Results in Mathematics

Borsuk Number for Planar Convex Bodies

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Abstract. By using some simple tools from graph theory, we obtain a characterization of the compact sets in $\mathbb{R}^n$ with Borsuk number equal to two. This result allows to give some examples of planar (convex) compact sets with Borsuk number equal to three. Moreover, we also prove that the unique centrally symmetric planar convex compact sets with Borsuk number equal to three are the Euclidean balls.

Mathematics Subject Classification. 52A10.

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1. Introduction

In 1933, K. Borsuk formulated the following question [5]:

Let C be a bounded set in $\mathbb{R}^n$. Is it possible to divide C into $n+1$ subsets with diameters strictly smaller than the diameter of C?

This question has become a classical problem in Geometry [8, Sect. D14], being deeply studied during the last century. In the planar case ($n = 2$), Borsuk gave an affirmative answer in his original paper [5], basing the proof in the following nice result by Pál [27], see also [2, Lemma 1.1]: any planar set C with diameter $h > 0$ is contained in a regular hexagon H of width equal to h. It is not difficult to check that H can be divided into three congruent subsets with diameters less than h, and consequently, the induced division for C will satisfy the same property. For $n = 3$, the answer is also affirmative and was proven by several authors with different techniques [10,14,16,28]. Moreover, the same property holds in $\mathbb{R}^n$ when C is compact, convex with smooth boundary [15],

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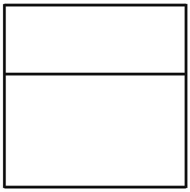

Figure 1. A division of a square into two subsets with strictly smaller diameters

or when C is compact, convex and centrally symmetric [31]. However, the answer to this question is negative in general, as the celebrated counterexample by Kahn and Kalai in 1993 exhibits [20]. In that work, they consider an equivalent discrete formulation of the problem (see [24]), and they deduced, by using a combinatorial result by Frankl and Wilson [12], that the answer to Borsuk's question is negative in $\mathbb{R}^n$ for $n \geq 2015$ (indicating also that the same happens in $\mathbb{R}^{1325}$). This was the initial point for a *sort of competition* searching for the least-dimension Euclidean space for which Borsuk's question does not hold [13,17,18,26,29,30]. Up to our knowledge, the last recent results in this direction show that the answer is negative in $\mathbb{R}^{65}$ [3] and in $\mathbb{R}^{64}$ [19]. In these last two works, the authors give elaborated counterexamples by considering a particular strongly regular graph and constructing a finite set of points in the corresponding sphere (in fact, the second reference is a refinement of the first one).

Particularizing in $\mathbb{R}^2$, the affirmative answer to Borsuk's question implies that any planar bounded set C can be divided into three (or less) subsets with diameters strictly smaller than the diameter of C. It is clear that there are planar sets which can be partitioned into *two* subsets with smaller diameters (for instance, consider a square, whose diameter is attained by the distance between any two non-consecutive vertices, and the division determined by any horizontal line, see Fig. 1).

This fact leads to the following reformulation of the problem, which will be stated for general dimension: for a given bounded set C in $\mathbb{R}^n$, let $\alpha(C)$ denote the minimal number of subsets with strictly smaller diameters into which C can be decomposed. This number is usually called *Borsuk number* of C. The original question by Borsuk can be thus rewritten as:

Let C be a bounded set in $\mathbb{R}^n$. Is it true that $\alpha(C) \leq n+1$?

The calculation of the precise value of Borsuk number for a given bounded set also constitutes an interesting issue and has been considered in different texts in literature, see for instance [2,32]. Moreover, in [22, Sect. 4] we can find a nice equivalent formulation of Borsuk's question in terms of the Minkowski sum of two convex compact sets of constant width, based on the precise values of the corresponding Borsuk numbers.

