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ON THE CONTROLLABILITY OF SOME EQUATIONS OF SOBOLEV-GALPERN TYPE

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ABSTRACT. In this paper we deal with the controllability problem for some Sobolev type equations. We show that the equations cannot be driven to zero if the control region is strictly supported within the domain. Nevertheless, we also prove that it is possible to control the equations using controls which have a moving support, under some assumptions on their movement.

Keywords: Controllability, Observability, Barenblatt-Zheltov-Kochina equation, Benjamin-Bona-Mahony equation, moving controls, gaussian beams.

Mathematics Subject Classification (2010): 93B05, 93B07, 93C05, 35M30

CONTENTS

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}^\star$) be a bounded domain whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and $\mathcal{O}$ be a nonempty open subset of $\Omega \times (0, T)$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.

In this paper we deal with controllability properties for some *pseudo-parabolic equations* of the form

$$(I - \gamma\mathcal{L})\partial_t y + \mathcal{M}y = f, \tag{1.1}$$

where γ is positive real number and $\mathcal{L}$ and $\mathcal{M}$ are linear partial differential operators of order $2l$ and m with $m \leqslant 2l$ in the spatial variable, respectively (see for instance [12, 20, 21, 22]). More precisely, we

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consider the following two problems

$$\left|\begin{array}{lll} y_t - \Delta y_t - \Delta y = v\chi_{\mathcal{O}} & \text{in} & Q, \\ y = 0 & \text{on} & \Sigma, \\ y(\cdot,0) = y_0 & \text{in} & \Omega \end{array}\right. \tag{1.2}$$

and

$$\left|\begin{array}{lll} y_t - \Delta y_t + \nabla\cdot(A(x,t)y) = v\chi_{\mathcal{O}} & \text{in} & Q, \\ y = 0 & \text{on} & \Sigma, \\ y(\cdot,0) = y_0 & \text{in} & \Omega, \end{array}\right. \tag{1.3}$$

where $A = (a_1, \dots, a_N)$ is a given regular vector field and $\chi_{\mathcal{O}} \in C^\infty(Q)$ with $\operatorname{supp}\chi_{\mathcal{O}} \subset \overline{\mathcal{O}}$.

Our goal in this paper is to investigate the *null controllability problem*:

given $T > 0$ and $y_0 \in H_0^1(\Omega)$ find a control $v \in L^2(\mathcal{O})$ such that the associated solution of (1.2) (*resp.* (1.3)) *satisfies:*

$$y(\cdot,T) = 0, \quad \text{in} \quad \Omega.$$

Equations such as (1.1) are a particular case of the so called equations of *Sobolev-Galpern type*, see [8, 23]. These type of equations appear for instance in the study of problems associated with the flow of certain viscous fluids, in the theory of seepage of homogeneous liquids in fissured rocks, see [3], and surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, see [4]. In particular, equations (1.2) and (1.3) are known as the *Barenblatt-Zheltov-Kochina* equation (in this case y represents the absolute value of the velocity of the fluid and η characterizes the fissured rock, increasing η corresponds to a decreasing degree of fissuring) and the multidimensional *Benjamin-Bona-Mahony equation*, respectively (see for instance [1, 2, 3, 4, 18]).

Regarding controllability for equations (1.2) and (1.3), as far as we know, the only results available in the literature were obtained in the one-dimensional setting. Indeed, in [17] it is proved that equation (1.3), with A being a constant, cannot be steered to zero if $\mathcal{O} = \omega \times (0,T)$ and $\omega \subsetneq \Omega$ is a proper subset. However, the proof given in [17] can be only performed in the $1d$ setting, since it relies on the moment method. For a positive controllability result for (1.2), we cite [24], where the authors consider the problem posed on the torus and prove that if one make the control to move in time covering the whole domain, it is possible to drive the solution exactly to zero. Also related to the controllability of (1.3), we cite [25, 26], where the unique continuation property is studied.

In this paper, we analyze the null controllability of equations (1.2) and (1.3) in the multi-dimensional setting. First, we show that both equations (1.2) and (1.3) cannot be steered to zero if the control is fixed and localized in a proper open subset of Ω. More precisely, we prove the following two negative results.

Theorem 1.1. *Let $T > 0$ and $\omega \subsetneq \Omega$ be a fixed open set. If $\mathcal{O} = \omega \times (0,T)$ then system* (1.2) *is not null controllable at time T, i.e., there exists $y_0 \in H^2(\Omega) \times H_0^1(\Omega)$ such that the null controllability of system* (1.2) *fails.*

Theorem 1.2. *Let $T > 0$, $\omega \subsetneq \Omega$ be a fixed open set and $A \in C^\infty(\overline{Q})$. If $\mathcal{O} = \omega \times (0,T)$ then system* (1.3) *is not null controllable at time T, i.e., there exists $y_0 \in H^2(\Omega) \times H_0^1(\Omega)$ for which the null controllability of* (1.3) *does not hold.*

It is worth to mention that Theorems 1.1 and 1.2 are closely related to the fact that the principal part of (1.2) and (1.3), given by $\partial_t\Delta$, has vertical characteristic hyperplanes which makes impossible to recover any information localized along these characteristics (see Section 2). In fact, the proof of both results relies on the construction of highly localized solutions (Gaussian beams). For Theorem 1.1 we construct such solutions by means of Fourier transform (for similar constructions see [14, 15]). On the other hand, since the vector field A in equation (1.3) depends on both the space and time variables, we can not use Fourier Transform to prove Theorem 1.2. Therefore, we will use a different approach based on asymptotic expansion of solutions.

The second main part of this paper is devoted to obtain positive null controllability results for equations (1.2) and (1.3). In fact, since the main obstruction to the null controllability with localized fixed controls is the existence of concentrated solutions, we ask the control to move so that we can see the information that would be lost otherwise, i.e., we make the control to move in time in order to cover the whole space domain. This idea of making the control to move in order make the system controllable has been used for many different problems in the past few years, see for instance [5, 6, 10, 11, 16]. Here, due to the techniques we shall employ, we consider two different types of movement.

The first type of movement, which has been introduced in [13] (see also [19]), will be used to deal with equation (1.2).

Definition 1.3 (Moving control region of type I)**.** *We say that an open set $\mathcal{O} \subset Q$ satisfies the Moving Geometric Control Condition of type I (MGCC-I) if for all $x_0 \in \Omega$, the vertical line $\{(s, x_0); s \in \mathbb{R}\}$ enters $\mathcal{O}$ before time T and*

$$L_{\mathcal{O}} = \inf_{x\in\Omega} \sup_{(t_1,t_2)\times\{x\}\subset\mathcal{O}} (t_2 - t_1) > 0.$$

Remark 1.4. *The condition in Definition 1.3 means that vertical rays which do not propagate in space also reach the control domain and stay in it during some time interval. In practice this means that the cross section of $\mathcal{O}$ moves, as the time evolves, covering the whole domain Ω.*

The positive controllability result we prove for equation (1.2) reads as follows:

Theorem 1.5. *Let $T > 0$ and assume that $\mathcal{O}$ satisfies MGCC-I. Then, for any $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ there exists a moving control $v \in L^2(\mathcal{O})$ such that the associated solution to (1.2) satisfies*

$$y(\cdot, T) = 0 \quad in \quad \Omega.$$

We prove Theorem 1.5 using a compactness-uniqueness argument which relies on an observability type inequality for an ODE (for which Definition 1.3 is necessary) and energy estimates for elliptic equations.

