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REMARKS ON THE CONTROL OF TWO-PHASE STEFAN FREE-BOUNDARY PROBLEMS*

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Abstract. This paper concerns the null controllability of the two-phase 1D Stefan problem with distributed controls. This is a free-boundary problem that models solidification or melting processes. In each phase, a parabolic equation, completed with initial and boundary conditions, must be satisfied; the phases are separated by a phase change interface, where an additional free-boundary condition is imposed (the so-called Stefan condition). We assume that two localized sources of heating/cooling controls act on the system (one in each phase). We prove the following local null controllability result: the temperatures can be steered to zero and, simultaneously, the interface can be steered to a prescribed location provided the initial data and the interface position are sufficiently close to the targets. The ingredients of the proofs are a compactness-uniqueness argument (which gives appropriate observability estimates adapted to constraints) and a fixed-point formulation and resolution of the controllability problem (which gives the result for the nonlinear system). We also prove a negative result corresponding to the case where only one control acts on the system and the interface does not collapse to the boundary.

Key words. free-boundary problems, two-phase Stefan problem, null controllability, Carleman inequalities, constrained observability inequalities

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1. Introduction.

The two-phase Stefan problem is a mathematical model, i.e., a coupled system composed of two PDEs and one ODE, used to describe liquid-solid phase transition. This physical phenomenon is frequently found in many processes in science and engineering, for example, continuous casting of steel [2], cryosurgical treatment of cancer [28], analysis of crystal growth [5], and design of lithium-ion batteries [4]. It is also important to highlight that, besides their use in thermodynamics processes, similar systems can be used to model other phenomena, such as analysis and computation of the flux of a fluid on a free surface [32, 21, 29], fluid-solid interaction [7, 26, 30], gas flow through a porous medium [1, 9, 31], and tumor growth [19, 18].

Let us recall the mathematical formulation of the two-phase 1D Stefan problem, and let us formulate the related control problem considered in this paper.

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Let $L > 0$, $T > 0$, and $\ell_l, \ell_0, \ell_r \in (0, L)$ be given with $\ell_l < \ell_0 < \ell_r$. We also consider two functions, $u_0 \in H_0^1(0, \ell_0)$ with $u_0 \geq 0$ and $v_0 \in H_0^1(\ell_0, L)$ with $v_0 \leq 0$, and two open sets, $\omega_l \subset\subset (0, \ell_l)$ and $\omega_r \subset\subset (\ell_r, L)$. At each t, the material domain is separated into two parts: $x \in [0, \ell(t))$ (the liquid phase) and $x \in (\ell(t), L]$ (the solid phase). Here, $\ell = \ell(t)$ is the position of the interface between liquid and solid phases; it satisfies $\ell(0) = \ell_0$ and $\ell(t) \in (\ell_l, \ell_r)$ for all t.

The aim of this paper is to study the controllability properties of the following *two-phase Stefan problem:*

$$(1) \qquad \begin{cases} u_t - d_l u_{xx} = h_l 1_{\omega_l} & \text{in} \quad Q_l, \\ v_t - d_r v_{xx} = h_r 1_{\omega_r} & \text{in} \quad Q_r, \\ u(0,t) = 0 & \text{in} \quad (0,T), \\ v(L,t) = 0 & \text{in} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,\ell_0), \\ v(\cdot,0) = v_0 & \text{in} \quad (\ell_0,L), \\ u(\ell(t),t) = v(\ell(t),t) = 0 & \text{in} \quad (0,T), \\ -\ell'(t) = d_l u_x(\ell(t),t) - d_r v_x(\ell(t),t) & \text{in} \quad (0,T). \end{cases}$$

Here and in what follows, d_l and d_r must be viewed as diffusion coefficients, and we use the notation

$$\begin{cases} Q := (0,L) \times (0,T), \\ Q_l := \{(x,t) \in Q : \ t \in (0,T), \ x \in (0,\ell(t))\}, \\ Q_r := \{(x,t) \in Q : \ t \in (0,T), \ x \in (\ell(t),L)\}, \\ \mathcal{O}_l = \omega_l \times (0,T), \text{ and } \mathcal{O}_r = \omega_r \times (0,T). \end{cases}$$

The main result in this paper is the following.

THEOREM 1. *Let $\ell_T \in (\ell_l, \ell_r)$. Then there exists $\delta > 0$ such that, for any $u_0 \in H_0^1(0, \ell_0)$ with $u_0 \geq 0$, any $v_0 \in H_0^1(\ell_0, L)$ with $v_0 \leq 0$, and any $\ell_0 \in (\ell_l, \ell_r)$ satisfying*

$$\|u_0\|_{H_0^1(0,\ell_0)} + \|v_0\|_{H_0^1(\ell_0,L)} + |\ell_0 - \ell_T| \leq \delta,$$

there exist controls $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and associated states (u, v, ℓ) with

$$\begin{cases} \ell \in H^1(0,T) \cap C^1((0,T]), \ \ell(t) \in (\ell_l, \ell_r) \ \ \forall\, t \in [0,T], \\ u, \, u_x, \, u_t, \, u_{xx} \in L^2(Q_l) \text{ and } v, \, v_x, \, v_t, \, v_{xx} \in L^2(Q_r), \end{cases}$$

such that

$$(2) \qquad \ell(T) = \ell_T, \ u(\cdot,T) = 0 \ \text{ in } \ (0,\ell_T), \ \text{ and } \ v(\cdot,T) = 0 \ \text{ in } \ (\ell_T, L).$$

Remark 1. We will see in section 5.1 that the maximum principle for parabolic equations implies that null controllability cannot hold if one of the controls (for instance, h_r) vanishes and the interface satisfies $0 < \ell(T) < L$. However, the possibility of getting a null control result with only one control when one of the phases is allowed to collapse to the boundary, that is, $\ell(T) = L$ or $\ell(T) = 0$, is open.

For completeness, let us mention some previous works on the control of (1) and similar models.

The analysis of the controllability properties for linear and nonlinear parabolic PDEs defined in cylindrical domains is a classical problem in control theory, and some of the main contributions are in the references [8, 10, 16, 20, 25]. On the other hand, the study of the controllability properties of free-boundary problems for PDEs has not been much explored, although some important results have been obtained in the past few years, especially for one-phase Stefan problems and variants; see [6, 13, 14, 12]. In [22], the authors study the controllability problem for the free-boundary viscous Burgers equation with one moving end point.

Regarding the two-phase Stefan problem, the best results to our knowledge concern *stabilization*. More precisely, it is proved in [23] that, under some assumptions, there exist Neumann boundary controls and associated states (u, v, ℓ) defined for all $t > 0$ such that

$$\lim_{t\to\infty} \|u(\cdot,t) - \mathcal{T}_m\|_{L^2(0,\ell(t))} = \lim_{t\to\infty} \|v(\cdot,t) - \mathcal{T}_m\|_{L^2(0,\ell(t))} = 0 \quad \text{and} \quad \lim_{t\to\infty} \ell(t) = \ell_T,$$

where $\mathcal{T}_m$ is a melting/solidification temperature.

A natural question is whether or not it is possible to drive both the temperature and the interface to prescribed targets at a finite time. In this paper we give a positive partial answer to this question. Recall that, in [7, 15, 26], a similar problem was considered for a 1D fluid-structure problem, with the following equations on the interface:

$$u(\ell(t),t) = v(\ell(t),t) = \ell'(t), \quad v_x(\ell(t),t) - u_x(\ell(t),t) = m\ell''(t) \quad \text{for} \quad t \in (0,T).$$

In contrast to previous works on free-boundary controllability, in this paper we deal with situations leading to new difficulties. Let us discuss some of these differences:

- *Control of two phases.* Obviously, the fact that we model a two-phase transition process greatly complicates the structure and properties of the state system and requires an appropriate analysis.
- *Control of the interface.* The aim is to control not only the temperature on both sides but also the interface between liquid and solid regions. This will bring an extra difficulty. The main strategy will rely on linearization, then reformulation as a constrained observability problem, and then resolution of a fixed-point equation.

The rest of this paper is organized as follows. In section 2.1, we will reformulate the free-boundary problem as a nonlinear parabolic system in a cylindrical domain. In section 3, we will present an improved observability inequality, which leads to the null controllability for a related linearized system subject to a linear constraint. In section 4, we will give a proof of Theorem 1. To this end, we will apply a fixed-point argument. Finally, in section 5, we will present some additional comments and questions.

2. Preliminaries.

2.1. Reformulation of the free-boundary problem. First, let us find a suitable diffeomorphism Φ that transforms the free-boundary problem for the parabolic system (1) into an equivalent problem for a nonlinear parabolic system in a cylindrical domain.

To do this, let us fix a function $\ell \in H^1(0,T) \cap C^1((0,T])$ such that $\ell(t) \in (\ell_l, \ell_r)$ for all $t \in [0,T]$, and let us take $\sigma > 0$ sufficiently small such that

$$\ell_l + \sigma < \ell(t) - \sigma \quad \text{and} \quad \ell(t) + \sigma < \ell_r - \sigma \quad \text{in} \quad [0,T].$$

Then, for any $\ell_l + 2\sigma < y < \ell_r - 2\sigma$, we build a function $m(\cdot, y) : \mathbb{R} \to \mathbb{R}$ by linear interpolation of the points $(\ell_l - \sigma, \ell_l - \sigma)$, $(\ell_l + \sigma, \ell_l + \sigma)$, $(y - \sigma, \ell_0 - \sigma)$, $(y + \sigma, \ell_0 + \sigma)$, $(\ell_r - \sigma, \ell_r - \sigma)$, and $(\ell_r + \sigma, \ell_r + \sigma)$ and then extend the extreme segments toward infinity. Specifically, we have the following definition for $m(\cdot, y)$:

$$m(x,y) := \begin{cases} x & \text{if} \quad x \le \ell_l + \sigma, \\ \ell_l + \sigma + \dfrac{(\ell_l - \ell_0 + 2\sigma)(x - \ell_l - \sigma)}{\ell_l + 2\sigma - y} & \text{if} \quad \ell_l + \sigma < x < y - \sigma, \\ x - y + \ell_0 & \text{if} \quad y - \sigma < x < y + \sigma, \\ \ell_0 + \sigma + \dfrac{(\ell_r - \ell_0 - 2\sigma)(x - y - \sigma)}{\ell_r - y - 2\sigma} & \text{if} \quad y + \sigma < x < \ell_r - \sigma, \\ x & \text{if} \quad x \ge \ell_r - \sigma. \end{cases}$$

Let us now consider a function $\eta \in C^\infty(\mathbb{R})$ such that

$$\operatorname{supp} \eta \subset (-\sigma, \sigma) \quad \int_{-\sigma}^{\sigma} \eta(x)\, dx = 1 \quad \text{and} \quad \eta(x) = \ \eta(-x) \quad \forall x \in \mathbb{R}.$$

Then, we can define a smooth function $G : \mathbb{R} \times (\ell_l + 2\sigma, \ell_r - 2\sigma) \mapsto \mathbb{R}$ as follows:

$$G(x, y) := [\eta * m(\cdot, y)](x).$$

A simple computation leads to the equalities

$$G(x, \ell_0) = x \ \ \forall x \in \mathbb{R}, \quad G(y, y) = \ell_0, \quad \partial_x G(y, y) = 1, \tag{3}$$

and

$$\nabla G(x, y) = \left([\eta' * m(\cdot, y)](x), [\eta * \partial_y m(\cdot, y)](x)\right),$$

where

$$\partial_y m(x,y) := \begin{cases} 0 & \text{if} \quad x \le \ell_l + \sigma, \\ \dfrac{(\ell_l - \ell_0 + 2\sigma)(x - \ell_l - \sigma)}{(\ell_l + 2\sigma - y)^2} & \text{if} \quad \ell_l + \sigma < x < y - \sigma, \\ -1 & \text{if} \quad y - \sigma < x < y + \sigma, \\ \dfrac{(\ell_r - \ell_0 - 2\sigma)(x - \ell_r + \sigma)}{(\ell_r - y - 2\sigma)^2} & \text{if} \quad y + \sigma < x < \ell_r - \sigma, \\ 0 & \text{if} \quad x \ge \ell_r - \sigma. \end{cases}$$

Let us introduce the mapping

$$\Phi : Q \mapsto Q, \quad \text{with} \quad \Phi(x, t) := (G(x, \ell(t)), t)\,.$$

