Topological minors in bipartite graphs

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Abstract

For a bipartite graph G on m and n vertices, respectively, in its vertices classes, and for integers s and t such that $2 \leq s \leq t$, $0 \leq m - s \leq n - t$, and $m + n \leq 2s + t - 1$, we prove that if G has at least $mn - (2(m - s) + n - t)$ edges then it contains a subdivision of the complete bipartite $K_{(s,t)}$ with s vertices in the m-class and t vertices in the n-class. Furthermore, we characterize the corresponding extremal bipartite graphs with $mn - (2(m-s)+n-t+1)$ edges for this topological Turan type problem.

Key words. bipartite graphs, extremal graph theory, topological minor

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1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Diestel (cf. [18]) for terminology and definitions.

Two well-known extensions of the Turán problem (cf. [1]) are the Turán topological problem and the Zarankiewicz problem. The former one consists of estimating the extremal function $ex(n, TK_p)$ which represents the maximum number of edges of a graph on n vertices free of a topological minor TK_p of a complete graph on p vertices (see Bollobás' excellent monograph (cf. [4]) devoted to this subject and the contributions on this topic (cf. [5, 9, 14, 19, 20, 22])). The second was stated by Zarankiewicz (cf. [2]) who studied the maximum size of a bipartite graph on (m, n) vertices, denoted by $z(m, n; s, t)$ that contains no bipartite complete $K_{(s,t)}$ subgraph with s vertices in the m-class and t vertices in the n-class. For a survey of this problem we also refer the reader to Section VI.2 of the book by Bollobás (cf. [4]). Most of the contributions are bounds for the function $z(m, n; s, t)$ when s, t are fixed and m, n are much larger than s, t (cf. [6, 7, 12]). Other contributions provide exact values of the extremal function (cf. [10, 13, 23]).

Recent results on some problems involving the existence of a complete bipartite graph or a subdivision of a complete bipartite graph as a subgraph can be found in the literature (cf. [15, 16, 17, 21, 24, 26, 27]). Böhme et al. (cf. [16]) studied the size of a k-connected graph free of either an induced path of a given length or a subdivision of a complete bipartite graph. Kühn and Osthus (cf. [17]) proved that for any graph H and for every integer s there exists a function $f = f(H, s)$ such that every graph of size at least f contains either a $K_{s,s}$ as a subgraph or an induced subdivision of H. Meyer (cf. [15]) also relates the size of a graph with the property of containing a minor of $K_{s,t}$. Other problems involving the existence of maximum matching in graphs are considered (cf. [26]). And involving the existence of 2-factors in bipartite graphs and k-factors in regular graphs can be found (cf. [21, 24, 28]).

Combining the topological version of the Turán problem for complete graphs with the Zarankiewicz problem, we introduce the extremal function $tz(m, n; s, t)$ as a natural extension. The function $tz(m, n; s, t)$ is defined as *the maximum size of a (m, n)-bipartite graph free of a topological minor $TK_{(s,t)}$ of a complete bipartite $K_{(s,t)}$ with s vertices in the m-class and t vertices in the n-class.* In 1998 Bollobás and Thomason (cf. [8]) proved that the function $ex(n; TK_p)$ is upper bounded by cp^2n. From this contribution together with the fact that the example, due to Jung (cf. [3]) and improved by Luczak (cf. [11]), giving a lower bound is a bipartite graph, it follows that the extremal function $tz(m, m; t, t)$ is asymptotically upper bounded by ct^2n. The objective of this paper is to obtain exact values for this extremal function $tz(m, n; s, t)$ and to characterize the corresponding extremal bipartite graphs for infinitely many related values of m, n, s, t. Namely, we determine the exact value of $tz(m, n; s, t)$ and we characterize the family $TZ(m, n; s, t)$ of extremal graphs for any values of m, n, s, t

satisfying $2 \le m-s \le n-t$ and $m+n \le 2s+t-1$.

1.1 Terminology and notations

A subdivision of a graph H is a graph TH obtained from H by replacing the edges of H with internally disjoint paths. The *branch vertices* of TH are all those vertices that correspond to vertices of H. The complete bipartite graph $K_{(s,t)}$ is said to be *a topological minor* of a bipartite graph G if $TK_{(s,t)} \subseteq G$.

Given two positive integers, m, n, a bipartite graph G with vertex classes X and Y of cardinalities $|X| = m$ and $|Y| = n$, is denoted by $G = (X, Y)$. The sets of vertices and edges of G are denoted by $V(G) = X \cup Y$ and $E(G)$, respectively, whereas $v(G)$ and $e(G)$ stand for the corresponding cardinalities.

For a bipartite graph $H = (X, Y)$, the degree of a vertex v in the graph H is denoted by $d_H(v)$ whereas $\Delta_X(H)$ (resp. $\Delta_Y(H)$) stand for the maximum degree among vertices in the first class (resp. second class). Thus, $\Delta(H) = \max\{\Delta_X(H), \Delta_Y(H)\}$ is the maximum degree of H. Let us consider two subsets of vertices $\{x_1, x_2, \dots, x_p\} \subseteq X$ and $\{y_1, y_2, \dots, y_p\} \subseteq Y$. Let us denote by $H_{0,0} = H$, $H_{1,0} = H - \{x_1\}$, $H_{1,1} = H_{1,0} - \{y_1\}$, and for all $i = 2, \dots, p$, let us denoted by $H_{i,i-1} = H_{i-1,i-1} - \{x_i\}$ and $H_{i,i} = H_{i,i-1} - \{y_i\}$. Next we introduce the notion of *decreasing sequence of vertices* in a bipartite graph $H = (X, Y)$.

Definition 1.1 *Given an integer $p \ge 1$ and a bipartite graph $H = (X, Y)$, a subset of vertices of H, $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$, with $\{x_1, \dots, x_p\} \subseteq X$ and $\{y_1, \dots, y_p\} \subseteq Y$, is called a decreasing sequence of H if the following assertions hold:*

(i) $d_{H_{i-1,i-1}}(x_i) = \Delta_X(H_{i-1,i-1})$, *for* $i = 1, \dots, p$.

(ii) $d_{H_{i,i-1}}(y_i) = \Delta_Y(H_{i,i-1})$, *for* $i = 1, \dots, p$.

