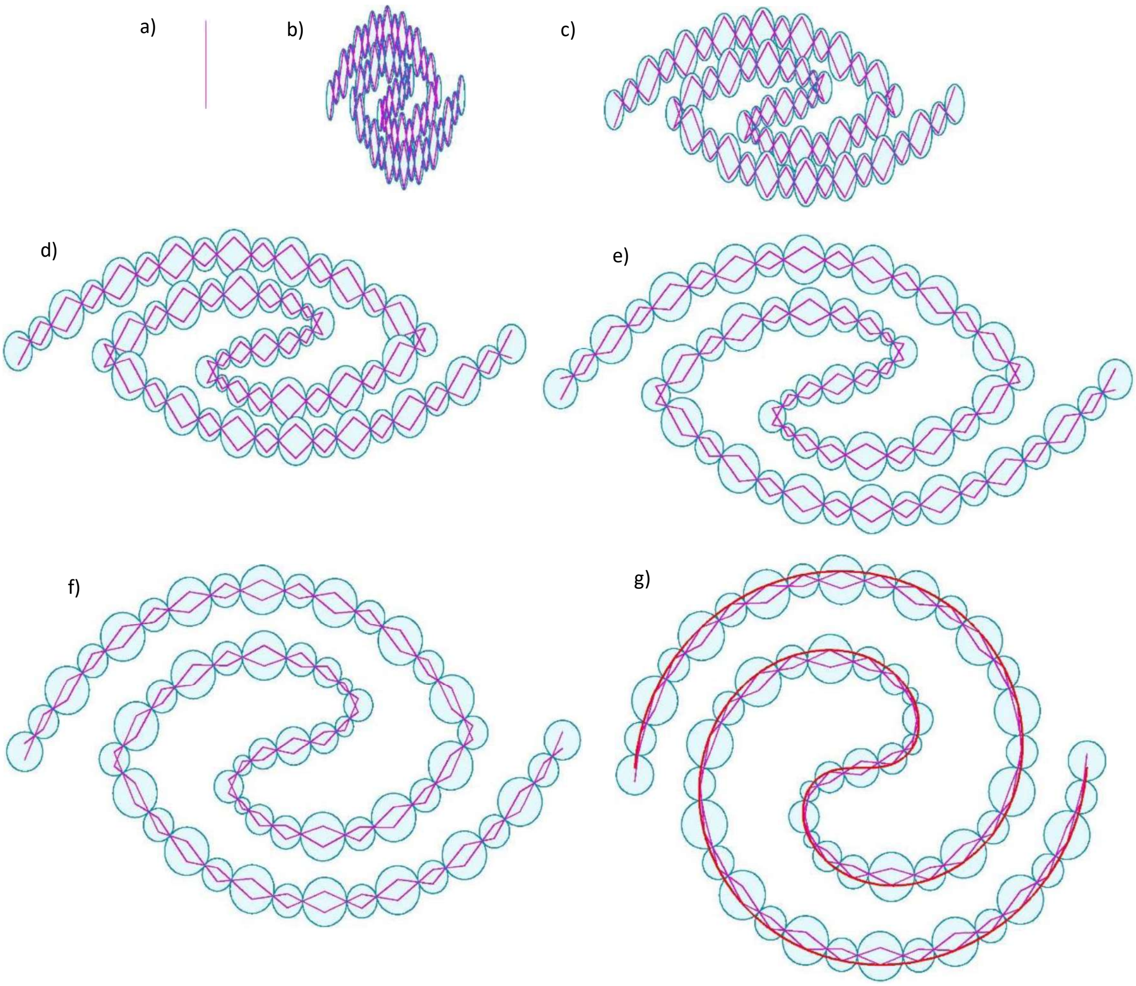


Fig. 16. (a) Deployment process: 0%; (b) Deployment process: 17%; (c) Deployment process: 34%; (d) Deployment process: 51%; (e) Deployment process: 68%; (f) Deployment process: 85%; (g) Deployment process: 100%.

2.4.2. Application example 2: Lemniscata

a) Definition: The Lemniscata or infinite curve is a plane and limited curve with a double mirror.

This curve can be obtained doing an inverse transformation of the hyperbola if the inverse circle is in the hyperbola center.

The discoverer of this curve was Jakob Bernoulli, and unlike with the ellipse (the addition between the focal distances is constant), in the Leminiscata the product between the focal distances is constant.

b) Equation:

$$\mathrm{d} = \pm \mathrm{a} \cdot \sqrt{2 \cdot \cos(2 \cdot \beta)} \tag{46}$$

c) Graphic representation (Fig. 17):

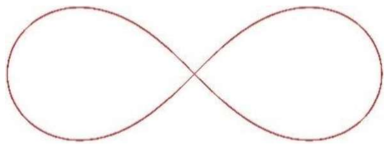


Fig. 17. Graphic representation of the Lemniscata.

d) Values of the deployment parameters:

- In the complete curve: l = -c2

- In the complete curve the focal distance is going to be perpendicular in each point of the curve where the scissor is drawn.

e) Deployable structure of a Lemniscata with $0 < \beta < 360^o$ (Fig. 18):

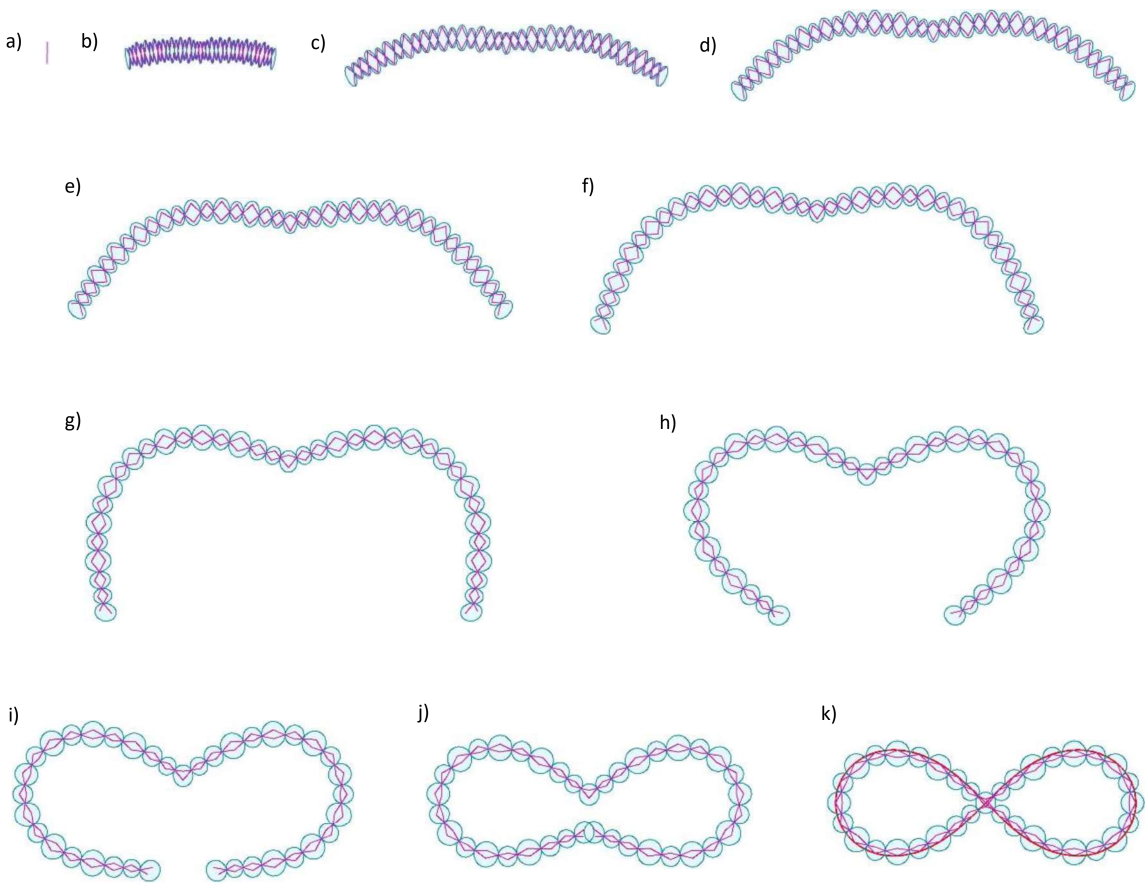


Fig. 18. (a) Deployment process: 0%; (b) Deployment process: 10%; (c) Deployment process: 20%; (d) Deployment process: 30%; (e) Deployment process: 40%; (f) Deployment process: 50%; (g) Deployment process: 60%; (h) Deployment process: 70%; (i) Deployment process: 80%; (j) Deployment process: 90%; (k) Deployment process: %.

