doi:10.7153/mia-2023-26-21

OPTIMAL DIVISIONS OF A CONVEX BODY

ANTONIO CAÑETE*, ISABEL FERNÁNDEZ AND ALBERTO MÁRQUEZ

(*Communicated by M. A. Hernández Cifre*)

Abstract. For a convex body C in $\mathbb{R}^d$ and a division of C into convex subsets $C_1,\ldots,C_n$, we can consider $max\{F(C_1),\ldots,F(C_n)\}$ (respectively, $min\{F(C_1),\ldots,F(C_n)\}$), where F represents one of these classical geometric magnitudes: the diameter, the minimal width, or the inradius. In this work we study the divisions of C minimizing (respectively, maximizing) the previous value, as well as other related questions.

1. Introduction

Finding the *best* division of a given set, from a geometric point of view, is an interesting non-trivial question deeply studied in different settings, specially in the last decades, which may yield striking results in some situations.

In this line, *Conway's fried potato problem* ([8, Problem C1]) looks for the division of a convex body C in $\mathbb{R}^d$ into n subsets (under the additional restriction of using $n-1$ successive hyperplane cuts) minimizing the largest inradius of the subsets. This problem was solved by A. Bezdek and K. Bezdek in 1995, proving that a minimizing division is given by means of $n-1$ parallel hyperplane cuts, equally spaced between hyperplanes providing the minimal width of a certain rounded body associated to C. We note that this construction is implicit, and the *optimal value* associated to this problem is determined theoretically [2, Th. 1].

A similar question for the diameter magnitude has been also considered in the planar setting in several joint works by one of the authors: for the family of centrally-symmetric planar convex bodies and arbitrary divisions into two subsets, necessary and sufficient conditions for being a minimizing division can be found in [6] (see also [13]). Moreover, for a k-rotationally symmetric planar convex body C, where $k\in\mathbb{N}$, $k\geqslant 3$, a minimizing k-partition (which is a particular type of division into k subsets, by means of k curves meeting at an interior point of C) is described in [5, Th. 4.5] for any $k\geqslant 3$, as well as a minimizing general division (without restrictions) into k subsets when

Mathematics subject classification (2020): 52A20, 52A40.

Keywords and phrases: Convex body, optimal division, diameter, minimal width, inradius.

The first author is supported by MICINN project PID2020-118180GB-I00, and by Junta de Andalucía grant FQM-325. The second author is supported by MICINN/FEDER project PID2020-118137GB-I00. The third author is supported by MICINN projects PID2019-103900GB-I00 and PID2019-104129GB-I00/AEI/10.13039/501100011033.

The authors would like to thank the anonymous reviewer, whose comments and suggestions have improved the final version of this paper.

* Corresponding author.

$k \leqslant 6$ [5, Th. 4.6]. Additionally, a related approach for general planar convex bodies has been treated in [4].

These two previous questions can be regarded as particular cases of the following *min-Max* and *Max-min* type problems:

PROBLEM. *Given a geometric magnitude F and a convex body $C \subset \mathbb{R}^d$, which are the divisions of C, if any, minimizing (resp., maximizing) the largest (resp., the smallest) value of F on the subsets of the division?*

The present work is devoted to study this problem when F is the diameter, the minimal width and the inradius. Following the original statement of Conway's fried potato problem, we will consider divisions determined by *successive hyperplane cuts* (see Section 2 or [2, §. 2] for a more precise description). We will address the question of the existence of an *optimal division*, as well as its *balancing behaviour* (in the sense that all the subsets in those divisions have the same value for the considered magnitude, see Section 2). We will also give the optimal value of the magnitude F when possible, or upper and lower bounds when not. Moreover, for the family of convex polygons, we will provide an algorithm for computing the optimal value (and consequently, an optimal division) for the min-Max problem for the inradius (Conway's fried potato problem), for which the solution was known only theoretically, as explained above. This algorithm is of quadratic order with respect to the number of sides of the polygon, see Subsection 3.3.1.

Our main results can be summarized as follows:

THEOREM A. (min-Max type problems) *Let C be a convex body in $\mathbb{R}^d$, F one of the following magnitudes: diameter (D), width (w), inradius (I), and $n \geqslant 2$. Then, there exists a division of C into n subsets (given by $n-1$ successive hyperplane cuts) minimizing the largest value of F on the subsets. This optimal division can be chosen to be* balanced. *Moreover:*

i) If $F = D$, lower and upper bounds for the optimal value are given by Equation (3).

ii) If $F = w$, any optimal division is balanced and the optimal value is $w(C)/n$.

iii) If $F = I$, the optimal value is given in terms of the width of some rounded body *associated to C (*[2, Th. 1]*), although an explicit sharp lower bound is given by Equation* (6).

In the last two cases, optimal divisions for convex polygons can be found by means of algorithms of linear and quadratic order with respect to the number of sides of the polygon, respectively.

THEOREM B. (Max-min type problems) *Let C be a convex body in $\mathbb{R}^d$, F one of the following magnitudes: diameter (D), width (w), inradius (I), and $n \geqslant 2$. Then, there exists a division of C into n subsets (given by $n-1$ successive hyperplane cuts) maximizing the smallest value of F on the subsets (except possibly when $d = 2$ and $F = D$). This optimal division can be chosen to be* balanced. *Moreover:*

i) If $F = D$, any optimal division is balanced and the optimal value is $D(C)$.

ii) If $F = w$, sharp lower and upper bounds for the optimal value are given by Equation (10).

iii) If $F = I$, a lower bound for the optimal value is given by Equation (11).

For the Max-min type problems, we also remark that any optimal division when $F = w$ is balanced for $n = 2$, and that the optimal value when $F = I$ can be expressed in an analogous way as in [2] (that is, in terms of the optimal value for the Max-min problem for the width for a certain rounded body associated to C, see Theorem 12).

The paper is organized as follows. Section 2 establishes the definitions and notation needed throughout the work. In Section 3 we consider the min-Max type problems, proving the results in Theorem A, whereas Section 4 is devoted to the corresponding Max-min type problems, proving Theorem B. Finally, Section 5 contains some related questions.

2. General setting and preliminaries

From now on, C will denote a convex body (compact convex set with non-empty interior) in $\mathbb{R}^d$, $d \geqslant 2$. We denote by ∂C and $int(C)$ the boundary and the interior of C respectively.

Following [2] (see also [15, Subsec. 2.2]), we consider the following definition.

DEFINITION 1. An *n-division* of a convex body $C \subset \mathbb{R}^d$ is a decomposition of C into n closed subsets $C_1, \ldots, C_n$, all of them with non-empty interior, given by $n-1$ *successive hyperplane cuts*: along the division process, only one subset is divided into two by each hyperplane cut (see Figure 1). In particular, all the subsets of an n-division are convex, and the intersection between two adjacent subsets is always a piece of hyperplane.

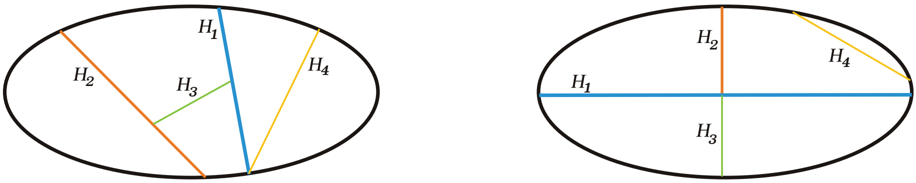

Figure 1: *Two 5-divisions of an ellipse, provided by four hyperplane cuts.*

REMARK 1. If $C, \widetilde{C} \subset \mathbb{R}^d$ are close enough (with respect to the Hausdorff distance) convex subsets, then any n-division P of C induces an n-division $\widetilde{P}$ of $\widetilde{C}$ obtained by the successive divisions given by the same hyperplanes, and in the same order, as in P (see Figure 2).

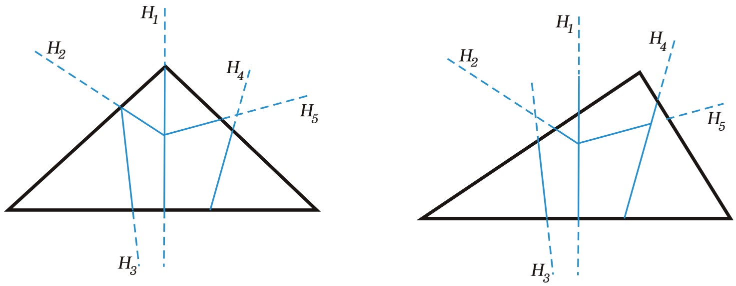

Figure 2: *A 6-division of a triangle C (on the left) and the induced 6-division of a triangle $\widetilde{C}$ close to C (on the right)*

Let F denote one of these three classical geometric magnitudes, defined for any compact set in $\mathbb{R}^d$:

– the diameter D, which is the largest distance between two points in the set,

– the (minimal) width w, which is the smallest distance between two parallel supporting hyperplanes of the set, and

– the inradius I, which is the largest radius of a ball entirely contained in the set.

Associated to the magnitude F, we consider the following *min-Max type problem*: determine the n-divisions P of C that provide the minimal possible value for

$$F(P) := \max\{F(C_1), \ldots, F(C_n)\},$$

where $C_1, \ldots, C_n$ are the subsets given by P, as well as finding that value:

$$F_n(C) = \inf\{F(P) : \ P \text{ is an } n\text{-division of } C\}. \tag{1}$$

The dual *Max-min type problem* seeks for the n-divisions P of C for which

$$\widetilde{F}(P) := \min\{F(C_1), \ldots, F(C_n)\}$$

agrees with

$$\widetilde{F}_n(C) = \sup\{\widetilde{F}(P) : \ P \text{ is an } n\text{-division of } C\}. \tag{2}$$

Any n-division P of C satisfying that $F(P) = F_n(C)$ or $\widetilde{F}(P) = \widetilde{F}_n(C)$ will be called an *optimal n-division* of C, and the values $F_n(C)$ and $\widetilde{F}_n(C)$ will be referred to as the *optimal values* of the considered problems. Additionally, we will say that an n-division of C into subsets $C_1, \ldots, C_n$ is *balanced* if $F(C_1) = \ldots = F(C_n)$.

