Depósito de investigación de la Universidad de Sevilla

https://idus.us.es/

Esta es la versión aceptada del artículo publicado en:

This is an accepted manuscript of a paper published in:

Discrete and Continuous Dynamical Systems - Series S (DCDS-S) (2023): Abril 2023

DOI:

Copyright:

El acceso a la versión publicada del artículo puede requerir la suscripción de la revista.

Access to the published version may require subscription.

"This article has been published in a revised form in Discrete and Continuous Dynamical Systems - Series S (DCDS-S) Discrete and Continuous Dynamical Systems - Series S (DCDS-S) 10.3934/dcdss.2023086*. This version is free to download for private research and study only. Not for redistribution, re-sale or use in derivative Works".*

Manuscript submitted to AIMS Journal

doi:10.3934/xx.xxxxxxx

REMARKS ON THE CONTROLLABILITY OF PARABOLIC SYSTEMS WITH NON-DIAGONALIZABLE DIFFUSION MATRIX

Michel Duprez[1] Manuel González-Burgos[2] Diego A. Souza[*2]

[1]MIMESIS team, Inria Nancy - Grand Est and MLMS team, Université de Strasbourg, France

[2]Dpto. Ecuaciones Diferenciales y Análisis Numérico and IMUS, Universidad de Sevilla, Spain

(Communicated by Handling Editor)

Abstract. The distributed null controllability for coupled parabolic systems with non-diagonalizable diffusion matrices with a reduced number of controls has been studied in the case of constant matrices. On the other hand, boundary controllability issues and distributed controllability with non-constant coefficients for this kind of systems is not completely understood. In this paper, we analyze the boundary controllability properties of a class of coupled parabolic systems with non-diagonalizable diffusion matrices in the constant case and the distributed controllability of a 2×2 non-diagonalizable parabolic system with space-dependent coefficients. For the boundary controllability problem, our strategy relies on the moment method. For the distributed controllability problem, our findings provide positive and negative control results by using the Fattorini-Hautus test and a fictitious control strategy.

1. **Introduction.** Let $T > 0$ be a fixed time and $\omega := (a, b)$ be a non empty open subinterval of $(0, \pi)$. Hereafter, we shall use the notation $Q_T := (0, T) \times (0, \pi)$. We denote by $M^\star$ the conjugate transpose of a matrix M and by e_i the i-th element of the canonical basis of $\mathbb{R}^n$ ($n \in \mathbb{N}^\star$, with $n \geq 2$, will be fixed later). On the other hand, we also denote $(\cdot, \cdot)_{\mathbb{R}^n}$ (resp., $(\cdot, \cdot)_{\mathbb{C}^n}$) the scalar product in $\mathbb{R}^n$ (resp., the hermitian product in $\mathbb{C}^n$) and $|\cdot|$ is the euclidean norm in $\mathbb{R}^n$ or $\mathbb{C}^n$.

In this work, we consider the following controlled parabolic systems in which the diffusion matrix is non-diagonalizable

$$\begin{cases} y_t - Dy_{xx} + Ay = B1_\omega u & \text{in } Q_T, \\ y(\cdot, 0) = y(\cdot, \pi) = 0 & \text{in } (0, T), \\ y(0, \cdot) = y^0 & \text{in } (0, \pi) \end{cases} \tag{1}$$

and

$$\begin{cases} z_t - Dz_{xx} + Az = 0 & \text{in } Q_T, \\ z(\cdot, 0) = Bv, \quad z(\cdot, \pi) = 0 & \text{in } (0, T), \\ z(0, \cdot) = z^0 & \text{in } (0, \pi), \end{cases} \tag{2}$$

where $y^0 \in L^2(0, \pi; \mathbb{R}^n)$ and $z^0 \in H^{-1}(0, \pi; \mathbb{R}^n)$ are the initial data, $u \in L^2(Q_T; \mathbb{R}^m)$ ($m \in \mathbb{N}^\star$) is the distributed control, $v \in L^2(0, T; \mathbb{R}^m)$ is the boundary control, $A \in \mathcal{L}(\mathbb{R}^n)$ is a zero order coupling matrix, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ is a matrix through which

2020 *Mathematics Subject Classification.* Primary: 93B05, 93C20 ; Secondary: 35K40.

Key words and phrases. Parabolic systems, Null controllability, Nondiagonalizable diffusion, Method of moments, Boundary controls.

*Corresponding author: Diego A. Souza.

the controls act on the system and $D \in \mathcal{L}(\mathbb{R}^n)$ is a non-diagonalizable diffusion matrix satisfying an ellipticity condition given by

$$(D\xi, \xi)_{\mathbb{R}^n} \geq \beta|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,$$

with $\beta > 0$.

It is well-known that for any initial datum $y^0 \in L^2(0, \pi; \mathbb{R}^n)$ (respectively, $z^0 \in H^{-1}(0, \pi; \mathbb{R}^n)$) and any control $u \in L^2(Q_T; \mathbb{R}^m)$ (resp., $v \in L^2(0, T; \mathbb{R}^m)$), system (1) (resp., system (2)) admits a unique weak solution y (resp., a unique solution by transposition z) with the regularity

$$\begin{aligned} &y \in L^2(0, T; H_0^1(0, \pi; \mathbb{R}^n)) \cap \mathcal{C}^0([0, T]; L^2(0, \pi; \mathbb{R}^n)) \\ &(\text{resp. } z \in L^2(Q_T; \mathbb{R}^n) \cap \mathcal{C}^0([0, T]; H^{-1}(0, \pi; \mathbb{R}^n))). \end{aligned}$$

For more details, see [31, p. 102] and [21, Prop. 2.2].

Let us recall the different concepts of controllability for (1) and (2) that we study in the present paper:

- system (1) (resp., system (2)) is approximately controllable at time T if for any y^0 and y^T in $L^2(0, \pi; \mathbb{R}^n)$ (resp., z^0 and z^T in $H^{-1}(0, \pi; \mathbb{R}^n)$) and any $\varepsilon > 0$, there exists a control $u \in L^2(Q_T; \mathbb{R}^m)$ (resp., $v \in L^2(0, T; \mathbb{R}^m)$) such that the associated solution y (resp., z) to system (1) (resp., system (2)) satisfies
$$\|y(T, \cdot) - y^T\|_{L^2(0,\pi;\mathbb{R}^n)} \leq \varepsilon, \quad (\text{resp., } \|z(T, \cdot) - z^T\|_{H^{-1}(0,\pi;\mathbb{R}^n)} \leq \varepsilon).$$
- system (1) (resp., system (2)) is null controllable at time T if for any $y^0 \in L^2(0, \pi; \mathbb{R}^n)$ (resp., $z^0 \in H^{-1}(0, \pi; \mathbb{R}^n)$), there exists a control $u \in L^2(Q_T; \mathbb{R}^m)$ (resp., $v \in L^2(0, T; \mathbb{R}^m)$) such that the associated solution y (resp., z) to system (1) (resp., system (2)) satisfies
$$y(T, \cdot) = 0 \ \text{ in } (0, \pi), \quad (\text{resp., } z(T, \cdot) = 0 \ \text{ in } (0, \pi)).$$

Concerning the case of systems of ordinary differential equations under the form

$$\begin{cases} y_t + Ay = Bu & \text{in} \quad (0, T), \\ y(0) = y^0, \end{cases} \tag{3}$$

where $y^0 \in \mathbb{R}^n$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, it is well-known that the exact controllability for (3) is equivalent to the so-called *Kalman rank condition*

$$\text{rank}\,[A \,:\, B] = n, \tag{4}$$

where $[A \,:\, B] \in \mathcal{L}(\mathbb{R}^{nm}; \mathbb{R}^n)$ is the matrix given by

$$[A \,:\, B] := [B \,|\, AB \,|\, \cdots \,|\, A^{n-1}B].$$

This result was proved in [28].

Concerning systems of partial differential equations, precisely PDEs of parabolic type, the first results about null controllability of the heat equation ((1) and (2) with $n = 1$), have been established in the one-dimensional case through the moment method by H.O. Fattorini and D.L. Russell, see [20]. The distributed null controllability of the heat equation in the multi-dimensional case, has been established later, simultaneously, by G. Lebeau and L. Robbiano in [29], using local elliptic Carlerman estimates, and by A. Fursikov and O. Yu. Imanuvilov in [23], using global parabolic Carleman estimates. Using an extension method, it is possible to prove that the internal null controllability and the boundary null controllability are equivalent for the heat equation and, in general, for scalar parabolic problems (see, for instance, [7]). It is interesting to point out that, in the context of scalar parabolic partial differential equations, the controllability properties are valid for any distributed or boundary control domain and for any time $T > 0$, i.e., there is no minimal time for controllability and no geometric restrictions on the internal or boundary control domains, contrarily to the wave equation and transport equation.

To the authors' knowledge, there are not many works devoted to the controllability of coupled parabolic systems. Unlike the scalar case, in [21], it was proved that the equivalence between the controllability of the systems of parabolic equations (1) and (2) does not hold (details will be provided below) and their controllability issues should be analyzed separably. Almost all the papers in the literature are devoted to the controllability of parabolic systems with distributed controls, acting on a small open region ω of the domain $\Omega \subset \mathbb{R}^N$; see, for example, [26, 4, 5, 22, 32]. About another kind of systems, for instance, some boundary controllability results for a system of wave equations and distributed controllability results for hyperbolic systems of first-order have been obtained in [2, 30, 1] and [3, 14], respectively.

Let us describe the state of the art in the case of parabolic systems with diagonalizable and non-diagonalizable diffusion matrices.

Diagonalizable diffusion matrices: In [5], the authors have proved, in the case of diagonalizable diffusion matrices D, that system (1) (constant coefficients and distributed controls) is null (resp., approximate) controllable at time T if and only if

$$\operatorname{rank}\,[\mu_k D + A \,:\, B] = n \quad \text{for all } k \in \mathbb{N}^\star, \tag{5}$$

where μ_k are the eigenvalues of $-\partial_{xx}$ in $(0, \pi)$ with homogeneous Dirichlet boundary conditions. When the matrix D is equal to the identity, conditions (4) and (5) are equivalent. It is surprising to obtain the same condition as in finite dimension. We refer to [4] for a study in the case of time dependent coupling matrices.