3. Application of the method for deployable structures of straight scissors to a surface

3.1. Extrapolation of the method from a curve to a surface

Eq. (38) was obtained in the geometric interpretation of the mathematical development of a curve. This equation is the equation of a displaced ellipse in its axis direction.

If we want to design a surface as deployable with straight scissors the initial ellipse is going to be an ellipsoid with a circle revolution. Consequently, the family of ellipses that satisfies the convergence for an exact value of “u” and “v” is also going to be a family of ellipsoids with a circle revolution.

Then the family of ellipsoids that satisfies the convergence for an ellipsoid in the space will have the following equation:

$$\left(\frac{z-f_1}{f_2}\right)^2+\left(\frac{x}{f_3}\right)^2+\left(\frac{y}{f_3}\right)^2=1 \tag{47}$$

The f_1, f_2 and f_3 equations have been already defined previously.

Also, the axis of the ellipse has the Y axis direction in Eq. (38). To work in a standard system the axis of the ellipsoid has the Z axis direction in Eq. (47).

Giving values to the “u” parameter and for an exact value of the “v” parameter a family of convergent ellipsoids will be obtained. This situation has been shown in Figure 19. In this Figure the initial ellipsoid is the gray ellipsoid and the rest of the ellipsoids are obtained giving values to the “u” parameter.

Consequently, Eq. (47) shows that this family of ellipsoids is going to be composed of proportional ellipsoids with a displacement in their axis direction (Fig. 19, b)):

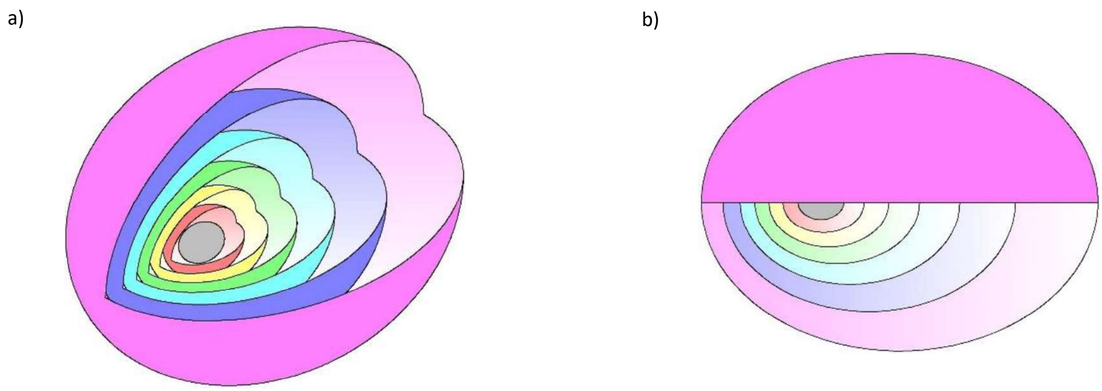


Fig. 19. (a) Perspective of the convergent ellipsoid family; (b) Top view of the convergent ellipsoid family.

However, the previous mathematical development is only focused on a single ellipsoid. When we want to design a surface as deployable, we always use two ellipsoids in the space.

Consequently, we must extrapolate the previous method to another method that satisfies the geometric convergence simultaneously between two ellipsoids in the space.

This geometric convergence will be satisfied in a set of space points where the focal distance obtained (c_2 value) has to be the same between the two ellipsoids. This situation means that we must intersect the convergent ellipsoids with the same “u” value of each initial ellipsoid (the “v” value will be always the same for both ellipsoid families).

The result of this process is a set of curves in the space. The distance between these curves is going to be very small if: $u_i - u_{i+1} \approx 0$.

The surface that is created using these curves will be a convergent surface. Then a point of this surface will have the same associated c_2 between the two ellipsoids and this c_2 value will satisfy the geometric convergence.

An example of this situation can be seen in Figure 20 where the red curves are obtained with the intersection of the convergent ellipsoids with the same "u" value. Also, the dark blue surfaces are the initial ellipsoids and the light blue surface is the convergent surface.

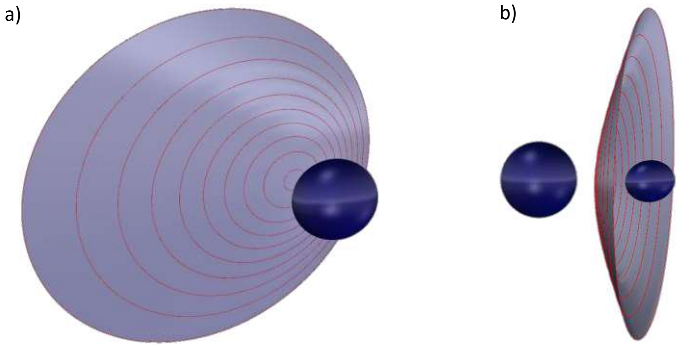


Fig. 20. (a) Perspective 1 of the convergent surface (b) Perspective 2 of the convergent surface.

Figure 21 shows this situation if we intersect both initial ellipsoids with a plane that contains the axes of these ellipsoids.

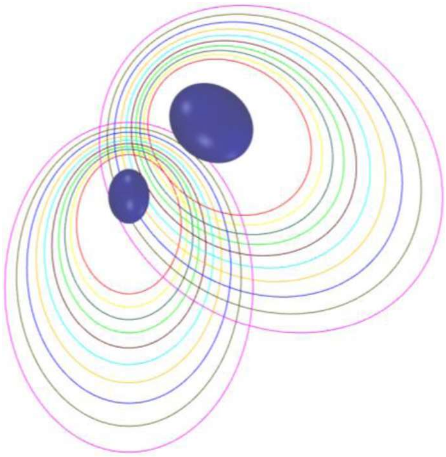


Fig. 21. Intersection between the plane that contains the axes of the two initial ellipsoids and the convergent surface. The ellipses with the same color are the ellipses that have the same "u" value.

3.2. Geometric relationship between the ellipsoids in the space to obtain a convergent surface

The previous section has developed that the intersection between two convergent ellipsoids (with the same "u" value) of two initial ellipsoids gives us a curve where a point of this curve will have the same c_2 value between the two initial ellipsoids. An example of this situation has been represented in Figure 22.

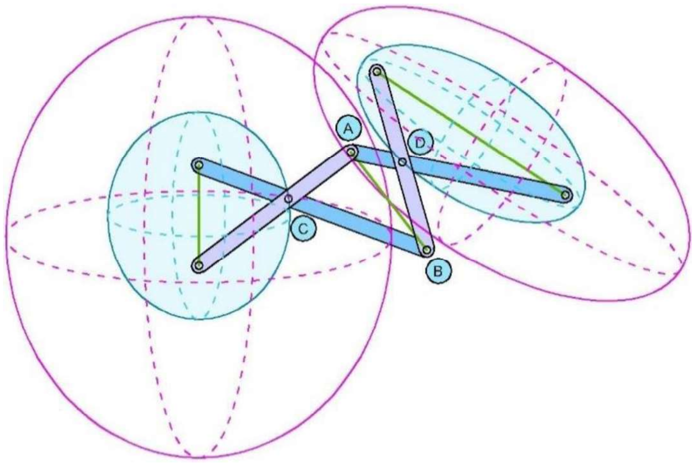


Fig. 22. Scissor for a point in the curve that has been obtained with the intersection between two convergent ellipsoids (purple ellipsoids) for a "u" value.