(iii) For each $i = 1, \dots, p$, *either* $x_i y_i \notin E(H)$ *or every vertex* $y \in V(H_{i,i-1}) \cap Y$ *with degree* $d_{H_{i,i-1}}(y) = \Delta_Y(H_{i,i-1})$ *is adjacent to vertex* x_i *in* H.

Note that

$$d_{H_{0,0}}(x_1) \ge d_{H_{1,1}}(x_2) \ge \dots \ge d_{H_{p-1,p-1}}(x_p) \ge \Delta_X(H_{p,p})$$

and

$$d_{H_{1,0}}(y_1) \ge d_{H_{2,1}}(y_2) \ge \dots \ge d_{H_{p,p-1}}(y_p) \ge \Delta_Y(H_{p,p}),$$

and furthermore,

$$e(H) = \sum_{i=1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \right) + e(H_{p,p}).$$

2 Exact values

Let G be a bipartite graph $G = (X, Y)$ on m and n vertices in X and Y respectively. We will henceforth use H to denote the bipartite complement of G, i.e., the bipartite graph $H = (X, Y) = K_{(m,n)} - E(G)$.

The problem of finding a $TK_{(s,t)}$ in a bipartite graph G can be formulated in terms of its bipartite complement H. Indeed, if $G = (X, Y)$ contains a $TK_{(s,t)}$ with set of branch vertices $S \cup T$, $S \subset X$, $T \subset Y$, then the edges of the graph $H[S \cup T]$ are missing in G and thus they must be replaced in G with internally disjoint paths passing through vertices of $X \setminus S$ and vertices of $Y \setminus T$. Since each of these paths must have odd length at least 3, it follows that $e\,(H[S \cup T]) \leq \min\{|X \setminus S|, |Y \setminus T|\}$. Hence, the following necessary but not sufficient condition on the induced subgraph $H[S \cup T]$ in order to determine whether $K_{(s,t)}$ is a topological minor of G is immediate.

Remark 2.1 *Let $G = (X, Y)$ be with $|X| = m$ and $|Y| = n$ and let H be the bipartite complement of G. If G contains a $TK_{(s,t)}$, then there exist $S \subseteq X$ and $T \subseteq Y$ with $|S| = s$, $|T| = t$, such that the number of edges of the subgraph induced by $S \cup T$ in the bipartite complement of G satisfies*

$$e\,(H[S \cup T]) \leq \min\{m - s, n - t\}.$$

■

Next, we obtain a lower bound on the maximum size of a (m, n)-bipartite graph free of a topological minor $TK_{(s,t)}$ of $K_{(s,t)}$.

Proposition 2.1 *Let m, n, s, t be integers such that $2 \leq s \leq t$, $0 \leq m - s \leq n - t$, and $m + n \leq 2s + t - 1$. Then the bipartite graph $G = K_{(m,n)} - M$, where M is any matching of cardinality $2(m - s) + n - t + 1$, does not contain $TK_{(s,t)}$ and therefore,*

$$tz(m, n; s, t) \geq mn - (2(m - s) + n - t + 1)\,.$$

Proof First, let us see that $K_{(m,n)}$ has a matching of cardinality $2(m - s) + n - t + 1$. This is clear because from $2 \leq s \leq t$ and $0 \leq m - s \leq n - t$, it follows that $m \leq n$, and from the hypothesis

$m+n \leq 2s+t-1$ it follows that $2(m-s)+n-t+1 = (m+n)+m-2s-t+1 \leq m \leq n$. Therefore, we may consider the bipartite graph $G=(X,Y)=K_{(m,n)}-M$ where M is a matching of cardinality $2(m-s)+n-t+1$ in $K_{(m,n)}$. Next let us see that $K_{(s,t)}$ is not a topological minor of G. For that, from Remark 2.1 it is enough to prove that $e\left(H[S \cup T]\right) > m-s$ for any subsets $S \subseteq X$ and $T \subseteq Y$ of cardinalities s and t, respectively, with $s \leq t$. Observe that the number of isolated vertices in the class Y of H is exactly $n-(2(m-s)+n-t+1)$. It follows that the number of edges of $H[X \cup T]$ is

$$e(H[X \cup T]) \geq t-(n-(2(m-s)+n-t+1)) = 2m-2s+1.$$

But since $e(H[(X \setminus S) \cup T]) \leq m-s$, then we have

$$\begin{aligned} e(H[S \cup T]) &= e(H[X \cup T]) - e(H[(X \setminus S) \cup T]) \\ &\geq 2m-2s+1-(m-s) \\ &= m-s+1 > m-s. \end{aligned}$$

Thus the result holds. ■

Lemma 2.1 *Let $p \geq 1$ be an integer and let $G=(X,Y)$ be a bipartite graph, with $|X| \geq p$ and $|Y| \geq p$, and denote by $H=(X,Y)$ the bipartite complement of G. Let $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be any decreasing sequence of H and denote by $r = e(H_{p,p})$. If $r \geq 1$ and $e(H) \leq 3p$, then:*

(i) $r \leq p$.

(ii) $\Delta(H_{p,p}) = 1$.

(iii) $\{x_{p-(r-1)}y_{p-(r-1)}, \ldots, x_p y_p\} \cap E(H) = \emptyset$.

(iv) $\{ay_{p-(r-1)}, \ldots, ay_p\} \cap E(H) = \emptyset$, *for each* $a \in X \setminus \{x_1, \ldots, x_p\}$ *of degree* $d_{H_{p,p}}(a) = 1$.

(v) If $r \geq 2$, *then* $\{x_{p-(r-2)}b, \ldots, x_p b\} \cap E(H) = \emptyset$, *for each* $b \in Y \setminus \{y_1, \ldots, y_p\}$ *of degree* $d_{H_{p,p}}(b) = 1$.

Proof Since $e(H_{p,p}) = r \geq 1$ we deduce $\Delta_X(H_{p,p}) \geq 1$, $\Delta_Y(H_{p,p}) \geq 1$, following that $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$, and therefore

$$e(H_{p,p}) = e(H) - \sum_{i=1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \right) \leq 3p - 2p = p,$$

thus item (i) is proved.