The case of coupling matrices depending on (t, x) is more intricate but in some particular parabolic systems it is possible to prove a null controllability result (cascade systems, see [26]). Let us describe the existing results for system (1) when $n = 2$, $m = 1$, $D = \operatorname{diag}(d_1, d_2)$ $(d_1, d_2 > 0)$ and

$$A = A(t, x) \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^2)) \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

i.e., for the 2×2 system:

$$\begin{cases} \partial_t y_1 - d_1 \partial_{xx} y_1 + a_{11} y_1 + a_{12} y_2 = 0 & \text{in } Q_T, \\ \partial_t y_2 - d_2 \partial_{xx} y_2 + a_{21} y_1 + a_{22} y_2 = 1_\omega u & \text{in } Q_T, \\ y_1(\cdot, 0) = y_2(\cdot, 0) = y_1(\cdot, \pi) = y_2(\cdot, \pi) = 0 & \text{in } (0, T), \\ y_1(0, \cdot) = y_1^0, \quad y_2(0, \cdot) = y_2^0 & \text{in } (0, \pi), \end{cases} \tag{6}$$

where $a_{ij} \in L^\infty(Q_T)$ are given functions $(1 \le i, j \le 2)$, $y^0 = (y_1^0, y_2^0)^\star \in L^2(0, \pi; \mathbb{R}^2)$ is the initial datum and $u \in L^2(Q_T)$ is the distributed control. Under the generic assumption

$$a_{12} \ge a_0 > 0 \text{ or } -a_{12} \ge a_0 > 0 \text{ in } (0, T) \times \omega_0, \quad \forall i : 2 \le i \le m. \tag{7}$$

for an open set $\omega_0 \subset \omega$, in [26] the authors prove a null controllability result at time T for system (6) which is independent of d_1, d_2, a_{11}, a_{21} and a_{22}.

The situation strongly changes if $\operatorname{supp} a_{12} \cap \omega = \emptyset$ (see for instance [13], [9] and [10]) or the coupling term $a_{12} y_2$ is changed by a first order coupling term (see [33, 18, 19]). In the first case, system (6) could have a minimal controllability time $T_0 \in [0, +\infty]$ such that if $T < T_0$ the system is not null controllable at time T and it is if $T > T_0$ (see also [17]). Moreover, this minimal time depends on the position of the open control set ω with respect to $\operatorname{supp} a_{12}$ (see [10]). In the second case, the null controllability result could depend on the coefficient a_{11} and also on the position of ω (see [18] and [19]).

Concerning the boundary controllability of systems of parabolic equations when D is a positive multiple of the identity matrix, i.e., $D = dI_n$, with $d > 0$, and $m < n$, the first results has been obtained in [21] in the case $n = 2$ and $m = 1$. In this paper, the authors prove that system (2) is approximately and null controllable at time $T > 0$ if and only if A and B satisfy the algebraic Kalman condition (4) and

$$\nu_1 - \nu_2 \neq d(\mu_j - \mu_k), \quad \forall k, j \in \mathbb{N}^\star \quad \text{with } k \neq j,$$

where $\nu_1, \nu_2 \in \mathbb{C}^2$ are the eigenvalues of A and $\mu_k := k^2$ are the eigenvalues of the Dirichlet-Laplace operator in $(0, \pi)$. The above condition shows the different nature of the null controllability problem for systems (1) and (2). When $D = dI_n$, with $d > 0$, the null controllability of system (1) is equivalent to condition (5) (which, in fact, is equivalent to (4) when D is a multiple of the identity matrix; see [5] or [4]).

The boundary null controllability of system (2) when $D = dI_n$ $(d > 0)$ has been generalized in [6] to the case $n \geq 2$ and $m \in \mathbb{N}^\star$. The authors prove that system (2) is null (resp., approximate) controllable at time T if and only if

$$\operatorname{rank}\,[\mathcal{L}_k \,:\, \mathcal{B}_k] = nk \ \text{ for all } k \in \mathbb{N}^\star,$$

where

$$\mathcal{B}_k := \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \text{ and } \mathcal{L}_k = \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_k \end{bmatrix},$$

with $L_k := \mu_k I_n + A$ (see [6, Theorem 1.1]).

Remark 1.1. In particular, in [21] and [6], the following property is proved ($D = dI_n$, with $d > 0$): *"Assume that condition* (4) *holds (or equivalently, assume system* (1) *is null controllable at time $T > 0$). Then, there exists a closed subspace $\mathcal{X}_0 \subset H^{-1}(0, \pi; \mathbb{R}^2)$, with finite codimension, which satisfies the following property: given $y^0 \in H^{-1}(0, \pi; \mathbb{R}^2)$, there exists a control $v \in L^2(0, T)$ such that the solution y of system* (2) *satisfies $y(T, \cdot) = 0$ in $(0, \pi)$ if and only if $y^0 \in \mathcal{X}_0$."*

Thus, we deduce that if system (1) is null controllable at time $T > 0$, then system (2) is also null controllable at time $T > 0$, apart from a finite-dimensional space.

The case where D is a diagonal matrix different from a multiple of the identity or the case in which the coupling matrices depend on the spatial variable are more delicate and new phenomena in the parabolic setting arise. For instance, a minimal time for the null controllability can appear, see [8], and [9].

Non-diagonalizable diffusion matrices: In the case of non-diagonalizable diffusion matrices D, only partial results about controllability of systems (1) or (2) have been established. In fact, in the distributed control setting, these results have been established in any spatial dimension for uniform elliptic time-independent operators L. More precisely:

- In [22] and under the following condition

$$\text{The dimensions of the } \textit{Jordan blocks} \text{ of the canonical form of } D \text{ are } \leq 4, \tag{8}$$

  the authors prove that system (1) is null (resp., approximately) controllable at time $T > 0$ with distributed controls if and only if (5) holds. The technical condition (8) is a restriction due to the method used to provide the characterization (global Carleman estimates for scalar parabolic operators; see [22, Remark 2.5 and Section 5]). In particular, under condition (8), the authors prove a general result of approximate and null controllability at time $T > 0$ for (1) and (2) if $m \geq n$ and condition $\operatorname{rank} B = n$ holds. This general result is also valid in the case of coupling matrices $A = A(t, x)$ which depend on t and x or uniform parabolic operators $L = L(t)$ depending on t.
- In [32], the authors provide a complete answer for the problem of controllability of system (1) in the constant case without imposing any extra assumption on the Jordan blocks of D. In fact, they prove that system (1) is null (resp., approximately) controllable at time $T > 0$ if and only if the constant matrices D, A and B satisfy the Kalman condition (5) (see [32] for more details). The approach followed in [32] (the Lebeau-Robbiano strategy together with a precise study of the cost of controllability for linear ordinary differential equations) cannot be applied to system (2). However, as a consequence of their controllability results, it is not difficult to deduce that system (2) is approximately and null controllable at any time T when $m \geq n$ and condition $\operatorname{rank} B = n$ holds.
- Finally, in [25] the authors study the boundary null controllability of a one-dimensional phase field system of Caginalp type which is a model describing the transition between the solid and liquid phases in solidification/melting processes of a material occupying the interval $(0, \pi)$. To this end, the authors prove the boundary null controllability of a linear 2×2 parabolic system with a non-diagonalizable diffusion matrix and a scalar control.

The general goal of this paper is to study the controllability properties of coupled parabolic systems with non-diagonalizable diffusion matrices. To this end, we shall consider two examples of parabolic systems whose controllability properties cannot be obtained as a consequence of the controllability results proved in [22] and [32].

First, we will consider system (2) when the matrices D, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathbb{R}^n$ $(n \geq 2)$ are given by the expressions

$$D = \begin{bmatrix} d & 1 & 0 & \cdots & 0 \\ 0 & d & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d & 1 \\ 0 & 0 & \cdots & 0 & d \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \alpha & 0 & \cdots & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{9}$$

for constants $d \geq 1$ and $\alpha \in \mathbb{R}$. Observe that since (2) is a boundary controllability problem, its controllability properties cannot be deduced from [32]. On the other hand, when $n \geq 5$, since D does not satisfy condition (8) we cannot apply the results in [22], not even if $m \geq n$ and $\operatorname{rank} B = n$.

Remark 1.2. It is not difficult to see that, when $d \geq 1$, the matrix D is positive definite. We can conclude that systems (1) and (2) are well-posed when, resp., $(y^0, u) \in L^2(0, \pi; \mathbb{R}^n) \times L^2(Q_T)$ and $(z^0, v) \in H^{-1}(0, \pi; \mathbb{R}^n) \times L^2(0, T)$.

Secondly, we will consider problem (1) in the case in which A is a matrix depending on x. To be precise, consider the following system

$$\begin{cases} \partial_t y_1 - \partial_{xx} y_1 + q(x) y_1 = \partial_{xx} y_2 & \text{in } Q_T, \\ \partial_t y_2 - \partial_{xx} y_2 = 1_\omega u & \text{in } Q_T, \\ y_1(\cdot,0) = y_2(\cdot,0) = y_1(\cdot,\pi) = y_2(\cdot,\pi) = 0 & \text{in } (0,T), \\ y_1(0,\cdot) = y_1^0, \quad y_2(0,\cdot) = y_2^0 & \text{in } (0,\pi), \end{cases} \tag{10}$$

where $q \in \mathcal{C}^\infty([0,\pi])$ is a given function, $y^0 = (y_1^0, y_2^0)^\star \in L^2(0,\pi;\mathbb{R}^2)$ is the initial datum and $u \in L^2(Q_T)$ is a scalar control. Observe that system (10) has the same structure as system (1) with a coupling matrix A depending on x. To be precise, $n = 2$, $m = 1$, D given in (9), with $d = 1$, and $A = q(x)A_0$ with

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Again, D is a definite positive matrix and, therefore, for any $y_1^0, y_2^0 \in L^2(0,\pi)$ and $u \in L^2(Q_T)$, system (10) has a unique solution $y \in L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2)) \cap \mathcal{C}^0([0,T];L^2(0,\pi;\mathbb{R}^2))$ which depends continuously on the data.

Remark 1.3. When $n = 2$ and q is a constant function, i.e., $q(x) = q_0$ for any $x \in (0,\pi)$, with $q_0 \in \mathbb{R}$, conditions (5) and (8) hold with $A = q_0 A_0$. In fact, the controllability matrix $[\mu_k D + A : B]$ does not depend on q_0. Thus, we can apply the results in [22] and [32] and deduce that system (10) is approximately and null controllable at any time $T > 0$. Nevertheless, these results in [22] and [32] cannot be applied when A depends on x. To our knowledge, the controllability properties of system (1) with coupling matrices depending on x are completely open.

