On the boundary controllability of incompressible Euler fluids with Boussinesq heat effects

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Abstract

This paper deals with the boundary controllability of inviscid incompressible fluids for which thermal effects are important and are modeled through the Boussinesq approximation. Almost all our results deal with zero heat diffusion. By adapting and extending some ideas from Coron and Glass, we establish the simultaneous global exact controllability of the velocity field and the temperature for 2D and 3D flows.

1 Introduction

Let Ω be a nonempty bounded open subset of $\mathbb{R}^N$ of class C^∞ ($N = 2$ or $N = 3$). We assume that Ω is connected and (for simplicity) simply connected. Let Γ_0 be a nonempty open subset of the boundary $\Gamma = \partial\Omega$. Bold letters will denote vector-valued functions; for instance, the vector function $\mathbf{v} \in \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^N)$ will be of the form $\mathbf{v} = (v_1, \dots, v_N)$, where $v_1, \dots, v_N \in C^0(\overline{\Omega}; \mathbb{R})$. Let us denote by $\mathbf{n}(\mathbf{x})$ the outward unit normal vector to Ω at any point $\mathbf{x} \in \Gamma$.

In this work, we are concerned with the boundary controllability of the system:

$$
\begin{cases}
\mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{in} \quad \Omega \times (0,T), \\
\nabla \cdot \mathbf{y} = 0 & \text{in} \quad \Omega \times (0,T), \\
\theta_t + \mathbf{y} \cdot \nabla\theta = \kappa\, \Delta\theta & \text{in} \quad \Omega \times (0,T), \\
\mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega,
\end{cases}
\tag{1}
$$

where:

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- $\mathbf{y}$ and the scalar function p stand for the velocity field and the pressure of an inviscid incompressible fluid in $\Omega \times (0,T)$.
- The function θ provides the temperature distribution of the fluid.
- $\vec{\mathbf{k}}\theta$ can be viewed as the *buoyancy force* density ($\vec{\mathbf{k}}$ is a nonzero vector of $\mathbb{R}^N$).
- $\kappa \geq 0$ is the heat diffusion coefficient.

When $\kappa = 0$, (1) is the *incompressible inviscid Boussinesq* system. On the other hand, when $\kappa > 0$, (1) is called the *incompressible, heat conductive, inviscid Boussinesq* system.

For now on, we assume that $\alpha \in (0,1)$ and we set

$$\begin{aligned} &\mathbf{C}_0^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) := \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) \,:\, \nabla\cdot\mathbf{u} = 0 \text{ in } \overline{\Omega} \text{ and } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}, \\ &\mathbf{C}(m,\alpha,\Gamma_0) := \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) \,:\, \nabla\cdot\mathbf{u} = 0 \text{ in } \overline{\Omega} \text{ and } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\backslash\Gamma_0 \,\}, \end{aligned} \tag{2}$$

where $\mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N)$ denotes the space of functions in $\mathbf{C}^m(\overline{\Omega};\mathbb{R}^N)$ whose m-th order derivatives are Hölder-continuous with exponent α.

When $\kappa = 0$, the exact boundary controllability problem for (1) can be stated as follows:

Given $\mathbf{y}_0,\, \mathbf{y}_1 \in \mathbf{C}(1,\alpha,\Gamma_0)$ *and* $\theta_0,\, \theta_1 \in C^{1,\alpha}(\overline{\Omega};\mathbb{R})$*, find* $\mathbf{y} \in C^0([0,T];\mathbf{C}(1,\alpha,\Gamma_0))$, $\theta \in C^0([0,T];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$ *and* $p \in \mathscr{D}'(\Omega\times(0,T))$ *such that* (1) *holds and*

$$\mathbf{y}(\mathbf{x},T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x},T) = \theta_1(\mathbf{x}) \quad in \quad \Omega. \tag{3}$$

If it is always possible to find $\mathbf{y}$, θ and p, we say that the incompressible inviscid Boussinesq system is *exactly controllable* for (Ω,Γ_0) at time T.

Remark 1 Let us first consider the case $\kappa = 0$. In order to determine without ambiguity a unique local in time regular solution to (1), it is sufficient to prescribe the normal component of the velocity vector on the boundary of the flow region and the full field $\mathbf{y}$ and the temperature θ only on the inflow section, i.e. only where $\mathbf{y}\cdot\mathbf{n} < 0$, see for instance [**?**]. Hence, in (1), we can assume that the controls are given by:

$$\begin{cases} \mathbf{y}\cdot\mathbf{n} \text{ on } \Gamma_0\times(0,T), \text{ with } \displaystyle\int_{\Gamma_0} \mathbf{y}\cdot\mathbf{n}\, d\Gamma = 0; \\ \mathbf{y} \text{ at any point of } \Gamma_0\times(0,T]) \text{ satisfying } \mathbf{y}\cdot\mathbf{n} < 0; \\ \theta \text{ at any point of } \Gamma_0\times(0,T) \text{ satisfying } \mathbf{y}\cdot\mathbf{n} < 0. \end{cases}$$

□

The meaning of the exact controllability property is that, when it holds, we can drive exactly the fluid, acting only on an arbitrary small part Γ_0 of the boundary during an arbitrary small time interval $(0,T)$, from any initial state $(\mathbf{y}_0,\theta_0)$ to any final state $(\mathbf{y}_1,\theta_1)$.

In the case $\kappa > 0$, the situation is a little bit different. Due to the *regularization effect* of the temperature equation we can not expect the exact controllability, at least for the temperature.

To propose a boundary controllability problem, let us consider Γ_1 a nonempty open subset of the boundary Γ. Then, the boundary controllability problem is the following:

Given $\mathbf{y}_0$, $\mathbf{y}_1 \in \mathbf{C}_0^{1,\alpha}(\overline{\Omega};\mathbb{R}^N)$ *and* $\theta_0 \in C^{1,\alpha}(\overline{\Omega};\mathbb{R})$*, with* $\theta_0 = 0$ *on* $\Gamma\backslash\Gamma_1$*, find* $\mathbf{y} \in C^0([0,T];\mathbf{C}_0^{1,\alpha}(\overline{\Omega};\mathbb{R}^N))$*,* $\theta \in C^0([0,T];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$*, with* $\theta = 0$ *on* $(\Gamma\backslash\Gamma_1) \times (0,T)$*, and* $p \in \mathscr{D}'(\Omega \times (0,T))$ *such that* (1) *holds and*

$$\mathbf{y}(\mathbf{x},T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x},T) = 0 \quad in \quad \Omega. \tag{4}$$

If it is always possible to find $\mathbf{y}$, θ and p, we say that the incompressible, heat conductive, inviscid Boussinesq system is *exact/null controllable* for $(\Omega, \Gamma_0, \Gamma_1)$.

Remark 2 Now, let us suppose that $\kappa > 0$. Then, there exists at most one solution to (1) if we provide the same boundary data for $\mathbf{y}$ and (for example) Dirichlet data for θ of the form

$$\theta = \theta_* 1_{\Gamma_1} \quad \text{on} \quad \Gamma \times (0,T).$$

Therefore, it can be assumed in this case that the controls are

$$\begin{cases} \mathbf{y}\cdot\mathbf{n} \text{ on } \Gamma_0 \times (0,T), \text{ with } \displaystyle\int_{\Gamma_0} \mathbf{y}\cdot\mathbf{n}\, d\Gamma = 0; \\ \mathbf{y} \text{ at any point of } \Gamma_0 \times (0,T]) \text{ satisfying } \mathbf{y}\cdot\mathbf{n} < 0; \\ \theta \text{ at any point of } \Gamma_1 \times (0,T). \end{cases}$$

□

The meaning of the exact/null controllability property is that, when it holds, we can drive exactly the fluid and the temperature of it, acting only on an arbitrary small parts Γ_0 and Γ_1 of the boundary during an arbitrary small time interval $(0,T)$, from any initial state $(\mathbf{y}_0, \theta_0)$ to any final state $(\mathbf{y}_1, 0)$.

The Boussinesq system is potentially relevant to the study of atmospheric and oceanographic turbulence, as well as to other astrophysical situations where rotation and stratification play a dominant role (see e.g. [**?**]). In fluid mechanics, (1) is used in the field of buoyancy-driven flow. It describes the motion of an incompressible inviscid fluid subject to convective heat transfer under the influence of gravitational forces, see [**?**].

The controllability of systems governed by (linear and nonlinear) PDEs has focused the attention of a lot of researchers the last decades. Some related results can be found in [**?**, **?**, **?**, **?**]. In the context of incompressible ideal fluids, this subject has been mainly investigated by Coron [**?**, **?**] and Glass [**?**, **?**, **?**].

In this paper, we are going to adapt the techniques and arguments of [**?**] and [**?**] to the situations modelled by (1).

The first main result is the following:

Theorem 1 *The incompressible inviscid Boussinesq system* (1) *($\kappa = 0$) is exactly controllable for* (Ω, Γ_0) *at any time* $T > 0$.

The proof of Theorem 1 relies on the extension and return methods. These have been applied in several different contexts to establish controllability; see the seminal work [**?**] and the contributions [**?**, **?**, **?**, **?**].

Let us give a sketch of the strategy:

- First, we construct a "good" trajectory connecting $\mathbf{0}$ to $\mathbf{0}$ (see sections 2.1 and 2.2).
- We apply the extension method of David L. Russell [**?**].
- We use a Fixed-Point Theorem to obtain a local exact controllability result.
- Finally, we use an appropriate scaling argument to deduce the desired global result.

In fact, Theorem 1 is a iconsequence of the following result:

Proposition 1 *Let us assume that* $\kappa = 0$. *There exists* $\delta > 0$ *such that, for any* $\theta_0 \in C^{1,\alpha}(\overline{\Omega};\mathbb{R})$ *and* $\mathbf{y}_0 \in \mathbf{C}(1,\alpha,\Gamma_0)$ *with*

$$\max\{\|\mathbf{y}_0\|_{1,\alpha}, \|\theta_0\|_{1,\alpha}\} < \delta,$$

there exist $\mathbf{y} \in C^0([0,1];\mathbf{C}(1,\alpha,\Gamma_0))$, $\theta \in C^0([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$ *and* $p \in \mathscr{D}'(\overline{\Omega} \times [0,1])$ *satisfying* (1) *in* $\Omega \times (0,1)$ *and*

$$\mathbf{y}(\mathbf{x},1) = \mathbf{0}, \;\; \theta(\mathbf{x},1) = 0 \text{ in } \Omega. \tag{5}$$

Our second main result is the following:

Theorem 2 *The incompressible, heat conductive, inviscid Boussinesq system* (1) *($\kappa > 0$) is partially local exact/null controllable for* $(\Omega, \Gamma_0, \Gamma_1)$ *at any time* $T > 0$. *More precisely, if* $T > 0$, $\mathbf{y}_0$, $\mathbf{y}_1 \in \mathbf{C}^{1,\alpha}_0(\overline{\Omega};\mathbb{R}^N)$ *are given, there exists* $\eta > 0$, *depending only on* $\mathbf{y}_0$, *such that for each* $\theta_0 \in C^{1,\alpha}(\overline{\Omega};\mathbb{R})$, *with* $\theta_0 = 0$ *on* $\Gamma\backslash\Gamma_1$ *and* $\|\theta_0\|_{C^{1,\alpha}} \leq \eta$, *we can find* $\mathbf{y} \in C^0([0,T];\mathbf{C}(1,\alpha,\Gamma_0))$, $\theta \in C^0([0,T];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$, *with* $\theta = 0$ *on* $(\Gamma\backslash\Gamma_1) \times (0,T)$, *and* $p \in \mathscr{D}'(\Omega \times (0,T))$ *satisfying* (1) *and* (4).

The rest of this paper is organized as follows. In Section 2, we recall the results needed to prove Theorems 1 and 2. In Section 3, we give the proof of Theorem 1. In Section 4, we prove Proposition 1 in the 2D case. It will be seen that the main ingredients of the proof are the construction of a nontrivial trajectory that starts and ends at $\mathbf{0}$ and a Fixed-Point Theorem (the key ideas of the return method). In Section 5, we give the proof of Theorem 1 in the 3D case. Section 6 contains the proof of Theorem 2. Finally, in Section **??**, we present some additional comments and open questions.

2 Preliminary results

In this section, we are going to recall some results used in the proofs of the main results. Also, we are going to indicate how to construct an appropriate trajectory in order to apply the return method.

The first one is an immediate consequence of the Banach Fixed-Point Theorem:

Theorem 3 *Let* $(B_1, \|\cdot\|_1)$ *and* $(B_2, \|\cdot\|_2)$ *be Banach spaces with* B_2 *continuously embedded in* B_1*. Let* B *be a subset of* B_2 *and let* $G : B \mapsto B$ *be a mapping such that*

$$\|G(u) - G(v)\|_1 \leq \gamma \|u - v\|_1 \quad \forall\ u, v \in B, \ \textit{for some}\ \gamma \in [0, 1).$$

Let us denote by $\tilde{B}$ *the closure of* B *for the norm* $\|\cdot\|_1$*. Then,* G *can be uniquely extended to a continuous mapping* $\tilde{G} : \tilde{B} \mapsto \tilde{B}$ *that possesses a unique fixed-point in* $\tilde{B}$*.*

The following lemma will be very important to deduce later appropriate estimates. The proof can be found in [**?**].