The following inequalities are almost straightforward from the previous definitions:

LEMMA 1. *Let C be a convex body in $\mathbb{R}^d$. Then, $0 < F_n(C) \leqslant F_m(C) \leqslant F(C)$ and $0 < \widetilde{F}_n(C) \leqslant F(C)$, for any $n \geqslant m \geqslant 2$.*

Proof. The second chain of inequalities is trivial. For the first one, suppose that $F_n(C) = 0$. As $F_n(C)$ is defined as an infimum, this implies that there exists a sequence $\{P_k\}_{k\in\mathbb{N}}$ of n-divisions of C such that $\{F(P_k)\}_{k\in\mathbb{N}}$ tends to zero. Let $C_1^k, \ldots, C_n^k$ be the subsets of C given by P_k. Without loss of generality, we can assume that $F(P_k) = F(C_1^k)$ for any $k \in \mathbb{N}$. We can apply successively Blaschke selection theorem [16, Th. 1.8.7] to the sequences $\{C_1^k\}_{k\in\mathbb{N}}, \ldots, \{C_n^k\}_{k\in\mathbb{N}}$ in order to obtain convex bodies $E_1, \ldots, E_n$ such that $C = E_1 \cup \ldots \cup E_n$. Since $F(E_1) = 0$ by continuity and, consequently, $F(E_i) = 0$, it follows that E_i has empty interior, for $i = 1, \ldots, n$, which yields a contradiction because C has non-empty interior. Finally, for $m \leqslant n$, let Q_m be an arbitrary m-division of C with subsets $C_1, \ldots, C_m$. Without loss of generality, we can assume that $F(Q_m) = F(C_1)$. By dividing the subset C_m into $n - m + 1$ subsets by successive hyperplane cuts, we will obtain an n-division Q_n of C with subsets $C_1, \ldots, C_{m-1}, C'_m, \ldots, C'_n$, for which $F(Q_m) = F(Q_n) \geqslant F_n(C)$, and therefore $F_m(C) \geqslant F_n(C)$. □

3. min-Max type problems

In this section we shall treat the min-Max type problems for the diameter, the width and the inradius. Recall that the optimal value for each of these problems will be denoted by $F_n(C)$, where F stands for the considered magnitude.

3.1. min-Max problem for the diameter

For this problem, Theorem 2 provides lower and upper bounds for the corresponding optimal value. Moreover, we will see that the existence of balanced optimal divisions is always assured (see Theorem 1), but not all optimal divisions are balanced, as shown in Example 1.

LEMMA 2. *Let $n \geqslant 2$. Assume that for any convex body $E \subset \mathbb{R}^d$ there exists an optimal n-division for the min-Max problem for the diameter. Then, the map $E \mapsto D_n(E)$ is continuous with respect to the Hausdorff distance.*

Proof. Let $\{E^k\}_{k\in\mathbb{N}}$ be a sequence of convex bodies converging to a fixed convex body E. Label P, Q^k the optimal n-divisions for E, E^k, respectively. Let $E_1^k, \ldots, E_n^k$ be the subsets given by the division Q^k, for any $k \in \mathbb{N}$. Consider the n-division $\widetilde{Q}^k$ of E induced by Q^k, for $k \in \mathbb{N}$ large enough (see Remark 1), with subsets $\widetilde{E}_1^k, \ldots, \widetilde{E}_n^k$.

By applying successively Blaschke selection theorem [16, Th. 1.8.7], we have that $\{\widetilde{E}_j^k\}_{k\in\mathbb{N}}$ will converge (up to a subsequence) to a convex body (maybe with empty interior) E_j^0, $j = 1, \ldots, n$. Subdividing conveniently if necessary, as in the proof of Lemma 1, we can assume that these sets provide an n-division Q of E with $\{D(\widetilde{Q}^k)\}_{k\in\mathbb{N}}$ converging to $D(Q)$. Consequently, $\{D(Q^k)\}_{k\in\mathbb{N}}$ also converges to $D(Q) \geqslant D(P) = D_n(E)$. In order to finish the proof, it suffices to check that $D(Q) = D(P)$, since $D(Q^k) = D_n(E^k)$ for any $k \in \mathbb{N}$.

Suppose that $D(Q) > D(P)$. Let P^k be the n-division of E^k induced by P (see Remark 1), for $k \in \mathbb{N}$ large enough. Since $\{D(P^k)\}_{k\in\mathbb{N}}$ converges to $D(P)$, we can find $k' \in \mathbb{N}$ such that $D_n(E^{k'}) = D(Q^{k'}) > D(P^{k'})$, which is impossible. Thus, $\{D_n(E^k)\}_{k\in\mathbb{N}}$ converges to $D(Q) = D(P) = D_n(E)$, which yields the statement. $\square$

THEOREM 1. *Let C be a convex body in $\mathbb{R}^d$. Then, there exists a balanced optimal n-division of C for the min-Max problem for the diameter.*

Proof. Let us first prove the existence of optimal divisions. As the optimal value $D_n(C)$ is defined as an infimum, let $\{P_k\}_{k\in\mathbb{N}}$ be a sequence of n-divisions of C such that $\lim_{k\to\infty} D(P_k) = D_n(C)$. Let $C_1^k,\ldots,C_n^k$ be the subsets provided by P_k, for any $k \in \mathbb{N}$, and assume that $D(P_k) = D(C_1^k)$ (consequently, $D(C_1^k) \geqslant D(C_j^k)$, for $j = 2,\ldots,n$, and $D_n(C) = \lim_{k\to\infty} D(C_1^k)$). By applying successively Blaschke selection theorem [16, Th. 1.8.7] for each $j \in \{1,\ldots,n\}$, we have that (a subsequence of) the sequence $\{C_j^k\}_{k\in\mathbb{N}}$ will converge to a convex body C_j^∞, which could have empty interior in some cases. Therefore, $C_1^\infty,\ldots,C_n^\infty$ will provide, in fact, an m-division P^∞ of C, with $m \leqslant n$, satisfying $D(P^\infty) = D(C_1^\infty) = \lim_{k\to\infty} D(C_1^k) = D_n(C)$ by construction, and so $D(P^\infty) = D_n(C)$. If $m = n$, then P^∞ is an optimal n-division of C, and if $m < n$, we can proceed as in the proof of Lemma 1 to obtain an n-division of C (by dividing properly the subset C_m^∞) such that $D(P) = D(C_1^\infty) = D_n(C)$, thus being optimal.

We will now prove that we can find a *balanced* optimal n-division of C by induction on the number n of subsets. For $n = 2$, let P be an optimal 2-division, which will be determined by just one hyperplane H, with subsets C_1, C_2. Assume that P is not balanced, say $D(P) = D(C_2) > D(C_1)$. Without loss of generality, we can also assume that H is not parallel to any flat piece of ∂C (if needed, we can consider another optimal division determined by a hyperplane $\widetilde{H}$ close (and non parallel) to H, with subsets $\widetilde{C}_1, \widetilde{C}_2$, satisfying that $\widetilde{C}_2 \subset C_2$ and $D(\widetilde{C}_1) < D(\widetilde{C}_2)$). Let H^t be the hyperplane parallel to H at distance $t \geqslant 0$ from C_1, which will yield a new 2-division P^t of C into subsets C_1^t, C_2^t, satisfying that $C_1 \subseteq C_1^t$, $C_2^t \subseteq C_2$. Let $t_1 > 0$ be the value for which $H^{t_1} \cap C$ reduces to a single point. Then we have

$$D(C_1^0) = D(C_1) < D(C_2) = D(C_2^0)$$

and

$$D(C_1^{t_1}) = D(C) > D(C_2^{t_1}) = 0.$$

By continuity, there exists $t_0 \in (0,t_1)$ such that $D(C_1^{t_0}) = D(C_2^{t_0})$. Then P^{t_0} is balanced and we have

$$D(P^{t_0}) = D(C_1^{t_0}) = D(C_2^{t_0}) \leqslant D(C_2) = D(P).$$

Note that the strict inequality above would contradict the optimality of P, so it follows that $D(P^{t_0}) = D(P)$ and P^{t_0} is also optimal. This proves the case $n = 2$.

Assume now $n > 2$ and let Q be a non-balanced optimal n-division of C. Consider a hyperplane cut H from Q dividing C into two different convex regions E_1, E_2, and label Q_i as the division of E_i into n_i subsets induced by Q, $i = 1,2$, with

$n_1 + n_2 = n$. By induction, there exists a balanced optimal n_i-division Q_i' of E_i, $i = 1,2$. Without loss of generality, we will distinguish three cases here:

- $D(Q_1) < D(Q_2)$. We proceed similarly as in the case $n = 2$. First, let us see that we can assume that H is not parallel to any flat piece of ∂C: indeed, if this is not the case, let $\widetilde{H}$ be a hyperplane close enough to H (but not parallel to H) dividing C into two subsets $\widetilde{E}_1, \widetilde{E}_2$ with $\widetilde{E}_2 \subset E_2$ and $E_1 \subset \widetilde{E}_1$. We can now consider the n-division $\widetilde{Q}$ of C given by $\widetilde{H}$ together with $\widetilde{Q}_1$ and $\widetilde{Q}_2$, where $\widetilde{Q}_i$ is the n_i-division of $\widetilde{E}_i$ induced by Q_i (see Remark 1), $i = 1,2$. Moreover, we can assume that $\widetilde{Q}$ satisfies
$$D(Q_1) \leqslant D(\widetilde{Q}_1) < D(\widetilde{Q}_2) \leqslant D(Q_2),$$
from where we infer that $\widetilde{Q}$ is also optimal.

  Thus, assume that H is not parallel to any flat piece of ∂C and consider the optimal balanced divisions Q_1', Q_2' defined above. If $D(Q_1') = D(Q_2')$, then Q_1', Q_2' yield a optimal balanced n-division Q' of C and we are done. If (say) $D(Q_1') < D(Q_2')$, let H^t be the hyperplane parallel to H at distance $t \geqslant 0$ from E_1 dividing C into two convex regions E_1^t, E_2^t, with $E_1 \subseteq E_1^t$ and $E_2^t \subseteq E_2$. For each $t \geqslant 0$ we can consider a balanced optimal n_i-division Q_i^t of E_i^t, $i = 1,2$. By continuity (observe that Lemma 2 holds, since the existence of optimal divisions has been already proved), there exists $t_0 > 0$ such that $D(Q_1^{t_0}) = D(Q_2^{t_0})$. These two divisions $Q_1^{t_0}$, $Q_2^{t_0}$ will yield a balanced n-division Q^{t_0} of C that satisfies
$$D(Q^{t_0}) = D(Q_2^{t_0}) = D_{n_2}(E_2^{t_0}) \leqslant D_{n_2}(E_2) = D(Q_2') \leqslant D(Q_2) = D(Q).$$
  Since Q is optimal, we necessarily have that $D(Q^{t_0}) = D(Q)$, and so Q^{t_0} is a (balanced) optimal n-division of C, as claimed.

- $D(Q_1) = D(Q_2)$ but at least one division, say Q_1, is not optimal. Let $\widehat{Q}_1$ be an optimal n_1-division of E_1. Then $D(\widehat{Q}_1) < D(Q_1)$ and the (optimal) n-division $\widehat{Q}$ of C obtained from $\widehat{Q}_1$ and Q_2 satisfies $D(\widehat{Q}_1) < D(Q_1) = D(Q_2)$, and we can proceed as in the previous case.