In Figure 22 we can see that the intersection between the rods is in the initial ellipses (C and D points) but we can also see that: $\overline{AC} + \overline{BC} \neq \overline{AD} + \overline{BD}$. Consequently, the Felix Escrig equation is not satisfied, and the scissors will have a limited deployment.

Then the question is: Why is this situation happening if the c_2 value has been obtained with the intersection of two convergent ellipsoids with the same "u" value?

The answer to this question is because the initial ellipsoids cannot be random ellipsoids. This situation means that there is a geometric relationship between the two initial ellipsoids.

Consequently, the goal will be to obtain the geometric relationship between the two initial ellipsoids.

Then, if we intersect two random ellipsoids in the space, we are going to have a curve with the shape of Figure 23 b).

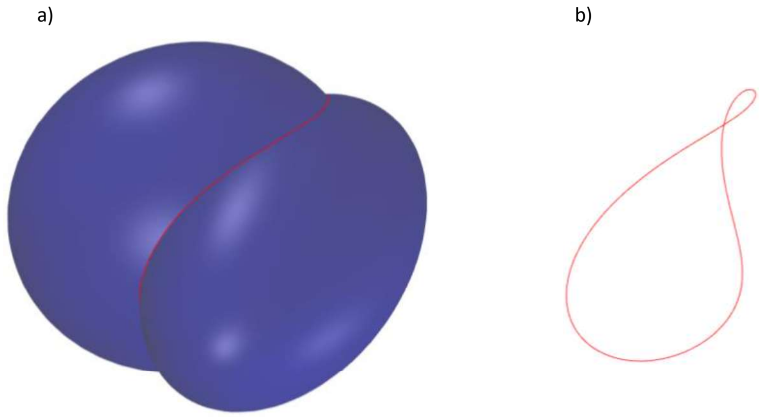


Fig. 23. (a) Two random ellipsoids in the space (b) Curve obtained from the intersection between the random ellipsoids in the space.

In Figure 23 we can see that the intersection curve is three-dimensional and it does not belong to a plane. However, if we obtain two ellipsoids that satisfy the geometric relationship sought using approximate numerical methods and we intersect the convergent ellipsoids (with the same "u" value) of these initial ellipsoids, we can see that the intersection curves for each "u" value are always contained in parallel planes. Also, these planes will be perpendicular to the plane that contains the axes of the initial ellipsoids. This situation is shown in Figure 24.

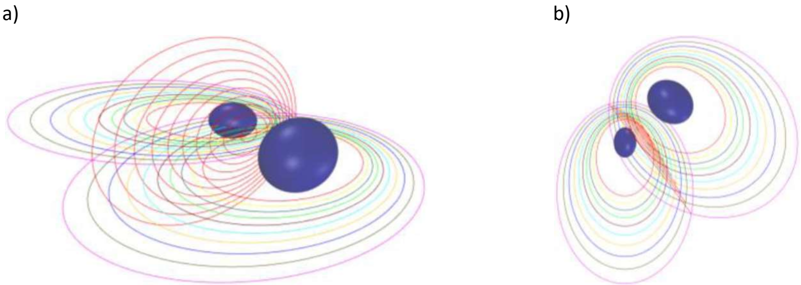


Fig. 24. (a) Perspective view of the curves that are obtained with the intersection of the convergent ellipsoid with the same "u" value (b) Top view of (a) where the reader can see the parallelism between the intersection curves.

Consequently, this property is going to be used to find the geometric relationship that the two initial ellipsoids must satisfy.

At first, we are going to define two random ellipsoids in the space in Figure 25.

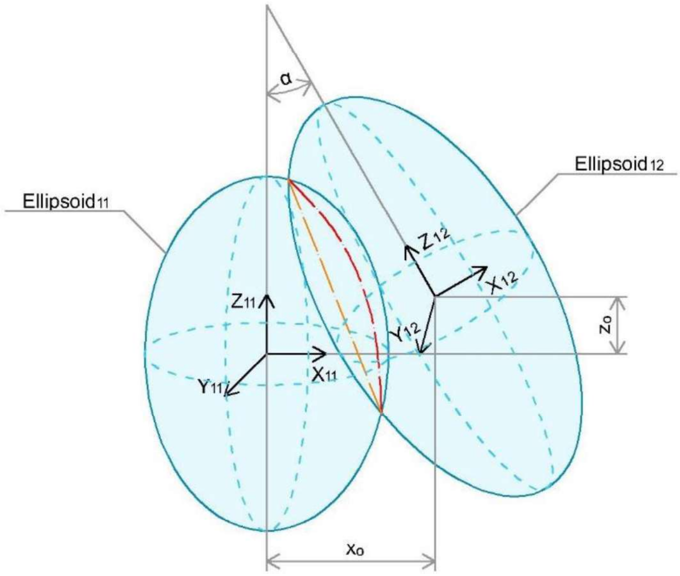


Fig. 25. Two random ellipsoids in the space.

The equation of ellipsoid 1 will be Eq. (48)

$$\left(\frac{x_{11}}{b1_1}\right)^2 + \left(\frac{y_{11}}{b_{11}}\right)^2 + \left(\frac{z_{11}}{a_{11}}\right)^2 = 1 \tag{48}$$

And the equation of ellipsoid 2 will be Eq. (49)

$$\left(\frac{x_{12}}{b_{12}}\right)^2 + \left(\frac{y_{12}}{b_{12}}\right)^2 + \left(\frac{z_{12}}{a_{12}}\right)^2 = 1 \tag{49}$$

In Figure 25 we can see that:

$$y_{12} = y_{11} \tag{50}$$

Consequently, Eq. (48) and Eq. (49) can be combined to obtain Eq. (51)

$$b_{11} \cdot \sqrt{1 - \left(\frac{x_{11}}{b_{11}}\right)^2 - \left(\frac{z_{11}}{a_{11}}\right)^2} = b_{12} \cdot \sqrt{1 - \left[\frac{x_{12}}{b_{12}}\right]^2 - \left[\frac{z_{12}}{a_{12}}\right]^2} \tag{51}$$

The next step is to convert the x_2 and z_2 variables to x_1 and z_1 variables with the following equations:

$$x_{12} = (x_{11} - x_o) \cdot \cos\alpha + (z_{11} - z_o) \cdot \sin\alpha \tag{52}$$

$$z_{12} = (z_{11} - z_o) \cdot \cos\alpha - (x_{11} - x_o) \cdot \sin\alpha \tag{53}$$

Also, we have Eq. (54):

$$a_{12}^2 = b_{12}^2 + c_{12}^2 \tag{54}$$

If Eq. (52), Eq. (53) and Eq. (54) are replaced in Eq. (51) we obtain Eq. (55):

$$t_1 \cdot z_{11} + t_2 \cdot x_{11} \cdot z_{11} + t_3 \cdot x_{11}^2 + t_4 \cdot z_{11}^2 = t_5 \cdot x_{11} + t_6 \tag{55}$$

Where:

$$t_1 = 2 \cdot z_o \cdot \left[1 - \left(\frac{c_{12}}{a_{12}} \cdot \cos\alpha\right)^2\right] + \left(\frac{c_{12}}{a_{12}}\right)^2 \cdot x_o \cdot \sin(2 \cdot \alpha) \tag{56}$$

$$t_2 = -\left(\frac{c_{12}}{a_{12}}\right)^2 \cdot \sin(2 \cdot \alpha) \tag{57}$$

$$t_3 = \left(\frac{c_{12}}{a_{12}} \cdot \sin\alpha\right)^2 \tag{58}$$

$$t_4 = -\left[1 - \left(\frac{c_{12}}{a_{12}} \cdot \cos\alpha\right)^2 - \left(\frac{b_{11}}{a_{11}}\right)^2\right] \tag{59}$$

$$t_5 = -\left[2 \cdot x_o \cdot \left[1 - \left(\frac{c_{12}}{a_{12}} \cdot \sin\alpha\right)^2\right] + \left(\frac{c_{12}}{a_{12}}\right)^2 \cdot z_o \cdot \sin(2 \cdot \alpha)\right] \tag{60}$$

$$t_6 = b_{11}^2 - a_{12}^2 + c_{12}^2 + (1 - t_3) \cdot x_o^2 + \left[\left(\frac{b_{11}}{a_{11}}\right)^2 - t_4\right] \cdot z_o^2 - x_o \cdot z_o \cdot t_2 \tag{61}$$

Eq. (55) is the projection of the intersection curve in the XZ plane for any geometric relationship between the two initial ellipsoids (the red curve of the Figure 25). The goal is to obtain the geometric relationship that allows converting this red curve into the orange curve in Figure 25: A line.