Lemma 1 *Let* m *be a nonnegative integer. Assume that* $u \in C^0([0,T]; C^{m+1,\alpha}(\overline{\Omega}; \mathbb{R}))$, $g \in C^0([0,T]; C^{m,\alpha}(\overline{\Omega}; \mathbb{R}))$ *and* $\mathbf{v} \in C^0([0,T]; \mathbf{C}^{m,\alpha}(\overline{\Omega}; \mathbb{R}^N))$ *are given, with* $\mathbf{v} \cdot \mathbf{n} = 0$ *on* $\Gamma \times (0,T)$ *and*

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = g \quad \textit{in} \quad \Omega \times (0,T). \tag{6}$$

Then,

$$\frac{d}{dt^+} \|u(\cdot,t)\|_{m,\alpha} \leq \|g(\cdot,t)\|_{m,\alpha} + K \|\mathbf{v}(\cdot,t)\|_{m,\alpha} \|u(\cdot,t)\|_{m,\alpha} \quad \textit{in} \quad (0,T), \tag{7}$$

where K *is a constant only depending on* α *and* m*.*

In the following sections, we will frequently use a technical lemma whose proof can be found in [**?**]:

Lemma 2 *Let us assume that*

$$\mathbf{w}_0 \in \mathbf{C}^{1,\alpha}(\bar{\Omega}; \mathbb{R}^N), \quad \nabla \cdot \mathbf{w}_0 = 0 \quad \textit{in} \quad \Omega,$$

$$\mathbf{u} \in C^0([0,T]; \mathbf{C}^{1,\alpha}(\bar{\Omega}; \mathbb{R}^N)), \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \textit{on} \quad \Gamma \times (0,T),$$

$$\mathbf{g} \in C^0([0,T]; \mathbf{C}^{0,\alpha}(\bar{\Omega}, \mathbb{R}^N)), \quad \nabla \cdot \mathbf{g} = 0 \quad \textit{in} \quad \Omega \times (0,T).$$

Let $\mathbf{w}$ *be a function in* $C^0([0,T]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^N))$ *satisfying*

$$\begin{cases} \mathbf{w}_t + (\mathbf{u} \cdot \nabla)\mathbf{w} = (\mathbf{w} \cdot \nabla)\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{w} + \mathbf{g} & \textit{in} \quad \Omega \times (0,T), \\ \mathbf{w}(\cdot, 0) = \mathbf{w}_0 & \textit{in} \quad \Omega. \end{cases}$$

Then

$$\nabla \cdot \mathbf{w} = 0 \quad \textit{in} \quad \Omega \times (0,T).$$

Moreover, there exists $\mathbf{v} \in C^0([0,T]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^N))$ *such that*

$$\mathbf{w} = \nabla \times \mathbf{v} \quad \textit{in} \quad \Omega \times (0,T).$$

Now, we will introduce a mollifier and we will state a result that will be useful in Sections 4 and 5. Let $\rho \in C^\infty(\mathbb{R}^N;\mathbb{R})$ be given by

$$\rho(\mathbf{x}) := \begin{cases} e^{\frac{|\mathbf{x}|^2}{1-|\mathbf{x}|^2}} & \text{if } |\mathbf{x}| < 1, \\ 0 & \text{if } |\mathbf{x}| \geq 1 \end{cases}$$

and let the $\rho_m \in C^\infty(\mathbb{R}^N;\mathbb{R})$ be defined as follows:

$$\rho_m(\mathbf{x}) := m^2 \rho(m\mathbf{x}).$$

Lemma 3 *For each $\mathbf{v}_0 \in \mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^i)$, let us set*

$$\mathbf{v}_0^m := (\rho_m * \pi(\mathbf{v}_0))\,\big|_{\overline{\Omega}},$$

where π is any linear and continuous extension operator that preserve regularity. Then $\mathbf{v}_0^m \to \mathbf{v}_0$ in $\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^i)$ as $m \to +\infty$ and

$$\|\mathbf{v}_0^m\|_{2,\alpha} \leq C\|\mathbf{v}_0\|_{2,\alpha} \quad \|\mathbf{v}_0^m\|_{3,\alpha} \leq mC\|\mathbf{v}_0\|_{2,\alpha}.$$

To end this Subsection, we will now recall a result dealing with the null controllability of general parabolic linear systems of the form

$$\begin{cases} u_t - \Delta u + \mathbf{w} \cdot \nabla u = v 1_{\boldsymbol{\omega}} & \text{in} \quad \Omega \times (0,T), \\ u = 0 & \text{on} \quad \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{8}$$

where $u_0 \in L^2(\Omega)$, $\mathbf{w} \in L^\infty(\Omega \times (0,T))$ and $v \in L^2(\boldsymbol{\omega} \times (0,T))$.

It is well known that there exists exactly one solution u to (8), with

$$u \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)).$$

Related to controllability result, we have the following:

Theorem 4 *The linear system (8) is null controllable at any time $T > 0$. In other words, for each $y_0 \in L^2(\Omega)$ there exists $v \in L^2(\boldsymbol{\omega} \times (0,T))$ such that the associated solution to (8) satisfies $u(\cdot,T) = 0$ in Ω. Furthermore, the extremal problem*

$$\begin{cases} \textit{Minimize} \ \ \dfrac{1}{2} \displaystyle\iint_{\boldsymbol{\omega}\times(0,T)} |v|^2\,dx\,dt \\ \textit{Subject to: } v \in L^2(\boldsymbol{\omega} \times (0,T)),\ (8),\ \ u(\cdot,T) = 0 \ \textit{in}\ \Omega \end{cases} \tag{9}$$

possesses exactly one solution $\hat{v}$ satisfying

$$\|\hat{v}\|_2 \leq C_0 \|y_0\|_2, \tag{10}$$

where

$$C_0 = e^{C_1(1+1/T+(1+T)\|A\|_\infty^2)}$$

and C_1 only depends on a, b and L.

2.1 Construction of a trajectory when $N = 2$

Our trajectory will be associated to a domain $\Omega \subset \mathbb{R}^2$. To do this, we will argue as in [?]. Thus, let Ω_1 be a bounded, Lipschitz contractible subset of $\mathbb{R}^2$ whose boundary consists of two disjoint closed line segments Γ^- and Γ^+ and two disjoint closed curves Σ' and Σ'' of class C^∞ such that $\partial\Sigma' \cup \partial\Sigma'' = \partial\Gamma^- \cup \partial\Gamma^+$. We also impose that there is a neighborhood U^- of Γ^- (resp. U^+ of Γ^+) such that $\Omega_1 \cap U^-$ (resp. $\Omega_1 \cap U^+$) coincides with the intersection of U^- (resp. U^+), an open semi-plane limited by the line containing Γ^- (resp. Γ^+) and the band limited by the two straight lines orthogonal to Γ^- (resp. Γ^+) and passing through $\partial\Gamma^-$ (resp. $\partial\Gamma^+$); see Figure 1.

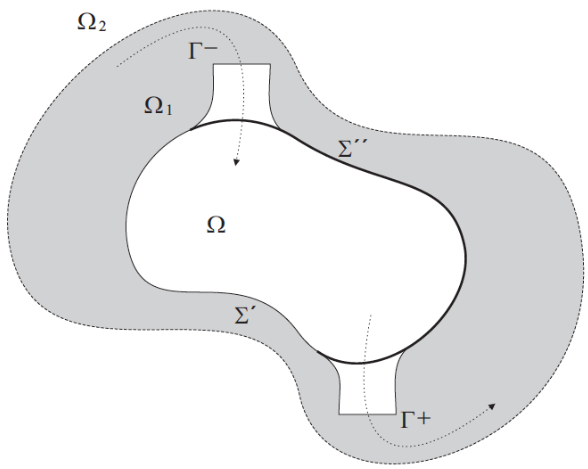

Figure 1: The domain Ω_1

Let φ be the solution to

$$\begin{cases} \Delta\varphi = 0 & \text{in} \quad \Omega_1, \\ \varphi = 1 & \text{on} \quad \Gamma^+, \\ \varphi = -1 & \text{on} \quad \Gamma^-, \\ \dfrac{\partial\varphi}{\partial\mathbf{n}_1} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{11}$$

where $\mathbf{n}_1$ is the outward unit normal vector field on $\partial\Omega_1$ and $\Sigma = \Sigma' \cup \Sigma''$. Then we have the following result from J.-M. Coron [?]:

Lemma 4 *One has $\varphi \in C^\infty(\bar{\Omega}_1; \mathbb{R})$, $-1 < \varphi(\mathbf{x}) < 1$ for all $\mathbf{x} \in \Omega_1$ and*

$$\nabla\varphi(\mathbf{x}) \neq \mathbf{0} \quad \forall \mathbf{x} \in \overline{\Omega}_1. \tag{12}$$

Let $\gamma \in C^\infty([0,1]; \mathbb{R}^+)$ be a non-zero function such that *supp* $\gamma \subset (0, 1/2) \cup (1/2, 1)$ and the subsets *supp* $\gamma \cap (0, 1/2)$ and *supp* $\gamma \cap (1/2, 1)$ are non-empty.

Let $M > 0$ (it will be chosen below) and set $\overline{\mathbf{y}}(\mathbf{x}, t) := M\gamma(t)\nabla\varphi(\mathbf{x})$, $\overline{p}(\mathbf{x}, t) := -M\gamma_t(t)\varphi(\mathbf{x}) - \frac{M^2}{2}\gamma(t)^2|\nabla\varphi(\mathbf{x})|^2$ and $\overline{\theta} \equiv 0$. Then, (1) is satisfied for $\mathbf{y} = \overline{\mathbf{y}}$, $p = \overline{p}$, $\theta =$

$\overline{\theta}$, $T = 1$, $\mathbf{y}_0 = \mathbf{0}$ and $\theta_0 = 0$. The solution $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta})$ is thus a trajectory of (1) that connects the zero state to itself.

Let Ω_3 be a bounded open set of class C^∞ such that $\Omega_1 \subset\subset \Omega_3$. We extend φ to $\overline{\Omega}_3$ as a C^∞ function with compact support in Ω_3 and we still denote this extension by φ. Let us introduce $\mathbf{y}^*(\mathbf{x}, t) := M\gamma(t)\nabla\varphi(\mathbf{x})$ (observe that $\overline{\mathbf{y}}$ is the restriction of $\mathbf{y}^*$ to $\overline{\Omega} \times [0, 1]$). Also, we consider the associated *flux function* $\mathbf{Y}^* : \overline{\Omega}_3 \times [0, 1] \times [0, 1] \mapsto \overline{\Omega}_3$, defined as follows:

$$\begin{cases} \mathbf{Y}^*_t(\mathbf{x}, t, s) = \mathbf{y}^*(\mathbf{Y}^*(\mathbf{x}, t, s), t) \\ \mathbf{Y}^*(\mathbf{x}, s, s) = \mathbf{x}. \end{cases} \tag{13}$$

The mapping $\mathbf{Y}^*$ contains all the information on the trajectories of the particles transported by the velocity field $\mathbf{y}^*$. It is customary to label the particles with the positions $\mathbf{x}$ at the beginning of observation ($t = s$). This system of ODEs express the fact that the particles travel with velocity $\mathbf{y}^*$. The flux $\mathbf{Y}^*$ is of class C^∞ in $\overline{\Omega}_3 \times [0, 1] \times [0, 1]$. Furthermore, $\mathbf{Y}^*(\cdot, t, s)$ is a diffeomorphism of $\overline{\Omega}_3$ onto itself for each $s, t \in [0, 1]$. Finally,

$$(\mathbf{Y}^*(\cdot, t, s))^{-1} = \mathbf{Y}^*(\cdot, s, t),\ \forall s, t \in [0, 1].$$

Remark 3 From the definition of $\mathbf{y}^*$ and the boundary conditions on Ω_1 satisfied by φ, we observe that:

- The condition $\mathbf{y}^* \cdot \mathbf{n}_1 = 0$ on Σ means that the particles cannot cross Σ;
- Since φ is constant on Γ^+, the gradient $\nabla\varphi$ is parallel to the normal vector on Γ^+. As φ attains a maximum at any point of Γ^+, we have $\nabla\varphi \cdot \mathbf{n}_1 > 0$ on Γ^+, whence, $\mathbf{y}^* \cdot \mathbf{n}_1 \geq 0$ on $\Gamma^+ \times [0, 1]$. Similary, $\mathbf{y}^* \cdot \mathbf{n} \leq 0$ on $\Gamma^- \times [0, 1]$.

Consequently, the particles moving with velocity field $\mathbf{y}^*$ can leave Ω_1 only through Γ^+ and can enter Ω_1 only through Γ^-. □

The following lemma shows that the particles that travel with velocity $\mathbf{y}^*$ and are inside $\overline{\Omega}_1$ at time $t = 0$ (resp. $t = 1/2$) will be outside $\overline{\Omega}_1$ at time $t = 1/2$ (resp. $t = 1$).

Lemma 5 *There exist $M > 0$ (large enough) and a bounded open subset Ω_2 satisfying $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ and*

$$\begin{aligned} \mathbf{x} \in \overline{\Omega}_2 \quad &\Longrightarrow \quad \mathbf{Y}^*(\mathbf{x}, 1/2, 0) \notin \overline{\Omega}_2, \\ \mathbf{x} \in \overline{\Omega}_2 \quad &\Longrightarrow \quad \mathbf{Y}^*(\mathbf{x}, 1, 1/2) \notin \overline{\Omega}_2. \end{aligned} \tag{14}$$

The proof is given in [**?**] and relies on the properties of $\mathbf{y}^*$ and, more precisely, on the fact that $t \mapsto \varphi(\mathbf{Y}^*(\mathbf{x}, t, s))$ is nondecreasing.

