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Fuzzy Sets and Systems 466 (2023) 108452

FUZZY sets and systems

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Sharing profits in formal fuzzy contexts

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Received 10 June 2022; received in revised form 15 November 2022; accepted 9 December 2022
Available online 15 December 2022

Abstract

Cooperative game theory is concerned with situations where a group of agents coordinate their actions to get a common benefit. An allocation rule for these situations is a way to share the common benefit among the agents. The search for a fair allocation rule may depend on the information one has about these agents. A formal context represents information about certain attributes of a set of objects in a table, and they have been used in the literature to describe information about the agents in a game. More recently, formal contexts are extended to the fuzzy setting. Now in this paper we establish a methodology to share the profits of a group of agents that have some information about them collected in a fuzzy formal context when those benefits depend on a set of attributes.

Keywords: Cooperative games; Fuzzy concept lattices; Shapley value; Fuzzy sets

1. Introduction

The theory of cooperative games analyzes situations in which a finite set of agents must share a common benefit or cost among themselves. For such a goal, the values that different subsets of agents (coalitions) could obtain in similar situations are used as data. So, a cooperative game of transferable utility for a set of agents is a function on the family of coalitions called the characteristic function. However, if some additional information about the agents and their relationships is known, then the sharing formulae are modified as the possibilities of cooperation between them change. The most widely used sharing rule is the Shapley value [24], introduced in 1953. Since then, numerous modifications of the rule have been proposed by introducing different kinds of information about the players: communication situations [22], a priori unions [26], conference systems [23], convex geometries [6], permission structures [8] or restricted formation of coalitions [20]. The definition of a sharing rule must be based on the verification of certain properties that make it reasonable. These properties are usually linked to the conditions imposed by the given information. Later, cooperative games have been extended to the fuzzy setting. Aubin uses fuzzy coalitions

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https://doi.org/10.1016/j.fss.2022.12.008

in [2], Mares works with fuzzy payoffs in [21] and Borkotokey mixes both tools in [7]. Also, the Shapley value is analyzed for cooperative games with additional fuzzy information about the agents. So, fuzzy communication situations [18], proximity relations among players [11], cohesion indices [12], fuzzy permission structures [13] or fuzzy authorization structures [14] have been studied. In all of them, fuzzy definitions of existing crisp structures were used.

A formal context represents, by means of a table (a binary relation), the relationships between a set of objects and a set of attributes applicable to those objects. A concept [15,16][1] in a context is a pair formed by a subset of objects and a subset of attributes in such a way that all attributes fulfilling all those objects and vice versa are agglutinated. Therefore, concepts represent those groups of objects and attributes that are well defined in the context. The family of concepts in a context has the structure of a lattice. Fuzzy contexts were introduced in 2002 [3] using a fuzzy binary relation between objects and attributes. There are numerous definitions of the fuzzy formal concept in the literature, in [5] several of them are compared. Some are defined as a pair of fuzzy sets of objects and attributes, others are mixed in the sense that they use a fuzzy set and a crisp set, and finally there are others that are crisp concepts but in the fuzzy setting. In this paper, we deal with a notion of concept fully fuzzy. Formal contexts and fuzzy formal contexts have been applied in several fields to analyzed data: software engineering, social network analysis, web services, security analysis, text mining, information retrieval, linguistics, ontology engineering, risk analysis or KDD process. Singh et al. [25] did a compilation of references before 2015 about applications of the concept analysis in both, crisp and fuzzy settings. More recently, Alwersh and Kovács [1] cited numerous applications of the fuzzy formal contexts in the above fields.

Game theory and formal concept analysis were related in two different ways. Ignatov and Kwuida in [17] used cooperative games to measure the importance of objects and attributes in the construction of a concept, namely they used game theory to study formal contexts. Faigle et al. in [10] introduced concept games. They proposed cooperative games whose characteristic function measures the benefits obtained by the concepts in a formal context. So, agents are the objects and attributes are those aspects that can diminish the profit obtained by these agents. The concept is then a stable coalition in the context with respect to the attributes. A Shapley value-kind solution was defined in [10] for this type of games and also for other values such as the Banzhaf value in [19].

Now, in this paper, our aim is to analyzed games in a fuzzy formal context following [10]. Particularly we look for an allocation rule similar to the Shapley value but for a particular family of concept games in a fuzzy formal context. For this goal, the paper is organized as follows. Section 2 presents an allocation problem to be solved that motivates the theory developed in the following sections. This example is used throughout the paper to illustrate definitions and results. Section 3 contains several preliminaries about cooperative games and fuzzy formal concepts. In Section 4 we define a Shapley value for the proposed situations, and in Section 5 we study several properties of our allocation rule. Also, there is a conclusion section at the end.

2. Motivation

The types of problem to be solved in this paper are similar to the following. Assume a set of agents cooperating to sell a software product that performs as many applications as possible at the best level for a given environment. The set of agents involved and the desired applications are therefore known as a given, as well as the profits obtained according to the applications that can be safely offered. But in reality, it is possible to sell intermediate products with certain levels of risk of failure in some of the applications, by lowering the price appropriately. Moreover, what we consider known regarding the relationship between agents and applications is the level of risk with which each of them is able to generate each application.

We consider three companies $N = \{1, 2, 3\}$ collaborating for the development of a certain software product. This software is intended to provide three different applications $M = \{a, b, c\}$ to users. The level of quality achieved in these three utilities will determine the final price of the product. The following table I shows for each company its ability to produce the product in terms of the quality it is capable of developing in each application through its error rate. So, for example, the company 2 is able to obtain a product without errors in terms of utility a, but is unable to make utility c work.

[1] There exists an older paper about formal concept analysis in German.

Table 2.1
Table I of risks of failure.

	a	b	c
1	0.1	0.6	0.8
2	0	0.3	1
3	0.5	0.1	0.8

The next function w, in Table 2.2, is the price of the product according to the services it has (assuming maximum quality). We assume that this price decreases if the quality of the applications is not maximal in proportion to their error levels. Also, we consider that if firms do not obtain useful software, then they sell their research to another external interested agent.

Table 2.2
Prices of the product depending on the applications not covered.

A	M	ab	ac	bc	a	b	c	$\emptyset$
$w(A)$	10	20	30	40	60	100	120	250

So, for instance $w(ab)$ is the price of a product contains only application c, $w(\emptyset)$ is the best price because all the applications work and $w(M)$ is the profit for selling the research work (the product does not work). Obviously w is decreasing in the family of subsets of M (according to the containment relationship) and non-negative.

We seek to distribute among agents the benefit that is obtained with the best product that they can build, adapting the price to the risks of failure of each application, according to the capacity that each agent has been contributing to the final project. Hence, the problem data are (N, M, I, w). The next sections present a methodology to carry out this distribution of benefits. The function w is seen as a cooperative game over M and the table I as a fuzzy formal context over (N, M). The fuzzy concepts of the context serve as a transmitter element of the information on the benefits given by attributes to obtain information on the involvement of each agent in these benefits.

3. Preliminaries

3.1. Lattices and formal contexts

Let $(L, \leq)$ be a (finite) partially ordered set with L a finite set and $x, y \in L$. The supremum of x, y exists if there is only one element of L, $x \vee y$, such that $x, y \leq x \vee y$ and any other $z \in L$ with $x, y \leq z$ satisfies $x \vee y \leq z$. The infimum is the only element $x \wedge y$ such that $x \wedge y \leq x, y$ and if $z \in L$ verifies $z \leq x, y$ then $z \leq x \wedge y$. A *lattice* is a partially ordered set $(L, \leq)$ in which each pair of elements has the supremum and infimum. The top of the lattice is an element $top \in L$ with $x \leq top$ for all $x \in L$, and the bottom is another element $bottom \in L$ with $bottom \leq x$ for all $x \in L$. We will use $x < y$ to say $x \leq y$ and $x \neq y$. Element y *covers* x in the lattice, $y \rhd x$, if $x < y$ and there is no $z \in L$ with $x < z < y$. The elements *coatoms* (*atoms*) in L are those elements $x \in L$ that satisfy $top \rhd x$ $(x \rhd bottom)$. A (maximal) *chain* in the lattice is an ordered set $C = \{x_0, x_1, ..., x_p\}$ such that $x_0 = bottom$, $x_p = top$ and $x_{k-1} \lhd x_k$ for all $k = 1, ..., p$. The family of chains of the lattice $(L, \leq)$ is denoted by $CH(L)$ and its cardinality $ch(L) = |CH(L)|$.

A *formal context* is a triple $\mathcal{C} = (N, M, I)$ where: N is a finite set whose elements are named objects, M is another finite set whose elements are called attributes, and I is a binary relation between N and M. Therefore, I is a $\{0, 1\}$-matrix such that $I(i, a) = 1$ if object i is related to attribute a and $I(i, a) = 0$ otherwise. The *derivation operators* in the formal context $\mathcal{C}$ are defined for each $S \subseteq N$ and $A \subseteq M$ as

$$S'_{\mathcal{C}} = \{a \in M : I(i, a) = 1\, \forall i \in S\} \quad \text{and} \quad A'_{\mathcal{C}} = \{i \in N : I(i, a) = 1\, \forall a \in A\}. \tag{1}$$

A *concept* in $\mathcal{C}$ is a pair (S, A) with $S \subseteq N$ and $A \subseteq M$ such that $S'_{\mathcal{C}} = A$ and $A'_{\mathcal{C}} = S$. If (S, A) is a concept, then S is called its extent and A its intent. The family of concepts of $\mathcal{C}$ is denoted by $L_{\mathcal{C}}$. If $(S, A), (T, B) \in L_{\mathcal{C}}$ then $(S, A) \sqsubseteq (T, B)$ if $S \subseteq T$ (or equivalently $B \subseteq A$). The pair $(L_{\mathcal{C}}, \sqsubseteq)$ is a lattice with supremum and infimum

$$(S,A)\sqcup(T,B)=\big((A\cap B)'_{\mathcal{C}},A\cap B\big)\quad\text{and}\quad(S,A)\sqcap(T,B)=\big(S\cap T,(S\cap T)'_{\mathcal{C}}\big).\tag{2}$$

The bottom of this lattice is $(M'_{\mathcal{C}}, M)$ and the top is $(N, N'_{\mathcal{C}})$.

3.2. Cooperative games and concept games

Consider N a finite set of agents (usually called players) in a cooperation situation, that is, a situation where they cooperate to get a common profit. We denote by $\mathbb{R}^N$ the $|N|$-dimensional real space using the labels of N in the axes. A payoff vector for N is an element $x \in \mathbb{R}^N$ such that x_i is understood as the individual outcome for player $i \in N$ from the common profit. The main goal in a cooperation situation is to find a way to divide the common profit among the agents.

We represent by 2^N the power set of N, this is the family of all subsets (called coalitions) of N. The Boolean algebra of N is the lattice $(2^N, \subseteq)$ formed with the elements of the power set ordered by the inclusion relation. It is known that $ch(2^N) = |N|!$. A *cooperative game* (with transferable utility) is a pair (N, v) where N is the set of players and v is a mapping $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ called characteristic function. Number $v(S)$ represents the worth of coalition S for the cooperation of its players. Cooperative games represent cooperation situations adding information about the different coalitions. A *value for cooperative games* ϕ is an application that obtains a payoff vector $\phi(N, v)$ for each game (N, v). A well-known value for cooperative games is the *Shapley value* [24], given (N, v) and $i \in N$ the outcome of i is

$$Sh_i(N,v)=\frac{1}{|N|!}\sum_{C\in CH(2^N)}\Big[v(S^i_C)-v(S^i_C\setminus\{i\})\Big]\tag{3}$$

where S^i_C is the first coalition in C from the bottom $\emptyset$ containing i. If $S \rhd T$ in 2^N then $v(S) - v(T)$ is called the contribution of S from T, this contribution is marginal because it is assignable to only one player, $S \setminus T = \{i\}$. Therefore, the Shapley value of i is the average of the marginal contributions obtained for this player i in each chain in 2^N. Given a nonempty coalition $T \subseteq N$, *unanimity game* u_T is defined as $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. The Shapley value of the unanimity game u_T for each player i is $Sh_i(N, u_T) = \dfrac{1}{|T|}$ if $i \in T$ and $Sh_i(N, u_T) = 0$ otherwise.