As we will see, unless $A = A(t)$, we cannot use a compact-uniqueness argument to deduce positive controllability results to equation (1.3) (see Section 4 for more details). For this reason, we use a different strategy based on Carleman estimates. Nevertheless, since Carleman estimates are heavily dependent on the construction of specific weight functions, we require stronger geometrical assumptions on the movement of the control region. In fact, we take the control domain determined by the evolution of a given reference subset $\omega \subset \mathbb{R}^N$ through a given flow and such that ω contains a smooth bounded domain $\omega_0 \subset \mathbb{R}^N$ which satisfies the following geometric requirements:

Assumption 1.6. *There exists a flow $X : \mathbb{R}^N \times [0, T] \times [0, T] \to \mathbb{R}^N$, which is generated by an admissible velocity field $F \in C([0, T]; W^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N))$, a curve $\Gamma \in C^\infty([0, T]; \mathbb{R}^N)$ and two times $t_1 < t_2$ in $(0, T)$*

such that:

$$\Gamma(t) \in X(\omega_0, t, 0) \cap \Omega, \quad \forall t \in [0,T]; \tag{1.4}$$

$$\overline{\Omega} \subset \bigcup_{t\in[0,T]} X(\omega_0, t, 0) = \{X(x,t,0); \quad (x,t) \in \omega_0 \times [0,T]\}; \tag{1.5}$$

$$\Omega\backslash\overline{X(\omega_0,t,0)} \text{ is nonempty and connected for } t \in [0,t_1] \cup [t_2,T]; \tag{1.6}$$

$$\Omega\backslash\overline{X(\omega_0,t,0)} \text{ has two (nonempty) connected components for } t \in (t_1,t_2); \tag{1.7}$$

$$\forall \theta \in C([0,T];\Omega), \ \exists t \in [0,T], \quad \theta(t) \in X(\omega_0,t,0). \tag{1.8}$$

Definition 1.7 (Moving control region of type II)**.** *A moving control region of type II (MGCC-II) is defined as* $\mathcal{O}_\omega := \bigcup_{t\in[0,T]} [X(\omega,t,0) \cap \Omega] \times \{t\}$ *where the reference control domain* $\omega \subset \mathbb{R}^N$ *contains a subset* ω_0 *which satisfies Assumptions 1.6 and* $\overline{\omega}_0 \subset \omega$*. For any* $t > 0$ *a time section is defined as* $\mathcal{O}_\omega(t) := X(\omega,t,0) \cap \Omega$*.*

The positive controllability result we prove for equation (1.3) is the following.

Theorem 1.8. *Let* $T > 0$*, let* $\mathcal{O}_\omega$ *satisfying MGCC-II and* $A \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^{N\times N})$*. Then, for any* $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$*, there exists a moving control* v *localized in* $\mathcal{O}_\omega$ *with* $v \in L^2(\mathcal{O}_\omega)$ *and such that the associated solution to* (1.3) *satisfies*

$$y(\cdot,T) = 0 \quad in \quad \Omega.$$

It is important to say that both Assumption 1.6 and Definition 1.7 were introduced in [5] to study the controllability of a wave equation with both viscous Kelvin-Voigt and frictional damping and the idea was to split the equation into a coupled parabolic-ODE system and prove new Carleman estimates for both the heat equation and ODE's when the control region moves as the time evolves. Here, to prove Theorem 1.8, we split equation (1.3) into a coupled elliptic-ODE system and prove new Carleman estimates for elliptic equations when the control region moves together with some suitable weighted energy inequalities combined with the Carleman inequality proved for ODE's given in [5] (see Section 3 for more details).

2. Negative Controllability Results

2.1. **Barenblatt-Zheltov-Kochina with fixed controls.** We prove Theorem 1.1. Here we assume that $\mathcal{O}$ is of the form $\omega \times (0,T)$, where ω is a proper open subset of Ω.

For analyzing the controllability of (1.2) we will make use of the following decomposition:

$$\left|\begin{array}{lll} u - \Delta u = w & \text{in} & Q, \\ w_t + w = u + v\chi_{\mathcal{O}} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ w(\cdot,0) = u_0 - \Delta u_0 & \text{in} & \Omega. \end{array}\right. \tag{2.1}$$

Indeed, the solution of equation (1.2) satisfies $u(\cdot,T) = 0$ if and only if the solution of system (2.1) satisfies $w(\cdot,T) = 0$.

From duality arguments, the null controllability for system (2.1) with control supported in $\omega \times (0,T)$ is equivalent to the existence of a constant $C > 0$ such that the observability inequality

$$\|\psi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C \int_0^T\!\!\int_\omega |\psi|^2 dxdt,$$

holds for all $\psi_T \in L^2(\Omega)$, where ψ, together with φ, is the solution of the adjoint system

$$\left| \begin{array}{lll} \varphi - \Delta\varphi = \psi & \text{in} & Q, \\ -\psi_t + \psi = \varphi & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \psi(T) = \psi_T & \text{in} & \Omega. \end{array} \right. \tag{2.2}$$

Theorem 1.1 is a direct consequence of the following proposition:

Proposition 2.1. *Let ω_0 be an open subset of Ω such that $\bar{\omega}_0 \subsetneq \Omega$. Then, there exist $\epsilon_0 > 0$ and $\psi_T^\epsilon \in L^2(\Omega)$ such that for any integer $k > N/4$ the corresponding solution of*

$$\left| \begin{array}{lll} \varphi^\epsilon - \Delta\varphi^\epsilon = \psi^\epsilon & \textit{in} & Q, \\ -\psi_t^\epsilon + \psi^\epsilon = \varphi^\epsilon & \textit{in} & Q, \\ \varphi^\epsilon = 0 & \textit{on} & \Sigma, \\ \psi^\epsilon(\cdot, T) = \psi_T^\epsilon & \textit{in} & \Omega \end{array} \right. \tag{2.3}$$

satisfies

$$\|\psi^\epsilon(\cdot,0)\|^2_{L^2(\Omega)} \geqslant C \quad \textit{and} \quad \|\psi^\epsilon\|^2_{L^2(0,T;L^2(\omega_0))} \leqslant C\epsilon^{k-N/4} \quad \forall \epsilon \in (0,\epsilon_0) \tag{2.4}$$

where C is a positive constant independent of ϵ.

Proof. Let us first consider the system (2.3) posed in $\mathbb{R}^N \times (0,T)$, i.e.

$$\left| \begin{array}{lll} \varphi - \Delta\varphi = \psi & \text{in} & \mathbb{R}^N \times (0,T), \\ -\psi_t + \psi = \varphi & \text{in} & \mathbb{R}^N \times (0,T), \\ \psi(\cdot, T) = \psi_T & \text{in} & \mathbb{R}^N, \end{array} \right. \tag{2.5}$$

with $\psi_T \in L^2(\mathbb{R}^N)$.

Taking the spatial Fourier transform, one verifies that

$$\hat{\psi}(\xi,t) = e^{-\frac{|\xi|^2}{(1+|\xi|^2)}(T-t)}\hat{\psi}_T(\xi) \quad \text{and} \quad \hat{\varphi}(\xi,t) = \frac{e^{-\frac{|\xi|^2}{(1+|\xi|^2)}(T-t)}}{1+|\xi|^2}\hat{\psi}_T(\xi)$$

solves

$$\left| \begin{array}{lll} (1+|\xi|^2)\hat{\varphi} = \hat{\psi} & \text{in} & \mathbb{R}^N \times (0,T), \\ -\hat{\psi}_t + \hat{\psi} = \hat{\varphi} & \text{in} & \mathbb{R}^N \times (0,T), \\ \hat{\psi}(\cdot, T) = \hat{\psi}_T & \text{in} & \mathbb{R}^N, \end{array} \right. \tag{2.6}$$

where $\hat{\psi}_T$ is the Fourier transform of ψ_T.

Now let θ be a real smooth function supported in $B_1(0)$ with $\|\theta\|_{L^2(\mathbb{R}^N)} = 1$ and for each $\epsilon > 0$ consider

$$\hat{\psi}_T^\epsilon(\xi) = \epsilon^{N/4}\theta\left(\sqrt{\epsilon}\left(\xi - \frac{\overline{\xi}}{\epsilon}\right)\right)e^{-ix_0\cdot\xi}, \tag{2.7}$$

where $\overline{\xi} \in \mathbb{R}^N$, $|\overline{\xi}| = 1$ and x_0 is a point around which we will localize our solution.

Let $(\hat{\psi}^\epsilon, \hat{\varphi}^\epsilon)$ be the solution of (2.6) associated to $\hat{\psi}_T^\epsilon$. Since $\hat{\psi}_T^\epsilon \in L^2(\mathbb{R}^N)$, let $(\check{\psi}^\epsilon, \check{\varphi}^\epsilon)$ be the solution of (2.5) with initial datum $\check{\psi}_T^\epsilon$, the inverse Fourier transform of $\hat{\psi}_T^\epsilon$.