The next step is to introduce appropriate extension mappings from Ω to Ω_3. More precisely, we have the following result from [**?**]:

Lemma 6 *For $i=1,2$, there exists $\pi_i : \mathbf{C}^0(\overline{\Omega};\mathbb{R}^i) \mapsto \mathbf{C}^0(\overline{\Omega}_3;\mathbb{R}^i)$, linear and continuous, such that*

$$\begin{cases} \pi_i(\mathbf{f}) = \mathbf{f} \quad \textit{in} \quad \overline{\Omega}, \ \forall \mathbf{f} \in \mathbf{C}^0(\overline{\Omega};\mathbb{R}^i), \\ \textit{supp } \pi_i(\mathbf{f}) \subset \Omega_2 \quad \forall \mathbf{f} \in \mathbf{C}^0(\overline{\Omega};\mathbb{R}^i), \\ \pi_i \textit{ maps continuously } \mathbf{C}^{n,\lambda}(\overline{\Omega};\mathbb{R}^i) \textit{ into } \mathbf{C}^{n,\lambda}(\overline{\Omega}_3;\mathbb{R}^i) \quad \forall n \geq 0, \quad \forall \lambda \in (0,1). \end{cases} \tag{15}$$

The next lemma says that (14) holds not only for $\mathbf{y}^*$ but also for any appropriate extension of any flow $\mathbf{z}$ sufficiently close to $\overline{\mathbf{y}}$:

Lemma 7 *For each $\mathbf{z} \in C^0([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R}^2))$, let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_2(\mathbf{z}-\overline{\mathbf{y}})$. There exists $\nu > 0$ such that, if $\|\mathbf{z}-\overline{\mathbf{y}}\|_0 \leq \nu$ and $\mathbf{Z}^*$ is the flux function associated to $\mathbf{z}^*$, then*

$$\begin{array}{lcl} \mathbf{x} \in \bar{\Omega}_2 & \Longrightarrow & \mathbf{Z}^*(\mathbf{x},1/2,0) \notin \overline{\Omega}_2 \\ \mathbf{x} \in \bar{\Omega}_2 & \Longrightarrow & \mathbf{Z}^*(\mathbf{x},1,1/2) \notin \overline{\Omega}_2. \end{array} \tag{16}$$

Proof: Let us set

$$\mathbf{F} = \left\{ \mathbf{Y}^*(\mathbf{x},1/2,0) \,:\, \mathbf{x} \in \overline{\Omega}_2 \right\} \cup \left\{ \mathbf{Y}^*(\mathbf{x},1,1/2) \,:\, \mathbf{x} \in \overline{\Omega}_2 \right\}.$$

Since $\mathbf{F}$ and $\overline{\Omega}_2$ are compact subsets of $\mathbb{R}^2$ and, in view of Lemma 5, $\mathbf{F} \cap \overline{\Omega}_2 = \emptyset$, we have $d := dist\left(\mathbf{F},\ \overline{\Omega}_2\right) > 0$.

Let us introduce $\mathbf{W} := \mathbf{Y}^* - \mathbf{Z}^*$. Then, in view of the Mean Value Theorem and the properties of π_2, we have:

$$\begin{aligned} |\mathbf{W}(\mathbf{x},t,s)| &\leq M \int_s^t \gamma(\sigma) |\nabla\varphi(\mathbf{Y}^*(\mathbf{x},\sigma,s)) - \nabla\varphi(\mathbf{Z}^*(\mathbf{x},\sigma,s))|\, d\sigma \\ &\quad + \int_s^t |\pi_2(\mathbf{z}-\overline{\mathbf{y}})(\mathbf{Z}^*(\mathbf{x},\sigma,s),\sigma)|\, d\sigma \\ &\leq M\|\nabla\varphi\|_0 \int_s^t \gamma(\sigma)|\mathbf{W}(\mathbf{x},\sigma,s)|\, d\sigma + \int_s^t \|\pi_2(\mathbf{z}-\overline{\mathbf{y}})\|_0(\sigma) d\sigma \\ &\leq M\|\nabla\varphi\|_0 \int_s^t \gamma(\sigma)|\mathbf{W}(\mathbf{x},\sigma,s)|\, d\sigma + C\int_s^t \|\mathbf{z}-\overline{\mathbf{y}}\|_0(\sigma) d\sigma, \end{aligned}$$

where $(\mathbf{x},t,s) \in \overline{\Omega}_3 \times [0,1] \times [0,1]$.

Hence, from Gronwall's Lemma, we see that

$$\begin{aligned} |\mathbf{W}(\mathbf{x},t,s)| &\leq C\left(\int_s^t \|\mathbf{z}-\overline{\mathbf{y}}\|_0(\sigma) d\sigma\right) \exp\left(M\|\nabla\varphi\|_0 \int_s^t \gamma(\sigma)\, d\sigma\right) \\ &\leq C e^{M\|\nabla\varphi\|_0 \|\gamma\|_0} \|\mathbf{z}-\overline{\mathbf{y}}\|_0 \end{aligned}$$

and, therefore, there exists $\nu > 0$ such that, if $\|\mathbf{z}-\overline{\mathbf{y}}\|_0 \leq \nu$, then

$$|\mathbf{W}(\mathbf{x},t,s)| \leq \frac{d}{2}, \quad \forall (\mathbf{x},t,s) \in \overline{\Omega}_3 \times [0,1] \times [0,1]. \tag{17}$$

Thanks to Lemma 5 and (17), we necessarily have (16) and the proof is done. □

2.2 Construction of a trajectory when $N = 3$

In this section, we will follow [?]. As in the rwo-dimensional case, $\overline{\mathbf{y}}$ will be of the potential form "$\nabla\varphi$", with the property that any particle travelling with velocity $\overline{\mathbf{y}}$ must leave $\overline{\Omega}$ at an appropriate time. The main difference is that, in this three-dimensional case, "$\nabla\varphi$" can no longer be chosen independent of t.

We first recall a lemma:

Lemma 8 *Let G be a regular bounded open set, with $G \supset \overline{\Omega}$. For each $\mathbf{a} \in \overline{\Omega}$, there exists $\phi^{\mathbf{a}} \in C^\infty(\overline{G} \times [0,1]; \mathbb{R})$ such that supp $\phi^{\mathbf{a}} \subset G \times (1/4, 3/4)$,*

$$\begin{cases} \Delta\phi^{\mathbf{a}} = 0 \quad \text{in} \quad \overline{\Omega} \times [0,1], \\ \dfrac{\partial \phi^{\mathbf{a}}}{\partial \eta} = 0 \quad \text{on} \quad (\Gamma \setminus \Gamma_0) \times [0,1], \end{cases} \tag{18}$$

and

$$\mathbf{X}^{\mathbf{a}}(\mathbf{a}, 1, 0) \in G \setminus \overline{\Omega},$$

where $\mathbf{X}^{\mathbf{a}} = \mathbf{X}^{\mathbf{a}}(\mathbf{x}, t, s)$ is the flux associated to $\nabla\phi^{\mathbf{a}}$, that is, the unique function that satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{X}^{\mathbf{a}}(\mathbf{x}, t, s) = \nabla\Phi^{\mathbf{a}}(\mathbf{X}^{\mathbf{a}}(\mathbf{x}, t, s), t), \\ \mathbf{X}^{\mathbf{a}}(\mathbf{x}, s, s) = \mathbf{x}. \end{cases} \tag{19}$$

The proof is given in [?].

With the help of these $\mathbf{X}^{\mathbf{a}}$, we can construct a vector field $\mathbf{y}^*$ in $G \times (0,1)$ that drives the particles out of Ω and then makes then come back the same way.

Indeed, from the continuity of the functions $\mathbf{X}^{\mathbf{a}}$ and the compactness of $\overline{\Omega}$, we can find $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ in $\bar{\Omega}$, real numbers $r_1, \dots, r_k$, smooth functions $\Phi^1 := \phi^{\mathbf{a}_1}, \dots, \Phi^k := \phi^{\mathbf{a}_k}$ satisfying Lemma 8 and a bounded open set G_0 with $\Omega \subset\subset G_0 \subset\subset G$, such that

$$\overline{\Omega} \subset \bigcup_{i=1}^{k} B^i \subset\subset G_0 \quad \text{and} \quad \mathbf{X}^i(\overline{B^i}, 1, 0) \subset G \setminus \bar{G}_0, \tag{20}$$

where $B^i := B(\mathbf{a}_i; r_i)$ and $\mathbf{X}^i := \mathbf{X}^{\mathbf{a}_i}$ for $i = 1, \dots, k$.

As in [?], the definition of $\mathbf{y}^*$ is as follows: let the t_i be given by

$$\begin{aligned} t_i &= \frac{1}{4} + \frac{i}{4k}, \quad i = 0, \dots, 2k, \\ t_{i+\frac{1}{2}} &= \frac{1}{4} + (i + \frac{1}{2})\frac{1}{4k}, \quad i = 0, \dots, 2k-1 \end{aligned} \tag{21}$$

and let us set

$$\Phi(\mathbf{x}, t) = \begin{cases} 0, & (\mathbf{x}, t) \in \bar{G} \times ([0, 1/4] \times [3/4, 1]), \\ 8k\Phi^j(\mathbf{x}, 8k(t - t_{j-1})), & (\mathbf{x}, t) \in \bar{G} \times \left[t_{j-1}, t_{j-\frac{1}{2}}\right], \\ -8k\Phi^j(\mathbf{x}, 8k(t_j - t)), & (\mathbf{x}, t) \in \bar{G} \times \left[t_{j-\frac{1}{2}}, t_j\right] \end{cases} \tag{22}$$

for $j = 1, \dots, 2k$; then, we set $\mathbf{y}^* = \nabla\Phi$, $\overline{\mathbf{y}} = \mathbf{y}^*|_{\overline{\Omega}\times[0,1]}$ and we denote by $\mathbf{X}^*$ the flux associated to $\nabla\Phi$.

If we set $\overline{p}(\mathbf{x},t) := -\Phi_t(\mathbf{x},t) - \frac{1}{2}|\nabla\Phi(\mathbf{x},t)|^2$ and $\overline{\theta} \equiv 0$, then (1) is verified for $\mathbf{y} = \overline{\mathbf{y}}$, $p = \overline{p}$, $\theta = \overline{\theta}$, $T = 1$, $\mathbf{y}_0 = \mathbf{y}_1 = \mathbf{0}$ and $\theta_0 = \theta_1 = 0$.

Thanks to (20) and (22), the particles of the fluid that travel with velocity $\mathbf{y}^*$ and are located in the ball B^i at $t = 0$ go out of $\bar{G}_0$ at $t = t_{i-\frac{1}{2}}$ and again at $t = t_{k+i-\frac{1}{2}}$. More precisely, one has:

Lemma 9 *The following properties hold for all $i = 1, ..., k$:*

$$\begin{array}{lll} \mathbf{x} \in \bar{B}^i & \Longrightarrow & \mathbf{X}^*(\mathbf{x}, t_{i-\frac{1}{2}}, 0) \in G \setminus \bar{G}_0, \\ \mathbf{x} \in \bar{B}^i & \Longrightarrow & \mathbf{X}^*(\mathbf{x}, t_{i+k-\frac{1}{2}}, 1/2) \in G \setminus \bar{G}_0. \end{array} \tag{23}$$

For the proof, it suffices to notice that $\mathbf{Y}^* = \mathbf{X}^0$ in $\bar{G} \times [1/4, 3/4] \times [1/4, 3/4]$, where

$$\mathbf{X}^0(\mathbf{x},t,s) := \begin{cases} \mathbf{X}^i(\mathbf{x}, 8k(t-t_{j-1}), 8k(s-t_{l-1}) & \text{if } (\mathbf{x},t,s) \in \bar{G}\times[t_{j-1}, t_{j-\frac{1}{2}}]\times[t_{l-1}, t_{l-\frac{1}{2}}], \\ \mathbf{X}^i(\mathbf{x}, 8k(t-t_{j-1}), 8k(t_l - s)) & \text{if } (\mathbf{x},t,s) \in \bar{G}\times[t_{j-1}, t_{j-\frac{1}{2}}]\times[t_{l-\frac{1}{2}}, t_l], \\ \mathbf{X}^i(\mathbf{x}, 8k(t_j - t), 8k(s-t_{l-1})) & \text{if } (\mathbf{x},t,s) \in \bar{G}\times[t_{j-\frac{1}{2}}, t_j]\times[t_{l-1}, t_{l-\frac{1}{2}}], \\ \mathbf{X}^i(\mathbf{x}, 8k(t_j - t), 8k(t_l - s)) & \text{if } (\mathbf{x},t,s) \in \bar{G}\times[t_{j-\frac{1}{2}}, t_j]\times[t_{l-\frac{1}{2}}, t_l] \end{cases}$$

for $l, j = 1, \dots, 2k$. Indeed, one has the following for each $\mathbf{x} \in \overline{B^i}$:

$$\mathbf{X}^*(\mathbf{x}, t_{i-\frac{1}{2}}, 0) = \mathbf{X}^*(\mathbf{x}, t_{i-\frac{1}{2}}, t_0) = \mathbf{X}^i(\mathbf{x}, 1, 0) \in G \setminus \bar{G}_0$$

and

$$\mathbf{X}^*\left(\mathbf{x}, t_{k+i-\frac{1}{2}}, \frac{1}{2}\right) = \mathbf{X}^*(\mathbf{x}, t_{k+i-\frac{1}{2}}, t_k) = \mathbf{X}^i(\mathbf{x}, 1, 0). \in G \setminus \bar{G}_0.$$

A result similar to Lemma 6 also holds here:

Lemma 10 *There exist π_1 and π_3 with*

$$\begin{cases} \pi_i(\mathbf{f})(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \ \forall \mathbf{x} \in \overline{\Omega}, \ \forall \mathbf{f} \in C^0(\overline{\Omega}; \mathbb{R}^i), \\ \textit{supp } \pi_i(\mathbf{f}) \subset G_0 \quad \forall \mathbf{f} \in C^0(\overline{\Omega}; \mathbb{R}^i), \\ \pi_i \textit{ maps continuously } C^{n,\lambda}(\overline{\Omega}; \mathbb{R}^i) \textit{ into } C^{n,\lambda}(\bar{G}; \mathbb{R}^i) \quad \forall n \geq 0, \quad \forall \lambda \in (0,1). \end{cases} \tag{24}$$

Finally, we also have a property similar to (11) for any flux corresponding to a velocity field sufficiently close to $\overline{\mathbf{y}}$:

