DE GRUYTER J. Inverse Ill-Posed Probl. 2017; 25 (4):445–468

Research Article

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Inverse problems for linear parabolic equations using mixed formulations – Part 1: Theoretical analysis

DOI: 10.1515/jiip-2015-0112
Received December 22, 2015; revised July 8, 2016; accepted August 1, 2016

Abstract: We consider the reconstruction of the solution of a parabolic equation posed in $\Omega \times (0, T)$, with a bounded open subset Ω of $\mathbb{R}^N$, from a partial distributed observation. We employ a least-squares technique and minimize the L^2-norm of the distance from the observation to any solution. Taking the parabolic equation as the main constraint, the optimality conditions are reduced to a mixed formulation involving both the state to reconstruct and a Lagrange multiplier. The well-posedness of this mixed formulation, in particular the inf-sup property, is a consequence of classical energy estimates. We then reproduce the arguments to a linear first-order system, involving the normal flux, equivalent to the linear parabolic equation. The method, valid in any spatial dimension N, may also be employed to reconstruct solutions from boundary observations. With respect to the hyperbolic case, the parabolic situation requires, due to regularization properties, the introduction of an appropriate weight function so as to make the reconstruction stable with respect to standard Sobolev spaces.

Keywords: Inverse problem, heat equation, Lagrangian variational formulation, Carleman estimate

MSC 2010: 35K05, 35R30, 49J20

1 Introduction: Inverse problems for linear parabolic equations

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary $\partial\Omega$ is regular enough. For any $T > 0$ we denote

$$Q_T := \Omega \times (0, T) \quad \text{and} \quad \Sigma_T := \partial\Omega \times (0, T).$$

We are concerned with inverse type problems for the following linear parabolic-type equation:

$$\begin{cases} y_t - \nabla \cdot (c(x)\nabla y) + d(x, t)y = f & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(x, 0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Let $\mathcal{M}_N(\mathbb{R})$ be the set of real square matrices of order N. We assume that $c := (c_{i,j}) \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R}))$ with $(c(x)\xi, \xi) \geq c_0|\xi|^2$ for any $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$ ($c_0 > 0$), $d \in L^\infty(Q_T)$ and $y_0 \in L^2(\Omega)$; here $f = f(x, t)$ is a *source* term (a function in $L^2(Q_T)$) and $y = y(x, t)$ is the associated state. For any $y_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$, there exists exactly one solution y to (1.1) with the regularity (see [5, 24])

$$y \in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)).$$

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In what follows, we shall use the following notations:

$$Ly := y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y, \quad L^\star \varphi := -\varphi_t - \nabla \cdot (c(x)\nabla \varphi) + d(x,t)\varphi.$$

Let now ω be any nonempty open subset of Ω and let $q_T := \omega \times (0,T) \subset Q_T$. A typical inverse problem for (1.1) (see [20, 32]) is the following one: from an *observation* or *measurement* y_{obs} on the open set q_T, we aim at recovering a solution y of the boundary value problem (1.1) which coincides with the observation on q_T. Assuming y_{obs} in $L^2(q_T)$ and introducing the operator $P : \mathcal{Y} \to L^2(Q_T) \times L^2(q_T)$ defined by $Py := (Ly, y_{|q_T})$ where the space $\mathcal{Y}$ is an appropriate Hilbert space (defined in Section 2.1), the problem is reformulated as the following:

$$\text{find a solution } y \in \mathcal{Y} \text{ of } Py = (f, y_{\text{obs}}). \tag{IP}$$

From the unique continuation property for (1.1), if y_{obs} is a restriction of a solution of (1.1) to q_T, then the problem is well-posed in the sense that the state y corresponding to the pair (f, y_{obs}) is unique, i.e. P is a bijective operator from $\mathcal{Y}$ to its range $\mathcal{R}(P)$. However, in view of the unavoidable uncertainties on the data y_{obs} (coming from measurements, numerical approximations, etc.), problem (IP) needs to be relaxed. In this respect, the approach widely used in practice consists in introducing the following extremal problem (of least-squares type):

$$\text{minimize } J(y_0) := \frac{1}{2}\|\rho_0^{-1}(y - y_{\text{obs}})\|^2_{L^2(q_T)} \text{ over } \mathcal{H}, \text{ where } y \text{ solves (1.1)}, \tag{LS}$$

since y is uniquely and fully determined from the data y_0. Here ρ_0 denotes an appropriate positive weight while $\mathcal{H}$ denotes a Hilbert space related to the space $\mathcal{Y}$; roughly, $\mathcal{H}$ is the set of initial data y_0 for which the solution of (1.1) satisfies $\rho_0^{-1} y \in L^2(q_T)$.

Here, the constraint $y - y_{\text{obs}} = 0$ in $L^2(q_T)$ is relaxed; however, if y_{obs} is a restriction to q_T of a solution of (1.1), then problems (LS) and (IP) coincide. A minimizing sequence for J in $\mathcal{H}$ is easily defined in terms of the solution of an auxiliary adjoint problem. From a numerical point of view, this extremal problem has mainly two independent drawbacks:

- First, it is in general not possible to minimize over a discrete subspace of the set $\{y; Ly - f = 0\}$ subject to the equality $Ly - f = 0$ (in $L^2(Q_T)$). Therefore, the minimization procedure first requires the discretization of the functional J and of equation (1.1). This raises the issue, when one wants to prove some convergence result of any discrete approximation of the uniform coercivity property (typically here some uniform discrete observability inequality for the adjoint solution) of the discrete functional with respect to the approximation parameter. As far as we know, this delicate issue has received answers only for specific and somehow academic situations (uniform Cartesian approximation of Ω, constant coefficients in (1.1), etc.). We refer to [4, 28].
- Second, in view of the regularization property of the heat kernel, the space of initial data $\mathcal{H}$ for which the corresponding solution of (1.1) belongs to $L^2(q_T)$ is a huge space. It contains in particular the negative Sobolev space $H^{-s}(\Omega)$ for any $s > 0$ and therefore is very hard to approximate numerically. For this reason, the reconstruction of the initial condition y_0 of (1.1) from a partial observation in $L^2(q_T)$ is therefore known to be numerically severely ill-posed and requires, within this framework, a regularization to enforce that the minimizer belongs to a standard Sobolev space (for instance $L^2(\Omega)$) which is easier to approximate (see [12]). The situation is analogous for the so-called backward heat problem, where the observation on q_T is replaced by a final time observation. We refer to [7, 30, 31] where this ill-posedeness is discussed.

The main reason of this work is to reformulate problem (LS) and show that the use of variational methods may overcome these two drawbacks. In the spirit of the works [1, 21] where Cauchy problems for parabolic equations are addressed (based on the quasi-reversibility method; see [22] and Remark 2.5), we explore the direct resolution of the optimality conditions associated to the extremal problem (LS), keeping the state y as the main variable of the problem instead of y_0. A Regularization method is not necessary anymore. This strategy, advocated in [30], avoids any iterative process and allows a stable numerical framework. It has been successfully applied in the closely related context of the exact controllability of (1.1) in [15, 27] and also to inverse problems for hyperbolic equations in [10, 11]. Keeping y as the main variable, the idea is to take into account the state constraint $Ly - f = 0$ with a Lagrange multiplier. This allows to derive explicitly the optimality systems associated to (LS) in terms of an elliptic mixed formulation and therefore reformulate the original problem. Well-posedness of such new formulation is related to classical energy estimates and unique continuation properties while the stability is guaranteed by some global observability inequality for the homogeneous parabolic equation.

The outline of this paper is as follows: In Section 2, we consider the least-squares problem (P) and reconstruct the solution of the parabolic equation from a partial observation localized on a subset q_T of Q_T. For that, in Section 2.1, we associate to (P) the equivalent mixed formulation (2.2) which relies on the optimality conditions of the problem. Using the unique continuation for equation (1.1), we show the well-posedness of this mixed formulation, in particular, we check the Babuska–Brezzi inf-sup condition (see Theorem 2.1). Interestingly, in Section 2.2, we also derive an equivalent dual extremal problem, which reduces the determination of the state y to the minimization of an elliptic functional with respect to the Lagrange multiplier. Then, in Section 3, we adapt these arguments to the first order mixed system (3.1), equivalent to the parabolic equation. There, the flux variable $\mathbf{p} := c(x)\nabla y$ appears explicitly and allows to reduce the order of regularity of the involved functional spaces. The underlying inf-sup condition is obtained by adapting a Carleman inequality due to Imanuvilov, Puel and Yamamoto (see [19]). The existence and uniqueness of a weak solution to this first-order system is studied in the Appendix. Section 4 concludes with some remarks and perspectives. In particular, we highlight why the mixed formulations developed and analyzed here are suitable at the numerical level to get a robust approximation of the variable y on the whole domain Q_T.

2 Recovering the solution from a partial observation: A second order mixed formulation

Assuming that the data y_0 is unknown, we address the inverse problem (IP). We introduce and analyze firstly a direct approach then an equivalent extremal problem. In view of the linearity of (1.1), we take for simplicity a zero source term f.

2.1 Direct approach: Minimal local weighted L^2-norm; a first mixed formulation

Let $\rho_\star \in \mathbb{R}_+^\star$ and let $\rho_0 \in \mathcal{R}$ with

$$\mathcal{R} := \{w : w \in C^0(Q_T),\ w \geq \rho_\star > 0 \text{ in } Q_T,\ w \in L^\infty(\Omega \times (\delta, T)) \text{ for all } \delta > 0\} \tag{2.1}$$

so that in particular, the weight ρ_0 may blow up as $t \to 0^+$. We define the space $\mathcal{Y}_0 := \{y \in C^2(\overline{Q_T}) : y = 0 \text{ on } \Sigma_T\}$ and for any $\eta > 0$ and any $\rho \in \mathcal{R}$, the bilinear form by

$$(y, \overline{y})_{\mathcal{Y}_0} := \langle \rho_0^{-1} y, \rho_0^{-1} \overline{y} \rangle_{L^2(q_T)} + \eta \langle \rho^{-1} Ly, \rho^{-1} L\overline{y} \rangle_{L^2(Q_T)} \quad \text{for all } y, \overline{y} \in \mathcal{Y}_0.$$

Here $\langle \cdot, \cdot \rangle_X$ denotes the usual scalar product over $X = L^2(q_T)$ or $X = L^2(Q_T)$. The introduction of the weight ρ which does not appear in the original problem will be motivated at the end of this section. From the unique continuation property for (1.1), this bilinear form defines a scalar product for any $\eta > 0$.

Let then $\mathcal{Y}$ be the completion of the space $\mathcal{Y}_0$ for this scalar product. We denote the norm over $\mathcal{Y}$ by $\|\cdot\|_{\mathcal{Y}}$ such that

$$\|y\|_{\mathcal{Y}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta \|\rho^{-1} Ly\|_{L^2(Q_T)}^2 \quad \text{for all } y \in \mathcal{Y}.$$

Finally, we define the closed subset $\mathcal{W}$ of $\mathcal{Y}$ by $\mathcal{W} := \{y \in \mathcal{Y} : \rho^{-1} Ly = 0 \text{ in } L^2(Q_T)\}$ and we endow $\mathcal{W}$ with the same norm as $\mathcal{Y}$.

For any $r \geq 0$ we then define the following extremal problem:

$$\text{Minimize } J_r(y) := \frac{1}{2} \|\rho_0^{-1}(y(x,t) - y_{\text{obs}}(x,t))\|_{L^2(q_T)}^2 + \frac{r}{2} \|\rho^{-1} Ly\|_{L^2(Q_T)}^2 \quad \text{subject to } y \in \mathcal{W}. \tag{P}$$

This problem is well-posed: the functional J_r is continuous, strictly convex and such that $J_r(y) \to +\infty$ as $\|y\|_{\mathcal{Y}} \to +\infty$. Note also that the solution of (P) does neither depend on η nor on ρ. Moreover, for any $y \in \mathcal{W}$, we have $Ly = 0$ a.e. in Q_T and $\|y\|_{\mathcal{Y}} = \|\rho_0^{-1} y\|_{L^2(q_T)}$ so that the restriction $y(\cdot, 0)$ belongs, by definition, to the abstract space $\mathcal{H}$. Consequently, extremal problems (LS) and (P) are equivalent.

In order to solve problem (P), we have to deal with the constraint equality $\rho^{-1} Ly = 0$ which appears in $\mathcal{W}$. Proceeding as in [10, 27], we introduce a Lagrange multiplier and the following mixed formulation: find a solution $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}, \lambda) = l(\overline{y}) & \text{for all } \overline{y} \in \mathcal{Y}, \\ \qquad\qquad\quad b(y, \overline{\lambda}) = 0 & \text{for all } \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{2.2}$$

where

$$\begin{aligned} a_r : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}, &\quad a_r(y, \overline{y}) := \langle \rho_0^{-1} y, \rho_0^{-1} \overline{y} \rangle_{L^2(q_T)} + r \langle \rho^{-1} Ly, \rho^{-1} L\overline{y} \rangle_{L^2(Q_T)}, \\ b : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}, &\quad b(y, \lambda) := \langle \rho^{-1} Ly, \lambda \rangle_{L^2(Q_T)}, \\ l : \mathcal{Y} \to \mathbb{R}, &\quad l(y) := \langle \rho_0^{-1} y, \rho_0^{-1} y_{\mathrm{obs}} \rangle_{L^2(q_T)}. \end{aligned}$$

System (2.2) corresponds to the optimality conditions of J_r while r stands as an augmentation parameter (see [16]). We have the following result.

Theorem 2.1. *Let $\rho_0 \in \mathcal{R}$, $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ and $r \geq 0$. Then the following hold:*
(i) *The mixed formulation (2.2) is well-posed.*
(ii) *The unique solution $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_r : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}$ defined by $\mathcal{L}_r(y, \lambda) := \frac{1}{2} a_r(y, y) + b(y, \lambda) - l(y)$.*
(iii) *The solution (y, λ) satisfies the estimates*

$$\|y\|_{\mathcal{Y}} \leq \|\rho_0^{-1} y_{\mathrm{obs}}\|_{L^2(q_T)}, \quad \|\lambda\|_{L^2(Q_T)} \leq 2 \sqrt{C_{\Omega,T}^2 \rho_\star^{-2} \|\rho\|_{L^\infty(Q_T)}^2 + \eta}\, \|\rho_0^{-1} y_{\mathrm{obs}}\|_{L^2(q_T)}$$

for some constant $C_{\Omega,T} > 0$.

Proof. We use classical results for saddle point problems (see [3, Chapter 4]).

We easily check the continuity of the symmetric and positive bilinear form a_r over $\mathcal{Y} \times \mathcal{Y}$, the continuity of the bilinear form b over $\mathcal{Y} \times L^2(Q_T)$ and the continuity of the linear form l over $\mathcal{Y}$. In particular, we get

$$\|l\|_{\mathcal{Y}'} \leq \|\rho_0^{-1} y_{\mathrm{obs}}\|_{L^2(q_T)}, \quad \|a_r\|_{\mathscr{L}^2(\mathcal{Y})} \leq \max\{1, \eta^{-1} r\}, \quad \|b\|_{\mathscr{L}^2(\mathcal{Y}, L^2(Q_T))} \leq \eta^{-1/2}, \tag{2.3}$$

where $\mathscr{L}^2(E, F)$ denotes the space of the continuous bilinear functions defined on the product Banach space $E \times F$; when $E = F$ we simply write $\mathscr{L}^2(E)$.

Moreover, the kernel

$$\mathcal{N}(b) := \{y \in \mathcal{Y} : b(y, \lambda) = 0 \text{ for all } \lambda \in L^2(Q_T)\}$$

coincides with $\mathcal{W}$: we have $a_r(y, y) = \|y\|_{\mathcal{Y}}^2$ for all $y \in \mathcal{N}(b) = \mathcal{W}$ leading to the coercivity of a_r over the kernel of b.

