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The convergence surface method for the design of deployable scissor structures

Abstract

In this paper, the operability of the most recent method to design bistable and non-bistable deployable scissors structures (the method of the convergence surface) is extended and this operability will be divided into two types of formulas: The exact formula and the approximate formulas. The exact formula involves the obtaining of the convergence surface using its own equations and this paper will prove that this surface is a triaxial two-leaf hyperboloid (for translational units) and a non-standard surface (for polar units). On the other hand, the approximate formulas are designed due to the need to obtain the convergence surface when the exact formula cannot be used. Finally, this research will demonstrate the potential of these approximate strategies to compete against the exact formulas and to boost an improvement in the mathematical results in terms of precision in the scissor design and speed in the calculation process.

Keywords Geometry; Deployable structure; Scissor; Mechanism; Folding; Kinematics; Bistable

Abbreviations

In this paper, the notation "CE" will sometimes be used due to the abbreviation (CE = Convergence ellipsoid).

1. Introduction

Historically, the first author who designed a deployable structure with straight scissors was Leonardo Da Vinci [1]. This inventor developed a scissor system that was controlled using a screw and it was used to lift a weight (Fig. 1). Although this structure was quite simple, this was the first deployable structure with straight scissors and to design this mechanism Leonardo used design concepts of symmetry or geometric compatibility.

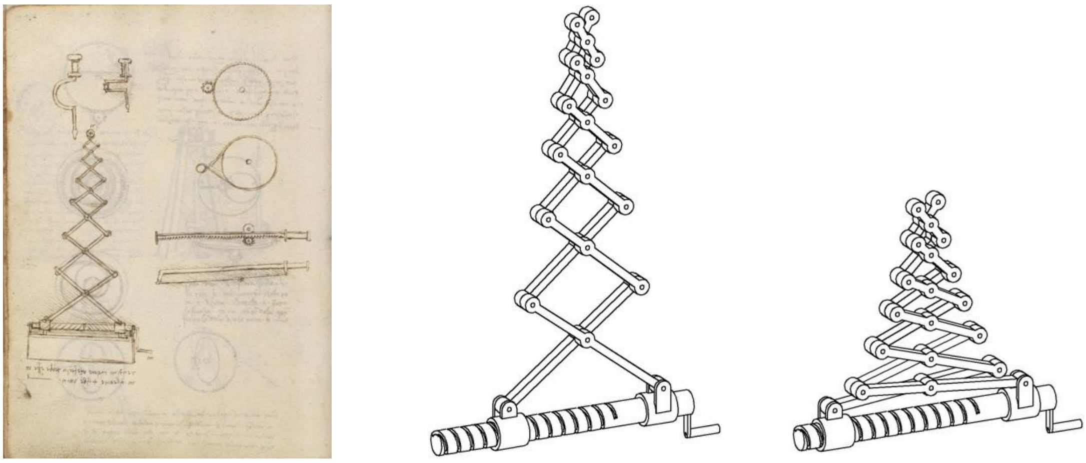

Fig. 1. Original drawings of the Leonardo Da Vinci design and three dimensional perspective of the mechanism [1].

Many centuries later, Emilio Pérez Piñero (1935 – 1972) revolutionized the realm of the deployable systems with his system of “Triaspas and Tetraspas” (Three-scissors Four-scissors) and [2] [3] and with such innovative projects as the famous “Teatro Ambulante” (Mobile Theater) [4] [5] [6] [7] or the order done by Salvador Dalí to design the “Vidriera Hipercúbica” (Hypercubic Stained Glass) (Fig. 2).

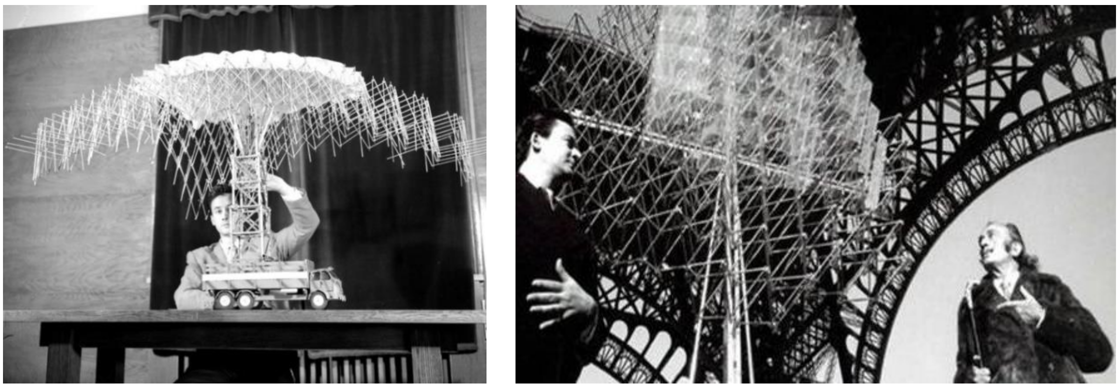

Fig. 2. (Left) Deployable structure with a doublé curvature for a Mobile Theater (Right) Emilio Pérez Piñero presenting the Hypercubic Stained Glass to Salvador Dalí.

The difference between the Three-scissors and the Four-scissors in comparison with the straight scissors is quite considerable. However, Emilio Pérez Piñero designed an effective design method that allowed him to create deployable structures in a relative short time period (without using any computer tools).

Some years later, in the 1990s, a new generation of authors began the extension of this method with the creation of new technologies: The straight scissors, whose main designer was Félix Escrig Pallarés (1950 - 2013) [8] [9] [10], and the angulated scissors, whose main designer was Chuck Hoberman (1956) [11] [12] (The study of the angulated scissors is beyond the topic of this paper).

The design methods that were later developed using the studies of Félix Escrig have the goal of obtaining a compactness level of 100%. This situation means that in the next scissors structure (Fig. 3):

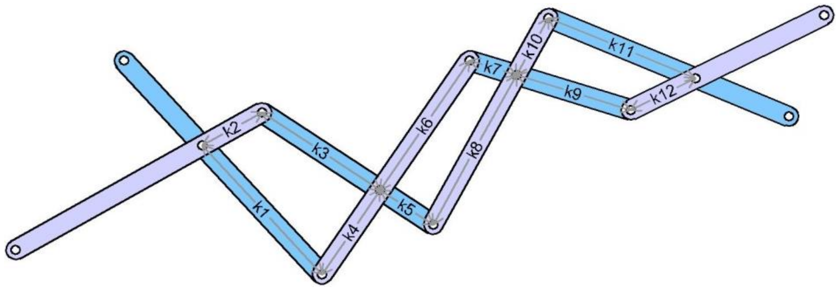

Fig. 3. Deployable structure composed with some straight scissors.

The following equations must be satisfied:

$$k_1 + k_2 = k_3 + k_4 \tag{1}$$

$$k_5 + k_6 = k_7 + k_8 \tag{2}$$

$$k_9 + k_{10} = k_{11} + k_{12} \tag{3}$$

This condition can be summarized in:

$$k_{i1} + k_{i2} = k_{j1} + k_{j2} \tag{4}$$

Another important aspect of this kind of mechanisms is the degrees of freedoms. The deployable structure will have 1 degree of freedom if the joints are defined using a point. However, if joints are defined with a displacement of each scissor the number of degrees of freedoms will be infinite (this situation could be avoid by blocking some axes of rotations). The study of the degrees of freedoms has been developed by Alexey Fomin in [13] and [14].

It is important to highlight that the previous information is only focused on the fulfilment of geometric conditions to guarantee a full deployment process. However, a different way to study this type of structures is from the energy point of view and, in consequence, two groups can be identified: bistable and non-bistable deployable structures. Basically, the main difference between both is the existence of geometric incompatibilities during the deployment process:

- Bistable structure: the rods are going to have elongations and reductions of the length due to a geometric incompatibility and, in consequence, the structure will accumulate energy of deformation during a part of the deployment process. This energy of deformation will be null in two positions of the deployment process (the folded position and the unfolded position) because the rods will not have variations of the length in these two cases. An example of a bistable deployable structure with the evolution of the energy of deformation has been represented in Figure 4 and more information about the optimisation of these structures can be found in [15] [16].

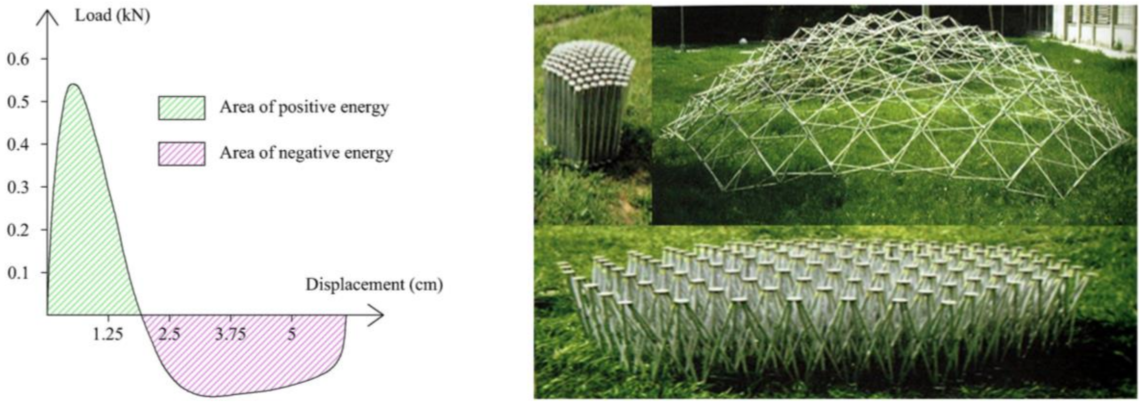

Fig. 4. (Left) Evolution of the energy of deformation in a spherical deployable structure (Right) Deployment process.

- Non-bistable structure: the rods will not have a deformation of the length during the deployment process and, in consequence, the energy of deformation will be null if the weight of the structure is not considered.

Thereby, a design method to create these deployable structures (bistable and non bidtables) is required. Firstly, this research will develop a brief description of the most important methods to obtain deployable structures with straight scissors and the advantages and disadvantages of each one will be underlined. This section will be called as "Previous methods to design deployable scissor structures".

The last method that will be introduced in this section is the Method of the convergence surface, which is not only quite useful but also complex. The research works that have been proposed previously only develop basic equations and general ideas about the practical applications but they don't consider important factors such as the automation of the method using an algorithm or the obtaining of approximate solutions where the loss of resolution is not important.

To improve and to enhance this situation, this research will develop the whole formulas that guarantee the operability and the practical use of the method of the convergence surface. These formulas can be divided into two groups:

- Exact formula: This process is based on the obtaining of the convergence surface equation. The paper will demonstrate that this equation is a two-leaf hyperboloid in case of translational units and there is not a standard geometric shape in case of polar units. The exact formula is useful when the designer wants a high level in the resolution of the scissor length.

- Approximate formulas: This section is the most important contribution of this research. The exact formula is important from a curious point of view but a calculation program must do a huge work, in terms of calculation, to obtain the convergence surface. The goal of the approximate formulas is to reduce and to simplify the obtaining of the convergence surface in exchange of a little loss of resolution. This research will develop two approximate formulas, the first one is focused on the resolution of a determinant and the second one is based on the creation of the convergence surface using level curves.

The use of these formulas allows the immediate application of external programs to obtain the convergence surface in a short time. Once this surface is created, all geometric solutions in the space that satisfy Eq. (4) are available.

2. Previous methods to design deployable scissor structures

2.1. Method of the spheres (1990) (Félix Escrig Pallarés, Jose Sánchez Sánchez and Juan Pérez Valcárcel):

This method [17] [18] is based on the assumption that the end of each scissor has an associated sphere (with radius r) with a center at the midpoint of its focal length (focal length = c). The cut point of the scissors will be in the tangency between two spheres. In the case of a cylindrical surface, it is possible to work with circles instead of spheres due to the parallelism between planes (Fig. 5).