Lemma 11 *For each $\mathbf{y} \in C([0,1], C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2))$, let us set $\tilde{\mathbf{y}} = \mathbf{y}^* + \pi_3(\mathbf{y} - \overline{\mathbf{y}})$. There exists $\nu > 0$ such that, if $\|\mathbf{y} - \overline{\mathbf{y}}\|_0 \leq \nu$, $\tilde{\mathbf{X}}$ is the flux associated to $\tilde{\mathbf{y}}$ and $1 \leq i \leq k$, one has:*

$$\begin{array}{llll} \mathbf{x} \in \bar{B}^i & \Longrightarrow & \tilde{\mathbf{X}}^*(\mathbf{x}, t_{i-\frac{1}{2}}, 0) & \in G \setminus \bar{G}_0, \\ \mathbf{x} \in \bar{B}^i & \Longrightarrow & \tilde{\mathbf{X}}^*(\mathbf{x}, t_{i+k-\frac{1}{2}}, 1/2) & \in G \setminus \bar{G}_0. \end{array}$$

Proof: It is easy to see that

$$\begin{aligned}\left|\widetilde{\mathbf{X}}(\mathbf{x},t,s)-\mathbf{Y}^*(\mathbf{x},t,s)\right| &= \left|\int_s^t (\widetilde{\mathbf{y}}(\widetilde{\mathbf{X}}(\mathbf{x},\tau,s),\tau)-\mathbf{y}^*(\mathbf{Y}^*(\mathbf{x},\tau,s),\tau))\,d\tau\right| \\ &\le \int_s^t |\mathbf{y}^*(\widetilde{\mathbf{X}}((\mathbf{x},\tau,s),\tau)-\mathbf{y}^*(\mathbf{Y}^*(\mathbf{x},\tau,s),\tau))|\,d\tau \\ &\quad+\int_s^t |\pi_3(\mathbf{y}-\overline{\mathbf{y}})(\widetilde{\mathbf{X}}(\mathbf{x},\tau,s),\tau))|\,d\tau \\ &\le C\int_s^t |\widetilde{\mathbf{X}}(\mathbf{x},\tau,s)-\mathbf{Y}^*(\mathbf{x},\tau,s)|\,d\tau + C\|\mathbf{y}-\overline{\mathbf{y}}\|_0.\end{aligned}$$

For Gronwall's Lemma, we have

$$\left|\widetilde{\mathbf{X}}(\mathbf{x},t,s)-\mathbf{X}^*(\mathbf{x},t,s)\right| \le C\|\mathbf{y}-\overline{\mathbf{y}}\|_0.$$

Let us set

$$\mathbf{F}=\bigcup_{i=1}^k\left(\widetilde{\mathbf{X}}(\overline{B^i},t_{i-\frac{1}{2}},0)\cup\widetilde{\mathbf{X}}\left(\overline{B^i},t_{k+i-\frac{1}{2}},\frac{1}{2}\right)\right).$$

Since $\mathbf{F}$ and $\bar{G}_0$ are compact and $\mathbf{F}\cap\bar{G}=\emptyset$, we have $d' := dist\left(\mathbf{F},\ \overline{\widetilde{\Omega}_1}\right)>0$. There exists $\nu>0$ such that, if $\|\mathbf{y}-\overline{\mathbf{y}}\|_0\le\nu$, then

$$\left|\widetilde{\mathbf{X}}(\mathbf{x},t,s)-\mathbf{Y}^*(\mathbf{x},t,s)\right|\le\frac{d'}{2}\qquad\forall(\mathbf{x},t,s)\in\bar{G}\times[0,1]\times[0,1]. \tag{25}$$

Consequently, for this ν we get the desired result. □

3 Proof of theorem 1

This section is devoted to prove the exact controllability result Theorem 1, we use a scaling argument. from Proposition 1.

Let $T>0$, θ_0, $\theta_1\in C^{1,\alpha}(\overline{\Omega};\mathbb{R})$ and $\mathbf{y}_0$, $\mathbf{y}_1\in\mathbf{C}(1,\alpha,\Gamma_0)$ be given. Let us see that, if

$$\|\mathbf{y}_0\|_{1,\alpha}+\|\mathbf{y}_1\|_{1,\alpha}+\|\theta_0\|_{1,\alpha}+\|\theta_1\|_{1,\alpha}$$

is small enough, we can construct a triplet $(\mathbf{y},p,\theta)$ satisfying (1) and (3).

If $\varepsilon\in(0,T/2)$ is sufficiently small to have

$$\{\varepsilon\|\mathbf{y}_0\|_{1,\alpha},\varepsilon^2\|\theta_0\|_{1,\alpha}\}<\delta\ \ (\text{resp. }\{\varepsilon\|\mathbf{y}_1\|_{1,\alpha},\ \varepsilon^2\|\theta_1\|_{1,\alpha}\}\le\delta),$$

then, thanks to Proposition 1, there exist $(\mathbf{y}^0,\ \theta^0)$ in $C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{N+1}))$ and a pressure p^0 (resp. $(\mathbf{y}^1,\ \theta^1)$ and p^1) solving (1), with $\mathbf{y}^0(\mathbf{x},0)\equiv\varepsilon\mathbf{y}_0(\mathbf{x})$ and $\theta^0(\mathbf{x},0)\equiv\varepsilon^2\theta_0$ (resp. $\mathbf{y}^1(\mathbf{x},0)\equiv-\varepsilon\mathbf{y}_1(\mathbf{x})$ and $\theta^1(\mathbf{x},0)=\varepsilon^2\theta_1(\mathbf{x})$) and satisfying (5).

Let us choose ε of this form and let us introduce $\mathbf{y} : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}^N$, $p : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}$ and $\theta : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}$ as follows:

$$\begin{cases} \mathbf{y}(\mathbf{x},t) = \varepsilon^{-1}\mathbf{y}^0(\mathbf{x},\varepsilon^{-1}t), \\ p(\mathbf{x},t) = \varepsilon^{-2}p^0(\mathbf{x},\varepsilon^{-1}t), \\ \theta(\mathbf{x},t) = \varepsilon^{-2}\theta^0(\mathbf{x},\varepsilon^{-1}t), \end{cases} \quad (\mathbf{x},t) \in \overline{\Omega} \times [0,\varepsilon],$$

$$\begin{cases} \mathbf{y}(\mathbf{x},t) = \mathbf{0}, \\ p(\mathbf{x},t) = 0, \\ \theta(\mathbf{x},t) = 0, \end{cases} \quad (\mathbf{x},t) \in \overline{\Omega} \times (\varepsilon, T-\varepsilon),$$

$$\begin{cases} \mathbf{y}(\mathbf{x},t) = -\varepsilon^{-1}\mathbf{y}^1(\mathbf{x},\varepsilon^{-1}(T-t)), \\ p(\mathbf{x},t) = \varepsilon^{-2}p^1(\mathbf{x},\varepsilon^{-1}(T-t)), \\ \theta(\mathbf{x},t) = \varepsilon^{-2}\theta^1(\mathbf{x},\varepsilon^{-1}(T-t)), \end{cases} \quad (\mathbf{x},t) \in \overline{\Omega} \times [T-\varepsilon, T].$$

Then, $(\mathbf{y},\theta) \in C^0([0,T];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{N+1})$ and the triplet $(\mathbf{y},p,\theta)$ satisfies (1) and (3).

4 Proof of Proposition 1. The 2D case

Let $\mu \in C^\infty([0,1];\mathbb{R})$ a function such that $\mu \equiv 1$ in $[0,1/4]$, $\mu \equiv 0$ in $[1/2,1]$ and $0 < \mu < 1$.

The proof of Proposition 1 is a consequence of the following result:

Proposition 2 *There exists $\delta > 0$ such that if* $\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \le \delta$, *then the system*

$$\begin{cases} \zeta_t + \mathbf{y}\cdot\nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta & in \quad \Omega \times (0,1), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0 & in \quad \Omega \times (0,1), \\ \mathbf{y}\cdot\mathbf{n} = (\overline{\mathbf{y}} + \mu\,\mathbf{y}_0)\cdot\mathbf{n} & on \quad \Gamma \times (0,1), \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\times\mathbf{y} = \zeta & in \quad \Omega \times (0,1), \\ \zeta(0) = \nabla\times\mathbf{y}_0, \ \theta(0) = \theta_0 & in \quad \Omega, \end{cases} \tag{26}$$

possesses at least one solution

$$(\zeta,\theta,\mathbf{y}) \in C^0([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R})) \times C^0([0,1];C^{2,\alpha}(\overline{\Omega};\mathbb{R})) \times C^0([0,1];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^2)) \tag{27}$$

such that $\theta(\mathbf{x},t) = 0$ in $\Omega \times [1/2,1]$ and $\zeta(\mathbf{x},1) = 0$ in Ω.

The reminder of this section is devoted to prove Proposition 2. We are going to adapt some ideas from Bardos and Frisch [?] and Kato [?] already used in [?] and [?].

Let us give a sketch.

We will start from an arbitrary flow $\mathbf{z} := \mathbf{z}(\mathbf{x}, t)$ in a suitable class $\mathbf{S}$ of continuous functions. Together with this $\mathbf{z}$, we will construct a scalar function θ verifying

$$\begin{cases} \theta_t + \mathbf{z} \cdot \nabla\theta = 0 & \text{in} \quad \overline{\Omega} \times [0, 1], \\ \theta(0) = \theta_0 & \text{in} \quad \overline{\Omega}. \end{cases} \tag{28}$$

and

$$\theta \equiv 0 \text{ in } \overline{\Omega} \times [1/2, 1].$$

Then, with this θ we construct a function ζ satisfying

$$\begin{cases} \zeta_t + \mathbf{z} \cdot \nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \overline{\Omega} \times [0, 1], \\ \zeta(0) = \nabla \times \mathbf{y}_0 & \text{in} \quad \overline{\Omega}. \end{cases} \tag{29}$$

and

$$\zeta(\cdot, 1) \equiv 0 \text{ in } \overline{\Omega}.$$

In this way, we shall have assigned ζ to each $\mathbf{z} \in \mathbf{S}$. Now, we can construct a flow $\mathbf{y}$ such that $\nabla \times \mathbf{y} = \zeta$ and, therefore, we will have defined a mapping F with $F(\mathbf{z}) = \mathbf{y}$. We will choose the class $\mathbf{S}$ in such a way that F maps $\mathbf{S}$ into itself.

Let $\mathbf{y}$ be the unique fixed-point of F in $\mathbf{S}$ and let θ and ζ be the associated *temperature* and *vorticity*. Then, the triplet $(\zeta, \theta, \mathbf{y})$ should solve (48) and satisfy (27).

Let us now give the details.

The good definition of $\mathbf{S}$ will be given below. First, let us introduce

$$\mathbf{S}' = \left\{ \mathbf{z} \in C^0([0, 1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2)) : \nabla \cdot \mathbf{z} = 0 \text{ in } \overline{\Omega} \times [0, 1],\ \mathbf{z} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\mathbf{y}_0) \cdot \mathbf{n} \text{ on } \Gamma \times [0, 1] \right\}.$$

For $\nu > 0$, we will denote by $\mathbf{S}_\nu$ the set

$$\mathbf{S}_\nu = \left\{ \mathbf{z} \in \mathbf{S}' : \|\mathbf{z}(\cdot, t) - \overline{\mathbf{y}}(\cdot, t)\|_{1,\alpha} \le \nu,\ \forall t \in [0, 1] \right\}.$$

Let $\nu > 0$, be the constant furnished by Lemma 7 and let us carry out the previous process for $\mathbf{z} \in \mathbf{S}_\nu$ ($\mathbf{S} = \mathbf{S}_\nu$).

First, let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_2(\mathbf{z} - \overline{\mathbf{y}})$. Then, we have the estimate

$$\|\mathbf{z}^*(\cdot, t)\|_{1,\alpha} \le \|\mathbf{y}^*(\cdot, t)\|_{1,\alpha} + C\|(\mathbf{z} - \overline{\mathbf{y}})(\cdot, t)\|_{1,\alpha}, \quad \forall t \in [0, 1] \tag{30}$$

and the following result:

Lemma 12 *There exists a unique global solution* $\mathbf{Z}^* \in C^1([0, 1] \times [0, 1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}_3; \mathbb{R}^2))$

$$\begin{cases} \mathbf{Z}^*_t(\mathbf{x}, t, s) = \mathbf{z}^*(\mathbf{Z}^*(\mathbf{x}, t, s), t), \\ \mathbf{Z}^*(\mathbf{x}, s, s) = \mathbf{x}, \end{cases} \tag{31}$$

with

$$\mathbf{Z}^*(\mathbf{x}, t, s) \in \overline{\Omega}_3 \quad \forall (\mathbf{x}, t, s) \in \overline{\Omega}_3 \times [0, 1] \times [0, 1].$$

For the proof, it is suffices to apply directly the well known (classical) existence, uniqueness and regularity theory of ODEs.

Since $\mathbf{Z}^* \in C^1([0,1]\times[0,1];\mathbf{C}^{2,\alpha}(\overline{\Omega}_3;\mathbb{R}^2))$ and $\theta_0 \in C^{2,\alpha}(\overline{\Omega};\mathbb{R})$, from the properties of π_1, we can obtain a unique solution $\theta^* \in C^0([0,1/2];C^{2,\alpha}(\overline{\Omega}_3;\mathbb{R}))$ to the problem

$$\begin{cases} \theta^*_t + \mathbf{z}^*\cdot\nabla\theta^* = 0 & \text{in} \quad \overline{\Omega}_3\times[0,1/2], \\ \theta^*(\mathbf{x},0) = \pi_1(\theta_0)(\mathbf{x}) & \text{in} \quad \overline{\Omega}_3. \end{cases} \tag{32}$$

Note that, in (32), boundary condition on θ does not appear. Obviously, this is because $\mathbf{z}^*$ has support contained in Ω_3.