Then, the equation of the curvature of a parameterized curve is:

$$\text{Curvature (f)} = \frac{|r'(f) \wedge r''(f)|}{|r'(f)|^3} \tag{62}$$

The curve sought has the shape of a line, consequently its curvature is 0:

$$\text{Curvature (f)} = \frac{|r'(f) \wedge r''(f)|}{|r'(f)|^3} = 0 \rightarrow |r'(f) \wedge r''(f)| = 0 \tag{63}$$

The next step is to find the expression of r'(f) and r''(f). If we use the "f" parameter to parameterize the curve obtained:

$$x_{11} = f \tag{64}$$

Then:

$$t_1 \cdot z_{11} + t_2 \cdot f \cdot z_{11} + t_3 \cdot f^2 + t_4 \cdot z_{11}^2 = t_5 \cdot f + t_6 \tag{65}$$

$$t_4 \cdot z_{11}^2 + (t_1 + t_2 \cdot f) \cdot z_{11} + t_3 \cdot f^2 - t_5 \cdot f - t_6 = 0 \tag{66}$$

This equation gives us two solutions. The correct solution is Eq. (67):

$$z_{11} = \frac{-(t_1 + t_2 \cdot f) - \sqrt{\Delta}}{2 \cdot t_4} \tag{67}$$

Where:

$$\Delta = (t_1 + t_2 \cdot f)^2 - 4 \cdot t_4 \cdot (t_3 \cdot f^2 - t_5 \cdot f - t_6) \tag{68}$$

Consequently, the parameterized curve is:

$$r(f) = \left(f, \frac{-(t_1 + t_2 \cdot f) - \sqrt{\Delta}}{2 \cdot t_4}\right) \tag{69}$$

r’(f) and r’’(f) will have the following equations:

$$r'(f) = \left(1, -\frac{t_2}{2 \cdot t_4} - \frac{2 \cdot t_2 \cdot (t_1 + t_2 \cdot f) + 4 \cdot t_4 \cdot (t_5 - 2 \cdot f \cdot t_3)}{4 \cdot t_4 \cdot \sqrt{\Delta}}\right) \tag{70}$$

$$r''(f) = \big(0, A(f) + B(f)\big) \tag{71}$$

Where:

$$A(f) = \frac{[2 \cdot t_2 \cdot (t_1 + t_2 \cdot f) + 4 \cdot t_4 \cdot (t_5 - 2 \cdot f \cdot t_3)]^2}{8 \cdot t_4 \cdot \sqrt{\Delta^3}} \tag{72}$$

$$B(f) = \frac{8 \cdot t_3 \cdot t_4 - 2 \cdot t_2^2}{4 \cdot t_4 \cdot \sqrt{\Delta}} \tag{73}$$

Solving the determinant of the vector product:

$$\begin{vmatrix} 1 & -\frac{t_2}{2 \cdot t_4} - \frac{2 \cdot t_2 \cdot (t_1 + t_2 \cdot f) + 4 \cdot t_4 \cdot (t_5 - 2 \cdot f \cdot t_3)}{4 \cdot t_4 \cdot \sqrt{\Delta}} \\ 0 & A(f) + B(f) \end{vmatrix} = 0 \tag{74}$$

The solution of the previous determinant is:

$$A(f) + B(f) = 0 \tag{75}$$

If the previous expression is simplified, we obtain Eq. (76):

$$[t_2 \cdot (t_1 + t_2 \cdot f) + 2 \cdot t_4 \cdot (t_5 - 2 \cdot f \cdot t_3)]^2 = (t_2^2 - 4 \cdot t_3 \cdot t_4) \cdot \Delta \tag{76}$$

We know that the curve sought has the shape of a line. Then, its curvature must be constant for any “f” value and consequently any “f” value will satisfy the previous equation. The “f” value that simplifies the previous expression is f = 0. If this value of “f” is replaced in Eq. (76) we obtain:

$$t_1^2 \cdot t_3 + t_4 \cdot t_5^2 + t_1 \cdot t_2 \cdot t_5 + 4 \cdot t_3 \cdot t_4 \cdot t_6 = t_6 \cdot t_2^2 \tag{77}$$

Eq. (77) is the geometric relationship that must satisfy two ellipsoids in the space for any orientation to obtain a convergent surface that satisfies the Felix Escrig equation.

If in Eq. (77) $\alpha = 0$ (Translational units):

$$t_2 = t_3 = 0 \tag{78}$$

$$t_5 = -2 \cdot x_o \tag{79}$$

If Eq. (78) and Eq. (79) are replaced in Eq. (77):

$$t_4(\alpha = 0) \cdot (-2 \cdot x_o)^2 = 0 \rightarrow t_4(\alpha = 0) = 0 \rightarrow 1 - \left(\frac{c_{12}}{a_{12}}\right)^2 - \left(\frac{b_{11}}{a_{11}}\right)^2 = 0 \tag{80}$$

If Eq. (54) is replaced in Eq. (80):

$$\frac{a_{11}}{a_{12}} = \frac{b_{11}}{b_{12}} = \frac{c_{11}}{c_{12}} \quad (81)$$

The meaning of Eq. (81) is that if $\alpha = 0$ (Translational units), the geometric relationship between the initial ellipsoids is the proportionality.

3.3. Step by step examples

3.3.1. Step by step example 1:

An ellipsoid in the space will have the following values:

a_{11} = 40 cm
b_{11} = 30 cm
c_{11} = 26.457513 cm

Tasks:

a) To obtain the properties of another ellipsoid that will allow the existence of a convergence surface. The values of the position and orientation parameters are:

x_o = 100 cm
z_o = 120 cm
$\alpha = 60^o$

Also, any scissor that is drawn in this new ellipsoid has to have the same focal distance as ellipsoid$_{11}$ ($c_{12} = c_{11}$)

b) To obtain the convergence surface that is created by the two previous ellipsoids with u = 0.

c) To draw a scissor in any point of the convergence surface.

d) To check the convergence for the scissor of c).

* Task a)

The situation of this problem is the most common situation that we can find during the design process. This situation means that one of the initial ellipsoids is fully defined and we know the position (the x_o and z_o parameters), the orientation (the α parameter) and the focal distance of the second ellipsoid. The focal distance is going to be half of the thickness of the deployable structure.

All the parameters of Eq. (77) give this problem with the exception of a_{12}. So, the strategy is to obtain this variable from Eq. (77). The result of this step is: a_{12} = 53.85711 cm

Also: $a_{12}^2 = b_{12}^2 + c_{12}^2 \rightarrow b_{12}$ = 46.910428 cm

Consequently, the parameters that define the ellipsoid soughthave been already obtained. The result is shown in Figure 26.