It can be seen that Φ is a diffeomorphism in Q; it coincides with the identity in the regions $(0, \ell_l + \sigma) \times (0, T)$ and $(\ell_r - \sigma, L) \times (0, T)$, and, moreover, $\Phi(\ell(t), t) = (\ell_0, t)$ for all $t \in [0, T]$. Let us introduce the sets $Q_{0,l} := (0, \ell_0) \times (0, T)$ and $Q_{0,r} := (\ell_0, L) \times (0, T)$, and let us define $p : Q_{0,l} \to \mathbb{R}$ and $q : Q_{0,r} \to \mathbb{R}$, with

$$p(\xi, t) := u(x, t) = u(\Phi^{-1}(\xi, t)) \ \text{ and } \ q(\xi, t) := v(x, t) = v(\Phi^{-1}(\xi, t)),$$

where $(\xi, t) := \Phi(x, t)$. Then, we have that the pair (p, q) satisfies

$$\begin{cases} p_t - d_l^\ell p_{\xi\xi} + b_l^\ell p_\xi = h_l 1_{\omega_l} & \text{in} \quad Q_{0,l}, \\ q_t - d_r^\ell q_{\xi\xi} + b_r^\ell q_\xi = h_r 1_{\omega_r} & \text{in} \quad Q_{0,r}, \\ p(0, \cdot) = q(L, \cdot) = 0 & \text{in} \quad (0, T), \\ p(\cdot, 0) = p_0 & \text{in} \quad (0, \ell_0), \\ q(\cdot, 0) = q_0 & \text{in} \quad (\ell_0, L), \\ p(\ell_0, \cdot) = q(\ell_0, \cdot) = 0 & \text{in} \quad (0, T), \\ d_l p_\xi(\ell_0, t) - d_r q_\xi(\ell_0, t) = -\ell'(t) & \text{in} \quad (0, T), \end{cases} \tag{4}$$

where $p_0 := u_0 \circ [G(\cdot, \ell_0)]^{-1} = u_0 \in H^1_0(0, \ell_0)$, $q_0 := v_0 \circ [G(\cdot, \ell_0)]^{-1} = v_0 \in H^1_0(\ell_0, L)$, and

$$
\begin{aligned}
&d^{\ell}_l(\cdot, t) := d_l \left(G_x \left([G(\cdot, \ell(t))]^{-1}, \ell(t)\right)\right)^2, \\
&d^{\ell}_r(\cdot, t) := d_r \left(G_x \left([G(\cdot, \ell(t))]^{-1}, \ell(t)\right)\right)^2, \\
&b^{\ell}_l(\cdot, t) := G_y \left([G(\cdot, \ell(t))]^{-1}, \ell(t)\right) \ell'(t) + d_l G_{xx} \left([G(\cdot, \ell(t))]^{-1}, \ell(t)\right), \\
&b^{\ell}_r(\cdot, t) := G_y \left([G(\cdot, \ell(t))]^{-1}, \ell(t)\right) \ell'(t) + d_r G_{xx} \left([G(\cdot, \ell(t))]^{-1}, \ell(t)\right).
\end{aligned} \tag{5}
$$

Remark 2. Since $\ell \in H^1(0, T)$, it is not difficult to deduce that $(d^{\ell}_l, d^{\ell}_r) \in L^\infty(Q_{0,l}) \times L^\infty(Q_{0,r})$ and $(b^{\ell}_l, b^{\ell}_r) \in L^2(0, T; L^\infty(0, \ell_0)) \times L^2(0, T; L^\infty(\ell_0, L))$. Moreover, there exist constants $K_1, K_2 > 0$, independent of ℓ and T, such that $\|(b^{\ell}_l, b^{\ell}_r)\|_{L^2(L^\infty)\times L^2(L^\infty)} \le K_1 \|\ell'\|_{L^2(0,T)} + K_2 T$.

This way, we have that Theorem 1 is equivalent to proving a local controllability result for (4). Actually, we will prove the following.

THEOREM 2. *Let $\ell_T \in (\ell_l, \ell_r)$. Then there exists $\delta > 0$ such that, for any $p_0 \in H^1_0(0, \ell_0)$ with $p_0 \ge 0$, any $q_0 \in H^1_0(\ell_0, L)$ with $q_0 \le 0$, and any $\ell_0 \in (\ell_l, \ell_r)$ satisfying*

$$\|p_0\|_{H^1_0(0,\ell_0)} + \|q_0\|_{H^1_0(\ell_0,L)} + |\ell_0 - \ell_T| < \delta,$$

there exist controls $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and associated solutions (p, q, ℓ) to (4) with

$$
\begin{cases}
\ell \in H^1(0, T) \cap C^1((0, T]), \quad \ell(t) \in (\ell_l, \ell_r) \quad \forall t \in [0, T], \\
p, p_\xi, p_t, p_{\xi\xi} \in L^2(Q_{0,l}) \quad \text{and} \quad q, q_\xi, q_t, q_{\xi\xi} \in L^2(Q_{0,r}),
\end{cases}
$$

such that

$$\ell(T) = \ell_T, \quad p(\cdot, T) = 0 \quad \text{in} \quad (0, \ell_0), \quad \text{and} \quad q(\cdot, T) = 0 \quad \text{in} \quad (\ell_0, L).$$

2.2. Well-posedness of the two-phase free-boundary problem. The aim of this section is to prove the local existence and uniqueness for the two-phase free-boundary problem (1). More precisely, we have the following result.

PROPOSITION 1. *Let $L, T > 0$ and $\ell_l < \ell_0 < \ell_r$ be given. Then, the system* (1) *is locally well-posed. In other words, for any $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and $(u_0, v_0) \in H^1_0(0, \ell_0) \times H^1_0(\ell_0, L)$, there exist a time $0 < \widehat{T} \le T$ and a unique strong solution to* (1) *in the time interval $(0, \widehat{T})$ such that*

$$
\begin{cases}
\ell \in H^1(0, \widehat{T}), \quad \ell(0) = \ell_0, \quad \ell(t) \in (\ell_l, \ell_r) \quad \forall t \in [0, \widehat{T}], \\
u,\, u_x,\, u_t,\, u_{xx} \in L^2(\widehat{Q}_l) \text{ and } v,\, v_x,\, v_t,\, v_{xx} \in L^2(\widehat{Q}_r),
\end{cases}
$$

where $\widehat{Q}_l := \{(x, t) \in Q : t \in (0, \widehat{T}),\, x \in (0, \ell(t))\}$ and $\widehat{Q}_r := \{(x, t) \in Q : t \in (0, \widehat{T}),\, x \in (\ell(t), L)\}$.

Thanks to the diffeomorphism $\Phi : Q \mapsto Q$, introduced in section 2.1, Proposition 1 is equivalent to the local existence and uniqueness of (4). More precisely, Proposition 1 is an immediate consequence of the following result.

PROPOSITION 2. *Let the conditions of Proposition* 1 *be satisfied. Then, the nonlinear system* (4) *is locally well-posed; i.e., if $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and $(p_0, q_0) \in$*

$H_0^1(0,\ell_0) \times H_0^1(\ell_0,L)$ are fixed, there exist $\widehat{T} \in (0,T)$ and a unique strong solution in $(0,\widehat{T})$ such that

$$\begin{cases} \ell \in H^1(0,\widehat{T}),\ \ell(0)=\ell_0, \quad \ell(t) \in (\ell_l,\ell_r) \quad \forall\, t \in [0,\widehat{T}], \\ p,\, p_\xi,\, p_t,\, p_{\xi\xi} \in L^2(\widehat{Q}_{0,l}) \text{ and } q,\, q_\xi,\, q_t,\, q_{\xi\xi} \in L^2(\widehat{Q}_{0,r}), \end{cases}$$

where $\widehat{Q}_{0,l} := (0,\ell_0) \times (0,\widehat{T})$ and $\widehat{Q}_{0,r} := (\ell_0,L) \times (0,\widehat{T})$.

Proof. First, let us introduce the spaces

$$X_l^T := L^2(0,T;H^2(0,\ell_0)) \cap H^1(0,T;L^2(0,\ell_0)) \quad \text{and}$$
$$X_r^T := L^2(0,T;H^2(\ell_0,L)) \cap H^1(0,T;L^2(\ell_0,L)).$$

For each fixed $\ell \in H^1(0,T)$, we can use the Faedo–Galerkin method to get a unique strong solution $(p^\ell,q^\ell) \in X_l^T \times X_r^T$ to $(4)_1$–$(4)_6$. Moreover, thanks to Remark 2, we find a positive constant C_1, independent of p_0, q_0, h_l, h_r, ℓ, and T, such that

$$\|(p,q)\|^2_{X_l^T \times X_r^T} \le C_1 \left(1+K+\sqrt{T}\right)\left[1+\left(K+\sqrt{T}\right)K+T\right] e^{C_1\left(1+\sqrt{T}\right)\left(K+\sqrt{T}\right)}\Pi^2, \tag{6}$$

where $K := \|\ell'\|_{L^2(0,T)}$ and $\Pi := \|(p_0,q_0)\|_{H_0^1 \times H_0^1} + \|(h_l,h_r)\|_{L^2 \times L^2}$.

Now, let us assume that $\ell_l < \widehat{\ell}_l < \ell_0 < \widehat{\ell}_r < \ell_r$ and $R>0$, and let us introduce the set

$$\mathcal{A}_{R,T} := \{\ell \in H^1(0,T) : \widehat{\ell}_l \le \ell(t) \le \widehat{\ell}_r\ \forall t \in [0,T], \quad \ell(0)=\ell_0,\ \|\ell'\|_{L^2(0,T)} \le R\}$$

and the mapping $\Lambda : \mathcal{A}_{R,T} \mapsto H^1(0,T)$, with

$$\Lambda(\ell) = \mathcal{L}_\ell \quad \text{and} \quad \mathcal{L}_\ell(t) := \ell_0 - \int_0^t \left[d_l p_\xi^\ell(\ell_0,\tau) - d_r q_\xi^\ell(\ell_0,\tau)\right] d\tau,$$

where $(p^\ell,q^\ell) \in X_l^T \times X_r^T$ is the unique strong solution to $(4)_1$–$(4)_6$. It is not difficult to see that $\mathcal{A}_{R,T}$ is a nonempty, closed, and convex subset of $H^1(0,T)$.

Let us check that, for some $0 < \widehat{T} \le T$, Λ satisfies the following assumptions of Banach's fixed-point theorem in $\mathcal{A}_{R,\widehat{T}}$:

- *There exists $\widetilde{T} \in (0,T]$ such that*

$$\Lambda(\mathcal{A}_{R,\tau}) \subset \mathcal{A}_{R,\tau} \quad \forall \tau \in (0,\widetilde{T}]. \tag{7}$$

  Indeed, $\mathcal{L}_\ell(0)=\ell_0$. Let us introduce

$$C(T,R) := C_1\left(1+R+\sqrt{T}\right)\left[1+\left(R+\sqrt{T}\right)R+T\right]e^{C_1\left(1+\sqrt{T}\right)\left(R+\sqrt{T}\right)}. \tag{8}$$

  Then, using the Hölder inequality, (6), and (8), we see that, for some $C_2>0$ independent of T, one has

$$\begin{aligned} |\mathcal{L}_\ell(t)-\ell_0| &\le \int_0^t |d_l p_\xi^\ell(\ell_0,\tau) - d_r q_\xi^\ell(\ell_0,\tau)|\, d\tau \\ &\le C_2 \widetilde{T}^{1/2} \|(p_\xi^\ell(\ell_0,\cdot), q_\xi^\ell(\ell_0,\cdot))\|_{L^2(0,\widetilde{T}) \times L^2(0,\widetilde{T})} \\ &\le C_2 \widetilde{T}^{1/2} \|(p^\ell,q^\ell)\|_{X_l^T \times X_r^T} \\ &\le C_2 \widetilde{T}^{1/2} C(T,R)^{1/2}\, \Pi \end{aligned}$$

for all $t \in [0, \widetilde{T}]$ and any $\widetilde{T} \in (0, T]$.

On the other hand, since the trace operator $\Gamma : X_l^T \times X_r^T \mapsto H^{1/4}(0,T) \times H^{1/4}(0,T)$ defined by $\Gamma(p,q) := (p_\xi^\ell(\ell_0, \cdot), q_\xi^\ell(\ell_0, \cdot))$ is continuous, thanks to the continuity of the embedding $H^{1/4}(0,T) \hookrightarrow L^4(0,T)$, we find $C_3 > 0$ (independent of T) such that

$$\|\mathcal{L}'_\ell\|_{L^2(0,\widetilde{T})} \leq C_3 \widetilde{T}^{1/4} C(T,R)^{1/2}\, \Pi.$$

It follows easily from the inequalities above that, if $\widetilde{T}$ is sufficiently small, (7) holds.