The solution to (32) verifies

$$\theta^*(\mathbf{Z}^*(\mathbf{x},t,0),t) = \pi_1(\theta_0)(\mathbf{x}) \quad \forall(\mathbf{x},t)\in\overline{\Omega}_3\times[0,1/2] \tag{33}$$

and, consequently,

$$supp\ \theta^*(\cdot,t) \subset \mathbf{Z}^*(\Omega_2,t,0) \quad \text{in } [0,1/2].$$

In particular, in view of the choice of ν, we get:

$$supp\ \theta^*(\cdot,1/2) \subset \mathbf{Z}^*(\Omega_2,1/2,0) \subset \Omega_3\setminus\overline{\Omega}_2.$$

Therefore, $\theta^*(\cdot,1/2)\equiv 0$ in $\overline{\Omega}_2$.

Now, applying Lemma 1 to the equation verified for θ^*, we have:

$$\frac{d}{dt^+}\|\theta^*(\cdot,t)\|_{1,\alpha} \le K\|\mathbf{z}^*(\cdot,t)\|_{1,\alpha}\|\theta^*(\cdot,t)\|_{1,\alpha}. \tag{34}$$

Then, from Gronwall's Lemma, we obtain:

$$\|\theta^*(\cdot,t)\|_{1,\alpha} \le \|\pi_1(\theta_0)\|_{1,\alpha}\exp\left(K\int_0^t \|\mathbf{z}^*(\cdot,\tau)\|_{1,\alpha}\,d\tau\right). \tag{35}$$

Let θ be the function

$$\theta(\mathbf{x},t) = \begin{cases} \theta^*(\mathbf{x},t), & (\mathbf{x},t)\in\overline{\Omega}\times[0,1/2], \\ 0, & (\mathbf{x},t)\in\overline{\Omega}\times[1/2,1]. \end{cases} \tag{36}$$

Then $\theta \in C^0([0,1];C^{2,\alpha}(\overline{\Omega};\mathbb{R}))$ and

$$\begin{cases} \theta_t + \mathbf{z}\cdot\nabla\theta = 0 & \text{in} \quad \Omega\times(0,1), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}), & \text{in} \quad \Omega. \end{cases} \tag{37}$$

Now, we set $\zeta^*_0 := \nabla\times(\pi_2(\mathbf{y}_0))$ and $\zeta^* \in C^0([0,1/2];C^{1,\alpha}(\overline{\Omega}_3;\mathbb{R}))$ unique solution to the problem

$$\begin{cases} \zeta^*_t + \mathbf{z}^*\cdot\nabla\zeta^* = -\vec{\mathbf{k}}\times\nabla\theta^* & \text{in} \quad \overline{\Omega}_3\times[0,1/2], \\ \zeta^*(\mathbf{x},0) = \zeta^*_0(\mathbf{x}) & \text{in} \quad \overline{\Omega}_3. \end{cases} \tag{38}$$

Applying again Lemma 1, we get:

$$\frac{d}{dt^+}\|\zeta^*(\cdot,t)\|_{0,\alpha} \le C\|\theta^*(\cdot,t)\|_{1,\alpha} + K\|\mathbf{z}^*(\cdot,t)\|_{1,\alpha}\|\zeta^*(\cdot,t)\|_{0,\alpha}. \tag{39}$$

From Gronwall's Lemma and (35), we obtain:

$$\|\zeta^*(\cdot,t)\|_{0,\alpha} \le C(\|\pi_2(\mathbf{y}_0)\|_{1,\alpha} + \|\pi_1(\theta_0)\|_{1,\alpha}) \exp\left(K\int_0^t \|\mathbf{z}^*(\cdot,\tau)\|_{1,\alpha}\, d\tau\right). \tag{40}$$

With this ζ^*, we define $\zeta^*_{1/2} \in C^{1,\alpha}(\overline{\Omega};\mathbb{R})$ given by $\zeta^*_{1/2}(\mathbf{x}) = \zeta^*(\mathbf{x},1/2)$ for all $\mathbf{x} \in \overline{\Omega}$. Then, we can obtain a unique solution $\zeta^{**} \in C^0([1/2,1];C^{1,\alpha}(\overline{\Omega}_3;\mathbb{R}))$ of the problem

$$\begin{cases} \zeta^{**}_t + \mathbf{z}^*\cdot\nabla\zeta^{**} = 0, & \text{in} \quad \overline{\Omega}_3 \times [1/2,1], \\ \zeta^{**}(\mathbf{x},1/2) = \pi_1(\zeta^*_{1/2})(\mathbf{x}), & \text{in} \quad \overline{\Omega}_3. \end{cases} \tag{41}$$

Also, we have that the solution of the problem (41) verifies

$$\zeta^{**}(\mathbf{Z}^*(\mathbf{x},t,1/2),t) = \pi_1(\zeta^*_{1/2})(\mathbf{x}), \quad \forall(\mathbf{x},t) \in \overline{\Omega}_3 \times [1/2,1]$$

and then

$$\mathit{supp}\ \zeta^{**}(\cdot,t) \subset \mathbf{Z}^*(\Omega_2,t,1/2) \quad \text{in } [1/2,1].$$

Again, by the choose of ν and the identity above, we have

$$\mathit{supp}\ \zeta^{**}(\cdot,t) \subset \mathbf{Z}^*(\Omega_2,1,1/2) \subset \Omega_3 \setminus \overline{\Omega}_2.$$

Therefore, $\zeta^{**}(\cdot,1) \equiv 0$ in $\overline{\Omega}_2$.

With similar arguments to get (35) and combining with (40), we have:

$$\|\zeta^{**}(\cdot,t)\|_{0,\alpha} \le C(\|\pi_2(\mathbf{y}_0)\|_{1,\alpha} + \|\pi_1(\theta_0)\|_{1,\alpha}) \exp\left(K\int_0^t \|\mathbf{z}^*(\cdot,\tau)\|_{1,\alpha}(\tau) d\tau\right), \tag{42}$$

for all $t \in [1/2,1]$.

So, we can define $\zeta \in C^0([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$ given by

$$\zeta(\mathbf{x},t) = \begin{cases} \zeta^*(\mathbf{x},t), & (\mathbf{x},t) \in \overline{\Omega} \times [0,1/2], \\ \zeta^{**}(\mathbf{x},t), & (\mathbf{x},t) \in \overline{\Omega} \times [1/2,1] \end{cases} \tag{43}$$

and we have that ζ is solution of

$$\begin{cases} \zeta_t + \mathbf{z}\cdot\nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta, & \text{in} \quad \overline{\Omega} \times [0,1], \\ \zeta(0) = \nabla \times \mathbf{y}_0, & \text{in} \quad \overline{\Omega}. \end{cases} \tag{44}$$

Therefore, thanks to (40) and (42), we obtain the for ζ:

$$\|\zeta(\cdot,t)\|_{0,\alpha} \le C(\|\mathbf{y}_0\|_{1,\alpha} + \|\theta_0\|_{1,\alpha}) \exp\left(K\int_0^t \|\mathbf{z}^*\|_{1,\alpha}(\tau) d\tau\right). \tag{45}$$

Whence, with this ζ we get a unique $\mathbf{y} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2))$ such that $\nabla \times \mathbf{y} = \zeta$ and $\mathbf{y} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu \mathbf{y}_0) \cdot \mathbf{n}$ on $\Gamma \times [0,1]$. In fact, let $\psi \in C^0([0,1]; C^{3,\alpha}(\overline{\Omega}; \mathbb{R}))$ a unique solution of the elliptic equation

$$\left| \begin{array}{ll} -\Delta \psi = \zeta - \mu \nabla \times \mathbf{y}_0, & \text{in } \Omega \times [0,1], \\ \psi = 0, & \text{on } \Gamma \times [0,1]. \end{array} \right. \tag{46}$$

Then, we define $\mathbf{y} := \nabla \times \psi + \overline{\mathbf{y}} + \mu \mathbf{y}_0$. Obviously, we have that $\mathbf{y}$ is a flow, $\mathbf{y} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2))$, $\mathbf{y} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu \mathbf{y}_0) \cdot \mathbf{n}$ on $\Gamma \times [0,1]$ and $\nabla \times \mathbf{y} = \zeta$. Also, easily we can see that $\mathbf{y}$ is unique.

Therefore, since $\mathbf{y}$ is determined by $\mathbf{z}$, we write

$$\begin{array}{rcl} F : \mathbf{S}_\nu & \to & \mathbf{S}' \\ \mathbf{z} & \mapsto & F(\mathbf{z}) := \mathbf{y}. \end{array} \tag{47}$$

The following result holds:

Lemma 13 *There exists $\delta > 0$ such that if*

$$\max \{ \|\mathbf{y}_0\|_{1,\alpha}, \|\theta_0\|_{1,\alpha} \} < \delta$$

then $F(\mathbf{S}_\nu) \subset \mathbf{S}_\nu$.

Proof: Let $\mathbf{z} \in \mathbf{S}_\nu$, then $F(\mathbf{z}) - \overline{\mathbf{y}} = \nabla \times \psi + \mu \mathbf{y}_0$ and we have:

$$\|F(\mathbf{z})(\cdot,t) - \overline{\mathbf{y}}(\cdot,t)\|_{1,\alpha} \leq C(\|\zeta(\cdot,t)\|_{0,\alpha} + \|\mathbf{y}_0\|_{1,\alpha}).$$

Therefore, using (45), (30) and the definition of $\mathbf{S}_\nu$, we obtain

$$\begin{aligned} \|F(\mathbf{z})(\cdot,t) - \overline{\mathbf{y}}(\cdot,t)\|_{1,\alpha} &\leq C_1(\|\mathbf{y}_0\|_{1,\alpha} + \|\theta_0\|_{1,\alpha}) \exp\left(C_2 \int_0^t \|\mathbf{z}(\cdot,\tau) - \overline{\mathbf{y}}(\cdot,\tau)\|_{1,\alpha} \, d\tau \right) \\ &\leq C_1(\|\mathbf{y}_0\|_{1,\alpha} + \|\theta_0\|_{1,\alpha}) \exp(C_2 \nu). \end{aligned}$$

So, we take $\delta > 0$ such that

$$C_1(\|\mathbf{y}_0\|_{1,\alpha} + \|\theta_0\|_{1,\alpha}) \exp(C_2 \nu) \leq \nu.$$

In this way, if $\mathbf{z} \in \mathbf{S}_\nu$ then we have that

$$\|F(\mathbf{z})(\cdot,t) - \overline{\mathbf{y}}(\cdot,t)\|_{1,\alpha} \leq \nu, \quad \forall t \in [0,1].$$

Then, $F : \mathbf{S}_\nu \to \mathbf{S}_\nu$ is a well-defined application. □

We will now construct a fixed-point to the application $F : \mathbf{S}_\nu \to \mathbf{S}_\nu$. To this end, we define $\mathbf{y}^0 = \overline{\mathbf{y}} + \mu \mathbf{y}_0 \in S_\nu$ and the sequences

$$(\zeta^{m+1}, \theta^{m+1}, \mathbf{y}^{m+1}) \in C^0([0,1]; C^{2,\alpha}(\overline{\Omega}; \mathbb{R})) \times C^0([0,1]; C^{3,\alpha}(\overline{\Omega}; \mathbb{R})) \times C^0([0,1]; \mathbf{C}^{3,\alpha}(\overline{\Omega}; \mathbb{R}^2))$$

given by

$$\begin{cases} \zeta_t^{m+1} + \mathbf{y}^m \cdot \nabla \zeta^{m+1} = -\vec{\mathbf{k}} \times \nabla \theta^{m+1} & \text{in} \quad \Omega \times (0,1), \\ \theta_t^{m+1} + \mathbf{y}^m \cdot \nabla \theta^{m+1} = 0 & \text{in} \quad \Omega \times (0,1), \\ \mathbf{y}^{m+1} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\, \mathbf{y}_0^{m+1}) \cdot \mathbf{n} & \text{on} \quad \Gamma \times (0,1), \\ \nabla \cdot \mathbf{y}^{m+1} = 0, \ \nabla \times \mathbf{y}^{m+1} = \zeta^{m+1} & \text{in} \quad \Omega \times (0,1), \\ \zeta^{m+1}(0) = \nabla \times \mathbf{y}_0^{m+1}, \ \theta^{m+1}(0) = \theta_0^{m+1} & \text{in} \quad \Omega. \end{cases} \tag{48}$$

This sequence is well-defined. Indeed, if we introduce

$$\mathbf{S}^{m+1} = \left\{ \mathbf{z} \in C^0([0,1]; \mathbf{C}^{3,\alpha}(\overline{\Omega}; \mathbb{R}^2)) : \nabla \cdot \mathbf{z} = 0, \ \mathbf{z} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu \mathbf{y}_0^{m+1}) \right\}$$

and for $\nu > 0$ (it was fixed before), we will denote by $\mathbf{S}_\nu$ the set

$$\mathbf{S}_\nu^{m+1} = \left\{ \mathbf{z} \in \mathbf{S}^{m+1} : \|\mathbf{z}(\cdot,t) - \overline{\mathbf{y}}(\cdot,t)\|_{1,\alpha} \leq \nu, \ \forall t \in [0,1] \right\}.$$

Then, as before, we can define an application $F^{m+1} : \mathbf{S}_\nu^m \mapsto \mathbf{S}'_{m+1}$ and, thanks to an analogous of Lemma 13 and the item c) of Lemma 3, there exists a constant $\delta > 0$ (independent of m) such that if

$$\max\{\|\mathbf{y}_0\|_{1,\alpha}, \|\theta_0\|_{1,\alpha}\} < \delta$$

then $F^{m+1}(\mathbf{S}_\nu^m) \subset \mathbf{S}_\nu^{m+1}$.