Let us now present our first main result concerning system (2):

Theorem 1.4. *Let us consider the matrices $D, A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathbb{R}^n$ given by (9), with $d \geq 1$ and $\alpha \in \mathbb{R}$. When $\alpha > 0$, we assume, in addition, that n is odd. Then,*

1. *If $\alpha < 0$, system (2) is null controllable at time T if and only if the family of eigenvalues of the operator $L^\star := -D^\star \partial_{xx} + A^\star$ has geometric multiplicity equal to 1 and*
$$\frac{\sqrt{|\alpha|}}{d^{n/2}} \notin \mathbb{N}^\star, \tag{11}$$
if n is odd.
2. *If $\alpha > 0$ (n is odd), system (2) is null controllable at time T if and only if the family of eigenvalues of the operator $L^\star := -D^\star \partial_{xx} + A^\star$ has geometric multiplicity equal to 1.*
3. *When $\alpha = 0$, system (2) is null controllable at time $T > 0$.*

As a consequence of the previous result, we also deduce:

Corollary 1.5. *Under assumptions of Theorem 1.4, one has that system (2) is approximately controllable at time T if and only it is null controllable at time T.*

Remark 1.6. Let us point out some consequences of Theorem 1.4 and Corollary 1.5:

- Unlike system (2), system (1) is approximately and null controllable at time $T > 0$ (even in the N-dimensional case) when D, A and B are given by (9). Indeed, it is easy to see that $[\mu_k D + A : B]$ is a squared matrix and
$$|\det[\mu_k D + A : B]| = \mu_k \mu_k^2 \dots \mu_k^{n-1} \neq 0, \quad \forall k \geq 1,$$
($\mu_k = k^2$, $k \geq 1$, are the eigenvalues of the Dirichlet-Laplace operator in $(0, \pi)$). Thus, condition (5) holds and system (1) is approximately and null controllable at time T. Again, this shows the important differences between the null controllability properties of systems (1) and (2).
- We will see that, under the assumptions of Theorem 1.4, the spectrum of the operator $L^\star := -D^\star \partial_{xx} + A^\star$ is simple, apart from a finite number of eigenvalues (see Proposition 2.7). Therefore, Theorem 1.4 implies that system (2) is always approximately and null controllable at any time $T > 0$ apart from a finite dimensional space of $H^{-1}(0, \pi; \mathbb{C}^n)$ (in particular, this is the case when n is odd).
- Condition (11) is related to the Fattorini-Hautus test for the operator $L^\star$ and is necessary to obtain the approximate controllability of system (2) when n is odd and the coefficient α of the matrix A (see (9)) is negative. Therefore, this condition is also necessary for the null controllability of (2). No additional condition on α and d is needed when n is odd and $\alpha > 0$.
- We will also see that, under assumptions of Theorem 1.4, we can obtain a positive result on null controllability at any time $T > 0$ for system
$$\begin{cases} y_t - D y_{xx} + A y = B \delta_{x_0} u & \text{in } Q_T, \\ y(\cdot, 0) = y(\cdot, \pi) = 0 & \text{in } (0, T), \\ y(0, \cdot) = y^0 & \text{in } (0, \pi), \end{cases} \tag{12}$$
where $u \in L^2(0, T)$, choosing very carefully the point $x_0 \in (0, \pi)$ where the control acts (δ_{x_0} is the Dirac distribution at point x_0). See Theorems 3.3 and 3.5.
- Observe that Theorem 1.4 and Corollary 1.5 do not provide any controllability result for system (2) when $n = 2p$, $p \in \mathbb{N}^\star$, and $\alpha > 0$. In this case, the operator $L^\star := -D^\star \partial_{xx} f + A^\star$ has, for any $k \geq 1$, exactly two real eigenvalues, given by
$$\lambda_{0,k} = dk^2 + \alpha^{1/n} k^{2-1/p} \quad \text{and} \quad \lambda_{p,k} = dk^2 - \alpha^{1/n} k^{2-1/p},$$
and $n-2$ complex eigenvalues (see Proposition 2.1). In this case, the real eigenvalues of $L^\star$ could concentrate. As a consequence, the controllability problem for system (2) could have a minimal time $T_0 \in [0, +\infty]$ of null controllability, which is indeed the condensation index of the sequence $\{\lambda_{0,k}, \lambda_{p,k}\}_{k \geq 1}$ (see [8], Figure 1, Figure 2 and Remark 2.9). In any case, condition (11) is also a necessary condition for the null controllability at time T for system (2) when $\alpha > 0$ and n is even (see (31)). The case $n = 2$ is simpler and will be completely analyzed in Theorem 1.7.
- The main advantage of the moment method, used in the present paper, is that it seems to be the best method to treat boundary null controllability problems with a reduced number of controls. Indeed, Theorem 1.4 is the first result dealing with the boundary controllability of non-diagonalizable systems of parabolic equations when $n \geq 3$ (see also [25] where a similar problem is considered when $n = 2$).

- One of the key ingredient for the proofs of Theorem 1.4, 1.7 and 3.5 is the spectral analysis of the operator $L^\star = -D^\star \partial_{xx} f + A^\star$ studied in Section 2. There, we prove that the eigenvectors of $L^\star$ form a Schauder basis and the controllability issues of system (2) are reduced to study the behavior of the eigenvalues of $L^\star$ (if there is or no condensation). Once this is established, in Section 3, for completeness, we apply the moment method to deduce the controllability results. Controllability results of parabolic systems with a minimal time of control in a more general framework have been obtain in, for instance, [8] or [11] (see [8, Theorem 2.5] or [11, Section 6.1, Theorem 1.11]). The main differences with respect the controllability result of Theorem 1.4 are: in [8] the authors add the hypothesis that eigenvectors of $L^\star$ form a Riesz basis. On the other hand, in the statement of [11, Theorem 1.11], the hypothesis on the behavior of the eigenvalues are different of the one proved here (see conditions (1.9) in [11] and (19)). □

Theorem 1.4 and Corollary 1.5 provide sufficient conditions on n and on the matrices $D, A \in \mathcal{L}(\mathbb{R}^n)$, given by (9), which guarantee the approximate and null controllability of system (2) at time $T > 0$. As we already mentioned in the previous remark, this theorem does not cover the case $n = 2p$, with $p \in \mathbb{N}^\star$, and $\alpha > 0$. In order to complete the study of the controllability problem for system (2), let us see the case $n = 2$ and $\alpha > 0$. One has:

Theorem 1.7. *Assume that $n = 2$ and let us consider the matrices $D, A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ given by (9), with $d \geq 1$ and $\alpha > 0$. Then, system (2) is null (resp., approximately) controllable in $H^{-1}(0, \pi; \mathbb{R}^2)$ at time $T > 0$ if and only if*

$$\frac{\sqrt{\alpha}}{d} \notin \mathbb{N}^\star.$$

Moreover, if $\sqrt{\alpha}/d \in \mathbb{N}^\star$, there exists a closed subspace $\mathcal{X} \subset H^{-1}(0, \pi; \mathbb{R}^2)$, with infinite codimension, which satisfies the following property: given $y^0 \in H^{-1}(0, \pi; \mathbb{R}^2)$, there exists a control $v \in L^2(0, T)$ such that the solution y of system (2) satisfies $y(T, \cdot) = 0$ in $(0, \pi)$ if and only if $y^0 \in \mathcal{X}$.

Remark 1.8. Even in the simplest case $n = 2$, Theorem 1.7 shows an important difference with respect to the results on boundary controllability proved in [21] and [6]: When $D = dI_n$, if system (1) is null controllable at a time $T_0 > 0$, then system (2) is also null controllable at any time $T > 0$, apart from a finite-dimensional space (see Remark 1.1). This property fails when D is not diagonalizable (even if D is equal to a unique Jordan block).

In order to obtain Theorems 1.4 and 1.7 we have used that the zero order coupling matrix A in system (2) is constant. Something similar occurs in [22] and [32]: the authors use in a fundamental way that the zero order coupling matrix A is constant. Let us now see that, if A depends on x, the controllability properties of system (1) can be strongly affected.

Our third and last result is related to the controllability properties of system (10). It reads as follows:

Theorem 1.9. *There exists a coefficient $q \in \mathcal{C}^\infty([0, \pi])$ such that:*

a) There exists an open interval $\omega \subset\subset (0, \pi)$ such that system (10) is never approximately controllable (then not null controllable) for any time $T > 0$.

b) There exists an open interval $\omega \subset\subset (0, \pi)$ such that system (10) is null controllable (then approximately controllable) at any time $T > 0$;

Remark 1.10. We can see system (10) as a cascade system (see system (6)) where the coupling term $a_{12}y_2$ has been changed by the second order term $a_{12}\partial_{xx}y_2$ and the coefficients are given by

$$d_1 = d_2 = 1, \quad a_{11} = q, \quad a_{12} = -1, \quad \text{and} \quad a_{21} = a_{22} = 0.$$

Observe that this choice of coefficients implies condition (7) and the null controllability of system (6) at any time $T > 0$ and for any control open set $\omega \subset (0, \pi)$. Comparing the controllability result for system (6) and system (10), Theorem 1.9 shows that the controllability results for coupled parabolic systems with non-diagonalizable diffusion matrices may be very different and the location of the control domain plays a key role. A similar result to Theorem 1.9 has been proved in [19] when the coupling term in system (10) is given by $\partial_x y_2$.

Remark 1.11. The proof of the Theorem 1.9 relies on the construction of a non constant coefficient q in $\mathcal{C}^\infty([0,\pi])$ which is constant in the interval $(5\pi/12, 7\pi/12)$. With this coefficient, we prove that system (10) is not approximately controllable at any time $T > 0$ when $\omega = (5\pi/12, 7\pi/12)$ and it is null controllable when we take $\omega \subset \operatorname{supp} q_x$ (i.e. q is not constant in ω). As stated in Remark 1.3, system (10) is null controllable at any time $T > 0$ when q is a constant function.

Remark 1.12. System (10) is null controllable at time $T > 0$ for any open interval $\omega \subset (0, \pi)$ and $q \in L^\infty(0, \pi)$ if two independent distributed controls are exerted, one on the right-hand side of the first equation and one on the right-hand side of the second equation, see [22].

The rest of the paper is organized as follows. In Section 2, we study the spectrum and eigenvectors associated to systems (1) and (2) and provide some properties needed to formulate the moment problem. In Section 3 we provide the proofs of Theorems 1.4 and 1.7. Finally, in Section 4 we prove Theorem 1.9.

2. **Preliminaries.**

2.1. **Spectral analysis.**

In the sequel, let us consider the following linear operator:

$$\begin{array}{rcl} L : D(L) \subset L^2(0,\pi;\mathbb{C}^n) & \longrightarrow & L^2(0,\pi;\mathbb{C}^n) \\ f & \mapsto & Lf := -D\partial_{xx} f + Af \end{array}$$

and its adjoint

$$\begin{array}{rcl} L^\star : D(L^\star) \subset L^2(0,\pi;\mathbb{C}^n) & \longrightarrow & L^2(0,\pi;\mathbb{C}^n) \\ f & \mapsto & L^\star f := -D^\star\partial_{xx} f + A^\star f, \end{array}$$

where $D(L) = D(L^\star) := H^2(0,\pi;\mathbb{C}^n) \cap H^1_0(0,\pi;\mathbb{C}^n)$ and $D, A \in \mathcal{L}(\mathbb{R}^n)$ are given by (9), with $n \geq 2$, $d \geq 1$ and $\alpha \in \mathbb{R}$.