As noted previously, we know that $\alpha(C)$ is less than or equal to three for any planar bounded set C (since Borsuk's question in $\mathbb{R}^2$ has an affirmative answer). As described in [2, Sect. 1.4], it is natural investigating when the equality to three occurs. The first characterization of the planar sets with Borsuk number equal to three is due to Boltyanskii [1], by means of the notion of *completion of constant width*: Pál proved that any set C (in arbitrary dimension) with diameter $h > 0$ can be covered by a compact set with constant width equal to h [27], see also [4, Sect. 15.64]. This constant-width set is called a completion of C, and is not unique in general. Boltyanskii's characterization (whose proof is purely geometrical) states that $\alpha(C) = 3$ if and only if the completion of C is unique [2, Theorem 1.3]. Unfortunately, it is not easy to check in practice whether the completion of a general set is unique, and so the applicability of this result is not very extensive. Later on, K. Kołodziejczyk provided a new characterization in the convex setting: for a given planar convex compact set C, a *diameter segment* of C will be a segment contained in C with length equal to the diameter of C. Then, $\alpha(C) = 2$ if and only if there exists a non-diameter segment in C which intersects the interior of any diameter segment of C [22, Theorem 3.1]. A definitive characterization holds when restricting to the family $\mathscr{C}_2$ of *centrally symmetric* planar convex compact sets: D. Kołodziejczyk proved that, if $C \in \mathscr{C}_2$, then $\alpha(C) = 3$ if and only if C is a Euclidean ball [21, Appendix]. We note that this nice result is stated for a wider family of sets, namely, the planar convex compact sets with all the diameter segments being concurrent at one point, and additionally, for general dimension.

In these notes we introduce an alternative approach to these questions, by using some tools from graph theory. This point of view has not been used when treating this kind of problems, and it might provide some advances in future. We have organized these notes into two separate main sections. Section 3 will be devoted to the family $\mathscr{C}_2$ of centrally symmetric planar convex compact sets, and in Sect. 4 we will consider general compact sets. We have chosen this structure because we think it will show the remarkable differences appearing due to the lack of symmetry.

In Sect. 3 we will recover the previous detailed results in $\mathscr{C}_2$ by means of the *diameter graph* of a set, as follows. For a given planar compact set C, we can consider the diameter graph $G_C = (V, E)$ associated to C, whose set of vertices V is composed by the endpoints of the diameter segments of C, and whose set of edges E is composed precisely by such diameter segments, see [9,11]. With this notation, our Lemma 3 provides a characterization for the sets in $\mathscr{C}_2$ with Borsuk number equal to two, expressed in terms of the associated diameter graph: for any $C \in \mathscr{C}_2$, we have that $\alpha(C) = 2$ if and only if $V \neq \partial C$. This result leads to the known fact that the only centrally symmetric planar convex compact sets with Borsuk number equal to three are the Euclidean balls (see Theorem 1).

Finally, in Sect. 4 we focus on general compact sets. Some similar reasonings involving the diameter graph will allow us to obtain Theorem 2, where a characterization for the compact sets in $\mathbb{R}^n$ with Borsuk number equal to two is established. In particular, this result easily provides several examples of planar (convex) sets with Borsuk number equal to three, different from a Euclidean ball (some of them are depicted in Figs. 3 and 4).

2. Some Basic Definitions

In this section we will give some simple definitions for compact sets which will be used along these notes.

Let $C \subset \mathbb{R}^n$ be a bounded set, and denote by d the Euclidean distance in $\mathbb{R}^n$. The *diameter* of C is defined by

$$D(C) = \sup\{d(x,y) : x, y \in C\}. \tag{1}$$

Throughout these notes, we will focus on compact sets (also referred to as bodies, as usual), and so the supremum in (1) can be replaced with the maximum, and the diameter of a compact set C will represent the maximal distance between two points of C. Any line segment $\overline{x\,y}$ with endpoints $x, y \in \partial C$ satisfying that $d(x,y) = D(C)$ will be called *a diameter segment* of C. This notion leads us to a particular graph associated to any compact set in the following way, see [6,9,11].