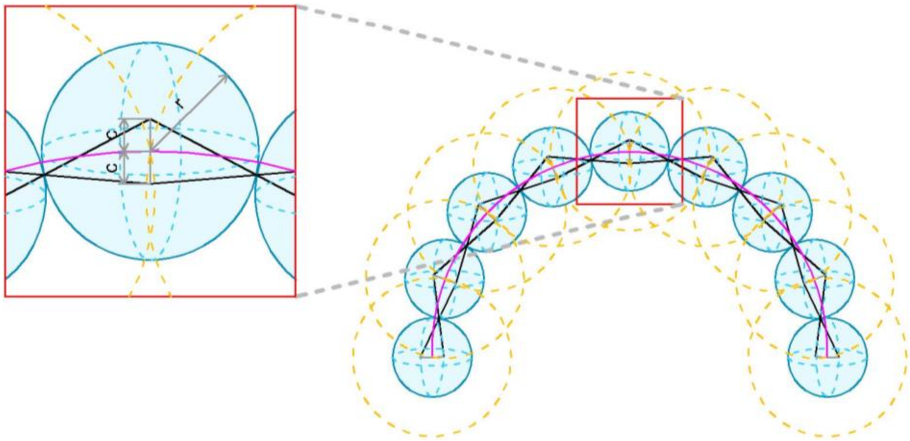

Fig. 5. Deployable structure of a circumference arc with the spheres method (Magenta curve = Curve to design as deployable; Orange curve = Convergence curve; Black lines = Scissors).

a) Advantages of the method:

- The value of the focal distance is independent of the value of the radius of the sphere.
- It is a very simple method because the designer only need to copy and paste the same spherical module and the focal distance of each scissor will be in the intersection point between the spherical modules and the original surface (surface that is going to be converted to a scissor mechanism).
- There is little previous knowledge required to apply this method because the designer only needs the commands “sphere”, “intersect”, “copy-paste” and “line” from a graphic program. Therefore, people who have never designed a deployable structure should begin with this method.

b) Disadvantages of the method:

- The sphere and the focal distance associated with the end of each rod must be the same in the complete structure. Otherwise, the structure will not fold entirely.
- It can only be applied in the following cases (the use of this method on any other surface not mentioned below will cause a non-complete deployment process):
    * Flat surfaces.
    * Straight extrusion surfaces with a circular guideline (cylinders with a circular base and with the focal distance oriented to the center of each scissor plane).
    * Spherical surfaces (With the focal distance oriented to the center of the sphere).
- The control of the relative position between the structure and the surface that is going to be designed as deployable is not allowed because the middle point of the focal distance will always be in this surface.

- The position of a scissor in a singular point (boundaries of the surface, supports of the surface, etc.) is not allowed and, in consequence, the tessellation cannot be regulated because for each pair of blades there is a unique mathematical solution.

2.2. Method of constant ellipsoids (1996) (Luis Sánchez Cuenca):

The constant ellipsoid method [19] [20] has been created as a consequence of the limited cases where the sphere method can be used. This new method considers that the end of each scissor has an associated ellipsoid with a circular revolution (a = major axis, b = minor axis, c = focal distance), where the focal distance of the ellipsoid coincides with the focal distance of the end of the scissor. To get a complete deployable process of the structure, the cross point between the two rods of each scissor must belong to the surface of the ellipsoid. This method uses the property of the ellipse (Eq. (5)) (Fig. 6) to satisfy Eq. (4).

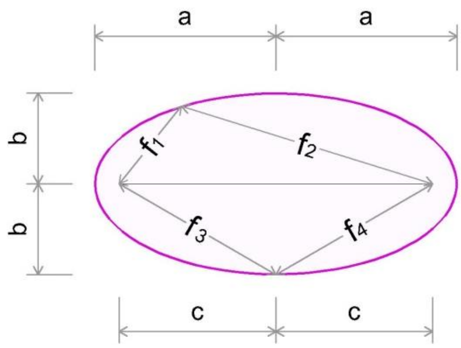

Fig. 6. Fundamental property of the ellipse.

$$f_1 + f_2 = f_3 + f_4 = 2 \cdot a \tag{5}$$

The ellipsoids must be constant because Luis Sánchez Cuenca only used the same ellipsoid which was repeated on the entire surface. An example of an application to a curve can be seen in Fig. 7 (in the case of a surface, the method is used with the same strategy but using two ellipsoids simultaneously).

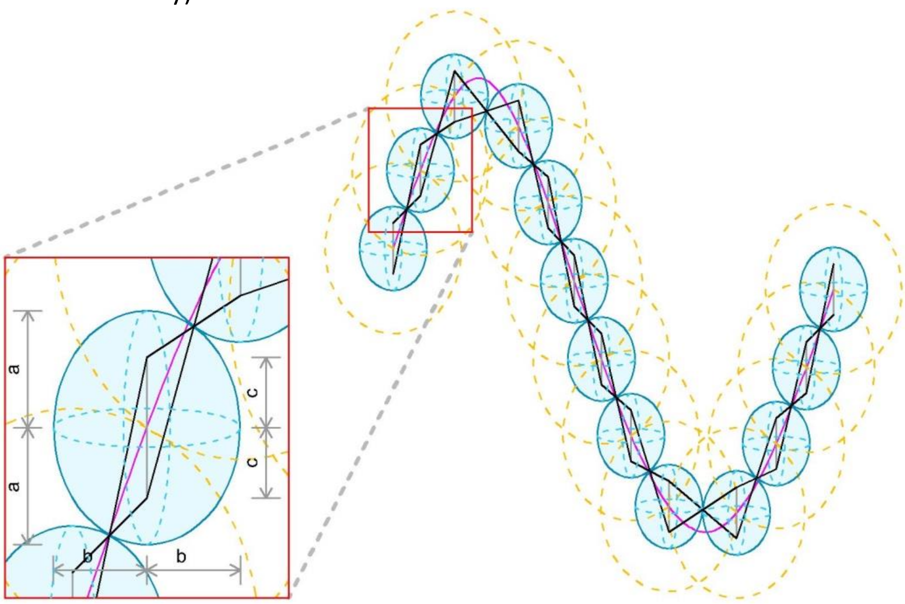

Fig. 7. Deployable structure of a random curve with the constant ellipsoids method (Magenta curve = Curve to design as deployable; Orange curve = Convergence curve; Black lines = Scissors).

a) Advantages of the method:

- It can be applied to any surface because the non-constant curvature of the ellipsoid allows the adaptation on any shape.
- As in the previous case, it is a very simple method to use: just copy and paste the same ellipsoid and the middle point of each scissor will be in the intersection points.
- Little previous knowledge is required to apply this method. The disadvantage that can be found here in comparison with the previous method is the use of ellipsoids instead of spheres (There are design programs where the command ellipsoid is not available).

b) Disadvantages of the method:

- The ellipsoid associated with the end of each rod must be the same in the complete structure. Otherwise, the structure will not fold entirely.
- The position of a scissor in a singular point (boundaries of the surface, supports of the surface, etc.) is not allowed and, in consequence, the tessellation cannot be regulated because for each pair of blades there is a unique mathematical solution.
- The control of the relative position between the structure and the surface that is going to be designed as deployable is not allowed because the middle point of the focal distance will always be in this surface.
- Only translational units can be used (the focal distance of all ellipsoids is parallel) and, in consequence, the size of the structure in the folded position will be bigger in comparison with the use of polar units.

2.3. Method of proportional ellipsoids (2017) (Niels De Temmerman and Kelvin Roovers):

An extension of the constant ellipsoid method is the proportional ellipsoid method [21] [22] [23]. With this new design system, the tessellation of the structure can be regulated by the designer (different mathematical solutions are obtained when changing the proportionality constant). Like its predecessor, this method can only be used for translational units. The reason for this situation is, as is going to be demonstrated later in the convergence surface method, if the units are translational it is mandatory that all the ellipsoids have to be proportional. An example of an application to a curve can be seen in Fig. 8 (in the case of a surface, the method is used with the same strategy but using two ellipsoids simultaneously).

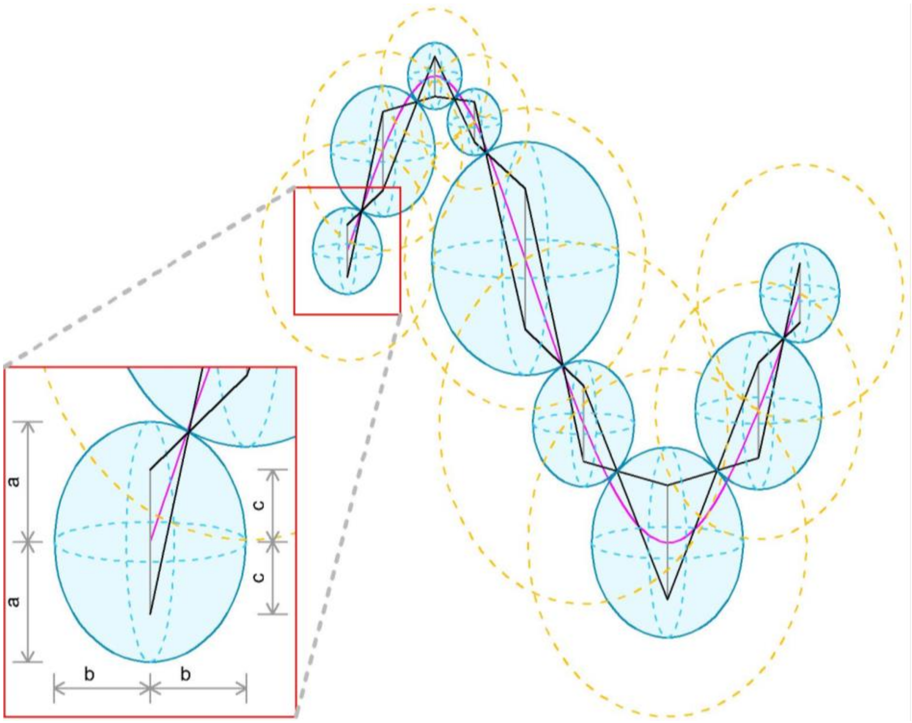

Fig. 8. Deployable structure of a random curve with the proportional ellipsoids method (Magenta curve = Curve to design as deployable; Orange curve = Convergence curve; Black lines = Scissors).

a) Advantages of the method:

- It can be applied to any surface because the non-constant curvature of the ellipsoid allows the adaptation on any shape.
- It is a very simple method to use. As in the previous cases, the designer has to copy and paste an ellipsoid but having in mind that he/she can change the size of this ellipsoid using a constant of proportionality.
- Little previous knowledge is required to apply this method. As in the previous case, the graphic program must have the command "ellipsoid" to use this method.
- The tessellation can be controlled: the variation of the constant of proportionality allows the change of the size of the ellipsoid and, in consequence, more than one mathematical solution can be obtained for one input.

b) Disadvantages of the method:

- The control of the relative position between the structure and the surface that is going to be designed as deployable is not allowed because the middle point of the focal distance will always be in this surface.
- Only translational units can be used (the focal distance of all ellipsoids is parallel) and, in consequence, the size of the structure in the folded position will be bigger in comparison with the use of polar units.
- The position of a scissor in a singular point (boundaries of the surface, supports of the surface, etc.) must be done using an iterative process and the final solution will never belong to the boundary of the surface, the support of the surface, etc. The reason for this situation is that this method only gives a unique mathematical solution for each proportionality constant.

2.4. Method of the convergence surface (2019) (Carlos José García Mora):

The convergence surface method [24] is the maximum generalization of the previous methods because it considers all solutions of the previous methods and the solutions that the previous methods are not able to calculate. This method allows the determination of the surface where each one of its points will have an associated focal distance value. This value will allow the satisfaction of (Eq. (4)) simultaneously between two ellipsoids in space. If a point that does not belong to this surface is chosen, there will be no focal distance value associated with this point that satisfies (Eq. (4)) simultaneously between two ellipsoids in space. Consequently, two ellipsoids in space will always define a unique convergence surface.

This method establishes that given an ellipsoid in the space, this ellipsoid will have associated a family of curves with the shape of ellipsoids proportional to the initial ellipsoid (convergence ellipsoids), where each one of these convergence ellipsoids will have associated a value of the focal distance of the final ellipsoid (for an ellipsoid of convergence, the focal distance of any of its points has the same value). In the case of spatial deployable structures, the design process is always done simultaneously between two ellipsoids in the space (ellipsoid 11 and ellipsoid 12). Consequently, the equation of the family of convergence ellipsoids for ellipsoid 11 and for ellipsoid 12 is Eq. (6) and Eq. (7) [24] (Fig. 9).

$$For\ ellipsoid\ 11 \rightarrow \left(\frac{z_{11} - f_{11}}{f_{21}}\right)^2 + \left(\frac{x_{11}}{f_{31}}\right)^2 + \left(\frac{y_{11}}{f_{31}}\right)^2 = 1 \quad (6)$$

$$For\ ellipsoid\ 12 \rightarrow \left(\frac{z_{12} - f_{12}}{f_{22}}\right)^2 + \left(\frac{x_{12}}{f_{32}}\right)^2 + \left(\frac{y_{12}}{f_{32}}\right)^2 = 1 \quad (7)$$

Where:

$$f_{11} = -\frac{h_1 \cdot u^2 + c_{11} \cdot u - h_1 \cdot v - h_1 \cdot v^2}{(1+v)^2 - u^2} \quad and \quad f_{12} = -\frac{h_2 \cdot u^2 + c_{12} \cdot u - h_2 \cdot v - h_2 \cdot v^2}{(1+v)^2 - u^2} \quad (8)$$

$$f_{21} = \left[\frac{a_{11}}{(1+v)^2 - u^2}\right] \cdot \left[1 + v + \frac{u \cdot h_1}{c_{11}}\right] \quad and \quad f_{22} = \left[\frac{a_{12}}{(1+v)^2 - u^2}\right] \cdot \left[1 + v + \frac{u \cdot h_2}{c_{12}}\right] \quad (9)$$

$$f_{31} = \left[\frac{b_{11}}{(1+v)^2 - u^2}\right] \cdot \left[1 + v + \frac{u \cdot h_1}{c_{11}}\right] \quad and \quad f_{32} = \left[\frac{b_{12}}{(1+v)^2 - u^2}\right] \cdot \left[1 + v + \frac{u \cdot h_2}{c_{12}}\right] \quad (10)$$

$$u = \frac{c_2}{n} \quad and \quad v = \frac{\ell}{n} \quad (11)$$

The figure that is associated with Eq. (6) and Eq. (7) is Fig. 9.