- $D(Q_1) = D(Q_2)$ and both Q_1 and Q_2 are optimal divisions (that is, $D(Q_i) = D_{n_i}(E_i)$, $i = 1,2$). Then the divisions Q_1', Q_2' satisfy $D(Q_1') = D(Q_2') = D(Q)$ and therefore they yield a balanced optimal n-division Q' of C. $\square$

The following example shows that not all optimal divisions for this problem are necessarily balanced.

EXAMPLE 1. For a given ball B in $\mathbb{R}^d$, it is clear that any division P of B into two subsets satisfies that $D(P) = D(C)$. Thus, $D_2(B) = D(B)$ and any 2-division of B is optimal (in particular, any non-balanced 2-division will be optimal in this case).

We will now focus on the computation of lower and upper bounds for $D_n(C)$. First, let us introduce the notion of *orthogonal widths* associated to a given convex body.

DEFINITION 2. Let C be a convex body in $\mathbb{R}^d$. The orthogonal widths associated to C, $w_1 \leqslant \ldots \leqslant w_d$, are defined recursively as follows:

- w_1 is the width of $C_1 := C$ measured in $\Pi_1 := \mathbb{R}^d$.
- For $i \in \{2,\ldots,d\}$, let Π_i be a supporting $(d-i+1)$-plane in Π_{i-1} of C_{i-1} determining its width (measured in Π_{i-1}), and let $\pi_i : \Pi_{i-1} \to \Pi_i$ be the associated orthogonal projection. Then, w_i is defined as the width of $C_i := \pi_i(C_{i-1})$ measured in Π_i.

As a consequence of the previous definition, any convex body C in $\mathbb{R}^d$ with orthogonal widths $w_1,\ldots,w_d$ is contained in a d-orthotope H_C with edge lengths $w_1,\ldots,w_d$ (see Figure 3).

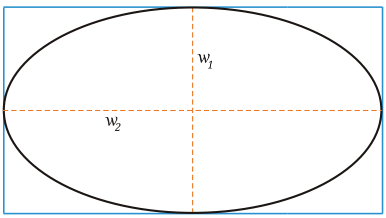

Figure 3: *Associated 2-orthotope given by the orthogonal widths of an ellipse*

THEOREM 2. *Let C be a convex body in $\mathbb{R}^d$ with orthogonal widths $w_1,\ldots,w_d$, and let $n \in \mathbb{N}$, $n \geqslant 2$. Then,*

$$\frac{1}{n} D(C) < D_n(C) \leqslant \min\left\{ D(C), \sqrt{\frac{w_1^2}{a_1^2} + \frac{w_2^2}{a_2^2} + \ldots + \frac{w_d^2}{a_d^2}} \right\}, \tag{3}$$

for any $a_1 \leqslant \ldots \leqslant a_d$ natural numbers such that $n \geqslant a_1 \cdot \ldots \cdot a_d$.

Proof. For the left-hand side of (3), let P be a balanced optimal n-division of C with subsets $C_1,\ldots,C_n$, in view of Theorem 1. Then $D_n(C) = D(P) = D(C_i)$ for all $i = 1,\ldots,n$. Fix a segment s in C with $\ell(s) = D(C)$, where ℓ represents the Euclidean length. If s is contained in a subset C_j, then $D_n(C) = D(C_j) = D(C)$ and the statement trivially holds. Assume now that P divides s into m segments $s_1,\ldots,s_m$, with $2 \leqslant m \leqslant n$, and $s_i \subset C_i$, $i = 1,\ldots,m$.

- If $m < n$, we have that

$$D(C) = \ell(s) = \sum_{i=1}^{m} \ell(s_i) \leqslant \sum_{i=1}^{m} D(C_i) < \sum_{i=1}^{n} D(C_i) = n\,D(P) = n\,D_n(C).$$

- If $m=n$, then

$$D(C)=\ell(s)=\sum_{i=1}^{n}\ell(s_i)<\sum_{i=1}^{n}D(C_i)=nD(P)=nD_n(C),$$

  where the strict inequality holds since s will be necessarily divided by the $n-1$ hyperplane cuts from P, and so $\ell(s_j)<D(C_j)$, for some $j\in\{1,\dots,n\}$: indeed, if we label $s_i=\overline{p_{i-1}p_i}$, for $i=1,\dots,n$, it is possible to find a point $p\in C$ (close to p_1) in the orthogonal line to s at p_1 such that either $p\in C_1$ and $d(p,p_0)>\ell(s_1)$ (and so $D(C_1)>\ell(s_1)$), or $p\in C_2$ and $d(p,p_2)>\ell(s_2)$ (and so $D(C_2)>\ell(s_2)$).

Both situations above yield $D_n(C)>D(C)/n$, as desired.

For the right-hand side of (3), let H_C be the d-orthotope containing C associated to the orthogonal widths $w_1,\dots,w_d$ of C. The facets of H_C are then determined by the boundary of d slabs, $B_1,\dots,B_d$. Fix $a_i\in\mathbb{N}$, $i=1,\dots,d$, with $a_1\leqslant\dots\leqslant a_d$ and such that $a_1\cdot\ldots\cdot a_d\leqslant n$. Then, for each $i\in\{1,\dots,d\}$, consider a_i-1 hyperplanes *equally spaced between the two hyperplanes in* ∂B_i (that is, hyperplanes parallel to ∂B_i and dividing the slab B_i into a_i slabs of the same width). These hyperplanes yield a mesh-type division P of H_C into $r=a_1\cdot\ldots\cdot a_d$ subsets $G_1,\dots,G_r$ (which can be seen as an r-division of H_C given by $r-1$ successive hyperplane cuts), where each G_i is a d-orthotope with edge lengths w_1/a_1, $w_2/a_2,\dots,w_d/a_d$. Then,

$$D(P)=D(G_i)=\sqrt{\frac{w_1^2}{a_1^2}+\frac{w_2^2}{a_2^2}+\dots+\frac{w_d^2}{a_d^2}},$$

which constitutes an upper bound for $D_n(C)$ in view of Lemma 1 (since P will induce an m-division of C, with $m\leqslant r\leqslant n$, by hypothesis). The proof finishes taking into account that $D_n(C)\leqslant D(C)$, again by Lemma 1. □

REMARK 2. The lower bound from Theorem 2 can be considered sharp in the following sense: for any $n\geqslant 2$ and any $\varepsilon>0$ small enough, there exists a convex body C in $\mathbb{R}^d$ such that $D_n(C)<D(C)/n+\varepsilon$ (it suffices to take C as an orthotope of lengths 1, $\varepsilon,\dots,\varepsilon$).

REMARK 3. The upper bound in Theorem 2 is obtained by means of a certain r-division of the d-orthotope containing C given by its orthogonal widths, where $r=a_1\cdot\ldots\cdot a_d\leqslant n$. Let us remark that a choice with $r=n$ does not always provide the best upper bound in Theorem 2, as can be observed in Table 1.

Moreover, in order to obtain the best upper bound using this result, the choice of the natural numbers $a_1,\dots,a_d$ will depend on the convex body C. Numerical simulations indicate that if C is, for example, a hypercube, then the right-hand side of (3) is attained for $a_i=\lfloor n^{1/d}\rfloor$, $i=1,\dots,d$, whereas for a long and narrow orthotope the minimum is given for $a_1=\dots=a_{d-1}=1$, $a_d=n$ (see Table 1 for some examples in the planar case).

	$C=[0,1]\times[0,M]$	(a_1,a_2)
$n=4$	$0<M\leqslant 2$	$(2,2)$
	$2\leqslant M$	$(1,4)$
$n=9$	$0<M\leqslant \frac{2\sqrt{5}}{\sqrt{7}}$	$(3,3)$
	$\frac{2\sqrt{5}}{\sqrt{7}}\leqslant M\leqslant \frac{18\sqrt{3}}{\sqrt{65}}$	$(2,4)$
	$\frac{18\sqrt{3}}{\sqrt{65}}\leqslant M$	$(1,9)$
$n=16$	$0<M\leqslant \frac{5\sqrt{7}}{9}$	$(4,4)$
	$\frac{5\sqrt{7}}{9}\leqslant M\leqslant \frac{20\sqrt{5}}{3\sqrt{39}}$	$(3,5)$
	$\frac{20\sqrt{5}}{3\sqrt{39}}\leqslant M\leqslant 8$	$(2,8)$
	$8\leqslant M$	$(1,16)$

Table 1: *Values of a_1, a_2 providing the best upper bound in Theorem 2 in the case of a rectangle C*

3.2. min-Max problem for the width

For this problem, we stress that the corresponding optimal value $w_n(C)$ for a given convex body C can be computed: we will see in Theorem 3 that such a value equals $w(C)/n$ (extending [4, Le. 4.1]), and we will obtain the existence of optimal divisions. Moreover, it also shows that all optimal divisions are balanced in this setting (which improves [4, Le. 2.3]). We will finish this subsection with some comments on the algorithm for determining the optimal value (and so an optimal division).

We start by recalling the following well-known result due to T. Bang [1].

LEMMA 3. ([1]) *Let C be a convex body in $\mathbb{R}^d$. Assume that $C\subset B_1\cup\ldots\cup B_m$, where B_i is a slab delimited by two parallel hyperplanes in $\mathbb{R}^d$, for $i=1,\ldots,m$. Then, $w(C)\leqslant w(B_1)+\ldots+w(B_m)$.*

THEOREM 3. *Let C be a convex body in $\mathbb{R}^d$. Then, there exists an optimal n-division for the min-Max problem for the width. Moreover,*

$$w_n(C)=w(C)/n, \tag{4}$$

and any optimal n-division of C is balanced.

Proof. For any n-division P of C into subsets $C_1,\ldots,C_n$, Lemma 3 gives that

$$w(C)\leqslant\sum_{i=1}^{n}w(C_i)\leqslant n\,w(P),$$

which implies that $w_n(C) \geqslant w(C)/n$. Let now B be a slab providing $w(C)$. We can construct an n-division P_0 of C by considering $n-1$ parallel hyperplanes, equally spaced between the two hyperplanes in ∂B, see Figure 4.