Definition 1. Let C be a compact set in $\mathbb{R}^n$. The diameter graph $G_C = (V, E)$ associated to C is the graph whose vertices are the points in ∂C which are endpoints of the diameter segments of C, and whose edges are the diameter segments. In other words,

$$V = \{x_i \in \partial C : \exists\ y_i \in \partial C \text{ with } d(x_i, y_i) = D(C)\}, \text{ and}$$
$$\overline{x_i\,x_j} \in E \text{ if and only if } d(x_i, x_j) = D(C).$$

Definition 2. Let C be a compact set in $\mathbb{R}^n$. The Borsuk number of C is defined as the minimal number $\alpha(C) \in \mathbb{N}$ satisfying that C can be divided into $\alpha(C)$ subsets, all of them with diameters strictly smaller than $D(C)$.

Remark 1. Recall that the answer to Borsuk's question in $\mathbb{R}^2$ is affirmative, and so $\alpha(C)$ is at most three for any planar compact set C.

3. Centrally Symmetric Planar Convex Bodies

Throughout this section we will focus on the family $\mathscr{C}_2$ of *centrally symmetric* planar convex bodies. We will see that the symmetry assumption allows us to completely characterize the sets of this family whose Borsuk number is equal to three.

Consider a planar convex body C which is centrally symmetric. Recall that this means that C is invariant under the action of the rotation of angle π

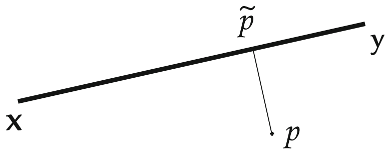

FIGURE 2. A diameter segment not containing p

centered at a point $p \in C$, which will be called the *center of symmetry* of C. We first prove some properties for the diameter graph associated to C in this setting.

Lemma 1. *Let $C \in \mathscr{C}_2$, with p its center of symmetry. Then, any diameter segment of C passes through p.*

Proof. Let $\overline{xy}$ be a diameter segment of C, and assume that it does not pass through p. Let $\widetilde{p}$ be the intersection point between $\overline{xy}$ and the line perpendicular to $\overline{xy}$ passing through p, see Fig. 2.

Without loss of generality, we can assume that $d(x,\widetilde{p}) \geq d(y,\widetilde{p})$. Denote by $x' \in \partial C$ the symmetric point of x with respect to p. Then,

$$d(x,x') = 2\,d(x,p) > 2\,d(x,\widetilde{p}) \geq d(x,\widetilde{p}) + d(y,\widetilde{p}) = d(x,y) = D(C),$$

which is a contradiction. Thus $\overline{xy}$ necessarily passes through p, as stated. □

Lemma 2. *Let $C \in \mathscr{C}_2$. Then, the diameter graph G_C associated to C is bipartite.*

Proof. From Lemma 1, any diameter segment of C passes through the center of symmetry of C, which necessarily implies that each vertex of G_C has a unique incident edge, and so it is a vertex of degree one. This directly gives that G_C is bipartite. □

Remark 2. Consider a centrally symmetric planar convex body C, with center of symmetry p and associated diameter graph $G_C = (V,E)$. Lemma 2 asserts that G_C is bipartite, which means that V can be decomposed into two disjoint subsets V_R, $V_B \subset \partial C$, satisfying that the endpoints of each edge of G_C lie in different subsets. Moreover, V_R and V_B are symmetric with respect to p, and without loss of generality, we can assume that the vertices in V_R are all placed together along ∂C, with no alternation with the vertices in V_B.

Remark 3. The reader may compare Lemma 2 above with [9, Theorem 1], where it is shown, in the general case (i.e., without the assumption of central symmetry), that the diameter graph minus one (particular) vertex is bipartite.

Remark 4. We note that Lemmas 1 and 2 immediately extend to the family of centrally symmetric compact sets in $\mathbb{R}^n$, $n \in \mathbb{N}$.

The following result characterizes the centrally symmetric planar convex bodies with Borsuk number equal to two, by means of the corresponding diameter graph.