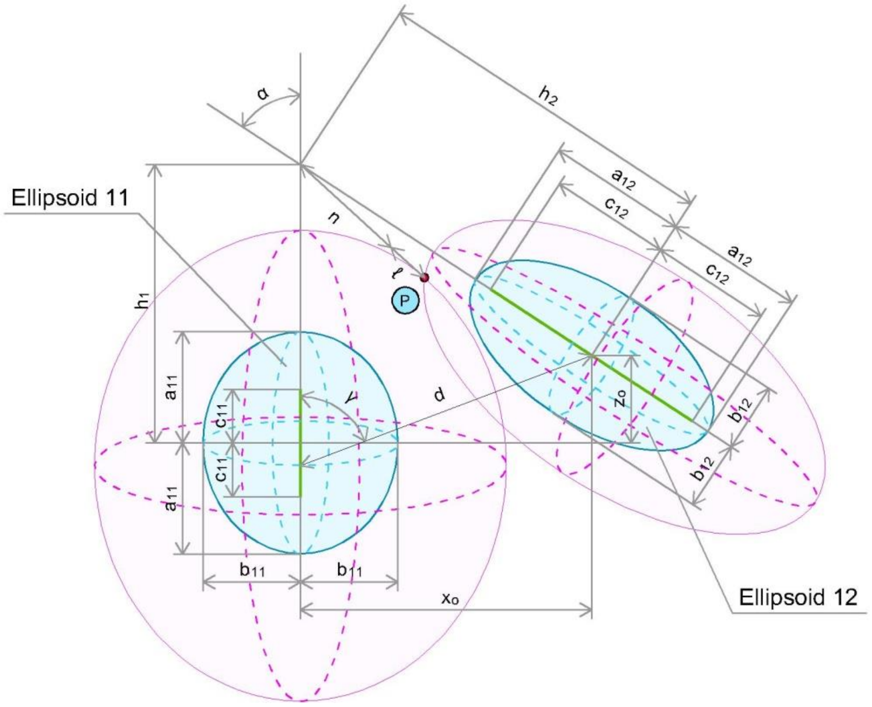

Fig. 9. Two ellipsoids in the space and their respective convergence ellipsoids for a value of the parameter "u".

Where:

- a_{11} and a_{12} = These variables are the semimajor axes of the initial ellipses.
- b_{11} and b_{12} = These variables are the semiminor axes of the initial ellipses.
- c_{11} and c_{12} = These variables are the focal distances of the initial ellipses.
- h_1 and h_2 = Distance from the center of the initial ellipsoids to the cross point of their axes.
- c_2 = Focal distance of the final ellipsoid.
- u = Parameter to iterate. For each value of "u", a different convergence ellipsoid with a value of c_2 is obtained.
- v = Parameter of position of the deployable structure with respect to the surface that is going to be designed as deployable. Controlling this parameter, the surface will be at the middle points of all scissors, at the top points of all scissors or at the bottom points of all scissors.
- α = Parameter of orientation between both ellipsoids. For α = 0 the situation is translational units (Axes of both ellipsoids are parallel) and for α = 0 the situation is polar units (Axes of both ellipsoids are not parallel).

It is important to mention that the use of the parameters "u" and "v" is more complex. More information about these variables is in [24].

Likewise, this method develops the relationship that two ellipsoids in the space must satisfy (irrespective of their orientations) for the convergence surface to be able to exist. This relationship is:

$$t_1^2 \cdot t_3 + t_4 \cdot t_5^2 + t_1 \cdot t_2 \cdot t_5 + 4 \cdot t_3 \cdot t_4 \cdot t_6 = t_6 \cdot t_2^2 \quad (12)$$

Where:

$$t_1 = 2 \cdot z_o \cdot \left[1 - \left(\frac{c_{12}}{a_{12}} \cdot \cos\alpha\right)^2\right] + \left(\frac{c_{12}}{a_{12}}\right)^2 \cdot x_o \cdot \sin(2 \cdot \alpha) \quad (13)$$

$$t_2 = -\left(\frac{c_{12}}{a_{12}}\right)^2 \cdot sin(2 \cdot \alpha) \quad (14)$$

$$t_3 = \left(\frac{c_{12}}{a_{12}} \cdot sin\,\alpha\right)^2 \quad (15)$$

$$t_4 = -\left[1 - \left(\frac{c_{12}}{a_{12}} \cdot cos\,\alpha\right)^2 - \left(\frac{b_{11}}{a_{11}}\right)^2\right] \quad (16)$$

$$t_5 = -\left[2 \cdot x_o \cdot \left[1 - \left(\frac{c_{12}}{a_{12}} \cdot sin\,\alpha\right)^2\right] + \left(\frac{c_{12}}{a_{12}}\right)^2 \cdot z_o \cdot sin(2 \cdot \alpha)\right] \quad (17)$$

$$t_6 = b_{11}^2 - a_{12}^2 + c_{12}^2 + (1 - t_3) \cdot x_o^2 + \left[\left(\frac{b_{11}}{a_{11}}\right)^2 - t_4\right] \cdot z_o^2 - x_o \cdot z_o \cdot t_2 \quad (18)$$

If in Eq. (12) the condition: α = 0 is satisfied (the axes of the ellipsoids are parallel= Translational units), the condition of proportionality between the ellipsoids is obtained (Eq. (19)):

$$\frac{a_{11}}{a_{12}} = \frac{b_{11}}{b_{12}} = \frac{c_{11}}{c_{12}} \quad (19)$$

Consequently, if there are infinite solutions of scissors for each ellipsoid of convergence where Eq. (4) is satisfied with the same value of c_2, the intersection between the convergence ellipsoids with the same value of "u" will give us a set of convergence curves for each value of "u". The surface that is composed with these curves is the convergence surface.

The intersection of the convergence surface with the surface that is going to be designed as deployable will give a curve that belongs to both surfaces.

Any point on this curve will give a scissor that satisfies Eq. (4) simultaneously between the initial ellipsoids and that will also belong to the surface that is going to be designed as deployable. In this paper, some application examples are going to be developed.

a) Advantages of the method:

- It can be applied to any surface because the non-constant curvature of the ellipsoid allows the adaptation on any shape.
- The tessellation can be controlled. This method gives all mathematical solutions in the space that satisfy Eq. (4) and, in consequence, the designer can choose the scissor that fits better in function of boundary conditions, behaviour of the structure, etc.
- The relative position between the structure and the surface that is going to be designed as deployable can be controlled because this method introduces the use of a new parameter: “ℓ” for translational units and “v” for polar units. This variable allows changing the position of scissors with respect of the surface that is going to be designed as deployable and, consequently, all superior or inferior points of scissors can belong to the surface instead of the middle point of the focal distance. More information about the use of these parameters can be found in [24].

- Translational units and polar units can be used. The use of this method guarantees Eq. (4) for any orientation of the ellipsoids and, in consequence, polar units can also be used. The design of a deployable structure with polar units allows a smaller size in the folded position in comparison with the use of translational units.
- The position of the scissor in a singular point (boundaries of the surface, supports of the surface, etc.) is obtained without any iterative process and the final solution will belong exactly to the boundary of the surface, the support of the surface, etc. This is one of the most important advantages of this method because the previous case (Method of proportional ellipsoids) required a huge number of iterations to get the solution and the Method of the convergence surface iterates only one time to get the final solution.

b) Disadvantages of the method:

- The method requires automation to be operational (this operability is going to be developed in this paper) because the most complicate step is the obtaining of the convergence surface. This surface needs a high resolution to be useful and the research developed in [24] does not guarantee enough accuracy. To solve that, this article will propose some formulas that can be easily introduced in a design program to avoid the manual process of creating the convergence surface.
- Considerable previous knowledge is required to use this method. This method gives all possible solution of the space and, in consequence, the designer should have a basic background in the field of deployable structures to know the meaning and the influence of each geometric parameter. Otherwise, the designer will not use the full potential of this Method.

In [24] the mathematical development of this method was defined in a conceptual way. However, the manual creation of the convergence surface is quite tedious and it takes so much time. In this article, two strategies are going to be proposed to solve this situation: the use of an exact formula and the used of approximate formulas.

The goal of the exact formula is to obtain the equation of the convergence surface, but, as the reader will find out in this paper, this equation cannot be always defined. On the other hand, the approximate formulas allow the obtaining of the convergence surface when the exact formula cannot be applied or when the designer does not want to use the analytic equation.

In this paper, two approximate formulas are going to be developed. The first supposes that the convergence surface has the shape of a two-leaf hyperboloid. As will be demonstrated later, this situation only can happen when the designer is using translational units. Although the shape of the convergence surface using polar units is not exactly a two-leaf hyperboloid, the similarity between both can be assumed because the difference is quite small. Authors such as the mathematicians Paul Breiding, Bernd Sturmfels and Sascha Timme propose a similar method in [25].

The second approximate formula is the obtaining of the convergence surface using convergence curves. In [24] it was developed that these curves are obtained with the iteration of the “u” parameter. In this paper, the maximum and minimum values of the “u” parameter for all possible orientations between two ellipsoids in the space are going to be obtained. If the designer uses a value of “u” that is out of this interval, the length of the rods will be negative. To obtain the values, many concepts of tangency between conics and the study of singular points are going to be used [26] [27] [28].

Lastly, it is important to highlight that this paper is focused on the extension of a recent design method based on a geometric and mathematical analysis. This situation means that no finite elements simulations [29] [30] have been running and that the joints are going to be designed as a point and not as a real joint [31] [32] during the whole study.

3. Exact formula of the convergence surface

3.1. For u = 0 (translational units = the axes of the ellipsoids are parallel)

If in Eq. (11) the next condition is satisfied:

$$u = 0 \rightarrow \frac{c_2}{n} = 0 \rightarrow n = \infty \tag{20}$$

Consequently:

$$v = \frac{l}{n} \rightarrow v = \frac{l}{\infty} = 0 \tag{21}$$

Also, if n = ∞ the value of h_1 is going to be ∞, and therefore, n = h_1. Substituting and avoiding the indeterminations of type 0 · ∞:

$$f_{11} = -\frac{h_1 \cdot \left(\frac{c_2}{h_1}\right)^2 + c_{11} \cdot \frac{c_2}{h_1} - h_1 \cdot \frac{l}{h_1} - h_1 \cdot \left(\frac{l}{h_1}\right)^2}{(1+0)^2 - 0^2} = -l \tag{22}$$

$$f_{21} = \left[\frac{a_{11}}{(1+0)^2 - 0^2}\right] \cdot \left[1 + 0 + \frac{\frac{c_2}{h_1} \cdot h_1}{c_{11}}\right] = a_{11} \cdot \left(1 + \frac{c_2}{c_{11}}\right) \tag{23}$$

$$f_{31} = \left[\frac{b_{11}}{(1+0)^2 - 0^2}\right] \cdot \left[1 + 0 + \frac{\frac{c_2}{h_1} \cdot h_1}{c_{11}}\right] = b_{11} \cdot \left(1 + \frac{c_2}{c_{11}}\right) \tag{24}$$

Substituting Eq. (22), Eq. (23) and Eq. (24) in Eq. (6):

$$\left[\frac{z_{11} - l}{a_{11} \cdot \left(1 + \frac{c_2}{c_{11}}\right)}\right]^2 + \left[\frac{x_{11}}{b_{11} \cdot \left(1 + \frac{c_2}{c_{11}}\right)}\right]^2 + \left[\frac{y_{11}}{b_{11} \cdot \left(1 + \frac{c_2}{c_{11}}\right)}\right]^2 = 1 \tag{25}$$

If the same process is done with Eq. (7):

$$\left[\frac{z_{12} - l}{a_{12} \cdot \left(1 + \frac{c_2}{c_{12}}\right)}\right]^2 + \left[\frac{x_{12}}{b_{12} \cdot \left(1 + \frac{c_2}{c_{12}}\right)}\right]^2 + \left[\frac{y_{12}}{b_{12} \cdot \left(1 + \frac{c_2}{c_{12}}\right)}\right]^2 = 1 \tag{26}$$

The next step will be to refer these two equations to the same coordinate system, for example, to the coordinate system of ellipsoid 11. In this case the axes of the ellipsoids are parallel and, consequently, the introduction of a rotation matrix will not be necessary (only a translational matrix is going to be needed) (Fig. 10).

$$x_{12} = x_{11} - x_o \quad with \quad y_{12} = y_{11} \quad and\ with \quad z_{12} = z_{11} - z_o \tag{27}$$

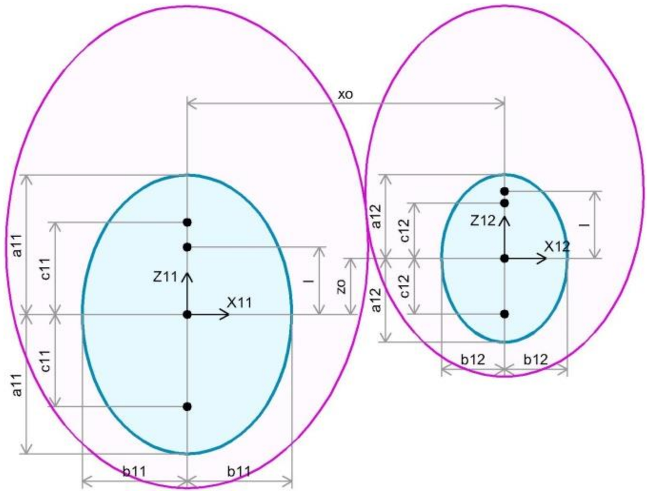

Fig. 10. Minimum geometric convergence situation for two ellipsoids in the space with u = 0.