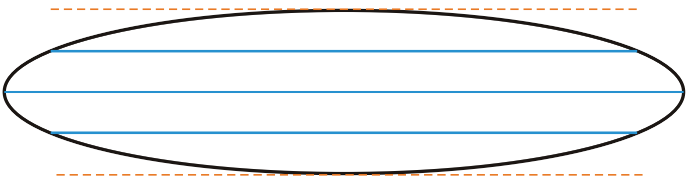

Figure 4: *An optimal 4-division of an ellipse*

It is clear that $w(P_0) = w(C)/n$, which gives that P_0 is optimal, as well as equality (4). Finally, let P be an optimal n-division of C dividing C into $C_1, \ldots, C_n$. If P is not balanced, we can assume without loss of generality that $w(P) = w(C_1) > w(C_2)$. By applying Lemma 3, it follows that

$$w(C) \leqslant \sum_{i=1}^{n} w(C_i) < n\,w(C_1) = n\,w(P),$$

which implies that $w_n(C) = w(P) > w(C)/n$. This contradicts (4), and so P must be balanced. $\square$

In view of Theorem 3, determining the width of a convex body C immediately leads to the optimal value for the corresponding min-Max problem, as well as to an optimal n-division of C (given as in the proof of Theorem 3). The algorithms for searching the width of a convex body are important components of modern algorithm theory, the complexity of which is polynomial for a fixed dimension (and linear for polygons in the planar case, see [18]). The main ideas of such algorithms were described in [9].

3.3. min-Max problem for the inradius

As pointed out in Section 1, the min-Max problem for the inradius is known as *Conway's fried potato problem* and has been deeply studied in [2] (see also [3]). In that paper, the authors proved that the optimal value of a given convex body C for this problem can be expressed in terms of the width of a certain *rounded body* associated to C, describing moreover an optimal division [2, Th. 1]. These results are stated in Theorem 4 for the sake of completeness (see also Definition 3). Our main contribution in this setting is providing an algorithm, based on the notion of *medial axis*, which leads to the optimal value for any convex polygon (see Subsection 3.3.1). We also give a general sharp lower bound for the optimal value in Corollary 1.

DEFINITION 3. Let C be a convex body in $\mathbb{R}^d$, and let $0 < \rho \leqslant I(C)$. The ρ-rounded body C^ρ of C is the union of all the balls of radius ρ which are contained in C. This construction can be extended to $\rho = 0$ by setting $C^0 = C$.

The notion of rounded body has been previously considered for different problems in the literature, such as the isoperimetric and Cheeger problems (see for instance [17, 12]). It is also related to inner parallel bodies, since C^{ρ} coincides with the Minkowski addition of the inner parallel body $C \div \rho B_d$ and ρB_d, where B_d is the unit ball in $\mathbb{R}^d$ (see [16] for a more detailed explanation and applications of this kind of constructions).

THEOREM 4. ([2, Th. 1]) *Let C be a convex body in $\mathbb{R}^d$. Then, $I_n(C)$ is the unique number $\widetilde{\rho}$ such that*

$$w(C^{\widetilde{\rho}}) = 2n\widetilde{\rho}. \tag{5}$$

Moreover, an optimal balanced n-division of C is given by $n-1$ parallel hyperplanes, equally spaced between the two hyperplanes delimiting a slab which provides $w(C^{\widetilde{\rho}})$.

The reader can find two different balanced optimal 3-divisions of an equilateral triangle for this problem in [2, Fig. 1] (which shows that, in general, the solution is not unique). An intriguing open question in this setting is investigating whether any optimal division is necessarily balanced, as it happens for the corresponding min-Max type problem for the width (Theorem 3).

REMARK 4. As a consequence of Theorems 3 and 4, we have that

$$2\widetilde{\rho} = w_n(C^{\widetilde{\rho}}),$$

for any convex body C (here, $\widetilde{\rho} = I_n(C)$). In particular, if C is *rounded enough* so that $C = C^{\widetilde{\rho}}$ (equivalently, if $\widetilde{\rho}B_d$ is a summand of C, see [16, Lemma 3.1.11]), the optimal values for the min-Max problems for the width and the inradius will coincide, up to a constant. However, we point out that the optimal divisions in these two situations will in general differ: for an equilateral triangle $\mathscr{T}$, the aforementioned [2, Fig. 1(b)] shows an optimal 3-division of $\mathscr{T}$ for the inradius (and so, also optimal for the $I_3(\mathscr{T})$-rounded body $\mathscr{T}^{I_3(\mathscr{T})}$ of $\mathscr{T}$), which is not optimal for $\mathscr{T}^{I_3(\mathscr{T})}$ when considering the width (in fact, such a 3-division of $\mathscr{T}^{I_3(\mathscr{T})}$ is not even balanced for the width).

Corollary 1 gives an explicit lower bound for $I_n(C)$ in terms of the inradius of the considered convex body C, by using the following result due to V. Kadets (which was originally stated in a general context).

LEMMA 4. ([11, Th. 2.1]) *Let C be a convex body in $\mathbb{R}^d$, and let P be an n-division of C into subsets $C_1, \ldots, C_n$. Then, $I(C) \leqslant \sum_{i=1}^{n} I(C_i)$.*

COROLLARY 1. *Let C be a convex body in $\mathbb{R}^d$. Then,*

$$I_n(C) \geqslant I(C)/n. \tag{6}$$

REMARK 5. Inequality (6) turns into an equality, for instance, for any convex body C whose inball B touches ∂C at exactly two points p, q. In that case, these two points will be necessarily antipodal in B, and an optimal n-division can be constructed by means of $n-1$ hyperplanes orthogonal to the segment $\overline{pq}$ and dividing it into n segments of the same length.

3.3.1. Algorithm for the optimal value of convex polygons

We will now describe a constructive procedure which will lead us to the optimal value for this problem when considering an arbitrary convex polygon (note that the solution given in [2] is obtained theoretically). In the following, *dist* will stand for the Euclidean planar distance.

DEFINITION 4. Given a convex polygon C and a side L of C, we will denote by $w_L(C)$ the *directional width* of C with respect to L. That is, $w_L(C)$ is the width of C when considering only slabs parallel to the direction determined by L. Analogously, for $\rho \in (0, I(C)]$, we will denote by $w_L(C^\rho)$ the directional width of C^ρ with respect to L.

The following lemma allows us to discretize the search space of the slopes of the slabs providing the width of any given rounded body associated to a polygon. As a result, the size of that search space is linear with respect to the number of edges of the polygon.

LEMMA 5. *Let C be a convex polygon, and let $0 < \rho \leqslant I(C)$. Then, there exists a slab providing $w(C^\rho)$, such that one side of C is contained in the boundary of the slab. That is, there exists a side L in C such that $w(C^\rho) = w_L(C^\rho)$.*

Proof. In this case, the boundary of C^ρ will be composed by circular arcs (of radius ρ) and, possibly, by some segments (which will be pieces of sides of C). Note that if ∂C^ρ does not contain any segment, then C^ρ will be necessarily a ball. This immediately implies the statement. So we can assume that ∂C^ρ contains, at least, one segment.

Let B be a slab determining $w(C^\rho)$, delimited by two parallel supporting lines h_1, h_2. Let $x_i \in h_i \cap C^\rho$, and assume that x_i lies in an arc σ_i of C^ρ, for $i = 1, 2$ (otherwise, the statement trivially follows).

Without loss of generality, we can assume that B is a horizontal slab. For any $t > 0$, let h_1^t, h_2^t be the two supporting lines of C^ρ with slope $-t$, and label B^t as the slab bounded by them.

Assume first that $dist(x_1, x_2) = w(C^\rho)$. Since $\rho \leqslant I(C^\rho) \leqslant w(C^\rho)/2$, we will discuss two possibilities.

On the one hand, if $\rho = w(C^\rho)/2$, it follows that σ_1, σ_2 will lie in the same ball. Then, the slab B^t will also provide $w(C^\rho)$, for $t \in [0, t_0]$, where either $h_1^{t_0}$ or $h_2^{t_0}$ contains a segment of ∂C^ρ, yielding the statement. On the other hand, if $\rho < w(C^\rho)/2$, it follows that the width of B^t will be smaller than $w(C^\rho)$, for $t > 0$ small, see Figure 5. Since B^t contains C^ρ, this gives a contradiction.

Finally, if $dist(x_1, x_2) > w(C^\rho)$, we can proceed similarly: the slab B^t, for $t > 0$ small, still contains C^ρ, but its width will be smaller than $w(C^\rho)$, yielding a contradiction again, see Figure 6.

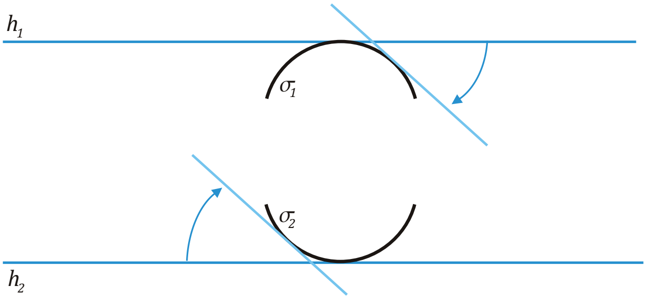

Figure 5: *If $\rho < w(C^\rho)/2$, a rotational argument yields a slab containing C^ρ, with width strictly smaller than $w(C^\rho)$*

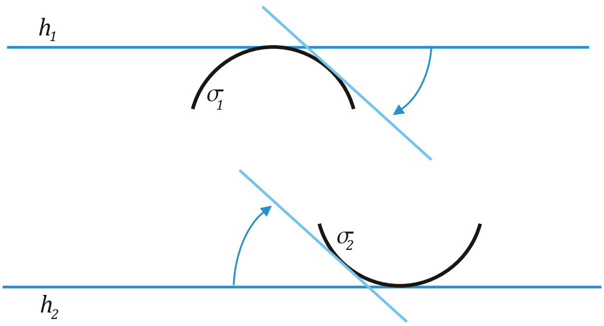

Figure 6: *If $dist(x_1,x_2) = w(C^\rho)$, an analogous rotational argument leads to a slab containing C^ρ, whose width is strictly smaller than $w(C^\rho)$*

□

In view of Theorem 4 and Lemma 5, in order to obtain the optimal value of this problem for a convex polygon C, it seems reasonable focusing on each side of C, finding the different values provided by Lemma 5. It follows that one of these values will be the desired optimal one. This approach will require some new definitions.

Let C be a convex polygon. The *medial axis* $M(C)$ of C is defined as the set of points of C having more than one closest side on the boundary of C, see Figure 7. Equivalently, $M(C)$ is the boundary of the Voronoi diagram associated to ∂C [10, §. 4], and so it will be composed by line segments (it is, in fact, a tree-like graph), see [14, 7]. From the computational point of view, it is known that $M(C)$ can be computed in linear time with respect to the number of sides of C [7, Co. 4.5].

Given a side L in the boundary of C, let L' be the supporting line of C, parallel to L, bounding the slab which provides $w_L(C)$. Any vertex of C contained in L' will be called an *antipodal vertex* to the side L. Notice that any side has at most two antipodal vertices. Each point in any segment s of $M(C)$ is equidistant to two sides in

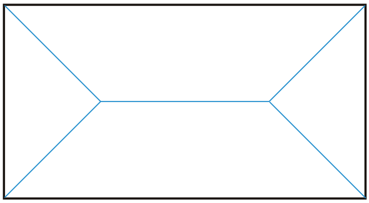

Figure 7: *The medial axis of a rectangle*

the boundary of C, that will be referred to as the *associated sides* to s. We will say that a segment s in $M(C)$ is an *antipodal segment* to L if each one of its associated sides contains an antipodal vertex to L, see Figure 8. Thus, each side will have at most three antipodal segments.