If $\Delta_X(H_{p,p}) \geq 2$, then $d_{H_{i-1,i-1}}(x_i) \geq 2$ for each $i = 1, \ldots, p$, hence,

$$e(H) = \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \big) + e(H_{p,p}) \geq 3p + r > 3p,$$

which is a contradiction. Analogously, we arrive at a contradiction if $\Delta_Y(H_{p,p}) \geq 2$. Thus, $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies $\Delta(H_{p,p}) = 1$, hence item (ii) is shown.

(iii) Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X \setminus \{x_1, \ldots, x_p\}$ and $b_i \in Y \setminus \{y_1, \ldots, y_p\}$, for $i = 1, \ldots, r$. By item (i) we know that $r \leq p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-1\}$ such that $x_{p-j}y_{p-j} \in E(H)$. First we claim that $d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$. Otherwise, if $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 2$ then $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \ldots, p-j$ and therefore, by (ii) we have

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \big) + \sum_{i=p-j+1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \big) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r \\ &= 3p + (r-j) \\ &> 3p, \end{aligned}$$

the last inequality due to the fact that $j \leq r-1$. Since this is a contradiction with the hypothesis, then $\Delta_Y(H_{p-j,p-j-1}) = d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$, yielding to $d_{H_{i,i-1}}(y_i) = 1$, for $i = p-j, \ldots, p$ and $d_{H_{p,p}}(b_i) = 1$, for $i = 1, \ldots, r$. As $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ is a decreasing sequence of H, it follows that x_{p-j} is adjacent in H to each one of the vertices of the set $\{y_{p-j}, \ldots, y_p, b_1, \ldots, b_r\}$ because of point (iii) of Definition 1.1. That is, $d_{H_{p-j-1,p-j-1}}(x_{p-j}) \geq j + 1 + r$, which means that $d_{H_{i-1,i-1}}(x_i) \geq j + 1 + r \geq 2$, for $i = 1, \ldots, p-j$ and therefore,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \big) + \sum_{i=p-j+1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \big) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r = 3p + (r-j) > 3p, \end{aligned}$$

again a contradiction. Thus $x_{p-j}y_{p-j} \notin E(H)$ for all $j \in \{0, \ldots, r-1\}$, hence item (iii) is valid.

(iv) Note that $r \geq 1$ implies $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-1\}$ such that $ay_{p-j} \in E(H)$ for a vertex $a \in X \setminus \{x_1, \ldots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$. Then $d_{H_{p-j-1,p-j-1}}(a) \geq 2$ and hence, $d_{H_{i-1,i-1}}(x_i) \geq 2$, for $i = 1, \dots, p-j$. Thus,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + \sum_{i=p-(j-1)}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r = 3p + (r-j) > 3p, \end{aligned}$$

because $j \leq r-1$, against the hypothesis.

(v) Since $r \geq 2$ then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \dots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \dots, r-2\}$ such that $x_{p-j}b \in E(H)$ for a vertex $b \in Y \setminus \{y_1, \dots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$. Then $d_{H_{p-j-1,p-j-2}}(b) \geq 2$ and hence, $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \dots, p-j-1$. Thus,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j-1} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + \sum_{i=p-j}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \\ &\geq 3(p-j-1) + 2(j+1) + r = 3p + (r-j-1) > 3p, \end{aligned}$$

because $j \leq r-2$, again a contradiction. ■

Lemma 2.2 *Let $p \geq 2$ be an integer. Let $G = (X, Y)$ be a bipartite graph with $|X| \geq p$ and $|Y| \geq p$, and denote by $H = (X, Y)$ the bipartite complement of G. Suppose that there exists a decreasing sequence of vertices $U = \{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ of H such that $E(H_{p,p}) = \{ab\}$ with $a \in X$ and $b \in Y$. If $e(H) \leq 3p$ then there exists an (a, b)-path in G with its internal vertices belonging to U.*

Proof Since $E(H_{p,p}) = \{ab\}$, then $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \dots, p$. If G contains the path a, y_p, x_p, b, then we are done. So assume that some of the edges ay_p, x_py_p, x_pb is an edge of H. We know by Lemma 2.1 (iii) that $x_py_p \notin E(H)$. If $ay_p \in E(H)$, then $d_{H_{p-1,p-1}}(a) \geq 2$, because $\{ay_p, ab\} \subset E(H_{p-1,p-1})$. Then $d_{H_{i-1,i-1}}(x_i) \geq 2$ and we get

$$e(H) = \sum_{i=1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \geq 3p + 1,$$

which is a contradiction. Therefore we can suppose that $x_pb \in E(H)$ and $ay_p \notin E(H)$. Then $\{x_pb, ab\} \subset E(H_{p-1,p-2})$, following that $d_{H_{p-1,p-2}}(b) \geq 2$, which implies that $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \dots, p-1$. Since $d_{H_{p,p-1}}(y_p) \geq 1$ and $d_{H_{i-1,i-1}}(x_i) \geq 1$ for $i = 1, \dots, p$, it follows that

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-1} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + \left(d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p)\right) + e(H_{p,p}) \\ &\geq 3(p-1) + 2 + 1 = 3p. \end{aligned}$$

This means that all the above inequalities become equalities, that is,

$$\begin{cases} d_{H_{i,i-1}}(y_i) = 2, \text{ for } i = 1, \ldots, p-1, \text{ and } d_{H_{p,p-1}}(y_p) = 1; \\ d_{H_{i-1,i-1}}(x_i) = 1, \text{ for } i = 1, \ldots, p. \end{cases} \tag{1}$$

Therefore we obtain that:

- $x_p y_{p-1} \notin E(H)$, because otherwise, $\{x_p b, x_p y_{p-1}\} \subset E(H_{p-2,p-2})$ and thus, $d_{H_{p-2,p-2}}(x_{p-1}) = \Delta_X(H_{p-1,p-1}) \geq 2$, contradicting (1).
- $x_{p-1} b \notin E(H)$, for if not, $\{x_{p-1}b, x_p b, ab\} \subset E(H_{p-2,p-3})$ and hence, $d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq 3$, against (1).
- $x_{p-1} y_{p-1} \notin E(H)$, because otherwise, $d_{H_{p-2,p-3}}(y_{p-1}) \geq 3$ and therefore, $d_{H_{p-2,p-3}}(y_{p-2}) \geq 3$, contradicting (1).