Lemma 14 *For $\mathbf{z}^1, \ \mathbf{z}^2 \in S_\nu$ we have the inequalities*

$$\|\zeta^1 - \zeta^2\|_{0+\alpha}(t) \leq C(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha}) e^{C(\|\mathbf{z}_\pi^1\|_{0,2+\alpha} + \|\mathbf{z}_\pi^2\|_{0,2+\alpha})} \int_0^t \|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(s)\, ds.$$

Proof: The equation verified by $\Theta^* = \theta^{1*} - \theta^{2*}$ is

$$\Theta_t^* + \mathbf{z}_\pi^1 \cdot \nabla \Theta^* = -(\mathbf{z}_\pi^1 - \mathbf{z}_\pi^2) \cdot \nabla \theta^{2*}. \tag{49}$$

So, applying the operator ∇ to the (52), we obtain

$$(\nabla \Theta^*)_t + (\mathbf{z}_\pi^1 \cdot \nabla)(\nabla \Theta^*) = -\nabla \mathbf{z}_\pi^1 \cdot \nabla \Theta^* - \nabla(\mathbf{z}_\pi^1 - \mathbf{z}_\pi^2) \cdot \nabla \theta^{2*} - ((\mathbf{z}_\pi^1 - \mathbf{z}_\pi^2) \cdot \nabla) \nabla \theta^{2*}.$$

Applying the lemma 1 to the equation above and using the Lemma **??**, we have

$$\begin{aligned} \frac{\partial}{\partial t^+} \|\nabla \Theta^*\|_{0+\alpha}(t) \leq\ & C\|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(t) \|\theta^{2*}\|_{2+\alpha,0} + C\|\mathbf{z}_\pi^1\|_{0,1+\alpha} \|\nabla \Theta^*\|_{0+\alpha}(t) \\ & C_\nu \|\theta_0\|_{2+\alpha} \|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(t) + C(1+\nu) \|\nabla \Theta^*\|_{0+\alpha}(t). \end{aligned}$$

Therefore, we define $\Theta = \theta^1 - \theta^2$ and then

$$\frac{\partial}{\partial t^+} \|\nabla \Theta\|_{0+\alpha}(t) \leq C_\nu \|\theta_0\|_{2+\alpha} \|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(t) + C(1+\nu) \|\nabla \Theta\|_{0+\alpha}(t). \tag{50}$$

The equation verified by $\Psi^* = \zeta^{1*} - \zeta^{2*}$ and $\Psi^{**} = \zeta^{1**} - \zeta^{2**}$ are

$$\Psi_t^* + \mathbf{z}_\pi^1 \cdot \nabla\Psi^* = -(\mathbf{z}_\pi^1 - \mathbf{z}_\pi^2) \cdot \nabla\zeta^{2*} - \vec{k} \times \nabla\pi_1(\Theta) \tag{51}$$

and

$$\Psi_t^{**} + \mathbf{z}_\pi^1 \cdot \nabla\Psi^{**} = -(\mathbf{z}_\pi^1 - \mathbf{z}_\pi^2) \cdot \nabla\zeta^{2**}. \tag{52}$$

respectively.

Applying the lemma 1 to the equations above and using the Lemma **??** and Lemma **??**, we have

$$\begin{aligned}\frac{\partial}{\partial t^+}\|\Psi^*\|_{0+\alpha}(t) &\le C\Big(\|\mathbf{z}^1 - \mathbf{z}^2\|_{0+\alpha}(t)\|\zeta^{2*}\|_{0,1+\alpha} + \|\mathbf{z}_\pi^1\|_0(t)\|\Psi^*\|_{0+\alpha}(t) + \|\nabla\Theta\|_{0+\alpha}(t)\Big)\\ &\le C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\|\mathbf{z}^1 - \mathbf{z}^2\|_{0+\alpha}(t) + C(1+\nu)\|\Psi^*\|_{0+\alpha}(t)\\ &\quad + C\|\nabla\Theta\|_{0+\alpha}(t)\end{aligned} \tag{53}$$

and

$$\begin{aligned}\frac{\partial}{\partial t^+}\|\Psi^{**}\|_{0+\alpha}(t) &\le C\|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(t)\|\zeta^{2**}\|_{1+\alpha,0} + \alpha\|\mathbf{z}_\pi^1\|_0\|\Psi^{**}\|_{0+\alpha}(t)\\ &\le C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\|\mathbf{z}^1 - \mathbf{z}^2\|_{0+\alpha}(t) + C(1+\nu)\|\Psi^{**}\|_{0+\alpha}(t).\end{aligned} \tag{54}$$

We sum the inequalities (50) and (53) to obtain

$$\begin{aligned}\frac{\partial}{\partial t^+}(\|\Psi^*\|_{0+\alpha} + \|\nabla\Theta\|_{0+\alpha})(t) &\le C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\|\mathbf{z}^1 - \mathbf{z}^2\|_{C^{1+\alpha}}(t)\\ &\quad + C(1+\nu)(\|\Psi^*\|_{0+\alpha} + \|\nabla\Theta\|_{0+\alpha})(t).\end{aligned}$$

Applying the Gronwall's lemma, we deduce

$$(\|\Psi^*\|_{0+\alpha} + \|\nabla\Theta\|_{0+\alpha})(t) \le C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\int_0^t \|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(\tau)\, d\tau.$$

Also, applying the Gronwall's Lemma to (54) and using the inequality above, we deduce

$$\begin{aligned}\|\Psi^{**}\|_{0+\alpha}(t) &\le C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\int_{1/2}^t \|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(\tau)\, d\tau + C_\nu\|\Psi^*(\cdot, 1/2)\|_{0+\alpha}\\ &\le C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\int_0^t \|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(\tau)\, d\tau.\end{aligned}$$

Therefore, we can define $\Psi = \zeta^1 - \zeta^2$. We can see that

$$\Psi(\mathbf{x}, t) = \begin{cases} \Psi^*(\mathbf{x}, t), & (\mathbf{x}, t) \in \overline{\Omega} \times [0, 1/2],\\ \Psi^{**}(\mathbf{x}, t), & (\mathbf{x}, t) \in \overline{\Omega} \times [1/2, 1] \end{cases} \tag{55}$$

and then

$$\|\Psi\|_{0+\alpha}(t) \le C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\int_0^t \|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(\tau)\, d\tau.$$

□

Note that $\mathbf{y}^1 - \mathbf{y}^2 = \nabla \times (\Phi^1 - \Phi^2)$. We can note that $\nabla \cdot (\nabla \times (\Phi^1 - \Phi^2)) = 0$, $\nabla \times (\nabla \times (\Phi^1 - \Phi^2)) = \zeta^1 - \zeta^2$ and $\nabla \times (\Phi^1 - \Phi^2) \cdot \mathbf{n}|_{\Gamma \times [0,1]} = 0$. Then we have that

$$|||\mathbf{y}^1 - \mathbf{y}^2|||_{1+\alpha}(t) = \|\zeta^1 - \zeta^2\|_{\alpha}(t)$$

is a norm equivalent to the norm $\|\cdot\|_{1+\alpha}$.

Lemma 15 *For each* $\mathbf{z}^1,\ \mathbf{z}^2 \in S_\nu$, *if* $\widetilde{C} = C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})$ *then we have that*

$$|||F^m(\mathbf{z}^1) - F^m(\mathbf{z}^2)|||_{1+\alpha}(t) \leq \frac{\widetilde{C}t^m}{m!}\|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1+\alpha} \tag{56}$$

for all $m \in \mathbb{N}$.

Proof: We are going to prove by induction.

▶ $m = 1$:

Thanks to the lemma 14;

▶ We suppose that it is true for $m = k$ and we want to prove for $m = k + 1$:

Applying the lemma 14 to $\mathbf{y}^1 = F^k(\mathbf{z}^1)$ and $\mathbf{y}^2 = F^k(\mathbf{z}^2)$ we have

$$|||F(\mathbf{z}^1) - F(\mathbf{z}^2)|||_{1+\alpha}(t) \leq \widetilde{C}\int_0^t \|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha}(s)ds.$$

so, using the induction hypothesis, we obtain

$$|||F^{k+1}(\mathbf{z}^1) - F^{k+1}(\mathbf{z}^2)|||_{1+\alpha}(t) \leq \widetilde{C}\|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha,0}\int_0^t \frac{\widetilde{C}s^k}{k!}ds = \frac{\widetilde{C}t^{k+1}}{(k+1)!}\|\mathbf{z}^1 - \mathbf{z}^2\|_{1+\alpha,0}.$$

Therefore, (77) holds for all $m = 1, 2, \ldots$. □

In this way, as

$$\|F(\mathbf{z}^1) - F(\mathbf{z}^2)\|_{0,1+\alpha} \leq C_1|||F^{k+1}(\mathbf{z}^1) - F^{k+1}(\mathbf{z}^2)|||_{0,1+\alpha} \leq \frac{C_1\widetilde{C}1^m}{m!}\|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1+\alpha}$$

So, how $C_1\widetilde{C} \leq K := K(\nu, \delta)$ we can take m large enough such that $\frac{K1^m}{m!} < 1$. We fix this m, so we have that $F^m : S_\nu \to S_\nu$ verifies

$$\|F^m(\mathbf{z}^1) - F^m(\mathbf{z}^2)\|_{0,1+\alpha} \leq \gamma\|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1+\alpha},\ \forall\ \mathbf{z}^1, \mathbf{z}^2 \in S_\nu, \tag{57}$$

where $\gamma \in (0, 1)$.

Then, defining $B := S_\nu$ and applying the theorem 3 to $G := F^m$ we obtain that F^m has a unique fixed-point $\mathbf{y} \in \overline{S_\nu}^{\|\cdot\|_{0,1+\alpha}}$. This ends the proof.

−−−

Therefore, $(\mathbf{y},\zeta,\theta)$ verifies

$$\begin{cases} \zeta_t + \mathbf{y}\cdot\nabla\zeta = -\vec{k}\times\nabla\theta, & \text{in} \quad \overline{\Omega}\times[0,1], \\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0, & \text{in} \quad \overline{\Omega}\times[0,1], \\ \mathbf{y}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = (\overline{\mathbf{y}}(\mathbf{x},t)+\mu(t)\mathbf{y}_0(\mathbf{x}))\cdot\mathbf{n}(\mathbf{x}), & \text{on} \quad \Gamma\times[0,1], \\ \nabla\cdot\mathbf{y} = 0,\ \nabla\times\mathbf{y} = \zeta & \text{in} \quad \overline{\Omega}\times[0,1], \\ \zeta(0) = \nabla\times\mathbf{y}_0,\ \theta(0) = \theta_0, & \text{in} \quad \overline{\Omega}, \end{cases} \tag{58}$$

From the first and fourth equation of (58), we have

$$\nabla\times\left(\mathbf{y}_t - (\mathbf{y}\cdot\nabla)\mathbf{y} - \vec{\mathbf{k}}\theta\right) = 0$$

and then there exists a pressure $p\in\mathscr{D}'(\overline{\Omega}\times[0,1])$ such that

$$\mathbf{y}_t - (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta,\ \text{in } \Omega\times[0,1].$$

From the fourth and fifth equation of (58), we have

$$\nabla\times(\mathbf{y}(0)-\mathbf{y}_0) = 0$$

and then there exists a function $g\in\mathscr{D}'(\overline{\Omega})$ such that

$$\mathbf{y}(0)-\mathbf{y}_0 = \nabla g.$$

How $\nabla\cdot\mathbf{y} = \nabla\cdot\mathbf{y}_0 = 0$ and $(\mathbf{y}(0)-\mathbf{y}_0)\cdot\mathbf{n} = 0$ in Γ we have that g is solution of the elliptic problem

$$\begin{cases} \Delta g = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial g}{\partial\mathbf{n}} = 0 & \text{on} \quad \Gamma, \end{cases}$$

Whence, g is constant and then $\mathbf{y}(0) = \mathbf{y}_0$.

From the fact that $\zeta = 0$ in $\overline{\Omega}\times[T_2,1]$ and the fourth equation of (58), we have for each $t\in[T_2,1]$

$$\nabla\times\mathbf{y}(t) = 0$$

and then there exists a function $h\in\mathscr{D}'(\overline{\Omega}\times[T_2,1])$ such that

$$\mathbf{y}(t) = \nabla h(t).$$

How $\nabla\cdot\mathbf{y} = 0$ and $\mathbf{y}(t)\cdot\mathbf{n} = 0$ in Γ we have that h is solution of the elliptic problem

$$\begin{cases} \Delta h = 0 & \text{in} \quad \Omega\times[T_2,1], \\ \dfrac{\partial h}{\partial\mathbf{n}} = 0 & \text{on} \quad \Gamma\times[T_2,1], \end{cases}$$

Whence, h is constant and then $\mathbf{y} = 0$ in $\overline{\Omega}\times[T_2,1]$.

5 Proof of theorem 2 - Tridimensional case

In this section we are going to prove the theorem 1 in the tridimensional case.

To do this let $\{\psi^i\}$ a partition of the unity associated to the balls B^i such that $\sum_{i=1}^{k} \psi^i = 1$ over $\overline{\Omega}$. We are going to denote $\boldsymbol{\omega}_0$ by $\boldsymbol{\omega}_0 = \nabla \times \pi_3(\mathbf{y}_0)$. The theorem 1 is a consequence of the following proposition:

Proposition 3 *There exists $\delta > 0$ such that if*

$$\max\{\|\mathbf{y}_0\|_{C^{2+\alpha}}, \|\theta_0\|_{C^{2+\alpha}}\} < \delta$$

then the system

$$\begin{cases} \boldsymbol{\omega}_t + (\mathbf{y}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{y} - \vec{k}\times\nabla\theta, & \textit{in} \quad \overline{\Omega}\times[0,1],\\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0, & \textit{in} \quad \overline{\Omega}\times[0,1],\\ \mathbf{y}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = (\overline{\mathbf{y}}(\mathbf{x},t) + \mu(t)\mathbf{y}_0(\mathbf{x}))\cdot\mathbf{n}(\mathbf{x}), & \textit{on} \quad \Gamma\times[0,1],\\ \nabla\cdot\mathbf{y} = 0,\ \nabla\times\mathbf{y} = \boldsymbol{\omega} & \textit{in} \quad \overline{\Omega}\times[0,1],\\ \boldsymbol{\omega}(0) = \nabla\times\mathbf{y}_0,\ \theta(0) = \theta_0, & \textit{in} \quad \overline{\Omega}, \end{cases} \tag{59}$$

has a unique solution $(\boldsymbol{\omega}, \theta, \mathbf{y})$, with

$$\boldsymbol{\omega} \in C^0([0,1]; C^{\alpha}(\overline{\Omega};\mathbb{R})),\ \theta \in [C^0([0,1]; C^{\alpha+1}(\overline{\Omega};\mathbb{R}))$$

and

$$\mathbf{y} \in [C^0([0,1]; C^{\alpha+1}(\overline{\Omega};\mathbb{R}^3)) \cap L^{\infty}([0,1]; C^{2+\alpha}(\overline{\Omega}))$$

such that $\theta(\mathbf{x},t) = 0$, for all $(\mathbf{x},t) \in \overline{\Omega}\times[t_{k-\frac{1}{2}},1]$ and $\boldsymbol{\omega}(\mathbf{x},t) = 0$, for all $(\mathbf{x},t) \in \overline{\Omega}\times[t_{2k-\frac{1}{2}},1]$.