Faigle et al. in [10] study games defined in formal contexts. A *concept game* is a pair $(\mathcal{C}, v)$ where $\mathcal{C} = (N, M, I)$ is a formal context and $v : L_{\mathcal{C}} \to \mathbb{R}$ with $v(M'_{\mathcal{C}}, M) = 0$ if $M'_{\mathcal{C}} = \emptyset$. In the context $\mathcal{C}$ the set N is the set of players and M is the set of handicaps that affect the worth of the collaborations. If $I(i, a) = 1$, then the player i satisfies the handicap a. The authors consider that the only well-defined coalitions are the extents of the different concepts; moreover, the worth can depend on the players and also on the handicaps. Hence, the characteristic function is restricted to the concept lattice. For each $(S, A) \in L_{\mathcal{C}}$, number $v(S, A)$ is understood as the profit obtained by S if the attributes (handicaps) of A occur and $v(N, N'_{\mathcal{C}})$ is considered as the common profit. So, concept games represent cooperative situations that add information from a formal context and a characteristic function. A *value for concept games* ϕ is a mapping that obtains a payoff vector for each concept game, that is, for every $(\mathcal{C}, v)$ we get $\phi(\mathcal{C}, v) \in \mathbb{R}^N$ if $\mathcal{C} = (N, M, I)$. The *extent Shapley value* is a version of the classic Shapley value for concept games defined in [10].[2] For any concept game $(\mathcal{C}, v)$ with $\mathcal{C} = (N, M, I)$ the payoff of a player $i \in N$ is

$$Sh^{ex}_i(\mathcal{C},v)=\begin{cases}\dfrac{v(M'_{\mathcal{C}},M)}{|M'_{\mathcal{C}}|} & \text{if } i\in M'_{\mathcal{C}}\\[2ex] \dfrac{1}{ch(\mathcal{C})}\displaystyle\sum_{C\in CH(\mathcal{C})}\frac{1}{|S^i_C\setminus T^i_C|}\Big[v\left(S^i_C,(S^i_C)'_{\mathcal{C}}\right)-v\left(T^i_C,(T^i_C)'_{\mathcal{C}}\right)\Big] & \text{otherwise}\end{cases}\tag{4}$$

where S^i_C is the extent of the first concept in C from the bottom $(M'_{\mathcal{C}}, M)$ containing i and T^i_C the extent of the concept covered by S^i_C in C. Thus, the extent Shapley value is similar to the Shapley value; it is an average of payoff vectors, one for each chain in the concept lattice. The extent Shapley value shares $v(N, N'_{\mathcal{C}})$ among players.

[2] Although the construction is feasible with all the concept games, they consider only formal contexts $\mathcal{C} = (N, M, I)$ such that there are not superfluous attribute, i.e. $N'_{\mathcal{C}} = \emptyset$.

3.3. Fuzzy sets and fuzzy formal contexts

We will use the triangular norm, its associated conorm and the implication of Gödel [9] over the interval [0, 1], this is for all $t_1, t_2 \in [0,1]$ respectively: $t_1 \wedge t_2 = \min(t_1, t_2)$, $t_1 \vee t_2 = \max(t_1, t_2)$ and $t_1 \to t_2 = \begin{Bmatrix} 1 & \text{if } t_1 \leq t_2 \\ t_2 & \text{if } t_1 > t_2 \end{Bmatrix}$.

Let L be a finite set. A *fuzzy set* τ of L is defined by its membership function $\mu_\tau : L \to [0,1]$. Without loss of generality, we will identify τ with μ_τ, so $\tau(x) = \mu_\tau(x)$ for all $x \in L$. The family of fuzzy sets of L is denoted by $[0,1]^L$. Classic sets are also fuzzy sets (called crisps), each $S \subseteq L$ is identified with $S \in [0,1]^L$ with $S(x) = 1$ if $x \in S$ and $S(x) = 0$ otherwise. We will use 0 to represent $\emptyset$. The *support* of $\tau \in [0,1]^L$ is the set $supp(\tau) = \{x \in L : \tau(x) > 0\}$, and the *image* of τ is supposed to be an ordered set, $im(\tau) = \{t \in [0,1] : \exists x \in L \text{ with } \tau(x) = t\} = \{t_1 < t_2 < \cdots < t_p\}$. If $t \in [0,1]$, then the *t-cut* of τ is the set $[\tau]_t = \{x \in L : \tau(x) \geq t\}$. The *minimum* and the *maximum* of τ are, respectively, the numbers $\wedge\tau = \wedge\{\tau(x) : x \in supp(\tau)\}$ ($\wedge 0 = 0$) and $\vee\tau = \vee\{\tau(x) : x \in L\}$. Next, we define several operations and relations for fuzzy sets. If $\tau, \sigma \in [0,1]^L$ then $\tau \leq \sigma$ if $\tau(x) \leq \sigma(x)$ for all $x \in L$. The intersection is $(\tau \cap \sigma)(x) = \tau(x) \wedge \sigma(x)$ and the union $(\tau \cup \sigma)(x) = \tau(x) \vee \sigma(x)$. The fuzzy sets τ, σ are *complementary* if $\tau(x) + \sigma(x) \leq 1$ for each $x \in L$. We can define the addition and the subtraction for particular cases. If τ, σ are complementary, then $(\tau + \sigma)(x) = \tau(x) + \sigma(x)$ is well defined; and if $\sigma \leq \tau$, then $(\tau - \sigma)(x) = \tau(x) - \sigma(x)$ is also well done. If $t \in [0,1]$, then $(t\tau)(x) = t\tau(x)$. Finally, if $\tau \in [0,1]^L$ and $L \subseteq \hat{L}$ then we will use $\tau^0 \in [0,1]^{\hat{L}}$ as the fuzzy set given by $\tau^0(x) = \begin{cases} \tau(x) & \text{if } x \in L \\ 0 & \text{if } x \in \hat{L} \setminus L \end{cases}$.

Let $v : 2^L \to \mathbb{R}$ be a function over the power set of L. The (non-monotonic) *Choquet integral* [28] of a fuzzy set $\tau \in [0,1]^L$ with regard to v is defined as

$$\int_c \tau \, dv = \sum_{k=1}^{p} (t_k - t_{k-1}) \left[v\left([\tau]_{t_k}\right) - v(\emptyset) \right], \tag{5}$$

where $im(\tau) \cup \{0,1\} = (t_0 < \cdots < t_p)$. Next, we show several known properties of the Choquet integral. Let $\tau, \sigma \in [0,1]^L$ and v, w are functions over 2^L: (P1) If $t_1, t_2 \in \mathbb{R}_+$ then $\int_c \tau \, d(t_1 v_1 + t_2 v_2) = t_1 \int_c \tau \, dv + t_2 \int_c \tau \, dw$, (P2) if $t \in [0,1]$ then $\int_c t\tau \, dv = t \int_c \tau \, dv$, (P3) if $S \subseteq N$ then $\int_c S \, dv = v(S) - v(\emptyset)$, and (P4) if $L \subseteq \bar{L}$ and v is a function over $2^{\bar{L}}$ then $\int_c \tau^0 \, dv = \int_c \tau \, dv|_{2^L}$. Generally $\int_c (\tau + \sigma) \, dv \neq \int_c \tau \, dv + \int_c \sigma \, dv$.

A *fuzzy formal context* is a triple $\mathcal{C} = (N, M, I)$; where N is a finite set of objects, M is a set of attributes and I is a fuzzy binary relation between N and M. So, I is now a [0, 1]-matrix, where $I(i,a)$ indicates the degree of fulfilment of the attribute a by the object i. For each $i \in N$ we represent by $I_i \in [0,1]^M$ the i-row of I and for each $a \in M$ we use $I^a \in [0,1]^N$ as the a-column of I. Obviously, a formal context is a fuzzy context with a {0, 1}-matrix, and we will call it *crisp*. Therefore, $\mathcal{C}$ is a fuzzy context *non-crisp* if there exists $(i,a) \in N \times M$ with $I(i,a) \in (0,1)$. The *derivation operators* are defined in the fuzzy setting following [3]. If $\sigma \in [0,1]^N$ and $\alpha \in [0,1]^M$ then $\sigma'_{\mathcal{C}} \in [0,1]^M$ and $\alpha'_{\mathcal{C}} \in [0,1]^N$ with

$$\sigma'_{\mathcal{C}}(a) = \bigwedge_{i \in N} [\sigma(i) \to I(i,a)] \quad \text{and} \quad \alpha'_{\mathcal{C}}(i) = \bigwedge_{a \in M} [\alpha(a) \to I(i,a)], \tag{6}$$

with $i \in N$ and $a \in M$. A *fuzzy concept*[3] for the fuzzy context $\mathcal{C}$ is a pair $(\sigma, \alpha) \in [0,1]^N \times [0,1]^M$ with $\sigma'_{\mathcal{C}} = \alpha$ and $\alpha'_{\mathcal{C}} = \sigma$. Let $L_{\mathcal{C}}$ be the set of fuzzy concepts for the fuzzy context $\mathcal{C}$, which is a finite set. If $(\sigma, \alpha) \in L_{\mathcal{C}}$ then $\sigma''_{\mathcal{C}} = (\sigma'_{\mathcal{C}})'_{\mathcal{C}} = \sigma$ and $\alpha''_{\mathcal{C}} = (\alpha'_{\mathcal{C}})'_{\mathcal{C}} = \alpha$. As in the crisp case, if we define $(\sigma, \alpha) \sqsubseteq (\tau, \beta)$ when $\sigma \leq \tau$ (or equivalent $\alpha \geq \beta$) then $(L_{\mathcal{C}}, \sqsubseteq)$ is a lattice with the following supremum and infimum

[3] There are several definitions of concepts in the fuzzy framework, Belohlavek in [5] comments on all the different notions. We use one of the most general definitions in the sense that both elements, extent and intent, are fuzzy.

$$(\sigma,\alpha)\sqcup(\tau,\beta)=\left((\alpha\cap\beta)'_{\mathcal{C}},\alpha\cap\beta\right)\quad\text{and}\quad(\sigma,\alpha)\sqcap(\tau,\beta)=\left(\sigma\cap\tau,(\sigma\cap\tau)'_{\mathcal{C}}\right).\tag{7}$$

Also, as in the crisp case, the bottom of this lattice is $(M'_{\mathcal{C}}, M)$ and the top $(N, N'_{\mathcal{C}})$. We use the algorithms in [4] to calculate fuzzy concepts.

4. Intent fuzzy concept games: the extent Shapley value

We define the model showed in Section 2 using the notions given in preliminaries. Following Section 2, observe that $\mathcal{C}=(N,M,I)$ with I the risks of failure given in Table 2.1 for the agents in N and the application in M is a fuzzy formal context.

Definition 4.1. An intent fuzzy concept game is a pair $(\mathcal{C}, w)$ where $\mathcal{C}=(N,M,I)$ is a fuzzy context and $w: 2^M \to \mathbb{R}_+$ is a function satisfying: $w(A)\le w(B)$ if $A\supseteq B$ and $w(M)=0$ if $M'_{\mathcal{C}}=0$. The family of this kind of games is denoted by Γ.

In the crisp case, when $\mathcal{C}$ is a formal context, we associate to $(\mathcal{C}, w)\in\Gamma$ a concept game (see Subsection 3.2) given by $v^w(S,A)=w(A)$ for any concept $(S,A)\in L_{\mathcal{C}}$. We get a solution for $(\mathcal{C}, w)$ applying formula (4) to the associated concept game,

$$Sh_i^{ex}(\mathcal{C},w)=\begin{cases}\dfrac{w(M)}{|M'_{\mathcal{C}}|}, & \text{if } i\in M'_{\mathcal{C}}\\ \dfrac{1}{ch(\mathcal{C})}\displaystyle\sum_{C\in CH(\mathcal{C})}\frac{1}{|S^i_C\setminus T^i_C|}\left[w\left((S^i_C)'_{\mathcal{C}}\right)-w\left((T^i_C)'_{\mathcal{C}}\right)\right], & \text{otherwise.}\end{cases}\tag{8}$$

The payoff vector $Sh^{ex}(\mathcal{C}, w)$ is an allocation of value $w(N'_{\mathcal{C}})$. The allocation is done as the average in the concept lattice of a payoff vector for each chain. In each step of a chain, we have two concepts $(S,A)\rhd(T,B)$ obtaining the contribution $w(A)-w(B)$. This contribution is not marginal for only one player and is allocated among the involved players $S\setminus T$, in an egalitarian way.

In this paper we deal with games defined in the environment of a fuzzy context, as in Section 2, following this idea, the intent fuzzy concept games. Hence, we study games where the potential value that players can obtain depends only on the requirements or handicaps (the attributes) that limit its attainment but considering the relation between players and attributes fuzzy.