Thus, it follows that $\{ay_p, x_p y_p, x_p y_{p-1}, x_{p-1} y_{p-1}, x_{p-1} b\} \cap E(H) = \emptyset$. Consequently, there exists in G the path $a, y_p, x_p, y_{p-1}, x_{p-1}, b$, hence the result holds. ■

Lemma 2.3 *Let m, n, p be integers such that $p \geq 2$, $m > p$ and $n > p$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$ and $|Y| = n$, and denote by $H = (X, Y)$ the bipartite complement of G. If $e(H) \leq 3p$, then $K_{(m-p,n-p)}$ is a topological minor of G.*

Proof Let $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be a decreasing sequence of H. The graph $H_{p,p}$ is a bipartite graph with vertex classes $X^* = X \setminus \{x_1, \ldots, x_p\}$ and $Y^* = Y \setminus \{y_1, \ldots, y_p\}$, so $|X^*| = m - p$ and $|Y^*| = n - p$. If $e(H_{p,p}) = 0$ then the bipartite complement of $H_{p,p}$ is $K_{(m-p,n-p)}$ and the result follows. We may henceforth assume that $e(H_{p,p}) > 0$, or in other words $\Delta_X(H_{p,p}) \geq 1$ and $\Delta_Y(H_{p,p}) \geq 1$, thus $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Then by Lemma 2.1 we have $e(H_{p,p}) = r \leq p$ and $\Delta(H_{p,p}) = 1$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1 b_1, \ldots, e_r = a_r b_r$, $a_i \in X^*$ and $b_i \in Y^*$, for $i = 1, \ldots, r$. In order to prove that G contains $TK_{(m-p,n-p)}$ with set of branch vertices $X^* \cup Y^*$, we will show the existence of vertex disjoint (a_i, b_i)-paths in G, $i = 1, \ldots, r$, with internal vertices from U. As $e(H) \leq 3p$, if $r = 1$ then the bipartite complement of $H_{p,p}$ is $K_{m-p,n-p} - e_1$. Thus, by Lemma 2.2, the bipartite graph G contains $TK_{m-p,n-p}$ and we are done. Hence assume that $2 \leq r \leq p$, then by Lemma 2.1 (*iii*), (*iv*), (*v*), for each $i = 1, \ldots, r$ and $j = 0, \ldots, r-2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Otherwise, since $x_{p-(r-1)}y_{p-(r-1)} \in E(G)$ and $a_i y_{p-(r-1)} \in E(G)$ for all $i = 1, \ldots, r$, because of Lemma 2.1, we deduce that $x_{p-(r-1)}b_i \in E(H)$ for all $i = 1, \ldots, r$, that is, $d_{H_{p-r,p-r}}(x_{p-(r-1)}) \geq r$ and therefore, $d_{H_{i-1,i-1}}(x_i) \geq r$ for $i = 1, \ldots, p-(r-1)$. Then since $2 \leq r \leq p$ it follows that

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-(r-1)} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + \sum_{i=p-(r-2)}^{p} \big(d_{H_{i-1,i-1}}(i_k) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq (r+1)(p-(r-1)) + 2(r-1) + r \\ &= 3p + 1 + (r-2)(p-r+1) \\ &> 3p \end{aligned}$$

which is a contradiction. Hence there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1, \ldots, r$. Thus, G contains $TK_{(m-p,n-p)}$ and this finishes the proof. ■

The following lemma gives a sufficient condition on the size of a bipartite graph in order to contain a complete bipartite graph as a topological minor.

Lemma 2.4 *Let m, n, s, t be integers such that $2 \leq m - s \leq n - t$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$, $|Y| = n$. If the bipartite complement H of G has size $e(H) \leq 2(m-s) + n - t$, then $K_{(s,t)}$ is a topological minor of G.*

Proof Set $p = m - s$ and $q = n - t$, then $2 \leq p \leq q$ and $e(H) \leq 2p + q$. First, suppose that $p = q$. Thus the bipartite graph H has size at most $3p$, and by Lemma 2.3, we obtain that $K_{(m-p,n-p)} = K_{(s,t)}$ is a topological minor of G. Hence, assume that $p < q$. Without loss of generality, we may assume that the vertices of the partite set Y are ordered in such a way that $d_H(y_1) \geq d_H(y_2) \geq \cdots \geq d_H(y_n)$. Set $Y' = \{y_1, \ldots, y_{q-p}\} \subseteq Y$ and let us consider the bipartite graph $H' = (X, Y \setminus Y')$. Observe that $|X| = m$ and $|Y \setminus Y'| = n - (q-p) = t + p$. If $e(H') = 0$ then the bipartite complement G' of H' is the complete bipartite graph $K_{(m,t+p)}$. Since G' is a subgraph of G and $K_{(s,t)} \subseteq K_{(m,t+p)}$, then G contains a $K_{(s,t)}$ and we are done. So, we may assume that $e(H') > 0$, which implies that $d_H(y_i) \geq 1$ for $i = 1, \ldots, q-p$. Hence, $e(H') = e(H) - \sum_{i=1}^{q-p} d_H(y_i) \leq 2p + q - (q-p) \leq 3p$, and therefore, from Lemma 2.3, it follows that $K_{(m-p,t+p-p)} = K_{(s,t)}$ is a topological minor of G. ■

Combining Proposition 2.1 and Lemma 2.4 the following theorem is immediate.

Theorem 2.1 *Let m, n, s, t be integers such that $2 \le s \le t$, $2 \le m-s \le n-t$, and $m+n \le 2s+t-1$. Then*

$$tz(m,n;s,t) = mn - (2(m-s)+n-t+1)\,.$$

3 Family of extremal graphs

Lemma 3.1 *Let $p \ge 2$ be an integer and let $G = (X,Y)$ be a bipartite graph with $|X| \ge p$ and $|Y| \ge p$, and denote by $H = (X,Y)$ the bipartite complement of G. Let $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be any decreasing sequence of H and denote by $r = e(H_{p,p})$. If $e(H) \le 3p+1$ and $\Delta_X(H) \ge 2$ then*

(i) $r \le p$.

(ii) $\Delta(H_{p,p}) \le 1$.