Therefore, in view of [3, Theorem 4.2.2], it remains to check the following so-called *inf-sup property*: there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(Q_T)} \sup_{y \in \mathcal{Y}} \frac{b(y, \lambda)}{\|y\|_{\mathcal{Y}} \|\lambda\|_{L^2(Q_T)}} \geq \delta. \tag{2.4}$$

We proceed as follows: For any fixed $\lambda^0 \in L^2(Q_T)$, using the fact that ρ is bounded in Q_T, we define the unique element y^0 as the solution of

$$\rho^{-1} L\, y^0 = \lambda^0 \text{ in } Q_T, \quad y^0 = 0 \text{ on } \Sigma_T, \quad y^0(\cdot, 0) = 0 \text{ in } \Omega.$$

Using energy estimates, we have

$$\|\rho_0^{-1} y^0\|_{L^2(q_T)} \leq \rho_\star^{-1} \|y^0\|_{L^2(Q_T)} \leq C_{\Omega,T} \rho_\star^{-1} \|\rho \lambda^0\|_{L^2(Q_T)} \leq C_{\Omega,T} \rho_\star^{-1} \|\rho\|_{L^\infty(Q_T)} \|\lambda^0\|_{L^2(Q_T)},$$

which proves that $y^0 \in \mathcal{Y}$ and that

$$\sup_{y\in\mathcal{Y}} \frac{b(y,\lambda^0)}{\|y\|_{\mathcal{Y}}\|\lambda^0\|_{L^2(Q_T)}} \geq \frac{b(y^0,\lambda^0)}{\|y^0\|_{\mathcal{Y}}\|\lambda^0\|_{L^2(Q_T)}} = \frac{\|\lambda^0\|_{L^2(Q_T)}}{(\|\rho_0^{-1}y^0\|^2_{L^2(q_T)} + \eta\|\lambda_0\|^2_{L^2(Q_T)})^{\frac{1}{2}}}.$$

Combining the above two inequalities, we obtain

$$\sup_{y\in\mathcal{Y}} \frac{b(y,\lambda_0)}{\|y\|_{\mathcal{Y}}\|\lambda_0\|_{L^2(Q_T)}} \geq \frac{1}{\sqrt{C^2_{\Omega,T}\rho_\star^{-2}\|\rho\|^2_{L^\infty(Q_T)} + \eta}},$$

and hence (2.4) holds with

$$\delta = (C^2_{\Omega,T}\rho_\star^{-2}\|\rho\|^2_{L^\infty(Q_T)} + \eta)^{-1/2}.$$

Point (ii) is due to the positivity and symmetry of the form a_r. Point (iii) is a consequence of classical estimates (see [3], Theorem 4.2.3), namely

$$\|y\|_{\mathcal{Y}} \leq \frac{1}{\alpha_0}\|l\|_{\mathcal{Y}'}, \quad \|\lambda\|_{L^2(Q_T)} \leq \frac{1}{\delta}\Big(1 + \frac{\|a_r\|_{\mathscr{L}^2(\mathcal{Y})}}{\alpha_0}\Big)\|l\|_{\mathcal{Y}'},$$

where $\alpha_0 := \inf_{y\in\mathcal{N}(b)} a_r(y,y)/\|y\|^2_{\mathcal{Y}}$. Estimates (2.3) and the equality $\alpha_0 = 1$ lead to the results. □

In order to get a global estimate of the reconstructed solution, we now recall the following important result.

Proposition 2.2 ([14, Lemma 3.1]). *Assume that Ω is at least of class C^2. We define the Carleman weights $\rho_c, \rho_{c,0}, \rho_{c,1} \in \mathcal{R}$ (see (2.1)) be defined as follows:*

$$\begin{aligned} &\rho_c(x,t) := \exp\Big(\frac{\beta(x)}{t}\Big), \qquad \beta(x) := K_1(e^{K_2} - e^{\beta_0(x)}),\\ &\rho_{c,0}(x,t) := t^{3/2}\rho_c(x,t), \quad \rho_{c,1}(x,t) := t^{1/2}\rho_c(x,t), \end{aligned} \tag{2.5}$$

with $\beta_0 \in C^\infty(\overline{\Omega})$ and where the positive constants K_i are sufficiently large (depending on T, c_0, $\|c\|_{C^1(\overline{\Omega})}$ and $\|d\|_{L^\infty(Q_T)}$) such that

$$\beta_0 > 0 \text{ in } \Omega, \quad \beta_0 = 0 \text{ on } \partial\Omega, \quad \nabla\beta_0(x) \neq 0 \text{ for all } x \in \overline{\Omega}\setminus\omega_0,$$

where ω_0 is an open subset of Ω such that $\overline{\omega}_0 \subset \omega$. Then there exists a constant $C > 0$, depending only on T, ω and Ω, such that

$$\|\rho_{c,0}^{-1}y\|_{L^2(Q_T)} + \|\rho_{c,1}^{-1}\nabla y\|_{L^2(Q_T)} \leq C\|y\|_{\mathcal{Y}_c} \quad \text{for all } y \in \mathcal{Y}_c, \tag{2.6}$$

where $\mathcal{Y}_c$ is the completion of $\mathcal{Y}_{0,c} := \mathcal{Y}_0$ with respect to the scalar product

$$(y,\overline{y})_{\mathcal{Y}_{0,c}} = \langle\rho_{c,0}^{-1}y, \rho_{c,0}^{-1}\overline{y}\rangle_{L^2(q_T)} + \eta\langle\rho_c^{-1}Ly, \rho_c^{-1}L\overline{y}\rangle_{L^2(Q_T)}.$$

Estimate (2.6) is a consequence of the celebrated global Carleman inequality satisfied by the solution of (1.1), introduced and popularized in [17]. We refer to the review [32] for applications to inverse problems. This result implies the following stability estimate which allows us to estimate a global norm of the solution y in term of the norm $\mathcal{Y}$.

Corollary 2.3. *Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume that there exists a positive constant K such that*

$$\rho_0 \leq K\rho_{c,0}, \quad \rho \leq K\rho_c \qquad \text{in } Q_T. \tag{2.7}$$

If (y,λ) is the solution of the mixed formulation (2.2), then there exists $C > 0$ such that

$$\|\rho_{c,0}^{-1}y\|_{L^2(Q_T)} \leq C\|y\|_{\mathcal{Y}}. \tag{2.8}$$

Proof. Hypothesis (2.7) implies that $\mathcal{Y} \subset \mathcal{Y}_c$. Therefore, estimate (2.6) implies that

$$\|\rho_{c,0}^{-1}y\|_{L^2(Q_T)} \leq C\|y\|_{\mathcal{Y}_c} \leq C\|y\|_{\mathcal{Y}} \leq C\|\rho_0^{-1}y_{\text{obs}}\|_{L^2(q_T)}.$$ □

Remark 2.4. The well-posedness of the mixed formulation (2.2), precisely the inf-sup property (2.4), is open in the case where the weight ρ is simply in $\mathcal{R}$, i.e. ρ may blow up as time $t \to 0^+$. In order to get (2.4), it suffices to prove that the function $z := \rho_0^{-1} y$ as the solution of the boundary value problem

$$\rho^{-1} L(\rho_0 z) = \lambda^0 \text{ in } Q_T, \quad z = 0 \text{ on } \Sigma_T, \quad z(\cdot, 0) = 0 \text{ in } \Omega$$

for any $\lambda^0 \in L^2(Q_T)$ satisfies the following estimate for some $C > 0$:

$$\|z\|_{L^2(q_T)} \le C \|\rho^{-1} L(\rho_0 z)\|_{L^2(Q_T)}.$$

In the cases of interest for which both ρ_0 and ρ blow up as $t \to 0^+$ (for instance given by $\rho_{c,0}$ and ρ_c), this estimate is open and does not seem to be a consequence of (2.6).

Let us now comment the introduction of the weight ρ_0 in problem (P). The space $\mathcal{Y}_c$, which contains the element y such that $\rho_c^{-1} L y \in L^2(Q_T)$ and $\rho_{c,0}^{-1} y \in L^2(q_T)$ satisfy the embedding $\mathcal{Y}_c \subset C^0([\delta, T], H_0^1(\Omega))$ for any $\delta > 0$ (see [14]). Under condition (2.7), the same embedding holds for $\mathcal{Y}$. In particular, there is no control of the restriction of the solution at time $t = 0$, which is due to the regularization property of the heat kernel. Consequently, from the observation $y_{\text{obs}} \in L^2(q_T)$ and the knowledge of $Ly \in L^2(Q_T)$, there is no hope to recover, for a Sobolev norm, the solution of y at the initial time $t = 0$. It is then suitable to add to the cost J a vanishing weight ρ_0^{-1} at time $t = 0$. The weight ρ is introduced here for similar reasons. Note that the solution y of (2.2) belongs to $\mathcal{W}$ and therefore does not depend on ρ (recall that ρ is strictly positive); this is in agreement with the fact that ρ does not appear in the equivalent problem (LS). However, very likely, a singular behavior for the $L^2(Q_T)$ function Ly occurs as well near $\Omega \times \{0\}$ so that the constraint $Ly = 0$ in $L^2(Q_T)$ is too "strong" and must be replaced, for numerical purposes, by the relaxed one $\rho^{-1} L y = 0$ in $L^2(Q_T)$ with ρ^{-1} small near $\Omega \times \{0\}$. This is actually the effect and the role of the Carleman weights ρ_c defined in (2.5). As a partial conclusion, the introduction of appropriate weights in the cost J allows us to use estimate (2.8) and to guarantee a Lipschitz stable reconstruction of the solution y on the whole domain except at the initial time.

We also emphasize that the mixed formulation (2.2) is still well defined with constant weights ρ and ρ_0 equal to one, but leading to weaker stability estimates and reconstruction results. We refer to [6, 7].

Remark 2.5. The first equation of the mixed formulation (2.2) reads as (recall that $y \in \mathcal{W}$)

$$\iint_{q_T} \rho_0^{-2}\, y \overline{y}\, dx\, dt + \iint_{Q_T} \rho^{-1} L\overline{y}\, \lambda\, dx\, dt = \iint_{q_T} \rho_0^{-2}\, y_{\text{obs}}\, \overline{y}\, dx\, dt \quad \text{for all } \overline{y} \in \mathcal{Y}$$

and means that $\rho^{-1}\lambda \in L^2(Q_T)$ is the solution of the parabolic equation in the transposition sense, i.e. $\rho^{-1}\lambda$ solves the problem

$$L^\star(\rho^{-1}\lambda) = -\rho_0^{-2}(y - y_{\text{obs}}) 1_\omega \text{ in } Q_T, \quad \rho^{-1}\lambda = 0 \text{ on } \Sigma_T, \quad (\rho^{-1}\lambda)(\cdot, T) = 0 \text{ in } \Omega \tag{2.9}$$

where 1_ω denotes the characteristic function associated to the open subset ω. However, since

$$-\rho_0^{-2}(y - y_{\text{obs}}) 1_\omega \in L^2(q_T),$$

we have that $\rho^{-1}\lambda$ is indeed a weak solution of (2.9) and

$$\rho^{-1}\lambda \in C^0([0, T]; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)).$$

- Moreover, if y_{obs} is the restriction of a solution of (1.1) to q_T, then the unique multiplier λ, a solution of (2.9), vanishes almost everywhere. In that case, we have

  $$\sup_{\lambda \in L^2(Q_T)} \inf_{y \in \mathcal{Y}} \mathcal{L}_r(y, \lambda) = \inf_{y \in \mathcal{Y}} \mathcal{L}_r(y, 0) = \inf_{y \in \mathcal{Y}} J_r(y).$$

  The corresponding variational formulation, well-posed for $r > 0$ is then the following: find $y \in \mathcal{Y}$ such that

  $$a_r(y, \overline{y}) = l(\overline{y}) \quad \text{for all } \overline{y} \in \mathcal{Y}.$$

- In the general case, (2.2) can be rewritten as follows: find a solution $(y,\lambda) \in \mathcal{Y} \times L^2(Q_T)$ of

$$\begin{cases} \langle P_r y, P_r \overline{y}\rangle_{L^2(Q_T)\times L^2(q_T)} + \langle \rho^{-1} L\overline{y}, \lambda\rangle_{L^2(Q_T)} = \langle (0, \rho_0^{-1} y_{\mathrm{obs}}), P_r\overline{y}\rangle_{L^2(Q_T)\times L^2(q_T)} & \text{for all } \overline{y} \in \mathcal{Y}, \\ \langle \rho^{-1} Ly, \overline{\lambda}\rangle_{L^2(Q_T)} = 0 & \text{for all } \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{2.10}$$

with $P_r y := (\sqrt{r}\rho^{-1} L\, y, \rho_0^{-1} y_{|q_T})$.
Formulation (2.10) may be seen as a generalization of the following quasi-reversibility formulation (initially introduced in [22]): for any Tikhonov like parameter $\varepsilon > 0$, find the solution $y_\varepsilon \in \mathcal{Y}$ of

$$\langle P y_\varepsilon, P\overline{y}\rangle_{L^2(Q_T)\times L^2(q_T)} + \varepsilon\langle y_\varepsilon, \overline{y}\rangle_{\mathcal{Y}} = \langle (f, y_{\mathrm{obs}}), P\overline{y}\rangle_{L^2(Q_T)\times L^2(q_T)} \quad \text{for all } \overline{y} \in \mathcal{Y}.$$

We refer to the book of Klibanov [21] and to the more recent work [1] where a Cauchy problem for the heat equation is addressed. In (2.10), the parameter ε is replaced by the Lagrange multiplier function λ, which is adjusted automatically to the situation, while the choice of ε is in general a delicate issue.

The optimality system (2.9) can be used to define an equivalent saddle-point formulation, very suitable at the numerical level. Precisely, we introduce, in view of (2.9), the space Λ given by

$$\Lambda := \{\lambda : \rho^{-1}\lambda \in C^0([0,T]; L^2(\Omega)),\ \rho_0\, L^\star(\rho^{-1}\lambda) \in L^2(Q_T),\ \rho^{-1}\lambda = 0 \text{ on } \Sigma_T,\ (\rho^{-1}\lambda)(\cdot, T) = 0\}.$$

Endowed with the scalar product

$$\langle \lambda, \overline{\lambda}\rangle_\Lambda := \langle \rho^{-1}\lambda, \rho^{-1}\overline{\lambda}\rangle_{L^2(Q_T)} + \langle \rho_0 L^\star(\rho^{-1}\lambda), \rho_0 L^\star(\rho^{-1}\overline{\lambda})\rangle_{L^2(Q_T)},$$

we check that Λ is a Hilbert space. Then, for any parameter $\alpha \in (0,1)$, we consider the following mixed formulation: find $(y,\lambda) \in \mathcal{Y} \times \Lambda$ such that

$$\begin{cases} a_{r,\alpha}(y,\overline{y}) + b_\alpha(\overline{y},\lambda) = l_{1,\alpha}(\overline{y}) & \text{for all } \overline{y} \in \mathcal{Y}, \\ b_\alpha(y,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) = l_{2,\alpha}(\overline{\lambda}) & \text{for all } \overline{\lambda} \in \Lambda, \end{cases} \tag{2.11}$$

where

$$\begin{aligned} a_{r,\alpha} &: \mathcal{Y}\times\mathcal{Y} \to \mathbb{R}, & a_{r,\alpha}(y,\overline{y}) &:= (1-\alpha)\langle \rho_0^{-1} y, \rho_0^{-1}\overline{y}\rangle_{L^2(q_T)} + r\langle \rho^{-1} Ly, \rho^{-1} L\overline{y}\rangle_{L^2(Q_T)}, \\ b_\alpha &: \mathcal{Y}\times\Lambda \to \mathbb{R}, & b_\alpha(y,\lambda) &:= \langle \rho^{-1} Ly, \lambda\rangle_{L^2(Q_T)} - \alpha\langle \rho_0 L^\star(\rho^{-1}\lambda), \rho_0^{-1} y\rangle_{L^2(q_T)}, \\ c_\alpha &: \Lambda\times\Lambda \to \mathbb{R}, & c_\alpha(\lambda,\overline{\lambda}) &:= \alpha\langle \rho_0 L^\star(\rho^{-1}\lambda), \rho_0 L^\star(\rho^{-1}\overline{\lambda})\rangle_{L^2(Q_T)}, \\ l_{1,\alpha} &: \mathcal{Y} \to \mathbb{R}, & l_{1,\alpha}(y) &:= (1-\alpha)\langle \rho_0^{-1} y_{\mathrm{obs}}, \rho_0^{-1} y\rangle_{L^2(q_T)}, \\ l_{2,\alpha} &: \Lambda \to \mathbb{R}, & l_{2,\alpha}(\lambda) &:= -\alpha\langle \rho_0^{-1} y_{\mathrm{obs}}, \rho_0\, L^\star(\rho^{-1}\lambda)\rangle_{L^2(q_T)}. \end{aligned}$$

From the symmetry of $a_{r,\alpha}$ and c_α, we easily check that this formulation corresponds to the saddle-point problem

$$\begin{cases} \sup\limits_{\lambda\in\Lambda} \inf\limits_{y\in\mathcal{Y}} \mathcal{L}_{r,\alpha}(y,\lambda), \\ \mathcal{L}_{r,\alpha}(y,\lambda) := \mathcal{L}_r(y,\lambda) - \dfrac{\alpha}{2}\|\rho_0 L^\star(\rho^{-1}\lambda) + \rho_0^{-1}(y - y_{\mathrm{obs}})1_\omega\|^2_{L^2(Q_T)}. \end{cases} \tag{2.12}$$

Proposition 2.6. *Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$. Then, for any $\alpha \in (0,1)$ and $r > 0$, formulation (2.11) is well-posed. Moreover, the unique pair (y,λ) in $\mathcal{Y}\times\Lambda$ satisfies*

$$\theta_1\|y\|^2_{\mathcal{Y}} + \theta_2\|\lambda\|^2_\Lambda \le \Big(\frac{(1-\alpha)^2}{\theta_1} + \frac{\alpha^2}{\theta_2}\Big)\|\rho_0^{-1} y_{\mathrm{obs}}\|^2_{L^2(q_T)} \tag{2.13}$$

with $\theta_1 := \min(1-\alpha, \eta^{-1} r)$ and $\theta_2 := \alpha\rho_\star/(\rho_\star + C_{\Omega,T})$ where $C_{\Omega,T}$ is the continuity constant so that

$$\|\rho^{-1}\lambda\|_{L^2(Q_T)} \le C_{\Omega,T}\|L^\star(\rho^{-1}\lambda)\|_{L^2(Q_T)}$$

for any $\lambda \in \Lambda$.