Claim 1. *There exist two constants $C_1, C_2 > 0$, independent of ϵ, such that*

$$C_1 \leqslant \|\check{\psi}^\epsilon(\cdot,0)\|_{L^2(\mathbb{R}^N)} \leqslant C_2.$$

Proof of Claim 1. We have

$$\check{\psi}^\epsilon(x,t) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-\frac{|\xi|^2}{(1+|\xi|^2)}(T-t)}\hat{\psi}^\epsilon_T(\xi)e^{ix\cdot\xi}d\xi \tag{2.8}$$

and by Parseval's identity

$$\|\check{\psi}^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{(2\pi)^{2N}}\int_{\mathbb{R}^N} e^{-2\frac{|\xi|^2}{(1+|\xi|^2)}(T-t)}|\hat{\psi}^\epsilon_T(\xi)|^2 d\xi.$$

Since $\|\hat{\psi}^\epsilon_T\|_{L^2(\mathbb{R}^N)} = 1$, it follows that

$$\frac{e^{-2T}}{(2\pi)^{2N}} \leqslant \|\check{\psi}^\epsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^n)} \leqslant \frac{1}{(2\pi)^{2N}}. \tag{2.9}$$

□

Claim 2. *Let $x_0 \in \mathbb{R}^N$. For any $\delta > 0$ there exists $C > 0$, independent of ϵ, such that*

$$\|\check{\varphi}^\epsilon\|^2_{L^2(0,T;H^1(|x-x_0|\geqslant\delta))} + \|\check{\psi}^\epsilon\|^2_{L^2(0,T;L^2(|x-x_0|\geqslant\delta))} \leqslant C\epsilon^{k-N/4}.$$

Proof of Claim 2. Let us show the estimate for $\check{\varphi}^\epsilon$. Similar arguments give the estimate for $\check{\psi}^\epsilon$.

Since

$$\check{\varphi}^\epsilon(x,t) = \frac{1}{(2\pi)^N}\iint\limits_{\mathbb{R}^N} \frac{e^{-\frac{|\xi|^2}{(1+|\xi|^2)}(T-t)}}{1+|\xi|^2}\hat{\psi}^\epsilon_T(\xi)e^{ix\cdot\xi}d\xi, \tag{2.10}$$

by the change of variables $\zeta = \sqrt{\epsilon}(\xi - \frac{\bar{\xi}}{\epsilon})$ we see that

$$\check{\varphi}^\epsilon(x,t) = \frac{\epsilon^{N/4-N/2}}{(2\pi)^N}\iint\limits_{|\zeta|\leqslant 1} \theta(\zeta)e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon})}\frac{e^{-\frac{|\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon}|^2}{(1+|\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon}|^2)}(T-t)}}{1+|\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon}|^2}d\zeta \tag{2.11}$$

From the fact that

$$\Delta^k_\zeta e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon})} = (-1)^k\left(\frac{|x-x_0|^2}{\epsilon}\right)^k e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon})} \quad k\in\mathbb{N},$$

for $|x-x_0| \geqslant \delta$ and for any integer $k > N/4$, we have

$$\check{\varphi}^\epsilon(x,t) = (-1)^k\frac{\epsilon^{k-N/4}}{(2\pi)^N|x-x_0|^{2k}}\iint\limits_{|\zeta|\leqslant 1} e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon})}\Delta^k_\zeta\Big(\frac{\theta(\zeta)e^{-\frac{|\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon}|^2}{(1+|\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon}|^2)}(T-t)}}{1+|\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\epsilon}|^2}\Big)d\zeta \tag{2.12}$$

For ϵ small, one can prove that the term in Δ^k_ζ in the above integral is bounded uniformly with respect to ϵ and then the following estimate holds

$$|\check{\varphi}^\epsilon(x,t)| \leqslant C\frac{\epsilon^{k-N/4}}{|x-x_0|^{2k}}. \tag{2.13}$$

Analogously, we have

$$|\nabla\check{\varphi}^\epsilon(x,t)| \leqslant C\frac{\epsilon^{k-N/4}}{|x-x_0|^{2k}} \tag{2.14}$$

and this gives the estimate for $\check{\varphi}^\epsilon$.

□

Claim 3. *Let $\hat{\psi}_T^\epsilon$ as (2.7) and $(\check{\varphi}^\epsilon, \check{\psi}^\epsilon)$ the associated solution of (2.5). Then,*

$$\|\check{\psi}^\epsilon(\cdot,0)\|^2_{L^2(|x-x_0|\leqslant\delta)} \geqslant C > 0.$$

Proof. From (2.13) for $t=0$, we get

$$\|\check{\psi}^\epsilon(\cdot,0)\|^2_{L^2(|x-x_0|\geqslant\delta)} \leqslant C\epsilon^{k-N/4}$$

and from Claim 1 we have

$$\frac{e^{-2T}}{(2\pi)^{2N}} \leqslant \|\check{\psi}^\epsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^n)},$$

which gives the result.

□

We now finish the proof of Proposition 2.1. To do that, consider $x_0 \in \Omega\backslash\bar{\omega}_0$ and

$$0 < \eta < \min\{\mathrm{dist}(x_0,\partial\Omega), \mathrm{dist}(x_0,\partial\omega_0)\}$$

such that $\{x : |x-x_0| \leqslant \eta\} \subset \Omega$.

As before, take $(\check{\psi}^\epsilon, \check{\varphi}^\epsilon)$ the solution of (2.5) associated to $\check{\psi}_T^\epsilon$, the inverse Fourier transform of $\hat{\psi}_T^\epsilon$.

Consider $(\bar{\psi}^\epsilon, \bar{\varphi}^\epsilon)$ the restriction of $(\check{\psi}^\epsilon, \check{\varphi}^\epsilon)$ to $\Omega\times(0,T)$. Thus,

$$\left|\begin{array}{lll} \bar{\varphi}^\epsilon - \Delta\bar{\varphi}^\epsilon = \bar{\psi}^\epsilon & \text{in} & Q, \\ -\bar{\psi}^\epsilon_t + \bar{\psi}^\epsilon = \bar{\varphi}^\epsilon & \text{in} & Q, \\ \bar{\varphi}^\epsilon = q^\epsilon & \text{on} & \Sigma, \\ \bar{\psi}^\epsilon(\cdot,T) = \psi_T^\epsilon & \text{in} & \Omega \end{array}\right. \tag{2.15}$$

where $\bar{\psi}_T^\epsilon := \check{\psi}_T^\epsilon|_{\Omega\times(0,T)}$ and $q^\epsilon := \check{\varphi}^\epsilon\big|_{\partial\Omega\times(0,T)}$.

From Claim 2 and Claim 3, we have that

$$\|\bar{\psi}^\epsilon\|^2_{L^2(0,T;L^2(\omega_0))} \leqslant C\epsilon^{k-N/4} \quad \text{and} \quad \|\bar{\psi}^\epsilon(\cdot,0)\|^2_{L^2(\Omega)} \geqslant C > 0, \tag{2.16}$$

respectively.

Now, let $(\varphi_\epsilon^\star, \psi_\epsilon^\star)$ be the solution of

$$\left|\begin{array}{lll} \varphi_\epsilon^\star - \Delta\varphi_\epsilon^\star = \psi_\epsilon^\star & \text{in} & Q, \\ -\psi^\star_{\epsilon,t} + \psi_\epsilon^\star = \varphi_\epsilon^\star & \text{in} & Q, \\ \varphi_\epsilon^\star = -q^\epsilon & \text{on} & \Sigma, \\ \psi_\epsilon^\star(\cdot,T) = 0 & \text{in} & \Omega. \end{array}\right.$$

Noticing that $q^\epsilon \in L^2(0,T;H^{1/2}(\partial\Omega))$, one can show that $\psi_\epsilon^\star \in H^1(0,T;L^2(\Omega))$ and the following estimate holds

$$\|\psi_\epsilon^\star\|_{H^1(0,T;L^2(\Omega))} \leqslant C\|q^\epsilon\|_{L^2(0,T;H^{1/2}(\partial\Omega))}.$$

Nevertheless, because $q^\epsilon := \check{\varphi}^\epsilon\big|_{\partial\Omega\times(0,T)}$, by trace estimate and Claim 2, we deduce that

$$\|\psi_\epsilon^\star\|_{H^1(0,T;L^2(\Omega))} \leqslant C\epsilon^{k-N/4}. \tag{2.17}$$

Finally, defining $(\psi^\epsilon, \varphi^\epsilon) = (\bar{\psi}_\epsilon + \psi_\epsilon^\star, \bar{\varphi}_\epsilon + \varphi_\epsilon^\star)$, we see that $(\psi^\epsilon, \varphi^\epsilon)$ solves (2.3) and by (2.16)–(2.17) we obtain (2.4). □

2.2. **Benjamin-Bona-Mahony with fixed controls.** We now prove Theorem 1.2. Here we assume that $\mathcal{O} = \omega \times (0,T)$, where ω is a proper open subset of Ω.

The null controllability for system (1.3) with control supported in $\omega \times (0,T)$ is equivalent to the existence of a constant $C > 0$ such that the observability inequality

$$\|\psi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C \int_0^T\int_\omega |\psi|^2 dxdt, \tag{2.18}$$

holds for all $\psi_T \in L^2(\Omega)$ and ψ is the solution of the adjoint equation

$$\left|\begin{array}{lll} -\psi_t + \Delta\psi_t - A\cdot\nabla\psi = 0 & \text{in} & Q,\\ \psi = 0 & \text{on} & \Sigma,\\ \psi(\cdot,T) = \psi_T & \text{in} & \Omega. \end{array}\right. \tag{2.19}$$

In order to prove Theorem 1.2, we show that the observability inequality (2.18) does not hold for every $\psi_T \in L^2(\Omega)$.