We consider[4] w proportional in the sense that if $A\subseteq M$ is satisfied at level $t\in[0,1]$ then the expected worth of $tA\in[0,1]^M$ is $tw(A)$. Hence, the Choquet integral is a good tool for managing fuzzy information. Therefore, the value by w of $\alpha\in[0,1]^M$ is defined by

$$w(\alpha)=w(\emptyset)+\int_c \alpha\,dw=\sum_{k=1}^{p}(t_k-t_{k-1})w\left([\alpha]_{t_k}\right),\tag{9}$$

with $im(\alpha)\cup\{0,1\}=(t_0<\cdots<t_p)$. Since property (P3) we have constructed an extension of the initial w. The top of the fuzzy concept lattice $L_{\mathcal{C}}$ is $(N, N'_{\mathcal{C}})$, so the profit to allocate, *top profit*, is $w(N'_{\mathcal{C}})$. The *bottom profit* is the worth of the bottom $(M'_{\mathcal{C}}, M)$, $w(M)$. We denote them by

$$w^{\top}(\mathcal{C})=w(N'_{\mathcal{C}})\text{ and }w^{\perp}(\mathcal{C})=w(M).\tag{10}$$

Example 4.1. Following the example in Section 2, the top profit using (9) and (10) is determined as

$$w^{\top}(\mathcal{C})=w(0,0.1,0.8)=0.1\cdot 40+0.7\cdot 120+0.2\cdot 250=138.$$

This number is the price to allocate among the players because it corresponds to the best product that they can obtain: a product with risk rate of 0.1 for application b and 0.8 for c. Also we can calculate the price of the bottom, the worst option: to do a useless job. This price following (10) is $w^{\perp}(\mathcal{C})=10$.

[4] This is a usual consideration in the fuzzy setting for games, see [2,27].

Our goal is to obtain a payoff vector for the players of $(\mathcal{C}, w) \in \Gamma$ following the philosophy of the classic Shapley value. A *value for intent fuzzy concept games* is a mapping ϕ over Γ that obtains a payoff vector $\phi(\mathcal{C}, w) \in \mathbb{R}^N$ for each $(\mathcal{C}, w) \in \Gamma$ with $\mathcal{C} = (N, M, I)$. Following formula (8), we go through the chains of the concept lattice distributing the value obtained by the increment of intent.

Example 4.2. Fig. 4.1 shows the fuzzy concept lattice $L_{\mathcal{C}}$ of the context in Table 2.1.

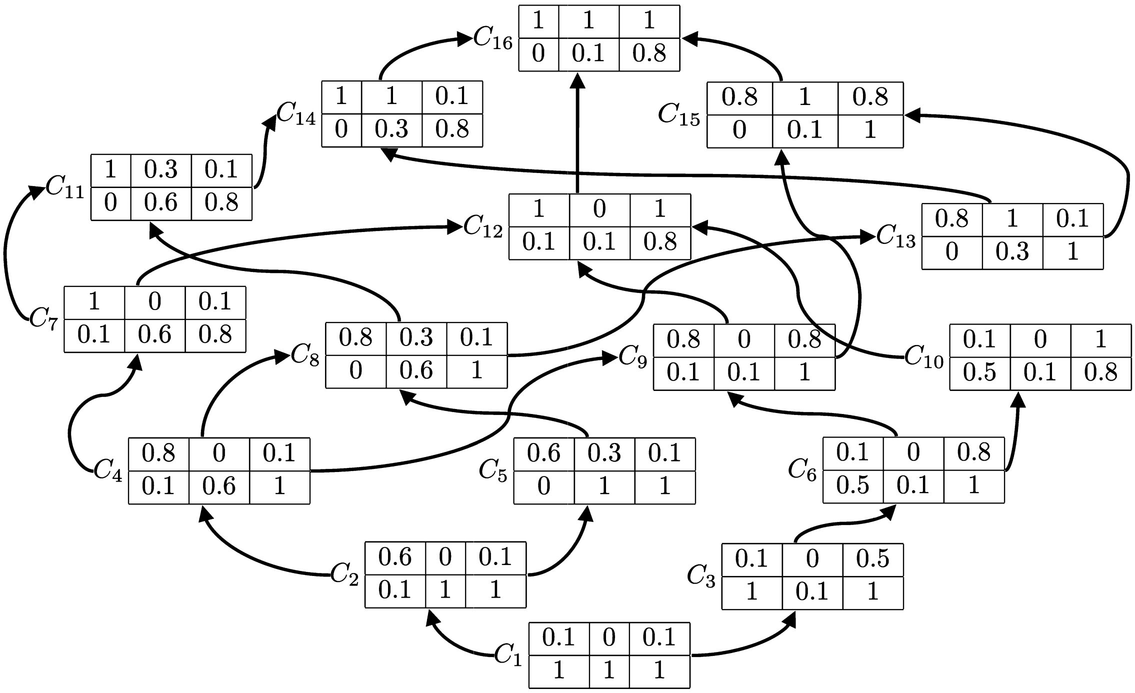

Fig. 4.1. Fuzzy concept lattice of context $\mathcal{C}$.

The fuzzy concepts in this example identify the possible products that can transfer information to the levels of participation of the players. Hence, the extents are fuzzy coalitions explaining the capacity with which each player engages in the collaboration. For example, it is possible to make a product with $\alpha = (1, 0.3, 1) \in [0, 1]^M$, namely a software with only the application b and with an error risk level of 0.3. But this fuzzy set is not the intent of a fuzzy concept in $L_{\mathcal{C}}$ (see Fig. 4.1) and this fact means that the participation levels of the firms given in $\alpha'_{\mathcal{C}} = (0.1, 0, 0.5)$ can get a better product, $\alpha''_{\mathcal{C}} = (1, 0.1, 1)$.

But now we need to define how to allocate this contribution among the players. The extent of each fuzzy concept represents the participation levels of the players required to get that intent, considering the marginalities of the players of the increment of extent.

We define the notion of contribution in each step of a chain in the fuzzy concept lattice as in formula (8). If $(\mathcal{C}, w) \in \Gamma$ and $(\sigma, \alpha) \triangleright (\tau, \beta)$ in $L_{\mathcal{C}}$ then the *contribution* of the first fuzzy concept to the second one is $w(\alpha) - w(\beta)$. As in the crisp case, this contribution is not always assignable to only one player, but neither to a crisp set of them. We need to explain how to allocate this contribution among the involved players taking into account their participation levels, the marginality. Suppose a unit of profit such that a particular set of players S is necessary to get it, the outcome of the classic Shapley value (3) for the unanimity game u_S (see Subsection 3.2) gives us the share for each player of this unit. This is the motivation of the next definition.

Definition 4.2. Let N be a set of players. The Shapley power function of a player $i \in N$ is the function $h_i : 2^N \to \mathbb{R}$ defined as $h_i(S) = Sh_i(u_S)$.

We use the Shapley power functions of the players to introduce a concept of marginality in an intent fuzzy concept game.

Definition 4.3. Let $(\mathcal{C}, w) \in \Gamma$ with $\mathcal{C} = (N, M, I)$. If $(\sigma, \alpha) \rhd (\tau, \beta)$ in $L_{\mathcal{C}}$ then the contribution of the first fuzzy concept to the second is $w(\alpha) - w(\beta)$. The marginality of player $i \in N$ for that contribution is given by

$$ma_{\tau}^{\sigma}(i, \mathcal{C}) = \frac{1}{\vee(\sigma - \tau)} \int_{c} (\sigma - \tau)\, dh_i.$$

Particularly, the marginality of each player i in the bottom $(M'_{\mathcal{C}}, M)$ is defined by

$$ma^{M'_{\mathcal{C}}}(i, \mathcal{C}) = \begin{cases} \dfrac{1}{\vee M'_{\mathcal{C}}} \displaystyle\int_{c} M'_{\mathcal{C}}\, dh_i & \text{if } M'_{\mathcal{C}} \neq 0 \\ \dfrac{1}{|N|} & \text{if } M'_{\mathcal{C}} = 0. \end{cases}$$

We compare these numbers with regard to the crisp case. Suppose $(\mathcal{C}, w)$ with $\mathcal{C}$ crisp. If we consider two concepts $(S, A) \rhd (T, B)$ in $L_{\mathcal{C}}$ as we said the contribution is $w(B) - w(A)$. In formula (8) this contribution is allocated among all players in $S \setminus T$ in the egalitarian way. Now, following Definition 4.2, the contribution by (P3) is the same if we see the context as a particular case of fuzzy one. Moreover, we have $S - T = S \setminus T$ and then the marginality for each player i is

$$ma_{T}^{S}(i, \mathcal{C}) = \frac{1}{\vee(S \setminus T)} \int_{c} (S \setminus T)\, dh_i = h_i(S \setminus T)$$

If $i \in S \setminus T$ then $h_i(S \setminus T) = \dfrac{1}{|S \setminus T|}$. So, the player i gets in this step

$$\frac{1}{|S \setminus T|}[w(A) - w(B)]$$

as in formula (8). If $i \notin S \setminus T$ then $h_i(S \setminus T) = 0$, and hence i gets nothing. Therefore, the marginality of a player for a contribution $w(\alpha) - w(\beta)$ generated at step $(\sigma, \alpha) \rhd (\tau, \beta)$ means his individual capacity to intervene in the generation of that contribution. This capacity may be different for the players involved, those in $supp(\sigma - \tau)$. The next proposition shows that the marginalities in each step always sum one.

Proposition 4.1. *Let $\mathcal{C} = (N, M, I)$ be a fuzzy context and $(\sigma, \alpha) \rhd (\tau, \beta)$ in $L_{\mathcal{C}}$. The marginalities of that step satisfy*

$$\sum_{i \in N} ma_{\tau}^{\sigma}(i, \mathcal{C}) = 1.$$

The same happened with the marginalities in the bottom.

Proof. We do the sum helped by property (P1),

$$\sum_{i \in N} ma_{\tau}^{\sigma}(i, \mathcal{C}) = \frac{1}{\vee(\sigma - \tau)} \sum_{i \in N} \int_{c} (\sigma - \tau)\, dh_i = \frac{1}{\vee(\sigma - \tau)} \int_{c} (\sigma - \tau)\, d\left(\sum_{i \in N} h_i\right)$$

For each non-empty coalition $S \subseteq N$ we have

$$\sum_{i \in N} h_i(S) = \sum_{i \in S} \frac{1}{|S|} = 1 \text{ and } \sum_{i \in N} h_i(\emptyset) = 0.$$

Let $im(\sigma - \tau) = (t_1 < \cdots < t_p)$ be the image of $\sigma - \tau$ where $t_p = \vee(\sigma - \tau)$, and $t_0 = 0$. Using (5) we get

$$\sum_{i\in N} ma_\tau^\sigma(i,\mathcal{C}) = \frac{1}{t_p}\sum_{k=1}^{p}(t_k - t_{k-1}) = 1.$$

The proof for the marginalities in the bottom is the same. □

We define now the extent Shapley value for intent fuzzy concept games. If $\mathcal{C}$ is a fuzzy context and $C \in CH(L_\mathcal{C})$ then we take

$$C = (\sigma_0, \sigma_1, ..., \sigma_p), \tag{11}$$

the ordered set from the bottom to the top of the intents in the chain. So $\sigma_0 = M'_\mathcal{C}$ and $\sigma_p = N$.

Definition 4.4. Let $(\mathcal{C}, w) \in \Gamma$ be an intent fuzzy concept game with $\mathcal{C} = (N, M, I)$. The extent fuzzy Shapley value of a player $i \in N$ is

$$SH_i^{ex}(\mathcal{C}, w) = ma^{M'_\mathcal{C}}(i,\mathcal{C})w(M) + \frac{1}{ch(L_\mathcal{C})}\sum_{\substack{C\in CH(L_\mathcal{C})\\ C=(\sigma_0,\sigma_1,...,\sigma_p)}}\sum_{k=1}^{p} ma_{\sigma_{k-1}}^{\sigma_k}(i,\mathcal{C})\left[w\left((\sigma_k)'_\mathcal{C}\right) - w\left((\sigma_{k-1})'_\mathcal{C}\right)\right].$$

The first term of the formula allocates the worth of the bottom in the fuzzy concept lattice, and later, for each chain, we allocate the contributions given by the intents of the steps considering the marginalities given by the extents. If we take a crisp context $\mathcal{C}$ then $SH^{ex}(\mathcal{C}, w) = Sh^{ex}(\mathcal{C}, w)$ following formula (8). We show the computation of the fuzzy extend Shapley value in an example.

Example 4.3. The extent Shapley value obtains an allocation of $w^\top(\mathcal{C})$ following the chains of the fuzzy concept lattice as the classic Shapley value. We have $ch(L_\mathcal{C}) = 13$. The Shapley power functions for the firms are in Table 4.1.