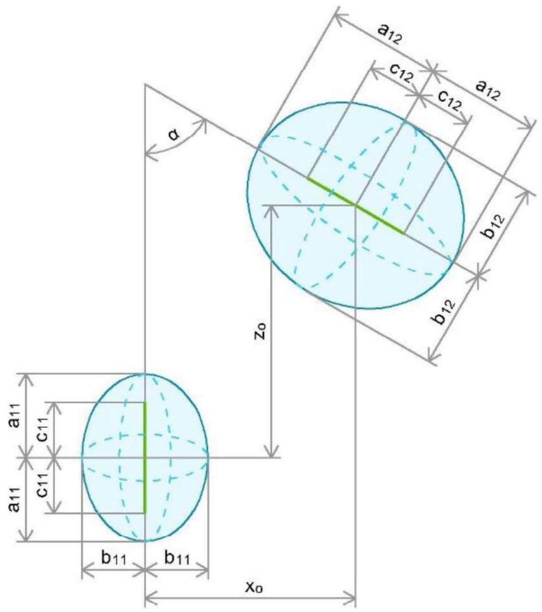


Fig. 26. Final ellipsoids of task a) of the 3.3.1. example.

* Task b)

To obtain the convergence surface between the ellipsoids of task a) we intersect the convergence ellipsoids of each initial ellipsoid with the same “u” value. The result of this process can be seen in Figure 27 where the red curves are the intersection curves.

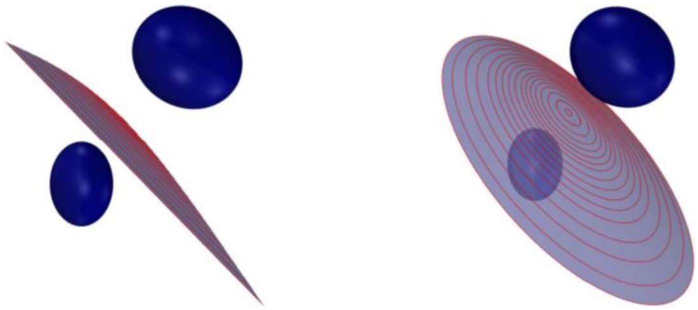


Fig. 27. Front view and perspective view of task b) of the 3.3.1. example.

* Task c)

Eq. (23) is used in a random point of the convergence surface. The c_2 value obtained using this equation has to be the same from the two initial ellipsoids. Also, we know that the orientation of c_2 has to be focused to the cross point between the axes of the initial ellipsoids (Fig. 28).

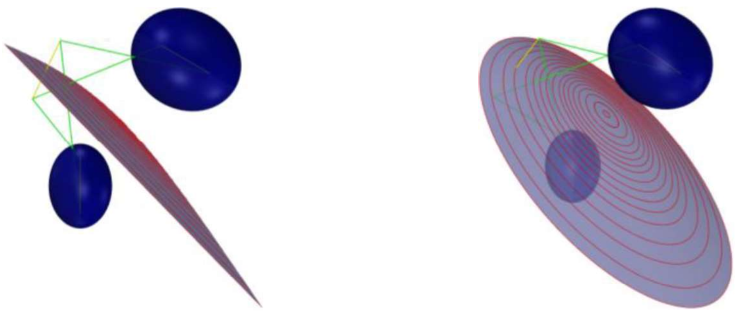


Fig. 28. Front view and perspective view of task c) of the 3.3.1. example.

* Task d)

To check the convergence, we measure the following distances:

$\overline{\mathrm{AH}} = 105.226452$ cm
$\overline{\mathrm{BH}} = 88.663468$ cm
$\overline{\mathrm{AG}} = 73.531606$ cm
$\overline{\mathrm{BG}} = 120.358314$ cm
$\overline{\mathrm{EG}} = 32.420717$ cm
$\overline{\mathrm{FG}} = 75.293501$ cm
$\overline{\mathrm{CH}} = 16.600288$ cm
$\overline{\mathrm{DH}} = 63.399712$ cm

We can see that: $\overline{\mathrm{AH}} + \overline{\mathrm{BH}} = \overline{\mathrm{AG}} + \overline{\mathrm{BG}}$; $\overline{\mathrm{EG}} + \overline{\mathrm{FG}} = 2 \cdot a_{12}$; $\overline{\mathrm{CH}} + \overline{\mathrm{DH}} = 2 \cdot a_{11}$. Consequently, the Felix Escrig equation is satisfied for both scissors (Fig. 29).

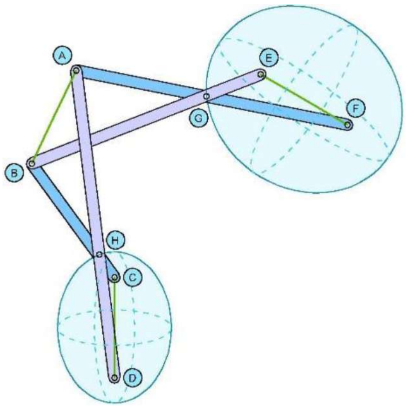


Fig. 29. Convergence in the scissors of the task d) of the 3.3.1. example.

3.3.2. Step by step example 2:

An ellipsoid in the space will have the following values:

$a_{11} = 50$ cm
$b_{11} = 25$ cm
$c_{11} = 43.3012702$ cm

a) To obtain the properties of another ellipsoid that will allow the existence of a convergence surface. The values of the position and orientation parameters are:

$x_o = 100$ cm
$z_o = 0$ cm
$\alpha = 0^o$

Also we want: $c_{12} = 1.5 \cdot c_{11}$

b) To obtain the convergence surface that is created by the two previous ellipsoids with u = 0.

c) To draw a scissor in any point of the convergence surface.

d) To check the convergence for the scissor of c).

* Task a)

As in the previous problem, we have in this problem all parameters of one of the initial ellipsoids, the parameters of position of the second ellipsoid and his orientation parameter. Also the value of the focal distance of the second ellipsoid is known.

To obtain the a_{12} and b_{12} parameters can be used the Eq. (77) (The same strategy that was done in the previous problem). However another option is to use Eq. (81) because in this problem $\alpha = 0^o$ (Translational units). This situation means that both initial ellipsoids have to be proportional.

Then:

$$\frac{a_{11}}{a_{12}} = \frac{c_{11}}{c_{12}} \rightarrow \frac{50}{a_{12}} = \frac{43.3012702}{1.5 \cdot 43.3012702} \rightarrow a_{12} = 75 \text{ cm}$$

Also: $a_{12}^2 = b_{12}^2 + c_{12}^2 \rightarrow b_{12} = 37.5$ cm

With the three parameters that define the second ellipsoid we can represent the following configuration (Fig. 30).

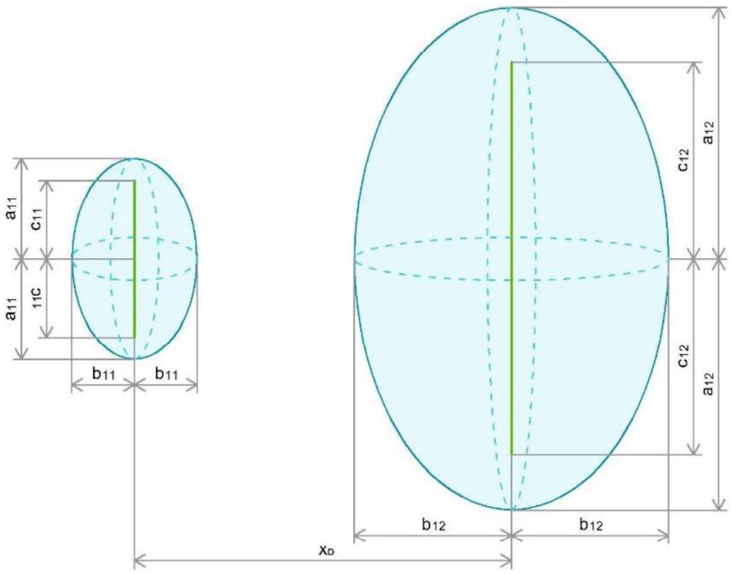


Fig. 30. Final ellipsoids of task a) of the 3.32. example.