Given $x_0 \in \Omega\backslash\overline{\omega}_0$, we set $\alpha(x) = x\cdot\xi_0 + i\frac{|x-x_0|^2}{2}$ with $\xi_0 \in \mathbb{R}^N\backslash\{0\}$ and let $\delta > 0$ be such that $B_\delta(x_0) \subset \Omega$ and $B_\delta(x_0) \cap \overline{\omega} = \varnothing$. For $h > 0$, we introduce the function

$$\psi_h(x,t) = e^{i\frac{\alpha(x)}{h}}\left(f_0(x) + hf_1(x,t) + h^2 f_2(x,t)\right),$$

where

$$\left\{\begin{array}{l} f_0 \in C_0^\infty(B_\delta(x_0)), \quad f_0 \equiv 1 \ \text{ in } \ (B_{\frac{\delta}{2}}(x_0),\\ f_1(x,t) = -if_0(x)\dfrac{\displaystyle\int_t^T A(x,\tau)d\tau\cdot\nabla\alpha(x)}{|\nabla\alpha(x)|^2},\\ f_2(x,t) = \dfrac{-\displaystyle\int_t^T A(x,\tau)d\tau\cdot\nabla f_0 - i\int_t^T f_1(x,\tau)A(x,\tau)d\tau\cdot\nabla\alpha - 2i\nabla f_1\cdot\nabla\alpha - if_1\Delta\alpha}{|\nabla\alpha(x)|^2}. \end{array}\right. \tag{2.20}$$

Remark 2.2. *Since $|\nabla\alpha(x)| \geqslant |\xi_0| \neq 0$ for all $x \in \overline{\Omega}$, f_1 and f_2 are well-defined and supp $f_1(\cdot,t) \subset$ supp f_0, supp $f_2(\cdot,t) \subset$ supp f_0 for all $t \in [0,T]$.*

It is easy to check that $\psi_h \in C^\infty(\overline{Q})$ satisfies

$$\left|\begin{array}{lll} -\psi_{h,t} + \Delta\psi_{h,t} - A\cdot\nabla\psi_h = R & \text{in} & Q,\\ \psi_h = 0 & \text{on} & \Sigma,\\ \psi_h(\cdot,T) = e^{i\frac{\alpha}{h}}f_0 & \text{in} & \Omega, \end{array}\right. \tag{2.21}$$

with

$$\begin{aligned} R &= e^{i\frac{\alpha}{h}}\Big[\big(-f_{1,t} + \Delta f_{1,t} - A\cdot\nabla f_1 + 2i\nabla\alpha\cdot\nabla f_{2,t} + i\Delta\alpha f_{2,t} - iA\cdot\nabla\alpha f_2\big)h\\ &\qquad + \big(-f_{2,t} + \Delta f_{2,t} - A\cdot\nabla f_2\big)h^2\Big]\\ &:= e^{i\frac{\alpha}{h}}\left(hR_1 + h^2R_2\right). \end{aligned} \tag{2.22}$$

Let now $\varphi \in H^1(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ be the unique solution of

$$\left|\begin{array}{lll} -\varphi_t + \Delta\varphi_t - A\cdot\nabla\varphi = -R & \text{in} & Q,\\ \varphi = 0 & \text{on} & \Sigma,\\ \varphi(\cdot,0) = 0 & \text{in} & \Omega. \end{array}\right. \tag{2.23}$$

The function $\psi = \psi_h + \varphi$ solves

$$\left|\begin{array}{lll} -\psi_t + \Delta\psi_t - A\cdot\nabla\psi = 0 & \text{in} & Q, \\ \psi = 0 & \text{on} & \Sigma, \\ \psi(\cdot, T) = e^{i\frac{\alpha}{h}} f_0 + \varphi(\cdot, T) & \text{in} & \Omega. \end{array}\right. \tag{2.24}$$

For h small enough, we have

$$\|R\|^2_{L^2(\Omega\times(0,T))} = h\int_0^T \int_{B_R(x_0)} e^{-\frac{|x-x_0|^2}{h}} \left|R_1(x,t) + hR_2(x,t)\right|^2 dx\, dt \sim O(h^{N/2+1}). \tag{2.25}$$

From standard energy estimates, one deduce that

$$\|\varphi\|^2_{L^2(\omega\times(0,T))} \leqslant \|R\|^2_{L^2(\Omega\times(0,T))} = O(h^{N/2+1}), \tag{2.26}$$

for h small enough.

Now, since $\psi_h\big|_{\omega\times(0,T)} = 0$, it follows that

$$\|\psi\|^2_{L^2(\omega\times(0,T))} \sim O(h^{N/2+1}). \tag{2.27}$$

On the other hand, we have

$$\begin{aligned} \|\psi(\cdot,0)\|^2_{L^2(\Omega)} &= \int_\Omega e^{-\frac{|x-x_0|^2}{h}} \left|f_0 + hf_1 + h^2 f_2\right|^2 dx \\ &\sim O(h^{N/2}). \end{aligned} \tag{2.28}$$

From (2.27) and (2.28), it follows that the observability inequality (2.18) cannot hold for every $\psi_T \in L^2(\Omega)$. This proves Theorem 1.2.

3. Positive Controllability Results

This section is devoted to prove Theorems 1.5 and 1.8.

3.1. Barenblatt-Zheltov-Kochina with moving controls.

In this section, we show the positive null controllability for equation (1.2) under the *MGCC-I*. In fact, Theorem 1.5 is a direct consequence of the following result.

Proposition 3.1. *Let $T > 0$ and assume that $\mathcal{O}$ satisfies MGCC-I. For any $z_0 \in L^2(\Omega)$, there exists a moving control $v \in L^2(\mathcal{O})$ such that the solution (y, z) of*

$$\left|\begin{array}{lll} y - \Delta y = z & \text{in} & Q, \\ z_t + z = y + v\chi_{\mathcal{O}} & \text{in} & Q, \\ y = 0 & \text{on} & \Sigma, \\ z(\cdot, 0) = z_0 & \text{in} & \Omega. \end{array}\right. \tag{3.1}$$

satisfies

$$y(\cdot, T) = z(\cdot, T) = 0 \quad \text{in} \quad \Omega.$$

Proof of Theorem 1.5. Given $y_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, we take $z_0 = y_0 - \Delta y_0 \in L^2(\Omega)$ and it follows from Proposition 3.1 that there exists $v \in L^2(\mathcal{O})$ such that the associated solution (y, z) to (3.1) satisfies $y(T) = z(T) = 0$. Replacing $(3.1)_1$ into $(3.1)_2$, we readily see that y, with the control v, solves the null controllability problem for the Barenblatt-Zheltov-Kochina equation (1.2). □

Let us now prove Proposition 3.1. Indeed, we only have to show the existence of a constant $C > 0$ such that

$$\|\psi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C \iint_{\mathcal{O}} |\psi|^2 dxdt, \quad \forall \psi_T \in L^2(\Omega), \tag{3.2}$$

where (φ,ψ) is the solution of

$$\left|\begin{array}{lll} \varphi - \Delta\varphi = \psi & \text{in} & Q, \\ -\psi_t + \psi = \varphi & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \psi(T) = \psi_T & \text{in} & \Omega. \end{array}\right. \tag{3.3}$$

Before proving Proposition 3.1, let us introduce some notation. For any $\epsilon > 0$ and any $A \subset \mathbb{R}^{N+1}$, let

$$\mathcal{M}_\epsilon(A) = \{z \in \mathbb{R}^{N+1};\ dist(z,A) < \epsilon\}$$

and

$$\mathcal{O}_\epsilon := \mathcal{O}\backslash\overline{\mathcal{M}_\epsilon(\partial\mathcal{O}\backslash\Sigma)}.$$

Remark 3.2. *Since $\mathcal{O}$ satisfies the MGCC-I, there exists $\epsilon_0 > 0$ such that $\mathcal{O}_{\frac{3}{2}\epsilon_0}$ (and hence $\mathcal{O}_{\epsilon_0}$) still fulfils the MGCC-I.*

