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Clustering Switching Regions for Feedback Controllers: A Convex Approach

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***Abstract*—Coalitional control groups dynamically local controllers into clusters that jointly determine their control actions to maximize control performance while minimizing the cooperation burden. This work presents linear matrix inequalities (LMIs) decision methods to set state-space regions where the switchings between network topologies satisfy properties of interest. In particular, *convexity* guarantees ellipsoidal switching sets with back-switchings avoided via a dwell time. Also, the convexity property is exploited to analyze the need for coordination at different points of the state space, leading to coordination effort zones. Moreover, by considering *invariance*, new ellipsoids that confine the state between topologies transitions are provided. Finally, we introduce additional conditions to attain *submodularity* and hence reduce the effort to find optimal solutions. A numerical example is given to illustrate the feasibility of the proposed approach.**

***Index Terms*—Distributed control, Distributed feedback gains, Control by clustering, Coalitional control, Linear control systems, Network topologies, Control systems design.**

I. Introduction

NON-centralized control strategies have gained attention due to the growing size and complexity of many systems [1]. The underlying idea is a division of the global system into a set of dynamically coupled subsystems so that the system-wide problem can be split into a set of smaller subproblems that are assigned to different control entities, the so-called *agents*, see [2], [3]. These strategies can broadly be divided into two groups: distributed approaches, where the set of agents uses a communication network to share local data so that the controllers' decisions are taken with some information of their neighboring subsystems [4], [5]; and decentralized approaches, when there is no information exchange between controllers [6], [7]. In this context, the availability of neighbors information in distributed control schemes commonly provides enhanced global performance, while decentralized strategies avoid any dependence on the communication infrastructure. In general, strongly coupled subsystems require denser communication to maintain a certain degree of performance, whereas the benefits of sharing data among weakly interacting subsystems may be negligible [8].

As a halfway approach, recent works propose control architectures where the network that interconnects local controllers becomes a manipulated *variable*. In this context, [9] introduces the so-called *coalitional control*, where the set of agents is dynamically divided into disjoint communication components or *coalitions*, i.e., clusters of cooperating controllers that operate in a decentralized manner with respect to the rest of the system. While other works embrace the same coalitional formulation [10]–[14], this type of approach also appears in the literature under different names, e.g., community detection [15], controller reconfiguration [16], sparsity-promoting control [17], coupling degree clustering [18], and dynamic partitioning [19]. These ideas are gaining relevance in the control of large-scale systems where the coupling conditions vary remarkably in time. For example, variable controller structures are proposed for controlling water levels in irrigation canals in [10]; improving oscillation damping in power systems in [17], [20]; and coordinating distributed energy resources in [21].

This work extends the control scheme proposed in [9]. Under this framework, the topology of the communication network is dynamically selected from the set of possible alternatives. Once a topology is active, the resulting coalitions are independently governed by a linear feedback controller, i.e., the coalitional input is computed using exclusively the state from the merged subsystems. In this regard, a block-structured global feedback is precomputed offline for each possible cooperation scenario by solving an optimization problem subject to linear matrix inequalities (LMIs). Also, [22]–[25] propose LMI-based controllers with structural constraints that foster a decentralization of the network and hence minimize the amount of data that needs to be transmitted. Our approach here consists of a decision procedure that preselects a subset of possible future topologies considering the current configuration of the network. To this end, a performance index evaluates the suitability of each topology as a function of the system state, considering both the expected improvement for the global behavior and the communication costs involved. Also, as pointed out in [9], the state space can be partitioned into subsets where one topology offers the best solution, and hence *dominates* the rest of possible alternatives. However, the resulting regions are non-convex in general, thus complicating their use for practical control purposes. In [26], a method to convexify the mentioned *dominance sets* was introduced by considering new LMI constraints in the controller design procedure. In this paper, this result is extended in the following directions:

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- First, a dwell-time approach for the controller in [26] is introduced to avoid switching back to prior communication topologies.
- Also, the convex dominant sets are refined by different design methods fulfilling *invariance and/or submodularity* properties.
- Finally, novelties regarding the partitioning of the state space in zones of cooperation effort are introduced.

The rest of the paper is organized as follows. Section II introduces the model of the system, describes the control architecture and states the control objective. Section III focuses on the logic for switching between topologies in coalitional control schemes, detailing the procedure for coalitions formation. Section IV presents the proposed controller design method, which assures the abovementioned properties. In Section V, the state space will be partitioned into coordination effort zones, to ease the topology selection. Section VI describes the control scheme and discusses its stability guarantees. Section VII illustrates the proposed approach by an academic example. Finally, concluding remarks are given in Section VIII.

II. System Description

In this section, we present the models used to describe the system dynamics and the communication infrastructure. Additionally, we introduce the control goals pursued by the coalitional strategy.

A. System Dynamics

Consider a class of linear systems that can be partitioned into a set $\mathcal{N} = \{1, 2, \dots, N\}$ of coupled subsystems, whose dynamics are modeled as

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_{ii}u_i(k) + d_i(k), \\ d_i(k) &= \sum_{j\in\mathcal{N}_i} [A_{ij}x_j(k) + B_{ij}u_j(k)], \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_{x_i}}$ and $u_i \in \mathbb{R}^{n_{u_i}}$ are respectively the state and input vectors of subsystem $i \in \mathcal{N}$, and d_i describes the coupling among subsystem i and its set of neighbours, defined as $\mathcal{N}_i = \{j \in \mathcal{N} \setminus \{i\} \mid (A_{ij}, B_{ij}) \neq \mathbf{0}\}$. Likewise, $A_{ij} \in \mathbb{R}^{n_{x_i}\times n_{x_j}}$ and $B_{ij} \in \mathbb{R}^{n_{x_i}\times n_{u_j}}$ are, respectively, the state transition and the input-to-state matrices for all $i, j \in \mathcal{N}$.

The global behavior can be modeled by matrices $A_\mathcal{N} = (A_{ij})_{i,j\in\mathcal{N}}$ and $B_\mathcal{N} = (B_{ij})_{i,j\in\mathcal{N}}$, which aggregate (1) for the N subsystems as

$$x_\mathcal{N}(k+1) = A_\mathcal{N}x_\mathcal{N}(k) + B_\mathcal{N}u_\mathcal{N}(k), \tag{2}$$

where $x_\mathcal{N} = [x_i]_{i\in\mathcal{N}} \in \mathbb{R}^{n_{x_\mathcal{N}}}$ and $u_\mathcal{N} = [u_i]_{i\in\mathcal{N}} \in \mathbb{R}^{n_{u_\mathcal{N}}}$ are the global state and input vectors, respectively.

B. Network Structure

The subsystems in $\mathcal{N}$ are individually governed by a set of local control agents interconnected through a communication network, whose structure is described by graph $(\mathcal{N}, \mathcal{L})$, where the nodes represent local agents, and set $\mathcal{L}$ contains the communication links, i.e.,

$$\mathcal{L} \subseteq \mathcal{L}^\mathcal{N} = \{\{i, j\}| i, j \in \mathcal{N}\}. \tag{3}$$

The state of the links can dynamically be switched between *enabled* and *disabled*, thus restricting the possibilities for sharing local data among control agents. Hereon, symbol Λ denotes in general sets of enabled links or *topologies* describing a configuration of the communication network, hence $\Lambda \subseteq \mathcal{L}$. Accordingly, $\Lambda(k)$ represents the network configuration at a given time instant k. Since the cardinality of $\mathcal{L}$ is $|\mathcal{L}|$, we can derive a set $\mathcal{T}$ of $2^{|\mathcal{L}|}$ possible communication topologies, i.e.,

$$\mathcal{T} = \{\overbrace{\Lambda_0}^{\Lambda_{\text{dec}}}, \Lambda_1, \dots, \overbrace{\Lambda_{2^{|\mathcal{L}|}-1}}^{\Lambda_{\text{cen}}}\}, \tag{4}$$

Each $\Lambda_i \in \mathcal{T}$ arranges the N subsystems into a set $\mathcal{N}/\Lambda_i$ of communication components, i.e., the *coalitions* or *clusters*. Therefore, sets $\mathcal{N}/\Lambda_i$ represent the possible partitions of $\mathcal{N}$, with cardinality ranging from one, when all agents are connected (i.e., $\Lambda_\text{cen} = \mathcal{L}$ and $\mathcal{N}/\Lambda_\text{cen} = \{\mathcal{N}\}$), to N, corresponding to a fully decentralized system (i.e., $\Lambda_\text{dec} = \emptyset$ and $\mathcal{N}/\Lambda_\text{dec} = \{\{1\}, \{2\}, ..., \{N\}\}$).

Dynamically, any coalition $\mathcal{C} \in \mathcal{N}/\Lambda_i$ can be considered as a single system modeled by

$$\begin{aligned} x_\mathcal{C}(k+1) &= A_{\mathcal{C}\mathcal{C}}x_\mathcal{C}(k) + B_{\mathcal{C}\mathcal{C}}u_\mathcal{C}(k) + d_\mathcal{C}(k), \\ d_\mathcal{C}(k) &= \sum_{\mathcal{D}\in\mathcal{N}_\mathcal{C}} [A_{\mathcal{C}\mathcal{D}}x_\mathcal{D}(k) + B_{\mathcal{C}\mathcal{D}}u_\mathcal{D}(k)], \end{aligned} \tag{5}$$

where $x_\mathcal{C} = [x_i]_{i\in\mathcal{C}}$ and $u_\mathcal{C} = [u_i]_{i\in\mathcal{C}}$ are respectively the aggregates of the states and inputs of subsystems $i \in \mathcal{C}$, and matrices $A_{\mathcal{C}\mathcal{C}}$ and $B_{\mathcal{C}\mathcal{C}}$ map the current coalition states and inputs to the successor state. Similarly, $d_\mathcal{C}$ models the effect of neighboring coalitions $\mathcal{D} \in \mathcal{N}_\mathcal{C}$, where matrices $A_{\mathcal{C}\mathcal{D}}$, $B_{\mathcal{C}\mathcal{D}}$ and set $\mathcal{N}_\mathcal{C}$ are defined analogously to the case of interacting subsystems. Note that if $\mathcal{C} = \mathcal{N}$, then $d_\mathcal{C} = \mathbf{0}$, i.e., there are no disturbances due to coupling because all system information is accounted for in the overall model as in (2).