It is well known that the operator $-\partial_{xx} : H^2(0,\pi;\mathbb{C}) \cap H^1_0(0,\pi;\mathbb{C}) \longrightarrow L^2(0,\pi;\mathbb{C})$ admits a sequence of positive eigenvalues and a sequence of normalized eigenfunctions $\{w_k\}_{k\geq 1}$, which is a Hilbert basis of $L^2(0,\pi)$, given by

$$\mu_k := k^2 \quad \text{and} \quad w_k(x) := \sqrt{\frac{2}{\pi}}\sin(kx), \quad x \in (0,\pi),\ k \geq 1.$$

Concerning the operator $L^\star$, we have the following description of its spectrum:

Proposition 2.1. *Let us consider the matrices $D, A \in \mathcal{L}(\mathbb{R}^n)$ given by (9), with $n \geq 2$, $d \geq 1$ and $\alpha \in \mathbb{R}$. Then, the following assertions hold:*

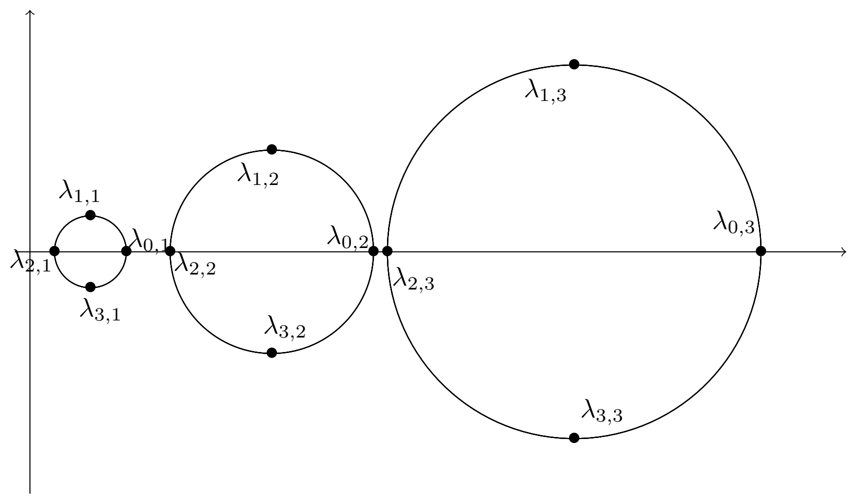

FIGURE 1. $\sigma(L^*)$ for $d = 2$, $n = 4$ and $\alpha = 2$

a) If $\alpha > 0$ then the spectrum of $L^\star$ is given by

$$\sigma(L^\star) = \left\{\lambda_{j,k} := dk^2 + \alpha^{1/n}k^{2-\frac{2}{n}}e^{\frac{2\pi j}{n}i} : k \in \mathbb{N}^\star, j \in \{0, ..., n-1\}\right\},$$

and, if $\alpha < 0$ then the spectrum of $L^\star$ is given by

$$\sigma(L^\star) = \left\{\lambda_{j,k} := dk^2 + |\alpha|^{1/n}k^{2-\frac{2}{n}}e^{\frac{(2j+1)\pi}{n}i} : k \in \mathbb{N}^\star, j \in \{0, ..., n-1\}\right\},$$

(see an example when $\alpha > 0$ in Figures 1 and 2). Moreover, for all $k \in \mathbb{N}^\star$ and $j \in \{0, \dots, n-1\}$, an eigenvector of $L^\star$, associated to the eigenvalue $\lambda_{j,k}$, is given by

$$\Phi_{j,k}(x) := V_{j,k}w_k(x),$$

where

$$V_{j,k} := \frac{\left(c^l_{j,k}\right)_{1\leq l\leq n}}{\left|\left(c^l_{j,k}\right)_{1\leq l\leq n}\right|} \quad \textit{with} \quad c^l_{j,k} = \begin{cases} \left[\alpha^{-\frac{1}{n}}k^{\frac{2}{n}}e^{\frac{-2\pi j}{n}i}\right]^{l-1} & \textit{if } \alpha > 0, \\ \left[|\alpha|^{-\frac{1}{n}}k^{\frac{2}{n}}e^{-\frac{(2j+1)\pi}{n}i}\right]^{l-1} & \textit{if } \alpha < 0. \end{cases} \tag{13}$$

b) If $\alpha = 0$ then the spectrum of $L^\star$ is given by

$$\sigma(L^\star) = \left\{\lambda_k := dk^2 : k \in \mathbb{N}^\star\right\}.$$

Moreover, for all $k \in \mathbb{N}^\star$, $\Phi_{0,k}(x) := e_n w_k$ is an eigenvector of $L^\star$ associated to the eigenvalue λ_k and $\Phi_{1,k}(x) := e_{n-1}w_k$, $\dots$, $\Phi_{n-1,k}(x) := e_1 w_k$ are generalized eigenvectors of $L^\star$ associated to λ_k.

Proof. **Case $\boldsymbol{\alpha \neq 0}$**: The goal is to solve the following eigenvalue problem:

$$L^\star\Psi = \lambda\Psi, \quad \lambda \in \mathbb{C}, \quad \Psi \in H^2(0, \pi; \mathbb{C}^n) \cap H^1_0(0, \pi; \mathbb{C}^n).$$

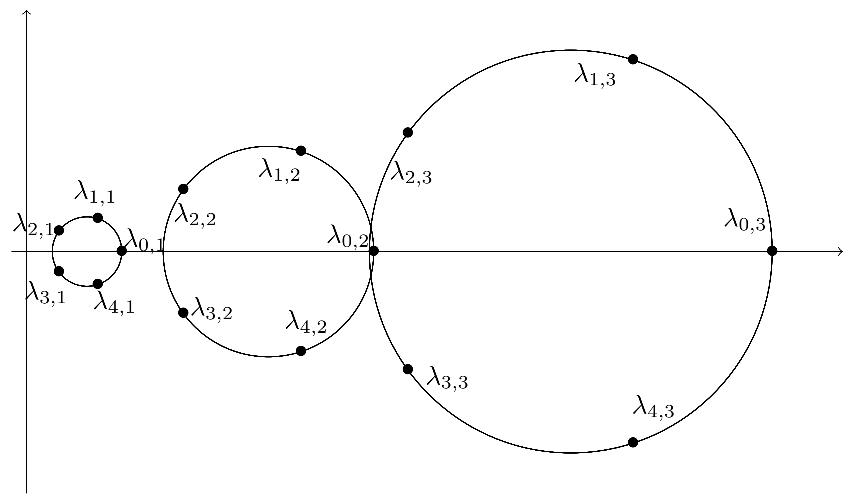

FIGURE 2. $\sigma(L^*)$ for $d = 2$, $n = 5$ and $\alpha = 2$

Considering $\Psi = (\psi_1, \ldots, \psi_n)$, the previous eigenvalue problem is equivalent to

$$\begin{cases} -d\partial_{xx}\psi_1 + \alpha\psi_n = \lambda\psi_1 \\ -\partial_{xx}\psi_1 - d\partial_{xx}\psi_2 = \lambda\psi_2 \\ \qquad\qquad \vdots \\ -\partial_{xx}\psi_{n-1} - d\partial_{xx}\psi_n = \lambda\psi_n. \end{cases}$$

It can be rewritten as an algebraic eigenvalue problem

$$\begin{cases} dk^2c_k^1 + \alpha c_k^n = \lambda c_k^1 \\ k^2c_k^1 + dk^2c_k^2 = \lambda c_k^2 \\ \qquad\qquad \vdots \\ k^2c_k^{n-1} + dk^2c_k^n = \lambda c_k^n, \end{cases}$$

for all $k \geq 1$. From the previous expression, we obtain

$$\begin{cases} (\lambda - dk^2)c_k^1 = \alpha c_k^n, \\ \left(\lambda - dk^2\right) c_k^l = k^2 c_k^{l-1}, \quad \forall l = 2, \ldots, n. \end{cases}$$

After some computations, we also get

$$c_k^l = k^{2(l-1)} \left(\lambda - dk^2\right)^{-(l-1)} c_k^1, \quad \forall l = 1, 2, \ldots, n. \tag{14}$$

Due to the identity for $l = n$, we necessarily have to impose the following condition

$$\alpha k^{2(n-1)} = (\lambda - dk^2)^n,$$

determining all the eigenvalues for the eigenvalue problem at the beginning of the proof. Taking into account that $\alpha \neq 0$, we deduce that the previous equation has n distinct solutions $\lambda_{j,k}$, $j = 0, \ldots, n-1$, which are given explicitly in item a).

Finally, for all $k \geq 1$ and $j = 0, \ldots, n-1$, from (14) and the expression of $\lambda_{j,k}$, it is not difficult to see that $\Phi_{j,k}$, with $c_{j,k}^l$ given in (13), is an eigenvector of $L^\star$ associated to the eigenvalue $\lambda_{j,k}$.

Case $\boldsymbol{\alpha = 0}$: The eigenvalue problem can be rewritten as the algebraic eigenvalue problem

$$\begin{cases} dk^2 c_k^1 = \lambda c_k^1 \\ k^2 c_k^1 + dk^2 c_k^2 = \lambda c_k^2 \\ \qquad \vdots \\ k^2 c_k^{n-1} + dk^2 c_k^n = \lambda c_k^n . \end{cases}$$

We deduce that the previous problem has a unique eigenvalue $\lambda_k := dk^2$ with algebraic multiplicity equal to n. An associated eigenvector is the vector e_n. Moreover the associated generalized eigenvectors are $e_1, \ldots, e_{n-1}$. This ends the proof. □

Let us consider the set

$$\mathcal{B}^\star := \{\Phi_{j,k} : \, k \in \mathbb{N}^\star, j \in \{0, ..., n-1\}\}, \tag{15}$$

where the functions $\Phi_{j,k}$ are given in Proposition 2.1. Then, we obtain the following result:

Lemma 2.2. *Under the assumptions of Proposition 2.1, the set $\mathcal{B}^\star$ is a Schauder basis of the spaces $L^2(0, \pi; \mathbb{C}^n)$ and $H_0^1(0, \pi; \mathbb{C}^n)$, normalized in $L^2(0, \pi; \mathbb{C}^n)$.*

Proof. Let us prove the result in the case $\alpha > 0$. The case $\alpha < 0$ can be deduced with a similar reasoning and the case $\alpha = 0$ is trivial. Consider the Schauder basis $\mathcal{B}_c$ of $L^2(0, \pi; \mathbb{C}^n)$ given by

$$\mathcal{B}_c := \{e_i w_k : \, k \in \mathbb{N}^\star, \; i \in \{1, ..., n\}\}.$$