Proof of Proposition 3.1. For any $t \in (0,T)$ and $x \in \mathcal{O}_{\epsilon_0}(t)$, where $\mathcal{O}_{\epsilon_0}(t)$ the cross section of $\mathcal{O}_{\epsilon_0}$ at time t, it follows from $(3.3)_2$ that

$$|\psi(s,x)|^2 \leqslant C\left(|\psi(t,x)|^2 + \int_0^T |\varphi(\tau,x)|^2 d\tau\right), \qquad \forall s \in (t,T). \tag{3.4}$$

Since $\mathcal{O}_{\epsilon_0}$ satisfies the *MGCC-I*, integrating (3.4) in $\mathcal{O}_{\epsilon_0}$ and using the definition of $L_{\mathcal{O}_{\epsilon_0}}$, we can show that

$$L_{\mathcal{O}_{\epsilon_0}} \int_0^T \int_\Omega |\psi(s,x)|^2 dxds \leqslant C\left(\iint_{\mathcal{O}_{\epsilon_0}} |\psi(t,x)|^2 dxdt + \iint_Q |\varphi(t,x)|^2 dxdt\right),$$

or equivalently

$$\|\psi\|^2_{L^2(Q)} \leqslant C\left(\|\psi\|^2_{L^2(\mathcal{O}_{\epsilon_0})} + \|\varphi\|^2_{L^2(Q)}\right). \tag{3.5}$$

By standard energy estimates applied to $(3.3)_1$, we also have that

$$\|\varphi\|^2_{L^2(Q)} + \|\nabla\varphi\|^2_{L^2(Q)} \leqslant C\|\psi\|^2_{L^2(Q)}. \tag{3.6}$$

Also, differentiating $(3.3)_1$ with respect to time and using energy estimates again, we get

$$\|\varphi_t\|^2_{L^2(Q)} + \|\nabla\varphi_t\|^2_{L^2(Q)} \leqslant C\left(\|\psi\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(Q)}\right). \tag{3.7}$$

Hence, from (3.5), (3.6) and (3.7), we obtain the estimate

$$\|\psi\|^2_{L^2(Q)} + \|\varphi\|^2_{H^1(Q)} \leqslant C\left(\|\psi\|^2_{L^2(\mathcal{O}_{\epsilon_0})} + \|\varphi\|^2_{L^2(Q)}\right). \tag{3.8}$$

In what follows we are going to get rid of the last term in (3.8) by a compactness-uniqueness argument. In fact, we will prove that

$$\|\psi\|^2_{L^2(Q)} + \|\varphi\|^2_{H^1(Q)} \leqslant C\|\psi\|^2_{L^2(\mathcal{O}_{\epsilon_0})}. \tag{3.9}$$

Indeed, if (3.9) does not hold, there exists $(\varphi^n,\psi^n) \subset H^1(Q)\times L^2(Q)$ solution of (3.3) such that

$$\|\psi^n\|^2_{L^2(Q)} + \|\varphi^n\|^2_{H^1(Q)} = 1 \tag{3.10}$$

and

$$\|\psi^n\|^2_{L^2(\mathcal{O}_{\epsilon_0})} \leqslant \frac{1}{n}. \tag{3.11}$$

Since (3.10) holds, there exists a subsequence (still denoted by the same index) such that

$$(\varphi^n, \psi^n) \text{ converges weakly to some } (\varphi^\star, \psi^\star) \text{ in } H^1(Q) \times L^2(Q).$$

It is not difficult to see that $(\varphi^\star, \psi^\star)$ is a weak solution of (3.3) and that

$$\varphi^n \text{ converges strongly to } \varphi^\star \text{ in } L^2(Q). \tag{3.12}$$

Using the weak convergence and (3.11), it follows that

$$\|\psi^\star\|^2_{L^2(\mathcal{O}_{\epsilon_0})} \leqslant \liminf_{n\to\infty} \|\psi^n\|^2_{L^2(\mathcal{O}_{\epsilon_0})} = 0$$

Therefore, we have that

$$\psi^\star = 0 \quad \text{in} \quad \mathcal{O}_{\epsilon_0} \tag{3.13}$$

and

$$\|\psi^\star\|^2_{L^2(Q)} + \|\varphi^\star\|^2_{H^1(Q)} \leqslant C\|\varphi^\star\|^2_{L^2(Q)}. \tag{3.14}$$

Since $(\psi^n - \psi^\star, \varphi^n - \varphi^\star)$ solves a problem like (3.3) and (3.8) and also (3.11)-(3.14) hold, we have that

$$\psi^n \text{ converges strongly to } \psi^\star \text{ in } L^2(Q). \tag{3.15}$$

From (3.8), (3.10) and (3.11), we see that

$$1 \leqslant \frac{C}{n} + C\|\varphi^n\|^2_{L^2(Q)}, \ \forall n \in \mathbb{N}. \tag{3.16}$$

According to (3.12) and (3.16), we get that

$$0 < \|\varphi^\star\|^2_{L^2(Q)}. \tag{3.17}$$

Thus, we conclude that $(\varphi^\star, \psi^\star)$ is not zero.

Let us introduce the linear space $E \subset H^1(Q) \times L^2(Q)$ given by

$$E := \left\{(\varphi, \psi) \in H^1(Q) \times L^2(Q) : \ (\varphi, \psi) \text{ satisfies } (3.3)_1 - (3.3)_2, \ \varphi\big|_\Sigma = 0 \text{ and } \psi = 0 \text{ in } \mathcal{O}_{\epsilon_0}\right\}. \tag{3.18}$$

Since $(\varphi^\star, \psi^\star) \in E$, we have that $E \neq \{0\}$. Let us now show that $E = \{0\}$, which is a contradiction.

Claim 4. $E \subset H^5(Q) \times H^3(Q)$.

Proof of Claim 4. Let $(\varphi, \psi) \in E$. Since $\psi = 0$ in $\mathcal{O}_{\epsilon_0}$, it follows from $(3.3)_1 - (3.3)_2$ that

$$-\Delta\varphi = 0 \text{ in } \mathcal{O}_{\epsilon_0}$$

and hence

$$\varphi \in H^{k+1}(\mathcal{O}_{\frac{3}{2}\epsilon_0}), \ \forall k \in \mathbb{N}.$$

Since $\mathcal{O}_{\frac{3}{2}\epsilon_0}$ also satisfies the *MGCC-I*, we can use similar arguments to those in (3.8) and equation $(3.3)_2$ to show that

$$\|\psi\|^2_{H^1(Q)} \leqslant C\left(\|\psi\|^2_{H^1(\mathcal{O}_{\frac{3}{2}\epsilon_0})} + \|\varphi\|^2_{H^1(Q)}\right) \tag{3.19}$$

$$\leqslant C\|\varphi\|^2_{H^1(Q)} \tag{3.20}$$

and then, by a bootstrap argument and elliptic regularity for $(3.3)_1$, it follows that

$$\varphi \in H^3(Q).$$

Arguing in the same way, we see that

$$\|\psi\|^2_{H^3(Q)} \leqslant C\left(\|\psi\|^2_{H^3(\mathcal{O}_{\frac{3}{2}\epsilon_0})} + \|\varphi\|^2_{H^3(Q)}\right) \tag{3.21}$$

$$\leqslant C\|\varphi\|^2_{H^3(Q)} \tag{3.22}$$

and that

$$\varphi \in H^5(Q),$$

which proves the Claim. □

Claim 5. *E is a finite dimensional vector space.*

Proof of Claim 5. Let $\{(\varphi^n, \psi^n)\}_{n=1}^{\infty} \subset E$ with

$$\|\psi^n\|^2_{L^2(Q)} + \|\varphi^n\|^2_{H^1(Q)} \leqslant 1, \ \forall n \in \mathbb{N}.$$

Then, (using the same index) we see that

$$(\varphi^n, \psi^n) \text{ converges weakly to some } (\widehat{\varphi}, \widehat{\psi}) \text{ in } H^1(Q) \times L^2(Q).$$

One can also see that $(\widehat{\varphi}, \widehat{\psi})$ is a weak solution of (3.3) and that

$$\varphi^n \text{ converges strongly to } \widehat{\varphi} \text{ in } L^2(Q).$$

From (3.8), we have that

$$\|\psi\|^2_{L^2(Q)} + \|\varphi\|^2_{H^1(Q)} \leqslant C\|\varphi\|^2_{L^2(Q)}, \ \forall (\varphi, \psi) \in E. \tag{3.23}$$

Therefore, it follows that

$$(\varphi^n, \psi^n) \text{ converges strongly to } (\widehat{\varphi}, \widehat{\psi}) \text{ in } H^1(Q) \times L^2(Q),$$

which proves that E is finite dimensional. □

Now, for any $(\varphi, \psi) \in E$, by Claim 4, and noting that $\mathcal{O}_{\epsilon_0}$ satisfies the *MGCC-I* and $\psi = 0$ in $\mathcal{O}_{\epsilon_0}$, we see that $\psi = 0$ on Σ and

$$\left|\begin{array}{lll} (I-\Delta)\varphi - \Delta((I-\Delta)\varphi) = (I-\Delta)\psi & \text{in} & Q, \\ -((I-\Delta)\psi)_t + (I-\Delta)\psi = (I-\Delta)\varphi & \text{in} & Q, \\ (I-\Delta)\varphi = 0 & \text{on} & \Sigma. \end{array}\right.$$

Thus, we have that $((I-\Delta)\varphi, (I-\Delta)\psi)$ belongs to E.