(iii) *If $r = 1$ then $\{x_{p-(r-1)}y_{p-(r-1)}, \ldots, x_p y_p\} \cap E(H) = \emptyset$.*

(iv) *If $r \ge 2$ then $\{ay_{p-(r-2)}, \ldots, ay_p\} \cap E(H) = \emptyset$, for each $a \in X \setminus \{x_1, \ldots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$, if any.*

(v) *If $r \ge 2$ then $\{x_{p-(r-2)}b, \ldots, x_p b\} \cap E(H) = \emptyset$, for each $b \in Y \setminus \{y_1, \ldots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$, if any.*

Proof If $e(H_{p,p}) = r = 0$, then both items (i) and (ii) hold. Hence we may assume that $0 < r = e(H_{p,p}) \le 3p+1$, which implies $\Delta_X(H_{p,p}) \ge 1$, $\Delta_Y(H_{p,p}) \ge 1$, following that $d_{H_{i-1,i-1}}(x_i) \ge 1$ for $i = 2, \ldots, p$ and $d_{H_{i,i-1}}(y_i) \ge 1$ for $i = 1, \ldots, p$. Moreover, $d_{H_{0,0}}(x_1) \ge 2$, because $\Delta_X(H) \ge 2$. Therefore

$$e(H_{p,p}) = e(H) - \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big) - \sum_{i=2}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \le 3p+1-3-2(p-1) = p,$$

thus item (i) is proved.

If $\Delta_X(H_{p,p}) \ge 2$, then $e(H_{p,p}) \ge 2$ and $d_{H_{i-1,i-1}}(x_i) \ge 2$ for each $i = 1, \ldots, p$, hence,

$$e(H) = \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \ge 3p + e(H_{p,p}) \ge 3p+2 > 3p+1,$$

which is a contradiction. Analogously, we arrive at a contradiction if $\Delta_Y(H_{p,p}) \geq 2$. Thus, $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies $\Delta(H_{p,p}) = 1$, hence item (ii) is shown.

(iii) From item (i) it follows that $r \leq p$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X \setminus \{x_1, \ldots, x_p\}$ and $b_i \in Y \setminus \{y_1, \ldots, y_p\}$, for $i = 1, \ldots, r$. Since $e(H_{p,p}) = r \geq 1$, then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-1\}$ such that $x_{p-j}y_{p-j} \in E(H)$. Then $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-j-1$, because $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 1$ and $x_{p-j}y_{p-j} \in E(H)$. We have two cases:

Case 1. Assume that $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 2$, then $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-j$. Since $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$ and $j \leq r-1$ it follows that

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 4 + 3(p-j-1) + 2j + r \\ &= 3p + 1 + (r-j) \\ &> 3p+1, \end{aligned}$$

which is a contradiction.

Case 2. Assume that $d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$, then $d_{H_{i,i-1}}(y_i) = 1$, for $i = p-j, \ldots, p$. Moreover, $d_{H_{p,p}}(b_i) = 1$, for $i = 1, \ldots, r$, because $\Delta(H_{p,p}) = 1$. As $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ is a decreasing sequence of H and $x_{p-j}y_{p-j} \in E(H)$, it follows that x_{p-j} is adjacent in H to each one of the vertices of the set $\{y_{p-j}, \ldots, y_p, b_1, \ldots, b_r\}$ because of point (iii) of Definition 1.1. That is, $d_{H_{p-j-1,p-j-1}}(x_{p-j}) \geq j+1+r$, which means that $d_{H_{i-1,i-1}}(x_i) \geq j+1+r$ for $i = 1, \ldots, p-j$. If $j = 0$ then $d_{H_{i-1,i-1}}(x_i) \geq 1+r$ for $i = 1, \ldots, p$, and therefore

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + (d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p)) + e(H_{p,p}) \\ &\geq (3+r)(p-1) + (r+2) + r \\ &= 3p + 1 + r(p+1) - 2 > 3p+1, \end{aligned}$$

because $r \geq 1$ and $p \geq 2$, which is a contradiction. If $j = r-1$ then $d_{H_{i-1,i-1}}(x_i) \geq j+1+r = 2r$ for

$i = 1, \ldots, p-(r-1)$, and therefore

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-r} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + (d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)})) \\ &+ \sum_{i=p-(r-2)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq (2r+2)(p-r) + (2r+1) + 2(r-1) + r \\ &= 3p+1+(2rp-2r^2-p+3r-2) \\ &\geq 3p+1+(p-1) \\ &> 3p+1, \end{aligned}$$

because $p \geq 2$, which also contradicts the hypothesis. Finally, if $1 \leq j \leq r-2$ then $d_{H_{i-1,i-1}}(x_i) \geq j+1+r \geq 3$ for $i = 1, \ldots, p-j$, and therefore

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + (d_{H_{p-j-1,p-j-1}}(x_{p-j}) + d_{H_{p-j,p-j-1}}(y_{p-j})) \\ &+ \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 5(p-j-1)+4+2j+r \\ &= 3p+1+(2p-3j-2+r) \\ &\geq 3p+1+(3r-3j-2) > 3p+1, \end{aligned}$$

because $p \geq r$ and $j \leq r-2$, again a contradiction.

Thus $x_{p-j}y_{p-j} \notin E(H)$ for all $j \in \{0, \ldots, r-1\}$, hence item (iii) is valid.

(iv) Assume $e(H_{p,p}) = r \geq 2$. Then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-2\}$ such that $ay_{p-j} \in E(H)$ for a vertex $a \in X \setminus \{x_1, \ldots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$. Then $d_{H_{p-j-1,p-j-1}}(a) \geq 2$ and hence, $d_{H_{i-1,i-1}}(x_i) \geq 2$, for $i = 1, \ldots, p-j$. Thus,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3(p-j)+2j+r = 3p+(r-j) > 3p+1, \end{aligned}$$

because $j \leq r-2$, against the hypothesis.