- *There exists $\widehat{T} \in (0, T]$ such that $\Lambda : \mathcal{A}_{R,\widehat{T}} \mapsto \mathcal{A}_{R,\widehat{T}}$ is a contraction.*

  It can be proved that there exists $D_1 = D_1(\tau)$ such that $D_1(\tau) \to 0$ as $\tau \to 0$ and

  (9)
  $$\|\mathcal{L}_{\ell_1} - \mathcal{L}_{\ell_2}\|_{H^1(0,\tau)} \leq D_1(\tau) \left(\|p^{\ell_1} - p^{\ell_2}\|_{X_l^\tau} + \|q^{\ell_1} - q^{\ell_2}\|_{X_r^\tau} \right) \quad \forall \ell_1, \ell_2 \in \mathcal{A}_{R,\tau}.$$

  Furthermore, using standard energy estimates, we get a positive $D_2 = D_2(\tau)$, similar to (8), such that

  $$\|p^{\ell_1} - p^{\ell_2}\|_{X_l^\tau} + \|q^{\ell_1} - q^{\ell_2}\|_{X_r^\tau} \leq D_2(\tau) \|(F_l^\ell, F_r^\ell)\|_{L^2(\widetilde{Q}_{0,l}) \times L^2(\widetilde{Q}_{0,r})}, \tag{10}$$

  where

  (11)
  $$F_l^\ell := (d_l^{\ell_1} - d_l^{\ell_2}) p_{\xi\xi}^{\ell_2} - (b_l^{\ell_1} - b_l^{\ell_2}) p_\xi^{\ell_2} \quad \text{and} \quad F_r^\ell := (d_r^{\ell_1} - d_r^{\ell_2}) q_{\xi\xi}^{\ell_2} - (b_r^{\ell_1} - b_r^{\ell_2}) q_\xi^{\ell_2}$$

  and $\widetilde{Q}_{0,l} := (0, \ell_0) \times (0, \tau)$ and $\widetilde{Q}_{0,r} := (\ell_0, L) \times (0, \tau)$.

  Then, using the fact that G and its inverse G^{-1} (defined in section 2.1) are smooth functions, we see that there exists $D_3 = D_3(\tau)$ ($D_3(s)$ is bounded for $0 \leq s \leq T$) such that

  $$\|(F_l^\ell, F_r^\ell)\|_{L^2(\widetilde{Q}_{0,l}) \times L^2(\widetilde{Q}_{0,r})} \leq D_3(\tau) \|(p^{\ell_2}, q^{\ell_2})\|_{X_l^\tau \times X_r^\tau} \|\ell_1 - \ell_2\|_{H^1(0,\tau)}. \tag{12}$$

  Combining (9)–(12), we deduce that

  $$\|\Lambda(\ell_1) - \Lambda(\ell_2)\|_{H^1(0,\tau)} \leq E(\tau) \|\ell_1 - \ell_2\|_{H^1(0,\tau)}, \tag{13}$$

  where $E(\tau) := D_1(\tau) D_2(\tau) D_3(\tau) \|(p^{\ell_2}, q^{\ell_2})\|_{X_l^\tau \times X_r^\tau}$. Since (p^{ℓ_2}, q^{ℓ_2}) is uniformly bounded in $X_l^\tau \times X_r^\tau$ for all $0 < \tau \leq \widetilde{T}$ provided $\ell_2 \in \mathcal{A}_{R,\tau}$, we find that $E(s) \to 0$ as $s \to 0$.

  As a consequence, there exists $\widehat{T} \in (0, \widetilde{T}]$ such that $\Lambda : \mathcal{A}_{R,\widehat{T}} \mapsto \mathcal{A}_{R,\widehat{T}}$ is a contraction.

Therefore, Λ possesses exactly one fixed-point in $\mathcal{A}_{R,\widehat{T}}$. This ends the proof. ☐

3. Approximate controllability of a linearized system. In this section, we are going to complete a first step in the proof of Theorem 2. More precisely, we are going to prove a controllability result for a suitable (natural) linearization of (4).

To do this, let us fix $\ell \in C^1([0,T])$ with $\ell(0) = \ell_0$ and $\ell([0,T]) \subset (\ell_l, \ell_r)$, and let us consider the system

$$\begin{cases} M_l^\ell(p) = h_l 1_{\omega_l} & \text{in} \quad Q_{0,l}, \\ M_r^\ell(q) = h_r 1_{\omega_r} & \text{in} \quad Q_{0,r}, \\ p(0,\cdot) = p(\ell_0,\cdot) = q(\ell_0,\cdot) = q(L,\cdot) = 0 & \text{in} \quad (0,T), \\ p(\cdot,0) = p_0 & \text{in} \quad (0,\ell_0), \\ q(\cdot,0) = q_0 & \text{in} \quad (\ell_0, L), \end{cases} \tag{14}$$

where the operators M_l^ℓ and M_r^ℓ are, respectively, defined by

$$M_l^\ell(p) := p_t - d_l^\ell p_{\xi\xi} + b_l^\ell p_\xi \quad \text{and} \quad M_r^\ell(q) := q_t - d_r^\ell q_{\xi\xi} + b_r^\ell q_\xi.$$

Also, let us introduce the function $\mathcal{L} : [0,T] \mapsto \mathbb{R}$ given by

$$\mathcal{L}(t) := \ell_0 - \int_0^t [d_l p_\xi(\ell_0,\tau) - d_r q_\xi(\ell_0,\tau)]\, d\tau.$$

Remark 3. Using the facts that G and G^{-1} are smooth and $\ell \in C^1([0,T])$, we can prove that d_l^ℓ and b_l^ℓ belong, respectively, to $C^1(\overline{Q}_{0,l})$ and $C^0(\overline{Q}_{0,l})$. Furthermore, the second spatial derivative of d_l^ℓ and the first spatial derivative of b_l^ℓ are functions in $C^0(\overline{Q}_{0,l})$. The same can be obtained for the coefficients d_r^ℓ and b_r^ℓ.

The main goal of this section is to obtain a (robust) approximate controllability result for (14) subject to the linear constraint on the states

$$\mathcal{L}(T) = \ell_T. \tag{15}$$

In other words, we want to find controls $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ such that the associated solutions to (14) satisfy (15).

Let us first reformulate (15). Thus, consider the *auxiliary* adjoint system

$$\begin{cases} (M_l^\ell)^*(\psi) = 0 & \text{in} \quad Q_{0,l}, \\ (M_r^\ell)^*(\zeta) = 0 & \text{in} \quad Q_{0,r}, \\ \psi(0,\cdot) = 0, \quad \psi(\ell_0,\cdot) = 1 & \text{in} \quad (0,T), \\ \zeta(\ell_0,\cdot) = 1, \quad \zeta(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \psi(\cdot,T) = 0 & \text{in} \quad (0,\ell_0), \\ \zeta(\cdot,T) = 0 & \text{in} \quad (\ell_0,L), \end{cases} \tag{16}$$

where the operators $(M_l^\ell)^*$ and $(M_r^\ell)^*$ are, respectively, defined by

$$(M_l^\ell)^*(\psi) := -\psi_t - (d_l^\ell \psi)_{\xi\xi} - (b_l^\ell \psi)_\xi \quad \text{and} \quad (M_r^\ell)^*(\zeta) := -\zeta_t - (d_r^\ell \zeta)_{\xi\xi} - (b_r^\ell \zeta)_\xi.$$

It is not difficult to check that (16) possesses a unique weak solution (ψ_ℓ, ζ_ℓ), with

$$\psi_\ell \in L^2(0,T;H^1(0,\ell_0)) \cap H^1(0,T;H^{-1}(0,\ell_0)),$$

$$\zeta_\ell \in L^2(0,T;H^1(\ell_0,L)) \cap H^1(0,T;H^{-1}(\ell_0,L)).$$

A crucial property of (ψ_ℓ, ζ_ℓ) is the following.

PROPOSITION 3. *Given $R > 0$, let us consider the set $\mathcal{B}_R := \{\ell \in C^1([0,T]);\ \|\ell'\|_{C^0([0,T])} \le R\}$. Then, there exists a positive constant C_0 depending only on ℓ_0, ℓ_l, ℓ_r, ω_l, ω_r, T, and R such that, for any $\ell \in \mathcal{B}_R$, one has*

$$\|\psi_\ell\|_{L^2(\mathcal{O}_l)} + \|\zeta_\ell\|_{L^2(\mathcal{O}_r)} \ge C_0.$$

Proof. We argue by contradiction. Thus, if the assertion were false, then there would exist $\ell_1, \ell_2, \dots$ and associated pairs $(\psi^1, \zeta^1), (\psi^2, \zeta^2), \dots$ (weak solutions to (16)), such that

$$\|\ell_n'\|_\infty \le R \ \text{ and } \ \|\psi^n\|_{L^2(\mathcal{O}_l)} + \|\zeta^n\|_{L^2(\mathcal{O}_r)} < \frac{1}{n} \ \ \forall\, n \ge 1. \tag{17}$$

Due to the smoothing effect of parabolic operators and the facts that the $(d_l^{\ell_n}, b_l^{\ell_n})$ are uniformly bounded in $C^1(\overline{Q}_{0,l}) \times C^0(\overline{Q}_{0,l})$ and the second spatial derivative of d_l^ℓ and the first spatial derivative of b_l^ℓ are uniformly bounded in $C^0(\overline{Q}_{0,l})$, there exists $\sigma > 0$ such that

$$\|\psi^n\|_{L^2(0,T-\sigma;H^2(0,\ell_0))} + \|\psi_t^n\|_{L^2(0,T-\sigma;L^2(0,\ell_0))} \le C \quad \forall\, n \ge 1,$$

with $C > 0$ depending only on ℓ_0, ℓ_l, ℓ_r, T, and R. Consequently, after extraction of a subsequence, we would have

$$\begin{cases} \ell_n \to \ell & \text{strongly in} \quad C^0([0,T-\sigma]), \\ \ell_n \to \ell & \text{weakly in} \quad H^1(0,T-\sigma), \\ \psi^n \to \psi & \text{weakly in} \quad L^2(0,T-\sigma;H^2(0,\ell_0)) \cap H^1(0,T-\sigma;L^2(0,\ell_0)), \end{cases}$$

and we would be able to pass to the limit in the equation and in the boundary condition satisfied by ψ^n to deduce that

$$\begin{cases} (M_l^\ell)^*(\psi) = 0 & \text{in} \quad (0,\ell_0) \times (0,T-\sigma), \\ \psi(0,\cdot) = 0, \ \psi(\ell_0,\cdot) = 1 & \text{in} \quad (0,T-\sigma). \end{cases} \tag{18}$$

But we would also have, by (17), that $\psi \equiv 0$ in $\omega_l \times (0,T-\sigma)$, which is impossible in view of the unique continuation property and $(18)_2$. This ends the proof. ∎

Let us multiply $(14)_1$ by ψ_ℓ, and let us integrate in $Q_{0,l}$ to obtain

$$\iint_{\mathcal{O}_l} h_l \psi_\ell \, d\xi \, dt = -\int_0^{\ell_0} p_0(\xi)\psi_\ell(\xi,0)\, d\xi - \int_0^T d_l p_\xi(\ell_0,\tau)\, d\tau. \tag{19}$$

Analogously, multiplying $(14)_2$ by ζ_ℓ and integrating in $Q_{0,r}$ we get

$$\iint_{\mathcal{O}_r} h_r \zeta_\ell \, d\xi \, dt = -\int_{\ell_0}^{L} q_0(\xi)\zeta_\ell(\xi,0)\, d\xi + \int_0^T d_r q_\xi(\ell_0,\tau)\, d\tau. \tag{20}$$

It follows from (19)–(20) that a pair of controls $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ are such that $\mathcal{L}(T) = \ell_T$ if and only if

(21)
$$\iint_{\mathcal{O}_l} h_l \psi_\ell \, d\xi \, dt + \iint_{\mathcal{O}_r} h_r \zeta_\ell \, d\xi \, dt = \ell_T - \ell_0 - \int_0^{\ell_0} p_0(\xi)\psi_\ell(\xi,0)\, d\xi - \int_{\ell_0}^{L} q_0(\xi)\zeta_\ell(\xi,0)\, d\xi.$$

Accordingly, we see that the role of the auxiliary adjoint problem (16) is to allow a reformulation of the approximate controllability problem for (14) subject to (15) (a constraint on the state) as a control-constrained approximate controllability problem for (14).