Since E is finite dimensional, the operator $I - \Delta$ must have an eigenvalue $\lambda \in \mathbb{C}$ and an eigenvector $(\varphi^\lambda, \psi^\lambda) \in E \backslash \{0\}$.

Claim 6. $\lambda \neq 0$.

Proof of Claim 6. Suppose $\lambda = 0$. Then, for any $t \in (0, T)$ we have that

$$\left|\begin{array}{lll} (I-\Delta)\varphi^\lambda(\cdot, t) = 0 & \text{in} & \Omega, \\ \varphi^\lambda(\cdot, t) = 0 & \text{on} & \partial\Omega, \end{array}\right.$$

which gives

$$\varphi^\lambda(\cdot, t) = 0 \text{ in } \Omega \text{ for all } t \in (0, T).$$

In particular, from the PDE equation we see that $\psi^\lambda = 0$ in Q. Then $(\varphi^\lambda, \psi^\lambda) = (0, 0)$, which is a contradiction. □

To finish the proof, we notice that since

$$\left|\begin{array}{lll} \psi^\lambda = \lambda\varphi^\lambda & \text{in} & Q, \\ -\psi^\lambda_t + \psi^\lambda = \varphi^\lambda & \text{in} & Q, \\ \varphi^\lambda = 0 & \text{on} & \Sigma, \\ \psi^\lambda = \varphi^\lambda = 0 & \text{in} & \mathcal{O}_{\epsilon_0}. \end{array}\right.$$

Now, for a fixed $t_0 \in (0,T)$ and $x_0 \in \mathcal{O}_{\epsilon_0}(t_0)$, it follows that $(\varphi^\lambda(x_0,.), \psi^\lambda(x_0,.))$ is the solution of

$$\left|\begin{array}{lll} \psi^\lambda(x_0,t) = \lambda\varphi^\lambda(x_0,t) & \text{in} & (0,T), \\ -\psi^\lambda_t(x_0,t) + \psi^\lambda(x_0,t) = \varphi^\lambda(x_0,t) & \text{in} & (0,T), \\ \varphi^\lambda(x_0,t_0) = 0, \quad \psi^\lambda(x_0,t_0) = 0 & & \end{array}\right.$$

and since $\lambda \neq 0$ and $MGCC$-I holds, we conclude that

$$\varphi^\lambda = \psi^\lambda = 0 \text{ in } Q.$$

This contradicts the fact that $(\varphi^\lambda, \psi^\lambda)$ is not zero.

Therefore, we have proved the observability inequality

$$\|\psi\|^2_{L^2(Q)} + \|\varphi\|^2_{H^1(Q)} \leqslant C\|\psi\|^2_{L^2(\mathcal{O}_{\epsilon_0})}. \tag{3.24}$$

and the proof of Proposition 3.1 is finished.

□

3.2. **Benjamin-Bona-Mahony with moving controls.** In this section, we prove the positive null controllability for the Benjamin-Bona-Mahony equation (1.3). Here we use an approach based on Carleman estimates.

In what follows, we assume that X and ω_0 satisfy Assumption 1.6, and for each open set $\omega \subset \mathbb{R}^N$, with $\overline{\omega}_0 \subset \omega$, we choose ω_1, ω_2 nonempty open sets in $\mathbb{R}^N$ such that

$$\overline{\omega}_0 \subset \omega_1, \; \overline{\omega}_1 \subset \omega_2, \; \overline{\omega}_2 \subset \omega.$$

The following weight function is constructed in [5].

Lemma 3.3 ([5]). *There exist a positive number $\tau \in (0, \min\{1, T/2\})$ and a function $\eta \in C^\infty(\overline{\Omega} \times [0,T])$ such that*

$$\nabla\eta(x,t) \neq 0, \qquad t \in [0,T], \; x \in \Omega\backslash\overline{\mathcal{O}_{\omega_1}(t)}, \tag{3.25}$$

$$\eta_t(x,t) \neq 0, \qquad t \in [0,T], \; x \in \Omega\backslash\overline{\mathcal{O}_{\omega_1}(t)}, \tag{3.26}$$

$$\eta_t(x,t) > 0, \qquad t \in [0,\tau], \; x \in \Omega\backslash\overline{\mathcal{O}_{\omega_1}(t)}, \tag{3.27}$$

$$\eta_t(x,t) < 0, \qquad t \in [T-\tau,T], \; x \in \Omega\backslash\overline{\mathcal{O}_{\omega_1}(t)}, \tag{3.28}$$

$$\frac{\partial\eta}{\partial\nu}(x,t) \leqslant 0, \qquad t \in [0,T], \; x \in \partial\Omega, \tag{3.29}$$

$$\eta(x,t) > \frac{3}{4}\|\eta\|_\infty, \qquad t \in [0,T], \; x \in \overline{\Omega}. \tag{3.30}$$

Next, we introduce a real function $r \in C^\infty(0,T)$, symmetric with respect to $t = \frac{T}{2}$ and such that for $\tau > 0$, as above,

$$r(t) = \begin{cases} \dfrac{1}{t} & \text{for} \quad 0 < t \leqslant \frac{\tau}{2}, \\ \text{strictly decreasing} & \text{for} \quad \frac{\tau}{2} < t < \tau, \\ 1 & \text{for} \quad \tau \leqslant t \leqslant \frac{T}{2}, \\ r(T-t) & \text{for} \quad \frac{T}{2} \leqslant t < T \end{cases}$$

and define the weights

$$\begin{aligned} \gamma(x,t) &= e^{\lambda\eta(x,t)} & (x,t) \in \Omega \times (0,T), \\ \alpha(x,t) &= r(t)(e^{2\lambda\|\eta\|_\infty} - \gamma(x,t)) & (x,t) \in \Omega \times (0,T), \\ \xi(x,t) &= r(t)\gamma(x,t) & (x,t) \in \Omega \times (0,T), \\ \alpha^*(t) &= \max_{x\in\overline{\Omega}} \alpha(x,t) & t \in (0,T), \\ \xi^*(t) &= \min_{x\in\overline{\Omega}} \xi(x,t) & t \in (0,T). \end{aligned}$$

where $\lambda > 0$ is a parameter that will be chosen large enough.

The following Carleman inequality was proved in [5].

Lemma 3.4. *Let $T > 0$ and let $\omega \subset \mathbb{R}^N$ satisfy Definition 1.7. There exist positive real numbers $\lambda_1 > 0$, $s_1 > 0$ and $C_1 > 0$ (depending on Ω and ω) such that for all $\lambda \geqslant \lambda_1$, all $s \geqslant s_1$ and all $q \in H^1(0,T;L^2(\Omega))$, the following inequality holds*

$$s\lambda^2 \iint\limits_Q \xi|q|^2 e^{-2s\alpha}\,dxdt \leqslant C_1 \left(\iint\limits_Q |q_t|^2 e^{-2s\alpha}\,dxdt + s^2\lambda^2 \int_0^T\!\!\int_{\mathcal{O}_{\omega_2}(t)} e^{-2s\alpha}\xi^2|q|^2\,dxdt \right).$$

We recall that $\mathcal{O}_{\omega_2}(t) = X(\omega_2,t,0) \cap \Omega$ (see Definition 1.7).

To our purposes, we prove the following new Carleman inequality for the Laplace operator.