C. Coalitional Control Objective

The coalitional control objective is double: optimizing the system performance and reducing communication costs. In general, the control problem we aim to solve is the following:

$$\begin{aligned} \min_{[\Lambda(k), u_\mathcal{N}(k)]_{k=0}^\infty} \quad & \sum_{k=0}^{\infty} \left(f(\Lambda(k)) + \sum_{\mathcal{C}\in\mathcal{N}/\Lambda(k)} \ell_\mathcal{C}(k) \right), \\ \text{s.t. } & (5),\ \forall \mathcal{C} \in \mathcal{N}/\Lambda(k), \\ & u_\mathcal{N}(k) = [u_\mathcal{C}(k)]_{\mathcal{C}\in\mathcal{N}/\Lambda(k)}, \\ & \Lambda(k) \in \mathcal{T}, \\ & k \in [0, 1, ..., \infty). \end{aligned} \tag{6}$$

The objective function in (6) weights both the stage performance index $\ell_\mathcal{C}(k)$ for all coalitions $\mathcal{C} \in \mathcal{N}/\Lambda(k)$ and the coordination costs incurred by using $\Lambda(k)$, i.e., $f(\Lambda(k))$, where $f(\cdot)$ is a suitable weighting function. For simplicity, we assume that all subsystems should be regulated towards the origin. Hence, we can define $\ell_\mathcal{C} : \mathbb{R}^{n_{x_\mathcal{C}}} \times \mathbb{R}^{n_{u_\mathcal{C}}} \to \mathbb{R}$ as

$$\ell_\mathcal{C}(k) = x_\mathcal{C}^\mathrm{T}(k)Q_\mathcal{C}x_\mathcal{C}(k) + u_\mathcal{C}^\mathrm{T}(k)R_\mathcal{C}u_\mathcal{C}(k), \tag{7}$$

where $Q_\mathcal{C} \geq 0$ and $R_\mathcal{C} > 0$ are weighting matrices.

The optimization of (6) belongs to the class of NP-complete problems [27], which restricts its applicability for real-time control to small networks unless some simplifications are introduced. Note that, at each time instant, global control action $u_{\mathcal{N}}(k)$ depends on the agents partition, and therefore, on variable $\Lambda(k)$. This fact leads to a nonconvex mixed integer optimization problem where, in turn, the number of possibilities for $\Lambda(k)$ experiences a combinatorial explosion with the number of network links. In this work, we make use of the coalitional control scheme proposed in [9], which provides a suboptimal solution for (6) based on periodical switchings between communication topologies.

In what follows, the switching criterion in [9] is modified to satisfy certain constraints that help us characterize the dominance regions of each topology, and hence the switchings. In particular, we will consider *convexity*, *invariance* and *submodularity* as desirable properties. The first two properties are useful to obtain coalitional convex invariant regions, which can be relevant in the context of coalitional model predictive control (MPC) approaches. Also, based on the convexity property, the state space will be partitioned into coordination effort zones, which simplifies the selection of the topology. Finally, submodularity provides us with a different means of obtaining the most suitable topology by enabling links in a sequential manner.

III. Topology Switching Features

In this section, we first describe the control law implemented by the clusters; secondly, we introduce the index used to assess the topologies performance; and finally, we discuss the method adopted to set the transitions between topologies.

A. Control Law

Consider that at time instant k the communication topology is given by Λ_i. Then, every coalition $\mathcal{C} \in \mathcal{N}/\Lambda_i$ implements a linear feedback control law satisfying $u_{\mathcal{C}}(k) = K_{\mathcal{C},\Lambda_i} x_{\mathcal{C}}(k)$, where $K_{\mathcal{C},\Lambda_i}$ is a constant matrix designed to optimize performance and to yield a stable closed-loop system. Globally, the controller is defined by

$$u_{\mathcal{N}}(k) = K_{\Lambda_i} x_{\mathcal{N}}(k), \tag{8}$$

where $K_{\Lambda_i} = (K_{\mathcal{C},\Lambda_i})_{\mathcal{C}\in\mathcal{N}/\Lambda_i}$ aggregates all coalitions feedback gains $K_{\mathcal{C},\Lambda_i}$ into a single matrix. Notice that constructing K_{Λ_i} as indicated, the communication constraints derived from Λ_i are met.

B. Performance

The switchings between topologies intend to find dynamically the most cost-efficient topology $\Lambda_i \in \mathcal{T}$ to maximize the system performance. In this regard, the criterion used to estimate the control performance in different communication scenarios is based on the following assumption:

Assumption 1. *Consider a static communication topology $\Lambda_i \in \mathcal{T}$ and a stabilizing control law (8). Then, there exists a symmetric positive definite matrix $P_{\Lambda_i} = (P_{\mathcal{C},\Lambda_i})_{\mathcal{C}\in\mathcal{N}/\Lambda_i}$ such that*

$$x_{\mathcal{N}}^{\mathrm{T}}(0) P_{\Lambda_i} x_{\mathcal{N}}(0) \geq \sum_{\mathcal{C}\in\mathcal{N}/\Lambda_i} \sum_{k=0}^{\infty} \ell_{\mathcal{C}}(k), \tag{9}$$

where $x_{\mathcal{N}}(0)$ is the system initial state.

Remark 1. *The ultimate goal of Assumption 1 is to provide a criterion to guide the switching decisions. In particular, if matrix P_{Λ_i} satisfies (9), then it provides an upper-bound estimate of the cost-to-go and also a Lyapunov function for the system, i.e., $V_{\Lambda_i}(x_{\mathcal{N}}(k)) = x_{\mathcal{N}}^{\mathrm{T}}(k) P_{\Lambda_i} x_{\mathcal{N}}(k)$. This assumption was taken in [9], [11], and also in [10] to estimate the best topology in a coalitional MPC scheme for irrigation canals. However, other criteria could equally be used as performance indicator of each $\Lambda_i \in \mathcal{T}$, e.g., by taking (8), it is possible to compute the performance costs of each Λ_i during a finite time horizon, which, combined with the coordination costs, also provides an index that can be used to guide the switchings.*

Considering Assumption 1 for each time instant, we define the *optimal* or *dominant* communication topology $\Lambda_i \in \mathcal{T}$ as the one that minimizes the bi-criteria objective $r(\Lambda_i, x_{\mathcal{N}}) : \mathcal{T} \times \mathbb{R}^{x_{\mathcal{N}}} \to \mathbb{R}$, with

$$r(\Lambda_i, x_{\mathcal{N}}(k)) = x_{\mathcal{N}}^{\mathrm{T}}(k) P_{\Lambda_i} x_{\mathcal{N}}(k) + f(\Lambda_i), \tag{10}$$

where the first term weights the system performance and the second one penalizes the communication effort. Hereafter we use $f_i = f(\Lambda_i)$ and assume $f_i > f_j$ for any topologies $\Lambda_i, \Lambda_j \in \mathcal{T}$ such that $|\Lambda_i| > |\Lambda_j|$ and $f_i = f_j$ if $|\Lambda_i| = |\Lambda_j|$.

Topology boundaries: Consider two different topologies Λ_i and Λ_j. Then, matrices P_{Λ_j} and P_{Λ_i} characterize the subset of the state space in which topology Λ_j dominates over Λ_i [26]

$$\Upsilon_{\Lambda_i}^{\Lambda_j} = \{x_{\mathcal{N}} \mid x_{\mathcal{N}}^{\mathrm{T}} \left(P_{\Lambda_j} - P_{\Lambda_i}\right) x_{\mathcal{N}} + \gamma_{ji} \leq 0\}, \tag{11}$$

where $\gamma_{ji} = f_j - f_i$. Hereon, matrices $P_{\Lambda_i}^{\Lambda_j} = P_{\Lambda_j} - P_{\Lambda_i}$ will be referred to as *transition matrices*. Note that if $P_{\Lambda_j} > P_{\Lambda_i}$ and $\gamma_{ji} < 0$, then set $\Upsilon_{\Lambda_i}^{\Lambda_j}$ represents an ellipsoid in $\mathbb{R}^{n_{x_{\mathcal{N}}}}$. In this case, it is possible to identify convex regions where a certain controller structure is the most suitable. Additionally, notice that sets $\Upsilon_{\Lambda_i}^{\Lambda_j}$ represent *fixed* regions of the state space since $P_{\Lambda_i}^{\Lambda_j}$ and γ_{ji} are constant parameters.

C. Coalitions Formation

In this work, the formation of clusters follows a sequential procedure where the communication structure at the moment prior to the switchings is taken into account to determine the subsequent transitions [26]. In this regard, we consider the following *relations* between topologies.

Definition 1. *The set of* children topologies *of any $\Lambda_i \in \mathcal{T}$, where $|\Lambda_i| \geq 1$, is defined as*

$$\mathcal{T}_{\Lambda_i} = \{\Lambda_j \in \mathcal{T} \mid \Lambda_j \subset \Lambda_i \;\; \text{and} \;\; |\Lambda_j| = |\Lambda_i| - 1\}. \tag{12}$$

That is, topologies Λ_j included in set $\mathcal{T}_{\Lambda_i}$ are those resulting from disabling *one link of Λ_i, hence $|\mathcal{T}_{\Lambda_i}| = |\Lambda_i|$.*

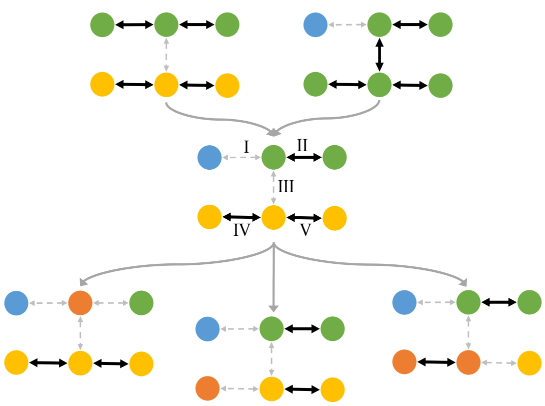


Fig. 1. *Parent* and *children* topologies for an example with $N = 6$ agents and $|\mathcal{L}| = 5$ links. The nodes with the same color represent clusters of controllers and the roman numerals are the indices for the links.