Let $h \in L^2(0, \pi; \mathbb{C}^n)$ (or $h \in H_0^1(0, \pi; \mathbb{C}^n)$). There exists a unique real sequence $\{\alpha_{i,k}\}_{k \in \mathbb{N}^\star, 1 \le i \le n}$ such that

$$h = \sum_{k=1}^{+\infty} \sum_{i=1}^{n} \alpha_{i,k} e_i w_k \quad \text{in } L^2(0, \pi; \mathbb{C}^n) \quad (\text{resp., in } H_0^1(0, \pi; \mathbb{C}^n)).$$

We remark that the matrix $\mathcal{V}_k := (\widetilde{V}_{0,k} | \cdots | \widetilde{V}_{n-1,k})$ is a Vandermonde matrix (and so, it is invertible), where $\widetilde{V}_{j,k} = \left(c_{j,k}^l\right)_{1 \le l \le n}$ for any $k \in \mathbb{N}^\star$ and $j \in \{0, \ldots, n-1\}$, see (13). Therefore, for each $k \in \mathbb{N}^\star$, there exist unique $\gamma_{0,k}, \ldots, \gamma_{n-1,k} \in \mathbb{C}^\star$ such that

$$\sum_{i=1}^{n} \alpha_{i,k} e_i = \sum_{j=0}^{n-1} \gamma_{j,k} V_{j,k}.$$

Finally, arguing by contradiction, we can obtain the sequence $\{\gamma_{p,k}\}_{k \in \mathbb{N}^\star, 0 \le p \le n-1}$ is unique. □

Remark 2.3. In the sequel, we will use the notation

$$\Lambda_0 := \{\lambda_{j,k} : k \in \mathbb{N}^\star, j \in \{0, ..., n-1\}\},$$

where $\lambda_{j,k}$ is given in Proposition 2.1.

2.2. **Biorthogonal family.** Given a complex sequence $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$, let us denote by p_k the complex function given by

$$p_k(t) := e^{-\Lambda_k t}, \quad \forall t \in (0,T). \tag{16}$$

We will see in Section 3 that the existence of a biorthogonal family in $L^2(0,T;\mathbb{C})$ to the sequence $\{p_k\}_{k\geq 1}$ will play a key role in the study of the controllability of systems (2) and (12). Recall that the sequence $\{q_k\}_{k\geq 1}$ is a biorthogonal family to the sequence $\{p_k\}_{k\geq 1}$ in $L^2(0,T;\mathbb{C})$ if

$$\int_0^T e^{-\Lambda_k t} q_j^\star(t)\,dt = \delta_{kj}, \quad \forall k,j \geq 1.$$

One has:

Theorem 2.4. *Let $T>0$ and consider a sequence $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying*

$$\begin{cases} \Lambda_i \neq \Lambda_k, \quad \forall i,k \in \mathbb{N}^\star \text{ with } i \neq k, \\ \displaystyle\sum_{k\geq 1} \frac{1}{|\Lambda_k|} < +\infty \quad \text{and} \quad \Re(\Lambda_k) \geq \delta |\Lambda_k| > 0, \quad \forall k \geq 1, \end{cases} \tag{17}$$

for some positive constant δ. Then, there exists a biorthogonal family $\{q_k\}_{k\geq 1}$ in $L^2(0,T;\mathbb{C})$ to the family $\{p_k\}_{k\geq 1}$, given in (16). Moreover, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|q_k\|_{L^2(0,T;\mathbb{C})} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon)\Re(\Lambda_k)}, \tag{18}$$

where $c(\Lambda) \in [0,+\infty]$ is the condensation index of the sequence Λ.

This result corresponds to [8, Proposition 4.1 and Remark 4.3]. See [8, Definition 3.1], for the definition of the condensation index.

In some situations, this index of condensation of the sequence $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ can be equal to zero. Let us consider two different situations:

Lemma 2.5. *Let us consider a sequence $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying (17). Let us assume that there exist a positive constant $\rho > 0$ and a positive integer k_1 such that one of the following conditions hold*

$$|\Lambda_k - \Lambda_l| \geq \rho |\Lambda_k|^{1/2}, \quad \forall k \geq k_1 \text{ and } l \neq k. \tag{19}$$

$$|\Lambda_k - \Lambda_l| \geq \rho |k-l|, \quad \forall k,l \in \mathbb{N}^\star. \tag{20}$$

Then,

$$c(\Lambda) = 0.$$

For a proof of the previous lemma, we refer to [34, Theorem 6].

In Section 3 we will also use a result on the existence of biorthogonal families to some complex matrix exponentials. In order to state the result, let us fix $\eta \geq 1$, a positive integer, and let us introduce the notation:

$$p_k^{(j)}(t) := t^j e^{-\Lambda_k t}, \quad \forall t > 0, \quad (k \geq 1 \text{ and } j : 0 \leq j \leq \eta - 1),$$

where $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ is a sequence of complex numbers.

Let us recall that the family $\{q_k^{(j)}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,T;\mathbb{C})$ is biorthogonal to the sequence $\{p_k^{(j)}\}_{k\geq 1, 0\leq j\leq \eta-1}$ if the equalities

$$\int_0^T t^j e^{-\Lambda_k t} q_m^{(l)\star}(t)\,dt = \delta_{km}\delta_{jl}, \quad \forall (j,k),(l,m) : k,m \geq 1,\ 0 \leq j,l \leq \eta - 1,$$

hold.

With the previous notation, one has:

Theorem 2.6. *Let us fix $\eta \geq 1$, a positive integer, and $T > 0$. Assume that $\{\Lambda_k\}_{k\geq 1}$ is a sequence of complex numbers satisfying (17) and (20) for two positive constants δ and ρ. Then, there exists a family $\left\{q_k^{(j)}\right\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,T;\mathbb{C})$ biorthogonal to $\left\{p_k^{(j)}\right\}_{k\geq 1, 0\leq j\leq \eta-1}$ such that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ for which*

$$\|q_k^{(j)}\|_{L^2(0,T;\mathbb{C})} \leq C_\varepsilon e^{\varepsilon \Re(\Lambda_k)}, \quad \forall (j,k) : k \geq 1,\ 0 \leq j \leq \eta - 1. \tag{21}$$

For a proof of this result, see [6, Theorem 1.2].

2.3. **Condensation index of the sequence Λ_0.** Our next objective is to check conditions (17) and (19) for the sequence

$$\Lambda_0 = \{\lambda_{j,k}\}_{0\leq j\leq n-1, k\geq 1}$$

in the case $\alpha \neq 0$. One has:

Proposition 2.7. *Let us consider the matrices $D, A \in \mathcal{L}(\mathbb{R}^n)$ given by (9), with $n \geq 2$, $d \geq 1$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. In addition, assume that $n := 2p+1$ ($p \in \mathbb{N}^\star$) when $\alpha > 0$. Then, there exist a positive integer k_1 and a constant $\rho > 0$, only depending on n, α and d, such that*

$$|\lambda_{j,k} - \lambda_{j',k'}| \geq \rho \left|\lambda_{j,k}\right|^{1/2},\ \forall k \geq k_1,\ \forall k' \geq 1,\ \forall j, j' \in \{0, ..., n-1\}, \tag{22}$$

with $(j,k) \neq (j',k')$. The expression of $\lambda_{j,k}$ ($k \geq 1$, $j : 0 \leq j \leq n-1$) is given in item a) of Proposition 2.1.

Proof. Let us prove that condition (22) holds when $\alpha \neq 0$. Fix $j \in \{0, ..., n-1\}$ and $k \in \mathbb{N}^\star$ and notice that

$$\left|\lambda_{j,k}\right|^{1/2} \leq k \left(d + |\alpha|^{\frac{1}{n}} k^{-\frac{2}{n}}\right)^{1/2} \leq C_1 k,$$

for all $k \geq 1$ and for some positive constant C_1 independent of j and k.

The goal is to prove that there exist a positive integer k_1 and a constant $C_2 > 0$, only depending on n, α and d, such that

$$|\lambda_{j,k} - \lambda_{j',k'}| \geq C_2 k, \quad \forall k \geq k_1, \quad \forall k' \geq 1, \quad \forall j, j' \in \{0, ..., n-1\}, \tag{23}$$

with $(j,k) \neq (j',k')$. Observe that condition (22) is a direct consequence of two previous inequality (23).

Case A: $\alpha > 0$. In this case, we assume that $n = 2p+1$, with $p \geq 1$.

Thanks to the fact that the function $dr^2 + \alpha^{\frac{1}{n}} \cos(\frac{2\pi j}{n}) r^{2-\frac{2}{n}}$ is increasing for r large enough, we deduce there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, $k' \in \mathbb{N}^\star$ (with $k' \neq k$) and $j \in \{0, \ldots, n-1\}$, we have (see item a) of Proposition 2.1)

$$\begin{aligned} |\lambda_{j,k} - \lambda_{j,k'}| &\geq \min\left\{|\Re(\lambda_{j,k}) - \Re(\lambda_{j,k-1})|, |\Re(\lambda_{j,k}) - \Re(\lambda_{j,k+1})|\right\}, \\ &\geq C_3 k, \end{aligned}$$

where $C_3 > 0$ is independent of j, k and k'. On the other hand, using that n is odd, we can also prove that if $k \neq k'$ we have $\lambda_{j,k} \neq \lambda_{j,k'}$. This proves that condition (23) holds choosing any $k_1 \geq 1$ when $j = j'$.

Consider now $k \geq 1$ and $j, j' \in \{0, ..., n-1\}$ such that $j' \neq j$. The goal is to prove inequality (23) for any $k \geq k_1$, $k' \geq 1$ and for any $j, j' \in \{0, ..., n-1\}$ with $j \neq j'$. Again, condition (22) will be a direct consequence of inequality (23).

Let us introduce the notation

$$c_j := \cos(2\pi j/n) \text{ and } s_j := \sin(2\pi j/n), \quad \forall j \in \{0, ..., n-1\}.$$

Observe that thanks to the assumption $n = 2p+1$, with $p \in \mathbb{N}^\star$, we can conclude that $s_j \neq 0$ for any $j : 1 \leq j \leq n-1$. On the other hand, with the previous notation, one has (recall that $\alpha > 0$)

$$\Re(\lambda_{j,k}) = dk^2 + \alpha^{\frac{1}{n}} c_j k^{2-\frac{2}{n}} \text{ and } \Im(\lambda_{j,k}) = \alpha^{\frac{1}{n}} s_j k^{2-\frac{2}{n}}, \quad \forall k \geq 1,\ \forall j \in \{0, ..., n-1\}.$$

In order to show (23), we distinguish three cases:

Case A.1: $j = 0$.

i. If $k' > k/2$ and $j' \neq 0$, we obtain

$$|\lambda_{0,k} - \lambda_{j',k'}| \geq |\Im(\lambda_{0,k} - \lambda_{j',k'})| = \alpha^{\frac{1}{n}} |s_{j'}| (k')^{2-\frac{2}{n}} \geq \alpha^{\frac{1}{n}} |s_{j'}| k' \geq \frac{1}{2}\alpha^{\frac{1}{n}} |s_{j'}| k;$$

ii. If $k' \leq k/2$ then we deduce

$$|\lambda_{0,k} - \lambda_{j',k'}| \geq \left| d(k^2 - (k')^2) + \alpha^{\frac{1}{n}} k^{2-\frac{2}{n}} - \alpha^{\frac{1}{n}} c_{j'} (k')^{2-\frac{2}{n}} \right| \geq d(k^2 - (k')^2) \geq \frac{3}{4} dk.$$

In both cases, (23) holds choosing any $k_1 \geq 1$ when $j = 0$.