Lemma 3.5. *Let $T > 0$ and let $\omega \subset \mathbb{R}^N$ satisfy Definition 1.7. There exist positive real numbers $\lambda_2 > 0$, $\tau_2 > 0$ and $C_2 > 0$, independent of t, such that for all $\lambda \geqslant \lambda_2$, all $\tau \geqslant \tau_2$ and all $(g,G) \in H^1(0,T;L^2(\Omega) \times L^2(\Omega)^N)$, the solution y of*

$$\left| \begin{array}{lll} -\Delta y = g + \nabla\cdot G & \text{in} & Q, \\ y = 0 & \text{on} & \Sigma, \end{array} \right. \tag{3.31}$$

satisfies

$$\int_\Omega \left[\lambda^2(\tau\gamma)^2|y|^2 + |\nabla y|^2\right] e^{2\tau\gamma}dx \leqslant C_2 \Bigg(\int_\Omega \left[\lambda^{-2}(\tau\gamma)^{-1}|g|^2 + (\tau\gamma)|G|^2\right] e^{2\tau\gamma}dx + \int_{\mathcal{O}_{\omega_2}(t)} \lambda^2(\tau\gamma)^2|y|^2 e^{2\tau\gamma}\,dx + \int_{\mathcal{O}_{\omega_2}(t)} e^{2\tau\gamma}|\nabla y|^2\,dx \Bigg), \tag{3.32}$$

for all $t \in [0,T]$.

For sake of completeness, we give a sketch of the proof of Lemma 3.5 in Appendix A.

Theorem 1.8 is a consequence of the following result:

Proposition 3.6. *Let $T > 0$, $A \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^{N\times N})$ and let $\omega \subset \mathbb{R}^N$ satisfy Definition 1.7. Then, for any $z_0 \in L^2(\Omega)$, there exists a moving control $v \in L^2(\mathcal{O}_\omega)$ such that the solution*

$$\left| \begin{array}{lll} y - \Delta y = z & in & Q, \\ z_t + \nabla \cdot (A(x,t)y) = v\chi_{\mathcal{O}_\omega} & in & Q, \\ y = 0 & on & \Sigma, \\ z(\cdot,0) = z_0 & in & \Omega. \end{array} \right. \tag{3.33}$$

satisfies

$$y(\cdot,T) = z(\cdot,T) = 0 \quad in \quad \Omega.$$

Proof of Theorem 1.8. Given $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, we consider $z_0 = y_0 - \Delta y_0 \in L^2(\Omega)$. From Proposition 3.6, there exists $v \in L^2(\mathcal{O}_\omega)$ such that the associated solution (y,z) to (3.33) satisfies $y(T) = z(T) = 0$. It is not difficult to see that y solves, together with the control v, the null controllability problem for the Benjamin-Bona-Mahony equation (1.3). □

The rest of this section is devoted to prove Proposition 3.6. As before, proving Proposition 3.6 is equivalent to find $C > 0$ such that

$$\|\psi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C \int_0^T \int_{\mathcal{O}_\omega(t)} |\psi|^2 dxdt, \quad \forall \psi_T \in L^2(\Omega), \tag{3.34}$$

for all (φ,ψ) solution of

$$\left| \begin{array}{lll} \varphi - \Delta\varphi = A\cdot\nabla\psi & \text{in} & Q, \\ -\psi_t = \varphi & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \psi(T) = \psi_T & \text{in} & \Omega. \end{array} \right. \tag{3.35}$$

Inequality (3.34) is a consequence of the following Carleman inequality:

Theorem 3.7. *Let $T > 0$, $A \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^{N\times N})$ and let $\omega \subset \mathbb{R}^N$ satisfy Definition 1.7. There exist positive constants s_0, $\lambda_0 \geqslant 1$ and C, only depending on Ω and ω_0, such that, for any $\psi_T \in L^2(\Omega)$, the solution (φ,ψ) to the adjoint system (3.35) satisfies:*

$$\begin{aligned} &\iint_Q e^{-2s\alpha}[|\nabla\varphi|^2 + s^2\lambda^2\xi^2|\varphi|^2]dxdt + s\lambda^2 \iint_Q \xi|\psi|^2 e^{-2s\alpha}dxdt \\ &+ \int_Q [s\lambda^2\xi^*|\nabla\varphi_t|^2 + s\lambda^2\xi^*|\varphi_t|^2]e^{-2s\alpha^*}dxdt \\ &\leqslant Cs^6\lambda^2 \int_0^T \int_{\mathcal{O}_\omega(t)} \xi^6 e^{-4s\alpha+2s\alpha^*}|\psi|^2 dxdt. \end{aligned}$$

for all $s \geqslant s_0(T+T^2)$ and for all $\lambda \geqslant \lambda_0$.

Proof. We begin applying the Carleman inequality given by Lemma 3.4 to $(3.35)_2$, which gives

$$\iint_Q s\lambda^2\xi|\psi|^2 e^{-2s\alpha}dxdt \leqslant \iint_Q |\varphi|^2 e^{-2s\alpha}dxdt + \int_0^T \int_{\mathcal{O}_{\omega_2}(t)} s^2\lambda^2\xi^2|\psi|^2 e^{-2s\alpha}dxdt. \tag{3.36}$$

Next, noticing that

$$A\cdot\nabla\psi = \nabla\cdot(A\psi) - \psi\nabla\cdot A$$

and applying the Carleman inequality given in Lemma 3.5 for $(3.35)_1$, we see that

$$\tau^2\lambda^2\int_\Omega e^{2\tau\gamma}\gamma^2|\varphi|^2\,dx+\int_\Omega e^{2\tau\gamma}|\nabla\varphi|^2\,dx\leqslant C\left(\frac{1}{\tau\lambda^2}\int_\Omega e^{2\tau\gamma}\frac{|\psi|^2}{\gamma}dx+\tau\int_\Omega e^{2\tau\gamma}\gamma|\psi|^2dx\right.$$
$$\left.+\tau^2\lambda^2\int_{\mathcal{O}_{\omega_2}(t)}e^{2\tau\gamma}\gamma^2|\varphi|^2\,dx+\int_{\mathcal{O}_{\omega_2}(t)}e^{2\tau\gamma}|\nabla\varphi|^2\,dx\right),$$

for $\lambda\geqslant\lambda_0$ and $\tau\geqslant\tau_0$.

To connect this elliptic estimate with (3.36), we set

$$\tau=sr(t),$$

multiply by

$$e^{-2sr(t)e^{2\|\eta\|_\infty}}$$

and integrate with respect to t in $(0,T)$. If we take $s_0\geqslant\tau_0$ then we have that $\tau\geqslant\tau_0$ and the following estimate holds

$$\begin{aligned}&s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt+\iint\limits_Q e^{-2s\alpha}|\nabla\varphi|^2\,dxdt\\&\leqslant C\left(\frac{1}{s\lambda^2}\iint\limits_Q e^{-2s\alpha}\frac{|\psi|^2}{\xi}\,dxdt+s\iint\limits_Q e^{-2s\alpha}\xi|\psi|^2\,dxdt\right.\\&\left.\quad+s^2\lambda^2\int_0^T\!\!\int_{\mathcal{O}_{\omega_2}(t)}e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt+\int_0^T\!\!\int_{\mathcal{O}_{\omega_2}(t)}e^{-2s\alpha}|\nabla\varphi|^2\,dxdt\right).\end{aligned}\tag{3.37}$$

Adding (3.36) and (3.37), and absorbing the lower order terms by taking λ large enough, we get

$$\begin{aligned}&s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi|\psi|^2dxdt+s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt+\iint\limits_Q e^{-2s\alpha}|\nabla\varphi|^2\,dxdt\\&\leqslant C\left(s^2\lambda^2\int_0^T\!\!\int_{\mathcal{O}_{\omega_2}(t)}e^{-2s\alpha}\xi^2|\psi|^2dxdt+\ s^2\lambda^2\int_0^T\!\!\int_{\mathcal{O}_{\omega_2}(t)}e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt\right.\\&\left.\quad+\int_0^T\!\!\int_{\mathcal{O}_{\omega_2}(t)}e^{-2s\alpha}|\nabla\varphi|^2\,dxdt\right).\end{aligned}$$

Now, let us introduce ω_3 such that $\overline{\omega}_2\subset\omega_3\subset\overline{\omega}_3\subset\omega$ and the function

$$\zeta(x,t):=\vartheta(X(x,0,t)),$$

where ϑ is a cut-off function satisfying

$$\vartheta\in C_0^\infty(\omega_3),\quad 0\leqslant\vartheta(x)\leqslant 1,\quad \vartheta\equiv 1\quad\text{in}\quad\omega_2.$$

This way, we have that

$$\int_0^T\!\!\int_{\mathcal{O}_{\omega_2}(t)}e^{-2s\alpha}|\nabla\varphi|^2\,dxdt\leqslant\iint\limits_Q\zeta e^{-2s\alpha}|\nabla\varphi|^2\,dxdt.\tag{3.38}$$