* Task b)

The strategy to solve this section is the same strategy used in the previous example: The convergence ellipsoids are calculated for different "u" values and the intersection between the convergence ellipsoids with the same "u" value give us the convergence curves. The surface that contains the intersected curves is the convergence surface (Fig. 31).

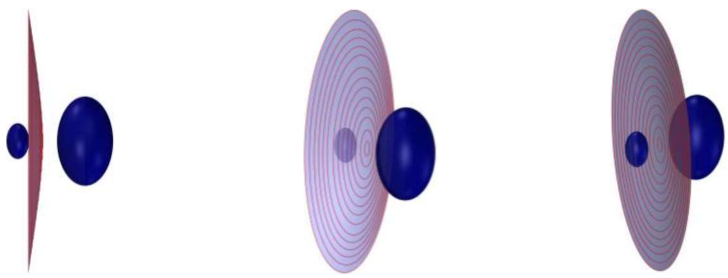


Fig.31. Front view and perspective view of task b) of the 3.3.1. example.

* Task c)

To solve this section a random point of the convergence surface is chosen. In this point Eq. (23) is used to obtain the c_2 value. Also the cross point between the initial ellipsoids is in the infinite so the orientation of c_2 is paralel with respect of the axes of the initial ellipsoids (Fig. 32).

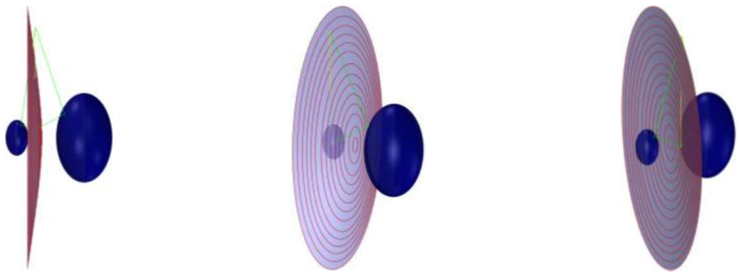


Fig. 32. Front view and perspective view of the task c) of the 3.3.2. example.

* Task d)

To check the convergence, we measure the following distances:

$\overline{AH} = 229.032808$ cm
$\overline{BH} = 66.115405$ cm
$\overline{AG} = 209.291155$ cm
$\overline{BG} = 85.857058$ cm
$\overline{EG} = 58.178945$ cm
$\overline{FG} = 141.821055$ cm
$\overline{CH} = 17.920598$ cm
$\overline{DH} = 62.079402$ cm

We can see that: $\overline{AH} + \overline{BH} = \overline{AG} + \overline{BG}$; $\overline{EG} + \overline{FG} = 2 \cdot a_{12}$; $\overline{CH} + \overline{DH} = 2 \cdot a_{11}$. Consequently, the Felix Escrig equation is satisfied for both scissors (Fig. 33).

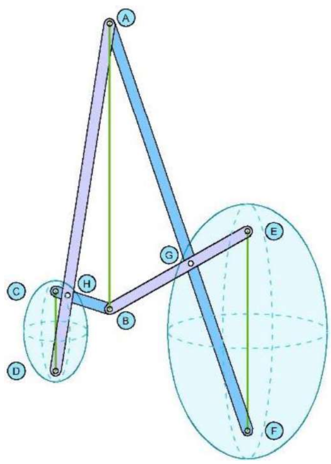


Fig. 33. Convergence in the scissors of the task d) of the 3.3.2. example.

3.4. Application examples

3.4.1. Manantiales Restaurant:

a) Information:
- Author: Félix Candela
- Year of construction: 1958
- Location: Xochimilco (México)
- Properties: The building is composed of the intersection with 4 hyperbolic paraboloids. The result of this intersection is the 8 sections of the building.
- Deployment parameters:

u = 0

l = c_2

b) Views of the geometry to design as deployable (Fig. 34):

Fig. 34. (a) Top view; (b) Front view; (c) Perspective view.

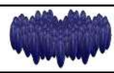
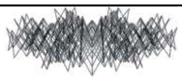
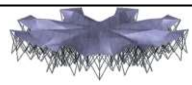
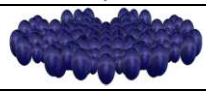
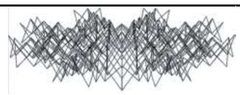
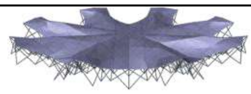
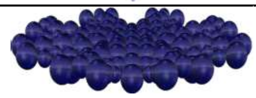
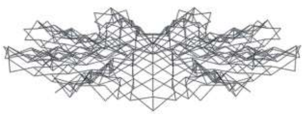
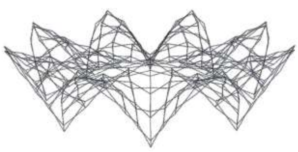
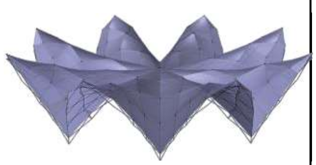
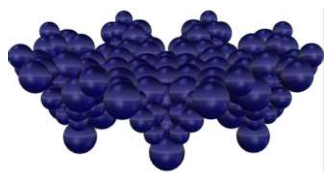
% deployed	Scissors structure	Membrane scissors structure	Convergence ellipsoids
0%			
20%			
40%			
60%			
80%			
100%			

Table 2. Deployment process of Manantiales Restaurant.

3.4.2. Exhibition Pavilion:

a) Information:
- Author: Paper author
- Year of design: 2018
- Location: Ephemeral building
- Properties: The building is composed of a hexagonal trimmed pyramid with a sphere as interior space.
- Deployment parameters :

u ≠ 0

v = -u

All superior joints belong to a plane surface.

All inferior joints belong to a spherical surface.

b) Views of the geometry to design as deployable (Fig. 35):

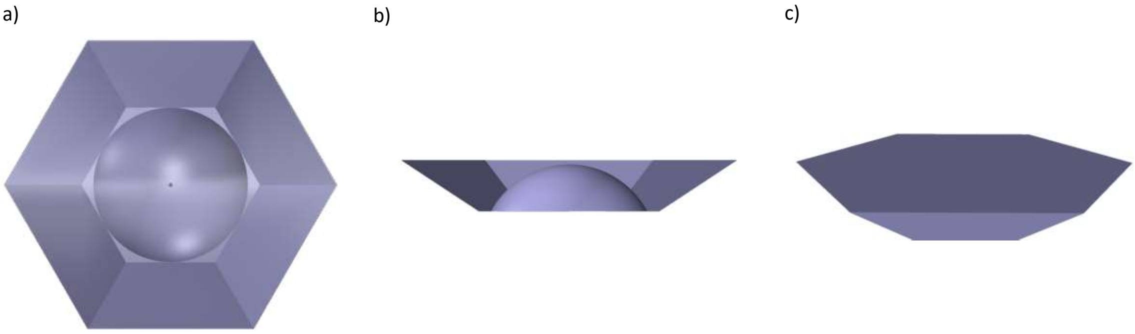


Fig. 35. (a) Top view; (b) Section view; (c) Perspective view

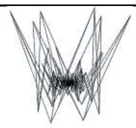
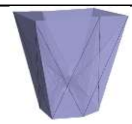
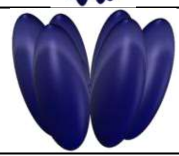
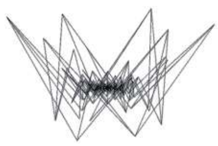
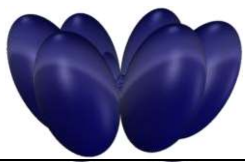
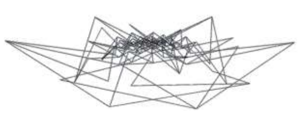
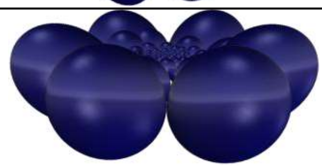
% deployed	Scissors structure	Membrane scissors structure	Convergence ellipsoids
0%			
25%			
50%			
75%			
100%			

Table 3. Deployment process of Exhibition Pavilion

4. Application of the method to a bistable structure and to a non bistable structure

To show how this method can be used in a bistable structure and in a non bistable structure, two examples are going to be developed.