Table 4.1
Shapley power functions.

S	∅	1	2	3	12	13	23	N
$h_1(S)$	0	1	0	0	1/2	1/2	0	1/3
$h_2(S)$	0	0	1	0	1/2	0	1/2	1/3
$h_3(S)$	0	0	0	1	0	1/2	1/2	1/3

The bottom profit is allocated using $M'_\mathcal{C} = (0.1, 0, 0.1)$ as

$$w^\perp(\mathcal{C})ma^{M'_\mathcal{C}}(i,\mathcal{C}) = 10\left(\frac{1}{2}, 0, \frac{1}{2}\right) = (5, 0, 5).$$

Consider for example the chain, given by its concepts ordered from the bottom,

$$C = (C_1, C_2, C_5, C_8, C_{13}, C_{15}, C_{16}).$$

In the first step of C, (C_1, C_2), we obtain by (9) the contribution

$$w(0.1, 1, 1) - w(M) = 27.$$

As $(0.6, 0, 0.1) - (0.1, 0, 0.1) = (0.5, 0, 0)$ then the marginalities of the players using Table 4.1 are

$$ma_{(0.1,0,0.1)}^{(0.6,0,0.1)} = (1, 0, 0),$$

namely player 1 gets the whole contribution, players get $(27, 0, 0)$. In the last step (C_{15}, C_{16}) we have the contribution

$$w(0, 0.1, 0.8) - w(0, 0.1, 1) = w^{\top}(\mathcal{C}) - w(0, 0.1, 1) = 138 - (0.1 \cdot 40 + 0.9 \cdot 120) = 26$$

but $N - (0.8, 1, 0.8) = (0.2, 0, 0.2)$. So, the marginalities are

$$ma^{N}_{(0.8,1,0.8)} = \left(\frac{1}{2}, 0, \frac{1}{2}\right).$$

Players get this last step in the chain $(13, 0, 13)$. Table 4.2 determines the contributions and the marginalities for each step in the chain.

Table 4.2
Calculating the allocation of the payoff for the chain C.

step	contribution	marginality	payoffs
C_1, C_2	27	$(1, 0, 0)$	$(27, 0, 0)$
C_2, C_5	3	$(0, 1, 0)$	$(0, 3, 0)$
C_5, C_8	32	$(1, 0, 0)$	$(32, 0, 0)$
C_8, C_{13}	24	$(0, 1, 0)$	$(0, 24, 0)$
C_{13}, C_{15}	16	$(0, 0, 1)$	$(0, 0, 16)$
C_{15}, C_{16}	26	$(1/2, 0, 1/2)$	$(13, 0, 13)$

Adding the last column to the table above, we obtain the allocation of $w^{\top}(\mathcal{C}) - w^{\perp}(\mathcal{C}) = 138 - 10 = 128$ among the players following the chain C, $(72, 27, 29)$. We repeat the process with the other chains. We add the payoff vectors of the chains and divide them by the number of chains in the structure. Finally, we add the share of $w^{\perp}(\mathcal{C})$, that is, $(5, 0, 5)$. Therefore, the fuzzy Shapley value of this intent fuzzy concept game is

$$SH^{ex}(\mathcal{C}, w) = (76.9, 15.8, 45.3).$$

Therefore, firms would earn for each unit of product 138 monetary units, of which 76.9 must be for firm 1, 15.8 for firm 2 and finally 45.3 for firm 3.

5. Properties of the proposed solution

We will see several interesting properties of our solution.

Theorem 5.1. *The extent fuzzy Shapley value is an efficient payoff vector for the top profit for all* $(\mathcal{C}, w) \in \Gamma$*, this is if* $\mathcal{C} = (N, M, I)$ *then*

$$\sum_{i \in N} SH_i^{ex}(\mathcal{C}, w) = w^{\top}(\mathcal{C}).$$

Proof. First, we calculate using Proposition 4.1, that

$$\sum_{i \in N} ma^{M'_{\mathcal{C}}}(i, \mathcal{C}) w(M) = w(M) \sum_{i \in N} ma^{M'_{\mathcal{C}}}(i, \mathcal{C}) = w(M).$$

Now we take any chain $C = (\sigma_0, \sigma_1, ..., \sigma_p) \in CH(L_{\mathcal{C}})$. Since Proposition 4.1 again, formula (10) and also the equalities $(\sigma_0)'_{\mathcal{C}} = M$ and $(\sigma_p)'_{\mathcal{C}} = N'_{\mathcal{C}}$, we do

$$\begin{aligned}\sum_{i \in N} \sum_{k=1}^{p} ma^{\sigma_k}_{\sigma_{k-1}}(i, \mathcal{C})[w\left((\sigma_k)'_{\mathcal{C}}\right) - w\left((\sigma_{k-1})'_{\mathcal{C}}\right)] &= \sum_{k=1}^{p}[w\left((\sigma_k)'_{\mathcal{C}}\right) - w\left((\sigma_{k-1})'_{\mathcal{C}}\right)] \sum_{i \in N} ma^{\sigma_k}_{\sigma_{k-1}}(i, \mathcal{C}) \\ &= \sum_{k=1}^{p}[w\left((\sigma_k)'_{\mathcal{C}}\right) - w\left((\sigma_{k-1})'_{\mathcal{C}}\right)] \\ &= w^{\top}(\mathcal{C}) - w(M).\end{aligned}$$

Finally, we get

$$\sum_{i\in N} SH_i^{ext}(\mathcal{C}, w) = \sum_{i\in N} ma^{M'_{\mathcal{C}}}(i) w(M)$$
$$+ \sum_{i\in N} \frac{1}{ch(L_{\mathcal{C}})} \sum_{C\in CH(L_{\mathcal{C}})} \sum_{k=1}^{p} ma_{\sigma_{k-1}}^{\sigma_k}(i)[w((\sigma_k)'_{\mathcal{C}}) - w\left(((\sigma_{k-1})'_{\mathcal{C}}\right)]$$
$$= w(M) + \frac{1}{ch(L_{\mathcal{C}})} \sum_{C\in CH(L_{\mathcal{C}})} [w^{\top}(\mathcal{C}) - w(M)] = w^{\top}(\mathcal{C}). \quad \square$$

The efficiency property says that the payoff vectors obtained are allocations of the top profit that the players can get in the game. Faigle et al. [10] argue that the bottom profit (10) in a concept game $(\mathcal{C}, v)$ with $\mathcal{C} = (N, M, I)$ should be shared only among the players in $M'_{\mathcal{C}}$. In a crisp concept lattice, players in $M'_{\mathcal{C}}$ do not participate in the rest of the contributions, so they propose to demand the condition for a value ϕ, $\sum_{i\in M'_{\mathcal{C}}} \phi_i(\mathcal{C}, w) = v(M'_{\mathcal{C}}, M)$. But now, in the fuzzy setting, the players in $M'_{\mathcal{C}}$ are also able to participate in the production of other contributions. Therefore, our solution should only verify the above condition in the case of a crisp bottom of players.

Theorem 5.2. *The extent fuzzy Shapley value satisfies that the bottom profit is a separable payoff if the bottom of the fuzzy concept lattice is crisp, this is for all* $(\mathcal{C}, w) \in \Gamma$ *with* $\mathcal{C} = (N, M, I)$ *and* $im(M'_{\mathcal{C}}) = \{0, 1\}$ *it holds*

$$\sum_{i\in M'_{\mathcal{C}}} SH_i^{ex}(\mathcal{C}, w) = w^{\perp}(\mathcal{C}).$$

Proof. Suppose then $im(M'_{\mathcal{C}}) = \{0, 1\}$, then $M'_{\mathcal{C}}$ is actually the crisp set $M'_{\mathcal{C}} = supp(M'_{\mathcal{C}})$. Moreover, we consider $M'_{\mathcal{C}} \neq 0$ because if not, the property is meaningless. If $M'_{\mathcal{C}}(i) = 0$ then $ma^{M'_{\mathcal{C}}}(i, \mathcal{C}) = 0$ and so by Proposition 4.1

$$\sum_{i\in M'_{\mathcal{C}}} ma^{M'_{\mathcal{C}}}(i) = 1.$$

Moreover, if $M'_{\mathcal{C}}(i) = 1$ then for all pairs of fuzzy concepts $(\sigma, \alpha) \rhd (\tau, \beta)$ we have

$$\sigma(i) = \tau(i) \geq M'_{\mathcal{C}}(i) = 1,$$

hence $ma_{\tau}^{\sigma}(i, \mathcal{C}) = 0$. The extent fuzzy Shapley value of these players i with $M'_{\mathcal{C}}(i) = 1$ is, by definition,

$$SH_i^{ex}(\mathcal{C}, w) = ma^{M'_{\mathcal{C}}}(i, \mathcal{C}) w(M),$$

and then

$$\sum_{i\in M'_{\mathcal{C}}} SH_i^{ex}(\mathcal{C}, w) = w(M) \sum_{i\in M'_{\mathcal{C}}} ma^{M'_{\mathcal{C}}}(i, \mathcal{C}) = w(M). \quad \square$$

The idea of a macro-player as a group of players with the same activity in the context was introduced in [10]. Now, we extend this definition to the fuzzy setting.

Definition 5.1. Let $\mathcal{C} = (N, M, I)$ be a fuzzy context. A set $K \subseteq N$ with $|K| > 1$ is a macro-player in $\mathcal{C}$ if each pair of players $i, j \in K$ satisfies $I(i, a) = I(j, a)$ for every $a \in M$.

There are no macro-players in the fuzzy context of Table 2.1. Similarly to the crisp case, we can find the macro-players in a fuzzy context looking for the equal rows in the matrix. If we modify Table 2.1 by the fuzzy context in Table 5.1 we see that $K = \{1, 3\}$ is a macro-player.

Table 5.1
Macro-player in a fuzzy context.

	a	b	c
1	0.1	0.6	0.8
2	0	0.3	1
3	0.1	0.6	0.8

We will demand that all the players in a macro-player obtain the same payoff. In the situation described in Section 2 this fact means that firms with exactly the same problems must obtain the same profit. The macro-player property is related to the condition of equal treatment for classic games.

Theorem 5.3. *Let $(\mathcal{C}, w) \in \Gamma$ and K be a macro-player in $\mathcal{C}$. The extent fuzzy Shapley value satisfies that $SH_i^{ex}(\mathcal{C}, w) = SH_j^{ex}(\mathcal{C}, w)$ for all $i, j \in K$.*

Proof. Let $(\mathcal{C}, w) \in \Gamma$ with $\mathcal{C} = (N, M, I)$ and $K \subseteq N$ macro-player in $\mathcal{C}$.

First, we will see that for all $\sigma \in [0,1]^N$ and $i, j \in N$ with $\sigma(i) = \sigma(j)$ we obtain the following:

$$\int_c \sigma \, dh_i = \int_c \sigma \, dh_j. \tag{12}$$

Let $im(\sigma) \cup \{0, 1\} = (t_0 < \cdots < t_p)$. As $\sigma(i) = \sigma(j)$, we have $i \in [\sigma]_{t_k}$ if and only if $j \in [\sigma]_{t_k}$ for each $k = 1, ..., p$. So, $h_i([\sigma]_{t_k}) = h_j([\sigma]_{t_k})$ for all $k = 1, ..., p$. Therefore, the definition of the Choquet integral (5) implies the equality.

Second, we will prove this claim: if K is a macro-player in $\mathcal{C}$, then $\sigma(i) = \sigma(j)$ for all $i, j \in K$ and $(\sigma, \alpha) \in L_{\mathcal{C}}$. Let (σ, α) be a fuzzy concept. As we said in Subsection 3.3 we have $\sigma''_{\mathcal{C}} = \sigma$, thus for all $i, j \in K$ the application of the derivation operator (6) gets

$$\sigma(i) = \sigma''_{\mathcal{C}}(i) = \bigwedge_{a \in M} [\sigma'_{\mathcal{C}}(a) \rightarrow I(i,a)] = \bigwedge_{a \in M} [\sigma'_{\mathcal{C}}(a) \rightarrow I(j,a)] = \sigma''_{\mathcal{C}} = \sigma(j).$$

Now, if we use the claim with $(\sigma, \alpha), (\tau, \beta) \in L_{\mathcal{C}}$ satisfying $(\sigma, \tau) \triangleright (\tau, \beta)$ we have $(\sigma - \tau)(i) = (\sigma - \tau)(j)$. Hence, the marginalities of both players are the same by (12), $ma_\tau^\sigma(i, \mathcal{C}) = ma_\tau^\sigma(j, \mathcal{C})$ and also $ma^{M'_{\mathcal{C}}}(i, \mathcal{C}) = ma^{M'_{\mathcal{C}}}(j, \mathcal{C})$. Since Definition 4.4 we obtain $SH_i^{ex}(\mathcal{C}, w) = SH_j^{ex}(\mathcal{C}, w)$. □

We introduce a concatenation operation for fuzzy contexts.