Then, since $\zeta e^{-2s\alpha}\nabla\varphi = \nabla(\zeta e^{-2s\alpha}\varphi) - \nabla(\zeta e^{-2s\alpha})\varphi$, we obtain

$$\begin{aligned}\iint\limits_Q \zeta e^{-2s\alpha}|\nabla\varphi|^2\,dxdt &= \iint\limits_Q \nabla(\zeta e^{-2s\alpha}\varphi)\cdot\nabla\varphi\,dxdt - \iint\limits_Q [\nabla(\zeta e^{-2s\alpha})\cdot\nabla\varphi]\varphi\,dxdt\\ &= \frac{1}{2}\iint\limits_Q \Delta(\zeta e^{-2s\alpha})|\varphi|^2\,dxdt + \iint\limits_Q \nabla(\zeta e^{-2s\alpha}\varphi)\cdot\nabla\varphi\,dxdt\\ &:= B_1 + B_2.\end{aligned}$$

Now, let us estimate the terms B_1 and B_2. For that, we use that

$$\Delta(\zeta e^{-2s\alpha}) := e^{-2s\alpha}\left\{\Delta\zeta + 4s\lambda\xi\left[\nabla\zeta\cdot\nabla\eta\right] + 2s\lambda\xi\zeta\left[\lambda|\nabla\eta|^2(2s\xi+1)+\Delta\eta\right]\right\},$$

to see that

$$B_1 \leqslant C\,s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt.$$

Since $A\cdot\nabla\psi := \nabla\cdot(A\psi) - \psi\nabla\cdot A \in H^{-1}(\Omega)$, the solution for $(3.35)_1$ satisfies the following weak formulation

$$(\varphi, w) + (\nabla\varphi, \nabla w) = -(A\psi, \nabla w) - (\psi\nabla\cdot A, w) \quad \forall w \in H_0^1(\Omega).$$

Using the previous formulation with $w = \zeta e^{-2s\alpha}\varphi$, we obtain

$$\begin{aligned}B_2 &= -\iint\limits_Q \zeta e^{-2s\alpha}|\varphi|^2\,dxdt - \iint\limits_Q (\nabla\cdot A)\,\zeta e^{-2s\alpha}\varphi\psi\,dxdt - \iint\limits_Q \psi\left[A\cdot\nabla(\zeta e^{-2s\alpha}\varphi)\right]\,dxdt\\ &:= B_2^1 + B_2^2 + B_2^3.\end{aligned}$$

Now, for B_2^1, we easily deduce that

$$|B_2^1| \leqslant C\,s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt.$$

For B_2^2, we notice that, for every $\delta > 0$, we obtain

$$|B_2^2| \leqslant \delta s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi|\psi|^2\,dxdt + C_\delta\,s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt.$$

Since $\nabla(\zeta e^{-2s\alpha}) = e^{-2s\alpha}(\nabla\zeta + 2s\lambda\xi\zeta\nabla\eta)$, for every $\delta > 0$, we have that

$$B_2^3 \leqslant \varepsilon\left(s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt + \iint\limits_Q e^{-2s\alpha}|\nabla\varphi|^2\,dxdt\right) + C_\delta\,s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\psi|^2\,dxdt.$$

This way, we get

$$\begin{aligned}&s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi|\psi|^2dxdt + s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt + \iint\limits_Q e^{-2s\alpha}|\nabla\varphi|^2\,dxdt\\ &\leqslant C\left(s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\psi|^2dxdt + s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt\right).\end{aligned} \tag{3.39}$$

Finally, to estimate the local integral of φ in the right-hand side of (3.39), we need to have some global integral of φ_t on the left-hand side. For that, we first take the time derivative in $(3.35)_1$, use the fact that $(A\cdot\nabla\psi)_t = \nabla\cdot(A_t\psi + A\psi_t) - \psi_t\nabla\cdot A - \psi\nabla\cdot A_t$ and $(3.35)_2$, to see that φ_t solves the following elliptic equation

$$\left|\begin{array}{lll} \varphi_t - \Delta\varphi_t = \nabla\cdot(A_t\psi - A\varphi) + \varphi\nabla\cdot A - \psi\nabla\cdot A_t & \text{in} & Q, \\ \varphi_t = 0 & \text{on} & \Sigma. \end{array}\right. \tag{3.40}$$

From (3.39) and energy estimates for (3.40), it is not difficult to see that

$$\begin{aligned} &\int_Q [s\lambda^2\xi^*|\nabla\varphi_t|^2 + s\lambda^2\xi^*|\varphi_t|^2]e^{-2s\alpha^*}dxdt \\ &\leqslant C\left(s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\psi|^2dxdt + s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt\right). \end{aligned} \tag{3.41}$$

Combining (3.39) and (3.41), we get

$$\begin{aligned} &\iint\limits_Q e^{-2s\alpha}[|\nabla\varphi|^2 + s^2\lambda^2\xi^2|\varphi|^2]dxdt + s\lambda^2\iint\limits_Q \xi|\psi|^2e^{-2s\alpha}dxdt \\ &+ \int_Q [s\lambda^2\xi^*|\nabla\varphi_t|^2 + s\lambda^2\xi^*|\varphi_t|^2]e^{-2s\alpha^*}dxdt \\ &\leqslant C\left(s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\psi|^2dxdt + s^2\lambda^2\int_0^T\int_{\mathcal{O}_{\omega_3}(t)} e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt\right). \end{aligned} \tag{3.42}$$

The proof of Theorem 3.7 is finished noticing that $-\psi_t = \varphi$ and using integration by parts to estimate the local integral in φ.

□

4. Comments and Open Problems

• We have proved Theorems 1.5 and 1.8 assuming that $y_0 \in H^2(\Omega)\cap H_0^1(\Omega)$. We do not know whether these results are true if we take only $y_0 \in H_0^1(\Omega)$. Indeed, in this case the initial condition for both decompositions (3.1) and (3.33) will be in $H^{-1}(\Omega)$ and it seems that both the compactness-uniqueness argument and the argument based on Carleman inequalities does not work.

• If the vector field A is regular enough and depends only on time, i.e., $A = A(t)$, we can weaken the hypothesis on the movement of the control for equation (1.3). In fact, we can use a compactness-uniqueness argument as in the proof of Theorem 1.5 and deduce the following result.

Theorem 4.1. *Let $T > 0$, $\mathcal{O}$ satisfying MGCC-I and $A \in C^\infty(0,T)$. Then, for any $y_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, there exists a moving control $v \in L^2(\mathcal{O})$ such that the associated solution to (1.3) satisfies*

$$y(\cdot,T) = 0 \quad in \quad \Omega.$$

• Concerning the geometrical requirements for the movement of controls given by the MGCC-II, i.e. Assumption 1.6, we do not know if all the conditions (1.4)-(1.8) are really necessary in order to get null controllability for equation (1.3) when A is x-dependent. Indeed, we use these conditions only to construct the function η given in Lemma 3.3. Removing any of the conditions (1.4)-(1.8) is interesting and, as far as we know, completely open.

• It would be interesting to study controllability issues for nonlinear pseudo-parabolic equations such as the nonlinear BBM equation. Indeed, it is not even clear whether nonlinear pseudo-parabolic equations can be driven or not to zero by means of controls applied to fixed control domains. Moreover, assuming that null controllability does not hold with fixed controls, it is a challenging problem to control nonlinear equations using moving controls and we are only aware of the paper [11] where a similar one-dimensional problem has been considered. We leave this study to a forthcoming paper.

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Appendix A. Carleman inequality for the Laplace operator

We give the sketch of the proof of Lemma 3.5, which is inspired by the arguments in [9].

For every $t \in [0, T]$, we set the function $\gamma(x,t) = e^{\lambda\eta(x,t)}$ and consider $w(x,t) = e^{\tau\gamma(x,t)} z(x,t)$.

We have the following decomposition

$$e^{\tau\gamma}\Delta z = e^{\tau\gamma}\Delta(e^{-\tau\gamma}w) = [\Delta w + \tau^2|\nabla\gamma|^2 w] - [2\tau\nabla\gamma\cdot\nabla w + \tau\Delta\gamma w] = e^{\tau\gamma}g + e^{\tau\gamma}\nabla\cdot G = e^{\tau\gamma}\tilde{g} + \nabla\cdot(e^{\tau\gamma}G),$$

where $\tilde{g} = g - \tau\nabla\gamma\cdot G$.

Multiplying the previous equation by w and integrating by parts, one can see that:

$$\int_\Omega |\nabla w|^2\,dx + \tau^2\int_\Omega |\nabla\gamma|^2|w|^2\,dx = \int_\Omega e^{\tau\gamma}\tilde{g}w\,dx - \int_\Omega e^{\tau\gamma}G\cdot\nabla w\,dx$$

which, together with the properties of the weigh function, gives the result.

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