After this change has been done, Eq. (28) is:

$$\left[\frac{z_{11}-l-z_o}{a_{12}\cdot\left(1+\frac{c_2}{c_{12}}\right)}\right]^2+\left[\frac{x_{11}-x_o}{b_{12}\cdot\left(1+\frac{c_2}{c_{12}}\right)}\right]^2+\left[\frac{y_{11}}{b_{12}\cdot\left(1+\frac{c_2}{c_{12}}\right)}\right]^2=1 \tag{28}$$

Consequently, and in function of the "l" value, the most important situations that can be found are:

3.1.1. l = Constant with l ≠ -c_2 and l ≠ c_2

If c_2 is isolated in Eq. (25) and in Eq. (28):

$$c_2=c_{11}\cdot\left[\sqrt{\left(\frac{z_{11}-l}{a_{11}}\right)^2+\left(\frac{x_{11}}{b_{11}}\right)^2+\left(\frac{y_{11}}{b_{11}}\right)^2}-1\right] \tag{29}$$

$$c_2=c_{12}\cdot\left[\sqrt{\left(\frac{z_{11}-l-z_o}{a_{12}}\right)^2+\left(\frac{x_{11}-x_o}{b_{12}}\right)^2+\left(\frac{y_{11}}{b_{12}}\right)^2}-1\right] \tag{30}$$

In addition, ellipsoids 11 and 12 must be proportional. In consequence:

$$\frac{c_{12}}{c_{11}}=\frac{a_{12}}{a_{11}}=\frac{b_{12}}{b_{11}}=k \tag{31}$$

Then, if Eq. (31) is replaced in Eq. (30) and after that Eq. (29) is equaled with Eq. (30):

$$k_1\cdot x_{11}^2+k_2\cdot y_{11}^2+k_3\cdot z_{11}^2+k_4\cdot x_{11}\cdot z_{11}+k_5\cdot x_{11}+k_6\cdot z_{11}=k_7 \tag{32}$$

Where:

$$k_1=\left[\frac{x_0}{b_{11}^2\cdot(1-k)}\right]^2-\frac{1}{b_{11}^2} \tag{33}$$

$$k_2=-\frac{1}{b_{11}^2} \tag{34}$$

$$k_3 = \left[\frac{z_0}{a_{11}^2 \cdot (1-k)}\right]^2 - \frac{1}{a_{11}^2} \tag{35}$$

$$k_4 = \frac{2 \cdot x_o \cdot z_o}{a_{11}^2 \cdot b_{11}^2 \cdot (1-k)^2} \tag{36}$$

$$k_5 = \frac{2 \cdot x_o}{b_{11}^2} \cdot \left[1 - \frac{\delta}{1-k}\right] \tag{37}$$

$$k_6 = \frac{2}{a_{11}^2} \cdot \left[l + z_o - \frac{\delta \cdot z_o}{1-k}\right] \tag{38}$$

$$k_7 = \left(\frac{x_o}{b_{11}}\right)^2 + \left(\frac{l + z_o}{a_{11}}\right)^2 - \delta^2 \tag{39}$$

Where:

$$\delta = \frac{1}{2 \cdot (1-k)} \cdot \left[\left(\frac{x_o}{b_{11}}\right)^2 + \left(\frac{z_o}{a_{11}}\right)^2 + \frac{2 \cdot l \cdot z_o}{a_{11}^2} + (1-k)^2\right] \tag{40}$$

As can be seen, Eq. (32) is the equation of a revolutionized conic whose axis is not parallel with respect to the axes of the reference system due to the term $x_{11} \cdot z_{11}$

To eliminate this rotation, a change of reference system must be done. The goal is to obtain the value of the angle "η" that guarantees the para parallelism between X'_{11} and the axis of the hyperboloid. Consequently:

$$x_{11} = x'_{11} \cdot cos(\eta) - z'_{11} \cdot sin(\eta) \tag{41}$$

$$z_{11} = x'_{11} \cdot sin(\eta) + z'_{11} \cdot cos(\eta) \tag{42}$$

$$y_{11} = y'_{11} \tag{43}$$

The parallelism between X'_{11} and the axis of the hyperboloid is going to be obtained for the value of "η" that makes: $X'_{11} \cdot Y'_{11} = 0$. Then:

$$-2 \cdot k_1 \cdot cos(\eta) \cdot sin(\eta) - k_4 \cdot sin^2(\eta) + k_4 \cdot cos^2(\eta) + 2 \cdot k_3 \cdot cos(\eta) \cdot sin(\eta) = 0 \tag{44}$$

Using the equations of the double angle for the sine and the cosine:

$$\frac{k_3 - k_1}{k_4} = \frac{sin^2(\eta) - cos^2(\eta)}{2 \cdot sin(\eta) \cdot cos(\eta)} = -\frac{cos(2 \cdot \eta)}{sin(2 \cdot \eta)} = -\frac{1}{tan(2 \cdot \eta)} \tag{45}$$

Consequently:

$$\eta = \frac{1}{2} \cdot tan^{-1}\left(\frac{k_4}{k_1 - k_3}\right) \tag{46}$$

If Eq. (41), Eq. (42), Eq. (43) and Eq. (46) are replaced in Eq. (32):

$$[k_1 \cdot cos^2(\eta) + k_3 \cdot sin^2(\eta) + k_4 \cdot cos(\eta) \cdot sin(\eta)] \cdot x_{11}'^2 + k_2 \cdot y_{11}'^2 + \\ +[k_1 \cdot sin^2(\eta) + k_3 \cdot cos^2(\eta) - k_4 \cdot cos(\eta) \cdot sin(\eta)] \cdot z_{11}'^2 + \\ +[k_5 \cdot cos(\eta) + k_6 \cdot sin(\eta)] \cdot x_{11}' + [k_6 \cdot cos(\eta) - k_5 \cdot sin(\eta)] \cdot z_{11}' = k_7 \quad (47)$$

The next step is to balance Eq. (47). If this process is done using the isolation of two auxiliary variables and these variables are added to Eq. (47) in a square form, the final equation is obtained: Eq. 51

$$\left[\frac{x_{11}' - d_1}{d_2}\right]^2 - \left[\frac{y_{11}'}{d_3}\right]^2 - \left[\frac{z_{11}' - d_4}{d_5}\right]^2 = 1 \quad (48)$$

Where:

$$d_1 = -\frac{k_5 \cdot cos(\eta) + k_6 \cdot sin(\eta)}{2 \cdot [k_1 \cdot cos^2(\eta) + k_3 \cdot sin^2(\eta) + k_4 \cdot cos(\eta) \cdot sin(\eta)]} \quad (49)$$

$$d_2 = \sqrt{\begin{array}{c} \left[\frac{1}{k_1 \cdot cos^2(\eta) + k_3 \cdot sin^2(\eta) + k_4 \cdot cos(\eta) \cdot sin(\eta)}\right] \cdot \\ \cdot \left[k_7 + \frac{[k_5 \cdot cos(\eta) + k_6 \cdot sin(\eta)]^2}{4 \cdot [k_1 \cdot cos^2(\eta) + k_3 \cdot sin^2(\eta) + k_4 \cdot cos(\eta) \cdot sin(\eta)]} + \right. \\ \left. + \frac{[k_6 \cdot cos(\eta) - k_5 \cdot sin(\eta)]^2}{4 \cdot [k_1 \cdot sin^2(\eta) + k_3 \cdot cos^2(\eta) - k_4 \cdot cos(\eta) \cdot sin(\eta)]}\right] \end{array}} \quad (50)$$

$$d_3 = \sqrt{\begin{array}{c} -\frac{1}{k_2} \cdot \left[k_7 + \frac{[k_5 \cdot cos(\eta) + k_6 \cdot sin(\eta)]^2}{4 \cdot [k_1 \cdot cos^2(\eta) + k_3 \cdot sin^2(\eta) + k_4 \cdot cos(\eta) \cdot sin(\eta)]} + \right. \\ \left. + \frac{[k_6 \cdot cos(\eta) - k_5 \cdot sin(\eta)]^2}{4 \cdot [k_1 \cdot sin^2(\eta) + k_3 \cdot cos^2(\eta) - k_4 \cdot cos(\eta) \cdot sin(\eta)]}\right] \end{array}} \quad (51)$$

$$d_4 = -\frac{k_6 \cdot cos(\eta) - k_5 \cdot sin(\eta)}{2 \cdot [k_1 \cdot sin^2(\eta) + k_3 \cdot cos^2(\eta) - k_4 \cdot cos(\eta) \cdot sin(\eta)]} \quad (52)$$

$$d_5 = \sqrt{\begin{array}{c} -\left[\frac{1}{k_1 \cdot sin^2(\eta) + k_3 \cdot cos^2(\eta) - k_4 \cdot cos(\eta) \cdot sin(\eta)}\right] \cdot \\ \cdot \left[k_7 + \frac{[k_5 \cdot cos(\eta) + k_6 \cdot sin(\eta)]^2}{4 \cdot [k_1 \cdot cos^2(\eta) + k_3 \cdot sin^2(\eta) + k_4 \cdot cos(\eta) \cdot sin(\eta)]} + \right. \\ \left. + \frac{[k_6 \cdot cos(\eta) - k_5 \cdot sin(\eta)]^2}{4 \cdot [k_1 \cdot sin^2(\eta) + k_3 \cdot cos^2(\eta) - k_4 \cdot cos(\eta) \cdot sin(\eta)]}\right] \end{array}} \quad (53)$$

Eq. (48) is the equation of a triaxial two-leaf hyperboloid. Likewise, if "l" were 0, the axis of the convergence surface would pass through the midpoint of the line that joins the centers of the two ellipsoids and the vertex of this hyperboloid would be in this point.

Another important aspect is that the “η” angle is an angle with orientation. Two situations can be defined:

- If the “η” value is positive, the orientation of this angle is counter-clockwise, taking the X_{11} axis as the origin of the angle and taking the X'_{11} axis as the end of the angle.
- If the “η” value is negative, the orientation of this angle is clockwise, taking the X_{11} axis as the origin of the angle and taking the X'_{11} axis as the end of the angle.