Case A.2: $j \neq 0$ and $j' \neq j$ with $\mathrm{sign}(s_j) \neq \mathrm{sign}(s_{j'})$. Then, we have

$$|\lambda_{j,k} - \lambda_{j',k'}| \geq |\Im(\lambda_{j,k} - \lambda_{j',k'})| \geq \alpha^{\frac{1}{n}} |s_j| k^{2-\frac{2}{n}} \geq \alpha^{\frac{1}{n}} |s_j| k;$$

In this case, we deduce (23) choosing any $k_1 \geq 1$.

Case A.3: $j \neq 0$ and $j' \neq j$ with $\mathrm{sign}(s_j) = \mathrm{sign}(s_{j'})$. First, notice that $j' \neq j$ and the fact that n is odd implies $s_j \neq 0$, $s_{j'} \neq 0$ and $|s_j| \neq |s_{j'}|$. Now, we define

$$\beta := \begin{cases} \left(\dfrac{|s_j|}{|s_{j'}|}\right)^{1/2} & \text{if} \quad \dfrac{|s_j|}{|s_{j'}|} > 1, \\ \left(\dfrac{|s_{j'}|}{|s_j|}\right)^{1/2} & \text{if} \quad \dfrac{|s_j|}{|s_{j'}|} < 1 \end{cases}$$

and the numbers:

$$\beta_1 := \left(\frac{1}{\beta}\frac{|s_j|}{|s_{j'}|}\right)^{\frac{n}{2(n-1)}} \qquad \text{and} \qquad \beta_2 := \left(\beta\frac{|s_j|}{|s_{j'}|}\right)^{\frac{n}{2(n-1)}}.$$

In any case, one has $\beta > 1$ and $\beta_1 < \beta_2$.

Then, we consider three different cases:

i. If $k' \leq \beta_1 k$, with $k \geq 1$, we use the imaginary part of $\lambda_{j,k} - \lambda_{j',k'}$ to deduce

$$\begin{aligned} |\lambda_{j,k} - \lambda_{j',k'}| &\geq \alpha^{\frac{1}{n}} \left(k^{2-\frac{2}{n}} |s_j| - (k')^{2-\frac{2}{n}} |s_{j'}|\right) \geq \alpha^{\frac{1}{n}} k^{2-\frac{2}{n}} \left(|s_j| - \beta_1^{\frac{2(n-1)}{n}} |s_{j'}|\right) \\ &= \alpha^{\frac{1}{n}} k^{2-\frac{2}{n}} \left(1 - \frac{1}{\beta}\right) |s_j| \geq \alpha^{\frac{1}{n}} \left(1 - \frac{1}{\beta}\right) |s_j| k. \end{aligned}$$

ii. Let us now assume that $k' \geq \beta_2 k$. Then a similar reasoning as before provides

$$|\lambda_{j,k} - \lambda_{j',k'}| \geq \alpha^{\frac{1}{n}} (\beta - 1) |s_j| k^{2-\frac{2}{n}} \geq \alpha^{\frac{1}{n}} (\beta - 1) |s_j| k.$$

In this particular case, we have (23) choosing any $k_1 \geq 1$.

iii. If now $k' = \gamma k$ with $\gamma \in (\beta_1, \beta_2)$, then we deduce

$$\begin{aligned} |\lambda_{j,k} - \lambda_{j',k'}| &\geq \left| d\,k^2 + \alpha^{\frac{1}{n}} c_j k^{2-\frac{2}{n}} - d\,\gamma^2 k^2 - \alpha^{\frac{1}{n}} c_{j'} \gamma^{2-\frac{2}{n}} k^{2-\frac{2}{n}} \right| \\ &= k^{2-\frac{2}{n}} \left| d(1-\gamma^2) k^{2/n} + \alpha^{\frac{1}{n}} (c_j - c_{j'} \gamma^{2-\frac{2}{n}}) \right|. \end{aligned}$$

Since

$$\begin{cases} \gamma^{2-2/n} \in \left(\dfrac{|s_j|^{1/2}}{|s_{j'}|^{1/2}}, \dfrac{|s_j|^{3/2}}{|s_{j'}|^{3/2}} \right) & \text{if } \dfrac{|s_j|}{|s_{j'}|} > 1, \\ \gamma^{2-2/n} \in \left(\dfrac{|s_j|^{3/2}}{|s_{j'}|^{3/2}}, \dfrac{|s_j|^{1/2}}{|s_{j'}|^{1/2}} \right) & \text{if } \dfrac{|s_j|}{|s_{j'}|} < 1, \end{cases}$$

we easily deduce that $\left|\gamma^2 - 1\right| \geq C > 0$ and then there exists a new positive integer $k_0 \in \mathbb{N}^\star$ such that for any $k \geq k_0$ we have

$$\left| d(1-\gamma^2) k^{2/n} + \alpha^{\frac{1}{n}} (c_j - c_{j'} \gamma^{2-\frac{2}{n}}) \right| \geq C > 0$$

and then, we obtain

$$|\lambda_{j,k} - \lambda_{j',k'}| \geq C k^{2-\frac{2}{n}} \geq C k.$$

This proves inequality (23) for any $k_1 \geq k_0$.

In conclusion, we have proved inequality (23) when n is odd and $\alpha > 0$.

Case B: $\alpha < 0$.

In this case, the eigenvalues of the operator $L^\star = -D^\star \partial_{xx} + A^\star$ are given by

$$\lambda_{j,k} = d\,k^2 + |\alpha|^{1/n} k^{2-\frac{2}{n}} e^{\frac{(2j+1)\pi}{n} i}, \quad k \geq 1, \quad j \in \{0, ..., n-1\}.$$

Observe that, in this case, $\lambda_{p,k} \in \mathbb{R}$ when $n = 2p+1$ ($p \geq 1$, an integer) and $\lambda_{j,k} \in \mathbb{C} \setminus \mathbb{R}$, otherwise. Again, our goal is to prove inequality (23) for a positive integer k_1 and a positive constant C_2 only depending on n, d and α.

Let us introduce the notation

$$\widetilde{c}_j := \cos\left(\frac{(2j+1)\pi}{n} \right) \quad \text{and} \quad \widetilde{s}_j := \sin\left(\frac{(2j+1)\pi}{n} \right), \quad \forall j \in \{0, ..., n-1\}.$$

With this notation, one has (recall that $\alpha < 0$)

$$\Re(\lambda_{j,k}) = dk^2 + |\alpha|^{\frac{1}{n}} \widetilde{c}_j k^{2-\frac{2}{n}} \text{ and } \Im(\lambda_{j,k}) = |\alpha|^{\frac{1}{n}} \widetilde{s}_j k^{2-\frac{2}{n}}, \; \forall k \geq 1, \; \forall j \in \{0, ..., n-1\}.$$

Using the fact that the function $dr^2 + |\alpha|^{\frac{1}{n}} \widetilde{c}_j r^{2-\frac{2}{n}}$ is increasing for r large enough, we deduce the existence of $\widetilde{k}_0 \in \mathbb{N}$ such that for any $k \geq \widetilde{k}_0$, $k' \in \mathbb{N}^\star$ (with $k' \neq k$) and $j \in \{0, \ldots, n-1\}$, we have

$$\begin{aligned} |\lambda_{j,k} - \lambda_{j,k'}| &\geq \min \left\{ |\Re(\lambda_{j,k}) - \Re(\lambda_{j,k-1})|, |\Re(\lambda_{j,k}) - \Re(\lambda_{j,k+1})| \right\}, \\ &\geq \widetilde{C}_3 k, \end{aligned}$$

where $\widetilde{C}_3 > 0$ is independent of j, k and k'. This proves inequality (23) choosing any $k_1 \geq \widetilde{k}_0$ and $j = j'$.

As in the case $\alpha > 0$, let us consider $k \geq 1$ and $j, j' \in \{0, ..., n-1\}$ such that $j' \neq j$ and let us prove inequality (23). We distinguish four cases:

Case B.1: $\lambda_{j,k} \in \mathbb{R}$, i.e., $n = 2p+1$ ($p \in \mathbb{N}^\star$) and $j = p$. In this case, $\lambda_{p,k} = dk^2 - |\alpha|^{\frac{1}{n}} k^{2-\frac{2}{n}}$. So, *i.* If $k' > k/2$ and $j' \neq p$, we obtain

$$|\lambda_{p,k} - \lambda_{j',k'}| \geq |\Im(\lambda_{p,k} - \lambda_{j',k'})| = |\alpha|^{\frac{1}{n}} |\widetilde{s}_{j'}| (k')^{2-\frac{2}{n}} \geq |\alpha|^{\frac{1}{n}} |\widetilde{s}_{j'}| k' \geq \frac{1}{2}|\alpha|^{\frac{1}{n}} |\widetilde{s}_{j'}| k;$$

ii. If $k' \leq k/2$, we deduce

$$\begin{aligned}|\lambda_{p,k} - \lambda_{j',k'}| &\geq d[k^2 - (k')^2] - |\alpha|^{\frac{1}{n}} k^{2-\frac{2}{n}} - |\alpha|^{\frac{1}{n}} |\widetilde{c}_{j'}| (k')^{2-\frac{2}{n}} \\ &\geq k^2 \left(\frac{3}{4}d - 2|\alpha|^{\frac{1}{n}} k^{-\frac{2}{n}}\right).\end{aligned}$$

From this expression, in this particular case, we deduce (23) for any $k_1 \geq 1$ and $C_2 > 0$ only depending on n, d and α.

Case B.2: $\lambda_{j,k} \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_{j',k'}$ is such that $\operatorname{sign}(\widetilde{s}_j) \neq \operatorname{sign}(\widetilde{s}_{j'})$. In this case, $\widetilde{s}_j \neq 0$ and we can write

$$|\lambda_{j,k} - \lambda_{j',k'}| \geq |\Im(\lambda_{j,k} - \lambda_{j',k'})| \geq |\alpha|^{\frac{1}{n}} |\widetilde{s}_j| k^{2-\frac{2}{n}} \geq |\alpha|^{\frac{1}{n}} |\widetilde{s}_j| k;$$

In this particular case, we deduce (23) choosing any $k_1 \geq 1$.