In section 3.2, we will establish the approximate controllability of (14) subject to the linear constraint (21). Before this, we will need an adequate (improved) observability inequality.

3.1. An improved observability inequality. To do this, let us first consider open sets $\omega_{0,l} \subset\subset \omega_l$, $\omega_{0,r} \subset\subset \omega_l$, and let us introduce the weight functions $\eta_{0,l} \in C^2([0,\ell_0])$ and $\eta_{0,r} \in C^2([\ell_0, L])$ satisfying

$$\begin{cases} \eta_{0,l} > 0 \text{ in } (0,\ell_0), \ \eta_{0,l}(0) = \eta_{0,l}(\ell_0) = 0, \text{ and } |\eta_{0,l}'| > 0 \text{ in } [0,\ell_0]\backslash\omega_{0,l}, \\ \eta_{0,r} > 0 \text{ in } (\ell_0, L), \ \eta_{0,r}(\ell_0) = \eta_{0,r}(L) = 0, \text{ and } |\eta_{0,r}'| > 0 \text{ in } [\ell_0, L]\backslash\omega_{0,r}. \end{cases}$$

Also, for any $\lambda > 0$, let us set

$$\begin{cases} \mu_l(\xi,t) := \dfrac{e^{\lambda\eta_{0,l}(\xi)}}{t(T-t)}, \ \ \alpha_l(\xi,t) := \dfrac{e^{2\lambda\|\eta_{0,l}\|_\infty} - e^{\lambda\eta_{0,l}(\xi)}}{t(T-t)}, \\[2ex] \mu_r(\xi,t) := \dfrac{e^{\lambda\eta_{0,r}(\xi)}}{t(T-t)}, \ \ \alpha_r(\xi,t) := \dfrac{e^{2\lambda\|\eta_{0,r}\|_\infty} - e^{\lambda\eta_{0,r}(\xi)}}{t(T-t)}. \end{cases}$$

Then, by using the regularity of the coefficients of the adjoint operators $(M_l^\ell)^*$ and $(M_r^\ell)^*$ (see Remark 3) and following the ideas in [11, 20], we get the following global Carleman estimates.

PROPOSITION 4. *Let $R > 0$, and assume that $\ell \in C^1([0,T])$ satisfies $\ell(0) = \ell_0$, $\ell([0,T]) \subset (\ell_l, \ell_r)$, and $\|\ell'\|_{C^0([0,T])} \le R$. Then, there exist positive constants λ_0, s_0, and C (depending on $\ell_l, \ell_r, R, \omega_l, \omega_r$, and T) such that, for any $s \ge s_0$ and $\lambda \ge \lambda_0$, we have*

$$\begin{aligned} (22) \quad & \iint_{Q_{0,l}} e^{-2s\alpha_l} \left[(s\mu_l)^{-1}\left(|\varphi_t|^2 + |\varphi_{\xi\xi}|^2\right) + \lambda^2(s\mu_l)|\varphi_\xi|^2 + \lambda^4(s\mu_l)^3|\varphi|^2\right] d\xi\, dt \\ & \le C\left[\iint_{Q_{0,l}} e^{-2s\alpha_l}\left|(M_l^\ell)^*(\varphi)\right|^2 d\xi\, dt + \iint_{\mathcal{O}_l} e^{-2s\alpha_l}\lambda^4(s\mu_l)^3|\varphi|^2\, d\xi\, dt\right] \end{aligned}$$

and

$$\begin{aligned} (23) \quad & \iint_{Q_{0,r}} e^{-2s\alpha_r} \left[(s\mu_r)^{-1}\left(|\phi_t|^2 + |\phi_{\xi\xi}|^2\right) + \lambda^2(s\mu_r)|\phi_\xi|^2 + \lambda^4(s\mu_r)^3|\phi|^2\right] d\xi\, dt \\ & \le C\left[\iint_{Q_{0,r}} e^{-2s\alpha_r}|(M_r^\ell)^*(\phi)|^2\, d\xi\, dt + \iint_{\mathcal{O}_r} e^{-2s\alpha_r}\lambda^4(s\mu_r)^3|\phi|^2\, d\xi\, dt\right] \end{aligned}$$

for any pair (φ, ϕ) in the Bochner–Sobolev space

$$[L^2(0,T;H_0^1(0,\ell_0)) \cap H^1(0,T;H^{-1}(0,\ell_0))] \times [L^2(0,T;H_0^1(\ell_0,L)) \cap H^1(0,T;H^{-1}(\ell_0,L))]$$

such that $\left((M_l^\ell)^*(\varphi), (M_r^\ell)^*(\phi)\right)$ belongs to $L^2(Q_{0,l}) \times L^2(Q_{0,r})$.

A straightforward argument, based on estimates (22)–(23), leads to the following observability inequality.

PROPOSITION 5. *Let $R > 0$, and assume that $\ell \in C^1([0,T])$ satisfies $\ell(0) = \ell_0$, $\ell([0,T]) \subset (\ell_l, \ell_r)$, and $\|\ell'\|_{C^0([0,T])} \le R$. There exist positive constants λ_0, s_0, and C, depending on ℓ_l, ℓ_r, R, ω_l, ω_r, and T, such that, for any $s \ge s_0$ and any $\lambda \ge \lambda_0$, we have*

$$(24) \qquad \|\varphi(\cdot,0)\|_{L^2(0,\ell_0)} \le C\|\varphi\|_{L^2(\mathcal{O}_l)} \quad \text{and} \quad \|\phi(\cdot,0)\|_{L^2(\ell_0,L)} \le C\|\phi\|_{L^2(\mathcal{O}_r)}$$

for any pair (φ, ϕ) in the Bochner–Sobolev space

$$[L^2(0,T;H_0^1(0,\ell_0)) \cap H^1(0,T;H^{-1}(0,\ell_0))] \times [L^2(0,T;H_0^1(\ell_0,L)) \cap H^1(0,T;H^{-1}(\ell_0,L))]$$

such that $\left((M_l^\ell)^*(\varphi), (M_r^\ell)^*(\phi)\right) = (0,0)$.

In order to present an improved observability inequality, let us introduce the linear projectors $\mathbb{P}_l^\ell : L^2(Q_{0,l}) \mapsto L^2(Q_{0,l})$ and $\mathbb{P}_r^\ell : L^2(Q_{0,r}) \mapsto L^2(Q_{0,r})$, respectively, given by

$$\mathbb{P}_l^\ell\varphi := \beta_l^\ell(\varphi)\psi_\ell \ \text{ and } \ \mathbb{P}_r^\ell\phi := \beta_r^\ell(\phi)\zeta_\ell,$$

where we have set

$$\beta_l^\ell(\varphi) := \frac{\iint_{\mathcal{O}_l} \psi_\ell \varphi \, d\xi \, dt}{\iint_{\mathcal{O}_l} |\psi_\ell|^2 \, d\xi \, dt} \quad \text{and} \quad \beta_r^\ell(\phi) := \frac{\iint_{\mathcal{O}_r} \zeta_\ell \phi \, d\xi \, dt}{\iint_{\mathcal{O}_r} |\zeta_\ell|^2 \, d\xi \, dt},$$

and (ψ_ℓ, ζ_ℓ) is the unique weak solution to (16).

Remark 4. Note that the ranges of $\mathbb{P}_l^\ell$ and $\mathbb{P}_r^\ell$ are 1D vector spaces. Therefore, these operators are compact.

For any $(\varphi_T, \phi_T) \in L^2(0, \ell_0) \times L^2(\ell_0, L)$, there exists a unique pair (φ, ϕ) satisfying

$$\begin{aligned} &\varphi \in L^2(0, T; H_0^1(0, \ell_0)) \cap H^1(0, T; H^{-1}(0, \ell_0)), \\ &\phi \in L^2(0, T; H_0^1(\ell_0, L)) \cap H^1(0, T; H^{-1}(\ell_0, L)) \end{aligned} \tag{25}$$

that solves in the weak sense the linear system

$$\begin{cases} (M_l^\ell)^*(\varphi) = 0 & \text{in} \quad Q_{0,l}, \\ (M_r^\ell)^*(\phi) = 0 & \text{in} \quad Q_{0,r}, \\ \varphi(0, \cdot) = \varphi(\ell_0, \cdot) = 0 & \text{in} \quad (0, T), \\ \phi(\ell_0, \cdot) = \phi(L, \cdot) = 0 & \text{in} \quad (0, T), \\ \varphi(\cdot, T) = \varphi_T & \text{in} \quad (0, \ell_0), \\ \phi(\cdot, T) = \phi_T & \text{in} \quad (\ell_0, L). \end{cases} \tag{26}$$

Accordingly, we can introduce the following functional in $L^2(0, \ell_0) \times L^2(\ell_0, L)$:

$$\begin{aligned} I(\varphi_T, \phi_T) := &\iint_{\mathcal{O}_l} |\varphi|^2 \, d\xi \, dt + \int_0^{\ell_0} |\varphi(\xi, 0)|^2 \, d\xi + |\beta_l^\ell(\varphi)|^2 \\ &+ \iint_{\mathcal{O}_r} |\phi|^2 \, d\xi \, dt + \int_{\ell_0}^L |\phi(\xi, 0)|^2 \, d\xi + |\beta_r^\ell(\phi)|^2, \end{aligned}$$

where (φ, ϕ) satisfies (25)–(26).

We can prove the following result.

Proposition 6. *Let $R > 0$, and let us assume that $\ell \in C^1([0, T])$ satisfies $\ell(0) = \ell_0$, $\ell([0, T]) \subset (\ell_l, \ell_r)$, and $\|\ell'\|_{C^0([0,T])} \le R$. Then, there exists a positive constant C, depending on ℓ_0, ℓ_l, ℓ_r, R, ω_l, ω_r, and T, such that, for any $(\varphi_T, \phi_T) \in L^2(0, \ell_0) \times L^2(\ell_0, L)$, the following holds:*

$$I(\varphi_T, \phi_T) \le C \left[\iint_{\mathcal{O}_l} |\varphi - \mathbb{P}_l^\ell \varphi|^2 \, d\xi \, dt + \iint_{\mathcal{O}_r} |\phi - \mathbb{P}_r^\ell \phi|^2 \, d\xi \, dt \right], \tag{27}$$

where (φ, ϕ) is the solution to (26).

Proof. The proof will be by contradiction. It is inspired by the results in [27].

Let us prove that there exists a constant $C_1 > 0$ (depending on $\ell_0, \ell_l, \ell_r, R, \omega_l, \omega_r$, and T) such that, for any pair of functions $(\varphi_T, \phi_T) \in L^2(0, \ell_0) \times L^2(\ell_0, L)$, one has

$$\begin{aligned} &\iint_{\mathcal{O}_l} |\varphi|^2 \, d\xi \, dt + \int_0^{\ell_0} |\varphi(\xi, 0)|^2 \, d\xi + \iint_{\mathcal{O}_r} |\phi|^2 \, d\xi \, dt + \int_{\ell_0}^L |\phi(\xi, 0)|^2 \, d\xi \\ &\quad \le C_1 \left[\iint_{\mathcal{O}_l} |\varphi - \mathbb{P}_l^\ell \varphi|^2 \, d\xi \, dt + \iint_{\mathcal{O}_r} |\phi - \mathbb{P}_r^\ell \phi|^2 \, d\xi \, dt \right]. \end{aligned} \tag{28}$$

If (28) does not hold, there must exist $(\varphi_{T,1},\phi_{T,1}),(\varphi_{T,2},\phi_{T,2}),\dots$ in $L^2(0,\ell_0)\times L^2(\ell_0,L)$ such that