Definition 5.2. Let $\mathcal{C}_1 = (N_1, M_1, I_1), \mathcal{C}_2 = (N_2, M_2, I_2)$ be two fuzzy contexts with $N_1 \cap N_2 = \emptyset$ and $M_1 \cap M_2 = \emptyset$. The concatenation $\mathcal{C}_1 \oplus \mathcal{C}_2 = (N, M, I)$ is a new fuzzy context defined as: $N = N_1 \cup N_2$, $M = M_1 \cup M_2$ and

$$I(i,a) = \begin{cases} I_1(i,a), & \text{if } i \in N_1,\ a \in M_1 \\ I_2(i,a), & \text{if } i \in N_2,\ a \in M_2 \\ 1, & \text{if } i \in N_1,\ a \in M_2 \\ 0, & \text{if } i \in N_2,\ a \in M_1 \end{cases}$$

The next result shows that the fuzzy concept lattice of the concatenation of two fuzzy contexts consists of linking their fuzzy concept lattices by the top of the first and the bottom of the second.

Lemma 5.4. *Let $\mathcal{C}_1 = (N_1, M_1, I_1)$, $\mathcal{C}_2 = (N_2, M_2, I_2)$ be two fuzzy contexts with $N_1 \cap N_2 = \emptyset$ and $M_1 \cap M_2 = \emptyset$. Let $(L_{\mathcal{C}_1}, \sqsubseteq_1)$ and $(L_{\mathcal{C}_2}, \sqsubseteq_2)$ be their corresponding fuzzy concept lattices. The fuzzy concept lattice of the concatenation $\big(L_{\mathcal{C}_1 \oplus \mathcal{C}_2}, \sqsubseteq\big)$ is isomorphic to the poset $(L_{12}, \sqsubseteq_{12})$ where*

$$L_{12} = \begin{cases} L_{\mathcal{C}_1} \cup L_{\mathcal{C}_2}, & \text{if } (M_2)'_{\mathcal{C}_2}, (N_1)'_{\mathcal{C}_1} \neq 0 \text{ or } (M_2)'_{\mathcal{C}_2} = (N_1)'_{\mathcal{C}_1} = 0 \\ \big[L_{\mathcal{C}_1} \cup L_{\mathcal{C}_2}\big] \setminus \{(0, M_2)\}, & \text{if } (M_2)'_{\mathcal{C}_2} = 0 \text{ and } (N_1)'_{\mathcal{C}_1} \neq 0 \\ \big[L_{\mathcal{C}_1} \cup L_{\mathcal{C}_2}\big] \setminus \{(N_1, 0)\}, & \text{if } (M_2)'_{\mathcal{C}_2} \neq 0 \text{ and } (N_1)'_{\mathcal{C}_1} = 0 \end{cases}$$

and $(\sigma,\alpha) \sqsubseteq_{12} (\tau,\beta)$ *if: both are in* $L_{\mathcal{C}_1}$ *and* $(\sigma,\alpha) \sqsubseteq_1 (\tau,\beta)$*, or both are in* $L_{\mathcal{C}_2}$ *and* $(\sigma,\alpha) \sqsubseteq_2 (\tau,\beta)$*, or* $(\sigma,\alpha) \in L_{\mathcal{C}_1}$ *and* $(\tau,\beta) \in L_{\mathcal{C}_2}$*.*

Proof. First we prove that if $(\sigma,\alpha) \in L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$ then: or $\sigma \leq N_1^0$ and $\alpha \geq M_2^0$, or $\sigma \geq N_1^0$ and $\alpha \leq M_2^0$. Indeed, when $\sigma \leq N_1^0$ then for every $a \in M_2$

$$\alpha(a) = \left[\bigwedge_{i \in N_1} \sigma(i) \to 1\right] \wedge \left[\bigwedge_{i \in N_2} 0 \to I_2(i,a)\right] = 1,$$

thus $\alpha \geq M_2^0$. Similarly, we test that if $\alpha \leq M_2^0$ then $\sigma(i) = 1$ for each $i \in N_1$ and also $\sigma \geq N_1^0$. However, if we consider that $\sigma \not\leq N_1^0$, namely there exists $j \in N_2$ with $\sigma(j) > 0$, we get for any $a \in M_1$

$$\alpha(a) = \sigma'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(a) = \left[\bigwedge_{i \in N_1} \sigma(i) \to I_1(i,a)\right] \wedge \left[\bigwedge_{i \in N_2} \sigma(i) \to 0\right] = 0$$

because $\sigma(j) \to 0 = 0$. Hence $\alpha \leq M_2^0$. Moreover, the only option to make both conditions joint is $(\sigma,\alpha) = (N_1^0, M_2^0)$. But $(N_1^0, M_2^0) \in L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$ if and only if $(N_1)'_{\mathcal{C}_1} = 0$ and also $(M_2)'_{\mathcal{C}_2} = 0$. We see that $(N_1^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2} = ((N_1)'_{\mathcal{C}_1})^0 \cup M_2^0$: if $a \in M_1$ then

$$(N_1^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(a) = \left[\bigwedge_{i \in N_1} 1 \to I_1(i,a)\right] \wedge \left[\bigwedge_{i \in N_2} 0 \to 0\right] = \bigwedge_{i \in N_1} I_1(i,a) = (N_1)'_{\mathcal{C}_1}(a),$$

and if $a \in M_2$

$$(N_1^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(a) = \left[\bigwedge_{i \in N_1} 1 \to 1\right] \wedge \left[\bigwedge_{i \in N_2} 0 \to I_2(i,a)\right] = 1.$$

Also we can repeat the reasoning obtaining $(M_2^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2} = N_1^0 \cup ((M_2)'_{\mathcal{C}_2})^0$. So, (N_1^0, M_2^0) is a fuzzy concept in the concatenation if and only if $(N_1)'_{\mathcal{C}_1} = 0$ and $(M_2)'_{\mathcal{C}_2} = 0$ as we said. We test that $\left(((N_1)'_{\mathcal{C}_1})^0 \cup M_2^0\right)'_{\mathcal{C}_1 \oplus \mathcal{C}_2} = N_1^0$ if and only if $(N_1)'_{\mathcal{C}_1} \neq 0$. In fact if we take $i \in N_1$ then the derivation is

$$((N_1)'_{\mathcal{C}_1})^0 \cup M_2^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(i) = \left[\bigwedge_{a \in M_1} (N_1)'_{\mathcal{C}_1}(a) \to I_1(i,a)\right] \wedge \left[\bigwedge_{a \in M_2} 1 \to 1\right] = 1$$

because $(N_1)'_{\mathcal{C}_1}(a) = \bigwedge_{i \in N_1} I_1(i,a)$, but if $i \in N_2$ we have

$$((N_1)'_{\mathcal{C}_1})^0 \cup M_2^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(i) = \left[\bigwedge_{a \in M_1} (N_1)'_{\mathcal{C}_1}(a) \to 0\right] \wedge \left[\bigwedge_{a \in M_2} 1 \to I_2(i,a)\right] = 0$$

because there exists $a \in M_1$ with $(N_1)'_{\mathcal{C}_1}(a) > 0$. In the particular case of $(M_2)'_{\mathcal{C}_2} \neq 0$ we test in a similar way that $\left(N_1^0 \cup ((M_2)'_{\mathcal{C}_2})^0, M_2^0\right) \in L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$.

In summary (see Fig. 5.1), we have seen that $L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$ can be divided into two groups:

$$D = \{(\sigma,\alpha) \in L_{\mathcal{C}_1 \oplus \mathcal{C}_2} : \sigma \leq N_1^0, \alpha \geq M_2^0\}$$
$$U = \{(\sigma,\alpha) \in L_{\mathcal{C}_1 \oplus \mathcal{C}_2} : \sigma \geq N_1^0, \alpha \leq M_2^0\}.$$

Generally $D \cap U = \emptyset$, except if $(N_1)'_{\mathcal{C}_1} = 0$ and $(M_2)'_{\mathcal{C}_2} = 0$. In that case $D \cap U = \{(N_1^0, M_2^0)\}$. If $(N_1)'_{\mathcal{C}_1} \neq 0$, then the top element in D, using the order of $L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$, is $(N_1^0, ((N_1)'_{\mathcal{C}_1})^0 \cup M_2^0)$. If $(M_2)'_{\mathcal{C}_2} \neq 0$, then the lowest element in U, using the order of $L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$, is $(N_1^0 \cup ((M_2)'_{\mathcal{C}_2})^0, M_2^0)$.

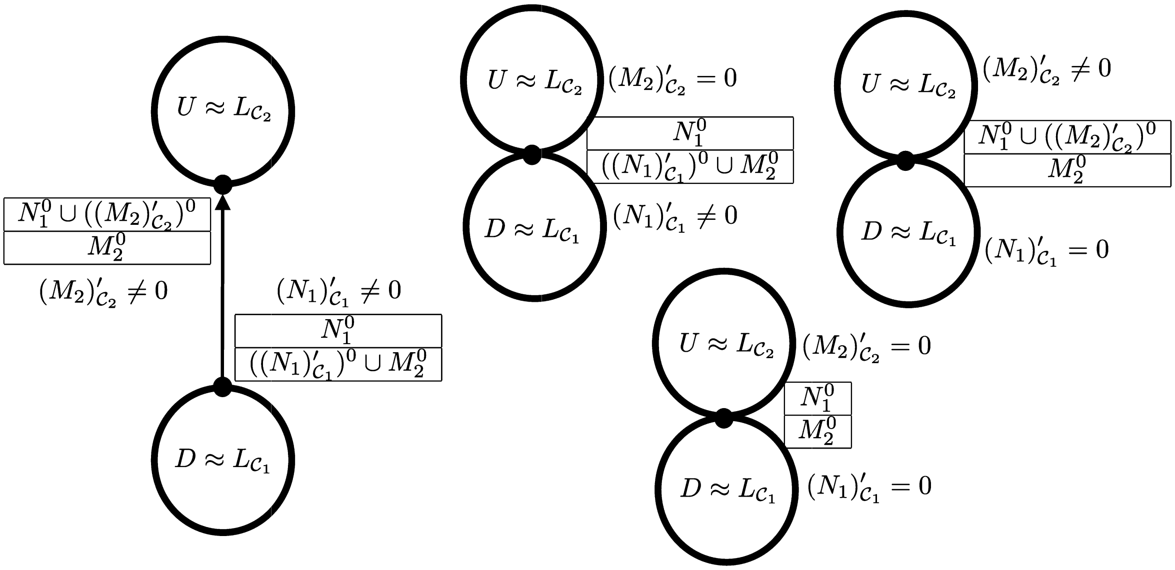

Fig. 5.1. Different options for the fuzzy concept lattice of the concatenation.