3.1.2. $l = -c_2$

Eq. (32) will be obtained if the previous process is repeated with this value of “l” parameter, but, in this case, the values of k_i are going to have the following equations:

$$k_1 = \left[\frac{a_{11} \cdot x_o}{a_{11}^2 \cdot (k-1) + c_{11} \cdot z_o}\right]^2 - 1 \tag{54}$$

$$k_2 = -1 \tag{55}$$

$$k_3 = \left[\frac{a_{11} \cdot [c_{11} \cdot (k-1) + z_o]}{a_{11}^2 \cdot (k-1) + c_{11} \cdot z_o}\right]^2 - 1 \tag{56}$$

$$k_4 = \frac{2 \cdot x_o \cdot a_{11}^2 \cdot [c_{11} \cdot (k-1) + z_o]}{[a_{11}^2 \cdot (k-1) + c_{11} \cdot z_o]^2} \tag{57}$$

$$k_5 = 2 \cdot x_o \cdot \left[1 - \frac{a_{11} \cdot \delta}{a_{11}^2 \cdot (k-1) + c_{11} \cdot z_o}\right] \tag{58}$$

$$k_6 = 2 \cdot \left[z_o + k \cdot c_{11} - \frac{a_{11} \cdot \delta \cdot [c_{11} \cdot (k-1) + z_o]}{a_{11}^2 \cdot (k-1) + c_{11} \cdot z_o}\right] \tag{59}$$

$$k_7 = x_o^2 + (z_o + k \cdot c_{11})^2 - \delta^2 \tag{60}$$

Where:

$$\delta = \frac{a_{11} \cdot [x_o^2 - c_{11}^2 + (z_o + k \cdot c_{11})^2]}{2 \cdot [a_{11}^2 \cdot (k-1) + c_{11} \cdot z_o]} + \frac{1}{2} \cdot \left[a_{11} \cdot (k-1) + \frac{c_{11} \cdot z_o}{a_{11}}\right] \tag{61}$$

3.1.3. $l = c_2$

Doing the same steps that have been developed in the previous section, the values of k_i will be:

$$k_1 = \left[\frac{a_{11} \cdot x_o}{a_{11}^2 \cdot (k-1) - c_{11} \cdot z_o}\right]^2 - 1 \tag{62}$$

$$k_2 = -1 \tag{63}$$

$$k_3 = \left[\frac{a_{11} \cdot [c_{11} \cdot (k-1) - z_o]}{a_{11}^2 \cdot (k-1) - c_{11} \cdot z_o}\right]^2 - 1 \tag{64}$$

$$k_4 = -\frac{2 \cdot x_o \cdot a_{11}^2 \cdot [c_{11} \cdot (k-1) - z_o]}{[a_{11}^2 \cdot (k-1) - c_{11} \cdot z_o]^2} \tag{65}$$

$$k_5 = 2 \cdot x_o \cdot \left[1 + \frac{a_{11} \cdot \delta}{a_{11}^2 \cdot (k-1) - c_{11} \cdot z_o}\right] \tag{66}$$

$$k_6 = 2 \cdot \left[z_o - k \cdot c_{11} - \frac{a_{11} \cdot \delta \cdot [c_{11} \cdot (k-1) - z_o]}{a_{11}^2 \cdot (k-1) - c_{11} \cdot z_o}\right] \quad (67)$$

$$k_7 = x_o^2 + (z_o - k \cdot c_{11})^2 - \delta^2 \quad (68)$$

Where:

$$\delta = \frac{a_{11} \cdot [x_o^2 - c_{11}^2 + (z_o - k \cdot c_{11})^2]}{2 \cdot [c_{11} \cdot z_o - a_{11}^2 \cdot (k-1)]} + \frac{1}{2} \cdot \left[\frac{c_{11} \cdot z_o}{a_{11}} - a_{11} \cdot (k-1)\right] \quad (69)$$

3.2. For u ≠ 0 (The axes of all the ellipsoids are not parallel)

If in Eq. (6) the “u” variable is isolated, a full 4-degree equation is obtained. Only one of these 4 values is going to be the correct one. To get this correct value the Descartes Method is going to be used:

a) The 4º degree term is modified with a multiplication by 1.
b) The 3º degree term is eliminated from the equation with a change of a variable.
c) The previous expression is factored to obtain a 6º degree equation with only even powers.
d) A variable change is made to convert the 6º degree equation from the previous section to a 3º degree grade equation with the full terms.
e) The 3º degree equation is solved using the Cardano Method.
f) With the results of this 3º degree equation, the calculation process must go back to find the solutions of the full 4-degree equation.

Once the value of “u” is isolated, this process must also be done with Eq. (7) in combination with Eq. (27) and, after that, the equation of the convergence surface will be obtained if the previous two equations are equalized. The final result will be:

- A huge equation with an extension of more than 200 pages (this equation is not operative).

- The shape of the convergence surface is similar in comparison with the shape of a triaxial two-leaf hyperboloid: the lower the angle between the ellipsoids (close to translational units), the better this approximation. This is the basis of the approximate formula 1.

4. Approximate formula 1 of the convergence surface: Numerical approximation

The approximate formula 1 can be used in the following cases:

- When the designer is working with translational units but he/she does not want to use the analytic equation.
- When the designer is working with polar units and he/she wants to approximate the convergence surface to a triaxial two-leaf hyperboloid (this is only recommended if the angle between the ellipsoids is small = less than 10°).

The goal of this formula is to obtain 6 random points that belong to the convergence surface using the intersection of convergence ellipsoids with the same "u" value (The relationship between these points cannot be a lineal combination). The next step is the calculation of a determinant to obtain the k_i parameters. Once these parameters are calculated, the values of the di parameters are automatic using Eq. (48), Eq. (49), Eq. (50), Eq. (51), Eq. (52), Eq. (53).

To obtain the value of k_i parameters, a system with 7 equations and 7 variables must be solved: Eq. (70)

$$\begin{pmatrix} x_{11_1}^2 & y_{11_1}^2 & z_{11_1}^2 & x_{11_1}\cdot z_{11_1} & x_{11_1} & z_{11_1} & -1 \\ x_{11_2}^2 & y_{11_2}^2 & z_{11_2}^2 & x_{11_2}\cdot z_{11_2} & x_{11_2} & z_{11_2} & -1 \\ x_{11_3}^2 & y_{11_3}^2 & z_{11_3}^2 & x_{11_3}\cdot z_{11_3} & x_{11_3} & z_{11_3} & -1 \\ x_{11_4}^2 & y_{11_4}^2 & z_{11_4}^2 & x_{11_4}\cdot z_{11_4} & x_{11_4} & z_{11_4} & -1 \\ x_{11_5}^2 & y_{11_5}^2 & z_{11_5}^2 & x_{11_5}\cdot z_{11_5} & x_{11_5} & z_{11_5} & -1 \\ x_{11_6}^2 & y_{11_6}^2 & z_{11_6}^2 & x_{11_6}\cdot z_{11_6} & x_{11_6} & z_{11_6} & -1 \\ x_{11_7}^2 & y_{11_7}^2 & z_{11_7}^2 & x_{11_7}\cdot z_{11_7} & x_{11_7} & z_{11_7} & -1 \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \end{pmatrix} = 0 \qquad (70)$$

However, the previous system just gives the trivial solution. To avoid this situation, the next determinant must be solved: Eq. (71)

$$\begin{vmatrix} x_{11}^2 & y_{11}^2 & z_{11}^2 & x_{11}\cdot z_{11} & x_{11} & z_{11} & 1 \\ x_{11_1}^2 & y_{11_1}^2 & z_{11_1}^2 & x_{11_1}\cdot z_{11_1} & x_{11_1} & z_{11_1} & 1 \\ x_{11_2}^2 & y_{11_2}^2 & z_{11_2}^2 & x_{11_2}\cdot z_{11_2} & x_{11_2} & z_{11_2} & 1 \\ x_{11_3}^2 & y_{11_3}^2 & z_{11_3}^2 & x_{11_3}\cdot z_{11_3} & x_{11_3} & z_{11_3} & 1 \\ x_{11_4}^2 & y_{11_4}^2 & z_{11_4}^2 & x_{11_4}\cdot z_{11_4} & x_{11_4} & z_{11_4} & 1 \\ x_{11_5}^2 & y_{11_5}^2 & z_{11_5}^2 & x_{11_5}\cdot z_{11_5} & x_{11_5} & z_{11_5} & 1 \\ x_{11_6}^2 & y_{11_6}^2 & z_{11_6}^2 & x_{11_6}\cdot z_{11_6} & x_{11_6} & z_{11_6} & 1 \end{vmatrix} = 0 \qquad (71)$$

If the previous determinant is calculated, a polynomial equation is going to be obtained and k_i parameters can be solved using the comparison with Eq. (32).

The last step is to calculate d_i parameters to get the equation of the hyperboloid. With all the terms of the hyperboloid, its geometry can be represented using a graphic program.

5. Approximate formula 2 of the convergence surface: Geometrical approximation

The goal of the approximate formula 2 is to make the proposed method in [24] operative: to find the interval of the "u" parameter where the length of the rods has a positive value.

To achieve this goal, the equation of the convergence ellipsoid 11 is going to be used (the results are the same if the equation of the convergence ellipsoid 12 is used instead of that of ellipsoid 11).

Likewise, if translational units are used, the limits will be those of the c_2 parameter and if polar units are used, the limits will be those of the "u" parameter.

5.1. Maximum value of the iteration interval: Maximum growth of the convergence surface

5.1.1. For u = 0 (the axes of the ellipsoids are parallel or translational units):

The maximum value of c_2 will cause the semi-axes of the convergence ellipsoid 11 to grow to ∞. If this situation is graphed, the following figure is obtained (Fig. 11) where the value of c_2 has been plotted against the semi-major axis of the convergence ellipsoid 11, keeping the values of a_{11}, c_{11} constant.

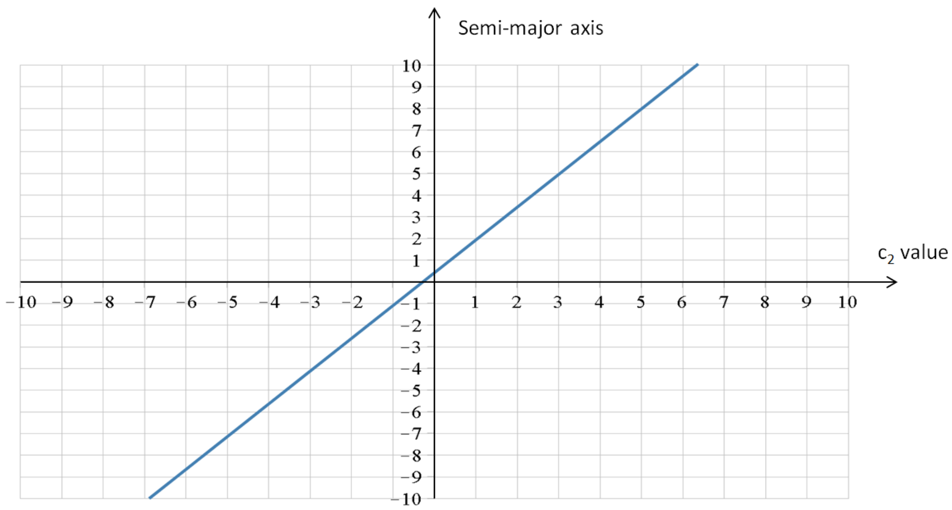

Fig. 11. Evolution of the semi-major axis of the convergence ellipsoid 11 as a function of c_2 value.

Consequently, using this curve the value of c_{2max} can be obtained with Eq. (72):

$$c_2 = c_{2max} \;\; if \lim_{a_{11}\cdot\left(1+\frac{c_2}{c_{11}}\right)\to\infty}\left[a_{11}\cdot\left(1+\frac{c_2}{c_{11}}\right)\right] \;\; and \;\; c_2 = c_{2max} \;\; if \lim_{b_{11}\cdot\left(1+\frac{c_2}{c_{11}}\right)\to\infty}\left[b_{11}\cdot\left(1+\frac{c_2}{c_{11}}\right)\right] \quad (72)$$

This equation implies that:

$$a_{11}\cdot\left(1+\frac{c_{2max}}{c_{11}}\right) = b_{11}\cdot\left(1+\frac{c_{2max}}{c_{11}}\right) = \infty \quad \rightarrow \quad c_{2max} = \infty \quad (73)$$

The reader can notice that this result is valid for -∞ < l < ∞ because the expression of the semi-major axis and the semi-minor axis do not depend on the parameter "l".

Furthermore, the displacement of the convergence ellipsoid 11 for $c_2 = c_{2max}$ will be the value of "l" because this function does not depend on the value of c_2.

5.1.2. For u ≠ 0 (the axes of the ellipsoids are not parallel or polar units)

a) If - ∞ ≤ v ≤ ∞ with v ≠ u and with v ≠ -u

The maximum value of "u" will cause the semi-axes of the convergence ellipsoid 11 to grow to ∞. This can be seen in the following graph where the value of "u" has been plotted against the semi-major axis of the convergence ellipsoid 11, keeping the values of a_{11}, c_{11} and h_1 constant and varying the value of the parameter "v" (Fig. 12).

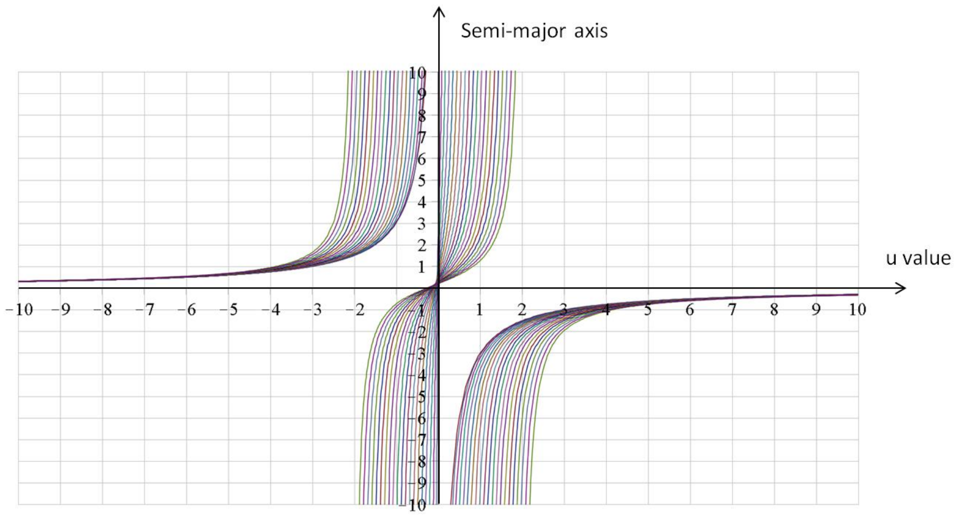

Fig. 12. Evolution of the semi-major axis of the convergence ellipsoid 11 as a function of the value of "u" for - ∞ ≤ v ≤ ∞ with v ≠ u and with v ≠ -u.