(v) Assume $e(H_{p,p}) = r \geq 2$. Then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Moreover, $d_{H_{0,0}}(x_1) \geq 2$, due to the fact that $\Delta_X(H) \geq 2$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-2\}$ such that $x_{p-j}b \in E(H)$ for a vertex $b \in Y \setminus \{y_1, \ldots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$. Then $d_{H_{p-j-1,p-j-2}}(b) \geq 2$ and hence, $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \ldots, p-j-1$. Thus,

$$\begin{aligned} e(H) &= (d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)) + \sum_{i=2}^{p-j-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-j}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 4 + 3(p-j-2) + 2(j+1) + r = 3p + (r-j) > 3p+1, \end{aligned}$$

because $j \leq r-2$, again a contradiction. ■

Lemma 3.2 *Let $p \geq 4$ be an integer. Let $G = (X, Y)$ be a bipartite graph with $|X| \geq p$ and $|Y| \geq p$, and denote by $H = (X, Y)$ the bipartite complement of G. Suppose that $\Delta_X(H) \geq 2$ and there exists a decreasing sequence of vertices $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ of H such that $E(H_{p,p}) = \{ab\}$ with $a \in X$ and $b \in Y$. If $e(H) \leq 3p+1$ then there exists an (a, b)-path in G with its internal vertices belonging to U.*

Proof Assume that $e(H) \leq 3p+1$. Note that $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$. Since $E(H_{p,p}) = \{ab\}$, then $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. If G contains the path a, y_p, x_p, b, then we are done. So assume that some of the edges ay_p, x_py_p, x_pb is an edge of H. We know by Lemma 3.1 that $x_py_p \notin E(H)$. So, let us distinguish two cases.

Case 1. Suppose that $ay_p \in E(H)$. Then $d_{H_{p-1,p-1}}(a) \geq 2$, because $ab \in E(H)$. Then $d_{H_{i-1,i-1}}(x_i) \geq 2$ and we get

$$e(H) = \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \geq 3p+1 \geq e(H).$$

Thus, all the inequalities become equalities, that is,

$$d_{H_{i-1,i-1}}(x_i) = 2 \text{ and } d_{H_{i,i-1}}(y_i) = 1, \text{ for } i = 1, \ldots, p. \tag{2}$$

Hence, we obtain that:

- $x_{p-1}y_{p-1} \notin E(H)$. Otherwise, since $\Delta_Y(H_{p-1,p-2}) = d_{H_{p-1,p-2}}(y_{p-1}) = 1$ and both y_p and b have also degree 1 in $H_{p-1,p-2}$, applying point *(iii)* of Definition 1.1, it follows that $\{x_{p-1}y_{p-1}, x_{p-1}y_p, x_{p-1}b\} \subset E(H)$ and therefore, $d_{H_{p-2,p-2}}(x_{p-1}) \geq 3$, which contradicts (2).

- $ay_{p-1} \notin E(H)$, because otherwise, $d_{H_{p-2,p-2}}(x_{p-1}) = \Delta_X(H_{p-2,p-2}) \geq d_{H_{p-2,p-2}}(a) \geq 3$, contradicting (2).

- $x_{p-1}b \notin E(H)$, for if not, $d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(b) \geq 2$, against (2).

As a consequence, we get that the path a, y_{p-1}, x_{p-1}, b of G connects the vertices a and b.

Case 2. Suppose that $x_p b \in E(H)$ and $ay_p \notin E(H)$. Thus, $d_{H_{p-1,p-2}}(y_p) \geq 2$, which implies that $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \ldots, p-1$. Since $d_{H_{p,p-1}}(y_p) \geq 1$, $d_{H_{0,0}}(x_1) \geq 2$ and $d_{H_{i-1,i-1}}(x_i) \geq 1$ for $i = 2, \ldots, p$, it follows that

$$\begin{aligned} e(H) &= (d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)) + \sum_{i=2}^{p-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ (d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p)) + e(H_{p,p}) \\ &\geq 4 + 3(p-2) + 2 + 1 = 3p + 1 = e(H), \end{aligned}$$

which means that all the above inequalities become equalities, that is,

$$\begin{cases} d_{H_{0,0}}(x_1) = 2 \text{ and } d_{H_{i-1,i-1}}(x_i) = 1 \text{ for } i = 2, \ldots, p; \\ d_{H_{i,i-1}}(y_i) = 2 \text{ for } i = 1, \ldots, p-1, \text{ and } d_{H_{p,p-1}}(y_p) = 1. \end{cases} \tag{3}$$

Therefore, we have:

- $x_{p-1}b \notin E(H)$, because on the contrary, $d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(b) \geq 3$ against (3).

- $x_p y_{p-1} \notin E(H)$, for if not, $d_{H_{p-3,p-3}}(x_{p-2}) = \Delta_X(H_{p-3,p-3}) \geq d_{p-3,p-3}(x_p) \geq 2$ and this contradicts (3), since $p \geq 4$.

- $x_{p-1}y_{p-1} \notin E(H)$, because otherwise, taking into account that $d_{H_{p-1,p-2}}(y_{p-1}) = 2$, we have $d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(y_{p-1}) \geq 3$, contradicting (3).

Thus, in this case, it follows that $\{ay_p, x_p y_p, x_p y_{p-1}, x_{p-1}y_{p-1}, x_{p-1}b\} \cap E(H) = \emptyset$. Consequently, there exists in G the path $a, y_p, x_p, y_{p-1}, x_{p-1}, b$, and the result also holds in this case. ■

Lemma 3.3 *Let m, n, p be integers such that $p \geq 4$, $m > p$ and $n > p$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$ and $|Y| = n$, and denote by $H = (X, Y)$ the bipartite complement of G. If $\Delta(H) \geq 2$ and $e(H) \leq 3p + 1$, then $K_{(m-p,n-p)}$ is a topological minor of G.*

Proof Without loss of generality we may assume that $\Delta(H) = \Delta_X(H)$ (otherwise it is enough to interchange the classes X with Y). Let $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be a decreasing sequence of H. The graph $H_{p,p}$ is a bipartite graph with vertex classes $X^* = X\backslash\{x_1, \ldots, x_p\}$ and $Y^* = Y\backslash\{y_1, \ldots, y_p\}$, so $|X^*| = m-p$ and $|Y^*| = n-p$. If $e(H_{p,p}) = 0$ then the bipartite complement of $H_{p,p}$ is $K_{(m-p,n-p)}$ and the result follows. So, we may henceforth assume that $e(H_{p,p}) \geq 1$ or in other words, $\Delta_X(H_{p,p}) \geq 1$ and $\Delta_Y(H_{p,p}) \geq 1$, thus $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Then by Lemma 3.1 we have $e(H_{p,p}) = r \leq p$ and $\Delta(H_{p,p}) = 1$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X^*$ and $b_i \in Y^*$, for $i = 1, \ldots, r$. In order to prove that G contains a $TK_{(m-p,n-p)}$ with set of branch vertices $X^* \cup Y^*$, we will show the existence of vertex disjoint (a_i, b_i)-paths in G, $i = 1, \ldots, r$, with internal vertices in U. As $e(H) \leq 3p+1$, we are done if $r = 1$ by applying Lemma 3.2, hence assume that $2 \leq r \leq p$.