4.1. Application in a bistable structure

The goal is to find the scissors that born from the peaks of the triangles modules and that end in the main direction of the surface (Orange curve) (Fig. 36). The obtained structure is going to be bistable.

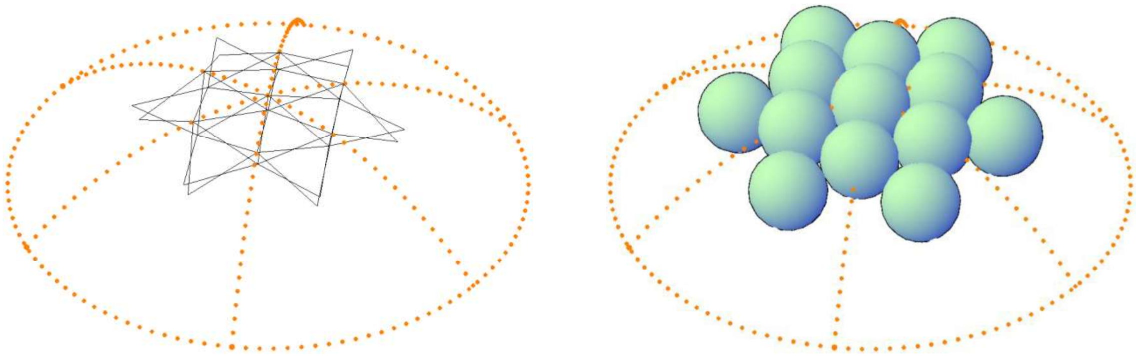


Fig. 36. Initial bistable structure where the method is going to be applied.

The first step is to obtain the convergence surface from the yellow ellipsoids (Fig. 37). The cross point between the convergence surface and the main direction of the surface that we want to design as deployable (Orange curve) is the solution of the problem.

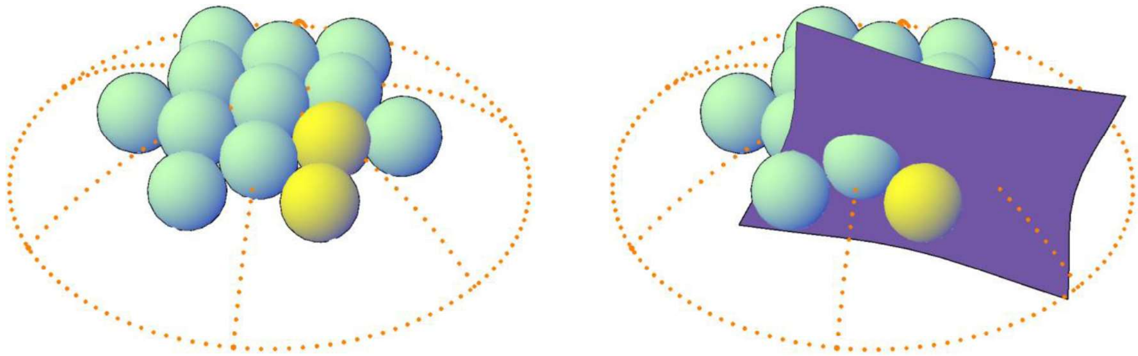


Fig. 37. Obtaining of the convergence surface from the yellow ellipsoids.

Finally the scissors are drawn and the obtained structure is bistable (Fig. 38).

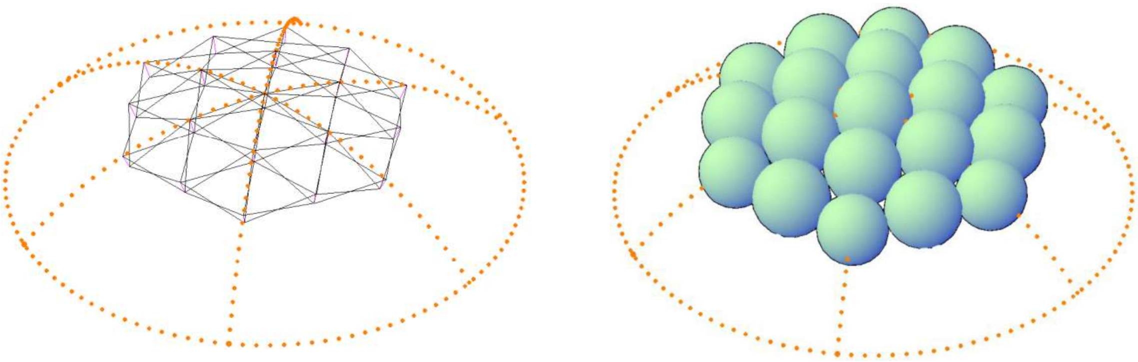


Fig. 38. Bistable structure (The length of the scissors of the main directions are going to be smaller during the deployment process).

4.2. Application in a non bistable structure

In this example, the scissors that close the module of the second frequency have to be obtained. Also, each scissor has to be contained in the same plane that the previous scissor (Fig. 39).

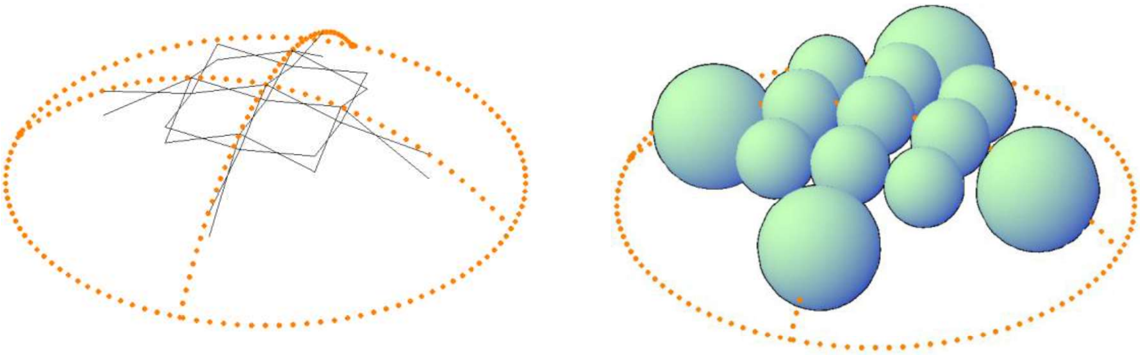


Fig. 39. Initial non bistable structure where the method is going to be applied.

Consequently, the convergende surface from the yellow ellipsoids is obtained. After that, the surface that we want to design as deployable is intersected with the convergence surface (Red surface) and the plane that contains the scissor of the previous frequency (Purple surface) (Fig. 40).