Lemma 3. *Let $C \in \mathscr{C}_2$, and let $G_C = (V, E)$ be the diameter graph associated to C. Then, $\alpha(C) = 2$ if and only if $V \neq \partial C$.*

Proof. Assume firstly that $\alpha(C) = 2$, and let $\{C_1, C_2\}$ be a division of C with $D(C_i) < D(C)$, $i = 1, 2$. There is a point $v \in \partial C$ which belongs to the closure of C_1 as well as that of C_2. Without loss of generality, we can assume that its symmetric point v' is contained in C_1. If $v' \in V$ then $D(C) = d(v, v')$ by Lemma 1, and by the choice of v we have $D(C_1) \geq d(v, v') = D(C)$, which is a contradiction. Thus, it follows that $v' \notin V$ (as well as $v \notin V$) and so $V \neq \partial C$.

Assume now that $V \neq \partial C$, and consider $x \in \partial C$ such that $x \notin V$. By using again Lemma 1, we have that its symmetric point x' will have the same property. The division of C given by the line segment $\overline{x\,x'}$ satisfies that the two corresponding subsets have diameter strictly less than $D(C)$, since no edge of G_C will be contained in any of them (in fact, $\overline{x\,x'}$ splits any diameter segment of C), and so $\alpha(C) = 2$. □

Remark 5. If a centrally symmetric planar convex body C has associated diameter graph with finite set of vertices, then Lemma 3 trivially holds, and so C will have Borsuk number equal to two.

We can now state the following result, which proves that the unique centrally symmetric planar convex bodies with Borsuk number equal to three are the Euclidean balls.

Theorem 1. *Let $C \in \mathscr{C}_2$. Then, $\alpha(C) = 3$ if and only if C is a Euclidean ball.*

Proof. It is well known that any planar Euclidean ball has Borsuk number equal to three [5]. On the other hand, consider the diameter graph $G_C = (V, E)$ associated to C, with $V = \partial C$ in view of Lemma 3. Hence, for any $z \in \partial C$, we have that $z \in V$ and so $d(z, z') = D(C)$, where $z' \in \partial C$ is the symmetric point of z with respect to the center of symmetry p of C. This implies that $d(z, p) = 1/2\, D(C)$, and so $z \in \partial B(p, 1/2\, D(C))$, where $B(q, r)$ denotes the Euclidean ball centered at q of radius r. This holds for all $z \in \partial C$, and consequently, $\partial C = \partial B(p, 1/2\, D(C))$, as desired. □

Remark 6. As noted in the Introduction, some characterizations in the spirit of Lemma 3 and Theorem 1 have appeared previously in literature. Boltyanskii [2, Theorem 1.3] proved that a planar set has Borsuk number equal to three if and only if the corresponding completion to a constant width set is unique. Unfortunately, this nice result seems hard to be applied (in general, given a planar set, it is difficult to check the uniqueness of a completion of this type). Later on, a result by K. Kołodziejczyk [22, Theorem 3.1] shows that a planar convex body C has Borsuk number equal to two if and only if there exists a non-diameter chord in C intersecting the interior of any diameter segment of C, which is equivalent to our Lemma 3 in the centrally symmetric case. The proof of this result is based on set theory and, in particular, on Zorn's Lemma [23, Appendix 2, Cor. 2.5]. More generally, D. Kołodziejczyk [21, Appendix] proved that $\alpha(C) = n + 1$ if and only if C is a Euclidean ball, for any convex body $C \subset \mathbb{R}^n$ whose diameter segments have a common point, which includes our Theorem 1.

Remark 7. A relative optimization problem involving the diameter functional, treated in [25] (see also [7]), has a direct connection with Theorem 1. Those papers focus on centrally symmetric planar convex bodies, searching for the divisions into two subsets that *minimize* the *maximum relative diameter* (which is defined as the maximum of the diameters of the two subsets of the division). Our Theorem 1 yields that the unique sets which are not suitable for this problem are the Euclidean balls: for any division $\{C_1, C_2\}$ of a given ball $\mathcal{D}$, it follows that $\max\{D(C_1), D(C_2)\} = D(\mathcal{D})$, since $\alpha(\mathcal{D}) = 3$. Therefore, the maximum relative diameter functional is constant for all the divisions of $\mathcal{D}$, and this minimization problem is not meaningful for this particular set (any division of $\mathcal{D}$ can be considered minimizing).