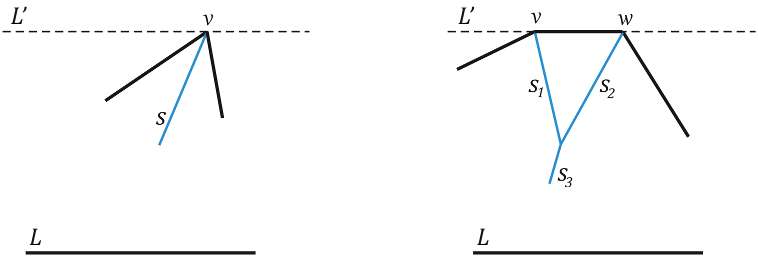

Figure 8: *Left-hand side: v is the antipodal vertex to the side L, and s is the antipodal segment to L. Right-hand side: v, w are the antipodal vertices to L, and s_1, s_2, s_3 are the antipodal segments to L*

We can now prove the following results on the directional width of a rounded body associated to a convex polygon.

LEMMA 6. *Let C be a convex polygon and $n \geqslant 2$. Then, for any side L of C there exists a unique value $\rho_L > 0$, that can be computed in linear time with respect to the number of sides of C, such that $w_L(C^{\rho_L}) = 2n\rho_L$.*

Proof. The existence and uniqueness of ρ_L can be proved by similar arguments as in [2, § 2], using the monotonic character of the continuous functions $\rho \mapsto 2n\rho$ and $\rho \mapsto w_L(C^\rho)$. To finish the proof, it suffices to show that the function $\rho \mapsto w_L(C^\rho)$, for $\rho \in [0, I(C)]$, can be computed in linear time with respect to the number of sides of C. This will follow from the fact that $\rho \mapsto w_L(C^\rho)$ is piecewise affine and its expression can be obtained by means of an iterative process involving a certain subset of segments of the medial axis $M(C)$, that can be computed in linear time.

Consider an antipodal vertex O to L, and label L_1, L_2 as the two sides in C containing O, and let $s = \overline{OR} \in M(C)$ be the corresponding antipodal segment. Set $\rho_1 := dist(R, L_1) = dist(R, L_2) > 0$. For each $\rho \in [0, \rho_1]$, label A_ρ as the unique point on s with $\rho = dist(A_\rho, L_1)$. Then $w_L(C^\rho) = dist(L, L_\rho)$, where L_ρ is the line parallel to L which is tangent to the circular arc of radius ρ centered at A_ρ (see Figure 9). If L has two antipodal vertices, then L_ρ is the extension of the side in C joining these two vertices, and therefore $w_L(C^\rho) = w_L(C)$ is constant for $\rho \in [0, \rho_1]$. Assume now that O is the unique antipodal vertex to L.

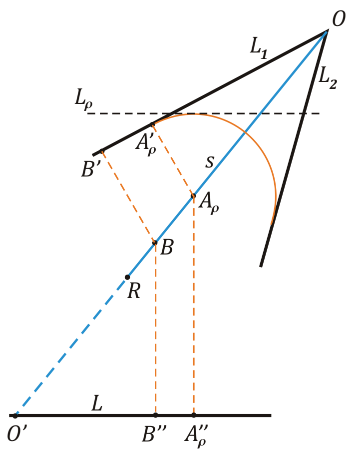

Figure 9: *Computing* $w_L(C^\rho)$, *where* $\rho = dist(A_\rho, A'_\rho)$

Fix a point B in s, $B \neq A_\rho$. Call A'_ρ the point in L_1 such that $\rho = dist(A_\rho, L_1) = dist(A_\rho, A'_\rho)$, and call A''_ρ the point in L such that $dist(A_\rho, L) = dist(A_\rho, A''_\rho)$, as Figure 9 shows. Define B' and B'' analogously. Then,

$$w_L(C^\rho) = dist(A''_\rho, L_\rho) = dist(A_\rho, A''_\rho) + \rho = dist(A_\rho, A''_\rho) + dist(A_\rho, A'_\rho). \tag{7}$$

Let O' be the intersection of the extensions of s and L. By considering the corresponding equivalent triangles we have

$$\frac{dist(O, A_\rho)}{dist(O, B)} = \frac{dist(O, A'_\rho)}{dist(O, B')} = \frac{dist(A_\rho, A'_\rho)}{dist(B, B')}$$

and

$$\frac{dist(O', A_\rho)}{dist(O', B)} = \frac{dist(O', A''_\rho)}{dist(O', B'')} = \frac{dist(A_\rho, A''_\rho)}{dist(B, B'')}.$$

Therefore,

$$dist(A_\rho, A''_\rho) = \lambda\,(dist(O, O') - dist(O, A_\rho)),$$
$$\rho = dist(A_\rho, A'_\rho) = \mu\, dist(O, A_\rho)$$

where $\lambda := dist(B, B'')/dist(O', B)$ and $\mu := dist(B, B')/dist(O, B)$ are independent from ρ. This implies that

$$\begin{aligned} w_L(C^\rho) &= dist(A_\rho, A''_\rho) + dist(A_\rho, A'_\rho) = (\mu - \lambda)\, dist(O, A_\rho) + \lambda\, dist(O, O') \\ &= (1 - \lambda/\mu)\rho + \lambda\, dist(O, O'). \end{aligned}$$

Since $dist(O,O')$ is independent of ρ, this gives an affine expression of the function $\rho \mapsto w_L(C^\rho)$ for $\rho \in [0,\rho_1]$.

The computation of $w_L(C^\rho)$ for larger ρ can be done with an analogous approach, by means of an iterative proccess considering the segments of $M(C)$ adjacent to s (and possibly the subsequent ones), until ρ reaches $I(C)$. Each one of these segments gives an affine expression for $w_L(C^\rho)$ on a suitable interval. □

Our Theorem 5 follows from the previous results and establishes that the optimal value for this problem can be found in quadratic time with respect to the number of sides of the polygon.

THEOREM 5. *Let C be a convex polygon and let $n \geqslant 2$. Consider the family $\mathscr{C} = \{\rho_L : L \text{ is a side of } C\}$ given by Lemma 6. Then, the optimal value for the corresponding Conway's fried potato problem, which is given by (5), coincides with* $\min\mathscr{C}$ *and can be computed in quadratic time with respect to the number of sides of C.*

Proof. Observe that, as a consequence of Lemma 6, the family $\mathscr{C}$ can be obtained in quadratic time with respect to the number of sides of C. This family will necessarily contain the desired optimal value, taking into account Theorem 4 and Lemma 5. We will now prove that the optimal value is $\rho_1 := \min\mathscr{C}$ (note that such a minimum can be determined in linear time). Label L_1 as the side in C satisfying $w_{L_1}(C^{\rho_1}) = 2n\rho_1$. Let L be the side in C given by Lemma 5 for $\rho = \rho_1$. Since $\rho_1 \leqslant \rho_L$, then $C^{\rho_L} \subseteq C^{\rho_1}$, and therefore

$$w(C^{\rho_1}) = w_L(C^{\rho_1}) \geqslant w_L(C^{\rho_L}) = 2n\rho_L \geqslant 2n\rho_1 = w_{L_1}(C^{\rho_1}).$$

As we trivially have $w(C^{\rho_1}) \leqslant w_{L_1}(C^{\rho_1})$, we conclude that $w(C^{\rho_1}) = w_{L_1}(C^{\rho_1}) = 2n\rho_1$, and so ρ_1 is the optimal value, by the uniqueness property from Theorem 4. □

4. Max-min type problems

In this Section we will treat the corresponding Max-min type problems for the diameter D, the width w and the inradius I. We recall that the optimal value for these problems will be denoted by $\widetilde{F}_n(C)$, where F stands for the considered magnitude (see Equation (2)).

4.1. Max-min problem for the diameter

For this problem, we will prove that the optimal value can be explicitly computed, being equal to the diameter of the considered convex body, and also that any optimal division must be balanced (Theorem 6). A remarkable fact is that the existence of optimal divisions is not assured in the planar setting (see Theorem 8 and Example 2), since it strongly depends on the location of the diameter segments of the set, see Definition 5 below.

DEFINITION 5. Let C be a convex body in $\mathbb{R}^d$. Any segment s in C with length equal to $D(C)$ will be called a *diameter segment* of C. If s is contained in ∂C we will say that s is a *boundary diameter segment* of C, and an *interior diameter segment* otherwise.

THEOREM 6. *Let C be a convex body in $\mathbb{R}^d$. Then,*

$$\widetilde{D}_n(C) = D(C).$$

Therefore, any optimal n-division of C for the Max-min problem for the diameter, if exists, is balanced.

Proof. Let s be a diameter segment of C and H_1 a hyperplane in $\mathbb{R}^d$ containing s. We can then consider hyperplanes $H_2,\dots,H_{n-1}$ parallel to H_1, and arbitrarily close to it, yielding an n-division P of C with $\widetilde{D}(P)$ arbitrarily close to $D(C)$. Hence $\widetilde{D}_n(C) \geqslant D(C)$, and therefore $\widetilde{D}_n(C) = D(C)$ (see Lemma 1).

Now, let P be an optimal n-division of C with subsets $C_1,\dots,C_n$. Then $\widetilde{D}(P) = \widetilde{D}_n(C) = D(C)$, and so $D(C_i) \geqslant D(C)$, which immediately gives $D(C_i) = D(C)$, for $i = 1,\dots,n$, finishing the proof. □

As a consequence of Theorem 6 it is immediate to check that there is no optimal n-division for this problem if, for example, C is a circle in $\mathbb{R}^2$ and $n > 2$. In the following results we study the existence of optimal divisions.

THEOREM 7. *Let C be a convex body in $\mathbb{R}^d$, where $d \geqslant 3$. Then, there exists an optimal n-division of C for the Max-min problem for the diameter.*

Proof. Let s be a diameter segment of C. We can consider $n-1$ distinct hyperplanes containing s, yielding an n-division P of C into subsets $C_1,\dots,C_n$. As each subset C_i contains the segment s, it follows that $D(C_i) = D(C)$, for $i = 1,\dots,n$. Then $\widetilde{D}(P) = D(C)$, and so P is optimal, in view of Theorem 6. □

In the planar case, observe that the proof of Theorem 7 cannot be applied, since there is only one hyperplane containing any fixed segment in $\mathbb{R}^2$. In order to study the existence of optimal divisions for a planar convex body C, we have to make some previous considerations. Theorem 6 states that all the subsets of an optimal division will have diameter equal to $D(C)$, which implies that all of them will necessarily contain a diameter segment of C. This suggests that enough diameter segments in C are needed to construct an optimal division (otherwise, it will not be possible to partition C into many subsets with diameter equal to $D(C)$). Therefore, in this planar setting, the existence of optimal divisions will strongly depend on the number of diameter segments of C, and more precisely, on how they are placed in C. In general, in order to construct an optimal division of C, each interior diameter segment of C will lead to two subsets of C with diameter equal to $D(C)$, by means of appropriate cuts (some of these cuts will be determined by the diameter segments, and the other ones will be done between the previous cuts, see Figure 10).