Proof. We easily get the continuity of the bilinear forms $a_{r,\alpha}$, b_α and c_α as

$$\begin{aligned}
&|a_{r,\alpha}(y,\overline{y})| \le \max(1-\alpha,\eta^{-1}r)\|y\|_{\mathcal{Y}}\|\overline{y}\|_{\mathcal{Y}} && \text{for all } y,\overline{y}\in\mathcal{Y},\\
&|b_\alpha(y,\lambda)| \le \max(\alpha,\eta^{-1/2}\|\rho\|_{L^\infty(Q_T)})\|y\|_{\mathcal{Y}}\|\lambda\|_\Lambda && \text{for all } y\in\mathcal{Y} \text{ and } \lambda\in\Lambda,\\
&|c_\alpha(\lambda,\overline{\lambda})| \le \alpha\|\lambda\|_\Lambda\|\overline{\lambda}\|_\Lambda && \text{for all } \lambda,\overline{\lambda}\in\Lambda,
\end{aligned}$$

and the continuity of the linear form $l_{1,\alpha}$ and $l_{2,\alpha}$ as

$$\|l_{1,\alpha}\|_{\mathcal{Y}'} \le (1-\alpha)\|\rho_0^{-1}y_{\mathrm{obs}}\|_{L^2(q_T)} \quad\text{and}\quad \|l_{2,\alpha}\|_{\Lambda'} \le \alpha\|\rho_0^{-1}y_{\mathrm{obs}}\|_{L^2(q_T)}.$$

Moreover, since $\alpha\in(0,1)$, we also obtain the coercivity of $a_{r,\alpha}$ and of c_α. Precisely, we check that $a_{r,\alpha}(y,y)\ge\theta_1\|y\|_{\mathcal{Y}}^2$ for all $y\in\mathcal{Y}$ while, for any $m\in(0,1)$, by writing

$$\begin{aligned}
c_\alpha(\lambda,\lambda) &= \alpha\|\rho_0L^\star(\rho^{-1}\lambda)\|_{L^2(Q_T)}^2 = \alpha m\|\rho_0L^\star(\rho^{-1}\lambda)\|_{L^2(Q_T)}^2 + \alpha(1-m)\|\rho_0L^\star(\rho^{-1}\lambda)\|_{L^2(Q_T)}^2\\
&\ge \alpha m\|\rho_0L^\star(\rho^{-1}\lambda)\|_{L^2(Q_T)}^2 + \frac{\alpha(1-m)\rho_\star}{C_{\Omega,T}}\|\rho^{-1}\lambda\|_{L^2(Q_T)}^2 \ge \alpha\min\Big(m,\frac{(1-m)\rho_\star}{C_{\Omega,T}}\Big)\|\lambda\|_\Lambda^2,
\end{aligned}$$

we get $c_\alpha(\lambda,\lambda)\ge\theta_2\|\lambda\|_\Lambda^2$ for all $\lambda\in\Lambda$ with $m=\rho_\star(\rho_\star+C_{\Omega,T})^{-1}$.

The result [3, Proposition 4.3.1] implies the well-posedness of the mixed formulation (2.11) and of estimate (2.13). □

The α-term in $\mathcal{L}_{r,\alpha}$ is a stabilization term. It ensures a coercivity property of $\mathcal{L}_{r,\alpha}$ with respect to the variable λ and automatically the well-posedness, assuming here $r>0$. In particular, there is no need to prove any inf-sup property for the application b_α.

Proposition 2.7. *For any $r>0$, The solutions of* (2.2) *and* (2.11) *coincide.*

Proof. For any $r>0$, let us check that the saddle-point $(y_r,\lambda_r)\in\mathcal{Y}\times L^2(Q_T)$ of $\mathcal{L}_r$ is also a saddle-point of $\mathcal{L}_{r,\alpha}$. From Remark 2.5, we first have $(y_r,\lambda_r)\in\mathcal{Y}\times\Lambda$. Moreover, for any $\mu\in\Lambda$, we have

$$\mathcal{L}_{r,\alpha}(y_r,\mu)\le\mathcal{L}_r(y_r,\mu)\le\mathcal{L}_r(y_r,\lambda_r)=\mathcal{L}_{r,\alpha}(y_r,\lambda_r)+\frac{\alpha}{2}\big\|\rho_0L^\star(\rho^{-1}\lambda_r)+\rho_0^{-1}(y_r-y_{\mathrm{obs}})\,1_\omega\big\|_{L^2(Q_T)}^2=\mathcal{L}_{r,\alpha}(y_r,\lambda_r)$$

since (y_r,λ_r) solves (2.9) for any $r\ge0$. Therefore, λ_r maximizes $\mu\mapsto\mathcal{L}_{r,\alpha}(y_r,\mu)$. Conversely, the functional $F:\mathcal{Y}\to\mathbb{R}$, given by $F(z)=\mathcal{L}_{r,\alpha}(z,\lambda_r)$, admits a unique extremal point for any $r>0$ and any $\alpha\in(0,1)$ (in view of the ellipticity of $a_{r,\alpha}$). Moreover, for all $\overline{z}\in\mathcal{Y}$, we compute that

$$\langle F'(z),\overline{z}\rangle_{\mathcal{Y}',\mathcal{Y}}=a_r(z,\overline{z})+b(\overline{z},\lambda_r)-l(\overline{z})-\alpha\big\langle\rho_0^{-1}\overline{z},\rho_0L^\star(\rho^{-1}\lambda_r)+\rho_0^{-1}(z-y_{\mathrm{obs}})1_\omega\big\rangle_{L^2(Q_T)}$$

and conclude in view of (2.2) and (2.9) that $\langle F'(y_r),\overline{z}\rangle_{\mathcal{Y}',\mathcal{Y}}=0$ for all $\overline{z}\in\mathcal{Y}$. Therefore, y_r minimizes the map $z\mapsto\mathcal{L}_{r,\alpha}(z,\lambda_r)$. Consequently, the pair (y_r,λ_r) is also a saddle-point for $\mathcal{L}_{r,\alpha}$. The conclusion follows from the uniqueness of the saddle-point. □

Remark 2.8. The least-squares functional $\mathcal{J}:\mathcal{Y}\times\Lambda\to\mathbb{R}$ defined by

$$\mathcal{J}(y,\lambda)=\|\rho^{-1}Ly\|_{L^2(Q_T)}^2+\big\|\rho_0L^\star(\rho^{-1}\lambda)+\rho_0^{-1}(y-y_{\mathrm{obs}})\,1_\omega\big\|_{L^2(q_T)}^2$$

is continuous, strictly convex and enjoys the property $\mathcal{J}(y,\lambda)\to\infty$ as $\|y\|_{\mathcal{Y}}+\|\lambda\|_\Lambda\to\infty$. Therefore, $\mathcal{J}$ admits a unique minimum (y_0,λ_0) for which $\mathcal{J}$ vanishes, i.e. $(y_0,\lambda_0)\in\mathcal{Y}\times\Lambda$ solves the optimality conditions (1.1) and (2.9). Therefore, the saddle point problem (2.12) is equivalent to the extremal problem $\inf_{(y,\lambda)\in\mathcal{Y}\times\Lambda}\mathcal{J}(y,\lambda)$.

2.2 Dual formulation of the extremal problem (2.2)

As discussed at length in [9], we may also associate to problem (P) an equivalent problem involving only the variable λ. This is very relevant at the numerical level. For any $r>0$ let $\mathcal{T}_r:L^2(Q_T)\to L^2(Q_T)$ be the linear operator defined by $\mathcal{T}_r\lambda:=\rho^{-1}Ly$ where $y\in\mathcal{Y}$ is the unique solution to

$$a_r(y,\overline{y})=b(\overline{y},\lambda)\quad\text{for all }\overline{y}\in\mathcal{Y}. \tag{2.14}$$

Remark that (2.14) is well-posed if and only if $r>0$. The following important lemma holds.

Lemma 2.9. *Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$. For any $r > 0$ the operator $\mathcal{T}_r$ is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.*

Proof. From the definition of a_r, we easily get that $\|\mathcal{T}_r\lambda\|_{L^2(Q_T)} \le r^{-1}\|\lambda\|_{L^2(Q_T)}$ and the continuity of $\mathcal{T}_r$. Next, consider any $\lambda' \in L^2(Q_T)$ and denote by $y' \in \mathcal{Y}$ the corresponding unique solution of (2.14) so that $\mathcal{T}_r\lambda' := \rho^{-1}Ly'$. Relation (2.14) with $\overline{y} = y'$ then implies that

$$\langle \mathcal{T}_r\lambda', \lambda\rangle_{L^2(Q_T)} = a_r(y, y'), \tag{2.15}$$

and therefore the symmetry and positivity of $\mathcal{T}_r$. The last relation with $\lambda' = \lambda$, and the unique continuation property for (1.1), and (2.14) imply that $\mathcal{T}_r$ is also positive definite. Finally, let us check the strong ellipticity of $\mathcal{T}_r$, equivalently that the bilinear functional $(\lambda, \lambda') \mapsto \langle \mathcal{T}_r\lambda, \lambda'\rangle_{L^2(Q_T)}$ is $L^2(Q_T)$-elliptic. Thus, we want to show that

$$\langle \mathcal{T}_r\lambda, \lambda\rangle_{L^2(Q_T)} \ge C\|\lambda\|^2_{L^2(Q_T)} \quad \text{for all } \lambda \in L^2(Q_T) \tag{2.16}$$

for some positive constant C. Suppose that (2.16) does not hold. Then there exists a sequence $\{\lambda_n\}_{n\ge 0}$ of $L^2(Q_T)$ such that $\|\lambda_n\|_{L^2(Q_T)} = 1$ for all $n \ge 0$ and $\lim_{n\to\infty}\langle \mathcal{T}_r\lambda_n, \lambda_n\rangle_{L^2(Q_T)} = 0$.

Let us denote by y_n the solution of (2.14) corresponding to λ_n. From (2.15), we then obtain that

$$\lim_{n\to\infty}\big(r\|\rho^{-1}Ly_n\|^2_{L^2(Q_T)} + \|\rho_0^{-1}y_n\|^2_{L^2(q_T)}\big) = 0. \tag{2.17}$$

From (2.14) with $y = y_n$ and $\lambda = \lambda_n$, we have

$$\langle r\rho^{-1}Ly_n - \lambda_n, \rho^{-1}L\overline{y}\rangle_{L^2(Q_T)} + \langle \rho_0^{-1}y_n, \rho_0^{-1}\overline{y}\rangle_{L^2(q_T)} = 0 \tag{2.18}$$

for every $\overline{y} \in \mathcal{Y}$. We define the sequence $\{\overline{y}_n\}_{n\ge 0}$ as follows:

$$\rho^{-1}L\overline{y}_n = r\rho^{-1}Ly_n - \lambda_n \text{ in } Q_T, \quad \overline{y}_n = 0 \text{ in } \Sigma_T, \quad \overline{y}_n(\cdot, 0) = 0 \text{ in } \Omega,$$

so that, for all $n \ge 0$, the term $\overline{y}_n$ is the solution of the heat equation with zero initial data and source term $r\rho^{-1}Ly_n - \lambda_n$ in $L^2(Q_T)$. Energy estimates imply that

$$\|\rho_0^{-1}\overline{y}_n\|_{L^2(q_T)} \le C_{\Omega,T}\rho_\star^{-1}\|\rho\|_{L^\infty(Q_T)}\|r\rho^{-1}Ly_n - \lambda_n\|_{L^2(Q_T)}$$

and that $\overline{y}_n \in \mathcal{Y}$. Then, using (2.18) with $\overline{y} = \overline{y}_n$, we get

$$\|r\rho^{-1}Ly_n - \lambda_n\|_{L^2(Q_T)} \le C_{\Omega,T}\rho_\star^{-1}\|\rho\|_{L^\infty(Q_T)}\|\rho_0^{-1}y_n\|_{L^2(q_T)}.$$

Then, from (2.17), we conclude that $\lim_{n\to+\infty}\|\lambda_n\|_{L^2(Q_T)} = 0$, leading to a contradiction (since $\|\lambda_n\|_{L^2(Q_T)} = 1$ for all $n \ge 0$) and to the strong ellipticity of the operator $\mathcal{T}_r$. □

The introduction of the operator $\mathcal{T}_r$ is motivated by the following proposition.

Proposition 2.10. *For any $r > 0$ let $y^0 \in \mathcal{Y}$ be the unique solution of*

$$a_r(y^0, \overline{y}) = l(\overline{y}) \quad \textit{for all } \overline{y} \in \mathcal{Y}$$

and let $J_r^{\star\star} : L^2(Q_T) \to L^2(Q_T)$ be the functional defined by

$$J_r^{\star\star}(\lambda) := \frac{1}{2}\langle \mathcal{T}_r\lambda, \lambda\rangle_{L^2(Q_T)} - b(y^0, \lambda).$$

The following equality holds:

$$\sup_{\lambda\in L^2(Q_T)}\inf_{y\in\mathcal{Y}}\mathcal{L}_r(y, \lambda) = -\inf_{\lambda\in L^2(Q_T)} J_r^{\star\star}(\lambda) + \mathcal{L}_r(y^0, 0).$$

The proof is standard and we refer for instance to [9] in a similar context. This proposition reduces the search for a solution y of problem (P) to the minimization of $J_r^{\star\star}$ with respect to λ. The well-posedness is a consequence of the ellipticity of the operator $\mathcal{T}_r$ stated in Lemma 2.9.

Remark 2.11. Let us assume that the domain Ω is of class C^2. The results of this section apply if the distributed observation on q_T is replaced by a Neumann boundary observation on the open subset γ of $\partial\Omega$ (i.e. assuming $y_{\mathrm{obs}} := \frac{\partial y}{\partial \nu} \in L^2(\gamma_T)$ is known on $\gamma_T := \gamma \times (0,T)$). This is due to the following Carleman inequality, proved in [17]: there exists a positive constant

$$C = C(\omega, \Omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(Q_T)})$$

such that

$$\|\tilde{\rho}_{c,0}^{-1} y\|^2_{L^2(Q_T)} + \|\tilde{\rho}_{c,1}^{-1} \nabla y\|^2_{L^2(Q_T)} \le C\|y\|^2_{\mathcal{Y}_0}$$

for any

$$y \in \mathcal{Y}_0 := \{y \in C^2(\overline{Q_T}) : y = 0 \text{ on } \Sigma_T \setminus \overline{\gamma}_T\},$$

where

$$(y, \overline{y})_{\mathcal{Y}_0} = \left\langle \tilde{\rho}_{c,1}^{-1} \frac{\partial y}{\partial \nu}, \tilde{\rho}_{c,1}^{-1} \frac{\partial \overline{y}}{\partial \nu} \right\rangle_{L^2(\gamma_T)} + \eta \langle \tilde{\rho}_c^{-1} Ly, \tilde{\rho}_c^{-1} L\overline{y}\rangle_{L^2(Q_T)},$$

$\|y\|^2_{\mathcal{Y}_0} = (y,y)_{\mathcal{Y}_0}$ and $\eta > 0$. Here, $\tilde{\rho}_c$, $\tilde{\rho}_{c,0}$ and $\tilde{\rho}_{c,1}$ are appropriate weight functions, similar to those in (2.5), associated to some $\tilde{\beta}_0 \in C^\infty(\overline{\Omega})$ such that $\tilde{\beta}_0 > 0$ in Ω, $\tilde{\beta}_0 = 0$ on $\partial\Omega \backslash \gamma$ and $\nabla \tilde{\beta}_0(x) \neq 0$ for all $x \in \overline{\Omega}$. It suffices to re-define the forms a_r and l in (2.2) by

$$\tilde{a}_r(y, \overline{y}) := \left\langle \tilde{\rho}_{c,1}^{-1} \frac{\partial y}{\partial \nu}, \tilde{\rho}_{c,1}^{-1} \frac{\partial \overline{y}}{\partial \nu} \right\rangle_{L^2(\gamma_T)} + r\langle \tilde{\rho}_c^{-1} L\, y, \tilde{\rho}_c^{-1} L\, \overline{y}\rangle_{L^2(Q_T)} \quad \text{for all } y, \overline{y} \in \tilde{\mathcal{Y}}$$

for any $r \ge 0$, and

$$\tilde{l}(y) := \left\langle \tilde{\rho}_{c,1}^{-1} \frac{\partial y}{\partial \nu}, \tilde{\rho}_{c,1}^{-1} y_{\mathrm{obs}} \right\rangle_{L^2(\gamma_T)} \quad \text{for all } y, \overline{y} \in \tilde{\mathcal{Y}},$$

where $\tilde{\mathcal{Y}}$ is the completion of $\tilde{\mathcal{Y}}_0$ with respect to the scalar product $(\cdot,\cdot)_{\tilde{\mathcal{Y}}_0}$.