4. General Compact Sets

In this section we will focus on general compact sets, with special emphasis in the planar case. The main difference with respect to Sect. 3 is that now the sets are not centrally symmetric. This more general setting presents some remarkable different properties. In particular, the associated diameter graph is not always bipartite: this can be easily seen by considering, for instance, an equilateral triangle (or a regular thetrahedron in $\mathbb{R}^3$). The following Theorem 2, which is stated for general dimension (and without the assumption of convexity), gives a characterization of the compact sets in $\mathbb{R}^n$ with Borsuk number equal to two. It will lead us to find examples in $\mathbb{R}^2$ with Borsuk number equal to three, which are different from the Euclidean balls.

Theorem 2. *Let C be a compact set in $\mathbb{R}^n$, and let $G_C = (V, E)$ be the diameter graph associated to C. Then, $\alpha(C) = 2$ if and only if G_C is bipartite, with a decomposition $V = V_R \cup V_B$ such that the closures of V_R and V_B have empty intersection (that is, $\overline{V_R} \cap \overline{V_B} = \emptyset$).*

Proof. Assume firstly that $\alpha(C) = 2$, and let $\{C_1, C_2\}$ be a division of C with $D(C_i) < D(C)$, $i = 1, 2$. Then, by considering $V_R = V \cap C_1$ and $V_B = V \cap C_2$, we will have a decomposition of V which implies that G_C is a bipartite graph: if an arbitrary edge $\overline{xy}$ in G_C has both vertices x, y in (say) V_R, then

$$D(C_1) \geq d(x, y) = D(C),$$

yielding a contradiction. Suppose now that $\overline{V_R} \cap \overline{V_B} \neq \emptyset$, and consider $v \in \overline{V_R} \cap \overline{V_B}$. In particular, there exists a sequence $\{x_R^n\}_n \subset V_R$ which converges to

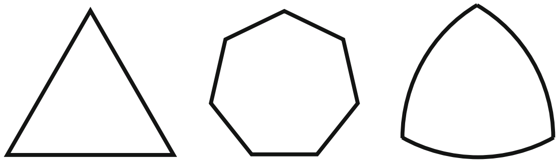

FIGURE 3. Some planar convex bodies with Borsuk number equal to three

v. Moreover, for each element x_R^n of the sequence, there exists $x_B^n \in V_B$ with

$$d(x_R^n, x_B^n) = D(C). \tag{2}$$

Note that the sequence $\{x_B^n\}_n$ is contained in ∂C, which is compact, so we can assume that $\{x_B^n\}_n$ converges to a certain $w \in \partial C$. Without loss of generality, we can assume that w belongs to C_1. Taking now limit in (2) when n tends to infinity, it follows that $D(C_1) \geq d(v, w) = D(C)$, which is again a contradiction. So necessarily $\overline{V_R} \cap \overline{V_B} = \emptyset$, as stated.

Now assume that G_C is bipartite with $V = V_R \cup V_B$ and $\overline{V_R} \cap \overline{V_B} = \emptyset$. Since $\overline{V_R}$ and $\overline{V_B}$ are two disjoint compact subsets in ∂C, the minimal distance ε between points from $\overline{V_R}$ and $\overline{V_B}$ is attained and positive. Let

$$C_1 = \{x \in C : \text{ there is a } y \in \overline{V_R} \text{ with } d(x, y) \leq \varepsilon/2\},$$

that is, the intersection of C and the $\varepsilon/2$–parallel set of $\overline{V_R}$, and let $C_2 = C \backslash C_1$. Then $\{C_1, C_2\}$ is a division of C with $V_R \cap \overline{C_2} = \emptyset$ and $V_B \cap \overline{C_1} = \emptyset$, where $\overline{C_i}$ stands for the closure of C_i, $i = 1, 2$. Since any diameter segment of C has an endpoint in V_R and another one in V_B, it follows that both C_1 and C_2 have diameters smaller than $D(C)$, and so $\alpha(C) = 2$. □