Definition 2. *Any topology Λ_j* descends *from Λ_i if it can be derived from Λ_i by disabling one or more links. Then, the set of all* descending topologies *of a given Λ_i results from applying recursively Definition 1.*

Definition 3. *The set of* parent topologies *of any $\Lambda_i \in \mathcal{T}$, where $|\Lambda_i| < |\mathcal{L}|$, is defined as*

$$\mathcal{T}^{\Lambda_i} = \{\Lambda_j \in \mathcal{T} \mid \Lambda_i \subset \Lambda_j \text{ and } |\Lambda_j| = |\Lambda_i| + 1\}. \quad (13)$$

That is, topologies Λ_j contained in set $\mathcal{T}^{\Lambda_i}$ are those resulting from adding *one link to Λ_i, hence $|\mathcal{T}^{\Lambda_i}| = |\mathcal{L}| - |\Lambda_i|$.*

Definition 4. *Any topology Λ_j* ascends *from Λ_i if it can be derived from Λ_i by enabling one or more additional links. Then, the set of all* ascending topologies *of a given Λ_i results from applying recursively Definition 3.*

Remark 2. *For the particular case of the centralized topology, the set of parent topologies is empty, i.e., $\mathcal{T}^{\Lambda_{\text{cen}}} = \emptyset$. Similarly, for topology Λ_{dec}, set $\mathcal{T}_{\Lambda_{\text{dec}}}$ is empty. Also, notice that all communication topologies descend from the fully connected case, and ascend from the decentralized system.*

On this basis, the procedure used for the topology switching is governed by the two following assumptions:

Assumption 2. *From any topology Λ_i, the communication network can switch to any topology that either descends from Λ_i, i.e., a lower level of cooperation, or ascends from Λ_i, i.e., a higher level of cooperation.*

Assumption 3. *The transitions are determined by a comparative appraisal to find the topology that minimizes (10) among the switching possibilities defined by Assumption 2.*

Therefore, the communication structure evolution is driven by movements within walks of a decision graph where the nodes are the set of topologies and the edges are defined by the *parent-children* relations (see Figure 1).

IV. Controller Design

In this section, we first introduce the main design method for matrices K_{Λ_i} and P_{Λ_i} [9]. Subsequently, new constraints are introduced to guarantee convexity, invariance and submodularity properties of the topologies boundaries.

A. Main Problem

Consider any topology $\Lambda_i \in \mathcal{T}$ and let

$$Q_{\mathcal{N}} = (Q_{\mathcal{C}})_{\mathcal{C}\in\mathcal{N}/\Lambda_i} \text{ and } R_{\mathcal{N}} = (R_{\mathcal{C}})_{\mathcal{C}\in\mathcal{N}/\Lambda_i}, \quad (14)$$

be the aggregate positive semi-definite and definite weighting matrices that define the global stage cost according to (7). Then, the problem for computing matrices K_{Λ_i}, which stabilize the overall system, and P_{Λ_i}, which provide a bound on the cost-to-go, is the following:

$$\min_{K_{\Lambda_i}, P_{\Lambda_i}} \ \text{tr}(P_{\Lambda_i})$$

s.t.

$$P_{\Lambda_i} - (A_{\mathcal{N}} + B_{\mathcal{N}} K_{\Lambda_i})^{\mathrm{T}} P_{\Lambda_i} (A_{\mathcal{N}} + B_{\mathcal{N}} K_{\Lambda_i}) > Q_{\mathcal{N}} + K_{\Lambda_i}^{\mathrm{T}} R_{\mathcal{N}} K_{\Lambda_i}. \quad (15a)$$

$$a \overset{\Lambda_i}{\not\leftrightarrow} b \Longrightarrow \begin{cases} P_{\Lambda_i,ab} = P_{\Lambda_i,ba} = \mathbf{0}, \\ K_{\Lambda_i,ab} = K_{\Lambda_i,ba} = \mathbf{0}, \end{cases} \quad (15b)$$

where

(i) Constraint (15a) implies

$$x_{\mathcal{N}}^{\mathrm{T}}(k) P_{\Lambda_i} x_{\mathcal{N}}(k) - x_{\mathcal{N}}^{\mathrm{T}}(k+1) P_{\Lambda_i} x_{\mathcal{N}}(k+1) > x_{\mathcal{N}}^{\mathrm{T}}(k) \left(Q_{\mathcal{N}} + K_{\Lambda_i}^{\mathrm{T}} R_{\mathcal{N}} K_{\Lambda_i}\right) x_{\mathcal{N}}(k), \quad \forall k \geq 0. \quad (16)$$

Therefore, inequality (9) is fulfilled.

(ii) Constraint (15b) entails that if two agents a and b are not physically connected by Λ_i, which is denoted by $a \overset{\Lambda_i}{\not\leftrightarrow} b$, the subblocks of matrices P_{Λ_i} and K_{Λ_i} that link both agents are null matrices of corresponding dimensions (see Figure 2).

Inequality (15a), which is bilinear on variables K_{Λ_i} and P_{Λ_i}, can be rewritten as an LMI on variables

$$W_{\Lambda_i} = P_{\Lambda_i}^{-1} \text{ and } Y_{\Lambda_i} = K_{\Lambda_i} W_{\Lambda_i}, \quad (17)$$

where $W_{\Lambda_i} = W_{\Lambda_i}^{\mathrm{T}}$ since P_{Λ_i} is symmetric [9]. In particular, pre- and post-multiplying (15a) by W_{Λ_i}, it is obtained

$$W_{\Lambda_i} - \left(W_{\Lambda_i} A_{\mathcal{N}}^{\mathrm{T}} + Y_{\Lambda_i}^{\mathrm{T}} B_{\mathcal{N}}^{\mathrm{T}}\right) P_{\Lambda_i} (A_{\mathcal{N}} W_{\Lambda_i} + B_{\mathcal{N}} Y_{\Lambda_i}) > W_{\Lambda_i} Q_{\mathcal{N}} W_{\Lambda_i} + Y_{\Lambda_i}^{\mathrm{T}} R_{\mathcal{N}} Y_{\Lambda}, \quad (18)$$

which, considering $Q_{\mathcal{N}} = Q_{\mathcal{N}}^{1/2} Q_{\mathcal{N}}^{1/2}$ and $R_{\mathcal{N}} = R_{\mathcal{N}}^{1/2} R_{\mathcal{N}}^{1/2}$, leads to

$$W_{\Lambda_i} - S_{\Lambda_i}^{\mathrm{T}} \begin{bmatrix} W_{\Lambda_i} & & \\ & I & \\ & & I \end{bmatrix}^{-1} S_{\Lambda_i} > 0, \quad (19)$$

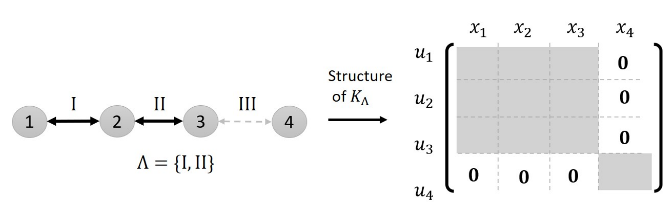


Fig. 2. Structure of feedback gain K_Λ for an example with $N = 4$ agents and $|\mathcal{L}| = 3$ links.

where $S_{\Lambda_i} = [\,W_{\Lambda_i} A_{\mathcal{N}}^{\mathrm{T}} + Y_{\Lambda_i}^{\mathrm{T}} B_{\mathcal{N}}^{\mathrm{T}} \quad W_{\Lambda_i} Q_{\mathcal{N}}^{1/2} \quad Y_{\Lambda_i}^{\mathrm{T}} R_{\mathcal{N}}^{1/2}\,]^{\mathrm{T}}$. Considering the above and applying the Schur complement [28], problem (15) can be reformulated as

$$\max_{Y_{\Lambda_i}, W_{\Lambda_i}} \ \mathrm{tr}(W_{\Lambda_i})$$

s.t.

$$\begin{bmatrix} W_{\Lambda_i} & W_{\Lambda_i} A_{\mathcal{N}}^{\mathrm{T}} + Y_{\Lambda_i}^{\mathrm{T}} B_{\mathcal{N}}^{\mathrm{T}} & W_{\Lambda_i} Q_{\mathcal{N}}^{1/2} & Y_{\Lambda_i}^{\mathrm{T}} R_{\mathcal{N}}^{1/2} \\ A_{\mathcal{N}} W_{\Lambda_i} + B_{\mathcal{N}} Y_{\Lambda_i} & W_{\Lambda_i} & 0 & 0 \\ Q_{\mathcal{N}}^{1/2} W_{\Lambda_i} & 0 & I & 0 \\ R_{\mathcal{N}}^{1/2} Y_{\Lambda_i} & 0 & 0 & I \end{bmatrix} > 0, \tag{20a}$$

$$a \overset{\Lambda_i}{\not\leftrightarrow} b \Longrightarrow \begin{cases} W_{\Lambda_i,ab} = W_{\Lambda_i,ba} = \mathbf{0}, \\ Y_{\Lambda_i,ab} = Y_{\Lambda_i,ba} = \mathbf{0}, \end{cases} \tag{20b}$$

where (20a) is the LMI associated with (19), the communication constraints are considered in (20b), and I is the identity matrix of suitable dimensions. That is, problem (20) provides a pair of matrices $(Y_{\Lambda_i}, W_{\Lambda_i})$ such that corresponding pair $(K_{\Lambda_i}, P_{\Lambda_i})$ satisfies constraints (15). In this way, for every communication structure Λ_i, we obtain a matrix P_{Λ_i} satisfying (9), and a feedback gain K_{Λ_i} that guarantees global closed-loop stability under the corresponding communication constraints.