We consider now the function $F : L_{12} \to L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$ with

$$F(\sigma, \alpha) = \begin{cases} (\sigma^0, \alpha^0 \cup M_2^0), & \text{if } (\sigma, \alpha) \in L_{\mathcal{C}_1} \\ (\sigma^0 \cup N_1^0, \alpha^0), & \text{if } (\sigma, \alpha) \in L_{\mathcal{C}_2}. \end{cases}$$

If $(\sigma, \alpha) \in L_{\mathcal{C}_1}$ then $F(\sigma, \alpha) \in D$, for each $a \in M_1$

$$(\sigma^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(a) = \left[\bigwedge_{i \in N_1} \sigma(i) \to I_1(i, a)\right] \wedge \left[\bigwedge_{i \in N_2} 0 \to 0\right] = \sigma'_{\mathcal{C}_1}(a) = \alpha(a),$$

and for each $a \in M_2$

$$(\sigma^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(a) = \left[\bigwedge_{i \in N_1} \sigma(i) \to 1\right] \wedge \left[\bigwedge_{i \in N_2} 0 \to I_2(i, a)\right] = 1.$$

On the other hand, if $i \in N_1$

$$(\alpha^0 \cup M_2^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(i) = \left[\bigwedge_{a \in M_1} \alpha(a) \to I_1(i, a)\right] \wedge \left[\bigwedge_{a \in M_2} 1 \to 1\right] = \alpha'_{\mathcal{C}_1}(i) = \sigma(i).$$

If $i \in N_2$

$$(\alpha^0 \cup M_2^0)'_{\mathcal{C}_1 \oplus \mathcal{C}_2}(i) = \left[\bigwedge_{a \in M_1} \alpha(a) \to 0\right] \wedge \left[\bigwedge_{a \in M_2} 1 \to I_2(i, a)\right] = \left[\bigwedge_{a \in M_1} \alpha(a) \to 0\right] \wedge (M_2)'_{\mathcal{C}_2}(i).$$

This quantity is not zero only if $\alpha = 0$ and $(M_2)'_{\mathcal{C}_2} \neq 0$, whose only option in $L_{\mathcal{C}_1}$ is $(N_1, 0)$ (that means $(N_1)'_{\mathcal{C}_1} = 0$ when $(M_2)'_{\mathcal{C}_2} \neq 0$). But observe that in that case $(N_1, 0) \notin L_{12}$. Similarly, we have that if $(\sigma, \alpha) \in L_{\mathcal{C}_2}$ then $(\sigma^0 \cup N_1^0, \alpha^0) \in L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$. Obviously F is into because there are not two fuzzy concepts with the same intent or extent. We prove that F is also onto. Suppose $(\sigma, \alpha) \in L_{\mathcal{C}_1 \oplus \mathcal{C}_2}$. If $(\sigma, \alpha) \in D$ then we take $\left(\sigma|_{N_1}, \alpha|_{M_1}\right)$ with $F\left(\sigma|_{N_1}, \alpha|_{M_1}\right) = (\sigma, \alpha)$. We need only to test that $\left(\sigma|_{N_1}, \alpha|_{M_1}\right) \in L_{\mathcal{C}_1}$. Let $a \in M_1$,

$$(\sigma|_{N_1})'_{\mathcal{C}_1}(a) = \bigwedge_{i\in N_1} \sigma(i) \to I_1(i,a) = \left[\bigwedge_{i\in N_1} \sigma(i) \to I_1(i,a)\right] \wedge \left[\bigwedge_{i\in N_2} 0 \to 0\right] = \sigma'_{\mathcal{C}_1\oplus\mathcal{C}_2} = \alpha(a).$$

Likewise if $i \in N_1$ we get $(\alpha|_{N_1})'_{\mathcal{C}_1}(i) = \sigma|_{N_1}(i)$. Finally, it is easy to test that if $(\sigma,\alpha) \sqsubseteq_{12} (\tau,\beta)$ then $F(\sigma,\alpha) \sqsubseteq F(\tau,\beta)$, being $\sqsubseteq$ the order in $L_{\mathcal{C}_1\oplus\mathcal{C}_2}$. □

The extent fuzzy Shapley value works well with the concatenation operation of fuzzy contexts, as the next theorem explains.

Theorem 5.5. *Let $(\mathcal{C}_1, w_1), (\mathcal{C}_2, w_2) \in \Gamma$ with $\mathcal{C}_1 = (N_1, M_1, I_1)$ and $\mathcal{C}_2 = (N_2, M_2, I_2)$. If $N_1 \cap N_2 = \emptyset$ and $M_1 \cap M_2 = \emptyset$ then the extent fuzzy Shapley value satisfies that*

$$SH_i^{ex}(\mathcal{C}_1 \oplus \mathcal{C}_2, w_{12}) = \begin{cases} SH_i^{ex}(\mathcal{C}_1, w_1), & \text{if } i \in N_1 \\ SH_i^{ex}(\mathcal{C}_2, w_2), & \text{if } i \in N_2 \end{cases}$$

being

$$w_{12}(A) = \begin{cases} w_1(A \cap M_1), & \text{if } A \supsetneq M_2 \\ w_1(A\cap M_1) + w_2(A \cap M_2), & \text{otherwise.} \end{cases}$$

Proof. We follow the notation in Lemma 5.4 (Fig. 5.1). Since that lemma we get

$$ch(L_{\mathcal{C}_1\oplus\mathcal{C}_2}) = ch(L_{\mathcal{C}_1})ch(L_{\mathcal{C}_2}).$$

Let

$$\bar{C} = (\bar{\sigma}_0, \bar{\sigma}_1, ..., \bar{\sigma}_q, \bar{\sigma}_{q+1}, ..., \bar{\sigma}_p) \in CH(L_{\mathcal{C}_1\oplus\mathcal{C}_2})$$

where $\bar{\sigma}_q, \bar{\sigma}_{q+1}$ represent the elements separating D and U in Fig. 5.1 (they can be the same depending on the situation).

First suppose $i \in N_1$. For $k = q, ..., p$ we are in zone U and $\bar{\sigma}_k(i) = 1$ always, so we have $ma^{\bar{\sigma}_k}_{\bar{\sigma}_{k-1}}(i) = 0$ for all $k = q+1, ..., p$. But for $k = 0, ..., q$ we are in the zone D and then we repeat the intents in a chain of $L_{\mathcal{C}_1}$, $C = (\sigma_0, \sigma_1, ..., \sigma_q) \in L_{\mathcal{C}_1}$, with $\bar{\sigma}_k = \sigma_k^0$. So, $ma^{\bar{\sigma}_k}_{\bar{\sigma}_{k-1}}(i) = ma^{\sigma_k}_{\sigma_{k-1}}(i)$. Moreover, for any $k = 0, ..., q$,

$$w_{12}((\bar{\sigma}_k)'_{\mathcal{C}_1\oplus\mathcal{C}_2}) = w_1((\sigma_k)'_{\mathcal{C}_1})$$

because $(\bar{\sigma}_k)'_{\mathcal{C}_1\oplus\mathcal{C}_2} = \left((\sigma_k)'_{\mathcal{C}_1}\right)^0 \cup M^0_{\mathcal{C}_2}$. Observe that the contributions for each $C \in CH(L_{\mathcal{C}_1})$ appear $ch(\mathcal{C}_2)$ times in $L_{\mathcal{C}_1\oplus\mathcal{C}_2}$. Since the Definition 4.4 of the extent fuzzy Shapley value we get

$$SH_i^{ex}(\mathcal{C}_1 \oplus \mathcal{C}_2, w_{12}) = SH_i^{ex}(\mathcal{C}_1, w_1).$$

Now, we take $i \in N_2$. In zone U there exists $C = (\sigma_q, ..., \sigma_p) \in L_{\mathcal{C}_2}$ such that we have $\bar{\sigma}_k = (\sigma_k)^0 \cup N_1^0$ for all $k = q, ..., p$. Therefore, $ma^{\bar{\sigma}_k}_{\bar{\sigma}_{k-1}}(i) = ma^{\sigma_k}_{\sigma_{k-1}}(i)$ for all those k. Moreover, for any $k = q, ..., p$,

$$w_{12}((\bar{\sigma}_k)'_{\mathcal{C}_1\oplus\mathcal{C}_2}) = w_2((\sigma_k)'_{\mathcal{C}_2})$$

because $(\bar{\sigma}_k)'_{\mathcal{C}_1\oplus\mathcal{C}_2} = \left((\sigma_k)'_{\mathcal{C}_2}\right)^0 \cup M^0_{\mathcal{C}_1}$. But marginality $ma^{\bar{\sigma}_k}_{\bar{\sigma}_{k-1}}(i) = 0$ if $k = 0, ..., q$. So,

$$SH_i^{ex}(\mathcal{C}_1 \oplus \mathcal{C}_2, w_{12}) = SH_i^{ex}(\mathcal{C}_2, w_2),$$

taking into account that the contributions in $C \in L_{\mathcal{C}_2}$ are repeated $ch(L_{\mathcal{C}_1})$. □

Example 5.1. We continue from Section 2. The intent fuzzy concept game defined there is denoted here as $(\mathcal{C}_1, w_1)$ where $\mathcal{C}_1 = (N_1, M_1, I_1)$ is given in Table 2.1 and w_1 in Table 2.2. Independently, two other different firms $N_2 = \{4, 5\}$ played a similar game with another product based on two applications $M_2 = \{d, e\}$ (different from the others). Table 5.2 represents the fuzzy context of this second problem.

Table 5.2
Fuzzy context $\mathcal{C}_2$.

	d	e
4	0.1	0.5
5	0.7	0.4

Table 5.3 contains the price of the second product. Worth $w_2(M_2)$ again means the sale of research on applications if the product does not work.

Table 5.3
Prices of the new product, w_2.

A	M_2	d	e	$\emptyset$
$w_2(A)$	10	80	100	200

In Fig. 5.2 we see the fuzzy concept lattice $L_{\mathcal{C}_2}$.

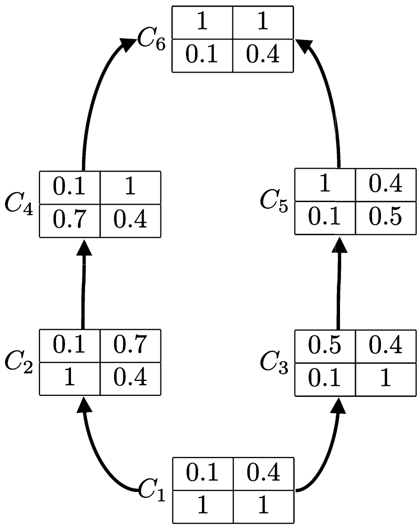

Fig. 5.2. Fuzzy concept lattice of context $\mathcal{C}_2$.

The top profit is

$$w_2^{\top}(\mathcal{C}) = w_2(\emptyset) + \int_c (0.1, 0.4)\, dw_2 = 151.$$

The solution of the extent Shapley value for this new intent fuzzy concept game is

$$SH^{ex}(\mathcal{C}_2, w_2) = (98.25, 52.75)$$

Suppose now that the first three firms decide to invite the two new ones to add their applications to the product to improve it being the price the sum of the independent prices. They are interested in this fact because they can increase the number of sales. Obviously the allocation of the price of one new product with all the five applications must be the same as in the independent problems. Concatenation expresses this idea in only one structure. Table 5.4 is the concatenation of the fuzzy contexts.

Table 5.4
Concatenation $\mathcal{C}_1 \oplus \mathcal{C}_2$.

	a	b	c	d	e
1	0.1	0.6	0.8	1	1
2	0	0.3	1	1	1
3	0.5	0.1	0.8	1	1
4	0	0	0	0.1	0.5
5	0	0	0	0.7	0.4

Following Fig. 5.1, as $(N_1)'_{\mathcal{C}_1} \neq 0$ and $(M_2)'_{\mathcal{C}_2} \neq 0$ the Fig. 5.3 shows the fuzzy concept lattice of the concatenation. Observe that U is isomorphic to $L_{\mathcal{C}_2}$ and D is isomorphic to $L_{\mathcal{C}_1}$.

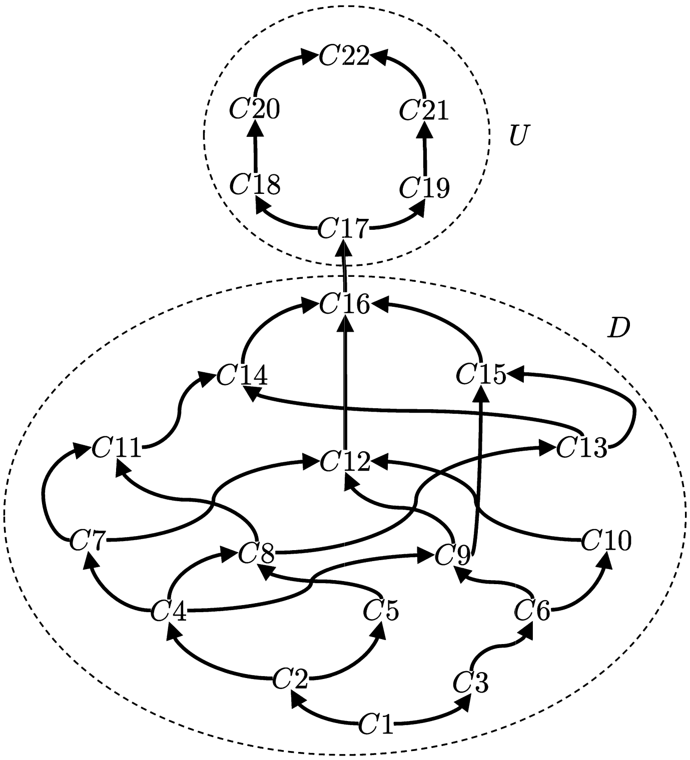

Fig. 5.3. Fuzzy concept lattice of $\mathcal{C}_1 \oplus \mathcal{C}_2$.

The list of fuzzy concepts of concatenation is in Table 5.5.