Proof: In this proof of a existence's result we are going to follow the ideas of Bardos and Frisch (see [**?**]).

First, we consider the calss

$$R' = \left\{\mathbf{z} \in C^0([0,1]; C^{2+\alpha}(\overline{\Omega}));\ \nabla\cdot\mathbf{z} = 0 \text{ in } \Omega\times[0,1] \text{ and } \mathbf{z}\cdot\mathbf{n} = (\overline{\mathbf{y}} + \mu\mathbf{y}_0)\cdot\mathbf{n} \text{ on } \Gamma\right\}.$$

For $\nu > 0$ we are going to denote by

$$R_\nu = R' \cap \{\mathbf{z} \in C^0([0,1]; C^{2+\alpha}(\overline{\Omega};\mathbb{R}^3)); \|\mathbf{z} - \overline{\mathbf{y}}\|_{0,2+\alpha} \le \nu\}.$$

We are going fix R_ν, where $\nu > 0$ is guaranteed by the lemma 11.

Now, we are going to define a application $F: R_\nu \to R_\nu$. We start from a arbitrary vector function $\mathbf{z} = \mathbf{z}(\mathbf{x},t)$ in R_ν. With this $\mathbf{z}$ we find θ^* verifying

$$\begin{cases} \theta^*_t + \widetilde{\mathbf{z}}\cdot\nabla\theta^* = 0, & \text{in} \quad \widetilde{\Omega}_2\times[0,1/2],\\ \theta(0) = \sum_{i=1}^{k}\psi^i\pi_1(\theta_0), & \text{in} \quad \widetilde{\Omega}_2,\\ \widetilde{\mathbf{z}} := \mathbf{y}^* + \pi_3(\mathbf{z} - \overline{\mathbf{y}}) \end{cases}$$

Now, we are going to decompose the solution θ^* in a sum of some function. To do this, we consider the solution of the problem

$$\begin{cases} \theta^i_t + \widetilde{\mathbf{z}} \cdot \nabla \theta^i = 0, & \text{in} \quad \widetilde{\Omega}_2 \times [0, \frac{1}{2}], \\ \theta(0) = \psi^i \pi_1(\theta_0), & \text{in} \quad \widetilde{\Omega}_2. \end{cases} \tag{60}$$

So,

$$\theta^* = \sum_{i=1}^{k} \theta^i.$$

The equality

$$\theta^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 0), t) = \psi^i(\mathbf{x})\pi_1(\theta_0)(\mathbf{x}) \tag{61}$$

implies that

$$\textit{supp}\ \theta^i(\cdot, t_{i-\frac{1}{2}}) \subset \mathbf{X}_{\widetilde{\mathbf{z}}}(B^i, t_{i-\frac{1}{2}}, 0). \tag{62}$$

Since $\mathbf{z} \in R_\nu$, we can use the Lemma 11 and the inclusion above to obtain

$$\begin{cases} \theta^i\left(\cdot, t_{i-\frac{1}{2}} - \beta(t_{i-\frac{1}{2}} - t_{i-1})\right) = 0, & \text{in} \quad \overline{\Omega}. \\ \theta^i\left(\cdot, t_{i-\frac{1}{2}} + \beta(t_i - t_{i-\frac{1}{2}})\right) = 0, & \text{in} \quad \overline{\Omega}. \end{cases} \tag{63}$$

for all $\beta \in [0, 1/4]$.

Then, we can define θ by

$$\theta(\mathbf{x}, t) := \theta^*(\mathbf{x}, t) \ \text{ in } \ \overline{\widetilde{\Omega}}_2 \times [0, 1/4], \tag{64}$$

then for $t \in [1/4, 1/2]$:

$$\theta_t + \widetilde{\mathbf{z}} \cdot \nabla \theta = 0, \ \text{ in } \ \widetilde{\Omega}_2 \times \left([1/4, 1/2] \setminus \bigcup_{i=1}^{k} t_{i-\frac{1}{2}}\right). \tag{65}$$

Thus to define θ properly, we have yet to define it at times $t_{i+\frac{1}{2}}$. To do this, and in the others points we can define θ by

$$\theta(\mathbf{x}, t_{i-\frac{1}{2}}) := \sum_{l=i+1}^{k} \theta^l(\mathbf{x}, t_{i-\frac{1}{2}}) \quad \text{in} \quad \widetilde{\Omega}_2,$$

with the convention

$$\theta(\mathbf{x}, t) := 0 \quad \text{in} \quad \widetilde{\Omega}_2 \times [t_{k-\frac{1}{2}}, 1].$$

By (60) and (65) we get

$$\theta(\mathbf{x}, t) = \sum_{l=i+1}^{k} \theta^l(\mathbf{x}, t) \quad \text{in} \quad \widetilde{\Omega} \times [t_{i-\frac{1}{2}}, t_{i+\frac{1}{2}}]. \tag{66}$$

Therefore, in particular we have

$$\begin{cases} \theta_t + \mathbf{z} \cdot \nabla \theta = 0, & \text{in} \quad \overline{\Omega} \times [0, \frac{1}{2}], \\ \theta(0) = \theta_0, & \text{in} \quad \overline{\Omega}. \end{cases} \tag{67}$$

with $\theta(\mathbf{x}, t) := 0$ in $\overline{\Omega} \times [t_{k-\frac{1}{2}}, 1]$.

By (63) and the fact that $\theta^*, \theta^i \in C^0([0,1]; C^{2+\alpha}(\overline{\widetilde{\Omega}_2}))$ we have that θ satisfies

$$\theta|_{\overline{\Omega} \times [0,1]} \in C^0([0,1]; C^{2+\alpha}(\overline{\Omega})).$$

Also, we have the following lemma:

Lemma 16 *For each $t \in [0, 1/2]$, we have the inequality*

$$\|\theta\|_{C^{\alpha+2}}(t) \leq C \|\theta_0\|_{C^{2+\alpha}} exp\left(C \int_0^t \|\widetilde{\mathbf{z}}\|_{C^{2+\alpha}}(\sigma) d\sigma \right).$$

Proof: Applying the lemma 1 to equation (60) like in the lemma **??**, we have

$$\|\theta^i\|_{C^{2+\alpha}}(t) \leq C \|\theta_0\|_{C^{2+\alpha}} exp\left(C \int_0^t \|\widetilde{\mathbf{z}}\|_{C^{2+\alpha}}(\sigma) d\sigma \right).$$

Then, thanks to the (64) and (66), we conclude the proof. □

With this θ we obtain $\boldsymbol{\omega}^*$ given by

$$\begin{cases} \boldsymbol{\omega}^*_t + (\widetilde{\mathbf{z}} \cdot \nabla) \boldsymbol{\omega}^* = (\boldsymbol{\omega}^* \cdot \nabla) \widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}}) \boldsymbol{\omega}^* - \overrightarrow{\mathbf{k}} \times \nabla \pi_1(\theta) & \text{in} \quad \widetilde{\Omega}_2 \times [0, 1/2], \\ \boldsymbol{\omega}^*(0) = \boldsymbol{\omega}_0 & \text{in} \quad \widetilde{\Omega}_2. \end{cases}$$

Using the same ideas to proof the Lemma **??**, we can prove the result:

Lemma 17 *Then, for each $t \in [0, 1/2]$ we have the inequality*

$$\|\boldsymbol{\omega}^*\|_{C^{1+\alpha}}(t) \leq C(\|\mathbf{y}_0\|_{C^{2+\alpha}} + \|\theta_0\|_{C^{2+\alpha}}) exp\left(C \int_0^t \|\widetilde{\mathbf{z}}\|_{C^{2+\alpha}}(\sigma) d\sigma \right).$$

Let $\widehat{\boldsymbol{\omega}}_{1/2} \in C^{1+\alpha}(\overline{\Omega})$ given by $\widehat{\boldsymbol{\omega}}_{1/2} = \boldsymbol{\omega}^*(\cdot, 1/2)$. We consider the solution of the problem

$$\begin{cases} \widehat{\boldsymbol{\omega}}_t + (\widetilde{\mathbf{z}} \cdot \nabla) \widehat{\boldsymbol{\omega}} = (\widehat{\boldsymbol{\omega}} \cdot \nabla) \widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}}) \widehat{\boldsymbol{\omega}}, & \text{in} \quad \widetilde{\Omega}_2 \times [1/2, 1], \\ \widehat{\boldsymbol{\omega}}(\cdot, 1/2) = \sum_{i=1}^{k} \psi^i \pi_1(\widehat{\boldsymbol{\omega}}_{1/2}), & \text{in} \quad \widetilde{\Omega}_2. \end{cases} \tag{68}$$

Now, we are going to decompose the solution $\widehat{\boldsymbol{\omega}}$ in a sum of some function. To do this, we consider the solution of the problem

$$\begin{cases} \widehat{\boldsymbol{\omega}}^i_t + (\widetilde{\mathbf{z}} \cdot \nabla) \widehat{\boldsymbol{\omega}}^i = (\widehat{\boldsymbol{\omega}}^i \cdot \nabla) \widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}}) \widehat{\boldsymbol{\omega}}^i, & \text{in} \quad \widetilde{\Omega}_2 \times [1/2, 1], \\ \widehat{\boldsymbol{\omega}}^i(\cdot, 1/2) = \psi^i \widehat{\boldsymbol{\omega}}_{1/2}, & \text{in} \quad \widetilde{\Omega}_2. \end{cases} \tag{69}$$

So,

$$\widehat{\boldsymbol{\omega}} = \sum_{i=1}^{k} \widehat{\boldsymbol{\omega}}^i.$$

The solution $\widehat{\boldsymbol{\omega}}^i$ can be written as

$$\widehat{\boldsymbol{\omega}}^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 1/2), t) = \widehat{\boldsymbol{\omega}}^i(\mathbf{x}, 1/2) + \int_{1/2}^{t} [(\widehat{\boldsymbol{\omega}}^i \cdot \nabla)\widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}})\widehat{\boldsymbol{\omega}}^i](\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, \sigma, 1/2), \sigma) d\sigma.$$

Then,

$$|\widehat{\boldsymbol{\omega}}^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 1/2), t)| \leq |\widehat{\boldsymbol{\omega}}^i(\mathbf{x}, 1/2)| + C\|\widetilde{\mathbf{z}}\|_{0,1} \int_{1/2}^{t} |\widehat{\boldsymbol{\omega}}^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, \sigma, 1/2), \sigma)| d\sigma.$$

Notice that if $\mathbf{x} \notin B^i$ we have

$$|\widehat{\boldsymbol{\omega}}^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 1/2), t)| \leq C\|\widetilde{\mathbf{z}}\|_{0,1} \int_{1/2}^{t} |\widehat{\boldsymbol{\omega}}^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, \sigma, 1/2), \sigma)| d\sigma.$$

For the Gronwall's Lemma, we see that

$$\widehat{\boldsymbol{\omega}}^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 1/2), t) = 0, \;\; (\mathbf{x}, t) \in (\widetilde{\Omega}_2 \setminus B^i) \times (0, T)$$

whence

$$supp \; \widehat{\boldsymbol{\omega}}^i(\cdot, t) \subset \mathbf{X}_{\widetilde{\mathbf{z}}}(B^i, t, 1/2).$$

In this way,

$$\widehat{\boldsymbol{\omega}}^i(\mathbf{x}, t_{k+i-\frac{1}{2}}) = 0, \;\; \forall \mathbf{x} \in \overline{\Omega}.$$

Then, we can define $\boldsymbol{\omega}$ by

$$\boldsymbol{\omega}(\mathbf{x}, t) := \boldsymbol{\omega}^*(\mathbf{x}, t) \;\; \text{in} \;\; \widetilde{\Omega}_2 \times [0, 1/2], \tag{70}$$

then for $t \in [1/2, 1]$:

$$\boldsymbol{\omega}_t + (\widetilde{\mathbf{z}} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}})\boldsymbol{\omega}, \;\; \text{in} \;\; \widetilde{\Omega}_2 \times \left([1/2, 1] \setminus \bigcup_{i=1}^{k} t_{k+i-\frac{1}{2}} \right). \tag{71}$$

Thus to define $\boldsymbol{\omega}$ properly, we have yet to define it at times $t_{k+i+\frac{1}{2}}$. To do this, in the others points we can define $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega}(\mathbf{x}, t_{k+i-\frac{1}{2}}) := \sum_{l=i+1}^{k} \widehat{\boldsymbol{\omega}}^l(\mathbf{x}, t_{k+i-\frac{1}{2}}) \quad \text{in} \quad \widetilde{\Omega}_2,$$

with the convention

$$\boldsymbol{\omega}(\mathbf{x}, t) := 0 \quad \text{in} \quad \widetilde{\Omega}_2 \times [t_{2k-\frac{1}{2}}, 1].$$