Theorem 1. *The set of LMIs resulting from (15) to calculate all K_{Λ_i}, P_{Λ_i} with $\Lambda_i \in \mathcal{T}$, has a solution iff the decentralized topology has a solution.*

Proof. We first demonstrate that the existence of a solution for the decentralized case is a sufficient condition. Let $(Y_{\Lambda_{\mathrm{dec}}}, W_{\Lambda_{\mathrm{dec}}})$ be a solution of problem (20) for topology $\Lambda_{\mathrm{dec}} = \emptyset$. Then, corresponding matrices $K_{\Lambda_{\mathrm{dec}}}$, $P_{\Lambda_{\mathrm{dec}}}$ satisfy (15) for all $\Lambda_i \in \mathcal{T}$ due to

(i) Constraint (15a) is equal for all $\Lambda_i \in \mathcal{T}$.
(ii) The decentralized communication constraints entail (15b) for all other $\Lambda_i \in \mathcal{T}$.

In particular, constraint (15b) becomes *less restrictive* for topologies with a greater number of enabled links. As a consequence, the existence of a feasible solution for the rest of topologies is guaranteed. Finally, the condition is also necessary because a feasible solution for the decentralized case, as for every topology managed by the coalitional controller, is also required. ∎

Remark 3. *Based on the above, if topology Λ_i has a solution, i.e., K_{Λ_i} and P_{Λ_i}, then the existence of a feasible solution for any of its ascending topologies is guaranteed.*

In the following subsections, problem (15) is extended to consider additional constraints providing properties of interest for the topology switchings. However, it must be noted that adding further restrictions to (15) may reduce the space of feasible solutions, and hence the optimality of matrices K_{Λ_i} and P_{Λ_i}. In order to assess the impact of the latter, we use the following index [11]:

$$\eta = \frac{\sum_{\Lambda_i \in \mathcal{T}} \mathrm{tr}(P_{\Lambda_i})}{2^{|\mathcal{L}|}\mathrm{tr}(P_{\mathrm{LQR}})}, \tag{21}$$

where P_{LQR} stems from the discrete-time Riccati equation associated with the centralized linear-quadratic regulator (LQR) controller, which satisfies (9) with an equality.

B. Convexity Constraints

Consider that the following inequalities are added to the set of constraints in problem (15):

$$P_{\Lambda_j} > P_{\Lambda_i}, \ \ \forall \Lambda_j \in \mathcal{T}_{\Lambda_i}, \ \Lambda_i \in \mathcal{T}. \tag{22}$$

Then, the *boundaries* between any *parent* and *children* topologies become ellipsoidal surfaces, and thus, sets $\Upsilon_{\Lambda_i}^{\Lambda_j}$ become ellipsoidal regions [26]. Since (22) can be rewritten as

$$W_{\Lambda_j} > W_{\Lambda_i}, \ \forall \Lambda_j \in \mathcal{T}^{\Lambda_i}, \ \Lambda_i \in \mathcal{T}, \tag{23}$$

then, the convexity constraint can be directly integrated into problem (20). In particular, we propose two alternatives for computing set of matrices $(Y_{\Lambda_i}, W_{\Lambda_i})$ for all $\Lambda_i \in \mathcal{T}$: on the one hand, it can be proceeded sequentially as described in Algorithm 1, i.e., starting from Λ_{cen}, for which (23) is omitted, and computing descending topologies pairs $(Y_{\Lambda_i}, W_{\Lambda_i})$; on the other hand, all matrices $(Y_{\Lambda_i}, W_{\Lambda_i})$ may be introduced as optimization variables in a single LMI to calculate all $\Lambda_i \in \mathcal{T}$ at once, i.e.,

$$\begin{aligned} &\max_{[Y_{\Lambda_i}, W_{\Lambda_i}]_{\Lambda_i \in \mathcal{T}}} \ \sum_{\Lambda_i \in \mathcal{T}} \mathrm{tr}(W_{\Lambda_i}) \\ &\text{s.t.} \quad (20a), (20b) \text{ and } (23), \ \forall \Lambda_i \in \mathcal{T}. \end{aligned} \tag{24}$$

Subsequently, using (17), all $(Y_{\Lambda_i}, W_{\Lambda_i})$ can be transformed into $(K_{\Lambda_i}, P_{\Lambda_i})$.

Algorithm 1 Sequential procedure for convexity

Let $\Gamma = \{\Lambda_{\mathrm{cen}}, \ldots, \Lambda_{\mathrm{dec}}\}$ be a sorted set arranging all $\Lambda_i \in \mathcal{T}$ in decreasing order according to their cardinality. Also, let Γ_t denote the t-th topology in Γ, hence, $\Gamma_1 = \Lambda_{\mathrm{cen}}$ and $\Gamma_{2^{|\mathcal{L}|}} = \Lambda_{\mathrm{dec}}$.

1: Compute matrices $(W_{\Lambda_{\mathrm{cen}}}, Y_{\Lambda_{\mathrm{cen}}})$ by solving (20).
2: **for** $t = 2$ to $t = 2^{|\mathcal{L}|}$ **do**
3: Set $\Lambda_i = \Gamma_t$.
4: Fix matrices W_{Λ_j} of topologies $\Lambda_j \in \mathcal{T}^{\Lambda_i}$ as constants.
5: Compute matrices $(W_{\Lambda_i}, Y_{\Lambda_i})$ by solving

$$\begin{aligned} &\max_{Y_{\Lambda_i}, W_{\Lambda_i}} \ \mathrm{tr}(W_{\Lambda_i}) \\ &\text{s.t.} \quad (20a), (20b) \text{ and } (23). \end{aligned} \tag{25}$$

6: **end for**

It is also possible to identify convex sets characterizing the *relative dominance* between different controller structures, i.e., where a certain topology is more suitable than others. For example:

$$\mathcal{R}^{\Lambda_j} = \{x_{\mathcal{N}} \mid r(\Lambda_j, x_{\mathcal{N}}) \leq r(\Lambda_i, x_{\mathcal{N}}), \ \forall \Lambda_i \in \mathcal{T}^{\Lambda_j}\}, \tag{26}$$

is the ellipsoidal set where topology Λ_j dominates all of its parent topologies. Additionally, notice that the convexity property of sets $\Upsilon_{\Lambda_i}^{\Lambda_j}$ applies equally for any pair Λ_i and Λ_j, with Λ_j descending from Λ_i.

Remark 4. *Condition (22) can be satisfied by adjusting the bound on the performance costs provided by matrices P_{Λ_i} for all $\Lambda_i \in \mathcal{T}$. For this reason, this constraint does not endanger the feasibility of the proposed set of LMIs.*

Dwell time to avoid back-switchings: Consider any switching from Λ_i to a descending topology Λ_j at any instant k'. Without further LMI constraints, we can assure that the global system state will evolve within ellipsoid $\Upsilon_{\Lambda_i}^{\Lambda_j}$ from instant $k' + T_{\mathrm{D}_{ij}}$ if topology Λ_j remains unchanged. Hence, after $T_{\mathrm{D}_{ij}}$ time steps, it is guaranteed that there are no *back-switchings* in the network, that is, the system will not switch back to the previous topology since Λ_j will always *dominate* Λ_i, i.e., $r(\Lambda_j, x_{\mathcal{N}}(k)) \leq r(\Lambda_i, x_{\mathcal{N}}(k))$ for all $k \geq k' + T_{\mathrm{D}_{ij}}$. By using the largest absolute value of the eigenvalues in $(A_{\mathcal{N}} + B_{\mathcal{N}} K_{\Lambda_j})$, say $\overline{\lambda}_j < 1$, an upper bound on this time interval can be found as

$$\left(\overline{\lambda}_j\right)^{T_{\mathrm{D_{ij}}}} R_{\Upsilon_{\Lambda_i}^{\Lambda_j}} \leq r_{\Upsilon_{\Lambda_i}^{\Lambda_j}}, \tag{27}$$

where $T_{\mathrm{D}_{ij}}$ is defined as the *dwell time*, and $R_{\Upsilon_{\Lambda_i}^{\Lambda_j}}$ and $r_{\Upsilon_{\Lambda_i}^{\Lambda_j}}$ are respectively the lengths of the semi-major and semi-minor axis of $\Upsilon_{\Lambda_i}^{\Lambda_j}$. Note that the worst scenario is the one where the system enters into set $\Upsilon_{\Lambda_i}^{\Lambda_j}$ through its furthest point from the origin and thus the reason for using $R_{\Upsilon_{\Lambda_i}^{\Lambda_j}}$ and $r_{\Upsilon_{\Lambda_i}^{\Lambda_j}}$.

C. Invariance Constraints

The evolution of the system state towards the origin is expected to progressively reduce the number of enabled links among control agents according to (10). In this context, besides the properties obtained from the constraints in (15) and (22), the invariance properties of the switching sets are of particular interest. To this end, let us consider

$$\begin{aligned} &P_{\Lambda_i}^{\Lambda_j} - \left(A_{\mathcal{N}} + B_{\mathcal{N}} K_{\Lambda_j}\right)^{\mathrm{T}} P_{\Lambda_i}^{\Lambda_j} \left(A_{\mathcal{N}} + B_{\mathcal{N}} K_{\Lambda_j}\right) > 0, \\ &\qquad \forall \Lambda_j \in \mathcal{T}_{\Lambda_i},\ \Lambda_i \in \mathcal{T}, \end{aligned} \tag{28}$$

as a new restriction on the transition matrices. By adding both (22) and (28) to problem (15), matrices $(K_{\Lambda_i}, P_{\Lambda_i})$ for all $\Lambda_i \in \mathcal{T}$ can be designed such that the switching sets are convex and invariant for any transition towards descending topologies, i.e., matrix $P_{\Lambda_i}^{\Lambda_j}$ provides an invariant set confining the system state when imposing topology Λ_j after Λ_i. Additionally, new Lyapunov functions for the system characterizing the ascending-to-descending topology transitions are obtained, i.e., $V_{\Lambda_j - \Lambda_i}(x_{\mathcal{N}}(k)) = x_{\mathcal{N}}^{\mathrm{T}}(k) P_{\Lambda_i}^{\Lambda_j} x_{\mathcal{N}}(k)$.