(29)
$$\begin{cases} \displaystyle\iint_{\mathcal{O}_l}|\varphi_n|^2\,d\xi\,dt+\int_0^{\ell_0}|\varphi_n(\xi,0)|^2\,d\xi+\iint_{\mathcal{O}_r}|\phi_n|^2\,d\xi\,dt+\int_{\ell_0}^{L}|\phi_n(\xi,0)|^2\,d\xi=1 \quad\text{and}\\ \displaystyle\iint_{\mathcal{O}_l}|\varphi_n-\mathbb{P}_l^{\ell}\varphi_n|^2\,d\xi\,dt+\iint_{\mathcal{O}_r}|\phi_n-\mathbb{P}_r^{\ell}\phi_n|^2\,d\xi\,dt\le\frac{1}{n}\end{cases}$$

for all $n\ge 1$. Noting that

$$\frac{1}{2}\iint_{\mathcal{O}_l}|\mathbb{P}_l^{\ell}\varphi_n|^2\,d\xi\,dt+\frac{1}{2}\iint_{\mathcal{O}_r}|\mathbb{P}_r^{\ell}\phi_n|^2\,d\xi\,dt$$
$$\le\iint_{\mathcal{O}_l}\left[|\varphi_n|^2+|\varphi_n-\mathbb{P}_l^{\ell}\varphi_n|^2\right]\,d\xi\,d+\iint_{\mathcal{O}_r}\left[|\phi_n|^2+|\phi_n-\mathbb{P}_l^{\ell}\phi_n|^2\right]\,d\xi\,dt,$$

we easily get from (29) that the $(\beta_l^{\ell}(\varphi_n),\beta_r^{\ell}(\phi_n))$ are uniformly bounded in $\mathbb{R}^2$. Consequently, there exist a subsequence (again indexed by n) and a pair $(\beta_l^*,\beta_r^*)\in\mathbb{R}^2$ such that

$$(\beta_l^{\ell}(\varphi_n),\beta_r^{\ell}(\phi_n))\to(\beta_l^*,\beta_r^*)\quad\text{in }\mathbb{R}^2. \tag{30}$$

It is clear from (22), (23), and $(29)_1$ that, at least for a new subsequence, one has

$$\begin{aligned}&\varphi_n\to\varphi\ \text{ weakly in }\ L^2(\sigma,T-\sigma;H^2(0,\ell_0)\cap H_0^1(0,\ell_0)),\\ &\varphi_{n,t}\to\varphi_t\ \text{ weakly in }\ L^2(\sigma,T-\sigma;H^{-1}(0,\ell_0)),\\ &\zeta_n\to\zeta\ \text{ weakly in }\ L^2(\sigma,T-\sigma;H^2(\ell_0,L)\cap H_0^1(\ell_0,L)),\\ &\zeta_{n,t}\to\zeta_t\ \text{ weakly in }\ L^2(\sigma,T-\sigma;H^{-1}(\ell_0,L))\end{aligned}$$

for all $\sigma>0$ small enough. Obviously, we have

$$\begin{cases}(M_l^{\ell})^*(\varphi)=0 & \text{in}\quad Q_{0,l},\\ (M_r^{\ell})^*(\phi)=0 & \text{in}\quad Q_{0,r},\\ \varphi(0,\cdot)=\varphi(\ell_0,\cdot)=0 & \text{in}\quad (0,T),\\ \phi(\ell_0,\cdot)=\phi(L,\cdot)=0 & \text{in}\quad (0,T).\end{cases} \tag{31}$$

Moreover, since $(\varphi_n,\phi_n)=(\varphi_n-\mathbb{P}_l^{\ell}\varphi_n,\phi_n-\mathbb{P}_r^{\ell}\phi_n)+(\mathbb{P}_l^{\ell}\varphi_n,\mathbb{P}_r^{\ell}\phi_n)$ in $\mathcal{O}_l\times\mathcal{O}_r$, using $(29)_2$ and (30), it is also true that

$$(\varphi_n,\phi_n)\to(\mathbb{P}_l^*\varphi,\mathbb{P}_r^*\phi)\quad\text{strongly in }\ L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r),$$

where $(\mathbb{P}_l^*\varphi,\mathbb{P}_r^*\phi)=(\beta_l^*\psi_\ell,\beta_r^*\zeta_\ell)$.

We have from (16) and (31) that $((M_l^{\ell})^*(\varphi-\mathbb{P}_l^*\varphi),(M_r^{\ell})^*(\phi-\mathbb{P}_r^*\phi))=(0,0)$ in $Q_{0,l}\times Q_{0,r}$ and also that, in view of (29) and (30), $(\varphi-\mathbb{P}_l^*\varphi,\phi-\mathbb{P}_l^*\phi)=(0,0)$ in $\mathcal{O}_l\times\mathcal{O}_r$. Then, by applying a classical *unique continuation* argument, we conclude that $(\varphi,\phi)=(\mathbb{P}_l^*\varphi,\mathbb{P}_r^*\phi)$ in $Q_{0,l}\times Q_{0,r}$. However, this implies $(\varphi,\phi)=(0,0)$ in $Q_{0,l}\times Q_{0,r}$, since

$$(0,0)=(\varphi(\ell_0,\cdot),\phi(\ell_0,\cdot))=(\beta_l^*\psi_\ell(\ell_0,\cdot),\beta_r^*\zeta_\ell(\ell_0,\cdot))=(\beta_l^*,\beta_r^*).$$

In other words,

$$(\varphi_n,\phi_n)\to(0,0)\quad\text{in }L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r).$$

Then, taking into account (24) and (29), we see that

$$\iint_{\mathcal{O}_l} |\varphi_n|^2 \, d\xi \, dt + \int_0^{\ell_0} |\varphi_n(\xi,0)|^2 \, d\xi + \iint_{\mathcal{O}_r} |\phi_n|^2 \, d\xi \, dt + \int_{\ell_0}^{L} |\phi_n(\xi,0)|^2 \, d\xi \to 0,$$

which is obviously absurd.

This proves (28). The remaining terms in $I(\varphi_T, \phi_T)$ can also be bounded by the right-hand side of (27) as an immediate consequence of Proposition 3. ∎

3.2. Approximate controllability problem with linear constraint. In this section, we prove the approximate controllability of (14) subject to the linear constraint (21). More precisely, the following holds.

PROPOSITION 7. *Assume that $R > 0$, $\ell_0 \in (\ell_l, \ell_r)$, and $\ell \in C^1([0,T])$ satisfy $\ell_l < \ell(t) < \ell_r$ for all $t \in [0,T]$, $\ell(0) = \ell_0$, and $\|\ell'\|_{C^0([0,T])} \leq R$. Then, for any $\varepsilon > 0$, any data $p_0 \in H_0^1(0,\ell_0)$ and $q_0 \in H_0^1(\ell_0, L)$, and any $\ell_T \in (\ell_l, \ell_r)$, there exist controls $(h^\ell_{l,\varepsilon}, h^\ell_{r,\varepsilon}) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and associated solutions to* (14)*, with*

$$\begin{cases} p \in L^2(0,T; H^2(0,\ell_0)) \cap H^1(0,T; L^2(0,\ell_0)), \\ q \in L^2(0,T; H^2(\ell_0,L)) \cap H^1(0,T; L^2(\ell_0,L)), \end{cases}$$

satisfying the approximate controllability condition

$$\|(p(\cdot,T), q(\cdot,T))\|_{L^2(0,\ell_0)\times L^2(\ell_0,L)} \leq \varepsilon \tag{32}$$

and the linear constraint (21)*. Furthermore, the controls can be chosen to satisfy*

$$\|((h^\ell_{l,\varepsilon} 1_{\omega_l}), (h^\ell_{r,\varepsilon} 1_{\omega_l}))\|_{L^2(Q_{0,l})\times L^2(Q_{0,r})} \leq C\left(\|(p_0,q_0)\|_{L^2\times L^2} + |\ell_0 - \ell_T|\right), \tag{33}$$

where the constant $C > 0$ depends only on $\ell_l, \ell_r, \omega_l, \omega_r, T$, and R.

Proof. Let us first introduce the notation

$$M_\ell := \ell_T - \ell_0 - \int_0^{\ell_0} p_0(\xi)\psi_\ell(\xi,0)\, d\xi - \int_{\ell_0}^{L} q_0(\xi)\zeta_\ell(\xi,0)\, d\xi,$$

where the pair (ψ_ℓ, ζ_ℓ) is the unique solution to (16).

Now, for any given $\varepsilon > 0$, let us introduce the functional $J_{\ell,\varepsilon} : L^2(0,\ell_0) \times L^2(\ell_0, L) \mapsto \mathbb{R}$, defined as follows: given $(\varphi_T, \phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0,L)$, we have

$$\begin{aligned} J_{\ell,\varepsilon}(\varphi_T,\phi_T) := & \iint_{\mathcal{O}_l} |\varphi - \mathbb{P}^\ell_l \varphi|^2 \, d\xi \, dt + \iint_{\mathcal{O}_r} |\phi - \mathbb{P}^\ell_r \phi|^2 \, d\xi \, dt + \frac{\varepsilon}{2}\|(\varphi_T,\phi_T)\|_{L^2\times L^2} \\ & - \int_0^{\ell_0} p_0(\xi)\varphi(\xi,0)\, d\xi - \int_{\ell_0}^{L} q_0(\xi)\phi(\xi,0)\, d\xi - \left[\beta^\ell_l(\varphi) + \beta^\ell_r(\phi)\right] \frac{M_\ell}{2}, \end{aligned} \tag{34}$$

where the pair (φ, ϕ) satisfies (25)–(26).

Using Hölder and Young inequalities, it is not difficult to check that $J_{\ell,\varepsilon}$ is a continuous, coercive, and strictly convex functional. Therefore, $J_{\ell,\varepsilon}$ possesses a unique minimizer $(\varphi^\varepsilon_T, \phi^\varepsilon_T) \in L^2(0,\ell_0) \times L^2(\ell_0,L)$. The corresponding solution to (26) will be denoted by $(\varphi_\varepsilon, \phi_\varepsilon)$. Then

$$J'_{\ell,\varepsilon}(\varphi^\varepsilon_T, \phi^\varepsilon_T)(\varphi_T,\phi_T) = 0 \quad \forall (\varphi_T,\phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0,L), \tag{35}$$

where

$$\begin{aligned} J'_{\ell,\varepsilon}(\varphi_{T,\varepsilon},\phi_{T,\varepsilon})(\varphi_T,\phi_T) &= \iint_{\mathcal{O}_l}[\varphi_\varepsilon - \mathbb{P}_l^\ell(\varphi_\varepsilon)]\varphi\,d\xi\,dt + \iint_{\mathcal{O}_r}[\phi_\varepsilon - \mathbb{P}_r^\ell(\phi_\varepsilon)]\phi\,d\xi\,dt \\ &\quad + \frac{\varepsilon}{2\|\varphi_T^\varepsilon\|_{L^2}}\int_0^{\ell_0}\varphi_T^\varepsilon(\xi)\varphi_T(\xi)\,d\xi + \frac{\varepsilon}{2\|\phi_T^\varepsilon\|_{L^2}}\int_{\ell_0}^{L}\phi_T^\varepsilon(\xi)\phi_T(\xi)\,d\xi \\ &\quad - \int_0^{\ell_0}p_0(\xi)\varphi(\xi,0)\,d\xi - \int_{\ell_0}^{L}q_0(\xi)\phi(\xi,0)\,d\xi - \left[\beta_l^\ell(\varphi)+\beta_r^\ell(\phi)\right]\frac{M_\ell}{2}. \end{aligned}$$

Here, we have used the facts that $\langle\varphi_\varepsilon - \mathbb{P}_l^\ell(\varphi_\varepsilon), \mathbb{P}_l^\ell(\varphi)\rangle_{L^2(\mathcal{O}_l)} = 0$ and $\langle\phi_\varepsilon - \mathbb{P}_r^\ell(\phi_\varepsilon), \mathbb{P}_r^\ell(\phi)\rangle_{L^2(\mathcal{O}_r)} = 0$.