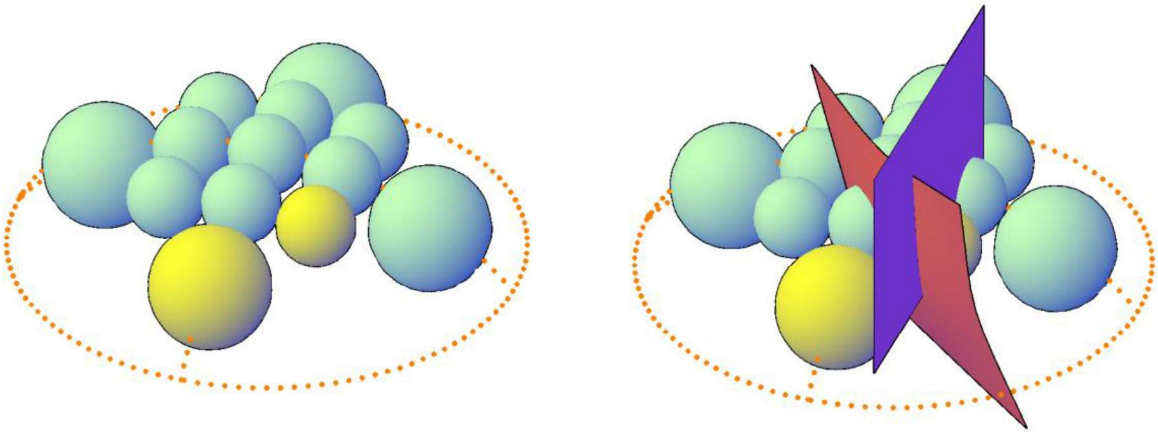


Fig. 40. Obtaining of the convergence surface from the yellow ellipsoids.

The result of this process is a point where the final scissor is obtained (Fig. 41).

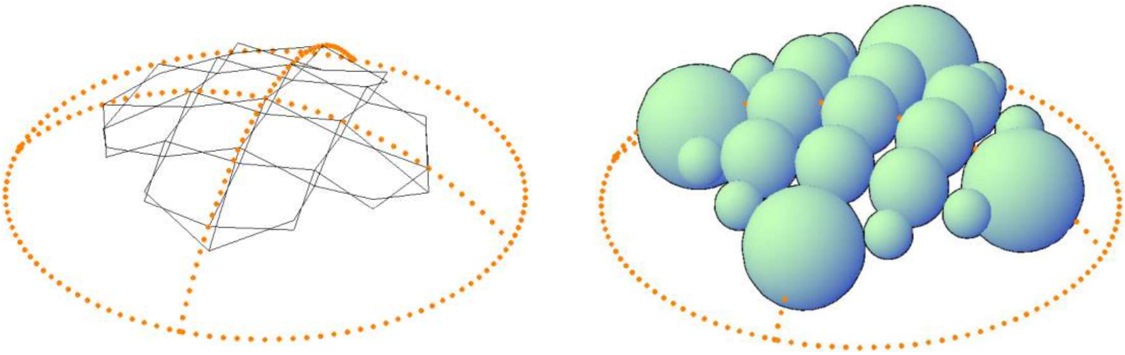


Fig. 41. Non bistable structure.

5. Built model and designed joint

The model that has been built to check the correct behavior of the method is a section of the Manantiales Restaurant. The material of the rods is acrylic using a laser machine for the cut.

The model has a size of 8 cm x 8 cm in its folded position and a size of 1.2 m x 0.50 m in its unfolded position (Fig. 42).

Fig. 42. Deployment process of the built model.

The designed joints are done using an element (a screw, a rivet, etc.) that is connected with some plastic tubes. Each rod will be in one plastic tube. This joint will allow all degrees of freedom using very cheap materials (Fig. 43) and (Fig. 44).

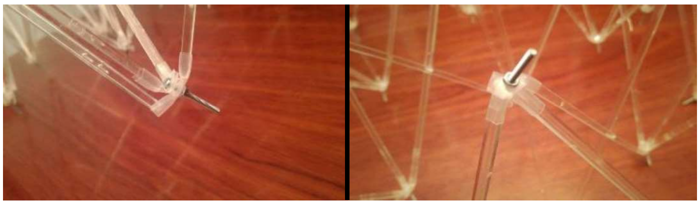


Fig. 43. Joint using a screw and a nut.

Fig. 44. Joint using a rivet.

If a real deployable structure is going to be built, the design of the joints has to be more developed. These joints need two free rotations and the connection between the rod and the joint has to be stronger. For this paper, three possible real joints have been developed.

The first one is composed of a part with a U shape and a support (Fig. 45). In this case, one of the rotations is between the support and the U shape part and the other one is between the U shape part and the rod.

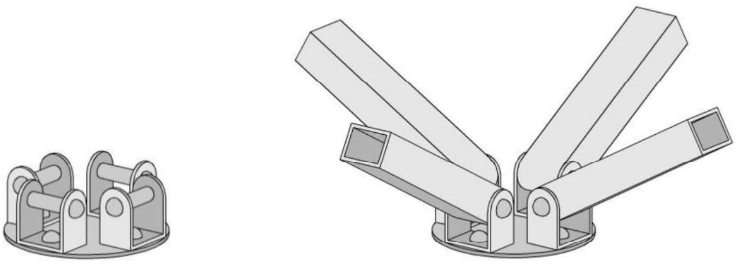


Fig. 45. Joint composed of a U shape part and a support.

The second one is the ball joint (Fig. 46). This connection has three free rotations in the space but the geometric restraints of the deployable structure remove the third rotation.

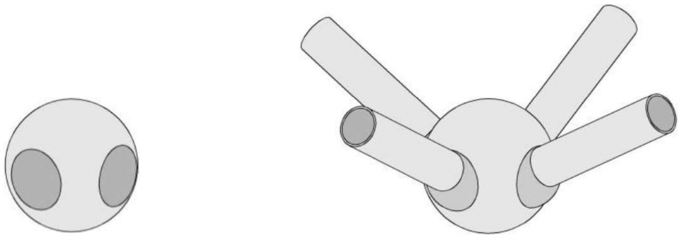


Fig. 46. Ball joint.

The third one is an elastic joint (Fig. 47). The rotations are possible because there is an elastic material between the rigid joint and the rod (Color green in Fig. 47). The connection between the elastic material and the rod is fixed.

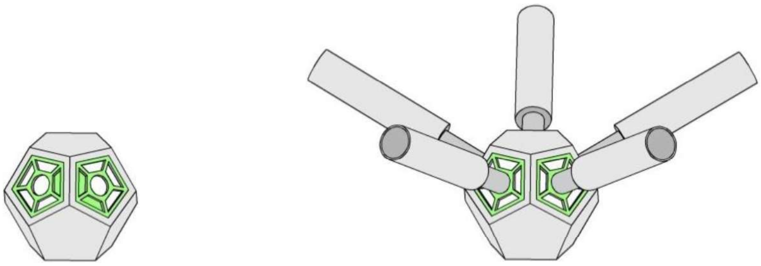


Fig. 47. Elastic joint.

6. Conclusions

The previous methods (Constant ellipsoids method or Luis Sanchez Cuenca method, proportional ellipsoids method or Niels De Temmerman method and spheres method or Felix Escrig Pallares method) are based on the calculation of a random point that satisfies the convergence on the surface that we want to design as deployable. If this point is not the point sought, we must do the method again. This situation means that the previous methods are iterative methods where we must repeat them until we obtain a solution that is close to the solution sought. This iterative process is quite slow, and it is difficult to automate. Consequently, the design of the deployable structure needs a lot of work hours using these methods.

On the other hand, the method that has been developed in this paper allows obtaining all points of the space that satisfy the convergence directly (without any iteration). The surface that contains these points is the convergence surface and the intersection between this surface and the surface that we want to design as deployable is a curve where all the points will satisfy the convergence between the two initial ellipsoids and will belong to the surface that we want to design as deployable.

Likewise, if we find a point that satisfies the convergence using the Cuenca, Temmerman or Escrig method this point will belong to the convergence surface. Consequently, the designer directly obtains the set of points of the surface that satisfies the convergence using the method only once. The goal of the designer is to choose one point of this set of points.

Also, the method is not an iterative method and it has an easy automation using a software.

Another novelty that has been developed in this paper is the use of the ℓ parameter (if $\alpha = 0$) and the "v" parameter (if $\alpha \neq 0$). Previously, the scissors were always drawn with the middle point between the extreme points in the surface. This situation means that the shape of a membrane over the structure will not be the shape that we want to design as deployable.

The use of these parameters solves this problem allowing a coincidence between the membrane shape and the original design shape.

7. Acknowledgements

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