Remark 2.12. The mixed formulation (2.2) is similar to the one we get when we address, using the same approach, the null controllability of (1.1): the control of minimal $L^2(q_T)$-norm which drives to rest the initial data $y_0 \in L^2(\Omega)$ given by $v = \rho_0^{-2} \varphi\, 1_{q_T}$, where $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$ solves the formulation

$$\begin{cases} a_r(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = l(\overline{\varphi}) & \text{for all } \overline{\varphi} \in \Phi, \\ b(\varphi, \overline{\lambda}) = 0 & \text{for all } \overline{\lambda} \in L^2(Q_T), \end{cases}$$

where

$$\begin{aligned} a_r : \Phi \times \Phi \to \mathbb{R}, &\quad a_r(\varphi, \overline{\varphi}) := \langle \rho_0^{-1}\varphi, \rho_0^{-1}\overline{\varphi}\rangle_{L^2(q_T)} + r\langle \rho^{-1} L^\star \varphi, \rho^{-1} L^\star \overline{\varphi}\rangle_{L^2(Q_T)}, \\ b : \Phi \times L^2(Q_T) \to \mathbb{R}, &\quad b(\varphi, \lambda) := \langle \rho^{-1} L^\star \varphi, \lambda\rangle_{L^2(Q_T)}, \\ l : \Phi \to \mathbb{R}, &\quad l(\varphi) := -(\varphi(\cdot, 0), y_0)_{L^2(\Omega)}. \end{aligned}$$

Here, the weights ρ and ρ_0 are taken in a space of functions that blow up at time $t = T$ and Φ is a complete space associated to these weights. Remark that an observability inequality (similar to (2.8)) is needed here to guarantee the continuity of l. We refer to [27].

3 Recovering the solution from a partial observation: A first order mixed formulation

We consider a first order mixed formulation of (1.1) introducing the flux variable $\mathbf{p} := c(x)\nabla y$. We then apply to this first-order system the procedure developed in the previous section and address the reconstruction of y and $\mathbf{p}$ from a distributed observation y_{obs}. The introduction of this equivalent first-order system is advantageous at the numerical level as it allows us to reduce the regularity order of the involved spaces.

3.1 Direct approach: Minimal local weighted L^2-norm

We rewrite the parabolic equation (1.1) into the following equivalent first-order system:

$$\begin{cases} y_t - \nabla\cdot\mathbf{p} + d\,y = f, \quad c(x)\nabla y - \mathbf{p} = 0 & \text{in } Q_T,\\ y = 0 & \text{on } \Sigma_T,\\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

The reformulation of (1.1) into a first-order system is standard and has been analyzed notably in [18, 23]. There, the existence and uniqueness of a solution for an associated $L^2 - H(\mathrm{div})$ weak formulation is proved, together with a priori estimates assuming that $y_0 \in H_0^1(\Omega)$. We use here instead the $H_0^1 - L^2$ weak formulation associated to (3.1) and refer to the appendix where the well-posedness of such a formulation is proved assuming $y_0 \in L^2(\Omega)$ (see notably Proposition A.2).

In the sequel, we use the following notations:

$$\mathcal{I}(y,\mathbf{p}) := y_t - \nabla\cdot\mathbf{p} + d(x,t)\,y, \quad \mathcal{J}(y,\mathbf{p}) := c(x)\nabla y - \mathbf{p}$$

and assume again for simplicity that $f = 0$. Then, in order to set up the least-squares approach, we preliminary define various spaces. First, let

$$\mathcal{U}_0 := \{(y,\mathbf{p}) \in C^1(\overline{Q_T}) \times \mathbf{C}^1(\overline{Q_T}) : y = 0 \text{ on } \Sigma_T\}$$

and for any $\eta_1, \eta_2 > 0$ and any $\rho, \rho_0, \rho_1 \in \mathcal{R}$, we define the bilinear form

$$((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))_{\mathcal{U}_0} := \langle \rho_0^{-1}y, \rho_0^{-1}\overline{y}\rangle_{L^2(q_T)} + \eta_1\langle \rho_1^{-1}\mathcal{J}(y,\mathbf{p}), \rho_1^{-1}\mathcal{J}(\overline{y},\overline{\mathbf{p}})\rangle_{\mathbf{L}^2(Q_T)} + \eta_2\langle \rho^{-1}\mathcal{I}(y,\mathbf{p}), \rho^{-1}\mathcal{I}(\overline{y},\overline{\mathbf{p}})\rangle_{L^2(Q_T)}$$

for all $(y,\mathbf{p}), (\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}_0$.

From unique continuation properties for parabolic equations (see also Proposition 3.2), this bilinear form defines a scalar product. We denote by $\mathcal{U}$ the completion of $\mathcal{U}_0$ for this scalar product and denote the norm over $\mathcal{U}$ by $\|\cdot\|_{\mathcal{U}}$ such that

$$\|(y,\mathbf{p})\|^2_{\mathcal{U}} := \|\rho_0^{-1}y\|^2_{L^2(q_T)} + \eta_1\|\rho_1^{-1}\mathcal{J}(y,\mathbf{p})\|^2_{\mathbf{L}^2(Q_T)} + \eta_2\|\rho^{-1}\mathcal{I}(y,\mathbf{p})\|^2_{L^2(Q_T)}.$$

Finally, we define the closed subset $\mathcal{V}$ of $\mathcal{U}$ by

$$\mathcal{V} := \{(y,\mathbf{p}) \in \mathcal{U} : \rho_1^{-1}\mathcal{J}(y,\mathbf{p}) = 0 \text{ in } \mathbf{L}^2(Q_T) \text{ and } \rho^{-1}\mathcal{I}(y,\mathbf{p}) = 0 \text{ in } L^2(Q_T)\}$$

and we endow $\mathcal{V}$ with the same norm as $\mathcal{U}$.

Within this setting, the analogue of the least-squares problem (P) reads as follows (without augmentation parameter in a first step, see Remark 3.5):

$$\text{Minimize } J(y,\mathbf{p}) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}|y(x,t) - y_{\mathrm{obs}}(x,t)|^2\,dx\,dt \text{ subject to } (y,\mathbf{p}) \in \mathcal{V}. \tag{3.2}$$

As in Section 2.1, this extremal problem, equivalent to (P), is well-posed in view of the definition of $\mathcal{V}$; there exists a unique pair $(y,\mathbf{p})$ which is a minimizer for J. With respect to (P), the scalar constraint $\rho^{-1}Ly = 0$ is now replaced by the $(N+1)$ constraints $\rho_1^{-1}\mathcal{J}(y,\mathbf{p}) = 0$ (in $\mathbf{L}^2(Q_T)$) and $\rho^{-1}\mathcal{I}(y,\mathbf{p}) = 0$ (in $L^2(Q_T)$). As before, these constraints are addressed by introducing Lagrange multipliers.

We set $\mathcal{X} := L^2(Q_T) \times \mathbf{L}^2(Q_T)$ and then we consider the following mixed formulation: find the solution $((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) \in \mathcal{U}\times\mathcal{X}$ of

$$\begin{cases} a((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) + b((\overline{y},\overline{\mathbf{p}}),(\lambda,\boldsymbol{\mu})) = l(\overline{y},\overline{\mathbf{p}}) & \text{for all } (\overline{y},\overline{\mathbf{p}}) \in \mathcal{U},\\ b((y,\mathbf{p}),(\overline{\lambda},\overline{\boldsymbol{\mu}})) = 0 & \text{for all } (\overline{\lambda},\overline{\boldsymbol{\mu}}) \in \mathcal{X}, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} &a : \mathcal{U}\times\mathcal{U} \to \mathbb{R}, \quad a((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) := \langle \rho_0^{-1}y, \rho_0^{-1}\overline{y}\rangle_{L^2(q_T)},\\ &b : \mathcal{U}\times\mathcal{X} \to \mathbb{R}, \quad b((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) := \langle \rho_1^{-1}\mathcal{J}(y,\mathbf{p}), \boldsymbol{\mu}\rangle_{\mathbf{L}^2(Q_T)} + \langle \rho^{-1}\mathcal{I}(y,\mathbf{p}), \lambda\rangle_{L^2(Q_T)},\\ &l : \mathcal{U} \to \mathbb{R}, \quad l(y,\mathbf{p}) := \langle \rho_0^{-1}y, \rho_0^{-1}y_{\mathrm{obs}}\rangle_{L^2(q_T)}. \end{aligned}$$

We have the following result.

Theorem 3.1. *Let $\rho_0 \in \mathcal{R}$ and $\rho, \rho_1 \in \mathcal{R} \cap L^\infty(Q_T)$. We have the following:*
(i) *The mixed formulation (3.3) is well-posed.*
(ii) *The unique solution $((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) \in \mathcal{U} \times \mathcal{X}$ is the unique saddle-point of the Lagrangian $\mathcal{L} : \mathcal{U} \times \mathcal{X} \to \mathbb{R}$ defined by*

$$\mathcal{L}((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) := \frac{1}{2} a((y,\mathbf{p}),(y,\mathbf{p})) + b((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) - l(y,\mathbf{p}). \tag{3.4}$$

(iii) *The unique solution $((y,\mathbf{p}),(\lambda,\boldsymbol{\mu}))$ satisfies the estimate*

$$\begin{aligned} &\|(y,\mathbf{p})\|_{\mathcal{U}} \le \|\rho_0^{-1} y_{\text{obs}}\|_{L^2(q_T)}, \\ &\|(\lambda,\boldsymbol{\mu})\|_{\mathcal{X}} \le 2\sqrt{\max\{C_{\Omega,T}\rho_\star^{-2}\|\rho_1\|^2_{L^\infty(Q_T)} + \eta_1, C_{\Omega,T}\rho_\star^{-2}\|\rho\|^2_{L^\infty(Q_T)} + \eta_2\}}\, \|\rho_0^{-1} y_{\text{obs}}\|_{L^2(q_T)}. \end{aligned} \tag{3.5}$$

Proof. The proof is similar to the proof of Theorem 2.1. From the definition, the bilinear form a is continuous over $\mathcal{U} \times \mathcal{U}$, symmetric and positive and the bilinear form b is continuous over $\mathcal{U} \times \mathcal{X}$. Furthermore, the linear form l is continuous over $\mathcal{X}$. In particular, we get

$$\|l\|_{\mathcal{X}'} \le \|\rho_0^{-1} y_{\text{obs}}\|_{L^2(q_T)}, \quad \|a\|_{\mathscr{L}^2(\mathcal{U})} \le 1, \quad \|b\|_{\mathscr{L}^2(\mathcal{U},\mathcal{X})} \le \max\{\eta_1^{-1/2}, \eta_2^{-1/2}\}. \tag{3.6}$$

Therefore, the well-posedness of formulation (3.3) is the consequence of two properties: First, the coercivity of the form a on the kernel

$$\mathcal{N}(b) := \{(y,\mathbf{p}) \in \mathcal{U} : b((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) = 0 \text{ for all } (\lambda,\boldsymbol{\mu}) \in \mathcal{X}\}.$$

This holds true since the kernel $\mathcal{N}(b)$ coincides with $\mathcal{V}$. Second, the existence of a constant $\delta > 0$ such that

$$\inf_{(\lambda,\boldsymbol{\mu})\in\mathcal{X}} \sup_{(y,\mathbf{p})\in\mathcal{U}} \frac{b((y,\mathbf{p}),(\lambda,\boldsymbol{\mu}))}{\|(y,\mathbf{p})\|_{\mathcal{U}}\|(\lambda,\boldsymbol{\mu})\|_{\mathcal{X}}} \ge \delta. \tag{3.7}$$

This inf-sup property holds true as follows: For any fixed $(\lambda^0,\boldsymbol{\mu}^0) \in \mathcal{X}$, we define the (unique) element $(y^0,\mathbf{p}^0)$ such that

$$\rho^{-1}\mathcal{I}(y^0,\mathbf{p}^0) = \lambda^0 \text{ in } Q_T, \quad \rho_1^{-1}\mathcal{J}(y^0,\mathbf{p}^0) = \boldsymbol{\mu}^0 \text{ in } Q_T, \quad y^0 = 0 \text{ on } \Sigma_T, \quad y^0(\cdot,0) = 0 \text{ in } \Omega.$$

The pair $(y^0,\mathbf{p}^0)$ is therefore the solution of a parabolic equation in the mixed form with source term $(\rho\lambda^0, \rho_1\boldsymbol{\mu}^0)$ in $L^2(0,T;L^2(\Omega)) \times L^2(0,T;\mathbf{L}^2(\Omega))$, null Dirichlet boundary condition and null initial state. From Proposition A.2 applied with $f = \rho\lambda^0 \in L^2(Q_T)$ and $\mathbf{F} = \rho_1\boldsymbol{\mu}^0 \in \mathbf{L}^2(Q_T)$, the weak solution satisfies

$$(y^0,\mathbf{p}^0) \in (L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))) \times \mathbf{L}^2(Q_T).$$

Moreover, from (A.2), there exists a constant $C_{\Omega,T}$ such that the unique pair $(y^0,\mathbf{p}^0)$ satisfies the inequality

$$\|\rho_0^{-1} y^0\|^2_{L^2(q_T)} \le C_{\Omega,T}\rho_\star^{-2}(\|\rho\|^2_{L^\infty(Q_T)}\|\lambda^0\|^2_{L^2(Q_T)} + \|\rho_1\|^2_{L^\infty(Q_T)}\|\boldsymbol{\mu}^0\|^2_{\mathbf{L}^2(Q_T)}), \tag{3.8}$$

which proves that $(y^0,\mathbf{p}^0) \in \mathcal{U}$. Consequently,

$$\sup_{(y,\mathbf{p})\in\mathcal{U}} \frac{b((y,\mathbf{p}),(\lambda^0,\boldsymbol{\mu}^0))}{\|(y,\mathbf{p})\|_{\mathcal{U}}\|(\lambda^0,\boldsymbol{\mu}^0)\|_{\mathcal{X}}} \ge \frac{b((y^0,\mathbf{p}^0),(\lambda^0,\boldsymbol{\mu}^0))}{\|(y^0,\mathbf{p}^0)\|_{\mathcal{U}}\|(\lambda^0,\boldsymbol{\mu}^0)\|_{\mathcal{X}}} = \frac{\|(\lambda^0,\boldsymbol{\mu}^0)\|_{\mathcal{X}}}{(\|\rho_0^{-1}y^0\|^2_{L^2(q_T)} + \eta_1\|\boldsymbol{\mu}^0\|^2_{\mathbf{L}^2(Q_T)} + \eta_2\|\lambda^0\|^2_{L^2(Q_T)})^{1/2}},$$

leading together with (3.8) to

$$\sup_{(y,\mathbf{p})\in\mathcal{U}} \frac{b((y,\mathbf{p}),(\lambda^0,\boldsymbol{\mu}^0))}{\|(y,\mathbf{p})\|_{\mathcal{U}}\|(\lambda^0,\boldsymbol{\mu}^0)\|_{\mathcal{X}}} \ge \delta,$$

with

$$\delta := (\max\{C_{\Omega,T}\rho_\star^{-2}\|\rho_1\|^2_{L^\infty(Q_T)} + \eta_1, C_{\Omega,T}\rho_\star^{-2}\|\rho\|^2_{L^\infty(Q_T)} + \eta_2\})^{-1/2}.$$

Hence, (3.7) holds.