Three curves can be differentiated from the previous graph:

- The curve on the left is negligible because "u" can never have negative values.
- The center curve is the only one that can give coherent values of "u" and of the semi-major axis of the ellipsoid.
- The curve on the right is also negligible because its interval is composed of negative values of the semi-major axis of the ellipsoid.

Consequently, the analysis is going to be focused on the center curve. In this curve there is a value of "u" where the axes of the convergence ellipsoid 11 grow to ∞ (This curve is asymptotic). Mathematically this can be represented with the following equation Eq. (74).

$$u = u_{max} \quad si \quad \lim_{f_{21}\to\infty} [f_{21}] = \lim_{f_{31}\to\infty} [f_{31}] \tag{74}$$

Consequently:

$$\left[\frac{a_{11}}{(1+v)^2 - u_{max}^2}\right] \cdot \left[1 + v + \frac{u_{max} \cdot h_1}{c_{11}}\right] = \infty \ and \ \left[\frac{b_{11}}{(1+v)^2 - u_{max}^2}\right] \cdot \left[1 + v + \frac{u_{max} \cdot h_1}{c_{11}}\right] = \infty \tag{75}$$

If the value of u_{max} is isolated:

$$u_{max} = |1 + v| \tag{76}$$

The previous expression must have an absolute value because u_{max} can only have positive values.

The displacement of the convergence ellipsoid 11 for u = u_{max} (f_{11max}) is:

$$f_{11max} = \lim_{\substack{-\infty \leq v \leq \infty \\ u \to |1+v|}} [f_{11}] = \lim_{\substack{-\infty \leq v \leq \infty \\ u \to |1+v|}} \left[-\frac{h_1 \cdot u^2 + c_{11} \cdot u - h_1 \cdot v - h_1 \cdot v^2}{(1+v)^2 - u^2} \right] = \infty \qquad (77)$$

b) If v = -u

As in the previous case, the maximum value of "u" is the value that causes a growth of the semi-axes of the convergence ellipsoids to ∞. The reason of this situation can be seen in the next graph where the value of "u" has been represented against the semi-major axis (Fig. 13).

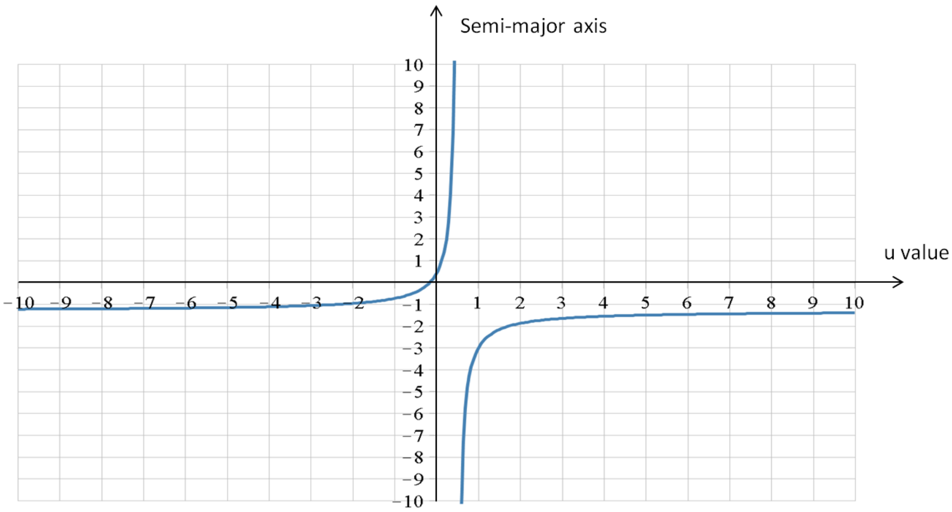

Fig. 13. Evolution of the semi-major axis of the convergence ellipsoid 11 as a function of the value of "u" for v = -u.

Two curves can be differentiated from the previous graph:

- The curve on the left can be divided into two parts: The first is composed of the negative values of "u" and, in consequence, this part of the curve is not acceptable ("u" cannot be negative). The second is composed of the positive values of "u" and, in consequence, this part of the curve is the right one.

- The curve on the right is not acceptable because in its interval the value of the semi-major axis is negative for any value of "u" and this situation is not possible.

Consequently, the analysis will focus on the part of the left curve where there are positive values of "u". Mathematically, the condition of this curve can be represented with the following equation Eq. (78).

$$u_{max} = \lim_{f_{21}(u=u_{max}) \to \infty} [f_{21}(u = u_{max})] = \lim_{f_{31}(u=u_{max}) \to \infty} [f_{31}(u = u_{max})] \qquad (78)$$

Eq. (79) implies that:

$$\left[\frac{a_{11}}{(1 - u_{max})^2 - u_{max}^2} \right] \cdot \left[1 - u_{max} + \frac{u_{max} \cdot h_1}{c_{11}} \right] = \infty \qquad (79)$$

$$\left[\frac{b_{11}}{(1-u_{max})^2-u_{max}^2}\right]\cdot\left[1-u_{max}+\frac{u_{max}\cdot h_1}{c_{11}}\right]=\infty \qquad (80)$$

If u_{max} is isolated:

$$u_{max}=0.5 \qquad (81)$$

The displacement of the convergence ellipsoid 11 for u = u_{max} (f_{11max}) is:

$$f_{11max}=\lim_{u\to 0.5}(f_{11})=\lim_{\substack{v\to -u\\ u\to 0.5}}\left[-\frac{h_1\cdot u^2+c_{11}\cdot u-h_1\cdot v-h_1\cdot v^2}{(1+v)^2-u^2}\right]=\infty \qquad (82)$$

c) If v = u

In this case, the parameter that is used is the maximum value of the semi-axes that causes a growth of “u” parameter to ꝏ. This situation can be observed in the next graph where the value of “u” has been represented against the semi-major axis (Fig. 14).

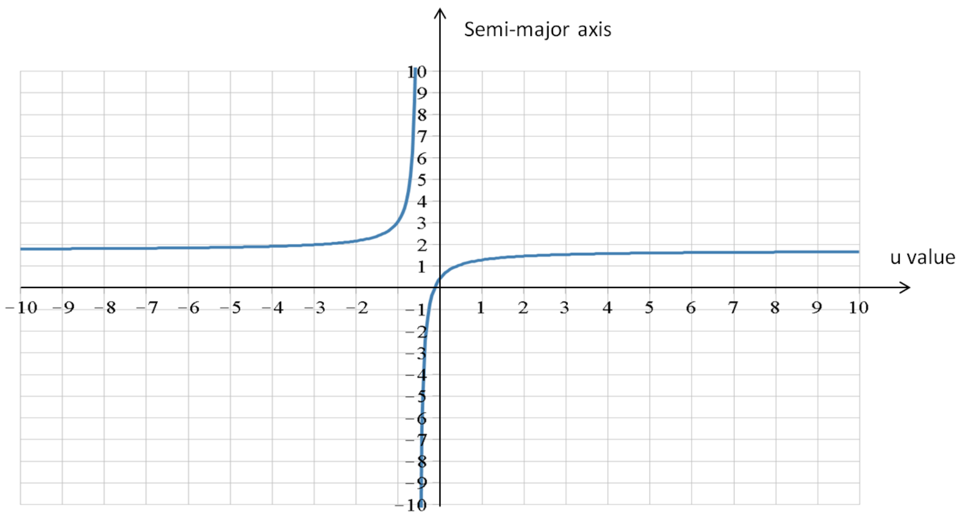

Fig. 14. Evolution of the semi-major axis of the convergence ellipsoid 11 as a function of the value of "u" for v = u.

Two curves can be differentiated from the previous graph:

- The curve on the left is not acceptable because all the values of “u” in this curve are negative and this parameter cannot be negative.

- The curve on the right can be divided into two parts: The first is composed of the negative values of the semi-major axis and, in consequence, this part of the curve is not acceptable (a semi-major axis cannot be negative). The second is composed of the positive values of this parameter and, in consequence, this part of the curve is the right one.

Consequently, the analysis will focus on the part of the right curve where there are positive values of the semi-major axis. Mathematically, the condition of this curve can be represented with the following equations: Eq. (83)

$$a_{max} del\ CE_{11}=\lim_{u\to\infty}[f_{21}(v=u)] \quad and \quad b_{max} del\ CE_{11}=\lim_{u\to\infty}[f_{31}(v=u)] \qquad (83)$$

If a_{max} and b_{max} are isolated:

$$a_{max} del\, CE_{11} = \frac{a_{11}}{2} \cdot \left(1 + \frac{h_1}{c_{11}}\right) \quad and \quad b_{max} del\, CE_{11} = \frac{b_{11}}{2} \cdot \left(1 + \frac{h_1}{c_{11}}\right) \tag{84}$$

In consequence, if v = u, the convergence surface is not going to grow to ∞ because it will have a limit value from which it will no longer grow.

The displacement of the convergence ellipsoid 11 for u = ∞ (f_{11max}) is:

$$f_{11max} = \lim_{\substack{v \to u \\ u \to \infty}} [f_{11}(v = u)] = \lim_{\substack{v \to u \\ u \to \infty}} \left[-\frac{h_1 \cdot u^2 + c_{11} \cdot u - h_1 \cdot v - h_1 \cdot v^2}{(1 + v)^2 - u^2} \right] = \frac{h_1 - c_{11}}{2} \tag{85}$$

5.2. Minimum value of the iteration interval: Minimum growth of the convergence surface

For any value of “v”, the minimum value of “u” is the value that will cause a tangency between the convergence ellipsoids 11 and 12.

The most complicated case to study will be the situation of two ellipsoids when they are neither tangent nor secant. After several iterations have been done, it could be observed that, if in this situation u = u_{min}, the geometric convergence always happens between the convergence ellipsoid 11 and the convergence ellipsoid 12; that is, if u = u_{min}, the cross point between the lines that link the extremes of the focal distances is going to be in the tangency between the convergence ellipsoids. Consequently, to know the value of u_{min} (or c_{2min} if u = 0), the strategy is to apply the equations of the convergence between the convergence ellipsoids 11 and 12.

5.2.1. For u = 0 (the axes of the ellipsoids are parallel or translational units)

a) If two initial ellipsoids (11 and 12) are neither tangent nor secant:

The convergence ellipsoids 11 and 12 for c_{2min} have been represented in Fig. 15:

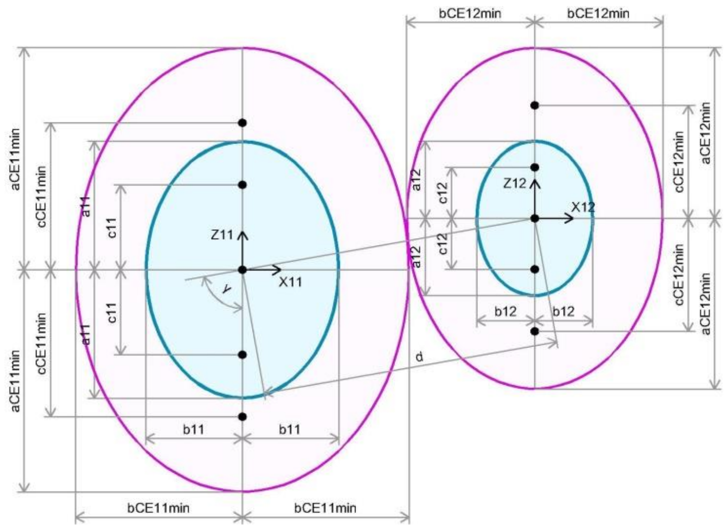

Fig. 15. Minimum geometric convergence situation for two ellipsoids in the space that are neither tangent nor secant and with u = 0.