Due to the nonlinearities arising when considering all conditions in (15), (22) and (28) at once, and the impossibility of rewriting all these constraints into LMIs, we propose the iterative design method described in Algorithm 2. Notice that at item 1, all feedback gains K_{Λ_i} and matrices P_{Λ_i} are designed using main problem (15). On the other hand, item 2 modifies matrices P_{Λ_i} to provide the convexity and invariance properties, which just affects the switching criterion. Since in this second item gains K_{Λ_i} are introduced as *constant*, problem (30) is formulated directly on variables P_{Λ_i}. Subsequently, gains K_{Λ_i} are updated at item 3 by minimizing variable $\xi > 0$ which weights all matrices P_{Λ_i}. In this regard, notice that constraints (22) and (28) particularized for matrices $P_{\Lambda_i}^{p+2} = \xi P_{\Lambda_i}^{p+1}$ simply imply

$$\begin{aligned} &\xi P_{\Lambda_i}^{\Lambda_j\, p+1} > 0 \quad \text{and} \\ &\xi P_{\Lambda_i}^{\Lambda_j\, p+1} - \left(A_{\mathcal{N}} + B_{\mathcal{N}} K_{\Lambda_j}\right)^{\mathrm{T}} \xi P_{\Lambda_i}^{\Lambda_j\, p+1} \left(A_{\mathcal{N}} + B_{\mathcal{N}} K_{\Lambda_j}\right) > 0, \\ &\quad \forall \Lambda_j \in \mathcal{T}_{\Lambda_i},\ \Lambda_i \in \mathcal{T}, \end{aligned} \tag{29}$$

where $P_{\Lambda_i}^{\Lambda_j\, p+1} = P_{\Lambda_j}^{p+1} - P_{\Lambda_i}^{p+1}$, with $P_{\Lambda_i}^{p+1}$ and $P_{\Lambda_j}^{p+1}$ being constant matrices satisfying (22) and (28), and ξ and K_{Λ_i} are variables. Following [11], constraints (15a) and (29) can be transformed into LMIs on variables (ξ, K_{Λ_i}) for all topologies $\Lambda_i \in \mathcal{T}$, hence problem (31) can be formulated as an optimization problem with LMI constraints.

Algorithm 2 Iterative design method

Let l be the iteration index and p a counter variable, and initialize $l = 1$ and $p = 0$.

1: For all $\Lambda_i \in \mathcal{T}$, compute matrices $(W_{\Lambda_i}^p, Y_{\Lambda_i}^p)$ by solving problem (20), and find $(K_{\Lambda_i}^p, P_{\Lambda_i}^p)$ using (17).

2: Fix matrices $K_{\Lambda_i}^{p+1} = K_{\Lambda_i}^p$ as constant matrices and obtain new $P_{\Lambda_i}^{p+1}$ by solving

$$\begin{aligned} &\min_{[P_{\Lambda_i}]_{\Lambda_i \in \mathcal{T}}} \sum_{\Lambda_i \in \mathcal{T}} \mathrm{tr}(P_{\Lambda_i}) \\ &\text{s.t.} \quad (15a), (15b), (22) \text{ and } (28),\ \forall \Lambda_i \in \mathcal{T}. \end{aligned} \tag{30}$$

3: Define matrices $P_{\Lambda_i}^{p+2} = \xi P_{\Lambda_i}^{p+1}$, where $P_{\Lambda_i}^{p+1}$ is a constant matrix and ξ is an optimization variable. Then, solve

$$\begin{aligned} &\min_{\xi, [K_{\Lambda_i}]_{\Lambda_i \in \mathcal{T}}} \xi \\ &\text{s.t.} \quad (15a), (15b), (29),\ \forall \Lambda_i \in \mathcal{T}, \\ &\qquad \xi > 0. \end{aligned} \tag{31}$$

and get new gains $K_{\Lambda_i}^{p+2}$.

4: Set $p = p + 2$ and $l = l + 1$ and go to Step 2 until a maximum number of iterations is reached or until convergence is attained.

Note that problem (30) in Algorithm 2 may be implemented via a single LMI or in a sequential fashion, i.e., starting from $\Lambda_i = \Lambda_{\mathrm{cen}}$, and solving

$$\begin{aligned} &\min_{P_{\Lambda_i}} \mathrm{tr}(P_{\Lambda_i}) \\ &\text{s.t.} \quad (15a), (15b), (22) \text{ and } (28), \end{aligned} \tag{32}$$

for descending topologies introducing P_{Λ_j} of all $\Lambda_j \in \mathcal{T}^{\Lambda_i}$ as constant matrices. Similarly, problem (31) can be implemented sequentially considering independent variables ξ_{Λ_i} for each topology $\Lambda_i \in \mathcal{T}$ with $\xi_{\Lambda_{\mathrm{cen}}} = 1$.

D. Submodularity Constraints

The problem of optimizing the system partition translates into finding a topology Λ^* such that

$$\Lambda^* = \arg \min_{\Lambda_i \in \mathcal{T}} r(\Lambda_i, x_{\mathcal{N}}). \tag{33}$$

Notice that the number of alternatives in $\mathcal{T}$ experiences a combinatorial explosion with the number of communication links, thus increasing the difficulty of the topology decision problem. As it is well known, in submodular functions optimization, greedy algorithms yield nearly optimal performance while notably reducing the space of possible solutions to evaluate [29]. In general, any set function $h : 2^{\mho} \to \mathbb{R}$ is submodular if the following holds:

$$h(\mathcal{A}\cup\{e\}) - h(\mathcal{A}) \geq h(\mathcal{B}\cup\{e\}) - h(\mathcal{B}). \quad \forall \mathcal{A}\subseteq\mathcal{B}\subseteq\mho,\ e\notin\mathcal{B}. \tag{34}$$

Commonly h represents an utility function, hence (34) implies that the marginal utility from adding a new element $\{e\}$ to a certain input set decreases as the size of the input set increases. Since $r(\Lambda_i, x_\mathcal{N}(k))$ does not provide directly the utility of topology Λ_i but its cost, function $-r(\Lambda_i, x_\mathcal{N}(k))$ will be considered. Then, applying (34), we obtain

$$r(\Lambda_j, x_\mathcal{N}(k)) - r(\Lambda_j\cup\{e\}, x_\mathcal{N}(k)) \geq r(\Lambda_i, x_\mathcal{N}(k)) - r(\Lambda_i\cup\{e\}, x_\mathcal{N}(k)), \quad \forall \Lambda_j\subseteq\Lambda_i\subseteq\Lambda_{\text{cen}},\ e\notin\Lambda_i, \tag{35}$$

which can be seen as an instance of the law of *diminishing returns* regarding the marginal contribution of link e, i.e., expression (35) entails that adding a new enabled link e to topology $\Lambda_j \in \mathcal{T}_{\Lambda_i}$ decreases $r(\Lambda_i, x_\mathcal{N}(k))$ by at least as much as adding link e to topology Λ_i. Introducing definition (10), inequality (35) can be formulated as

$$x_\mathcal{N}^{\mathrm{T}}(k)\left(P_{\Lambda_j} - P_{\Lambda_j\cup\{e\}} + P_{\Lambda_i\cup\{e\}} - P_{\Lambda_i}\right)x_\mathcal{N}(k) \geq (f(\Lambda_i\cup\{e\}) - f(\Lambda_i) - f(\Lambda_j\cup\{e\}) + f(\Lambda_j)), \quad \forall\Lambda_j\subseteq\Lambda_i,\ e\notin\Lambda_i. \tag{36}$$

Notice that for the particular case of linear coordination costs $f(\Lambda_i) = c|\Lambda_i|$ for all $\Lambda_i \in \mathcal{T}$, where $c \in \mathbb{R}^+\backslash\{0\}$ is a cost per enabled link, condition (36) leads to

$$P_{\Lambda_j} - P_{\Lambda_j\cup\{e\}} \geq P_{\Lambda_i} - P_{\Lambda_i\cup\{e\}}, \quad \forall\Lambda_j\subseteq\Lambda_i,\ e\notin\Lambda_i, \tag{37}$$

since $f(\Lambda_j\cup\{e\}) - f(\Lambda_j) = f(\Lambda_i\cup\{e\}) - f(\Lambda_i)$.

Following similar steps as in Algorithm 2, it is possible to design the criterion for switching to satisfy the submodularity condition and hence problem (33) can be solved in a greedy fashion. In particular, for the linear coordination costs case, constraints (22) and/or (28) can be replaced by/combined with (37) in problem (30). Likewise, in problem (31), constraint (29) may be omitted since the submodularity condition will hold for all $\xi > 0$.

Remark 5. *Each iteration of Algorithm 2 involves two optimization problems comprising all topologies variables and constraints, which may limit the applicability of this design method. However, consider communication network $(\mathcal{N},\mathcal{L})$ (from which set $\mathcal{T}$ is derived) and let $(\mathcal{N}^-,\mathcal{L}^-)\subset(\mathcal{N},\mathcal{L})$ be a certain subnetwork. Then, all submodularity constraints (36) associated with $(\mathcal{N}^-,\mathcal{L}^-)$ are contained in the set of constraints (36) corresponding to $(\mathcal{N},\mathcal{L})$. The latter allows us to define a sequential procedure for finding matrices fulfilling (36), i.e., starting from a subnetwork with two links, it may be possible to go towards larger subnetworks by setting the previously computed matrices as constant. Additionally, it is possible to assess how far is the system for being submodular by computing factor ε such that*

$$P_{\Lambda_j} - P_{\Lambda_j\cup\{e\}} + P_{\Lambda_i\cup\{e\}} - P_{\Lambda_i} + \varepsilon I \geq (f(\Lambda_i\cup\{e\}) - f(\Lambda_i) - f(\Lambda_j\cup\{e\}) + f(\Lambda_j))I, \quad \forall\Lambda_j\subseteq\Lambda_i,\ e\notin\Lambda_i. \tag{38}$$

V. Coordination Effort Regions

In this section, we exploit the convexity property provided by (22) to find ellipsoidal *approximations* of the state-space regions where topologies with a given number of enabled links dominate those involving a higher/lower level of coordination.