In the case of the function w_{12}, let us look at some examples of its evaluation. If $A = \{a, b, d, e\}$, then $w_{12}(A) = w_1(\{a, b\}) = 20$ because $A \supsetneq M_2$. If $A = \{d\}$, then $w_{12} = w_1(\emptyset) + w_2(d) = 330$. Finally, although this option does not appear in the calculation of the value, if $A = \{a, e\}$ then $w_{12}(A) = w_1(a) + w_2(e) = 160$. The concatenation property satisfied by SH^{ex} says that

$$SH^{ex}(\mathcal{C}_1 \oplus \mathcal{C}_2, w_1 2) = (76.9, 15.8, 45.3, 98.25, 52.75).$$

Finally, we will see that under certain conditions it is feasible to determine the solution by decomposition of the fuzzy concept lattice into groups of chains. Let $\mathcal{C} = (N, M, I)$ be a fuzzy context with $N'_{\mathcal{C}} = 0$. For each $a \in M$, we take the fuzzy set in N,

$$\sigma_a^{\mathcal{C}}(i) = \begin{cases} 1, & \text{if } I(i,a) \neq 0 \\ 0, & \text{if } I(i,a) = 0 \end{cases} \tag{13}$$

Table 5.5
Fuzzy concepts of the concatenation $\mathcal{C}_1 \oplus \mathcal{C}_2$.

Concept	Extent	Intent	Concept	Extent	Intent
$C1$	(0.1, 0, 0.1, 0, 0)	(1, 1, 1, 1, 1)	$C12$	(1, 0, 1, 0, 0)	(0.1, 0.1, 0.8, 1, 1)
$C2$	(0.6, 0, 0.1, 0, 0)	(0.1, 1, 1, 1, 1)	$C13$	(0.8, 1, 0.1, 0, 0)	(0, 0.3, 1, 1, 1)
$C3$	(0.1, 0, 0.5, 0, 0)	(1, 0.1, 1, 1, 1)	$C14$	(1, 1, 0.1, 0, 0)	(0, 0.3, 0.8, 1, 1)
$C4$	(0.8, 0, 0.1, 0, 0)	(0.1, 0.6, 1, 1, 1)	$C15$	(0.8, 1, 0.8, 0, 0)	(0, 0.1, 1, 1, 1)
$C5$	(0.6, 0.3, 0.1, 0, 0)	(0, 1, 1, 1, 1)	$C16$	(1, 1, 1, 0, 0)	(0, 0.1, 0.8, 1, 1)
$C6$	(0.1, 0, 0.8, 0, 0)	(0.5, 0.1, 1, 1, 1)	$C17$	(1, 1, 1, 0.1, 0.4)	(0, 0, 0, 1, 1)
$C7$	(1, 0, 0.1, 0, 0)	(0.1, 0.6, 0.8, 1, 1)	$C18$	(1, 1, 1, 0.1, 0.7)	(0, 0, 0, 1, 0.4)
$C8$	(0.8, 0.3, 0.1, 0, 0)	(0, 0.6, 1, 1, 1)	$C19$	(1, 1, 1, 0.5, 0.4)	(0, 0, 0, 0.1, 1)
$C9$	(0.8, 0, 0.8, 0, 0)	(0.1, 0.1, 1)	$C20$	(1, 1, 1, 0.1, 1)	(0, 0, 0, 0.7, 0.4)
$C10$	(0.1, 0, 1, 0, 0)	(0.5, 0.1, 0.8, 1, 1)	$C21$	(1, 1, 1, 1, 0.4)	(0, 0, 0, 0.1, 0.5)
$C11$	(1, 0.3, 0.1, 0, 0)	(0, 0.6, 0.8, 1, 1)	$C22$	(1, 1, 1, 1, 1)	(0, 0, 0, 0.1, 0.4)

Observe that $\sigma_a^{\mathcal{C}} \neq N$ because as $N'_{\mathcal{C}} = 0$ then there exists i with $I(i,a) = 0$. We say that $a \sim b$ with $a, b \in M$ if $\sigma_a^{\mathcal{C}} = \sigma_b^{\mathcal{C}}$ and let $M_{\mathcal{C}}$ the set of equivalence classes in M by $\sim$. If $A \in M_{\mathcal{C}}$ then $\sigma_A^{\mathcal{C}} = \sigma_a^{\mathcal{C}}$ for any $a \in A$. We endow $M_{\mathcal{C}}$ with the following partial order: for all $A, B \in M_{\mathcal{C}}$ we have $B < A$ if $\sigma_B^{\mathcal{C}} < \sigma_A^{\mathcal{C}}$.

Definition 5.3. Let $\mathcal{C} = (N, M, I)$ be a fuzzy context with $N'_{\mathcal{C}} = 0$. The set of separators of $\mathcal{C}$ is

$$\mathrm{Sep}\,(\mathcal{C}) = \{A \in M_{\mathcal{C}} : A \text{ maximal in } (M_{\mathcal{C}}, <)\}.$$

Observe that $\sigma_A^{\mathcal{C}} = 0$ if $I(i,a) = 0$ for all $i \in N$ and any $a \in A$, so or $M \notin \mathrm{Sep}\,(\mathcal{C})$ or $\mathrm{Sep}\,(\mathcal{C}) = \{M\}$ and we have the trivial case $I = 0$. The following results will explain the idea of the separator notion in a context.

Lemma 5.6. *Let $\mathcal{C} = (N, M, I)$ be a fuzzy context with $N'_{\mathcal{C}} = 0$. The set of coatoms of $L_{\mathcal{C}}$ is*

$$\left\{(\sigma_A^{\mathcal{C}}, \alpha_A^{\mathcal{C}}) : A \in \mathrm{Sep}\,(\mathcal{C})\right\},$$

being $\alpha_A^{\mathcal{C}}(b) = \bigwedge_{\{i \in N : I(i,a) \neq 0\}} I(i,b)$ for all $b \in M$ and any $a \in A$.

Proof. Let $(\sigma_A^{\mathcal{C}}, \alpha_A^{\mathcal{C}})$ with $A \in M_{\mathcal{C}}$.

First, we prove the claim $(\sigma_A^{\mathcal{C}}, \alpha_A^{\mathcal{C}}) \in L_{\mathcal{C}}$. Let $a \in A$. For all $b \in M$,

$$\begin{aligned}(\sigma_A^{\mathcal{C}})'_{\mathcal{C}}(b) &= \bigwedge_{i \in N} \sigma_a^{\mathcal{C}}(i) \to I(i,b) \\ &= \left(\bigwedge_{\{i \in N : I(i,a) \neq 0\}} 1 \to I(i,b)\right) \wedge \left(\bigwedge_{\{i \in N : I(i,a) = 0\}} 0 \to I(i,b)\right) \\ &= \bigwedge_{\{i \in N : I(i,a) \neq 0\}} I(i,b) = \alpha_A^{\mathcal{C}}(b).\end{aligned}$$

We define the fuzzy set $\bar{\alpha}_a \in [0,1]^M$ with

$$\bar{\alpha}_a^{\mathcal{C}}(b) = \begin{cases} \bigwedge_{\{i \in N : I(i,a) \neq 0\}} I(i,a), & \text{if } b = a \\ 0, & \text{otherwise.} \end{cases}$$

It holds for any $i \in N$

$$(\bar{\alpha}_a^{\mathcal{C}})'_{\mathcal{C}}(i) = \bigwedge_{b \in M} \bar{\alpha}_a(b) \to I(i,b) = \left(\bigwedge_{\{i \in N : I(i,a) \neq 0\}} I(i,a)\right) \to I(i,a) = \sigma_A^{\mathcal{C}}(i).$$

Therefore, $\alpha_A^{\mathcal{C}} = (\bar{\alpha}_a^{\mathcal{C}})''_{\mathcal{C}}$ and the claim is true.

Now, we consider $A \in \mathrm{Sep}(\mathcal{C})$. Suppose there exists another fuzzy concept $(\sigma, \alpha) \rhd (\sigma_A^{\mathcal{C}}, \alpha_A^{\mathcal{C}})$ with $(\sigma, \alpha) \neq (N, 0)$. As $\alpha \neq 0$ there is $b \in M$ with $\bar{\alpha}_b^{\mathcal{C}} \leq \alpha$. So,

$$(\bar{\alpha}_b^{\mathcal{C}})'_{\mathcal{C}} \geq \sigma > \sigma_A.$$

But we have already proved that $(\bar{\alpha}_b^{\mathcal{C}})'_{\mathcal{C}} = \sigma_B^{\mathcal{C}}$ for some $B \in M_{\mathcal{C}}$, so A is not maximal in $(M_{\mathcal{C}}, <)$. □

Each separator defines a new fuzzy context such that its fuzzy concept lattice is a certain group of chains in the old fuzzy concept lattice.

Definition 5.4. Let $\mathcal{C} = (N, M, I)$ be a fuzzy context with $N'_{\mathcal{C}} = 0$. For each $A \in \mathrm{Sep}(\mathcal{C})$ the context restricted by A is $\mathcal{C}_A = (N, M, I_A)$ with

$$I_A(i,b) = \begin{cases} I(i,b), & \text{if } I(i,a) \neq 0 \\ 0, & \text{if } I(i,a) = 0 \end{cases}$$

for all $i \in N$, $b \in M$ and any $a \in A$.

Lemma 5.7. *Let $\mathcal{C} = (N, M, I)$ be a fuzzy context with $N'_{\mathcal{C}} = 0$. It holds*

$$L_{\mathcal{C}_A} = \{(\sigma, \alpha) \in L_{\mathcal{C}} : \alpha \geq \alpha_A^{\mathcal{C}}\}.$$

Proof. If $I = 0$ then $\mathrm{Sep}(\mathcal{C}) = \{M\}$ and obviously $\mathcal{C}_M = \mathcal{C}$. Suppose then $I \neq 0$ and then $\sigma_A^{\mathcal{C}} \neq 0$ for all $A \in M(\mathcal{C})$. First, we see that given $b \in M$, for all $i \in N$ and any $a \in A$

$$\sigma_b^{\mathcal{C}_A}(i) = \begin{cases} \sigma_b^{\mathcal{C}}(i), & \text{if } I(i,a) \neq 0 \\ 0, & \text{if } I(i,a) = 0. \end{cases} \tag{14}$$

If $i \in N$ verifies $I(i,a) \neq 0$, then we get $I_A(i,b) = I(i,b)$. So, in that case, $\sigma_b^{\mathcal{C}_A}(i) = \sigma_b^{\mathcal{C}}(i)$. If $I(i,a) = 0$, then $I_A(i,b) = 0$ and we get $\sigma_b^{\mathcal{C}_A}(i) = 0$. Hence we have from (14) that $\sigma_A^{\mathcal{C}_A} = \sigma_A^{\mathcal{C}}$ and $\sigma_B^{\mathcal{C}_A} \leq \sigma_B^{\mathcal{C}}$ for every $B \in M \setminus \{A\}$. Following Lemma 5.6, for all $b \in M$ and any $a \in A$

$$\alpha_A^{\mathcal{C}_A}(b) = \bigwedge_{\{i \in N : I_A(i,a) \neq 0\}} I_A(i,b) = \bigwedge_{\{i \in N : I(i,a) \neq 0\}} I(i,b) = \alpha_A^{\mathcal{C}}(b)$$

because $I_A(i,a) \neq 0$ if and only if $I(i,a) \neq 0$, and, moreover $I_A(i,b) = I(i,b)$ when $I(i,a) \neq 0$. We obtain the result that the only coatom of $L_{\mathcal{C}_A}$ is $(\sigma_A^{\mathcal{C}}, \alpha_A^{\mathcal{C}})$.

Let $\sigma \in [0,1]^N$ be a fuzzy set of players with $\sigma \leq \sigma_A^{\mathcal{C}}$. As $\sigma \leq \sigma_A^{\mathcal{C}}$ then $\sigma(i) = 0$ if $I(i,a) = 0$ for $a \in A$. We calculate the derivation in both contexts, $\mathcal{C}$ and $\mathcal{C}_A$. Let $b \in M$ and $a \in M$,

$$\sigma'_{\mathcal{C}}(b) = \bigwedge_{i \in N} \sigma(i) \to I(i,b) = \bigwedge_{\{i \in N : I(i,a) \neq 0\}} \sigma(i) \to I(i,b),$$

$$\sigma'_{\mathcal{C}_A}(b) = \bigwedge_{i \in N} \sigma(i) \to I_A(i,b) = \bigwedge_{\{i \in N : I(i,a) \neq 0\}} \sigma(i) \to I_A(i,b) = \bigwedge_{\{i \in N : I(i,a) \neq 0\}} \sigma(i) \to I(i,b)$$

because $I_A(i,b) = I(i,b)$ when $I(i,a) \neq 0$.