Point (ii) is due to the positivity and symmetry of the form a. Point (iii) is a consequence of classical estimates (see [3], Theorem 4.2.3), namely

$$\|(y,\mathbf{p})\|_{\mathcal{U}} \le \frac{1}{\alpha_0}\|l\|_{\mathcal{U}'}, \quad \|(\lambda,\boldsymbol{\mu})\|_{\mathcal{X}} \le \frac{1}{\delta}\Big(1 + \frac{\|a\|_{\mathscr{L}^2(\mathcal{U})}}{\alpha_0}\Big)\|l\|_{\mathcal{U}'},$$

where $\alpha_0 := \inf_{y\in\mathcal{N}(b)} a((y,\mathbf{p}),(y,\mathbf{p}))/\|(y,\mathbf{p}\|^2_{\mathcal{U}}$. Estimates (3.6) and the equality $\alpha_0 = 1$ lead to the results. □

Again, we emphasize that the solution of (3.3) does not depend on the parameters η_1, η_2 only introduced in order to construct a scalar product over $\mathcal{U}_0$. In particular, η_1 and η_2 can be arbitrarily small.

Now, let us recall the following important result, analogue of Proposition 2.2, which provides a global estimate of y, the solution of a parabolic equation with right-hand side $L^2(0,T;H^{-1}(\Omega))$, from a local (in q_T) observation.

Proposition 3.2 (Theorem 2.2 in [19]). *Suppose that Ω is at least of class C^2 and assume that the weights $\rho_p, \rho_{p,0}, \rho_{p,1} \in \mathcal{R}$ (see (2.1)) are defined as follows:*

$$\begin{aligned} &\rho_p(x,t) := \exp\Big(\frac{\beta(x)}{t^2}\Big), \qquad \beta(x) := K_1(e^{K_2} - e^{\beta_0(x)}),\\ &\rho_{p,0}(x,t) := t\rho_p(x,t), \qquad \rho_{p,1}(x,t) := t^{-1}\rho_p(x,t), \qquad \rho_{p,2}(x,t) := t^{-2}\rho_p(x,t) \end{aligned} \tag{3.9}$$

with $\beta_0 \in C^\infty(\overline{\Omega})$ and K_i as in Proposition 2.2. Then there exists a constant $C = C(T,\omega,\Omega) > 0$ such that the following inequality holds:

$$\|\rho_{p,0}^{-1}y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\nabla y\|^2_{L^2(Q_T)} \le C\big(\|\rho_p^{-1}\mathbf{G}\|^2_{\mathbf{L}^2(Q_T)} + \|\rho_{p,2}^{-1}g\|^2_{L^2(Q_T)} + \|\rho_{p,0}^{-1}y\|^2_{L^2(q_T)}\big), \tag{3.10}$$

where y belongs to

$$\mathcal{K} := \{y \in L^2(0,T;H_0^1(\Omega)) : y_t \in L^2(0,T;H^{-1}(\Omega))\}$$

and satisfies $Ly = g + \nabla\cdot\mathbf{G}$ in Q_T, with $g \in L^2(Q_T)$ and $\mathbf{G} \in \mathbf{L}^2(Q_T)$.

This proposition allows us to get the following second global estimate.

Proposition 3.3. *Assume that Ω is at least of class C^2 and let $\rho_p, \rho_{p,0}, \rho_{p,1} \in \mathcal{R}$ be the weights defined by* (3.9). *There exists a constant $C > 0$, depending only on ω, Ω, T such that*

$$\|\rho_{p,0}^{-1}y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\nabla y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\mathbf{p}\|^2_{L^2(Q_T)} \le C\|(y,\mathbf{p})\|^2_{\mathcal{U}_p} \quad \text{for all } (y,\mathbf{p}) \in \mathcal{U}_p, \tag{3.11}$$

where $\mathcal{U}_p$ is the completion of $\mathcal{U}_{0,p} := \mathcal{U}_0$ with respect to the scalar product

$$((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))_{\mathcal{U}_{0,p}} = \langle\rho_{p,0}^{-1}y, \rho_{p,0}^{-1}\rangle_{L^2(q_T)} + \eta_1\langle\rho_{p,1}^{-1}\mathcal{J}(y,\mathbf{p}), \rho_{p,1}^{-1}\mathcal{J}(\overline{y},\overline{\mathbf{p}})\rangle_{\mathbf{L}^2(Q_T)} + \eta_2\langle\rho_p^{-1}\mathcal{I}(y,\mathbf{p}), \rho_p^{-1}\mathcal{I}(\overline{y},\overline{\mathbf{p}})\rangle_{L^2(Q_T)}.$$

Proof. First, let us prove this inequality for $(y,\mathbf{p}) \in \mathcal{U}_0$ for which $y \in \mathcal{K}$. We denote $\mathbf{G} := \mathcal{J}(y,\mathbf{p})$ and $g := \mathcal{I}(y,\mathbf{p})$, leading to $Ly = g - \nabla\cdot\mathbf{G}$ in Q_T. The Carleman inequality (3.10) then provides

$$\|\rho_{p,0}^{-1}y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\nabla y\|^2_{L^2(Q_T)} \le C\big(\|\rho_p^{-1}\mathbf{G}\|^2_{\mathbf{L}^2(Q_T)} + \|\rho_{p,2}^{-1}g\|^2_{L^2(Q_T)} + \|\rho_{p,0}^{-1}y\|^2_{L^2(q_T)}\big).$$

Moreover, noting that $\mathbf{p} = c(x)\nabla y - \mathbf{G}$, we get

$$\|\rho_{p,1}^{-1}\mathbf{p}\|^2_{L^2(Q_T)} \le 2\big(\|\rho_{p,1}^{-1}c\nabla y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\mathbf{G}\|^2_{\mathbf{L}^2(Q_T)}\big).$$

Finally, since $\rho_{p,1}^{-1} \le T\rho_p^{-1}$, we combine the last two inequalities to obtain

$$\|\rho_{p,0}^{-1}y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\nabla y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\mathbf{p}\|^2_{L^2(Q_T)} \le C\|(y,\mathbf{p})\|^2_{\mathcal{U}_{0,p}} \quad \text{for all } (y,\mathbf{p}) \in \mathcal{U}_0.$$

A standard density argument leads to (3.11). □

Eventually, assuming that the weights ρ_0, ρ_1, ρ (from (3.3)) are related to the *Carleman*-type weights $\rho_{p,0}, \rho_{p,1}, \rho_p$ so that $\mathcal{U} \subset \mathcal{U}_p$, we get the following stability result and a global estimate, analogue to (2.8), of any pair $(y,\mathbf{p}) \in \mathcal{U}$ in terms of the norm $\|(y,\mathbf{p})\|_{\mathcal{U}}$ (in particular for the solution of (3.3)).

Corollary 3.4. *Let $\rho_0 \in \mathcal{R}$ and $\rho, \rho_1 \in \mathcal{R} \cap L^\infty(Q_T)$ and assume that there exists a constant $K > 0$ such that*

$$\rho_0 \le K\rho_{p,0}, \quad \rho_1 \le K\rho_{p,1}, \quad \rho \le K\rho_{p,2} \qquad \text{in } Q_T. \tag{3.12}$$

If $((y,\mathbf{p}),(\lambda,\mu)) \in \mathcal{U}\times\mathcal{X}$ is the solution of the mixed formulation (3.3), *then there exists $C > 0$ such that*

$$\|\rho_{p,0}^{-1}y\|_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\mathbf{p}\|_{L^2(Q_T)} \le C\|(y,\mathbf{p})\|_{\mathcal{U}}.$$

Proof. Hypothesis (3.12) implies that $\mathcal{U} \subset \mathcal{U}_p$. Therefore, estimates (3.5) and (3.11) lead to

$$\|\rho_{p,0}^{-1}y\|_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\mathbf{p}\|_{L^2(Q_T)} \le C\|(y,\mathbf{p})\|_{\mathcal{U}_p} \le C\|(y,\mathbf{p})\|_{\mathcal{U}} \le C\|\rho_0^{-1}y_{\mathrm{obs}}\|_{L^2(q_T)}.$$ □

Again, the functions $\rho_{p,0}^{-1}$, $\rho_{p,1}^{-1}$ vanish at time $t = 0$, so that the variable y and the flux $\mathbf{p}$ are reconstructed from the observation y_{obs} everywhere in Q_T except on the set $\Omega \times \{0\}$. Similarly, the weights ρ_1 and ρ are introduced in the definition of $\mathcal{V}$ in order to reduce the effect of the "singularity" of the variable y and $\mathbf{p}$ in the neighborhood of $\Omega \times \{t = 0\}$. We refer to the discussion at the end of Section 2.1.

Remark 3.5. As in Section 2, it is convenient to augment the Lagrangian $\mathcal{L}$ defined in (3.4) as follows:

$$\begin{cases} \mathcal{L}_{\boldsymbol{r}}((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) := \dfrac{1}{2} a_{\boldsymbol{r}}((y, \mathbf{p}), (y, \mathbf{p})) + b((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) - l(y, \mathbf{p}), \\ a_{\boldsymbol{r}}((y, \mathbf{p}), (y, \mathbf{p})) := a((y, \mathbf{p}), (y, \mathbf{p})) + r_1 \|\rho_1^{-1} \mathcal{J}(y, \mathbf{p})\|^2_{\mathbf{L}^2(Q_T)} + r_2 \|\rho^{-1} \mathcal{I}(y, \mathbf{p})\|^2_{L^2(Q_T)} \end{cases}$$

for any $\boldsymbol{r} = (r_1, r_2) \in (\mathbb{R}^+)^2$. The two Lagrangian $\mathcal{L}$ and $\mathcal{L}_r$ share the same saddle-point, since the solution of (3.3) satisfies the constraint $\rho_1^{-1} \mathcal{J}(y, \mathbf{p}) = \mathbf{0}$ and $\rho^{-1} \mathcal{I}(y, \mathbf{p}) = 0$.

Remark 3.6. Similarly to Remark 2.5, the first equation of (3.3) reads as follows:

$$\iint\limits_{q_T} \rho_0^{-2} y \overline{y}\, dx\, dt + \iint\limits_{Q_T} \rho_1^{-1} \mathcal{J}(\overline{y}, \overline{\mathbf{p}}) \cdot \boldsymbol{\mu}\, dx\, dt + \iint\limits_{Q_T} \rho^{-1} \mathcal{I}(\overline{y}, \overline{\mathbf{p}}) \lambda\, dx\, dt = \iint\limits_{q_T} \rho_0^{-2} \overline{y}\, y_{\text{obs}}\, dx\, dt \quad \text{for all } (\overline{y}, \overline{\mathbf{p}}) \in \mathcal{U}.$$

But, according to Definition A.4, this means that the pair

$$(\varphi, \boldsymbol{\sigma}) := (\rho^{-1} \lambda, c\rho_1^{-1} \boldsymbol{\mu}) \in L^2(Q_T) \times \mathbf{L}^2(Q_T)$$

is a solution of the parabolic equation in the mixed form in the transposition sense, i.e. $(\varphi, \boldsymbol{\sigma})$ solves the problem:

$$\begin{cases} \mathcal{I}^\star(\varphi, \boldsymbol{\sigma}) = -\rho_0^{-2} (y - y_{\text{obs}}) 1_\omega, \quad \mathcal{J}(\varphi, \boldsymbol{\sigma}) = 0 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \tag{3.13}$$

where

$$\mathcal{I}^\star(\varphi, \boldsymbol{\sigma}) := -\varphi_t - \nabla \cdot \boldsymbol{\sigma} + d(x, t)\varphi.$$

Moreover, since

$$\rho_0^{-1}(y - y_{\text{obs}}) 1_\omega \in L^2(q_T),$$

we have that $(\varphi, \boldsymbol{\sigma})$ is a weak solution and

$$(\varphi, \boldsymbol{\sigma}) \in C([0, T], L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \times \mathbf{L}^2(Q_T).$$

System (3.13) also means that the pair $(\lambda, \boldsymbol{\mu})$, a solution of a backward mixed system, vanishes if y_{obs} is the restriction to q_T of a solution of (3.1). In this context, the rest of Remark 2.5, in particular (2.10), can be adapted to (3.3) as follows: the two multipliers λ and $\boldsymbol{\mu}$ measure how good the observation y_{obs} is to reconstruct the state y satisfying $y = y_{\text{obs}}$ on q_T under the constraints $\rho_1^{-1} \mathcal{J}(y, \mathbf{p}) = 0$ in $\mathbf{L}^2(Q_T)$ and $\rho^{-1} \mathcal{I}(y, \mathbf{p}) = 0$ in $L^2(Q_T)$.

Eventually, as in Section 2, we emphasize that the additional optimality system (3.13) can be used to define an equivalent saddle-point formulation. Precisely, in view of (3.13), we introduce the space Ψ defined by

$$\Psi := \Big\{(\varphi, \boldsymbol{\sigma}) \in (C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))) \times \mathbf{L}^2(Q_T) : \rho_0 \mathcal{I}^\star(\varphi, \boldsymbol{\sigma}) \in L^2(Q_T),\ \varphi(\cdot, T) = 0\Big\}.$$

Endowed with the scalar product

$$\langle(\varphi, \boldsymbol{\sigma}), (\overline{\varphi}, \overline{\boldsymbol{\sigma}})\rangle_\Psi := \langle \boldsymbol{\sigma}, \overline{\boldsymbol{\sigma}} \rangle_{\mathbf{L}^2(Q_T)} + \langle \rho^{-1} \nabla\varphi, \rho^{-1} \nabla\overline{\varphi} \rangle_{\mathbf{L}^2(Q_T)} + \big\langle \rho_0\, \mathcal{I}^\star(\varphi, \boldsymbol{\sigma}), \rho_0 \mathcal{I}^\star(\overline{\varphi}, \overline{\boldsymbol{\sigma}}) \big\rangle_{L^2(Q_T)},$$

we first check that Ψ is a Hilbert space. Then, for any parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in (0, 1)^2$ and $\boldsymbol{r} = (r_1, r_2) \in (\mathbb{R}_\star^+)^2$, we consider the following mixed formulation: find $((y, \mathbf{p}), (\varphi, \boldsymbol{\sigma})) \in \mathcal{U} \times \Psi$ such that

$$\begin{cases} a_{\boldsymbol{r},\boldsymbol{\alpha}}((y, \mathbf{p}), (\overline{y}, \overline{\mathbf{p}})) + b_{\boldsymbol{\alpha}}((\overline{y}, \overline{\mathbf{p}}), (\varphi, \boldsymbol{\sigma})) = l_{1,\boldsymbol{\alpha}}(\overline{y}, \overline{\mathbf{p}}) & \text{for all } \overline{y}, \overline{\mathbf{p}} \in \mathcal{U} \\ b_{\boldsymbol{\alpha}}((y, \mathbf{p}), (\overline{\varphi}, \overline{\boldsymbol{\sigma}})) - c_{\boldsymbol{\alpha}}((\varphi, \boldsymbol{\sigma}), (\overline{\varphi}, \overline{\boldsymbol{\sigma}})) = l_{2,\boldsymbol{\alpha}}(\overline{\varphi}, \overline{\boldsymbol{\sigma}}) & \text{for all } \overline{\varphi}, \overline{\boldsymbol{\sigma}} \in \Psi, \end{cases} \tag{3.14}$$

where

$$a_{\boldsymbol{r},\boldsymbol{\alpha}} : \mathcal{U}\times\mathcal{U}\to\mathbb{R}, \quad b_{\boldsymbol{\alpha}} : \mathcal{U}\times\Psi\to\mathbb{R}, \quad c_{\boldsymbol{\alpha}} : \Psi\times\Psi\to\mathbb{R}, \quad l_{1,\boldsymbol{\alpha}} : \mathcal{U}\to\mathbb{R}, \quad l_{2,\boldsymbol{\alpha}} : \Psi\to\mathbb{R}$$

are defined as

$$\begin{aligned}
a_{\boldsymbol{r},\boldsymbol{\alpha}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) &:= (1-\alpha_1)\langle\rho_0^{-1}y,\rho_0^{-1}\overline{y}\rangle_{L^2(q_T)} + r_1\big\langle\rho_1^{-1}\mathcal{J}(y,\mathbf{p}),\rho_1^{-1}\mathcal{J}(\overline{y},\overline{\mathbf{p}})\big\rangle_{\mathbf{L}^2(Q_T)}\\
&\quad + r_2\big\langle\rho^{-1}\mathcal{I}(y,\mathbf{p}),\rho^{-1}\mathcal{I}(\overline{y},\overline{\mathbf{p}})\big\rangle_{L^2(Q_T)},\\
b_{\boldsymbol{\alpha}}((y,\mathbf{p}),(\varphi,\boldsymbol{\sigma})) &:= \langle\mathcal{J}(y,\mathbf{p}),\boldsymbol{\sigma}\rangle_{\mathbf{L}^2(Q_T)} + \langle\mathcal{I}(y,\mathbf{p}),\varphi\rangle_{L^2(Q_T)} - \alpha_1\big\langle\rho_0\mathcal{I}^\star(\varphi,\boldsymbol{\sigma}),\rho_0^{-1}y\big\rangle_{L^2(q_T)},\\
c_{\boldsymbol{\alpha}}((\varphi,\boldsymbol{\sigma}),(\overline{\varphi},\overline{\boldsymbol{\sigma}})) &:= \alpha_1\big\langle\rho_0\mathcal{I}^\star(\varphi,\boldsymbol{\sigma}),\rho_0\mathcal{I}^\star(\overline{\varphi},\overline{\boldsymbol{\sigma}})\big\rangle_{L^2(Q_T)} + \alpha_2\big\langle\mathcal{J}(\varphi,\boldsymbol{\sigma}),\mathcal{J}(\overline{\varphi},\overline{\boldsymbol{\sigma}})\big\rangle_{\mathbf{L}^2(Q_T)}\\
l_{1,\boldsymbol{\alpha}}(y,\mathbf{p}) &:= (1-\alpha_1)\langle\rho_0^{-1}y_{\mathrm{obs}},\rho_0^{-1}y\rangle_{L^2(q_T)},\\
l_{2,\boldsymbol{\alpha}}(\varphi,\boldsymbol{\sigma}) &:= -\alpha_1\big\langle\rho_0^{-1}y_{\mathrm{obs}},\rho_0\mathcal{I}^\star(\varphi,\boldsymbol{\sigma})\big\rangle_{L^2(q_T)}.
\end{aligned}$$

Similarly to Proposition 2.6 we have the following result.