Consider ellipsoid $\mathcal{S}(G,g) = \{x_\mathcal{N} \in \mathbb{R}^{n_{x_\mathcal{N}}} : x_\mathcal{N}^{\mathrm{T}} G x_\mathcal{N} + g \leq 0\}$, where $G \geq 0$ and $g \leq 0$, and note that, by (22), there exist real numbers $\alpha_{ij} \geq 0$ and $\beta_{ij} \geq 0$ satisfying

$$\beta_{ij} I \leq P_{\Lambda_j} - P_{\Lambda_i} \leq \alpha_{ij} I, \qquad \forall\Lambda_i\in\mathcal{T}, \Lambda_j\in\mathcal{T}_{\Lambda_i}, \tag{39}$$

being I the identity matrix of corresponding dimensions.

Remark 6. *Taking α_{ij} and β_{ij} respectively as the maximum and minimum eigenvalues of $P_{\Lambda_i}^{\Lambda_j}$, it is possible to obtain the maximum inscribed ball and the minimum enclosing ball of dominance set Υ_{ij}, hence, $\mathcal{S}(\alpha_{ij}I, f_{ij}) \subseteq \Upsilon_{ij} \subseteq \mathcal{S}(\beta_{ij}I, f_{ij})$.*

Additionally, let us define set

$$\Gamma_\ell = \{\Lambda_i : \Lambda_i \in \mathcal{T} \text{ and } |\Lambda_i| = \ell\}, \quad \forall \ell = 0, ..., |\mathcal{L}|, \tag{40}$$

which contains all topologies in $\mathcal{T}$ with ℓ enabled links, and let $\Delta f_\ell = f(\ell) - f(\ell+1)$. Finally, consider the following optimization problems in positive definite symmetric variables M_ℓ and N_ℓ:

$$M_\ell = \arg\min_M \ \mathrm{tr}(M) \quad \text{s.t. } P_{\Lambda_j} - P_{\Lambda_i} < M,\ \forall\Lambda_j\in\Gamma_\ell,\ \Lambda_i\in\mathcal{T}^{\Lambda_j}, \tag{41}$$

and

$$N_\ell = \arg\max_N \ \mathrm{tr}(N) \quad \text{s.t. } N < P_{\Lambda_j} - P_{\Lambda_i},\ \forall\Lambda_j\in\Gamma_\ell,\ \Lambda_i\in\mathcal{T}^{\Lambda_j},\quad N > 0. \tag{42}$$

Then, the following conclusions can be drawn:

(i) If $x_\mathcal{N}(k) \notin \mathcal{S}(N_\ell, \Delta f_\ell)$, topologies with $\ell+1$ links prevail over those with ℓ links.

(ii) If $x_\mathcal{N}(k) \in \mathcal{S}(N_\ell, \Delta f_\ell)$ but $x_\mathcal{N}(k) \notin \mathcal{S}(M_\ell, \Delta f_\ell)$, the system is in a *transition zone* where either topologies with ℓ or $\ell+1$ links may dominate each other.

(iii) If $x_\mathcal{N}(k) \in \mathcal{S}(M_\ell, \Delta f_\ell)$, it is guaranteed that all topologies with ℓ links dominate those with $\ell+1$ links.

In particular, consider any pair of topologies Λ_i and Λ_j such that $|\Lambda_i| = \ell+1$ and $|\Lambda_j| = \ell$, and assume that state $x_\mathcal{N}$ belongs to set $\mathcal{S}(M_\ell, \Delta f_\ell)$. Then,

$$x_\mathcal{N}^{\mathrm{T}}(k) M_\ell x_\mathcal{N}(k) + \Delta f_\ell \leq 0, \tag{43}$$

and, since $P_{\Lambda_j} - P_{\Lambda_i} < M_\ell$ and $\Delta f_\ell = \gamma_{ji}$, the following holds:

$$\begin{aligned} x_{\mathcal{N}}^{\mathrm{T}}(k)\left(P_{\Lambda_j} - P_{\Lambda_i}\right)x_{\mathcal{N}}(k) + \gamma_{ji} \\ < x_{\mathcal{N}}^{\mathrm{T}}(k) M_\ell x_{\mathcal{N}}(k) + \gamma_{ji} \leq 0. \end{aligned} \tag{44}$$

That is, $x_{\mathcal{N}}(k)$ is contained in ellipsoid $\Upsilon_{\Lambda_i}^{\Lambda_j}$. Similarly, if state $x_{\mathcal{N}}(k) \notin \mathcal{S}(N_\ell, \Delta f_\ell)$, then, $x_{\mathcal{N}}^{\mathrm{T}}(k) N_\ell x_{\mathcal{N}}(k) + \gamma_{ij} > 0$. Therefore,

$$x_{\mathcal{N}}^{\mathrm{T}}(k)\left(P_{\Lambda_j} - P_{\Lambda_i}\right)x_{\mathcal{N}}(k) + \gamma_{ji} > 0, \tag{45}$$

i.e., $x_{\mathcal{N}}(k) \notin \Upsilon_{\Lambda_i}^{\Lambda_j}$.

Remark 7. *Constraints on the structure of matrices N_ℓ and M_ℓ can be added to problems (41) and (42). In particular, if $N_\ell = n_\ell I$ and $M_\ell = m_\ell I$, with $n_\ell, m_\ell \geq 0$, then sets $\mathcal{S}(M_\ell, \Delta f_\ell)$ and $\mathcal{S}(N_\ell, \Delta f_\ell)$ represent balls in the state space satisfying the properties above. In this regard, notice that problems (41) and (42) are feasible since it is possible to take $n_\ell = \max_{ij} \alpha_{ij}$ and $m_\ell = \min_{ij} \beta_{ij}$, for all i, j such that $\Lambda_j \in \Gamma_\ell, \Lambda_i \in \mathcal{T}^{\Lambda_j}$.*

VI. Control Scheme

In this section, a description of the proposed coalitional control scheme is provided. In particular, the control algorithm is based on a double-sample rate strategy that allows periodical switchings between a set of predefined topologies according to Assumption 2. After the formal statement, we briefly discuss the stability properties of this control scheme.

Control Scheme 1

Let T_{top} be a number of time steps denoting the operation period of any communication topology. Consider also set $\mathcal{K} = \{pT_{\text{top}} \mid p \in \mathbb{N}^+\}$. Then, starting from any communication topology $\Lambda_i \in \mathcal{T}$, at each sample time k:

1: **if** $k \in \mathcal{K}$ **then**
2: Set $\mathcal{T}^+$ of possible future topologies is determined following Assumption 2.
3: The agents share their state, and cost function (10) is evaluated for all possibilities in $\mathcal{T}^+$.[1]
4: Topology Λ^* that results in the minimum cost is selected and new coalitions are formed if necessary, then $\Lambda_i \leftarrow \Lambda^*$.
5: **else**
6: Within each communication component, the state of the grouped subsystems is shared.
7: **end if**
8: Control action $u_{\mathcal{C}}(k) = K_{\mathcal{C},\Lambda_i} x_{\mathcal{C}}(k)$ is applied to all clusters $\mathcal{C} \in \mathcal{N}/\Lambda_i$, where matrices $K_{\mathcal{C},\Lambda_i}$ are determined by one of the two algorithms introduced in Section IV.

[1]The exchange of the local states should be sufficient to compute (10) for all topologies in $\mathcal{T}^+$. In this sense, it can be considered that all agents share their state with a top layer that computes all indices $r(\Lambda_i, x_{\mathcal{N}}(k))$. Alternatively, given the structure of matrices P_Λ, any coalition $\mathcal{C} \in \mathcal{N}/\Lambda$ can compute independently its contribution to (10) if just the agents in $\mathcal{C}$ exchange their state. In the latter case, function (10) may be computed as a sum of the clusters contributions.

Notice that the network topology switchover is based on an assessment procedure in which the agents share information to make a collective choice of the new communication network, hence all nodes should be accessible. Also, note that the control scheme is independent of the design method used to compute matrices K_{Λ_i} and P_{Λ_i}.

Remark 8. *The dwell-time constraint can be easily considered by modifying the condition at item 1 of Control Scheme 1, i.e., if any topology Λ_i is imposed at instant k, then items 2 to 4 are not implemented again until instant $k + T_{D_{ij}}$.*

Remark 9. *Conditions (i) and (iii) in Section V can be integrated at item 3 of Control Scheme 1 to fasten the topology decision. For example, if the current topology has ℓ links and $x_{\mathcal{N}} \in \mathcal{S}(M_\ell, \Delta f_\ell)$, then there is no need for evaluating function (10) for any topology in $\mathcal{T}_{\ell+1}$. In this regard, notice that, since it is needed to alter problem (15) to satisfy the convexity constraints, the cooperation effort zones are suboptimal. However, if convexity can be guaranteed with just a minor change in matrices P_{Λ_i}, then the difference of (10) between the selected topology and the optimal one according to [9] should be small. Note also that matrices M_ℓ and N_ℓ can be designed to be block diagonal in order to facilitate a distributed computation of conditions (i) to (iii).*

A. Stability Properties

In this subsection, stability of Control Scheme 1 is first established by proving that function (10) goes to zero asymptotically. In particular, once we fix a particular topology Λ_i, the global system state evolves within the contractive invariant set

$$\begin{aligned} &\Omega_{\Lambda_i}(x_{\mathcal{N}}(k)) = \\ &\quad \left\{x_{\mathcal{N}}(k') \mid x_{\mathcal{N}}^{\mathrm{T}}(k') P_{\Lambda_i} x_{\mathcal{N}}(k') \leq \rho_{\Lambda_i}(x_{\mathcal{N}}(k))\right\}, \end{aligned} \tag{46}$$

where $k' \geq k$ and $\rho_{\Lambda_i}(k) = x_{\mathcal{N}}^{\mathrm{T}}(k) P_{\Lambda_i} x_{\mathcal{N}}(k)$. As a consequence, function (10), i.e., $r(\Lambda_i, x_{\mathcal{N}}(k)) = x_{\mathcal{N}}^{\mathrm{T}}(k) P_{\Lambda_i} x_{\mathcal{N}}(k) + f_i$, decreases in time. With this in mind, we focus on the dynamical switchings between communication topologies.