Let us introduce

(36)
$$h_{l,\varepsilon}^\ell := \left[\mathbb{P}_l^\ell(\varphi_\varepsilon) - \varphi_\varepsilon\right] + \frac{M_\ell}{2}\frac{\psi_\ell}{\|\psi_\ell\|^2_{L^2(\mathcal{O}_l)}} \quad\text{and}\quad h_{r,\varepsilon}^\ell := \left[\mathbb{P}_r^\ell(\phi_\varepsilon) - \phi_\varepsilon\right] + \frac{M_\ell}{2}\frac{\zeta_\ell}{\|\zeta_\ell\|^2_{L^2(\mathcal{O}_r)}}.$$

Let $(\varphi_T,\phi_T)\in L^2(0,\ell_0)\times L^2(\ell_0,L)$ be given, and let (p,q) be the solution to (14) associated to the control pair $(h_{l,\varepsilon}^\ell, h_{r,\varepsilon}^\ell)$. Then, multiplying (14) by the solution (φ,ϕ) to (26) and integrating in $Q_{0,l}$ and $Q_{0,r}$, we obtain

$$\begin{aligned} \iint_{\mathcal{O}_l}h_{l,\varepsilon}^\ell\varphi\,d\xi\,dt + \iint_{\mathcal{O}_r}h_{r,\varepsilon}^\ell\phi\,d\xi\,dt &= \int_0^{\ell_0}[p(\xi,T)\varphi(\xi,T) - p_0(\xi)\varphi(\xi,0)]\,d\xi \\ &\quad + \int_{\ell_0}^{L}[q(\xi,T)\phi(\xi,T) - q_0(\xi)\phi(\xi,0)]\,d\xi. \end{aligned} \tag{37}$$

Taking into account (36) and comparing (35) with (37), we get

$$\begin{aligned} &\int_0^{\ell_0}p(\xi,T)\varphi_T(\xi)\,d\xi + \int_{\ell_0}^{L}q(\xi,T)\phi_T(\xi)\,d\xi \\ &\quad = \frac{\varepsilon}{2}\left(\int_0^{\ell_0}\frac{\varphi_T^\varepsilon(\xi)}{\|\varphi_T^\varepsilon\|_{L^2}}\varphi_T(\xi)\,d\xi + \int_{\ell_0}^{L}\frac{\phi_T^\varepsilon(\xi)}{\|\phi_T^\varepsilon\|_{L^2}}\phi_T(\xi)\,d\xi\right) \end{aligned}$$

for all $(\varphi_T,\phi_T)\in L^2(0,\ell_0)\times L^2(\ell_0,L)$. Therefore, the approximate controllability condition (32) follows. Since we also have

$$\begin{aligned} \iint_{\mathcal{O}_l}h_{l,\varepsilon}^\ell\psi_\ell\,d\xi\,dt + \iint_{\mathcal{O}_r}h_{r,\varepsilon}^\ell\zeta_\ell\,d\xi\,dt &= \iint_{\mathcal{O}_l}\left[\mathbb{P}_l^\ell(\varphi_\varepsilon)-\varphi_\varepsilon\right]\psi_\ell\,d\xi\,dt + \frac{M_\ell}{2} \\ &\quad + \iint_{\mathcal{O}_r}\left[\mathbb{P}_r^\ell(\phi_\varepsilon)-\phi_\varepsilon\right]\zeta_\ell\,d\xi\,dt + \frac{M_\ell}{2} \\ &= M_\ell, \end{aligned}$$

the pair $(h_{l,\varepsilon}^\ell, h_{r,\varepsilon}^\ell)$ satisfies (21) and, consequently, $\mathcal{L}(T)=\ell_T$.

Finally, due to the fact that $(\varphi_{T,\varepsilon},\phi_{T,\varepsilon})$ is the minimum of $J_{\ell,\varepsilon}$, we have the inequality $J_{\ell,\varepsilon}(\varphi_T^\varepsilon,\phi_T^\varepsilon)\le J_{\ell,\varepsilon}(0,0)=0$. Using this fact and the definition of M_ℓ and (27), we deduce that there exist positive constants C (depending on $\ell_l,\ell_r,R,\omega_l,\omega_r$, and T) such that

$$\begin{aligned} &\|(\varphi_\varepsilon - \mathbb{P}_l^\ell(\varphi_\varepsilon))\|_{L^2(\mathcal{O}_l)} + \|(\phi_\varepsilon - \mathbb{P}_r^\ell(\phi_\varepsilon))\|_{L^2(\mathcal{O}_r)} \\ &\quad \le C\left(\|p_0\|_{L^2(0,\ell_0)} + \|q_0\|_{L^2(\ell_0,L)} + |\ell_0-\ell_T|\right) \end{aligned}$$

and

$$\begin{aligned}\|h^{\ell}_{l,\varepsilon}\|_{L^2(\mathcal{O}_l)} + \|h^{\ell}_{l,\varepsilon}\|_{L^2(\mathcal{O}_r)} &\le C\left(\|(\varphi_\varepsilon - \mathbb{P}^{\ell}_l(\varphi_\varepsilon))\|_{L^2(\mathcal{O}_l)} + \|(\phi_\varepsilon - \mathbb{P}^{\ell}_r(\phi_\varepsilon))\|_{L^2(\mathcal{O}_r)} + |M_\ell|\right)\\ &\le C\left(\|p_0\|_{L^2(0,\ell_0)} + \|q_0\|_{L^2(\ell_0,L)} + |\ell_0 - \ell_T|\right).\end{aligned}$$

This ends the proof. ∎

4. Controllability of the two-phase Stefan problem. In this section we prove Theorem 2. The proof relies on a fixed-point argument. First, it will be convenient to recall some regularity properties for linear parabolic systems.

Due to Propositions 1 and 2, the smoothing effect of (4) implies that we can assume that $(p_0, q_0) \in W^{1,4}_0(0,\ell_0) \times W^{1,4}_0(\ell_0, L)$, and we can consider smallness assumptions of (p_0, q_0) in this space.

4.1. A regularity property. First, let us assume that $(p_0, q_0) \in W^{1,4}_0(0,\ell_0) \times W^{1,4}_0(\ell_0, L)$. For any open interval $I \subset \mathbb{R}$, let us introduce the Banach space

$$X^4(0,T;I) := L^4(0,T;W^{2,4}(I)) \cap W^{1,4}(0,T;L^4(I)).$$

On the other hand, let us consider the cylinder $G_l := (\ell_l, \ell_0) \times (0,T)$, the Hölder seminorms

$$\langle u\rangle^{\kappa}_{\xi,G_l} := \sup_{\substack{(\xi,t),(\xi',t)\in\overline{G}_l\\ \xi\neq\xi'}} \frac{|u(\xi,t) - u(\xi',t)|}{|\xi-\xi'|^{\kappa}}$$

and

$$\langle u\rangle^{\kappa}_{t,G_l} := \sup_{\substack{(\xi,t),(\xi,t')\in\overline{G}_l\\ t\neq t'}} \frac{|u(\xi,t) - u(\xi,t')|}{|t-t'|^{\kappa}},$$

where $0 < \kappa < 1$, and the space $C^{\kappa,\kappa/2}(\overline{G}_l)$ formed by the functions $u \in C^0(\overline{G}_l)$ whose corresponding $\langle u\rangle^{\kappa}_{\xi,G_l}$ and $\langle u\rangle^{\kappa/2}_{t,G_l}$ are finite. It is known that $C^{\kappa,\kappa/2}(\overline{G}_l)$ is a Banach space (see [24]) with the following norm:

$$\|u\|_{\kappa,\kappa/2;\overline{G}_l} := \|u\|_{C^0(\overline{G}_l)} + \langle u\rangle^{\kappa}_{\xi,G_l} + \langle u\rangle^{\kappa/2}_{t,G_l}.$$

Finally, let us introduce the Banach space

$$C^{1+\kappa,(1+\kappa)/2}(\overline{G}_l) := \{u \in C^0(\overline{G}_l) : u_\xi \in C^{\kappa,\kappa/2}(\overline{G}_l), \quad \langle u\rangle^{(1+\kappa)/2}_{t,G_l} < +\infty\}.$$

Obviously, we can introduce similar quantities and spaces for functions defined in $G_r := (\ell_0, \ell_r) \times (0,T)$. The following result holds.

LEMMA 1. *Let us assume that $\ell_0, \ell_T \in (\ell_l, \ell_r)$ and $(p_0, q_0) \in W^{1,4}_0(0,\ell_0) \times W^{1,4}_0(\ell_0, L)$. Then, the states (p,q), furnished by Proposition 7, satisfy*

$$(p,q) \in C^{1+\kappa,(1+\kappa)/2}(\overline{G}_l) \times C^{1+\kappa,(1+\kappa)/2}(\overline{G}_r) \quad \textit{for } \kappa = 1/4.$$

Furthermore, there exists $C > 0$, depending on ℓ_l, ℓ_r, ω_l, ω_r, T, and R, such that

$$\|p\|_{1+\kappa,(1+\kappa)/2;\overline{G}_l} + \|q\|_{1+\kappa,(1+\kappa)/2;\overline{G}_r} \le C\left(\|(p_0,q_0)\|_{W^{1,4}_0\times W^{1,4}_0} + |\ell_0 - \ell_T|\right). \tag{38}$$

Proof. Clearly, due to the regularity of p_0, there exists a function $f \in X^4(0,T;(0,\ell_0))$ such that $f(0,t) = f(\ell_0,t) = 0$, for $t \in (0,T)$, and $f(\xi,0) = p_0(\xi)$, for $\xi \in (0,\ell_0)$. Consequently, the state p, provided by Proposition 7, can be written in the form

$p = y + f$, where $y \in L^2(0,T;H^2(0,\ell_0)) \cap H^1(0,T;L^2(0,\ell_0))$ is the unique strong solution of the problem

$$\begin{cases} y_t - d_l^\ell y_{\xi\xi} + b_l^\ell y_\xi = F & \text{in} \quad Q_{0,l}, \\ y(0,\cdot) = y(\ell_0,\cdot) = 0 & \text{in} \quad (0,T), \\ y(\cdot,0) = 0 & \text{in} \quad (0,\ell_0), \end{cases} \tag{39}$$

where $F = h_{l,\varepsilon}^\ell 1_{\omega_l} - f_t + d_l^\ell f_{\xi\xi} - b_l^\ell f_\xi$.

Now, let $\sigma > 0$ be such that $\omega_l \subset\subset (0,\ell_l - \sigma)$, and, moreover, let $G_l^\sigma := (\ell_l - \sigma, \ell_l + \sigma) \times (0,T) \subset Q_{0,l}$. We can easily check that $F \in L^4(0,T;L^4(\ell_l - \sigma, \ell_l + \sigma))$. Therefore, from local parabolic regularity results, we obtain that $y \in X^4(0,T;(\ell_l - \sigma/2, \ell_l + \sigma/2))$ and

$$\begin{aligned} &\|y\|_{X^4(0,T;(\ell_l-\sigma/2,\ell_l+\sigma/2))} \\ &\quad \le C\left(\|F\|_{L^4(0,T;L^4(\ell_l-\sigma,\ell_l+\sigma))} + \|y\|_{L^2(0,T;H^2(0,\ell_0))\cap H^1(0,T;L^2(0,\ell_0))}\right), \end{aligned}$$

where C depends only on $\|d_l^\ell\|_\infty$, $\|b_l^\ell\|_\infty$, ℓ_l, ℓ_0, and σ.

Next, using standard parabolic energy estimates and (33), we get

$$\|y\|_{X^4(0,T;(\ell_l-\sigma/2,\ell_l+\sigma/2))} \le C\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}} + |\ell_0 - \ell_T|\right)$$

for some $C > 0$ as above. Here, we have used the fact that $\|d_l^\ell\|_\infty$ and $\|b_l^\ell\|_\infty$ are bounded in terms of R. Finally, using this inequality, the regularity of the trace $y(\ell_l,\cdot)$, the fact that y is a strong solution to (39), and [33, Propositions 9.2.3 and 9.2.5], we conclude that $y \in X^4(0,T;(\ell_l,\ell_0))$, and, moreover,

$$\|y\|_{X^4(0,T;(\ell_l,\ell_0))} \le C\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}} + |\ell_0 - \ell_T|\right) \tag{40}$$

for a new $C > 0$.

In a similar way, we can write $q = z + g$, where $g \in X^4(0,T;(\ell_0,L))$ is a shift function for the initial data q_0, and $z \in X^4(0,T;(\ell_0,\ell_r))$ satisfies

$$\|z\|_{X^4(0,T;(\ell_0,\ell_r))} \le C\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}} + |\ell_0 - \ell_T|\right). \tag{41}$$

Then, the estimate in (38) is an immediate consequence of (40)–(41) and the following embedding from [3, Lemma 2.2]:

$$X^4(0,T;(\ell_l,\ell_0)) \times X^4(0,T;(\ell_0,\ell_r)) \hookrightarrow C^{1+\kappa,(1+\kappa)/2}(\overline{G}_l) \times C^{1+\kappa,(1+\kappa)/2}(\overline{G}_r),$$

where $\kappa = 1/4$. $\square$

Let us introduce the function $\theta : [0,T] \mapsto \mathbb{R}$, given by

$$\theta(t) = d_r q_\xi(\ell_0,t) - d_l p_\xi(\ell_0,t). \tag{42}$$

Then, as an immediate consequence of (38), we get that $\theta \in C^{1/8}([0,T])$, and, moreover, there exists a positive constant C (depending on $\ell_l, \ell_r, \omega_l, \omega_r, T$, and R) such that

$$\|\theta\|_{C^{1/8}([0,T])} \le C\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}} + |\ell_0 - \ell_T|\right). \tag{43}$$

4.2. A fixed-point argument. In this section we will achieve the proof of Theorem 2. It will be a consequence of the following uniform approximate controllability result.