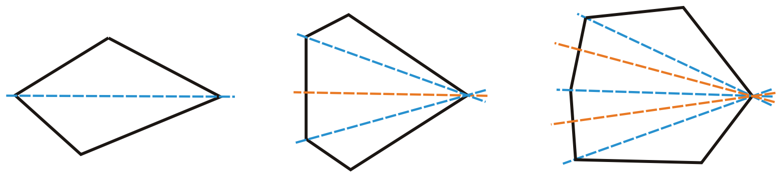

Figure 10: *The cuts in an optimal division of C can be done in the following way: each interior diameter segment determines a cut (in blue), and for any pair of consecutive interior diameter segments, a new cut can be done between them (in orange). This will guarantee that all the subsets have diameter equal to $D(C)$*

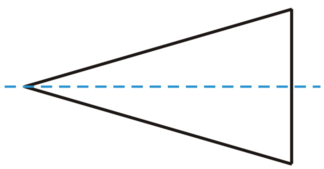

Figure 11: *In an optimal division, any boundary diameter segment will belong to a unique subset (delimited by the dashed line) with diameter equal to $D(C)$*

Besides, note that each boundary diameter segment of C can be only contained in one subset with diameter equal to $D(C)$, see Figure 11.

In order to state the existence result, let us first introduce the following definition. It is easy to check that any pair of diameter segments of a convex body C will necessarily intersect at one point, and such an intersection point will be either an endpoint of both segments or an interior point of both segments. In particular, for a set J_C of diameter segments of C *with disjoint interiors*, one of the following two possibilities holds (see Figure 12):

i) J_C consists on three diameter segments forming an equilateral triangle. In this case, we will say that J_C is *triangle-type*.

ii) All the diameter segments of J_C share a common endpoint, and J_C will be called *fan-type*.

We note that a given convex body C may have different sets of diameter segments with disjoint interiors, each of them of different type, see Figure 12.

All this taken into account, we have the following existence result in the planar case.

THEOREM 8. *Let C be a convex body in $\mathbb{R}^2$. Then, there exists an optimal n-division of C for the Max-min problem for the diameter if and only if there exists a set J_C of diameter segments of C with disjoint interiors such that*

$$n \leqslant 2a + b - \delta, \tag{8}$$

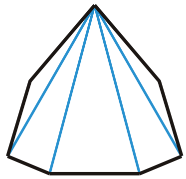

Figure 12: *A planar convex body with two different sets of diameter segments with disjoint interiors*

where a is the number of interior diameter segments of J_C and b is the number of boundary diameter segments of J_C, and $\delta = 1$ if J_C is triangle-type, or $\delta = 0$ if J_C is fan-type.

Proof. Assume firstly that (8) holds for some set J_C of diameter segments of C with disjoint interiors, and let us construct an optimal n-division. On the one hand, if J_C is fan-type, then $b \leqslant 2$ and we can proceed as follows: if $a = 0$, then necessarily $b = 2$, which implies that $n = 2$ and any cut between the two boundary diameter segments will provide an optimal 2-division of C. Otherwise, if $a > 0$, each interior diameter segment of C will determine a cut, and each boundary diameter segment will give an additional cut (placed between the boundary diameter segment and the adjacent interior diameter segment). Finally, for each two consecutive interior ones, we can consider the corresponding bisector as a new cut. This procedure gives $2a - 1 + b$ cuts, yielding a division P of C into at most $2a + b$ subsets, all of them with diameter equal to $D(C)$. Then, P is optimal. On the other hand, if J_C is triangle-type, it follows that we can have four different possibilities, depending on the number of interior diameter segments of J_C. For each possibility we can find an optimal division into at most $2a + b - 1$ subsets, as shown in Figure 13.

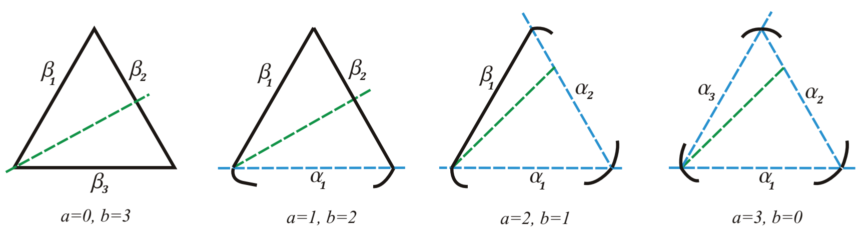

Figure 13: *Optimal divisions of C into $2a + b - 1$ subsets when J_C is triangle-type (the dashed lines indicate the cuts for each division, being α_i the interior diameter segments and β_i the boundary diameter segments, $i = 1, 2, 3$)*

Conversely, let P be an optimal n-division of C. By Theorem 6, each subset of P must contain a diameter segment of C. Let J_C be the set composed by one diameter segment from each subset of P. The same reasoning as before yields that the maximum number of subsets in P will be $2a + b - \delta$, which finishes the proof. $\square$

EXAMPLE 2. As an application of Theorem 8 it follows that, for example, for the planar convex body of Figure 12 there are no optimal n-divisions for the Max-min problem for the diameter for any $n > 8$.

4.2. Max-min problem for the width

For this problem, Theorem 9 below guarantees the existence of optimal divisions, proving also that all optimal divisions into $n = 2$ subsets are balanced. We will also obtain sharp lower and upper bounds for the corresponding optimal value in Theorem 10. We start with the following auxiliary lemmas.

LEMMA 7. *Let C be a convex body in $\mathbb{R}^d$, and let P be a 2-division of C given by a hyperplane H. Label C_1, C_2 as the two subsets provided by P, and assume $w(C_1) < w(C)$. For any $t > 0$, let P_t be the 2-division of C into subsets C_1^t, C_2^t, given by the hyperplane H^t parallel to H at distance t such that $C_1 \subset C_1^t$. Then, $w(C_1) < w(C_1^t)$.*

Proof. If $w(C_1^t) = w(C)$, the statement trivially holds. So we can assume that $w(C_1^t) < w(C)$. Let B be a slab determining the width of C_1^t. Then $w(B) = w(C_1^t) < w(C)$, and so there necessarily exists a point $q_2 \in C_2^t \subset C$ such that $q_2 \notin B$. Call H_1 the hyperplane in ∂B which is closer to q_2 (in particular, $H_1 \cap \partial C_1^t \neq \emptyset$).

Let us show that $H_1 \cap \partial C_1^t \subset H^t$. For any $q_1 \in H_1 \cap \partial C_1^t$, the segment $\overline{q_1 q_2}$ is clearly contained in C and $\overline{q_1 q_2} \cap B = \{q_1\}$. As $q_1 \in C_1^t$ and $q_2 \in C_2^t$, then there is a point $x \in \overline{q_1 q_2} \cap H^t \subset C_1^t$. If $x \neq q_1$, the segment $\overline{q_1 x} - \{q_1\}$ would be contained in C_1^t by convexity but not in B, which gives a contradiction. Thus, $q_1 \in H^t$.

Finally, if $H_1 \cap \partial C_1 \neq \emptyset$, then any intersection point q_1' would be also in ∂C_1^t, and the previous argument would imply that $q_1' \in H^t$, which is a contradiction. Therefore, $H_1 \cap \partial C_1 = \emptyset$ and so there exists a slab B' containing C_1 which is strictly contained in B. Then, $w(C_1) \leqslant w(B') < w(B) = w(C_1^t)$, as stated. □

LEMMA 8. *Let $n \geqslant 2$. Assume that for any convex body $E \subset \mathbb{R}^d$ there exists an optimal n-division for the Max-min problem for the width. Then, the map $E \mapsto \widetilde{w}_n(E)$ is continuous with respect to the Hausdorff distance.*

Proof. We can follow here a similar argument as in the proof of Lemma 2. The only difference is the case in which Q is a m-division with $m < n$. If this holds, then $\widetilde{w}(Q^k)$ converges to zero (and not to $\widetilde{w}(Q)$ as happens if $m = n$). Since $\widetilde{w}_n(E) > 0$ (see Lemma 1), the contradiction follows in the same way as in Lemma 2. □

THEOREM 9. *Let C be a convex body in $\mathbb{R}^d$. Then, there exists a balanced optimal n-division of C for the Max-min problem for the width. Moreover, any optimal 2-division is balanced.*

Proof. Let us first prove the existence of an optimal n-division of C. Let $\{P_k\}_{k\in\mathbb{N}}$ be a sequence of n-divisions of C such that $\{\widetilde{w}(P_k)\}_{k\in\mathbb{N}}$ converges to $\widetilde{w}_n(C)$, and let

$C_1^k,\dots,C_n^k$ be the subsets of C provided by P_k, $k\in\mathbb{N}$. By applying Blaschke selection theorem [16, Th. 1.8.7] successively, we can assume that, for each $i=1,\dots,n$, the sequence $\{C_i^k\}_{k\in\mathbb{N}}$ converges to a subset C_i^∞ with non-empty interior: if C_j^∞ has empty interior for some $j\in\{1,\dots,n\}$, then $0=w(C_j^\infty)=\lim_{k\to\infty} w(C_j^k)$ and so $\widetilde{w}_n(C)=0$, which contradicts Lemma 1. Thus, the subsets $C_1^\infty,\dots,C_n^\infty$ yield a new n-division P^∞ of C with $\widetilde{w}(P^\infty)=\lim_{k\to\infty}\widetilde{w}(P_k)=\widetilde{w}_n(C)$, which implies that P^∞ is optimal.

Let us now check that for $n=2$ any optimal division is balanced. Let P be an optimal 2-division of C into subsets C_1, C_2, and assume that P is not balanced, say $w(C_1)<w(C_2)$. By Lemma 7 and the continuity of the width functional, we can find a 2-division P^t of C with subsets C_1^t, C_2^t such that $w(C_1)<w(C_1^t)\leqslant w(C_2^t)$. Then $\widetilde{w}(P^t)=w(C_1^t)>w(C_1)=\widetilde{w}(P)$, which contradicts the optimality of P. Therefore, P must be balanced.