Proposition 1. *Index $r(\cdot)$ is a decreasing function over time regardless the switchings between topologies.*

Proof. Let us consider that the system starts from state $x_{\mathcal{N}}(k) \neq 0$ with topology Λ_i and that a transition to topology Λ_j takes place at time $k^* > k$. Hence, it follows that $r(\Lambda_i, x_{\mathcal{N}}(k^*)) \leq r(\Lambda_i, x_{\mathcal{N}}(k))$. Additionally, the condition for switching is

$$r(\Lambda_j, x_{\mathcal{N}}(k^*)) < r(\Lambda_i, x_{\mathcal{N}}(k^*)), \tag{47}$$

that is, any transition entails an instantaneous decrease of $r(\cdot)$. Likewise, this index will decrease over time once we impose Λ_j. The extension of this result for all topology transitions leads us to conclude that (10) is a decreasing function in time. Finally, when the system reaches the origin, the term dependent on the system state becomes zero and $r(\cdot)$ will be defined by the cost of the enabled links, which will foster transitions towards the most decentralized topology. ■

Remark 10. *Index $r(\cdot)$ can be written as $r(\Lambda_i, x_\mathcal{N}(k)) = V_{\Lambda_i}(x_\mathcal{N}(k)) + f_i$. Then, a transition from Λ_i to Λ_j at instant k takes place iff $V_{\Lambda_j}(x_\mathcal{N}(k)) - V_{\Lambda_i}(x_\mathcal{N}(k)) \leq f_i - f_j$. Additionally, since $V_{\Lambda_j}(x_\mathcal{N}(k+1)) \leq V_{\Lambda_j}(x_\mathcal{N}(k))$, we have that $V_{\Lambda_j}(x_\mathcal{N}(k+1)) - V_{\Lambda_i}(x_\mathcal{N}(k)) \leq f_i - f_j$. That is, any transition entails a decrease of $r(\Lambda_i, x_\mathcal{N}(k))$, so that the increases of $V_{\Lambda_i}(x_\mathcal{N}(k))$ due to the switchings are bounded.*

Also, the following proposition introduces an alternative approach for proving stability of the proposed coalitional controller:

Proposition 2. *Let $\Lambda(k)$ be the current topology at time instant k. If the number of topology switchings along time is equal to $|\mathcal{L}|$, there are at most $|\mathcal{L}|$ bounded increases in function $V_{\Lambda(k)}(x_\mathcal{N}(k))$, hence this function is bounded. Additionally, after the last switching event, $V_{\Lambda(k)}(x_\mathcal{N}(k))$ will go to zero asymptotically.*

The proof of the proposition above is straightforward by considering Assumption 1 and Remark 10.

Finally, note that if the design was successful for all topologies, then stability between switchings may also be proven if a common Lyapunov function can be found for all topologies. In particular, let us fix feedback gains K_{Λ_i}, then, we need to find a common matrix $P > 0$ satisfying the set of LMIs

$$P - (A_\mathcal{N} + B_\mathcal{N} K_{\Lambda_i})^\mathrm{T} P (A_\mathcal{N} + B_\mathcal{N} K_{\Lambda_i}) > 0, \tag{48}$$

for all $\Lambda_i \in \mathcal{T}$. If such P exists, then function $x_\mathcal{N}^\mathrm{T} P x_\mathcal{N}$ is a common Lyapunov function for all $\Lambda_i \in \mathcal{T}$. Notice that the inequalities above constitutes an LMI that can be solved *a posteriori* since gains K_{Λ_i} are constant matrices.

VII. Simulation Example

In this section, we apply the coalitional scheme to a modified version of the system proposed in [30], [31]. In particular, we consider four trucks, i.e., $\mathcal{N} = \{1, 2, 3, 4\}$, coupled via springs and dumpers as shown in Figure 3. The dynamics of each truck are modeled by

$$\begin{bmatrix} \dot{r}_i \\ \dot{v}_i \end{bmatrix} = A_{ii} \begin{bmatrix} r_i \\ v_i \end{bmatrix} + B_{ii} u_i + \sum_{j \in \mathcal{N}_i} A_{ij} \begin{bmatrix} r_j \\ v_j \end{bmatrix}, \tag{49}$$

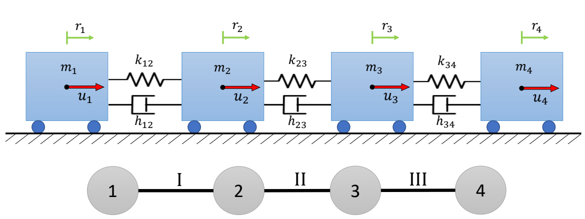


Fig. 3. Four-truck system and related simplified network.

with

$$A_{ii} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{m_i} \sum_{j \in \mathcal{N}_i} k_{ij} & -\frac{1}{m_i} \sum_{j \in \mathcal{N}_i} h_{ij} \end{bmatrix},$$
$$A_{ij} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_i} \sum_{j \in \mathcal{N}_i} k_{ij} & \frac{1}{m_i} \sum_{j \in \mathcal{N}_i} h_{ij} \end{bmatrix}, \tag{50}$$
$$\text{and } B_{ii} = \begin{bmatrix} 0 \\ 50 \end{bmatrix}, \quad \forall i \in \mathcal{N},$$

where m_i, k_{ij} and h_{ij} represent the masses, spring stiffnesses and damping factors, respectively. The state x_i of each subsystem is formed by the displacement r_i from the equilibrium point and by the instantaneous velocity v_i. Additionally, the agents can apply a longitudinal force $F_i = 50u_i$ [N], where u_i is the control action. The continuous-time dynamics are discretized using zero-order hold and a sampling time of 0.15s. The parameters that characterize the system are given in Table I, the stage performance function ℓ_i is defined by matrices $Q_i = I$ and $R_i = 100$ for all $i \in \mathcal{N}$, and the initial states are $x_1 = [-2.6,\ 1.0]^\mathrm{T}$, $x_2 = [2.8,\ -2.0]^\mathrm{T}$, $x_3 = [3.0,\ -1.6]^\mathrm{T}$ and $x_4 = [-4.4,\ 1.4]^\mathrm{T}$.

The four agents are connected by a network of three bidirectional links, i.e., $\mathcal{L} = \{\mathrm{I, II, III}\}$, as shown in Figure 3. Also, we have assumed indirect connections, i.e., any two agents can communicate if there exists a path of one or more links connecting them. In Table II, we specify the eight network topologies that can be imposed, specifying the links that are enabled in each of them.

Control Scheme 1 has been implemented using $T_\mathrm{top} = 5$ and an initial centralized topology $\Lambda_\mathrm{cen} = \Lambda_7$. For the controller design, we have used Matlab® `LMI Control Toolbox` [32] and, in particular, the solver `mincx`, in a 1.8 GHz quad-core Intel® Core™ i7/8 GB RAM computer. In what follows, we illustrate graphically the results obtained when matrices $(K_{\Lambda_i}, P_{\Lambda_i})$ for all $\Lambda_i \in \mathcal{T}$ satisfy, besides conditions in (15), the constraints in the following scenarios:

(i) (22), i.e., the convexity condition.
(ii) (22) and (28), i.e., convexity and invariance.
(iii) (22) and (36), i.e., convexity and submodularity.

For the matrices computation, we have followed Algorithm 1 in case (i) and Algorithm 2 in cases (ii) and (iii). The

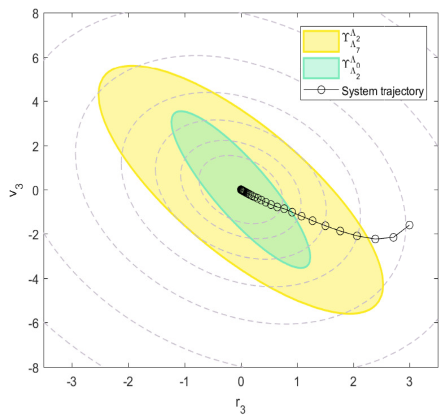
TABLE I
Masses, Spring Stiffnesses and Damping Factors

Masses	m_1	m_2	m_3	m_4
Value [kg]	2.5	1.0	1.5	2.0

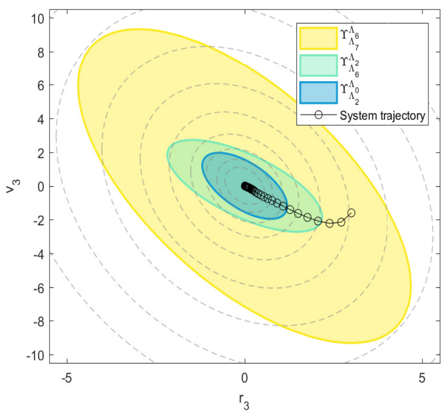
Spring stiffnesses	Value [N/m]	Damping factors	Value [N/(m·s)]
k_{12}	0.75	h_{12}	0.40
k_{23}	1.50	h_{23}	0.50
k_{34}	1.0	h_{34}	0.45