Proposition 3.7. *Let $\rho_0\in\mathcal{R}$ and $\rho,\rho_1\in\mathcal{R}\cap L^\infty(Q_T)$. Then, for any $\alpha_1,\alpha_2\in(0,1)$ and $\boldsymbol{r}=(r_1,r_2)\in(\mathbb{R}_+^\star)^2$, formulation* (3.14) *is well-posed. Moreover, the unique pair $((y,\mathbf{p}),(\varphi,\boldsymbol{\sigma}))$ in $\mathcal{U}\times\Psi$ satisfies*

$$\theta_1\|(y,\mathbf{p})\|_{\mathcal{U}}^2 + \theta_2\|(\varphi,\boldsymbol{\sigma})\|_{\Psi}^2 \le \Big(\frac{(1-\alpha_1)^2}{\theta_1} + \frac{\alpha_1^2}{\theta_2}\Big)\|\rho_0^{-1}y_{\mathrm{obs}}\|_{L^2(q_T)}^2, \tag{3.15}$$

with $\theta_1 := \min(1-\alpha_1, r_1\eta_1^{-1}, r_2\eta_2^{-1})$ and θ_2 given by (3.16).

Proof. We get the continuity of the bilinear forms $a_{\mathbf{r},\boldsymbol{\alpha}}$, $b_{\boldsymbol{\alpha}}$ and $c_{\boldsymbol{\alpha}}$ from

$$\begin{aligned}
|a_{\boldsymbol{r},\boldsymbol{\alpha}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))| &\le \max\{1-\alpha_1, r_1\eta_1^{-1}, r_2\eta_2^{-1}\}\|(y,\mathbf{p})\|_{\mathcal{U}}\|(\overline{y},\overline{\mathbf{p}})\|_{\mathcal{U}},\\
|b_{\boldsymbol{\alpha}}((y,\mathbf{p}),(\varphi,\boldsymbol{\sigma}))| &\le \max\{\alpha_1, \eta_1^{-1/2}\|\rho_1\|_{L^\infty(Q_T)}, \eta_2^{-1/2}\|\rho\|_{L^\infty(Q_T)}\}\|(y,\mathbf{p})\|_{\mathcal{U}}\|(\varphi,\boldsymbol{\sigma})\|_{\Psi},\\
|c_{\boldsymbol{\alpha}}((\varphi,\boldsymbol{\sigma}),(\overline{\varphi},\overline{\boldsymbol{\sigma}}))| &\le \max\{\alpha_1,\alpha_2\}\max\{2, 1+\|c\rho\|_{L^\infty(Q_T)}^2\}\|(\varphi,\boldsymbol{\sigma})\|_{\Psi}\|(\overline{\varphi},\overline{\boldsymbol{\sigma}})\|_{\Psi}
\end{aligned}$$

for all $(y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})\in\mathcal{U}$ and for all $(\varphi,\boldsymbol{\sigma}),(\overline{\varphi},\overline{\boldsymbol{\sigma}})\in\Psi$. Also, we can easily deduce the continuity of the linear forms $l_{1,\boldsymbol{\alpha}}$ and $l_{2,\boldsymbol{\alpha}}$ from

$$\|l_{1,\boldsymbol{\alpha}}\|_{\mathcal{U}'} \le (1-\alpha_1)\|\rho_0^{-1}y_{\mathrm{obs}}\|_{L^2(q_T)} \quad \text{and} \quad \|l_{2,\boldsymbol{\alpha}}\|_{\Psi'} \le \alpha_1\|\rho_0^{-1}y_{\mathrm{obs}}\|_{L^2(q_T)}.$$

Moreover, the two symmetric forms $a_{\boldsymbol{r},\boldsymbol{\alpha}}$ and $c_{\boldsymbol{\alpha}}$ are coercive since

$$a_{\boldsymbol{r},\boldsymbol{\alpha}}((y,\mathbf{p}),(y,\mathbf{p})) \ge \theta_1\|(y,\mathbf{p})\|_{\mathcal{U}}^2 \quad \text{for all } (y,\mathbf{p})\in\mathcal{U}.$$

Also, using Proposition (A.2) for the pair $(\varphi,\boldsymbol{\sigma})$, we get that there exists a continuity constant $C_{\Omega,T}$ such that

$$\begin{aligned}
\|\nabla\varphi\|_{L^2(Q_T)}^2 + \|\boldsymbol{\sigma}\|_{L^2(Q_T)}^2 &\le C_{\Omega,T}\big(\|\mathcal{I}^\star(\varphi,\boldsymbol{\sigma})\|_{L^2(Q_T)}^2 + \|\mathbf{G}\|^2\big)\\
&\le C_{\Omega,T}\max\{\|\rho_0^{-1}\|_{L^\infty(Q_T)}^2\alpha_1^{-1}, \alpha_2^{-1}\}\big(\alpha_1\|g\|_{L^2(Q_T)}^2 + \alpha_2\|\mathbf{G}\|^2\big),
\end{aligned}$$

where we denote $g := \rho_0\mathcal{I}^\star(\varphi,\boldsymbol{\sigma})$, $\mathbf{G} := \mathcal{J}(\varphi,\boldsymbol{\sigma})$ and where we have used that $\varphi(\cdot,T)=0$. Consequently, for any $m\in(0,1)$, we may write

$$\begin{aligned}
c_{\boldsymbol{\alpha}}((\varphi,\boldsymbol{\sigma}),(\varphi,\boldsymbol{\sigma})) &= \alpha_1\|g\|_{L^2(Q_T)}^2 + \alpha_2\|\mathbf{G}\|_{L^2(Q_T)}^2\\
&\ge m\big(\alpha_1\|g\|_{L^2(Q_T)}^2 + \alpha_2\|\mathbf{G}\|_{L^2(Q_T)}^2\big)\\
&\qquad + (1-m)\big(C_{\Omega,T}\max\{\|\rho_0^{-1}\|_{L^\infty(Q_T)}^2\alpha_1^{-1},\alpha_2^{-1}\}\big)^{-1}\big(\|\nabla\varphi\|_{L^2(Q_T)}^2 + \|\boldsymbol{\sigma}\|_{L^2(Q_T)}^2\big)\\
&\ge m\alpha_1\|g\|_{L^2(Q_T)}^2 + (1-m)\big(C_{\Omega,T}\max\{\|\rho_0^{-1}\|_{L^\infty(Q_T)}^2\alpha_1^{-1},\alpha_2^{-1}\}\big)^{-1}\big(\rho_\star^2\|\rho^{-1}\nabla\varphi\|_{L^2(Q_T)}^2 + \|\boldsymbol{\sigma}\|_{L^2(Q_T)}^2\big),
\end{aligned}$$

leading to $c_{\boldsymbol{\alpha}}((\varphi,\boldsymbol{\sigma}),(\varphi,\boldsymbol{\sigma})) \ge \theta_2\|(\varphi,\boldsymbol{\sigma})\|_{\Psi}^2$ for all $(\varphi,\boldsymbol{\sigma})\in\Psi$ with

$$\theta_2 := \min\Big(m\alpha_1, \frac{(1-m)\min\{1,\rho_\star^2\}}{C_{\Omega,T}\max\{\|\rho_0^{-1}\|_{L^\infty(Q_T)}^2\alpha_1^{-1},\alpha_2^{-1}\}}\Big). \tag{3.16}$$

The result [3, Proposition 4.3.1] now implies the well-posedness of the mixed formulation (3.14) and of estimate (3.15). □

Moreover, in view of the symmetry of the bilinear forms $a_{\mathbf{r},\boldsymbol{\alpha}}$ and $c_{\boldsymbol{\alpha}}$, formulation (3.14) corresponds to the following saddle point problem:

$$\begin{cases} \sup\limits_{(\varphi,\boldsymbol{\sigma})\in\Psi}\ \inf\limits_{(y,\mathbf{p})\in\mathcal{U}} \mathcal{L}_{\boldsymbol{r},\boldsymbol{\alpha}}((y,\mathbf{p}),(\varphi,\boldsymbol{\sigma})), \\ \mathcal{L}_{\mathbf{r},\boldsymbol{\alpha}}((y,\mathbf{p}),(\varphi,\boldsymbol{\sigma})) := \mathcal{L}_{\mathbf{r}}((y,\mathbf{p}),(\rho\varphi,\rho_1 c^{-1}\boldsymbol{\sigma})) - \dfrac{\alpha_2}{2}\|\mathcal{J}(\varphi,\boldsymbol{\sigma})\|^2_{L^2(Q_T)} \\ \qquad\qquad - \dfrac{\alpha_1}{2}\|\rho_0\mathcal{I}^\star(\varphi,\boldsymbol{\sigma}) + \rho_0^{-1}(y-y_{\mathrm{obs}})1_\omega\|^2_{L^2(Q_T)}. \end{cases}$$

The $\boldsymbol{\alpha}$-terms in $\mathcal{L}_{\mathbf{r},\boldsymbol{\alpha}}$ are stabilization terms; they ensure the ellipticity of $\mathcal{L}_{\mathbf{r},\boldsymbol{\alpha}}$ with respect to the variables $(\varphi,\boldsymbol{\sigma})$ and automatically the well-posedness. In particular, there is no need to prove any inf-sup property for the bilinear form $b_{\boldsymbol{\alpha}}$.

Eventually, since the pair of multipliers $(\lambda,\boldsymbol{\mu})\in\mathcal{X}$, a solution of (3.3) belongs indeed to Ψ, then arguing as in Proposition 2.7, we have the following result.

Proposition 3.8. *If $\boldsymbol{r}\in(\mathbb{R}_+^\star)^2$ and $\alpha_1,\alpha_2\in(0,1)$, then the solution of* (3.3) *and* (3.14) *coincide.*

3.2 Dual formulation of the extremal problem (3.3)

For any $\boldsymbol{r}=(r_1,r_2)\in(\mathbb{R}_+^\star)^2$ we define the linear operator $\mathcal{T}_{\boldsymbol{r}}$ from $\mathcal{X}:=L^2(Q_T)\times\mathbf{L}^2(Q_T)$ into $\mathcal{X}$ by

$$\mathcal{T}_{\boldsymbol{r}}(\lambda,\boldsymbol{\mu}) := (\rho_1^{-1}\mathcal{J}(y,\mathbf{p}),\rho^{-1}\mathcal{I}(y,\mathbf{p})),$$

where $(y,\mathbf{p})\in\mathcal{U}$ solves, for any $\boldsymbol{r}=(r_1,r_2)\in(\mathbb{R}_+^\star)^2$, the following equation:

$$a_{\boldsymbol{r}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) = b((\overline{y},\overline{\mathbf{p}}),(\lambda,\boldsymbol{\mu})) \quad \text{for all } (\overline{y},\overline{\mathbf{p}})\in\mathcal{U}. \tag{3.17}$$

Similarly to Lemma 2.9, the following lemma holds true.

Lemma 3.9. *For any $\boldsymbol{r}=(r_1,r_2)\in(\mathbb{R}_+^\star)^2$ the operator $\mathcal{T}_{\boldsymbol{r}}$ is a strongly elliptic, symmetric isomorphism from $\mathcal{X}$ into $\mathcal{X}$.*

Proof. From the definition of $a_{\boldsymbol{r}}$ we get that

$$\|\mathcal{T}_{\boldsymbol{r}}(\lambda,\boldsymbol{\mu})\|_{\mathcal{X}} \le \min(r_1,r_2)^{-1}\|(\lambda,\boldsymbol{\mu})\|_{\mathcal{X}},$$

leading to the continuity of $\mathcal{T}_{\boldsymbol{r}}$. Next, consider any $(\lambda',\boldsymbol{\mu}')\in\mathcal{X}$ and denote by $(y',\mathbf{p}')$ the corresponding solution of (3.17) so that $\mathcal{T}_{\boldsymbol{r}}(\lambda',\boldsymbol{\mu}')=(\rho_1^{-1}\mathcal{J}(y',\mathbf{p}'),\rho^{-1}\mathcal{I}(y',\mathbf{p}'))$. Relation (3.17) with $(\overline{y},\overline{\mathbf{p}})=(y',\mathbf{p}')$ implies that

$$\langle\mathcal{T}_{\boldsymbol{r}}(\lambda',\boldsymbol{\mu}'),(\lambda,\boldsymbol{\mu})\rangle_{\mathcal{X}} = a_{\boldsymbol{r}}((y,\mathbf{p}),(y',\mathbf{p}')) \tag{3.18}$$

and therefore the symmetry and positivity of $\mathcal{T}_{\boldsymbol{r}}$. The last relation with $(\lambda',\boldsymbol{\mu}')$ and the unique continuation property for (3.1) and (3.17) imply that the operator $\mathcal{T}_{\boldsymbol{r}}$ is also positive definite. Actually, as announced, we can check that $\mathcal{T}_{\boldsymbol{r}}$ is strongly elliptic, i.e. there exists a constant $C>0$ such that

$$\langle\mathcal{T}_{\boldsymbol{r}}(\lambda,\boldsymbol{\mu}),(\lambda,\boldsymbol{\mu})\rangle_{\mathcal{X}} \ge C\|(\lambda,\boldsymbol{\mu})\|^2_{\mathcal{X}} \quad \text{for all } (\lambda,\boldsymbol{\mu})\in\mathcal{X}.$$

We argue by contradiction and suppose that there exists a sequence $\{(\lambda_n,\boldsymbol{\mu}_n)\}_{n\ge0}$ of $\mathcal{X}$ such that

$$\|(\lambda_n,\boldsymbol{\mu}_n)\|_{\mathcal{X}} = 1 \text{ for all } n\ge0 \quad \text{and} \quad \lim_{n\to\infty}\langle\mathcal{T}_{\boldsymbol{r}}(\lambda_n,\boldsymbol{\mu}_n),(\lambda_n,\boldsymbol{\mu}_n)\rangle_{\mathcal{X}} = 0. \tag{3.19}$$

We denote by $(y_n,\mathbf{p}_n)$ the solution of (3.17) corresponding to $(\lambda_n,\boldsymbol{\mu}_n)$. From (3.18) we then obtain that

$$\lim_{n\to\infty}\Big(\|\rho_0^{-1}y_n\|^2_{L^2(q_T)} + r_1\|\rho_1^{-1}\mathcal{J}(y_n,\mathbf{p}_n)\|^2_{\mathbf{L}^2(Q_T)} + r_2\|\rho^{-1}\mathcal{I}(y_n,\mathbf{p}_n)\|^2_{L^2(Q_T)}\Big) = 0. \tag{3.20}$$