To finish the proof, we will now show the existence of a balanced optimal n-division by induction on the number of subsets $n\geqslant 2$. If $n=2$, it has been already shown that any optimal 2-division is balanced. Fix now $n>2$, and assume that for any convex body in $\mathbb{R}^d$, there exists a balanced optimal m-division for $m<n$. Let Q be an optimal n-division of C, whose existence we have already shown. Let H be a hyperplane cut from Q dividing C into two convex regions E_1, E_2, and let Q_i be the n_i-division of E_i induced by Q, $i=1,2$, with $n=n_1+n_2$. Taking into account the induction hypothesis, there exists a balanced optimal n_i-division Q_i' of E_i, $i=1,2$. Observe that $\widetilde{w}(Q_i)\leqslant\widetilde{w}(Q_i')=\widetilde{w}_{n_i}(E_i)$, $i=1,2$, and so

$$\widetilde{w}_n(C)=\widetilde{w}(Q)=\min\{\widetilde{w}(Q_1),\widetilde{w}(Q_2)\}\leqslant\min\{\widetilde{w}(Q_1'),\widetilde{w}(Q_2')\}=\widetilde{w}(Q'),\tag{9}$$

where Q' is the n-division of C determined by Q_1', Q_2'. Observe that (9) implies that Q' is also an optimal n-division of C. On the one hand, if $\widetilde{w}(Q_1')=\widetilde{w}(Q_2')$, then Q' is balanced by construction, and the statement holds. On the other hand, if (say) $\widetilde{w}(Q_1')>\widetilde{w}(Q_2')$, let H^t be the hyperplane parallel to H at distance $t\geqslant 0$ from E_2, and let E_1^t, E_2^t be the two convex regions into which H^t divides C (which satisfy $E_2\subset E_2^t$, $E_1^t\subset E_1$). Observe that

$$\widetilde{w}_{n_1}(E_1^0)=\widetilde{w}_{n_1}(E_1)=\widetilde{w}(Q_1')>\widetilde{w}(Q_2')=\widetilde{w}_{n_2}(E_2)=\widetilde{w}_{n_2}(E_2^0)$$

and

$$\widetilde{w}_{n_1}(E_1^{t_1})=0<\widetilde{w}_{n_2}(C)=\widetilde{w}_{n_2}(E_2^{t_1}),$$

for certain $t_1>0$ large enough. From Lemma 8 and the existence of optimal divisions previously proved, there is $t_0>0$ such that $\widetilde{w}_{n_1}(E_1^{t_0})=\widetilde{w}_{n_2}(E_2^{t_0})$.

By considering a balanced optimal n_i-division $Q_i^{t_0}$ of $E_i^{t_0}$, which exists by the induction hypothesis and satisfies $\widetilde{w}(Q_i^{t_0})=\widetilde{w}_{n_i}(E_i^{t_0})$, $i=1,2$, it follows that the n-division Q^{t_0} of C, determined by $Q_1^{t_0}$, $Q_2^{t_0}$, is balanced by construction, and also optimal: since $E_2\subset E_2^{t_0}$, we have

$$\begin{aligned}\widetilde{w}(Q')&=\min\{\widetilde{w}(Q_1'),\widetilde{w}(Q_2')\}=\widetilde{w}(Q_2')=\widetilde{w}_{n_2}(E_2)\leqslant\widetilde{w}_{n_2}(E_2^{t_0})\\&=\widetilde{w}(Q_2^{t_0})=\min\{\widetilde{w}(Q_1^{t_0}),\widetilde{w}(Q_2^{t_0})\}=\widetilde{w}(Q^{t_0}),\end{aligned}$$

and so equality above must hold to avoid a contradiction with the optimality of Q'. $\square$

EXAMPLE 3. The optimal division of a given convex body for this problem is not unique in general, as it can be seen with the following example. Consider an isosceles triangle T of sides l_1, l_2, l_3 (being l_1 the shortest one), *relatively close* to be equilateral (for instance, let the side lengths be $4,5,5$), and let v_2 be one of the endpoints of l_1, as shown in Figure 14. Let P_T be the 2-division of T determined by the bisector of the angle at the vertex v_2.

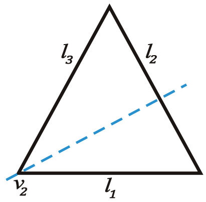

Figure 14: *For the isosceles triangle T, the 2-division P_T is optimal*

Call p the intersection point of that bisector and the opposite side l_2. Then, P_T is balanced and $\widetilde{w}(P_T) = dist(p,l_1) = dist(p,l_3)$. We claim that P_T is also optimal. Let P be an arbitrary 2-division of T into subsets C_1, C_2. It can be checked that one of the subsets C_i (or its symmetral with respect to the bisector of the side l_1) will be contained in one of the two slabs providing $\widetilde{w}(P_T)$. This implies that

$$\widetilde{w}(P) = \min\{w(C_1), w(C_2)\} \leqslant w(C_i) \leqslant \widetilde{w}(P_T),$$

which yields the optimality of P_T. Now, let q be the point on l_3 with the same height as p, see Figure 15.

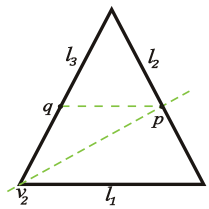

Figure 15: *Any 2-division of T given by a segment joining p and a point of $\overline{qv_2}$ is optimal*

Then, any 2-division Q of T given by a segment joining p with a point of the segment $\overline{qv_2}$ is also optimal, since $\widetilde{w}(Q)$ will coincide with $\widetilde{w}(P_T)$ by construction.

We will now focus on obtaining lower and upper bounds for the optimal value for this problem.

THEOREM 10. *Let C be a convex body in $\mathbb{R}^d$. Then,*

$$\frac{w(C)}{n} \leqslant \widetilde{w}_n(C) \leqslant \min\left\{w(C), \frac{D(C)}{2}\right\}. \tag{10}$$

Proof. By Theorem 9, there exists a balanced optimal n-division P of C into subsets $C_1,\dots,C_n$. By considering the slabs determining $w(C_i)$, $i=1,\dots,n$, and applying Lemma 3, it follows that

$$w(C) \leqslant \sum_{i=1}^{n} w(C_i) = n\,\widetilde{w}_n(C),$$

which gives the left-hand side inequality in (10). Let now H be a hyperplane cut from P dividing C into two convex regions. Consider two hyperplanes H_1, H_2 parallel to H and tangent to ∂C. Let p_i be a point from $\partial C \cap H_i$, and let B_i be the slab delimited by H and H_i, $i=1,2$. Observe that any subset C_i will be contained in either B_1 or in B_2, and so we can assume, without loss of generality, that $C_i \subset B_i$, $i=1,2$. Then,

$$D(C) \geqslant dist(p_1,p_2) \geqslant w(B_1)+w(B_2) \geqslant w(C_1)+w(C_2) = 2\,\widetilde{w}(P) = 2\,\widetilde{w}_n(C).$$

As a consequence, $\widetilde{w}_n(C) \leqslant D(C)/2$, which together with Lemma 1 gives the right-hand side inequality in (10). □

We point out that if $n=2$, the equality in the left-hand side of (10) is attained when C is a constant-width body. Sharpness of the right-hand side of (10) is shown in Example 4.

EXAMPLE 4. Let C be a (sufficiently) long and narrow orthotope in $\mathbb{R}^d$. It is possible to construct a balanced n-division P of C, by using $n-1$ parallel hyperplanes, such that the width of all the subsets given by P equals $w(C)$, see Figure 16. Thus, Theorem 10 implies that P is optimal and $\widetilde{w}_n(C) = w(C)$. Additionally, let B be a planar ball of radius $r>0$, and let P be one of the (balanced) n-divisions of B from Figure 17, for $n=2,3,4$. Then $\widetilde{w}(P)=r$, which in particular gives that P is optimal and $\widetilde{w}_n(B) = D(B)/2$ (the analogous example holds for a ball in $\mathbb{R}^d$ and $n \leqslant 2^d$).

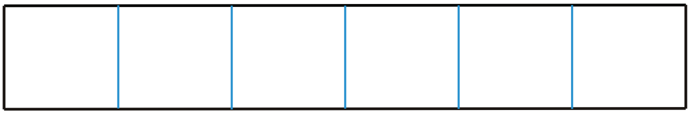

Figure 16: *An optimal 6-division of a long and narrow rectangle ($d=2$)*

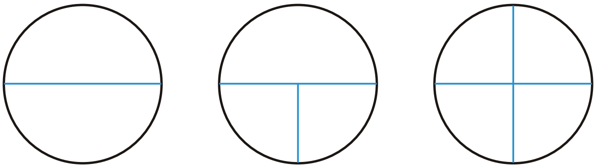

Figure 17: *Optimal n-divisions of a planar ball for $n=2,3,4$*

4.3. Max-min problem for the inradius

The Max-min problem for the inradius shares several features with the Max-min problem for the width from Subsection 4.2: Theorem 11 follows by using analogous techniques as in the width case. Moreover, the optimal value for this problem when considering divisions of a convex body C into $n=2$ subsets can be expressed in terms of the optimal value for the Max-min problem for the width of a certain rounded body of C (see Theorem 12). We point out that several issues remain open for this problem, such as refining the bounds for the optimal value or deciding whether any optimal division is balanced.

THEOREM 11. *Let C be a convex body in $\mathbb{R}^d$. Then, there exists a balanced optimal n-division of C for the Max-min problem for the inradius. Moreover,*

$$\widetilde{I}_n(C) \geqslant I(C)/n. \tag{11}$$

Proof. For the existence of optimal divisions, it can be checked that the proof of Theorem 9 still holds if we consider the inradius instead of the minimal width. Let us see that we can always find a balanced optimal n-division by induction on the number n of subsets.

For $n=2$, let P be an optimal 2-division of C into subsets C_1, C_2, determined by a hyperplane H. Assume that P is not balanced, say $I(C_1) < I(C_2)$. For each $t \geqslant 0$, consider the hyperplane H^t parallel to H at distance t from C_1, and let P^t be the 2-division of C into subsets C_1^t, C_2^t determined by H^t. Since $I(C_1^0) = I(C_1) < I(C_2) = I(C_2^0)$, and $I(C_1^{t_1}) = I(C) > 0 = I(C_2^{t_1})$, for certain large enough $t_1 > 0$, it follows by continuity that there exists $t_0 \in (0,t_1)$ such that $I(C_1^{t_0}) = I(C_2^{t_0})$. Thus, the 2-division P^{t_0} of C is balanced and satisfies that $\widetilde{I}(P^{t_0}) = I(C_1^{t_0}) \geqslant I(C_1) = \widetilde{I}(P)$, since $C_1 \subset C_1^{t_0}$. This implies that P^{t_0} is also optimal, as desired. And for an arbitrary $n > 2$, we can proceed as in the proof of Theorem 9 in order to obtain a balanced optimal n-division of C.