TABLE II
Network Topologies for the Four Trucks System

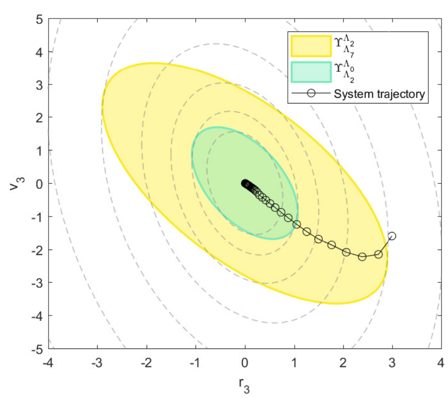
Topology	Links	Topology	Links
$\Lambda_\mathrm{dec} = \Lambda_0$	$\emptyset$	Λ_4	{I,II}
Λ_1	{I}	Λ_5	{I,III}
Λ_2	{II}	Λ_6	{II,III}
Λ_3	{III}	$\Lambda_\mathrm{cen} = \Lambda_7$	$\mathcal{L}$ = {I,II,III}



(a) Case (i)



(b) Case (ii)

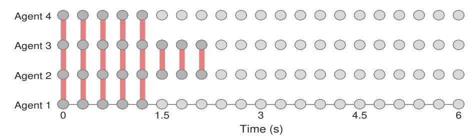


(c) Case (iii)

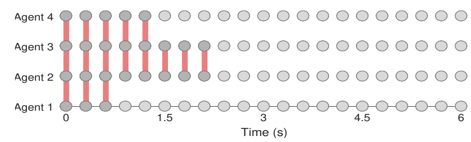
Fig. 4. Boundaries between topologies in the plane defined by $x_1 = [0\ 0]^{\mathrm{T}}$, $x_2 = [0\ 0]^{\mathrm{T}}$ and $x_4 = [0\ 0]^{\mathrm{T}}$. The circle marked line represents the evolution of subsystem 3 state components and the dashed lines are level curves of $V_{\Lambda_i} = x_{\mathcal{N}}^{\mathrm{T}} P_{\Lambda_i} x_{\mathcal{N}}$. Notice that according to the properties provided by constraint (28), the colored ellipsoids in figure (b) are also invariant sets.

TABLE III
PERFORMANCE COSTS AND COMMUNICATION EFFORT

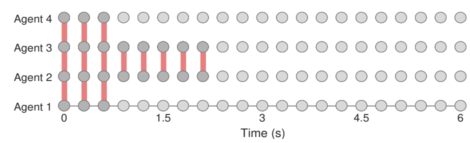
	Case (i)		Case (ii)		Case (iii)		Problem (15)	
c	Perf.	Comm.	Perf.	Comm.	Perf.	Comm.	Perf.	Comm.
2.5	280.63	17	280.31	17	280.29	17	280.50	17
1	280.19	32	280.31	17	280.17	22	280.22	27
0.6	280.19	32	280.14	27	280.17	22	280.22	27
0.45	280.19	32	282.14	27	280.13	27	280.22	27
0.15	280.11	47	282.11	37	280.11	32	280.12	42
Λ_{dec}	284.47	0	283.28	0	282.38	0	284.24	0
Λ_{cen}	280.09	120	280.09	120	280.09	120	280.09	120



(a) Case (i)



(b) Case (ii)



(c) Case (iii)

Fig. 5. Network topologies across time for cases (i), (ii) and (iii).

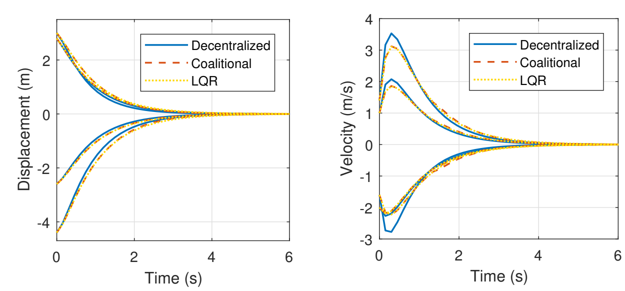


Fig. 6. Displacement and velocity of the four trucks in case (i), and corresponding decentralized and LQR results.

stop criterion in Algorithm 2 is $|\eta(l) - \eta(l-1)| < 0.001$, being $\eta(l)$ the value of (21) obtained at iteration l. Since the system evolves in the eight-dimensional space, a plane has been selected to allow for a bidimentional illustration of the boundaries that characterize the topology transitions. In this regard, Figure 4 shows the plane $[r_3, v_3]$ for cases (i) to (iii) when the communication costs are linear, i.e., $f_i = c|\Lambda_i|$ in (10) for all $\Lambda_i \in \mathcal{T}$, with $c = 0.6$. This figure shows the ellipsoidal shape of these borders, which enclose dominance sets where a certain topology improves the value of (10) over the previously established one on the network (recall (11) and note that constraint (22) is satisfied in all cases). The network topologies used over time in these simulations are specified in Figure 5. In this regard, the topology sequence shown in scenario (iii) equals the solution obtained in a greedy fashion, i.e., when Λ^* is built up from Λ_{dec} by incrementally adding those links providing greater benefits. Also, as example, Figure 6 illustrates the evolution of the system state obtained in case (i) and the corresponding decentralized and LQR results.

Table III shows the cumulative performance costs (i.e., $\sum_k \sum_{\mathcal{C}} \ell_{\mathcal{C}}(k)$, where k is the time index) and the communication effort (calculated as $\sum_k |\Lambda(k)|$) for the coalitional controllers of the proposed scenarios for different values of parameter c. Also, the costs of the decentralized and centralized controllers are given, together with the results that would have been obtained with the original design method [9], i.e., just

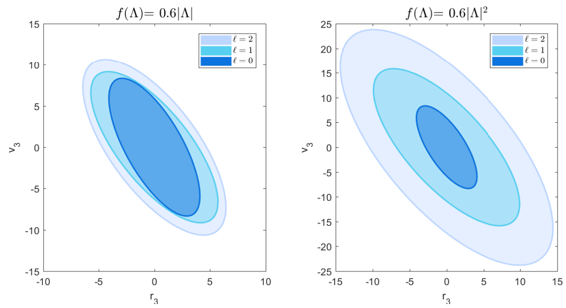


Fig. 7. Sets $\mathcal{S}(N_\ell, \Delta f_\ell)$ in the plane defined by $x_1 = [0\ 0]^{\mathrm{T}}$, $x_2 = [0\ 0]^{\mathrm{T}}$ and $x_4 = [0\ 0]^{\mathrm{T}}$ for linear and quadratic costs $f(\Lambda)$. In this figure, matrices P_Λ were designed according to Design Method 2.

considering problem and constraints (15). In this regard, notice that adding these new constraints to the controller design do not significantly impact the overall performance costs. Also, Table IV compares the value of suboptimality index (21) obtained for different sets of constraints and design methods. Note that when η is close to 1, then the sum of the traces of matrices P_{Λ_i} is close to the corresponding LQR result.

Additionally, for the case (i), we have computed the dwell time under the conservative approach proposed in Subsection IV-B, where the values of all $T_{\mathrm{D}_{ij}}$ calculated are specified in Table V. Note that using a switching period T_{top} lower than the dwell time neither endangers stability nor the convexity properties of the boundaries.

Finally, sets $\mathcal{S}(N_\ell, \Delta f_\ell)$ for $\ell = 2, 1, 0$ are illustrated in Figure 7. This figure compares the results when using linear and quadratic functions for weighting the coordination costs. Also, matrices P_{Λ_i} were designed according to case (ii). Note that $\mathcal{S}(N_2, \Delta f_2)$ shows the region where at least one topology with two links starts dominating the centralized topology.

VIII. Conclusions

In this paper, we deal with topology switching boundaries in a coalitional control scheme. In particular, we have considered a multi-agent network where the communication structure among the set of control entities changes with time to deal with the varying interaction between subsystems. In this context, different communication topologies have been assessed in real time rewarding improvements of the system performance and penalizing overuses of communication resources.

Our main contribution are design procedures for coalitional control that guarantee convexity and invariance of the switching sets, thus simplifying the selection of the topology. New conditions to attain submodularity and reduce the effort to find optimal solutions have also been presented. Likewise, a dwell-time approach to prevent back-switchings has been introduced, and the convexity property is exploited to analyze the need for coordination at different points of the state space. Two different algorithms have been proposed with the aim of combining the aforementioned properties at the designer's choice. All these strategies allow us to identify convex sets characterizing regions in the system state space where a certain controller structure is the most suitable.

The proposed design methods may result of interest in distributed model predictive control (MPC) schemes as they provide convex terminal invariant sets identifying regions where the network topology can be modified, hence paving the way for novel coalitional MPC applications. For this reason, future research should extend this study to the MPC framework and provide results in large-scale systems.

TABLE IV
Suboptimality index η

	Constraints	η
Algorithm 1	(15), (22)	1.0675
Algorithm 2	(15), (22)	1.0404
	(15), (22), (28)	1.0414
	(15), (36)	1.0413
	(15), (22), (36)	1.0413
	(15), (22), (28), (36)	1.0424

TABLE V
Dwell time in the allowed transitions

Transition	Dwell time	Transition	Dwell time	Transition	Dwell time
$\Lambda_7 \to \Lambda_6$	18	$\Lambda_6 \to \Lambda_2$	26	$\Lambda_4 \to \Lambda_1$	20
$\Lambda_7 \to \Lambda_5$	22	$\Lambda_5 \to \Lambda_3$	24	$\Lambda_3 \to \Lambda_0$	19
$\Lambda_7 \to \Lambda_4$	22	$\Lambda_5 \to \Lambda_1$	21	$\Lambda_2 \to \Lambda_0$	20
$\Lambda_6 \to \Lambda_3$	30	$\Lambda_4 \to \Lambda_2$	18	$\Lambda_1 \to \Lambda_0$	21

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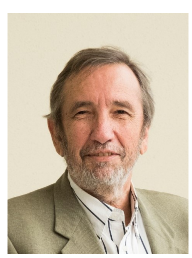

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