Case B.3: $\lambda_{j,k} \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_{j',k'}$ is such that $\widetilde{s}_j = \widetilde{s}_{j'}$, with $j \neq j'$. In this case, notice that $\widetilde{c}_j = -\widetilde{c}_{j'} \neq 0$, $\widetilde{s}_j \neq 0$ and n should be an even number greater or equal than 4. On the other hand, we also have:

$$(x+M)^{2-\frac{2}{n}} - x^{2-\frac{2}{n}} \geq \left(2 - \frac{2}{n}\right) M x^{1-\frac{2}{n}}, \quad \forall x, M > 0, \quad (n \geq 2). \tag{24}$$

Let us fix η, the positive root of the equation $dx^2 + 2dx - |\alpha|^{\frac{1}{n}} |\widetilde{c}_j| / 2 = 0$, i.e.

$$\eta = \frac{-2d + \sqrt{4d^2 + 2d|\alpha|^{\frac{1}{n}} |\widetilde{c}_j|}}{2d}.$$

We divide the proof into three cases:

i. If $k' \geq k + \eta k^{2/n}$, we obtain

$$\begin{aligned}|\lambda_{j,k} - \lambda_{j',k'}| &\geq |\Im(\lambda_{j,k} - \lambda_{j',k'})| = |\alpha|^{\frac{1}{n}} |\widetilde{s}_j| \left[(k')^{2-\frac{2}{n}} - k^{2-\frac{2}{n}}\right] \\ &\geq |\alpha|^{\frac{1}{n}} |\widetilde{s}_j| \left[\left(k + \eta k^{2/n}\right)^{2-\frac{2}{n}} - k^{2-\frac{2}{n}}\right] \geq |\alpha|^{\frac{1}{n}} |\widetilde{s}_j| \eta \left(2 - \frac{2}{n}\right) k.\end{aligned}$$

In the previous inequalities we have used (24) with $x = k$ and $M = \eta k^{2/n}$. In this particular case, inequality (23) holds choosing any $k_1 \geq 1$.

ii. Let us take $k' \in (k - \eta k^{2/n}, k + \eta k^{2/n})$. Let us take $\widehat{k}_0$ large enough such that $k - \eta k^{2/n} > 0$ for all $k \geq \widehat{k}_0$. If $k \geq \widehat{k}_0$ and using that $n \geq 4$ and

$$1 + \frac{2}{n} \leq 2 - \frac{2}{n}, \quad \frac{4}{n} \leq 2 - \frac{2}{n},$$

we deduce

$$\begin{aligned}|\lambda_{j,k}-\lambda_{j',k'}| &\geq |\Re(\lambda_{j,k}-\lambda_{j',k'})| \\ &\geq |\alpha|^{\frac{1}{n}}|\widetilde{c}_j|\left[k^{2-\frac{2}{n}}+(k')^{2-\frac{2}{n}}\right]-d\left|k^2-(k')^2\right| \\ &\geq |\alpha|^{\frac{1}{n}}|\widetilde{c}_j|\,k^{2-\frac{2}{n}} \\ &\quad - d\max\left\{k^2-\left(k-\eta k^{2/n}\right)^2,\left(k+\eta k^{2/n}\right)^2-k^2\right\} \\ &\geq |\alpha|^{\frac{1}{n}}|\widetilde{c}_j|\,k^{2-\frac{2}{n}}-2d\eta k^{1+\frac{2}{n}}-d\eta^2k^{\frac{4}{n}} \\ &\geq \left[|\alpha|^{\frac{1}{n}}|\widetilde{c}_j|-2d\eta-d\eta^2\right]k^{2-\frac{2}{n}}=\frac{1}{2}|\alpha|^{\frac{1}{n}}|\widetilde{c}_j|\,k^{2-\frac{2}{n}}.\end{aligned}$$

In the last equality, we have used the expression of η. Then, we obtain inequality (23) for any $k_1 \geq \widehat{k}_0$.

iii. Finally, let us consider k, k' such that $1 \leq k' \leq k-\eta k^{2/n}$. In this case we can repeat the previous arguments. Indeed, if we choose $k \geq \check{k}_0$ large enough, one has:

$$\left\{\begin{aligned}|\lambda_{j,k}-\lambda_{j',k'}| &\geq |\Im(\lambda_{j,k}-\lambda_{j',k'})| = |\alpha|^{\frac{1}{n}}|\widetilde{s}_j|\left[k^{2-\frac{2}{n}}-(k')^{2-\frac{2}{n}}\right] \\ &\geq |\alpha|^{\frac{1}{n}}|\widetilde{s}_j|\left[k^{2-\frac{2}{n}}-\left(k-\eta k^{2/n}\right)^{2-\frac{2}{n}}\right] \\ &\geq |\alpha|^{\frac{1}{n}}|\widetilde{s}_j|\,\eta\left(2-\frac{2}{n}\right)k^{\frac{2}{n}}\left(k-\eta k^{2/n}\right)^{1-\frac{2}{n}} \geq \widetilde{C}k,\end{aligned}\right.$$

where $\widetilde{C}$ is a new positive constant only depending on n, α and d. In the previous inequalities we have used that $n \geq 4$ and inequality (24) with $x = k-\eta k^{2/n}$ and $M = \eta k^{2/n}$. This proves inequality (23) for any $k_1 \geq \check{k}_0$.

Case B.4: $\lambda_{j,k} \in \mathbb{C}\setminus\mathbb{R}$ and $\lambda_{j',k'}$ is such that $\widetilde{s}_j \neq \widetilde{s}_{j'}$ and $\mathrm{sign}(\widetilde{s}_j) = \mathrm{sign}(\widetilde{s}_{j'})$. Observe that in this case, $\widetilde{s}_j$ $\widetilde{s}_{j'}$ are non null and we can repeat the same proof as **Case A.3**.

Therefore, taking $k_1 = \max\{k_0, \widetilde{k}_0, \widehat{k}_0, \check{k}_0\}$, one obtains inequality (23). This ends the proof. ☐

Remark 2.8. Under the conditions of Proposition 2.7 we infer that

$$m(\lambda_{j,k}) = 1,^{1} \quad \forall k \geq k_1, \quad \forall j : 0 \leq j \leq n-1$$

and it is easy to see that there exists $\widetilde{k}_0 \in \mathbb{N}^\star$ such that $\Re(\lambda_{j,k}) > 0$, for any $k \geq \widetilde{k}_0$ and any $j \in \{0, 1, \ldots, n-1\}$. Thus, if we define the set

$$\mathcal{J} := \{(j,k) \in \{0,\ldots,n-1\}\times\mathbb{N}^\star : m(\lambda_{j,k}) = 1, \quad \Re(\lambda_{j,k}) > 0\}, \tag{25}$$

then, under conditions of Proposition 2.7, there exists $k_0 \in \mathbb{N}^\star$ such that

$$\{0,\ldots,n-1\}\times\{k_0, k_0+1,\ldots\} \subset \mathcal{J}.$$

On the other hand, the situation is very different when n is even and $\alpha > 0$. For instance, when $n = 2$ and d and α does not satisfy (11), the operator $L^\star$ has an infinite number of eigenvalues with geometric multiplicity equal to 2 (see (41)).

Remark 2.9. As said in Remark 1.6, when $n = 2p$, $p \in \mathbb{N}^\star$, and $\alpha > 0$, the operator $L^\star := -D^\star\partial_{xx}f + A^\star$ has, for any $k \geq 1$, exactly two real eigenvalues, $\lambda_{0,k}$

[1] $m(\lambda)$ is the geometric multiplicity of $\lambda \in \sigma(L^\star)$

and $\lambda_{p,k}$, and $n-2$ complex eigenvalues (see Proposition 2.1). In this case the real eigenvalues are given by

$$\lambda_{0,k} = dk^2 + \alpha^{1/n} k^{2-1/p} \quad \text{and} \quad \lambda_{p,k} = dk^2 - \alpha^{1/n} k^{2-1/p}, \quad \forall k \geq 1.$$

In this case and in view of the proof of Proposition 2.7, we can conclude that inequality (22) holds except for $(j, j') = (0, p)$ or $(j, j') = (p, 0)$.

As a direct consequence of Proposition 2.7, Lemma 2.5 and Remark 2.8, we have:

Corollary 2.10. *Under assumptions of Proposition 2.7, the subsequence of eigenvalues $\Lambda = \{\lambda_{j,k}\}_{(j,k)\in\mathcal{J}}$, with $\mathcal{J}$ given by* (25), *satisfies* (17) *and* $c(\Lambda) = 0$.

3. **Boundary null controllability.** In this section, we will prove the boundary null controllability results stated in Theorems 1.4 and 1.7.

3.1. **Proof of Theorem 1.4.** The proof of Theorem 1.4 follows the well-known moment method and is a consequence of Proposition 2.7. For the sake of completeness it will be included here. The proof will be developed in three sections. In the first section we will prove the sufficient condition stated in item 1. The second section is devoted to the proof of the necessary condition in item 1. Finally, we will prove item 2 in Section 3.1.2.

3.1.1. *Proof of Items 1 and 2 of Theorem 1.4.* Recall that the matrices $D, A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathbb{R}^n$ are given by (9), with $d \geq 1$ and $\alpha \in \mathbb{R}^\star$, and the expression of the eigenvalues of $L^\star = -D^\star \partial_{xx} + A^\star$ is given in item a) of Proposition 2.1. Remember also that in the case $\alpha > 0$ the dimension n of system (2) is odd and, therefore, (22) holds.

Let us first observe that, under assumptions of Theorem 1.4, one has

$$\lambda_{j,k} \neq \lambda_{j',k'}, \quad \forall k, k' \in \mathbb{N}^\star, \quad \forall j, j' \in \{0, ..., n-1\} \text{ with } (j,k) \neq (j',k') \tag{26}$$

and condition (11) holds when $\alpha < 0$ and n is odd. Then, the goal is to prove that system (2) is null (resp., approximately) controllable at time T.