Finally, let P be a balanced optimal n-division of C into subsets $C_1,\dots,C_n$. Lemma 4 implies that

$$I(C) \leqslant \sum_{i=1}^{n} I(C_i) = \sum_{i=1}^{n} \widetilde{I}_n(C) = n\widetilde{I}_n(C),$$

which yields the statement. $\square$

We will now focus on estimating the optimal value of a given convex body C for this problem when considering divisions into $n=2$ subsets. The two following technical results will lead us to Theorem 12, which establishes implicitly the optimal value for this problem when $n=2$, following the same spirit as in [2, Th. 1] for the min-Max problem for the inradius, see Subsection 3.3.

LEMMA 9. *Let C be a convex body in $\mathbb{R}^d$, and let P be a 2-division of C into subsets C_1, C_2. Assume that $C = C^\rho$, where $\rho = I(C_1)$. Then, $w(C_1) = 2\rho$.*

Proof. Recall that $2I(E) \leqslant w(E)$ holds for any convex set E in $\mathbb{R}^d$. Thus, to prove the statement it suffices to check that $w(C_1) \leqslant 2\rho$.

Let B_ρ be an inball of C_1 and label H as the hyperplane providing the 2-division P. If there are two points in $B_\rho \cap \partial C_1$ which are antipodal in B_ρ, then we are done. Assume now the contrary: we cannot find a pair of antipodal points in $B_\rho \cap \partial C_1$. We are going to see that this assumption yields a contradiction.

Denote by $S_\rho = \partial B_\rho$ and let $D = (\partial C_1 \cap S_\rho) \setminus H \subset S_\rho$. We claim that D is a convex subset of S_ρ, that is, for any two points p_1, $p_2 \in D$, the shortest great-circle arc γ of S_ρ joining p_1 and p_2 is contained in D.

Indeed, let $p_3 \in \gamma$, and call H_{p_i} the hyperplane tangent to B_ρ at p_i, $i = 1,2,3$. Note that H_{p_1}, H_{p_2} are supporting hyperplanes of C. Since $C = C^\rho$, if $p_3 \in int(C)$, then there exists a ball $B'_\rho \subset C$ of radius ρ containing p_3 as an interior point. But any such ball must intersect H_{p_1} or H_{p_2}, which is impossible and proves that $p_3 \in \partial C$. As a consequence, H_{p_3} is also a supporting hyperplane of C. If $p_3 \in H$, then $H_{p_3} = H$ contradicting the previous affirmation. Therefore $p_3 \in D$ and D is a convex subset of S_ρ containing no antipodal points.

In particular, D is contained in some open half-sphere $S^+_\rho \subset S_\rho$ (see [10, Le. 3.4][1]). Since $D \subset S^+_\rho$ and B_ρ is an inball of C_1, there necessarily exists $p_0 \in H \cap S_\rho$. By assumption, p_0 is not antipodal to any point in D. But this means that $\partial C_1 \cap S_\rho = D \cup \{p_0\}$ is contained in some open half-sphere of S_ρ, contradicting that B_ρ is an inball of C_1. □

LEMMA 10. *Let C be a convex body in $\mathbb{R}^d$. Assume that $C = C^\rho$, where $\rho = \widetilde{I}_2(C)$. Then, $\widetilde{w}_2(C) = 2\rho$.*

Proof. Let P be a balanced optimal 2-division of C for the Max-min problem for the width, with subsets C_1, C_2, in view of Theorem 9. Then, $\widetilde{I}(P) \leqslant \rho$, and so $I(C_i) \leqslant \rho$ for some $i \in \{1,2\}$. This implies that $C^\rho \subseteq C^{I(C_i)}$, and since $C = C^\rho$, it follows that $C = C^{I(C_i)}$. By using now Lemma 9, we conclude that $w(C_i) = 2I(C_i) \leqslant 2\rho$, which gives $\widetilde{w}_2(C) = \widetilde{w}(P) = w(C_i) \leqslant 2\rho = 2\widetilde{I}_2(C)$. The reverse inequality is an immediate consequence of the general property $2I(E) \leqslant w(E)$, for any convex body $E \subset \mathbb{R}^d$, which finishes the proof. □

THEOREM 12. *Let C be a convex body in $\mathbb{R}^d$. Then, $\widetilde{I}_2(C)$ is the unique number $\widetilde{\rho}$ such that*

$$2\widetilde{\rho} = \widetilde{w}_2(C^{\widetilde{\rho}}). \tag{12}$$

Proof. The existence and uniqueness of $\widetilde{\rho}$ can be proved as in [2, § 2], taking into account the monotonic character of the continuous functions $\rho \mapsto 2\rho$ and $\rho \mapsto \widetilde{w}_2(C^\rho)$.

We will now check that $\widetilde{I}_2(C)$ satisfies (12). Call $\tau := \widetilde{I}_2(C)$ for simplicity. On the one hand, let P be a balanced optimal 2-division of C for the Max-min type problem for the inradius, with subsets C_1, C_2 (see Theorem 11). Observe that P will induce a

[1] Although the result in [10] is stated for $d = 3$, the proof can be mimicked in any dimension

2-division P' of C^τ into subsets C_1', C_2'. Moreover, it follows that $I(C_i') = I(C_i) = \tau$ for $i = 1, 2$, and therefore

$$\widetilde{I}_2(C^\tau) \geqslant \widetilde{I}(P') = \tau.$$

Since $2I(E) \leqslant w(E)$ holds for any convex body E in $\mathbb{R}^d$, then

$$2\widetilde{I}_2(C^\tau) \leqslant \widetilde{w}_2(C^\tau).$$

The two above inequalities give that

$$2\tau \leqslant \widetilde{w}_2(C^\tau). \tag{13}$$

On the other hand, $\widetilde{I}_2(C^\tau) \leqslant \tau$ since $C^\tau \subseteq C$, and so C^τ will coincide with its associated $\widetilde{I}_2(C^\tau)$-rounded body. Then, by applying Lemma 10 to C^τ we have that $\widetilde{w}_2(C^\tau) = 2\widetilde{I}_2(C^\tau) \leqslant 2\tau$, which completes the proof, taking into account (13). $\square$

5. Related problems

Apart from the questions which have not been completely solved in the previous sections, the corresponding min-Max and Max-min type problems can be considered with other magnitudes F, as the *circumradius* (which represents the smallest radius of a ball containing the original convex body) or the *perimeter*. In this first case, it can be proved for instance that not all optimal divisions for the Max-min type problem are balanced. The second case, with the additional restriction that the subsets of the divisions enclose a prescribed quantity of volume, is related to the isoperimetric tilings problem [8, Problem C15].

A nice variant of the problems treated in this work is described in [2, Re. 3]: for an n-division P of a convex body C into subsets $C_1, \dots, C_n$, we can consider the quantity $F(C_1) + \dots + F(C_n)$, where F is one fixed geometric magnitude. In this setting, the question is determining the n-division of C minimizing (or maximizing) that quantity, as well as the corresponding optimal value. Lemma 3 (by T. Bang) and Lemma 4 (by V. Kadets) provide lower bounds for the optimal values in the case of the minimal width and the inradius, respectively.

Finally, a possible generalization of our work can be posed by considering general divisions, not necessarily determined by hyperplane cuts. In that case, the subsets provided by those divisions may not be convex, yielding more complicated situations.

REFERENCES

[1] T. BANG, *A solution of the "plank problem"*, Proc. Amer. Math. Soc., **2**: 990–993, 1951.

[2] A. BEZDEK AND K. BEZDEK, *A solution of Conway's fried potato problem*, Bull. London Math. Soc., **27** (5): 492–496, 1995.

[3] A. BEZDEK AND K. BEZDEK, *Conway's fried potato problem revisited*, Arch. Math. (Basel), **66** (6): 522–528, 1996.

[4] A. CAÑETE AND B. GONZÁLEZ MERINO, *On the isodiametric and isominwidth inequalities for planar bisections*, Rev. Mat. Iberoam., **37** (4): 1247–1275, 2021.

[5] A. CAÑETE, U. SCHNELL, AND S. SEGURA GOMIS, *Subdivisions of rotationally symmetric planar convex bodies minimizing the maximum relative diameter*, J. Math. Anal. Appl., **435** (1): 718–734, 2016.

[6] A. Cañete and S. Segura Gomis, *Bisections of centrally symmetric planar convex bodies minimizing the maximum relative diameter*, Mediterr. J. Math., **16**, Paper No. 151, 19 pp., 2019.

[7] F. Chin, J. Snoeyink, and C. A. Wang, *Finding the medial axis of a simple polygon in linear time*, Discrete Comput. Geom., **21**, (3): 405–420, 1999.

[8] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved Problems in Geometry*, Springer, Berlin, 1991.

[9] F. Eisenbrand, *Integer programming and algorithmic geometry of numbers*, In *50 Years of Integer Programming*, paes 1958–2008, Springer, 2010.

[10] C. I. Grima and A. Márquez, *Computational geometry on surfaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[11] V. Kadets, *Coverings by convex bodies and inscribed balls*, Proc. Amer. Math. Soc., **133**, (5): 1491–1495, 2005.

[12] B. Kawohl and T. Lachand-Robert, *Characterization of Cheeger sets for convex subsets of the plane*, Pacific J. Math., **225**, (1): 103–118, 2006.

[13] C. Miori, C. Peri, and S. Segura Gomis, *On fencing problems*, J. Math. Anal. Appl., **300**, (2): 464–476, 2004.

[14] F. P. Preparata, *The medial axis of a simple polygon*, In *Mathematical foundations of computer science (Proc. Sixth Sympos., Tatranská Lomnica, 1977)*, pages 443–450, Lecture Notes in Comput. Sci., Vol. 53, 1977.

[15] E. Roldán-Pensado and P. Soberón, *A survey of mass partitions*, Bull. Amer. Math. Soc., **59**, (2): 227–267, 2022.

[16] R. Schneider, *Convex bodies: the Brunn-Minkowski theory. Second edition*, Cambridge University Press, Cambridge, 2014.

[17] E. Stredulinsky and W. P. Ziemer, *Area minimizing sets subject to a volume constraint in a convex set*, J. Geom. Anal., **7**, (4): 653–677, 1997.

[18] G. T. Toussaint, *Solving geometric problems with the rotating calipers*, In *Proceedings of the IEEE MELECON'83*, 1983.

(Received January 28, 2022)

Antonio Cañete
Departamento de Matemática Aplicada I and IMUS
Universidad de Sevilla
e-mail: `antonioc@us.es`

Isabel Fernández
Departamento de Matemática Aplicada I and IMUS
Universidad de Sevilla
e-mail: `isafer@us.es`

Alberto Márquez
Departamento de Matemática Aplicada I and IMUS
Universidad de Sevilla
e-mail: `almar@us.es`