First, suppose that $2 \leq r \leq p-1$. Then, by Lemma 3.1 *(iii)*, *(iv)*, *(v)*, for each $i = 1, \ldots, r$ and $j = 0, \ldots, r-2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. We reason by way of contradiction supposing that for all $i = 1, \ldots, r$ the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G. From Lemma 3.1 it follows that $x_{p-(r-1)}, y_{p-(r-1)} \in E(G)$, thus $a_iy_{p-(r-1)} \in E(H)$ or $x_{p-(r-1)}b_i \in E(H)$ for each $i = 1, \ldots, r$. We will distinguish three possible cases:

Case 1. Assume that $x_{p-(r-1)}b_i \in E(H)$ for all $i = 1, \ldots, r$, then $d_{H_{p-r,p-r}}(x_{p-(r-1)}) \geq r$ and thus, $d_{H_{j-1,j-1}}(x_j) \geq r$ for $j = 1, \ldots, p-(r-1)$. Moreover, $d_{H_{p-r,p-(r+1)}}(y_{p-r}) = \Delta_Y(H_{p-r,p-(r+1)}) \geq d_{H_{p-r,p-(r+1)}}(b_i) \geq 2$, which means that $d_{H_{j,j-1}}(y_j) \geq 2$ for $j = 1, \ldots, p-r$. Thus,

$$\begin{aligned} e(H) &= \sum_{j=1}^{p-r} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + \big(d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)})\big) \\ &+ \sum_{j=p-(r-2)}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\ &\geq (r+2)(p-r) + (r+1) + 2(r-1) + r \\ &= 3p + 1 + (r-2)(p-r) + p - 2 \\ &> 3p+1, \end{aligned}$$

since $2 \leq r < p$ and $p > 2$, which is a contradiction.

Case 2. Assume that $a_iy_{p-(r-1)} \in E(H)$ for all $i = 1, \ldots, r$, then, reasoning as in Case 1, we have

$d_{H_{j,j-1}}(y_j) \geq r$ for $j = 1, \ldots, p-(r-1)$, and $d_{H_{j-1,j-1}}(x_j) \geq 2$ for $j = 1, \ldots, p-(r-1)$. Thus,

$$\begin{aligned} e(H) &= \sum_{j=1}^{p-(r-1)} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + \sum_{j=p-(r-2)}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\ &\geq (r+2)(p-(r-1)) + 2(r-1) + r \\ &= 3p + 1 + (r-2)(p-r) + p - 1 \\ &> 3p + 1, \end{aligned}$$

since $2 \leq r < p$ and $p > 1$, which is a contradiction.

Case 3. Assume that there exist $i_0, j_0 \in \{1, \ldots, r\}$ such that $x_{p-(r-1)}b_{i_0} \notin E(H)$ and $a_{j_0}y_{p-(r-1)} \notin E(H)$. Clearly $i_0 \neq j_0$, because $x_{p-(r-1)}y_{p-(r-1)} \notin E(H)$ (by Lemma 3.1) and by hypothesis, the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G for all $i = 1, \ldots, r$. Since $x_{p-(r-1)}y_{p-(r-1)} \notin E(H)$, it follows that $x_{p-(r-1)}b_{j_0} \in E(H)$, for if not, we find in G the path $a_{j_0}, y_{p-(r-1)}, x_{p-(r-1)}, b_{j_0}$ against our assumption. Analogously, $a_{i_0}x_{p-(r-1)} \in E(H)$. Observe that $\{a_{i_0}x_{p-(r-1)}, a_{i_0}b_{i_0}\} \subset E(H_{p-r,p-r})$ and therefore, $d_{H_{p-r,p-r}}(x_{p-(r-1)}) = \Delta_X(H_{p-r,p-r}) \geq d_{H_{p-r,p-r}}(a_{i_0}) \geq 2$, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 2$ for $i = 1, \ldots, p-(r-1)$. Moreover, observe also that $\{y_{p-(r-1)}b_{j_0}, a_{j_0}b_{j_0}\} \subset E(H_{p-r,p-(r+1)})$ and therefore, $d_{H_{p-r,p-(r+1)}}(y_{p-r}) = \Delta_X(H_{p-r,p-(r+1)}) \geq d_{H_{p-r,p-(r+1)}}(b_{j_0}) \geq 2$, which means that $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-r$. Hence,

$$\begin{aligned} e(H) &= \sum_{j=1}^{p-r} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + \big(d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)})\big) \\ &+ \sum_{j=p-(r-2)}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\ &\geq 4(p-r) + 3 + 2(r-1) + r \\ &= 4p + 1 - r \\ &= 3p + 1 + (p-r) \\ &> 3p + 1, \end{aligned}$$

since $r \leq p-1$. Then, if $2 \leq r \leq p-1$, in all the possible cases, we arrive at a contradiction with the assumption that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G for all $i = 1, \ldots, r$. Thus, if $2 \leq r \leq p-1$ there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1, \ldots, r$.