The goal is to obtain the value of c_{2min} that causes the situation of the previous figure. Using Eq. (25) the next equations are satisfied:

$$a_{CE_{11}min} = a_{11} \cdot \left(1 + \frac{c_{2min}}{c_{11}}\right) \quad and \quad b_{CE_{11}min} = b_{11} \cdot \left(1 + \frac{c_{2min}}{c_{11}}\right) \tag{86}$$

Eq. (87) is obtained by applying the Pythagorean Theorem between the axes of the ellipse:

$$c_{CE_{11}min} = \sqrt{a^2_{CE_{11}min} - b^2_{CE_{11}min}} = c_{11} \cdot \left(1 + \frac{c_{2min}}{c_{11}}\right) \tag{87}$$

The same is done with the convergence ellipsoid 12:

$$c_{CE_{12}min} = \sqrt{a^2_{CE_{12}min} - b^2_{CE_{12}min}} = c_{12} \cdot \left(1 + \frac{c_{2min}}{c_{12}}\right) \tag{88}$$

The position of the tangent point (x_c, z_c) between two ellipses that satisfies the geometry convergence was obtained in [24]: Eq. (89) and Eq. (90) (in [24] the next equation is in function of "x" and "y", in this paper this equation is going to depend on "x" and "z").

$$x_c = \frac{c_{CE_{11}min} \cdot \left[(d \cdot \sin\gamma)^2 + \left(l^2 - c^2_{CE_{12}min}\right) \cdot (\sin\alpha)^2 + 2 \cdot d \cdot l \cdot \sin\gamma \cdot \sin\alpha\right]}{c_{CE_{11}min} \cdot (d \cdot \sin\gamma + l \cdot \sin\alpha) + c_{CE_{12}min} \cdot d \cdot \sin(\alpha + \gamma)} \tag{89}$$

$$\begin{aligned} z_c = {} & \frac{c_{CE_{11}min}}{2 \cdot c_{CE_{11}min} \cdot (d \cdot \sin\gamma + l \cdot \sin\alpha) + 2 \cdot c_{CE_{12}min} \cdot d \cdot \sin(\alpha + \gamma)} \cdot \\ & \cdot \left[c^2_{CE_{12}min} \cdot \sin(2 \cdot \alpha) + 2 \cdot c_{GCE_{11}min} \cdot c_{CE_{12}min} \cdot \sin\alpha + \right. \\ & \left. + d^2 \cdot \sin(2 \cdot \gamma) + 2 \cdot d \cdot l \cdot \sin(\alpha - \gamma) - l^2 \cdot \sin(2 \cdot \alpha)\right] \end{aligned} \tag{90}$$

The previous equations are for a general situation. In this case: u = 0 → α = 0. In addition, the "l" parameter depends on the reference system of the initial ellipsoids but it does not depend on the reference system of the convergence ellipsoids. Consequently, the condition l = 0 is satisfied.

Then, Eq. (89) and Eq. (90) are transformed to Eq. (91) and Eq. (92):

$$x_c = \left(\frac{c_{CE_{11}min}}{c_{CE_{11}min} + c_{CE_{12}min}}\right) \cdot d \cdot \sin(\gamma) = \left(\frac{c_{11} + c_{2min}}{c_{11} + c_{12} + 2 \cdot c_{2min}}\right) \cdot d \cdot \sin(\gamma) \tag{91}$$

$$z_c = \left(\frac{c_{CE_{11}min}}{c_{CE_{11}min} + c_{CE_{12}min}}\right) \cdot d \cdot \cos(\gamma) = \left(\frac{c_{11} + c_{2min}}{c_{11} + c_{12} + 2 \cdot c_{2min}}\right) \cdot d \cdot \cos(\gamma) \tag{92}$$

The point (x_c, z_c) must belong to the convergence ellipsoids 11 and 12 simultaneously and that situation will imply the satisfaction of Eq. (93), where the convergence ellipsoid equation 11 has been used (the use of the convergence ellipsoid equation 12 would give the same result).

$$\left[\frac{z_c}{a_{11} \cdot \left(1 + \frac{c_{2min}}{c_{11}}\right)}\right]^2 + \left[\frac{x_c}{b_{11} \cdot \left(1 + \frac{c_{2min}}{c_{11}}\right)}\right]^2 = 1 \tag{93}$$

If Eq. (91) and Eq. (92) are replaced in Eq. (93) and c_{2min} is isolated using Eq. (31), Eq. (94) is obtained:

$$c_{2min} = \frac{c_{11}}{2} \cdot \left[d \cdot \sqrt{\left(\frac{cos(\gamma)}{a_{11}}\right)^2 + \left(\frac{sin(\gamma)}{b_{11}}\right)^2} - \left(\frac{k+1}{k}\right) \right] \quad (94)$$

It is important to highlight that Eq. (94) does not depend on the "l" parameter. This situation means that the value of c_{2min} does not have a relationship with the relative position between the scissor and the deployable surface and, in consequence, c_{2min} is going to be constant.

b) If two initial ellipsoids (11 and 12) are tangent:

The convergence ellipsoids 11 and 12 for c_{2min} in this situation have been represented in Fig. 16:

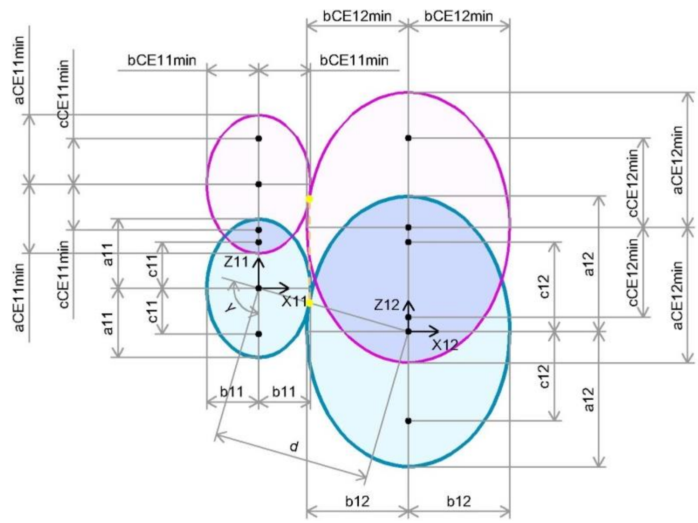

Fig. 16. Minimum geometric convergence situation for two ellipsoids in the space that are tangent with l = 0.5 and with u = 0.

To know the minimum value of c_2, the orientation with c_{2min} = 0 will be obtained. If this condition is used in Eq. (91) and Eq. (92):

$$x_c = \left(\frac{c_{11}}{c_{11}+c_{12}}\right) \cdot d \cdot sin(\gamma) \quad and \quad z_c = \left(\frac{c_{11}}{c_{11}+c_{12}}\right) \cdot d \cdot cos(\gamma) \quad (95)$$

The value of x_c, with a geometric convergence between two ellipsoids in a plane and with $\alpha = 0$ is:

$$x_c = \frac{c_{11} \cdot [(d \cdot sin\,\gamma)^2 + (l^2 - c_{12}^2) \cdot (sin\,\alpha)^2 + 2 \cdot d \cdot l \cdot sin\,\gamma \cdot sin\,\alpha]}{c_{11} \cdot (d \cdot sin\,\gamma + l \cdot sin\,\alpha) + c_{12} \cdot d \cdot sin(\alpha + \gamma)} = \left(\frac{c_{11}}{c_{11}+c_{12}}\right) \cdot d \cdot sin(\gamma) \quad (96)$$

As can be seen, Eq. (95) and Eq. (96) are the same equations. On the other hand, if the expression of y_c of [24] is used instead of the expression of x_c, Eq. (95) would be obtained. Consequently, if c_{2min} = 0, the condition of geometric convergence between two ellipsoids in the space is satisfied. In addition, Eq. (95) and Eq. (96) do not depend on the "l" parameter, so this minimum value of c_2 can be applied to any value of "l".

c) If two initial ellipsoids (11 and 12) are secants:

The convergence ellipsoids 11 and 12 for c_{2min} have been represented in Fig. 17:

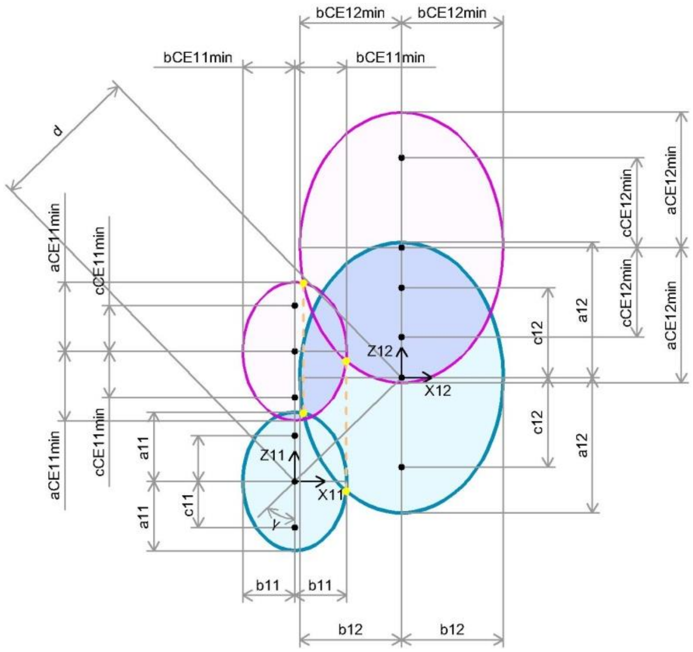

Fig. 17. Minimum geometric convergence situation for two ellipsoids in the space that are secants with l = 0.75 and with u = 0.

As can be seen in Fig. 17, if the c_2 value is smaller, the convergence ellipsoid of each initial ellipsoid is also going to be smaller. Finally, both ellipsoids (the initial ellipsoid and its convergence ellipsoid) would be the same.

When this situation happens, the intersection between the convergence ellipsoid of each initial ellipsoid will be the interception between the initial ellipsoids (for values of "l" different from 0, this intersection will be displaced in the direction of the axes of the ellipsoids). For lower values of c_2, this variable will be negative and this situation cannot happen.

Consequently, if two ellipsoids are secant, $c_{2min} = 0$. In addition, the convergence surface will not be a continuous surface because it will have a hole.

When a sequential design of a deployable structure is done, the situation of secant ellipsoids is not common in the first frequencies. However, if the frequencies of the structure are higher, the curvature of the surface could cause this phenomenon.

5.2.2. For u ≠ 0 (the axes of the ellipsoids are not parallel or polar units)

a) If two initial ellipsoids (11 and 12) are neither tangent nor secant:

The convergence ellipsoids 11 and 12 for u_{min} in this situation have been represented in Fig. 18:

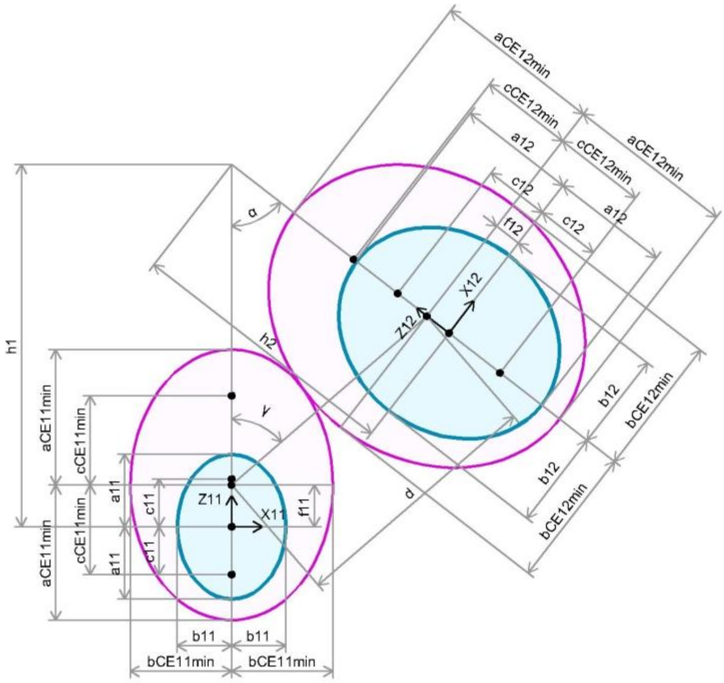

Fig. 18. Minimum geometric convergence situation for two ellipsoids in the space that are neither tangent nor secant and with $u \neq 0$.

The goal is to obtain the value of u_{min} that causes the situation of the previous figure. In addition, in Figure 9 the next condition is satisfied:

$$\frac{sin(\propto)}{d} = \frac{sin(\gamma)}{h_2 - f_{12} - l} = \frac{sin(180 - \propto - \gamma)}{h_1 - f_{11}} \tag{97}$$

The reader can notice that in Eq. (97), the "l" parameter has been used. The reason is to balance the translations in the initial ellipsoids because this translation will cause a displacement in the convergence ellipsoids.