Moreover, from (3.17) with $(y,\mathbf{p}) = (y_n,\mathbf{p}_n)$ and $(\lambda,\boldsymbol{\mu}) = (\lambda_n,\boldsymbol{\mu_n})$, we get the equality

$$\langle(\rho_1^{-1}\mathcal{J}(\overline{y},\overline{\mathbf{p}}), r_1\rho_1^{-1}\mathcal{J}(y_n,\mathbf{p}_n)-\boldsymbol{\mu_n}\rangle_{\mathbf{L}^2(Q_T)}\langle\rho^{-1}\mathcal{I}(\overline{y},\overline{\mathbf{p}}), r_2\rho^{-1}\mathcal{I}(y_n,\mathbf{p}_n)-\lambda_n\rangle_{L^2(Q_T)}+\langle\rho_0^{-1}y_n,\rho_0^{-1}\overline{y}\rangle_{L^2(q_T)} = 0 \quad (3.21)$$

for every $(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}$. Then, in order to get a contradiction, we define the sequence $\{(\overline{y}_n,\overline{\mathbf{p}}_n)\}_{n\geq 0}$ as follows:

$$\begin{cases} \rho_1^{-1}\mathcal{J}(\overline{y}_n,\overline{\mathbf{p}}_n) = r_1\rho_1^{-1}\mathcal{J}(y_n,\mathbf{p}_n) - \boldsymbol{\mu_n} & \text{in } Q_T,\\ \rho^{-1}\mathcal{I}(\overline{y}_n,\overline{\mathbf{p}}_n) = r_2\rho^{-1}\mathcal{I}(y_n,\mathbf{p}_n) - \lambda_n & \text{in } Q_T,\\ \overline{y}_n = 0 & \text{on } \Sigma_T,\\ \overline{y}_n(x,0) = 0 & \text{in } \Omega, \end{cases}$$

so that, for all $n \geq 0$, the pair $(\overline{y}_n,\overline{\mathbf{p}}_n)$ is the solution of a first-order system as discussed in the appendix with zero initial data and source term in $\mathcal{X}$. The energy estimate (A.2) implies that

$$\begin{aligned}\|\rho_0^{-1}\overline{y}_n\|_{L^2(q_T)} \leq C_{\Omega,T}\rho_\star^{-1}\Big(&\|\rho_1\|_{L^\infty(Q_T)}\|r_1\rho_1^{-1}\mathcal{J}(y_n,\mathbf{p}_n) - \boldsymbol{\mu_n}\|_{\mathbf{L}^2(Q_T)}\\ &+ \|\rho\|_{L^\infty(Q_T)}\|r_2\rho^{-1}\mathcal{I}(y_n,\mathbf{p}_n) - \lambda_n\|_{L^2(Q_T)}\Big)\end{aligned}$$

for some constant $C_{\Omega,T}$ and that $(\overline{y}_n,\overline{\mathbf{p}}_n) \in \mathcal{U}$. Then, using this inequality and (3.21) with $(\overline{y},\overline{\mathbf{p}}) = (\overline{y},\overline{\mathbf{p}}_n)$, we get that

$$\begin{aligned}&\|r_1\rho_1^{-1}\mathcal{J}(y_n,\mathbf{p}_n) - \boldsymbol{\mu_n}\|_{\mathbf{L}^2(Q_T)} + \|r_2\rho^{-1}\mathcal{I}(y_n,\mathbf{p}_n) - \lambda_n\|_{L^2(Q_T)}\\ &\quad\leq 2C_{\Omega,T}\rho_\star^{-1}\max(\|\rho_1\|_{L^\infty(Q_T)},\|\rho\|_{L^\infty(Q_T)})\|\rho_0^{-1}y_n\|_{L^2(q_T)}.\end{aligned}$$

Eventually, from (3.20), we conclude that

$$\lim_{n\to\infty}\|\lambda_n\|_{L^2(Q_T)} = \lim_{n\to\infty}\|\boldsymbol{\mu_n}\|_{\mathbf{L}^2(Q_T)} = 0,$$

which is a contradiction to the first hypothesis of (3.19). □

Again, the introduction of the operator $\mathcal{T}_{\boldsymbol{r}}$ is motivated by the following proposition, which reduces the determination of the solution $(y,\mathbf{p})$ of problem (3.2) to the unconstrained minimization of an elliptic functional.

Proposition 3.10. *For any $\boldsymbol{r} = (r_1,r_2) \in (\mathbb{R}_+^\star)^2$ let $(y_0,\mathbf{p}_0) \in \mathcal{U}$ be the unique solution of*

$$a_{\boldsymbol{r}}((y_0,\mathbf{p}_0),(\overline{y},\overline{\mathbf{p}})) = l(\overline{y},\overline{\mathbf{p}}) \quad \textit{for all } (\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}$$

and let $J_{\boldsymbol{r}}^{\star\star} : \mathcal{X} \to \mathcal{X}$ be the functional defined by

$$J_{\boldsymbol{r}}^{\star\star}(\lambda,\boldsymbol{\mu}) = \frac{1}{2}\langle\mathcal{T}_r(\lambda,\boldsymbol{\mu}),(\lambda,\boldsymbol{\mu})\rangle_{\mathcal{X}} - b((y_0,\mathbf{p}_0),(\lambda,\boldsymbol{\mu})).$$

Then the following equality holds:

$$\sup_{(\lambda,\boldsymbol{\mu})\in\mathcal{X}}\inf_{(y,\mathbf{p})\in\mathcal{U}}\mathcal{L}_{\boldsymbol{r}}((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) = -\inf_{(\lambda,\boldsymbol{\mu})\in\mathcal{X}}J_r^{\star\star}(\lambda,\boldsymbol{\mu}) + \mathcal{L}_r((y_0,\mathbf{p}_0),(0,\mathbf{0})),$$

where the lagrangian $\mathcal{L}_{\boldsymbol{r}}$ is defined in Remark 3.5.

A similar procedure may be conducted for the stabilized Lagrangian $\mathcal{L}_{\boldsymbol{r},\boldsymbol{\alpha}}$.

4 Concluding remarks and perspectives

The mixed formulations introduced to address inverse problems for linear parabolic equations correspond to the optimality systems associated to weighted least-squares type functionals. These formulations depend on both the state to reconstruct and a Lagrange multiplier, introduced to take into account the state constraint $Ly - f = 0$. The multiplier turns out to be a measure of how good the observation data is for reconstructing the solution. This approach, recently used in a controllability context in [27], leads to variational problems defined over time-space Hilbert spaces, without distinction between the time and the space variable. The main ingredient is the unique continuation property leading to well-posedness in appropriate constructed Hilbert spaces. Moreover, global Carleman estimates then allow to precise in which norm the full solution can be reconstructed. For these reasons, the method can be applied to many systems for which such estimates are available, as in [10] for linear hyperbolic equations, or as in Section 3 for a first-order system. In the parabolic situation, in view of the regularization property, the method requires the introduction of exponentially vanishing weights at the initial time; this guarantees a stable Lipschitz reconstruction of the solution on the whole domain, the initial condition excepted.

From the theoretical standpoint, the minimization of the L^2-weighted least-squares norm with respect either to $y \in \mathcal{W}$ (problem (P), Section 2.1), or to the initial data $y_0 \in \mathcal{H}$ (problem (LS), Section 1) are equivalent. However, the completed space $\mathcal{W}$ embedded in the space $C([\delta, T], H_0^1(\Omega))$ is a priori much more "practical" than the huge space $\mathcal{H}$, a fortiori since from the definition of the cost, the variable of interest is not y but

$$\rho_0^{-1} y \in C([0, T], H_0^1(\Omega)) \quad \text{with} \quad \rho_0^{-1}(\cdot, t = 0) = 0 \text{ in } \Omega.$$

Therefore, from a practical (i.e. numerical) viewpoint, as enhanced in [19, 29] and recently used in [1, 2] for inverse problems and in [26, 27] in the close controllability context (see Remark 2.12), variational methods where the state y is kept unknown are very appropriate and lead to robust approximations. Moreover, as detailed in [10], the space-time framework allows us to use classical approximation and interpolation theory leading to strong convergence results with error estimates, again without the need of proving any discrete Carleman inequalities. We refer to the second part [25] of this work where the numerical approximation of the mixed formulations (2.2) in (y, λ) and (3.3) in $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu}))$ is examined, implemented and compared with the standard minimization of the cost with respect to the initial data. As observed in [27, Section 3.2] for the related control problem, described in Remark 2.12, an appropriate preliminary change (renormalization) of variable, i.e. $\tilde{y} := \rho_0^{-1} y$, so as to eliminate (by compensation) the exponential behavior of the coefficient in $\rho^{-1} L y = \rho^{-1} L(\rho_0 \tilde{y})$, leads to an impressive low condition number of the corresponding discrete system. We also emphasize, that the second mixed formulation (3.3), apparently more involved with more variables allows us to use (standard) continuous finite dimensional approximation spaces for $\mathcal{U}$, in contrast to formulation (2.2) which requires continuously differentiable approximation spaces.

Eventually, we also emphasize that such a direct method may be used to reconstruct the state as well as a source term. By the assumption that the source $f(x, t) = \sigma(t)\mu(x)$ with $\sigma \in C^1([0, T])$, $\sigma(0) \neq 0$ and $\mu \in L^2(\Omega)$, it is shown in [8] that the knowledge of $\partial_t(\partial_\nu u) \in L^2(\partial\Omega \times (0, T))$ allows one to reconstruct uniquely the pair (y, μ) satisfying the state equation $Ly - \sigma\mu = 0$. This allows one to construct appropriate Hilbert spaces, associate a least-squares functional in (y, μ) and the corresponding optimality system. The (logarithmic) stability estimate proved in [8, Theorem 1.2] guarantees the reconstruction of the solution. We refer to [11, Section 3], where this strategy is implemented in the simpler case of the wave equation.

A Appendix: Well-posedness of parabolic equations in the mixed form

The aim of this appendix is to study the existence and uniqueness of a solution for the following linear boundary value problem, which appears in Section 3: find $(y, \mathbf{p})$ such that

$$\begin{cases} y_t - \nabla \cdot \mathbf{p} + d\, y = f, \quad c(x)\nabla y - \mathbf{p} = \mathbf{F} & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(x, 0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{A.1}$$

We assume that the initial datum y_0 belongs to $L^2(\Omega)$ and that the source terms f and $\mathbf{F}$ belong to $L^2(Q_T)$ and $\mathbf{L}^2(Q_T)$, respectively. The functions c and d enjoy the regularity described in the introduction, namely $c := (c_{i,j}) \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R}))$ with $(c(x)\xi, \xi) \geq c_0|\xi|^2$ for any $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$ $(c_0 > 0)$ and $d \in L^\infty(Q_T)$. Moreover, we assume that c is symmetric and $(c^{-1}(x)\xi, \xi) \geq c_0|\xi|^2$ for any $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$ $(c_0 > 0)$.

Let us introduce a definition of weak solution in accordance to the classical definition of weak solution for the *standard* parabolic equation (1.1).

Definition A.1. We say that a pair $(y, \mathbf{p})$ satisfying

$$\mathbf{p} \in \mathbf{L}^2(Q_T), \quad y \in L^2(0,T;H_0^1(\Omega)) \text{ with } y_t \in L^2(0,T;H^{-1}(\Omega))$$

is a weak solution of the parabolic equation in the mixed form (A.1) if and only if the following hold:
(i) $\langle y_t, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\mathbf{p}, \nabla w) + (d\,y, w) = (f, w)$ for all $w \in H_0^1(\Omega)$ and a.e. time $t \in [0,T]$,
(ii) $(\nabla y, \mathbf{u}) - (c^{-1}\mathbf{p}, \mathbf{u}) = (c^{-1}\mathbf{F}, \mathbf{u})$ for all $\mathbf{u} \in \mathbf{L}^2(\Omega)$ and a.e. time $t \in [0,T]$,
(iii) $y(\cdot, 0) = y_0$.

In this way, we have the following result.

Proposition A.2. *There exists a unique weak solution for the parabolic equation in the mixed form* (A.1). *Moreover, there exists a constant $C > 0$ such that*

$$\|y_t\|_{L^2(0,T;H^{-1}(\Omega))} + \|y\|_{L^2(0,T;H_0^1(\Omega))} + \|\mathbf{p}\|_{\mathbf{L}^2(Q_T)} \le C\big(y_{L^2(\Omega)} + \|f\|_{L^2(Q_T)} + \|\mathbf{F}\|_{\mathbf{L}^2(Q_T)}\big). \tag{A.2}$$

Proof. Following [18], we proof the existence of a solution relying on the *Faedo–Galerkin method* and we divide it into several steps.

(a) *Galerkin approximations.* We first introduce some notations. Let $\{w_k : k \in \mathbb{N}\}$ be an orthogonal basis of $H_0^1(\Omega)$ (which is orthonormal in $L^2(\Omega)$) and let $\{\mathbf{u}_k : k \in \mathbb{N}\}$ be an orthonormal basis of $\mathbf{L}^2(\Omega)$. For each pair $(m,n) \in \mathbb{N} \times \mathbb{N}$ we look for a pair $(y_n, \mathbf{p}_m) : [0,T] \to H_0^1(\Omega) \times \mathbf{L}^2(\Omega)$ of the form

$$y_n(t) = \sum_{k=1}^{n} a_n^k(t) w_k \quad \text{and} \quad \mathbf{p}_m(t) = \sum_{k=1}^{m} b_m^k(t)\mathbf{u}_k, \tag{A.3}$$

as a solution of the weak formulation

$$\begin{cases} (y_n', w_i) + (\mathbf{p}_m, \nabla w_i) + (d\,y_n, w_i) = (f, w_i) & (0 \le t \le T,\ i = 1,\dots,n), \\ (\nabla y_n, \mathbf{u}_j) - (c^{-1}\mathbf{p}_m, \mathbf{u}_j) = (c^{-1}\mathbf{F}, \mathbf{u}_j) & (0 \le t \le T,\ j = 1,\dots,m) \end{cases} \tag{A.4}$$

(the prime $'$ stands for the derivation in time). We denote by $(a_n^k)_{k=1}^n$ and $(b_m^k)_{k=1}^m$ some time functions from $[0,T]$ to $\mathbb{R}$ for each pair $(m,n) \in \mathbb{N} \times \mathbb{N}$. We assume that the a_n^k satisfy

$$a_n^k(0) = (y_0, w_k) \quad (k = 1,\dots,n). \tag{A.5}$$

We also denote

$$(\mathbf{f}_n(t))_i = (f(t), w_i), \quad (\mathbf{Y}_0)_i = (y_0, w_i), \quad (\mathbf{A}_n)_{ij} = (w_j, w_i), \quad (\mathbf{D}_n(t))_{ij} = (d(\cdot,t)\,w_j, w_i)$$

for all $i, j = 1,\dots,n$,

$$(\mathbf{B}_m)_{ij} = (c^{-1}\mathbf{u}_j, \mathbf{u}_i), \quad (\mathbf{F}_m(t))_i = (c^{-1}\mathbf{F}(t), \mathbf{u}_i)$$

for all $i, j = 1,\dots,m$, and

$$(\mathbf{E}_{nm})_{ij} = (\mathbf{u}_j, \nabla w_i)$$

for all $i = 1,\dots,n$ and $j = 1,\dots,m$.

Eventually, we also denote by $\mathbf{Y}_n(t)$ the vector formed by a_n^k $(k = 1,\dots,n)$ and by $\mathbf{P}_m(t)$ the vector formed by b_m^k $(k = 1,\dots,m)$. With these notations, (A.4) and (A.5) may be rewritten as

$$\begin{cases} \mathbf{Y}_n'(t) + \mathbf{E}_{nm}\mathbf{P}_m(t) + \mathbf{D}_n(t)\mathbf{Y}_n(t) = \mathbf{f}_n(t), \\ \mathbf{E}_{nm}^T\mathbf{Y}_n(t) - \mathbf{B}_m\mathbf{P}_m(t) = \mathbf{F}_m(t), \\ \mathbf{Y}_n(0) = \mathbf{Y}_0. \end{cases} \tag{A.6}$$

From the positivity and the symmetry of c^{-1}, we obtain that $\mathbf{B}_m$ is a symmetric and positive definite square matrix of order m. Therefore, $\mathbf{B}_m$ is invertible and the second equation of (A.6) implies the relation

$$\mathbf{P}_m(t) = \mathbf{B}_m^{-1}(\mathbf{E}_{nm}^T\mathbf{Y}_n(t) - \mathbf{F}_m(t)).$$

Using this equality in the first equation of (A.6), we obtain

$$\begin{cases} \mathbf{Y}_n'(t) + (\mathbf{E}_{nm}\mathbf{B}_m^{-1}\mathbf{E}_{nm}^T + \mathbf{D}_n(t))\mathbf{Y}_n(t) = \mathbf{f}_n(t) + \mathbf{E}_{nm}\mathbf{B}_m^{-1}\mathbf{F}_m(t) \quad \text{for a.e. } t \in [0, T], \\ \mathbf{Y}_n(0) = \mathbf{Y}_0. \end{cases} \tag{A.7}$$

System (A.7) is a system of n linear ODEs of order 1. Hence, from standard theory for ODEs, there exists a unique absolutely continuous $\mathbf{Y}_n : [0, T] \to \mathbb{R}^n$ satisfying (A.7). Consequently, the pair $(y_n, \mathbf{p}_m)$ given by (A.3) is the unique solution of (A.4) and (A.5).

(b) *A priori estimates.* We now derive some uniform estimates for the pair $(y_n, \mathbf{p}_m)$ with respect to m and n. This allows us to see that the sequences $\{y_n\}_{n>0}$, $\{\mathbf{p}_m\}_{m>0}$ converge to y and $\mathbf{p}$, respectively, such that $(y, \mathbf{p})$ is the weak solution of (A.1).