THEOREM 3. *Assume that $R > 0$ is given. Then, there exists $\delta > 0$ such that, for any $p_0 \in W_0^{1,4}(0, \ell_0)$ with $p_0 \geq 0$, any $q_0 \in W_0^{1,4}(\ell_0, L)$ with $q_0 \leq 0$, any $\ell_0, \ell_T \in (\ell_l, \ell_r)$ satisfying*

$$\|p_0\|_{W_0^{1,4}(0,\ell_0)} + \|q_0\|_{W_0^{1,4}(\ell_0,L)} + |\ell_0 - \ell_T| \leq \delta,$$

and any $\varepsilon > 0$, there exist controls $(h_l^\varepsilon, h_r^\varepsilon) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and associated solutions to (4), with

$$\left\{\begin{array}{l} \ell_\varepsilon \in C^1([0,T]) \text{ and } \ell_\varepsilon(t) \in (\ell_l, \ell_r)\, \forall\, t \in [0,T], \ \|\ell_\varepsilon'\|_{C^0([0,T])} \leq R, \\ p_\varepsilon \in L^2(0,T;H^2(0,\ell_0)) \cap H^1(0,T;L^2(0,\ell_0)), \\ q_\varepsilon \in L^2(0,T;H^2(\ell_0,L)) \cap H^1(0,T;L^2(\ell_0,L)), \end{array}\right. \tag{44}$$

satisfying the exact-approximate controllability condition

$$\ell_\varepsilon(T) = \ell_T \quad \text{and} \quad \|(p_\varepsilon(\cdot,T), q_\varepsilon(\cdot,T))\|_{L^2(0,\ell_0)\times L^2(\ell_0,L)} \leq \varepsilon. \tag{45}$$

Moreover, the controls can be found satisfying the following uniform estimate with respect to ε:

$$\|(h_l^\varepsilon 1_{\omega_l}, h_r^\varepsilon 1_{\omega_r})\|_{L^2(Q_{0,l})\times L^2(Q_{0,r})} \leq C\left(\|(p_0,q_0)\|_{W^{1,4}\times W^{1,4}} + |\ell_0 - \ell_T|\right) \tag{46}$$

for some positive C (depending on $\ell_l, \ell_r, \omega_l, \omega_r, T$, and R).

Proof. Given $\ell_l < \tilde{\ell}_l < \tilde{\ell}_r < \ell_r$ and $R > 0$, we define the set

$$\mathcal{A}_R := \{\ell \in C^1([0,T]) : \tilde{\ell}_l \leq \ell(t) \leq \tilde{\ell}_r \ \forall t \in [0,T], \ \ \ell(0) = \ell_0, \ \|\ell'\|_{C^0([0,T])} \leq R\}.$$

Obviously, $\mathcal{A}_R$ is a nonempty, closed, and convex subset of $C^1([0,T])$. Let us also introduce the mapping $\Lambda_\varepsilon : \mathcal{A}_R \mapsto C^1([0,T])$, given by

$$\Lambda_\varepsilon(\ell) = \mathcal{L}, \quad \text{with} \quad \mathcal{L}(t) := \ell_0 - \int_0^t \left[d_l p_\xi(\ell_0,\tau) - d_r q_\xi(\ell_0,\tau)\right] d\tau,$$

where (p,q) is the state associated to the control pair $(h_{l,\varepsilon}^\ell, h_{r,\varepsilon}^\ell)$ constructed as in the proof of Proposition 7 (recall Lemma 1) and, therefore, $\mathcal{L}(T) = \ell_T$. Thanks to (42) and (43), we have that $\mathcal{L} \in C^1([0,T])$.

Let us check that Λ_ε satisfies the following conditions of Schauder's fixed-point theorem:

- Λ_ε *is continuous.* Indeed, let the $\ell_n (n \geq 1)$ and ℓ belong to $\mathcal{A}_R$, and assume that $\ell_n \to \ell$ in $C^1([0,T])$. We must prove that $\Lambda_\varepsilon(\ell_n) \to \Lambda_\varepsilon(\ell)$ in $C^1([0,T])$. To that end, we will first prove that the corresponding solutions to (16) satisfy

$$(\psi_{\ell_n}, \zeta_{\ell_n}) \to (\psi_\ell, \zeta_\ell) \quad \text{strongly in} \quad L^2(Q_{0,l}) \times L^2(Q_{0,r}). \tag{47}$$

  Let $f \in L^2(0,T;H^2(0,\ell_0)) \times H^1(0,T;L^2(0,\ell_0))$ be such that $f(0,\cdot) = 0$ and $f(\ell_0,\cdot) = 1$ on $(0,T)$, and let us put $\psi_{\ell_n} = \Psi_{\ell_n} + f$ and $\psi_\ell = \Psi_\ell + f$. It is then clear that $y_{\ell_n} := \Psi_{\ell_n} - \Psi_\ell$ is the unique weak solution to

$$\left\{\begin{array}{lll} (M_l^{\ell_n})^*(y_{\ell_n}) = F_{\ell_n} & \text{in} & Q_{0,l}, \\ y_{\ell_n}(0,\cdot) = y_{\ell_n}(\ell_0,\cdot) = 0 & \text{in} & (0,T), \\ y_{\ell_n}(\cdot,0) = 0 & \text{in} & (0,\ell_0), \end{array}\right.$$

where $F_{\ell_n} \in L^2(0,T;H^{-1}(0,\ell_0))$ is given by

$$F_{\ell_n} := (d^{\ell}_{l,\xi\xi} - d^{\ell_n}_{l,\xi\xi})f + 2(d^{\ell}_{l,\xi} - d^{\ell_n}_{l,\xi})f_\xi + (d^{\ell}_{l} - d^{\ell_n}_{l})f_{\xi\xi} + (b^{\ell}_{l,\xi} - b^{\ell_n}_{l,\xi})f \\ + (b^{\ell}_{l} - b^{\ell_n}_{l})f_\xi + ((d^{\ell}_{l} - d^{\ell_n}_{l})\Psi_\ell)_{\xi\xi} + ((b^{\ell}_{l} - b^{\ell_n}_{l})\Psi_\ell)_{\xi}.$$

Then, using the facts that $(d^{\ell_n}_l, b^{\ell_n}_l)$ are uniformly bounded in the space $C^1(\overline{Q}_{0,l}) \times C^0(\overline{Q}_{0,l})$ and $(d^{\ell_n}_{l,\xi\xi}, b^{\ell_n}_{l,\xi})$ are uniformly bounded in $C^0(\overline{Q}_{0,l}) \times C^0(\overline{Q}_{0,l})$, the standard parabolic energy estimates, and the regularity of the function G (as well as the regularity of its inverse G^{-1}), we get that $y_{\ell_n} \to 0$ strongly in $L^2(Q_{0,l})$ which, in turn, implies

$$\psi_{\ell_n} \to \psi_\ell \ \text{ strongly in } \ L^2(Q_{0,l}).$$

Analogously, we can prove that $\zeta_{\ell_n} \to \zeta_\ell$ strongly in $L^2(Q_{0,r})$.

Now, we recall that, for each $\varepsilon > 0$, there exists a unique $(\varphi^n_{T,\varepsilon}, \phi^n_{T,\varepsilon})$ in $L^2(0,\ell_0) \times L^2(\ell_0, L)$ that minimizes the functional $J_{\ell_n,\varepsilon}$, defined in (34). Due to the facts that $J_{\ell_n,\varepsilon}(\varphi^n_{T,\varepsilon}, \phi^n_{T,\varepsilon}) \le 0$ and the constant appearing in the right side of (27) does not depend on n, we get that the minimizers are uniformly bounded with respect to n in the space $L^2(0,\ell_0) \times L^2(\ell_0, L)$, and the corresponding $(\varphi^n_\varepsilon, \phi^n_\varepsilon)$, solutions to (26), are uniformly bounded spaces given in (25). Therefore, there exist $(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon})$ in $L^2(0,\ell_0) \times L^2(\ell_0, L)$ and $(\varphi_\varepsilon, \phi_\varepsilon)$ in $L^2(Q_{0,l}) \times L^2(Q_{0,r})$ such that, at least for a subsequence, one has

(48)
$$\begin{cases} (\varphi^n_{T,\varepsilon}, \phi^n_{T,\varepsilon}) \to (\varphi_{T,\varepsilon}, \phi_{T,\varepsilon}) \ \text{ weakly in } \ L^2(0,\ell_0) \times L^2(\ell_0, L), \\ (\varphi^n_\varepsilon(\cdot,0), \phi^n_\varepsilon(\cdot,0)) \to (\varphi_\varepsilon(\cdot,0), \phi_\varepsilon(\cdot,0)) \text{ weakly in } L^2(0,\ell_0) \times L^2(\ell_0, L), \ \text{ and} \\ (\varphi^n_\varepsilon, \phi^n_\varepsilon) \to (\varphi_\varepsilon, \phi_\varepsilon) \ \text{ strongly in } \ L^2(Q_{0,l}) \times L^2(Q_{0,r}). \end{cases}$$

We will show now that $(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon})$ is the unique minimizer of the functional $J_{\ell,\varepsilon}$. Indeed, we first note from the convergences in (47) and $(48)_3$ that

$$(\mathbb{P}^{\ell_n}_l(\varphi^n_\varepsilon), \mathbb{P}^{\ell_n}_r(\phi^n_\varepsilon)) \to (\mathbb{P}^{\ell}_l(\varphi_\varepsilon), \mathbb{P}^{\ell}_r(\phi_\varepsilon)) \quad \text{strongly in} \quad L^2(Q_{0,l}) \times L^2(Q_{0,r}).$$

Then, using this fact and the weak convergences $(48)_{1,2}$, we easily get

$$J_{\ell,\varepsilon}(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon}) \le \liminf_n J_{\ell_n,\varepsilon}(\varphi^n_{T,\varepsilon}, \phi^n_{T,\varepsilon}). \tag{49}$$

Now, let (φ_T, ϕ_T) be given in $L^2(0,\ell_0) \times L^2(\ell_0, L)$, and let the (φ^n, ϕ^n) be the solutions to the system (26), with ℓ replaced by ℓ_n, for $n = 1, 2, \ldots$. Then, using the same ideas that led to (47), we can ensure that the (φ^n, ϕ^n) converge strongly in $L^2(Q_{0,l}) \times L^2(Q_{0,r})$ to the solution (φ, ϕ) to (26) and the $(\varphi^n(\cdot,0), \phi^n(\cdot,0))$ converge weakly in $L^2(0,\ell_0) \times L^2(\ell_0, L)$ to $(\varphi(\cdot,0), \phi(\cdot,0))$. Therefore, from (47) and (49) we deduce that

(50)
$$J_{\ell,\varepsilon}(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon}) \le \liminf_n J_{\ell_n,\varepsilon}(\varphi^n_{T,\varepsilon}, \phi^n_{T,\varepsilon}) \le \liminf_n J_{\ell_n,\varepsilon}(\varphi_T, \phi_T) = J_{\ell,\varepsilon}(\varphi_T, \phi_T).$$

Since $(\varphi_T, \phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0, L)$ is arbitrary, we conclude that $(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon})$ minimizes $J_{\ell,\varepsilon}$.