Remark 8. Any planar (convex) body which does not satisfy the conditions stated in Theorem 2 will have Borsuk number equal to three. This is the case for any regular polygon with an odd number of vertices, or any Reuleaux polygon (it is easy to check that the corresponding diameter graphs are not bipartite). Moreover, any small variation of the previous sets preserving the corresponding diameter segments will have the same property (Fig. 3).

The following Example 1 describes the construction of a family of planar convex bodies with Borsuk number equal to three.

Example 1. Let C_1 be a circle centered at the origin $o = (0, 0)$ with radius $r > 0$. Let $a = (r, 0)$, and consider C_2 the circle centered at a with radius r. Call L the symmetric lens given by the intersection of the circles C_1 and C_2. Choose two arbitrary points $b \in L \cap C_2$, and $c \in L \cap C_1$, both of them in the upper half-plane, and let d be the intersection point of the two circles centered at b and c with radius r, which is contained in L. Then, the convex hull C of points $\{o, a, b, c, d\}$ has five diameter segments, namely $\overline{oa}$, $\overline{ab}$, $\overline{bd}$, $\overline{cd}$ and $\overline{oc}$, and the associated diameter graph G_C is not bipartite. By applying

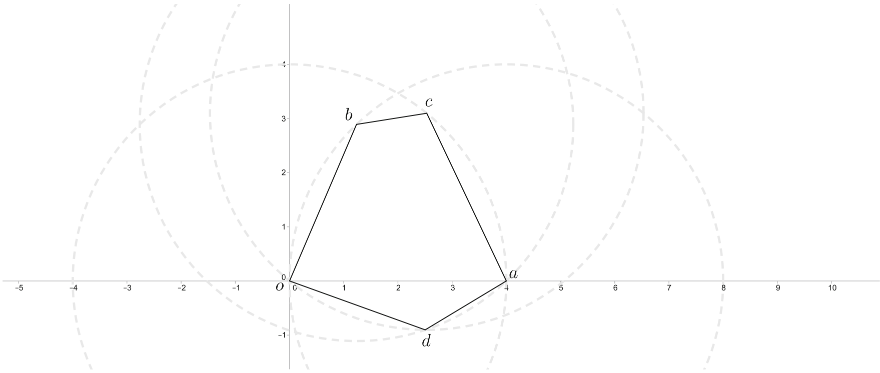

Figure 4. A planar convex body with Borsuk number equal to three

Theorem 2, we conclude that $\alpha(C) = 3$. Figure 4 below shows a particular set of this family for radius $r = 4$. We also note that slight variations of C will provide new non-polygonal examples with Borsuk number equal to three.

Remark 9. Although Borsuk's original question is not stated in a convex setting, the following related problem can be posed: given a *convex* body C in $\mathbb{R}^n$, is it possible to divide C into $n+1$ *convex* subsets with strictly smaller diameters than C? We want to note that the answer is affirmative in $\mathbb{R}^2$. Recall that, in this case, $\alpha(C) \in \{2,3\}$. If $\alpha(C) = 3$, the original reasoning by Borsuk (already outlined in the Introduction) proves the claim: C is contained in the regular hexagon H of width equal to $D(C)$, and H can be divided into three appropriate congruent subsets by using three line segments. Then, the induced division of C will consist of three convex subsets with smaller diameters, due to the convexity of C. And if $\alpha(C) = 2$, then by [22, Theorem 3.1] there exists a non-diameter segment in C intersecting any diameter segment of C, which provides a division of C into two convex subsets with smaller diameters (taking into account again that C is convex).

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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