Let $\alpha \in [0,1]^M$ be a fuzzy set of attributes with $\alpha \geq \alpha_A^{\mathcal{C}}$. As $\alpha \geq \alpha_A^{\mathcal{C}}$ then

$$\alpha(b) \geq \bigwedge_{\{i \in N : I(i,a) \neq 0\}} I(i,b)$$

for all $b \in M$ and any $a \in A$. In particular, if $a \in M$ then $\alpha(a) > 0$. We calculate the derivation in both contexts, $\mathcal{C}$ and $\mathcal{C}_A$. Let $i \in N$ and $a \in M$. If $I(i,a) \neq 0$ then $I_A(i,b) = I(i,b)$ and, obviously, $\alpha'_{\mathcal{C}_A}(i) = \alpha'_{\mathcal{C}}(i)$. If $I(i,a) = 0$, and then $I_A(i,a) = 0$, we obtain $\alpha(a) \to 0 = 0$. Hence

$$\alpha'_{\mathcal{C}}(i) = \bigwedge_{b \in M} \alpha(b) \to I(i,b) = 0 = \bigwedge_{b \in M} \alpha(b) \to I_A(i,b) = \alpha'_{\mathcal{C}_A}(i). \quad □$$

Theorem 5.8. *Let* $(\mathcal{C}, w) \in \Gamma$ *with* $\mathcal{C} = (N, M, I)$ *and* $N'_{\mathcal{C}} = 0$*. The extent fuzzy Shapley value satisfies*

$$ch(\mathcal{C})SH^{ex}(\mathcal{C}, w) = \sum_{A \in \text{Sep}(\mathcal{C})} ch(\mathcal{C}_A)SH_i^{ex}(\mathcal{C}_A, w).$$

Proof. Let $i \in N$. As $M'_{\mathcal{C}_A} = M'_{\mathcal{C}}$ then $ma^{M'_{\mathcal{C}_A}}(i, \mathcal{C}_A) = ma^{M'_{\mathcal{C}}}(i, \mathcal{C})$ for all $A \in \text{Sep}(\mathcal{C})$. From the above proposition we have

$$CH(\mathcal{C}) = \bigcup_{A \in \text{Sep}(\mathcal{C})} CH(\mathcal{C}_A),\ CH(\mathcal{C}_A) \cap CH(\mathcal{C}_B) = \emptyset\ \forall A, B \in \text{Sep}(\mathcal{C}).$$

Let $C \in CH(\mathcal{C}_A)$ with $C = \{\sigma_0, \sigma_1, ..., \sigma_p\}$ and $A \in \text{Sep}(\mathcal{C})$. Also, from Lemma 5.7 we get for all $k = 1, ..., p$ that

$$ma_{\sigma_{k-1}}^{\sigma_k}(i, \mathcal{C}) = ma_{\sigma_{k-1}}^{\sigma_k}(i, \mathcal{C}_A).$$

Finally, $(\sigma_k)'_{\mathcal{C}_A} = (\sigma_k)'_{\mathcal{C}}$ implies

$$ch(\mathcal{C})\, SH_i^{ex}(\mathcal{C}, w) = \sum_{A \in \text{Sep}(\mathcal{C})} ch(\mathcal{C}_A)\, SH_i^{ex}(\mathcal{C}_A, w). \quad \square$$

We explain the property of decomposition seen in the above theorem.

Example 5.2. The fuzzy context of Section 2, see Table 2.1, does not satisfy the condition $N'_{\mathcal{C}} = 0$. Following the literature of the Example, that condition means that all the applications of the products can be reached by the firms without risk of error. We change the matrix looking for the condition. Now consider the fuzzy context of Table 5.6.

Table 5.6
Fuzzy context $\mathcal{C}$ in Example 5.2.

	a	b	c
1	0.1	0.6	0
2	0	0	1
3	0.5	0.1	0.8

Now $N'_{\mathcal{C}} = 0$. Fig. 5.4 shows the fuzzy concept lattice of this new context.

We determine the set of separators in $\mathcal{C}$. As $\sigma_a = (1, 0, 1)$, $\sigma_b = (1, 0, 1)$ and $\sigma_c = (0, 1, 1)$ then $a \equiv b$ and following Definition 5.3

$$\text{Sep}(\mathcal{C}) = \{\{a, b\}, \{c\}\}.$$

We can test Lemma 5.6 seeing that the fuzzy concepts

$$(\sigma_{\{a,b\}}^{\mathcal{C}}, \alpha_{\{a,b\}}^{\mathcal{C}}) = ((1, 0, 1), (0.1, 0.1, 0))$$
$$(\sigma_c^{\mathcal{C}}, \alpha_c^{\mathcal{C}}) = ((0, 1, 1), (0, 0, 0.8))$$

are the coatoms of $L_{\mathcal{C}}$ in Fig. 5.4. Now, in Table 5.7 we construct the fuzzy contexts $\mathcal{C}_{\{a,b\}}$ and $\mathcal{C}_c$.

Table 5.7
Fuzzy contexts $\mathcal{C}_{\{a,b\}}$ and $\mathcal{C}_c$.

$\mathcal{C}_{\{a,b\}}$	a	b	c
1	0.1	0.6	0
2	0	0	0
3	0.5	0.1	0.8

$\mathcal{C}_c$	a	b	c
1	0	0	0
2	0	0	1
3	0.5	0.1	0.8

Fig. 5.5 shows the fuzzy concept lattices of these two new fuzzy contexts.

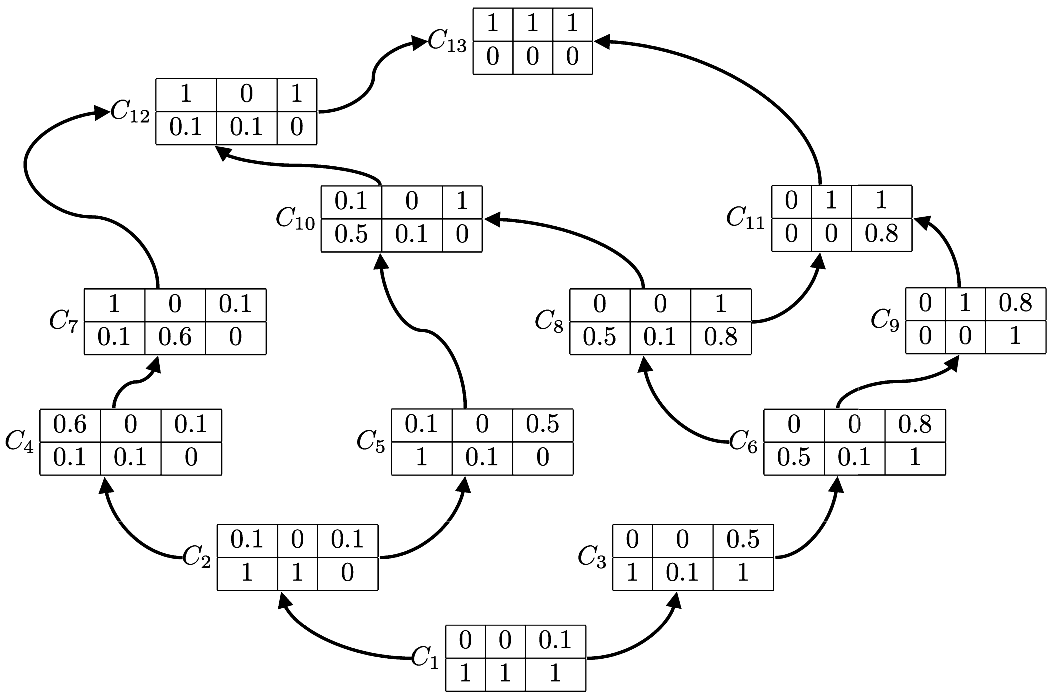

Fig. 5.4. Fuzzy concept lattice of context $\mathcal{C}$ in Example 4.2.

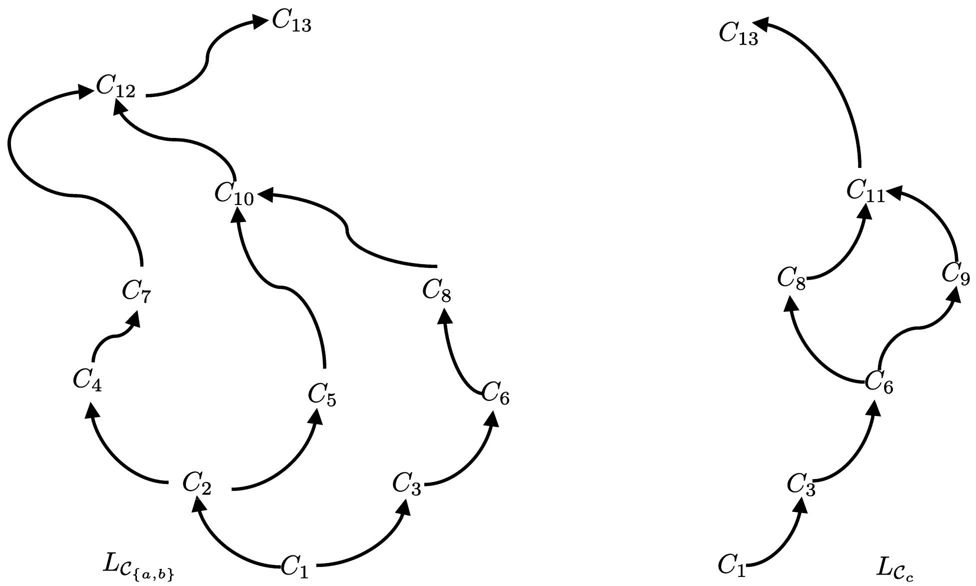

Fig. 5.5. Fuzzy concept lattices $L_{\mathcal{C}_{\{a,b\}}}$ and $L_{\mathcal{C}_c}$.

Theorem 5.8 allows for the decomposition of the game by chains if $N'_{\mathcal{C}} = 0$, moreover, the decomposition is easy to find by looking at the coatoms of the fuzzy concept lattice. But it is not true that we can always guarantee a proper decomposition if we have only one coatom, we have only one separator, and the result says nothing. If $N'_{\mathcal{C}} \neq 0$ then it is hard to identify the fuzzy contexts that can be properly decomposed and also to find the decomposition. For example, consider the fuzzy context $\mathcal{C}_2$ in Table 5.2 using two players $\{4, 5\}$ and two attributes $\{d, e\}$ with the fuzzy concept lattice in Fig. 5.2. We will see that it is not possible to divide the lattice in chains using fuzzy formal contexts although it has two coatoms. The concept lattice $L_{\mathcal{C}_2}$ only has two maximal chains, so we can only separate the lattice one by one. If we take the lattice formed by the chain $L = \{C_1, C_2, C_4, C_6\}$ in Fig. 5.2 then we will test that there is no fuzzy context $\mathcal{C}$ with table I such that $L_{\mathcal{C}} = L$. The context $\mathcal{C}$ must contain in I^4 the number 0.1, and, moreover, $\wedge I^4 = 0.1$. The raw I^5 must contain the numbers 0.4 and 0.7. Column I_d numbers 0.1 and 0.7, and finally column I_e number 0.4. So I has the following structure,

	d	e
4	0.1	p
5	0.7	0.4

with $p \geq 0.4$

Therefore, $(p, 0.4) \in [0, 1]^{\{4,5\}}$ verifies $(p, 0.4)''_{\mathcal{C}} = (0.1, 1)'_{\mathcal{C}} = (p, 0.4)$, but then $((p, 0.4), (0.1, 1)) \in L_{\mathcal{C}}$ and this is not true. The above fact shows an important difference between the analysis of concepts in the fuzzy setting and this analysis in the crisp case. In the crisp setting [10] it is clear that any chain (any closure system really) represents the concept lattice of a formal context, but now we have proved that in the fuzzy case there exist chains which are not the fuzzy concept lattice of a fuzzy context.

6. Conclusions

In this paper we have introduced an allocation rule for a family of profit-sharing problems with fuzzy data. In these problems (see Section 2) we have a set of agents, a fuzzy formal context that relates them to certain properties, and a profit function that depends on these properties. The proposed allocation rule follows the Shapley value philosophy. In Section 5, we have shown that the solution is well constructed in the sense that it meets reasonable conditions (Theorems 5.1, 5.2 and 5.3), as well as having good operational properties on data expansion or decomposition (Theorems 5.5 and 5.8). We have chosen the Choquet integral as a tool for aggregating information, but it is an open problem to define similar solutions using other integrals.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

This research work has been funded by the grant FQM-237 of the Junta de Andalucía (Spain).

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