Second, assume that $r = p$. Then, from Lemma 3.1 it follows that

$$\begin{cases} \{x_1y_1, \ldots, x_py_p\} \cap E(H) &= \emptyset; \\ \{a_iy_2, \ldots, a_iy_p\} \cap E(H) &= \emptyset \text{ for } i = 1, \ldots, p; \\ \{x_2b_i, \ldots, x_pb_i\} \cap E(H) &= \emptyset \text{ for } i = 1, \ldots, p. \end{cases} \tag{4}$$

This means that for each $i = 1, \ldots, p$ and $j = 0, \ldots, p-2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in \{1, \ldots, p\}$ such that the path a_i, y_1, x_1, b_i is contained in G. We reason by way of contradiction supposing that for all $i = 1, \ldots, p$ the path a_i, y_1, x_1, b_i does not exist in G. Since $x_1y_1 \in E(G)$ we deduce that for each $i = 1, \ldots, p$, $a_iy_1 \in E(H)$ or $x_1b_i \in E(H)$. If $\{a_iy_1, a_{i^*}y_1\} \subset E(H)$ for two indices $i, i^* \in \{1, \ldots, p\}$, with $i \neq i^*$, then $d_{H_{1,0}}(y_1) \geq 2$. Since $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$ we have

$$\begin{aligned} e(H) &= \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big) + \sum_{j=2}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\ &\geq 4 + 2(p-1) + p \\ &= 3p + 2 \\ &> 3p + 1, \end{aligned}$$

a contradiction. Thus, in the set $\{a_1, \ldots, a_p\}$ there is at most one vertex adjacent to y_1 in H, which means that x_1 must be adjacent in H to at least $p-1$ vertices of the set $\{b_1, \ldots, b_p\}$, due to the fact that for each $i = 1, \ldots, p$, $a_iy_1 \in E(H)$ or $x_1b_i \in E(H)$. Then $d_{H_{0,0}}(x_1) \geq p-1$ and therefore,

$$\begin{aligned} e(H) &= \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big) + \sum_{j=2}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\ &\geq p + 2(p-1) + p \\ &= 3p + 1 + (p-3) \\ &> 3p + 1, \end{aligned}$$

since $p \geq 4$, again a contradiction with the hypothesis. Hence, there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1, \ldots, r$, and the result holds. ■

Theorem 3.1 *Let m, n, s, t be integers such that $2 \leq s \leq t$, $4 \leq m-s \leq n-t$, and $m+n \leq 2s+t-1$.*

Then $G = (X, Y) \in TZ(m, n; s, t)$ *iff* $G = K_{(m,n)} - M$ *where* M *is any matching of cardinality* $2(m-s)+n-t+1$.

Proof By Proposition 2.1 and Theorem 2.1, if $G = K_{(m,n)} - M$ where M is any matching of cardinality $2(m-s)+n-t+1$, then $G \in TZ(m, n; s, t)$. Thus, we only must show that there are no more extremal bipartite graphs. For that, it is enough to prove that the bipartite complement $H = (X, Y)$ of every extremal bipartite graph $G = (X, Y) \in TZ(m, n; s, t)$ has maximum degree $\Delta(H) = 1$.

Let $G = (X, Y) \in TZ(m, n; s, t)$ satisfy the hypothesis of the theorem and let us denote by $H = (X, Y)$ the bipartite complement of G. Set $p = m - s$ and $q = n - t$, then $4 \le p \le q$ and $e(H) = 2p + q + 1$. If $p = q$ then $\Delta(H) = 1$, follows from Lemma 3.3. Thus, assume that $p < q$. Without loss of generality, we may assume that the vertices of the partite set Y are ordered in such a way that $d_H(y_1) \ge d_H(y_2) \ge \cdots \ge d_H(y_n)$. Set $Y' = \{y_1, \ldots, y_{q-p}\} \subseteq Y$ and let us consider the bipartite graph $H' = (X, Y \setminus Y')$. Observe that $|X| = m$ and $|Y \setminus Y'| = n - (q - p) = t + p$. If $e(H') = 0$ then the bipartite complement G' of H' is the complete bipartite graph $K_{(m,t+p)}$. Since G' is a subgraph of G and $K_{(s,t)} \subseteq K_{(m,t+p)}$, then G contains a $K_{(s,t)}$, against the assumption. So, we may assume that $e(H') > 0$, which means that $d_H(y_i) \ge 1$ for $i = 1, \ldots, q - p$. Hence,

$$e(H') = e(H) - \sum_{i=1}^{q-p} d_H(y_i) \le 2p + q + 1 - (q - p) \le 3p + 1. \quad (5)$$

Then the following facts can be concluded:

- $E(H') = 3p + 1$. Otherwise if $E(H') < 3p + 1$ then, from Lemma 2.3, it follows that G' contains $TK_{(m-p,n-(q-p)+p)} = TK_{(m-p,n-q)} = TK_{(s,t)}$, but this contradicts the fact that $G \in TZ(m, n; s, t)$.
- $d_H(y_i) = 1$, for $i = 1, \ldots, q - p$, thus $\Delta_Y(H) = 1$, because $\Delta_Y(H) = d_H(y_1)$. This is directly derived because all the inequalities (5) become equalities since $E(H') = 3p + 1$.

Next let us see that $\Delta_X(H) = 1$. Otherwise, there is a vertex $x \in X$ having two distinct neighbors $y, y^* \in N_H(x)$. Since $\Delta_Y(H) = 1$, then $N_H(y) = N_H(y^*) = \{x\}$, and besides, there are exactly $e(H) = 2p + q + 1 > q - p + 2$ vertices of degree 1 in the class Y. Let us consider the bipartite graph $G^* = (X^*, Y^*)$ whose bipartite complement $H^* = (X^*, Y^*)$ is obtained from H by removing any $q - p$ vertices of $Y \setminus \{y, y^*\}$ of degree 1. The graph H^* satisfies that $|X^*| = |X| = m > p$, $|Y^*| = |Y| - (q - p) = t + p > p$, $e(H^*) = e(H) - (q - p) = 3p + 1$. Further, observe that $d_{H^*}(x) \ge 2$, because $\{y, y^*\} \subset Y^*$ and $\{xy, xy^*\} \subset E(H^*)$, which means that $\Delta(H^*) \ge 2$. Then, by applying Lemma 3.3, the bipartite complement G^* of H^* contains a $TK_{(m-p,t+p-p)} = TK_{(s,t)}$. Since G^* is a subgraph of G, we deduce that G contains $TK_{(s,t)}$, and this contradicts the fact that $G \in TZ(m,n;s,t)$. Hence, $\Delta(H) = \min\{\Delta_X(H), \Delta_Y(H)\} = 1$ and this proves the result. ∎

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