If γ is isolated from Eq. (97):

$$cos(\gamma) = \frac{1}{d} \cdot [h_1 - f_{11} - (h_2 - f_{12} - l) \cdot cos(\propto)] \tag{98}$$

Likewise, the following expressions are obtained from Eq. (8):

$$h_1 - f_{11} = \frac{h_1 + u_{min} \cdot c_{11} + v \cdot h_1}{(1+v)^2 - u_{min}^2} \quad and \quad h_2 - f_{12} = \frac{h_2 + u_{min} \cdot c_{12} + v \cdot h_2}{(1+v)^2 - u_{min}^2} \tag{99}$$

If Eq. (6) with $u = u_{min}$, Eq. (7) with $u = u_{min}$, Eq. (98) and Eq. (99) are replaced in Eq. (89):

$$x_c = \frac{[c_{11} \cdot (1+v) + u_{min} \cdot h_1] \cdot (h_2^2 - c_{12}^2) \cdot sin(\propto)}{\begin{bmatrix} [c_{11} \cdot (1+v) + u_{min} \cdot h_1] \cdot [h_2 \cdot (1+v) + u_{min} \cdot c_{12}] + \\ +[c_{12} \cdot (1+v) + u_{min} \cdot h_2] \cdot [h_1 \cdot (1+v) + u_{min} \cdot c_{11}] \end{bmatrix}} \tag{100}$$

Finally, if Eq. (6) with $u = u_{min}$, Eq. (7) with $u = u_{min}$, Eq. (98) and Eq. (99) are replaced in Eq. (90):

$$z_c = A \cdot \frac{x_c}{sin(\propto)} + B \tag{101}$$

Where:

$$A = \left[\frac{(h_1 - c_{11}) \cdot (1 + v - u_{min})}{(h_2 + c_{12}) \cdot (1 + v + u_{min})} - cos(\propto)\right] \tag{102}$$

$$B = \left[\frac{c_{11}}{(1+v)^2 - u_{min}^2}\right] \cdot \left[1 + v + \frac{u_{min} \cdot h_1}{c_{11}}\right] \tag{103}$$

The next condition must be satisfied:

$$\forall (x_c, z_c) \in \left[\frac{z_c}{f_{21}}\right]^2 + \left[\frac{x_c}{f_{31}}\right]^2 = 1 \tag{104}$$

If Eq. (9) and Eq. (10) are replaced in Eq. (104) with u = u_{min}:

$$b_{11}^2 \cdot z_c^2 + a_{11}^2 \cdot x_c^2 = \left[\frac{a_{11} \cdot b_{11}}{(1+v)^2 - u_{min}^2}\right]^2 \cdot \left[1 + v + \frac{u_{min} \cdot h_1}{c_{11}}\right]^2 \tag{105}$$

The final step is the substitution of Eq. (100) and Eq. (101) in Eq. (105). The result of this process is a 5-degree equation that only depends on u_{min}.

This equation has been represented in Fig. 19:

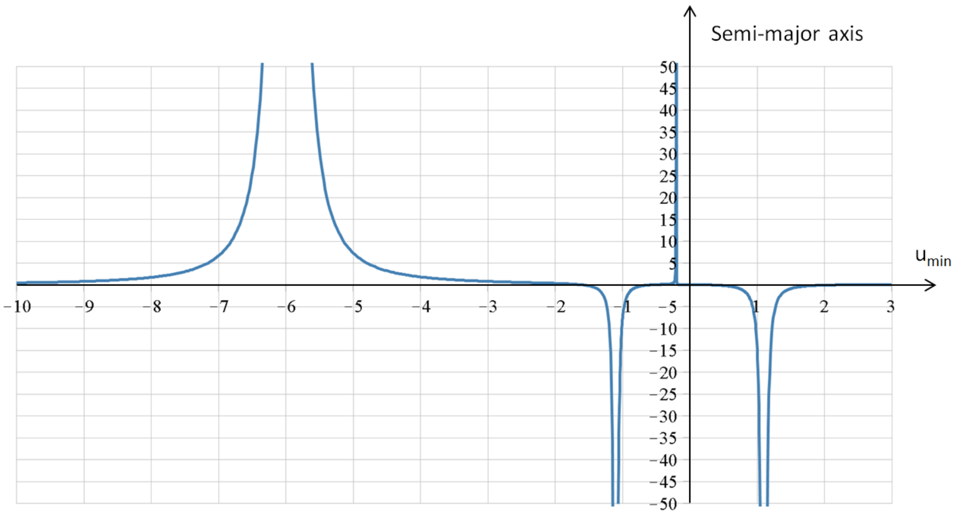

Fig. 19. Evolution of the u_{min} parameter (Horizontal axis).

As can be seen in the previous figure, there are 5 cuts between the blue curve and the horizontal axis (there are two cuts between 0 and -1, but they are very close). However, just 1 of these 5 values is going to be always correct. The rest of the values will be negative or greater in comparison with the value of u_{max}.

b) If two initial ellipsoids (11 and 12) are tangent:

The convergence ellipsoids 11 and 12 for u_{min} have been represented in Fig. 20:

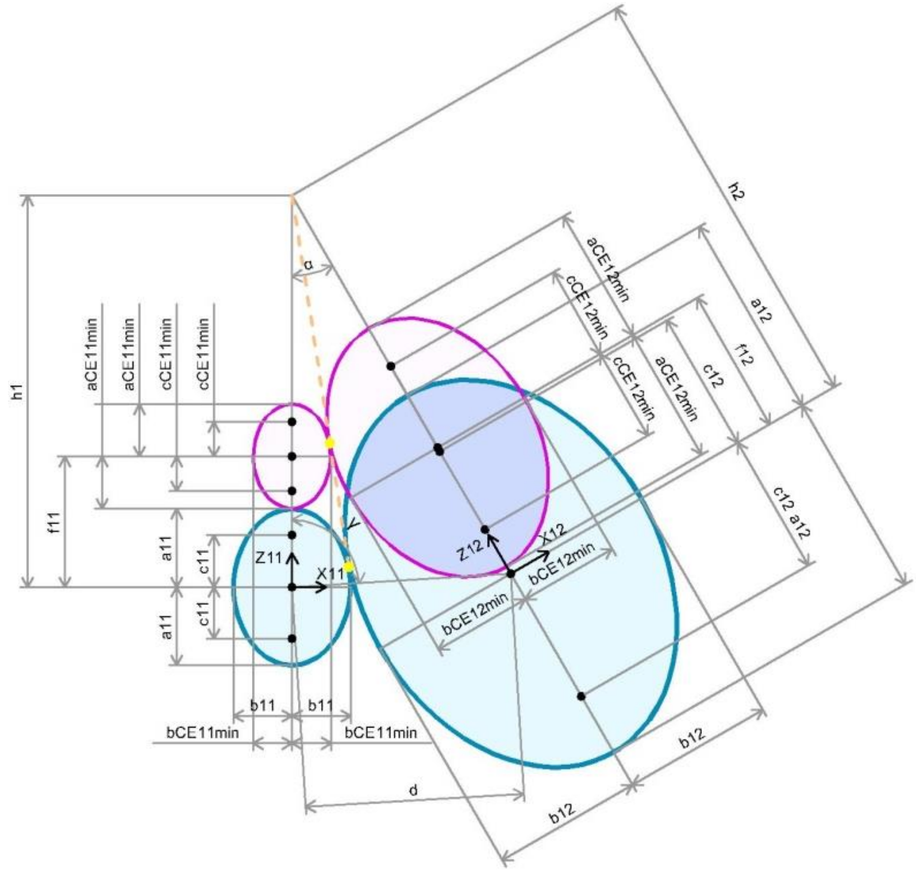

Fig. 20. Minimum geometric convergence situation for two ellipsoids in the space that are tangent with v = 0.5 and with u ≠ 0.

To find out the minimum value of "u", the orientation between the ellipsoids with $u_{min} = 0$ is going to be developed. If this value of u_{min} is replaced in Eq. (100):

$$x_c = \frac{c_{11} \cdot (h_2^2 - c_{12}^2) \cdot sin(\propto)}{[c_{11} \cdot h_2 + c_{12} \cdot h_1] \cdot (1 + v)} \tag{106}$$

Also, the next condition is satisfied in Fig. 9:

$$\frac{sin(\propto)}{d} = \frac{sin(\gamma)}{h_2 - l} = \frac{sin(180 - \propto - \gamma)}{h_1} \tag{107}$$

If Eq. (107) is replaced in Eq. (106) and the variables h_1 and h_2 are converted to other known variables, Eq. (89) is obtained. If this process is done with the equation of z_c, the equation of y_c of [24] is obtained. In [24], the equations of x_c and y_c were developed to satisfy the tangency between the ellipsoids and, in consequence, if the ellipsoids are tangent → $u_{min} = 0$.

c) If two initial ellipsoids (11 and 12) are secants:

The convergence ellipsoids 11 and 12 for u_{min} in this situation have been represented in Fig. 21:

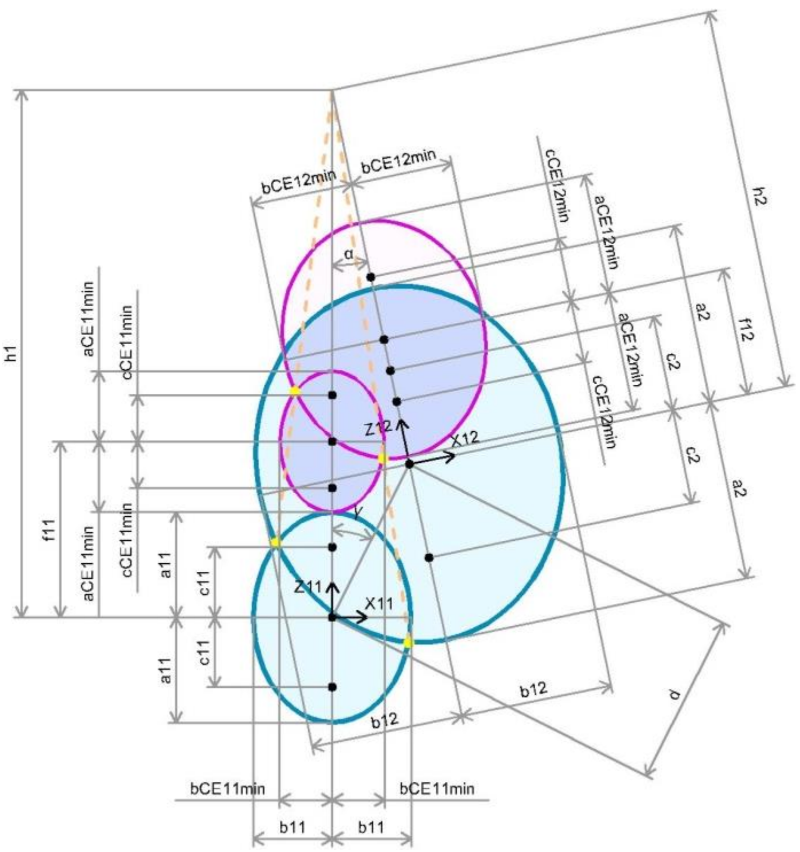

Fig. 21. Minimum geometric convergence situation for two ellipsoids in the space that are secants with v = 0.5 and with u ≠ 0.

To find out the value of u_{min}, a geometric argumentation is going to be used due to the complexity of the equations of this case.

If the value of “u” is lower, the geometry of the convergence ellipsoid is going to be smaller according to Eq. (6), Eq. (7), Eq. (8), Eq. (9), Eq. (10) and Eq. (11). When “u” has the value of 0, the convergence ellipsoid will only depend on the “v” parameter. In consequence, this problem can be solved with the analysis of the cases of “v”:

The first is v = 0. In this case, the convergence ellipsoid would have the same geometry and position in comparison with the initial ellipsoid. Consequently, $u_{min} = 0$.

The second is v ≠ 0 (as the situation that is represented in Fig. 21). In this case, the convergence ellipsoid would have a different geometry and a different position in comparison with the initial ellipsoid. However, v = l/n, where “n” is the distance between the intersection of the two convergence ellipsoids and the “l” parameter”. Consequently, if “u” is going to be 0, the values of “l” are going to be in the intersection points between the initial ellipsoids.

This situation means that the focal distance with a value of 0 will belong to the intersection between the initial ellipsoids. The “u” parameter cannot be negative, so if the initial ellipsoids are secants → $u_{min} = 0$.

6. Conclusions

Before the onset of the convergence surface Method, the obtaining of two scissors from two ellipsoids in the space had to be done using the intersection of 2 proportional ellipsoids and the result was a unique mathematical solution. If the designer wanted other mathematical solution, a different peer of ellipsoids had to be intersected using other proportional constant. In comparison with this situation, the convergence surface Method not only gives all possible solutions with just one mathematical operation but also this Method allows the use of polar units and, in consequence, the size of the deployable structure in the folded position will be smaller. Likewise, the consideration of all solutions of the space enables the design of bistable deployable structures.

However, the main drawback of the convergence surface Method is the obtaining of this surface. To balance this situation, the research of this article has completed the equations that control the Method of [24] and the results can be summarised into 2 types of formulas: the exact formula (triaxial two-leaf hyperboloid) and the approximate formulas (determinant and level curves).

Before the outcomes of this article, the convergence surface had to be created with the manual intersection of ellipsoids without knowing the start point of the intersection process and the solution that limits the superior interval of valid results. Moreover, the standard shape of the convergence surface was a mystery and, consequently, the use of mathematical programs (MatLab, Maple, etc.) to create this surface was quite tedious.

The results of this paper have solved these disadvantages so the Method of [24] is now more operative and friendly. Once the convergence surface has been obtained, the rest of the design process is based on the intersection of this surface with the design surface and the final curve will be all possible points where there is a mathematical solution. The aim of the designer is the selection of a point of this curve in function of boundary conditions, structural behaviour, etc.

Finally, a further extension intended for future work is the incorporation of the formulas developed in this research into a design program where the designer will have the convergence surface just clicking a button.

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8. Formatting of funding sources

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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