By (69), (70) and (71) we get

$$\boldsymbol{\omega}(\mathbf{x}, t) = \sum_{l=i+1}^{k} \widehat{\boldsymbol{\omega}}^l(\mathbf{x}, t) \quad \text{in} \quad \widetilde{\Omega} \times [t_{k+i-\frac{1}{2}}, t_{k+i+\frac{1}{2}}]. \tag{72}$$

Using the Lemma 16, Lemma 17 and analogues lemmas to $\widehat{\boldsymbol{\omega}}^i$, we have the result

Lemma 18 *Then, for each $t \in [0,1]$ we have the inequality*

$$\|\boldsymbol{\omega}\|_{C^{1+\alpha}}(t) \leq C(\|\mathbf{y}_0\|_{C^{2+\alpha}} + \|\theta_0\|_{C^{2+\alpha}})exp\Big(C\int_0^t \|\widetilde{\mathbf{z}}\|_{C^{2+\alpha}}(\sigma)d\sigma\Big).$$

This way, we get that the restriction of $\boldsymbol{\omega}$ to $\overline{\Omega}\times[0,1]$ is $C^0([0,1];C^{1+\alpha}(\overline{\Omega}))$-regular and that we have in $\overline{\Omega}\times[0,1]$ the relation

$$\begin{cases} \boldsymbol{\omega}_t + (\widetilde{\mathbf{z}}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\widetilde{\mathbf{z}} - (\nabla\cdot\widetilde{\mathbf{z}})\boldsymbol{\omega} - \overrightarrow{\mathbf{k}}\times\nabla\theta & \text{in} \quad \Omega\times[0,1], \\ \theta_t + \widetilde{\mathbf{z}}\cdot\nabla\theta = 0 & \text{in} \quad \Omega\times[0,1], \\ \theta(0) = \theta_0, \;\; \boldsymbol{\omega}(0) = \nabla\times\mathbf{y}_0 & \text{in} \quad \Omega \end{cases}$$

is such that $\boldsymbol{\omega}(\mathbf{x},t) := 0$ in $\widetilde{\Omega}\times[t_{2k-\frac{1}{2}},1]$ and $\theta(\mathbf{x},t) := 0$ in $\widetilde{\Omega}\times[t_{k-\frac{1}{2}},1]$.

Firstly, thanks to the Lemma 2, $\boldsymbol{\omega}$ stays divergence-free in $\overline{\widetilde{\Omega}_2}\times[0,1]$.

It is known that there exists a vector function Ψ such that

$$\begin{cases} \nabla\times\Psi = \boldsymbol{\omega}^* - \mu\nabla\times\mathbf{y}_0 & \text{in} \quad \overline{\Omega}\times[0,1], \\ \nabla\cdot\Psi = 0 & \text{in} \quad \Omega\times[0,1], \\ \Psi\cdot\mathbf{n} = 0 & \text{on} \quad \partial\Omega\times[0,1]. \end{cases} \tag{73}$$

Also, there exists a vector function Ψ^* such that

$$\begin{cases} \nabla\times\Psi^* = \Psi & \text{in} \quad \overline{\Omega}\times[0,1], \\ \nabla\cdot\Psi^* = 0 & \text{in} \quad \Omega\times[0,1], \\ \Psi^*\cdot\mathbf{n} = 0 & \text{on} \quad \partial\Omega\times[0,1]. \end{cases} \tag{74}$$

We note that Ψ^* can by characterized as the unique solution of the problem

$$\begin{cases} -\Delta\Psi^* = \nabla\times\Psi = \boldsymbol{\omega} - \mu\nabla\times\mathbf{y}_0, & \text{in} \quad \Omega, \\ \nabla\cdot\Psi^* = 0 & \text{in} \quad \Omega\times[0,1/2], \\ (\nabla\times\Psi^*)\cdot\mathbf{n} = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

So, with this Ψ^*, there exists a unique $\mathbf{y}$ in $C([0,1];C^{2+\alpha}(\overline{\Omega}))$ such that

$$\begin{cases} \nabla\times\mathbf{y} = \boldsymbol{\omega} & \text{in} \quad \overline{\Omega}\times[0,1], \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad \Omega\times[0,1], \\ \mathbf{y}\cdot\mathbf{n} = (\mu\mathbf{y}_0 + \overline{\mathbf{y}})\cdot\mathbf{n} & \text{on} \quad \partial\Omega\times[0,1]. \end{cases} \tag{75}$$

Therefore, since $\mathbf{y}$ is uniquely determined by $\mathbf{z}$, we write

$$\begin{array}{rcl} F : R_\nu & \rightarrow & R' \\ \mathbf{z} & \mapsto & F(\mathbf{z}) = \mathbf{y}. \end{array} \tag{76}$$

Thanks to the lemma above, we can take the initial data small enough such that $F(R_\nu)\subset R_\nu$. More precisely,

Lemma 19 *There exists $\delta > 0$ such that if $\{\|\mathbf{y}_0\|_{2+\alpha}, \|\theta_0\|_{2+\alpha}\} \leq \delta$ $F : R_\nu \to R_\nu$ is well defined.*

In order to prove that F has a fixed-point, we need to prove that, in some sense, F is a contraction. For this purpose, we will apply the Lemma 3, as was done before.

With the same ideas of the Lemma 15, we can prove the result:

Lemma 20 *For each $\mathbf{z}^1$, $\mathbf{z}^2 \in R_\nu$, if $\widetilde{C} = C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})$ then we have that*

$$|||F^m(\mathbf{z}^1) - F^m(\mathbf{z}^2)|||_{1+\alpha}(t) \leq \frac{\widetilde{C}t^m}{m!}\|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1+\alpha} \tag{77}$$

for all $m \in \mathbb{N}$.

Then, if m is great enough, we have that $F^m : R_\nu \to R_\nu$ verifies

$$\|F^m(\mathbf{z}^1) - F^m(\mathbf{z}^2)\|_{0,1+\alpha} \leq \gamma\|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1+\alpha},\ \forall\ \mathbf{z}^1, \mathbf{z}^2 \in S_\nu, \tag{78}$$

where $\gamma \in (0, 1)$.

Therefore, applying the theorem 3 to F^m we obtain that F^m has a unique fixed-point $\mathbf{y} \in \overline{R_\nu}^{\|\cdot\|_{0,1+\alpha}}$. This ends the proof. □

6 Proof of Theorem 2

We reduce the proof of this Theorem to the following result:

Proposition 4 *For each $\mathbf{y}_0 \in \mathbf{C}_0^{2,\alpha}(\overline{\Omega}; \mathbb{R}^N)$ there exist a time $T^* \in (0, T)$ and $\delta > 0$ such that if $\|\theta_0\|_{2,\alpha} \leq \delta$, then the system*

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{in} \quad \Omega\times(0,T^*), \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad \Omega\times(0,T^*), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = \kappa\,\Delta\theta & \text{in} \quad \Omega\times(0,T^*), \\ \mathbf{y}\cdot\mathbf{n} = 0 & \text{on} \quad \Gamma\times(0,T^*) \\ \theta = 0 & \text{on} \quad (\Gamma\backslash\Gamma_1)\times(0,T^*) \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega, \end{cases} \tag{79}$$

possesses at least one solution $\mathbf{y} \in C^0([0,T^*]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^N))$, $\theta \in C^0([0,T^*]; C^{2,\alpha}(\overline{\Omega}; \mathbb{R}))$ and $p \in \mathscr{D}'(\Omega\times(0,T^*))$ such that $\theta(\mathbf{x}, T^*) = 0$ in Ω.

Proof: For simplicity, let us consider the case $N = 2$ and suppose that $\theta_0 \in C^{2,\alpha'}(\overline{\Omega})$, $\alpha' > \alpha$. We will apply a fixed point argument to guarantee the existence of solution to the problem (79).

We will start from an arbitrary thermal source term $\tilde{\theta} := \tilde{\theta}(\mathbf{x}, t)$ in $C^0([0, T/2]; C^{2,\alpha}(\overline{\Omega}))$. Together with this $\tilde{\theta}$, we will construct a flow $\mathbf{y} \in C^0([0, T/2]; \mathbf{C}^{2,\alpha}(\overline{\Omega}))$ verifying

$$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\tilde{\theta} & \text{in} \quad \Omega \times (0, T/2), \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad \Omega \times (0, T/2), \\ \mathbf{y} \cdot \mathbf{n} = 0 & \text{on} \quad \Gamma \times (0, T/2), \\ \mathbf{y}(\cdot, 0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{80}$$

verifying the inequality

$$\|\mathbf{y}(\cdot, t)\|_{2,\alpha} \leq C(\|\mathbf{y}_0\|_{2,\alpha} + \|\tilde{\theta}\|_{C^0(C^{2,\alpha})}).$$

Let $\widetilde{\Omega}$ be given, with $\Omega \subset \widetilde{\Omega}$ and $\partial\widetilde{\Omega} \cap \Gamma = \Gamma \setminus \gamma$ such that $\partial\widetilde{\Omega}$ is of class C^2 (see Fig. 2); let $\boldsymbol{\omega} \subset \widetilde{\Omega} \setminus \overline{\Omega}$ be a non-empty open subset.

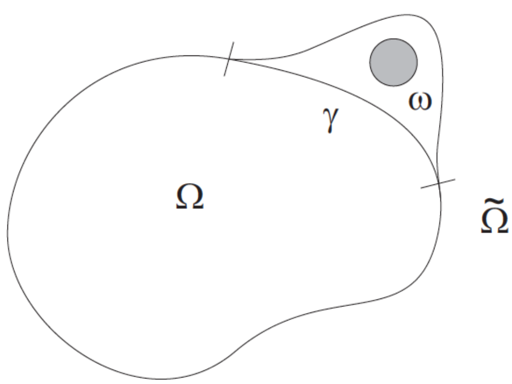

Figure 2: The domain $\widetilde{\Omega}$

Then, with this flow $\mathbf{y}$ we get a unique pair $(\overline{\theta}, \overline{v})$, see Theorem 4, satisfying

$$\begin{cases} \overline{\theta}_t + \pi(\mathbf{y}) \cdot \nabla\overline{\theta} = \kappa\Delta\overline{\theta} + \overline{v}1_{\boldsymbol{\omega}} & \text{in} \quad \overline{\widetilde{\Omega}} \times [0, T/2], \\ \overline{\theta} = 0 & \text{on} \quad \widetilde{\Gamma} \times [0, T/2], \\ \overline{\theta}(\cdot, 0) = \tilde{\pi}(\theta_0), \quad \overline{\theta}(\cdot, T/2) = 0, & \text{in} \quad \overline{\widetilde{\Omega}}, \end{cases} \tag{81}$$

where π and $\tilde{\pi}$ are extension operator of Ω to $\tilde{\Omega}$ that preserve regularity. So, we define $\theta := \tilde{\theta}|_{\overline{\Omega} \times [0, T/2]}$ and then θ satisfies:

$$\begin{cases} \theta_t + \mathbf{y} \cdot \nabla\theta = \kappa\Delta\theta & \text{in} \quad \overline{\Omega} \times [0, T/2], \\ \theta = 0 & \text{on} \quad \Gamma \times [0, T/2], \\ \theta(\cdot, 0) = \theta_0, \quad \theta(\cdot, T/2) = 0, & \text{in} \quad \overline{\Omega}, \end{cases} \tag{82}$$

Moreover, the following inequalities hold:

$$\|\theta(\cdot,t)\|_{0,\alpha'} \leq \|\theta_0\|_{2,\alpha'} e^{C(\|\mathbf{y}_0\|_{2,\alpha}+\|\tilde{\theta}\|_{C^0(C^{2,\alpha})})},$$

$$\|\theta_t\|^2_{C^0(C^{2,\alpha'})} + \|\theta\|^2_{C^0(C^{2,\alpha'})} \leq C(\|\theta_0\|^2_{2,\alpha'} + \|h\|^2_{C^0(C^{2,\alpha'})}) e^{C\|\mathbf{y}\|^2_{C^0(C^{2,\alpha})}}.$$

Now, let us set

$$W = \{\, \theta \in C^0([0,T/2]; C^{2,\alpha'}(\overline{\Omega})) : \theta_t \in C^0([0,T/2]; C^{0,\alpha'}(\overline{\Omega}))\,\}$$

and let us consider the closed ball

$$R = \{\, \tilde{\theta} \in C^0([0,T/2]; C^{2,\alpha}(\overline{\Omega})) : \|\tilde{\theta}\|_{C^0(C^{2,\alpha})} \leq 1 \,\}$$

and the mapping $\tilde{F}$, with $\tilde{F}(\tilde{\theta}) = \theta$ for all $\tilde{\theta} \in C^0([0,T/2]; C^{2,\alpha}(\overline{\Omega}))$. Obviously, $\tilde{F}$ is well defined; furthermore, in view of the inequalities above, it maps the whole space $C^0([0,T/2]; C^{2,\alpha}(\overline{\Omega}))$ into W.

Notice that, if $\mathbf{U}$ is bounded set of W then it is relatively compact in the space $C^0([0,T/2]; C^{2,\alpha}(\overline{\Omega}))$, in view of the classical results of the Aubin-Lions kind, see for instance [**?**].

Let us denote by F the restriction to R of $\tilde{F}$. Then, if $\|\theta_0\|_{2,\alpha} \leq \varepsilon$, F maps R into itself.

Moreover, it is clear that $F : R \mapsto R$ satisfies the hypotheses of Schauder's Theorem. Indeed, this nonlinear mapping is continuous and compact (the latter is a consequence of the fact that, if $\mathbf{B}$ is bounded in $C^0([0,T/2]; C^{2,\alpha}(\overline{\Omega}))$, then $\tilde{F}(\mathbf{B})$ is bounded in W). Consequently, F possesses at least one fixed point in R, and this ends the proof of Proposition.
□

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