Multiplying the first equation of (A.4) by $a_n^k(t)$ and summing over $k = 1, \ldots, n$ and multiplying the second equation of (A.4) by $-b_m^k(t)$ and summing over $k = 1, \ldots, m$, we obtain the relations

$$\begin{cases} (y_n', y_n) + (\mathbf{p}_m, \nabla y_n) + (d\, y_n, y_n) = (f, y_n) & (0 \le t \le T), \\ -(\nabla y_n, \mathbf{p}_m) + (c^{-1}\mathbf{p}_m, \mathbf{p}_m) = -(c^{-1}\mathbf{F}, \mathbf{p}_m) & (0 \le t \le T). \end{cases}$$

Adding these two equations and applying the Cauchy–Bunyakovski–Schwarz inequality, we get

$$\frac{d}{dt}\|y_n(t)\|_{L^2(\Omega)}^2 + \|\mathbf{p}_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \le C\big(\|y_n(t)\|_{L^2(\Omega)}^2 + \|f(\cdot, t)\|_{L^2(\Omega)}\|y_n(t)\|_{L^2(\Omega)} + \|\mathbf{F}(\cdot, t)\|_{\mathbf{L}^2(\Omega)}\|\mathbf{p}_m(t)\|_{\mathbf{L}^2(\Omega)}\big)$$

for some positive constant $C = C(\|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(Q_T)})$. Then, by using the Gronwall's Lemma, we deduce, that for all $m, n > 0$, we have

$$\|y_n\|_{L^\infty(0,T;L^2(\Omega))}^2 \le C\big(\|y_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_T)}^2 + \|\mathbf{F}\|_{\mathbf{L}^2(Q_T)}^2\big) \tag{A.8}$$

and

$$\|\mathbf{p}_m\|_{\mathbf{L}^2(Q_T)}^2 \le C\big(\|y_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_T)}^2 + \|\mathbf{F}\|_{\mathbf{L}^2(Q_T)}^2\big). \tag{A.9}$$

Now we derive a uniform estimate for y_n'. To do this, fix any $w \in H_0^1(\Omega)$ with $\|w\|_{H_0^1(\Omega)} \le 1$. Notice that we can decompose w as $w = w^1 + w^2$ with $w^1 \in \operatorname{span}\{w_k\}_{k=1}^n$ and $(w^2, w_k) = 0$ for $k = 1, \ldots, n$. Using the first equation of (A.4), we can deduce, for a.e. $0 \le t \le T$, that

$$(y_n', w^1) + (\mathbf{p}_m, \nabla w^1) + (d\, y_n, w^1) = (f, w^1).$$

Then, using that $(w^2, w_k) = 0$ for $k = 1, \ldots, n$ and $y_n'(t) = \sum_{k=1}^n (a_n^k)'(t) w_k$, we write

$$(y_n', w) = (y_n', w^1 + w^2) = (y_n', w^1) = (f, w^1) - (\mathbf{p}_m, \nabla w^1) - (d\, y_n, w^1).$$

Consequently,

$$\begin{aligned} |(y_n'(t), w)| &\le \|f(\cdot, t)\|_{L^2(\Omega)}\|w^1\|_{L^2(\Omega)} + \|\mathbf{p}_m(t)\|_{\mathbf{L}^2(\Omega)}\|\nabla w^1\|_{\mathbf{L}^2(\Omega)} + \|d\|_{L^\infty(Q_T)}\|y_n(t)\|_{L^2(\Omega)}\|w^1\|_{L^2(\Omega)} \\ &\le C\big(\|f(\cdot, t)\|_{L^2(\Omega)} + \|\mathbf{p}_m\|_{\mathbf{L}^2(\Omega)} + \|y_n\|_{L^2(\Omega)}\big) \end{aligned}$$

since $\|w^1\|_{H_0^1(\Omega)} \le 1$. Finally, using (A.8), (A.9) and the identification of the duality

$$\langle y_n'(t), w\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} := (y_n'(t), w),$$

we get

$$\|y_n'\|_{L^2(0,T;H^{-1}(\Omega))}^2 \le C\big(\|y_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_T)}^2 + \|\mathbf{F}\|_{\mathbf{L}^2(Q_T)}^2\big). \tag{A.10}$$

To end this second step, let us prove a uniform estimate for ∇y_n. To do this, let us fix $n \ge 1$ and as $\nabla y_n(t) \in \mathbf{L}^2(\Omega)$, we can write

$$\nabla y_n(t) = \sum_{k=1}^{+\infty} \xi_n^k(t)\mathbf{u}_k \quad \text{for a.e. } 0 \le t \le T, \tag{A.11}$$

where ξ_n^k denotes a time function for $[0,T] \to \mathbb{R}$ for each k. Then, fixing $m \geq 1$, multiplying the second equation of (A.4) by $\xi_n^k(t)$ and summing over $k = 1, \ldots, m$, we deduce

$$\Big(\nabla y_n(t), \sum_{k=1}^{m} \xi_n^k(t)\mathbf{u}_k\Big) \leq \frac{C}{2}\big(\|\mathbf{F}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{p}_m(t)\|^2_{\mathbf{L}^2(\Omega)}\big) + \frac{1}{2}\Big\|\sum_{k=1}^{m} \xi_n^k(t)\mathbf{u}_k\Big\|^2_{\mathbf{L}^2(\Omega)}.$$

Integrating with respect to the time variable and recalling (A.9), we find

$$\int_0^T \Big(\nabla y_n(t), \sum_{k=1}^{m} \xi_n^k(t)\mathbf{u}_k\Big)\,dt \leq C\big(\|\mathbf{F}\|^2_{\mathbf{L}^2(Q_T)} + \|f\|^2_{L^2(Q_T)} + \|y_0\|^2_{L^2(\Omega)}\big) + \frac{1}{2}\int_0^T \Big\|\sum_{k=1}^{m} \xi_n^k(t)\mathbf{u}_k\Big\|^2_{\mathbf{L}^2(\Omega)}\,dt.$$

Let $m \to +\infty$ and using (A.11), we finally obtain

$$\|\nabla y_n\|^2_{L^2(Q_T)} \leq C\big(\|y_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q_T)} + \|\mathbf{F}\|^2_{\mathbf{L}^2(Q_T)}\big). \tag{A.12}$$

(c) *Building a weak solution.* Let us pass to the limit in the sequence $(y_n, \mathbf{p}_m)$.

From the a priori estimates (A.8), (A.9), (A.10) and (A.12), there exist subsequences $(y_{nl})_{l=1}^{\infty} \subset (y_n)_{n=1}^{\infty}$ and $(\mathbf{p}_{ml})_{l=1}^{\infty} \subset (\mathbf{p}_m)_{m=1}^{\infty}$ and functions $\mathbf{p} \in \mathbf{L}^2(Q_T)$ and $y \in L^2(0,T;H_0^1(\Omega))$ with $y_t \in L^2(0,T;H^{-1}(\Omega))$ such that

$$\begin{aligned} y_{nl} &\to y \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)),\\ y'_{nl} &\to y' \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)),\\ \mathbf{p}_{ml} &\to \mathbf{p} \quad \text{weakly in } L^2(Q_T). \end{aligned} \tag{A.13}$$

Next, let $w \in C^1([0,T];\operatorname{span}\{w_k\}_{k=1}^{r})$ with $r \leq n$ and let $\mathbf{u} \in C^1([0,T];\operatorname{span}\{\mathbf{u}_k\}_{k=1}^{s})$ with $s \leq m$. Taking w and $\mathbf{u}$ as the test functions in (A.4) and integrating with respect to time, we obtain

$$\begin{cases} \displaystyle\int_0^T [\langle y'_n, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\mathbf{p}_m, \nabla w) + (d\,y_n, w)]\,dt = \int_0^T (f,w)\,dt,\\ \displaystyle\int_0^T [(\nabla y_n, \mathbf{u}) - (c^{-1}\mathbf{p}_m, \mathbf{u})]\,dt = \int_0^T (c^{-1}\mathbf{F}, \mathbf{u})\,dt. \end{cases} \tag{A.14}$$

Taking $n = nl$ and $m = ml$ in the above equalities and passing to the limit, we obtain,

$$\begin{cases} \displaystyle\int_0^T [\langle y', w\rangle + (\mathbf{p}, \nabla w) + (d\,y, w)]\,dt = \int_0^T (f,w)\,dt,\\ \displaystyle\int_0^T [(\nabla y, \mathbf{u}) - (c^{-1}\mathbf{p}, \mathbf{u})]\,dt = \int_0^T (c^{-1}\mathbf{F}, \mathbf{u})\,dt. \end{cases} \tag{A.15}$$

in view of the weak convergence in (A.13). Eventually, by a density property of $C^1([0,T];\operatorname{span}\{w_k\}_{k=1}^{r})$ and $C^1([0,T];\operatorname{span}\{\mathbf{u}_k\}_{k=1}^{s})$ in $L^2(0,T;H_0^1(\Omega))$ and $\mathbf{L}^2(Q_T)$, respectively, it follows from (A.15) that items (i) and (ii) of Definition A.1 hold true for the pair $(y, \mathbf{p})$.

(d) *Initial datum.* We now check that item (iii) of Definition A.1 holds also true. First, since $y \in L^2(0,T;H_0^1(\Omega))$ with $y_t \in L^2(0,T;H^{-1}(\Omega))$, we can deduce that $y \in C^0([0,T];L^2(\Omega))$ (see [13, Theorem 3, p. 303]). Hence, from (A.15), we deduce that

$$\int_0^T [-\langle y, w'\rangle + (\mathbf{p}, \nabla w) + (d\,y, w)]\,dt = \int_0^T (f,w)\,dt + (y(\cdot,0), w(\cdot,0)) \tag{A.16}$$

for all $w \in C^1([0,T];H_0^1(\Omega))$ such that $w(\cdot,T) = 0$.

And from (A.14) we have the same for the sequence (y_{ml}, p_{ml}), namely

$$\int_0^T [-\langle y_{nl}, w'\rangle + (\mathbf{p}_{ml}, \nabla w) + (d\,y_{nl}, w)]\,dt = \int_0^T (f,w)\,dt + (y_{nl}(\cdot,0), w(\cdot,0)).$$

Taking the limit, we obtain

$$\int_0^T [-\langle y, w' \rangle + (\mathbf{p}, \nabla w) + (d\,y, w)]\,dt = \int_0^T (f, w)\,dt + (y_0, w(\cdot\,, 0)). \tag{A.17}$$

Comparing (A.16) and (A.17), we conclude $y(\cdot\,, 0) = y_0$.

(e) *Uniqueness.* The uniqueness is deduced from the energy estimate (A.2). □

Remark A.3. According to Definition A.1, the unique weak solution for (A.1) gives us the weak solution for the standard parabolic equation

$$\begin{cases} y_t - \nabla \cdot (c(x)\nabla y) + d\,y = f - \nabla \cdot \mathbf{F} & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(x, 0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{A.18}$$

Indeed, using the fact that c is symmetric and taking $\mathbf{u} = c\,\nabla w$ for $w \in H_0^1(\Omega)$ and summing the equations in (i) and (ii), we obtain the classical definition of weak solution for (A.18).

Now, let us introduce another concept of solution for (A.1) weaker than the previous one.

Definition A.4. We say that the pair $(y, \mathbf{p}) \in L^2(Q_T) \times \mathbf{L}^2(Q_T)$ is a solution by transposition of (A.1) if and only if the following hold:

(i) We have the identity

$$\iint_{Q_T} (y(x, t), \mathbf{p}(x, t)) \cdot (g(x, t), \mathbf{G}(x, t))\,dx\,dt = \mathbf{M}(g, \mathbf{G}) \quad \text{for all } (g, \mathbf{G}) \in L^2(Q_T) \times \mathbf{L}^2(Q_T),$$

with $\mathbf{M} : L^2(Q_T) \times \mathbf{L}^2(Q_T) \to \mathbb{R}$ given by

$$\mathbf{M}(g, \mathbf{G}) := \iint_{Q_T} f(x, t)\varphi(x, t)\,dx\,dt + (y_0, \varphi(\cdot\,, 0)) + \iint_{Q_T} c^{-1}\mathbf{F}(x, t) \cdot \boldsymbol{\sigma}(x, t)\,dx\,dt,$$

where $(\varphi, \boldsymbol{\sigma})$ is the unique strong solution of

$$\begin{cases} -\varphi_t - \nabla \cdot \boldsymbol{\sigma} + d\,\varphi = g, \quad \nabla\varphi - c^{-1}\boldsymbol{\sigma} = \mathbf{G} & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(x, T) = 0 & \text{in } \Omega. \end{cases} \tag{A.19}$$

(ii) $y(\cdot\,, 0) = y_0$.

Now we can deduce the following existence/uniqueness result.

Proposition A.5. *There exists a unique solution by transposition for* (A.1). *Moreover, there exists a constant* $C > 0$ *such that*

$$\|y\|_{L^2(Q_T)} + \|\mathbf{p}\|_{\mathbf{L}^2(Q_T)} \le C\big(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)} + \|\mathbf{F}\|_{\mathbf{L}^2(Q_T)}\big).$$

Proof. Firstly, notice that $\mathbf{M} : L^2(Q_T) \times \mathbf{L}^2(Q_T) \to \mathbb{R}$ is a linear form. Then, since $(\varphi, \boldsymbol{\sigma})$ is the unique weak solution, we obtain

$$\|\varphi\|^2_{L^2(Q_T)} + \|\varphi(\cdot\,, 0)\|^2_{H_0^1(\Omega)} + \|\boldsymbol{\sigma}\|^2_{\mathbf{L}^2(Q_T)} \le C\|(g, \mathbf{G})\|^2_{L^2(Q_T)\times\mathbf{L}^2(Q_T)}.$$

in view of Proposition A.2. This implies that the linear form $\mathbf{M}$ is continuous. Therefore, by the Riesz representation Theorem, there exists a unique pair

$$(y, \mathbf{p}) \in L^2(Q_T) \times \mathbf{L}^2(Q_T)$$

such that

$$\iint_{Q_T} (y(x, t), \mathbf{p}(x, t)) \cdot (g(x, t), \mathbf{G}(x, t))\,dx\,dt = \mathbf{M}(g, \mathbf{G}) \quad \text{for all } (g, \mathbf{G}) \in L^2(Q_T) \times \mathbf{L}^2(Q_T).$$

Furthermore,

$$\|(y, \mathbf{p})\|_{L^2(Q_T\times \mathbf{L}^2(Q_T)} = \|\mathbf{M}\|_{(L^2(Q_T\times \mathbf{L}^2(Q_T))'}.$$

The uniqueness is obtained by *Du Bois–Reymond's Lemma*. □

Remark A.6. According to Definition A.4, the solution by transposition for (A.1) gives us the unique solution by transposition for the standard parabolic equation (A.18). Indeed, taking $\mathbf{G} = \mathbf{0}$ we have that $\boldsymbol{\sigma} = c\nabla\varphi$ in $\mathbf{L}^2(Q_T)$ and then

$$\iint_{Q_T} y(x,t)g(x,t)\,dx\,dt = \iint_{Q_T} f(x,t)\varphi(x,t)\,dx\,dt + (y_0, \varphi(\,\cdot\,,0)) + \iint_{Q_T} \mathbf{F}(x,t)\cdot\nabla\varphi(x,t)\,dx\,dt$$

for any $g \in L^2(Q_T)$, where, recalling Remark A.3, the function φ is the associated solution for

$$\begin{cases} -\varphi_t - \nabla\cdot(c(x)\nabla\varphi) + d\,\varphi = g, & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Then we can use a similar argument from Proposition A.2 and a regularization of the initial datum to deduce item (ii). This is in fact the classical definition of solution by transposition for equation (A.18).

Remark A.7. The concept of solution by transposition appears due to the low regularity on the data. In fact, this kind of solution can be viewed as a generalization of weak solution, in the sense that every weak solution is a solution by transposition. This way, we can see that if $(y, \mathbf{p})$ is a weak solution, in agreement with Definition A.1, then $(y, \mathbf{p})$ is a solution by transposition, in agreement with Definition A.4. Indeed, taking $(w, \mathbf{u}) = (\varphi, \boldsymbol{\sigma})$ in the weak formulation for (A.1) and taking $(w, \mathbf{u}) = (y, \mathbf{p})$ in the weak formulation for (A.19), we can deduce that $(y, \mathbf{u})$ is a solution by transposition in agreement with Definition A.4.

Acknowledgment: This work has been partially done while the second author was visiting the Blaise Pascal University (Clermont-Ferrand, France).

Funding: The second author was partially supported by grant MTM2013-41286-P (DGI-MICINN, Spain) and ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7) and by the ERC advanced grant 668998 (OCLOC) under the EU's H2020 research program.

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