Now, let us consider, for each n, the pair $(h^{\ell_n}_{l,\varepsilon}1_{\omega_l}, h^{\ell_n}_{r,\varepsilon}1_{\omega_r})$ associated by Proposition 7 to ℓ_n. It follows easily from (47), $(48)_3$, and (50) that

$$(51)\qquad (h^{\ell_n}_{l,\varepsilon}1_{\omega_l}, h^{\ell_n}_{r,\varepsilon}1_{\omega_r}) \to (h^{\ell}_{l,\varepsilon}1_{\omega_l}, h^{\ell}_{r,\varepsilon}1_{\omega_r}) \text{ strongly in } L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r),$$

where $(h^{\ell}_{l,\varepsilon}1_{\omega_l}, h^{\ell}_{r,\varepsilon}1_{\omega_r})$ is the control corresponding to ℓ. Let us denote by $(p^n_\varepsilon, q^n_\varepsilon)$ and $(p_\varepsilon, q_\varepsilon)$ the solutions to (14) associated, respectively, to $(h^{\ell_n}_{l,\varepsilon}1_{\omega_l}, h^{\ell_n}_{r,\varepsilon}1_{\omega_r})$ and $(h^{\ell}_{l,\varepsilon}1_{\omega_l}, h^{\ell}_{r,\varepsilon}1_{\omega_r})$. Then, if we set $(y^n, z^n) := (p^n_\varepsilon - p_\varepsilon, q^n_\varepsilon - q_\varepsilon)$ and $(w^n_l 1_{\omega_l}, w^n_r 1_{\omega_r}) := (h^{\ell_n}_{l,\varepsilon}1_{\omega_l} - h^{\ell}_{l,\varepsilon}1_{\omega_l}, h^{\ell_n}_{r,\varepsilon}1_{\omega_r} - h^{\ell}_{r,\varepsilon}1_{\omega_r})$, we find that

$$(52)\qquad \begin{cases} y^n_t - d^{\ell_n}_l y^n_{\xi\xi} + b^{\ell_n}_l y^n_\xi = w^n_l 1_{\omega_l} + F^n_l & \text{in } Q_{0,l},\\ z^n_t - d^{\ell_n}_r z^n_{\xi\xi} + b^{\ell_n}_r z^n_\xi = w^n_r 1_{\omega_r} + F^n_r & \text{in } Q_{0,r},\\ y^n(0,\cdot) = y^n(\ell_0,\cdot) = z^n(\ell_0,\cdot) = z^n(L,\cdot) = 0 & \text{in } (0,T),\\ y^n(\cdot,0) = 0 & \text{in } (0,\ell_0),\\ z^n(\cdot,0) = 0 & \text{in } (\ell_0, L),\end{cases}$$

where

$$F^n_l := (d^{\ell_n}_l - d^{\ell}_l)p_{\varepsilon,\xi\xi} - (b^{\ell_n}_l - b^{\ell}_l)p_{\varepsilon,\xi} \quad\text{and}\quad F^n_l := (d^{\ell_n}_l - d^{\ell}_l)p_{\varepsilon,\xi\xi} - (b^{\ell_n}_l - b^{\ell}_l)p_{\varepsilon,\xi}.$$

Recall that $(p_0, q_0) \in W^{1,4}_0(0,\ell_0)\times W^{1,4}_0(\ell_0, L)$. Therefore, arguing as in section 4.1 and Lemma 1, we first deduce that $(F^n_l, F^n_r) \in L^4((\ell_l,\ell_0)\times(0,T))\times L^4((\ell_0,\ell_r)\times(0,T))$ and $(y^n_\xi(\ell_0,\cdot), z^n_\xi(\ell_0,\cdot)) \in C^{1/8}([0,T])\times C^{1/8}([0,T])$, and also that

$$\begin{aligned}&\|y^n_\xi(\ell_0,\cdot)\|_{C^{1/8}} + \|z^n_\xi(\ell_0,\cdot)\|_{C^{1/8}}\\ &\quad\le C\left(\|(F^n_l, F^n_r)\|_{L^4(L^4)\times L^4(L^4)} + \|(y^n,z^n)\|_{L^2(H^2)\times L^2(H^2)}\right)\end{aligned}$$

for some $C > 0$, independent of n.

It is not difficult to check that, in this inequality, the first term in the right-hand side goes to 0 when $n\to\infty$. From standard parabolic estimates applied to (52) and (51), we also have the convergence to zero of the second term in the right-hand side. Therefore, we deduce that $(p^n_{\varepsilon,\xi}(\ell_0,\cdot), q^n_{\varepsilon,\xi}(\ell_0,\cdot)) \to (p_{\varepsilon,\xi}(\ell_0,\cdot), q_{\varepsilon,\xi}(\ell_0,\cdot))$ in $C^{1/8}([0,T])$, which implies the continuity of Λ_ε.

- Λ_ε *is compact.* Note that $\Lambda_\varepsilon(\ell)'(t) = \theta(t)$ for all $\ell\in\mathcal{A}_R$ and all $t\in[0,T]$, where θ is the function defined in (42). Thus, we conclude easily from (43) that $\Lambda_\varepsilon(\mathcal{A}_R)$ is a bounded subset of $C^{1+1/8}([0,T])$, which is a compact subset of $C^1([0,T])$.
- There exists $\delta > 0$, such that, whenever $(p_0,q_0)\in W^{1,4}_0(0,\ell_0)\times W^{1,4}_0(\ell_0,L)$ and
$$\|(p_0,q_0)\|_{W^{1,4}_0\times W^{1,4}_0} + |\ell_0 - \ell_T| \le \delta,$$
$\Lambda_\varepsilon(\mathcal{A}_R)\subset\mathcal{A}_R$. Indeed, it follows easily from (43) that there exists $C > 0$ (depending on $\ell_l, \ell_r, \omega_l, \omega_r, T$, and R) such that
$$|\mathcal{L}(t) - \ell_0| \le CT\left(\|(p_0,q_0)\|_{W^{1,4}_0\times W^{1,4}_0} + |\ell_0 - \ell_T|\right)\quad \forall t\in[0,T]$$
and
$$|\mathcal{L}'(t)| \le C\left(\|(p_0,q_0)\|_{W^{1,4}_0\times W^{1,4}_0} + |\ell_0 - \ell_T|\right)\quad \forall t\in[0,T].$$
Thus, we get the result by taking
$$\delta \le \min\left\{\frac{R}{C}, \frac{\ell_0 - \tilde{\ell}_l}{CT}, \frac{\tilde{\ell}_r - \ell_0}{CT}\right\}.$$

Consequently, for initial data p_0, q_0, and ℓ_0 satisfying the above conditions, Schauder's fixed-point theorem guarantees that there exists $\ell_\varepsilon \in \mathcal{A}_R$ such that $\Lambda_\varepsilon(\ell_\varepsilon) = \ell_\varepsilon$. It is easy to see that this is sufficient to achieve the proof of the result. ∎

Now, we are in position to prove Theorem 2. Indeed, since the fixed points ℓ_ε and controls $(h_l^\varepsilon, h_r^\varepsilon)$ furnished by Theorem 3 are uniformly bounded, respectively, in $C^{1+1/8}([0,T])$ and $L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$, there exist ℓ and (h_l, h_r) such that, at least for a subsequence, we have

$$\left\{\begin{array}{l} \ell_\varepsilon \to \ell \ \text{ strongly in } \ C^1([0,T]) \ \text{ and} \\ (h_l^\varepsilon, h_r^\varepsilon) \to (h_l, h_r) \ \text{ weakly in } \ L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r). \end{array}\right. \tag{53}$$

Since the coefficients $(d_l^{\ell_\varepsilon}, b_l^{\ell_\varepsilon})$ and $(d_r^{\ell_\varepsilon}, b_r^{\ell_\varepsilon})$ are uniformly bounded, respectively, in $L^\infty(Q_{0,l}) \times L^\infty(Q_{0,l})$ and $L^\infty(Q_{0,r}) \times L^\infty(Q_{0,r})$, we can conclude from energy estimates and (53) that there exists (p, q) with

$$\left\{\begin{array}{lll} p_\varepsilon \to p & \text{weakly in} & L^2(0,T;H^2(0,\ell_0) \cap H_0^1(0,\ell_0)) \cap H^1(0,T;L^2(0,\ell_0)), \\ q_\varepsilon \rightharpoonup q & \text{weakly in} & L^2(0,T;H^2(0,\ell_0) \cap H_0^1(\ell_0,L)) \cap H^1(0,T;L^2(\ell_0,L)), \end{array}\right. \tag{54}$$

where the $(p_\varepsilon, q_\varepsilon)$ are associated to the $(h_l^\varepsilon, h_r^\varepsilon)$. Then, (p, q) is the solution to (14), associated to (h_l, h_r). Moreover, from (45), it is clear that $\ell(T) = \ell_T$ and $(p(\cdot, T), q(\cdot, T)) = (0, 0)$ on $(0, T)$.

Furthermore, as a consequence of (54) and the embeddings

$$H^2(0,\ell_0) \overset{c}{\hookrightarrow} C^1([0,\ell_0]) \hookrightarrow L^2(0,\ell_0) \quad \text{and} \quad H^2(\ell_0,L) \overset{c}{\hookrightarrow} C^1([\ell_0,L]) \hookrightarrow L^2(\ell_0,L),$$

we find that, for any given $t \in [0,T]$, the following holds:

$$\begin{aligned} \ell(t) = \lim_{\varepsilon\to 0} \ell_\varepsilon(t) &= \lim_{\varepsilon\to 0} \left(\ell_0 - \int_0^t \left[d_l p_{\varepsilon,\xi}(\ell_0,\tau) - d_r q_{\varepsilon,\xi}(\ell_0,\tau) \right] d\tau \right) \\ &= \ell_0 - \int_0^t \left[d_l p_\xi(\ell_0,\tau) - d_r q_\xi(\ell_0,\tau) \right] d\tau. \end{aligned}$$

This implies that the Stefan condition $(4)_7$ is satisfied by (ℓ, p, q) and ends the proof of Theorem 2. ∎

5. Additional comments.

5.1. Lack of controllability with only one control. In the next result it is proved that, if h_l or h_r vanishes and the interface does not collapse to the boundary, then null controllability cannot hold.

THEOREM 4. *Assume that $u_0 \in H_0^1(0,\ell_0)$ with $u_0 \geq 0$, $v_0 \in H_0^1(\ell_0, L)$ with $v_0 \leq 0$, and $v_0 \not\equiv 0$. Then, if $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$, $h_r \equiv 0$, and the associated strong solution to* (1) *satisfies $\ell(t) < L$ for all $t \in [0,T]$, we necessarily have*

$$v(\cdot, T) \not\equiv 0 \quad \textit{in} \quad (\ell(T), L).$$

Proof. Let us assume, by contradiction, that (1) is null controllable with $h_r \equiv 0$, i.e., $u(\cdot, T) \equiv 0$ in $(0, \ell(T))$ and $v(\cdot, T) \equiv 0$ in $(\ell(T), L)$.

Then, considering the diffeomorphism Φ and the function $q = v \circ \Phi^{-1}$, defined in section 2.1, we get easily that q is the solution to

$$\left\{\begin{array}{lll} q_t - d_r^\ell q_{\xi\xi} + b_r^\ell q_\xi = 0 & \text{in} & Q_{0,r}, \\ q(\ell_0, \cdot) = q(L, \cdot) = 0 & \text{in} & (0,T), \\ q(\cdot, 0) = q_0 & \text{in} & (\ell_0, L), \end{array}\right.$$

where $q_0 := v_0 \circ [G(\cdot,\ell_0)]^{-1} \in H_0^1(\ell_0, L)$ and, obviously, $q_0 \leq 0$ and $q_0 \not\equiv 0$. We also have

$$q(\cdot, T) \equiv 0 \quad \text{in} \quad (\ell_0, L). \tag{55}$$

We can apply the weak maximum principle to q in $Q_{0,r}$. Obviously, this gives $q \leq 0$ in $Q_{0,r}$.

Thus, if (55) holds, we get, from the strong maximum principle and the fact that $q \leq 0$ in $Q_{0,r}$, that $q \equiv 0$ in $\overline{Q}_{0,r}$, which contradicts $q_0 \not\equiv 0$. ◻

Remark 5. Note that the previous argument also shows that the null controllability for (1) cannot be achieved for solutions which preserve the signs of the initial conditions in each phase region. In other words, in order to drive the solution to zero at time T, the liquid and solid states must penetrate each other before T.

5.2. Boundary controllability and other extensions. We can prove local boundary controllability results similar to Theorem 1. Thus, let us introduce the system

$$\begin{cases} u_t - d_l u_{xx} = 0 & \text{in} \quad Q_l, \\ v_t - d_r v_{xx} = 0 & \text{in} \quad Q_r, \\ u(0,t) = k_l(t), \ v(L,t) = k_r(t) & \text{in} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,\ell_0), \\ v(\cdot,0) = v_0 & \text{in} \quad (\ell_0,L), \\ u(\ell(t),t) = v(\ell(t),t) = 0 & \text{in} \quad (0,T), \\ -\ell'(t) = d_l u_x(\ell(t),t) - d_r v_x(\ell(t),t) & \text{in} \quad (0,T), \end{cases} \tag{56}$$

where (k_l, k_r) stands for the boundary control pair.

Then, using a domain extension technique and Theorem 1, it is easy to prove that, if u_0 and v_0 are sufficiently small, and ℓ_0 is sufficiently close to ℓ_T, there exist controls (k_l, k_r) and associated solutions to (56) that satisfy $\ell(T) = \ell_T$, $u(\cdot,T) = 0$ in $(0,\ell_T)$ and $v(\cdot,T) = 0$ in (ℓ_T, L).

Let us finally mention that the arguments and results in this paper can also be used to solve other variants of the two-phase Stefan controllability problem. Thus, we can prove results similar to Theorem 1 when the controls are Neumann data, and we can assume that the equations contain lower order terms or even appropriate nonlinearities.

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