On the other hand, without loss of generality, we can assume that $\Re(\lambda_{j,k}) > 0$ for any $k \geq 1$ and $j : 0 \leq j \leq n-1$. Indeed, taking into account that $\lim_{k\to\infty} \Re(\lambda_{j,k}) = \infty$, for any $j : 0 \leq j \leq n-1$, we can conclude the existence of a positive constant $M > 0$ such that

$$\Re(\lambda_{j,k} + M) > 0, \quad \forall k \in \mathbb{N}^\star, \quad \forall j \in \{0, ..., n-1\}.$$

Performing the change $\widetilde{z} = e^{-Mt} z$ in system (2), the controllability properties of this system at time T are equivalent to the corresponding properties of system

$$\begin{cases} \widetilde{z}_t - D\widetilde{z}_{xx} + (A + MI_n)\widetilde{z} = 0 & \text{in } Q_T, \\ \widetilde{z}(\cdot, 0) = Be^{-Mt} v, \quad \widetilde{z}(\cdot, \pi) = 0 & \text{in } (0,T), \\ \widetilde{z}(0, \cdot) = z^0 & \text{in } (0,\pi), \end{cases}$$

with $z^0 \in H^{-1}(0, \pi; \mathbb{R}^n)$ and $v \in L^2(0,T)$. It is clear that

$$\sigma(-D^\star \partial_{xx} + A^\star + MI_n) = \{\lambda + M : \lambda \in \sigma(-D^\star \partial_{xx} + A^\star)\},$$

($I_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix) and then $\Re(\lambda) > 0$ for any $\lambda \in \sigma(-D^\star \partial_{xx} + A^\star + MI_n)$.

In order to prove that system (2) is null controllable at time $T > 0$, let us first present an equivalent property to the null controllability of the system. Let us introduce the following adjoint system to (2):

$$\begin{cases} -\varphi_t - D^\star \varphi_{xx} + A^\star \varphi = 0 & \text{in } Q_T, \\ \varphi(\cdot, 0) = \varphi(\cdot, \pi) = 0 & \text{on } (0, T), \\ \varphi(T, \cdot) = \varphi^T & \text{in } (0, \pi), \end{cases} \tag{27}$$

where $\varphi^T \in H_0^1(0, \pi; \mathbb{C}^n)$ and matrices D, A and B are defined in (9). For any $\varphi^T \in H_0^1(0, \pi; \mathbb{C}^n)$ system (27) has a unique solution

$$\varphi \in L^2(0, T; H^2(0, \pi; \mathbb{C}^n) \cap H_0^1(0, \pi; \mathbb{C}^n)) \cap \mathcal{C}^0(0, T; H_0^1(0, \pi; \mathbb{C}^n))$$

which depends continuously on the initial data φ^T. In fact, if $\varphi^T \in H_0^1(0, \pi; \mathbb{C}^n)$, $v \in L^2(0, T; \mathbb{C})$ and $y^0 \in H^{-1}(0, \pi; \mathbb{C}^n)$, the corresponding solutions z and φ of systems (2) and (27) satisfy

$$\langle y(T, \cdot), \varphi^T \rangle_{H^{-1}, H_0^1} - \langle y^0, \varphi(0, \cdot) \rangle_{H^{-1}, H_0^1} = \int_0^T (v(t), B^\star D^\star \varphi_x(t, 0))_{\mathbb{C}} \, dt, \tag{28}$$

where $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$ is the duality pairing between $H^{-1}(0, \pi; \mathbb{C}^n)$ and $H_0^1(0, \pi; \mathbb{C}^n)$.

From identity (28), the null controllability problem of system (2) can be reformulated as a *moment problem*. More precisely, system (2) is null controllable at time $T > 0$ if and only if for any initial data $y^0 \in H^{-1}(0, \pi; \mathbb{C}^n)$ there exists a control $v \in L^2(0, T; \mathbb{C})$ such that

$$-\langle y^0, \varphi(0, \cdot) \rangle_{H^{-1}, H_0^1} = \int_0^T (v(t), B^\star D^\star \varphi_x(t, 0))_{\mathbb{C}} \, dt, \quad \forall \varphi^T \in H_0^1(0, \pi; \mathbb{C}^n),$$

with φ the solution of the adjoint problem (27) associated to φ^T. Since $\mathcal{B}^\star$ is a Schauder basis of $H_0^1(0, \pi; \mathbb{C}^n)$ (see (15) and Lemma 2.2), the null controllability of system (2) is equivalent to the following property:

Property: For any initial data $y^0 \in H^{-1}(0, \pi; \mathbb{C}^n)$, there exists a control $v \in L^2(0, T; \mathbb{C})$ such that

$$-\langle y^0, \varphi_{j,k}(0, \cdot) \rangle_{H^{-1}, H_0^1} = \int_0^T (v(t), B^\star D^\star \partial_x \varphi_{j,k}(t, 0))_{\mathbb{C}} \, dt, \tag{29}$$

for all $(j, k) \in \{0, ..., n-1\} \times \mathbb{N}^\star$, where $\varphi_{j,k}$ is the solution of system (27) associated to the initial data $\varphi^T = \Phi_{j,k}$.

A simple computation leads to the formula

$$\varphi_{j,k}(t, x) = e^{-\lambda_{j,k}(T-t)} \Phi_{j,k}(x), \quad (t, x) \in Q_T,$$

whence

$$\begin{cases} \varphi_{j,k}(0, x) = e^{-\lambda_{j,k} T} \Phi_{j,k}(x), \quad x \in (0, \pi), \\ \partial_x \varphi_{j,k}(t, 0) = k \sqrt{\dfrac{2}{\pi}} e^{-\lambda_{j,k}(T-t)} V_{j,k}, \quad \forall t \in (0, T), \end{cases}$$

where the vector $V_{j,k} \in \mathbb{C}^n$ is given in Proposition 2.1. Thus, using these expressions in problem (29) we can conclude that system (2) is null controllable at time $T > 0$ if and only if

$$\begin{cases} \text{there is } v \in L^2(0, T; \mathbb{C}) \text{ such that for all } k \in \mathbb{N}^\star \text{ and } j \in \{0, \dots, n-1\}, \\ (B^\star D^\star V_{j,k})^\star \displaystyle\int_0^T v(T-t) e^{-\lambda_{j,k}^\star t} \, dt = -\frac{1}{k} \sqrt{\frac{\pi}{2}} e^{-\lambda_{j,k}^\star T} \langle y^0, \Phi_{j,k} \rangle_{H^{-1}, H_0^1}. \end{cases} \tag{30}$$

Notice that a necessary condition to solve the moment problem (30) is that $(B^\star D^\star V_{j,k})^\star \neq 0$ for all $k \in \mathbb{N}^\star$ and $j \in \{0,\dots,n-1\}$ and $\lambda^\star_{j,k} \neq \lambda^\star_{j',k'}$ for all $k,k' \in \mathbb{N}^\star$ and $j,j' \in \{0,\dots,n-1\}$ with $(j,k) \neq (j',k')$.

Let us analyse the expression $B^\star D^\star V_{j,k}$. From (9) and Proposition 2.1, we deduce that, for each $k \in \mathbb{N}^\star$ and $j : 0 \le j \le n-1$:

$$B^\star D^\star V_{j,k} = 0 \iff \begin{cases} 1 + d\alpha^{-\frac{1}{n}} k^{\frac{2}{n}} e^{-\frac{2\pi j}{n} i} = 0, & \text{if } \alpha > 0, \\ 1 + d\,|\alpha|^{-\frac{1}{n}} k^{\frac{2}{n}} e^{-\frac{(2j+1)\pi}{n} i} = 0, & \text{if } \alpha < 0. \end{cases} \tag{31}$$

If $\alpha > 0$, thanks to the assumption $n = 2p+1$, with $p \in \mathbb{N}^\star$, we can conclude that $B^\star D^\star V_{j,k} \neq 0$ for every $k \in \mathbb{N}^\star$ and $j : 0 \le j \le n-1$. On the other hand, if $\alpha < 0$, $B^\star D^\star V_{j,k} = 0$ if and only if $n = 2p+1$, with $p \in \mathbb{N}^\star$, $j = p$ and $k =: \sqrt{|\alpha|}/d^{n/2} \in \mathbb{N}^\star$. Thanks to assumption (11) we can also conclude that $B^\star D^\star V_{j,k} \neq 0$ for every $k \in \mathbb{N}^\star$ and $j : 0 \le j \le n-1$. This proves the necessary condition for the null controllability of system (2) at time T in items 1 and 2 of Theorem 1.4.

Let us now prove the sufficient condition for the null controllability in items 1 and 2 of Theorem 1.4. Under these conditions, problem (30) is equivalent to:

$$\begin{cases} \text{find } v \in L^2(0,T;\mathbb{C}) \text{ such that for all, } k \in \mathbb{N}^\star \text{ and } j \in \{0,\dots,n-1\}, \\ \displaystyle\int_0^T v(T-t) e^{-\lambda^\star_{j,k} t}\, dt = e^{-\lambda^\star_{j,k} T} M_{j,k}(y^0), \end{cases} \tag{32}$$

where

$$M_{j,k}(y^0) := -\frac{1}{k}\sqrt{\frac{\pi}{2}} \frac{\langle y^0, \Phi_{j,k}\rangle_{H^{-1},H^1_0}}{V^\star_{j,k} DB}.$$

This is the moment problem associated to the boundary null controllability of system (2).

In order to solve the moment problem (32), we will apply Theorem 2.4 to the sequence $\Lambda := \{\lambda_{j,k}\}_{k\ge 1, 0\le j\le n-1}$ of eigenvalues of the operator $L^\star = -D^\star \partial_{xx} f + A^\star$. Thanks to condition (26), the sequence Λ satisfies condition (17). If we use the notation

$$p_{j,k}(t) := e^{-\lambda_{j,k} t}, \quad \forall t \in (0,T), \tag{33}$$

we can apply Theorem 2.4 and deduce the existence of a biorthogonal family $\{q_{j,k}\}_{k\ge 1, 0\le j\le n-1}$ to $\{p_{j,k}\}_{k\ge 1, 0\le j\le n-1}$ in $L^2(0,T;\mathbb{C})$ which satisfies (18), i.e., a family such that

$$\int_0^T p_{j,k}(t) q^\star_{l,m}(t)\, dt = \delta_{km}\delta_{jl}, \quad \forall k,m \ge 1, \quad \forall j,l : 0 \le j,l \le n-1.$$

Moreover, under the assumptions of Theorem 1.4, we can also apply Proposition 2.7 and Lemma 2.5 to obtain that $c(\Lambda) = 0$. Therefore, the biorthogonal family $\{q_{j,k}\}_{k\ge 1, 0\le j\le n-1}$ satisfies the following property: for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|q_{j,k}\|_{L^2(0,T;\mathbb{C})} \le C_\varepsilon e^{\varepsilon \Re(\lambda_{j,k})}, \quad \forall k \ge 1, \quad \forall j : 0 \le j \le n-1. \tag{34}$$

We are in conditions to solve problem (32). The function

$$v(t) = \sum_{k=1}^\infty \sum_{j=0}^{n-1} e^{-\lambda^\star_{j,k} T} M_{j,k}(y^0) q_{j,k}(T-t)$$

provides a formal solution to this problem. Let us see that, in fact, it is a solution to the moment problem, i.e., let us see that $v \in L^2(0,T;\mathbb{C})$.

First, since $y^0 \in H^{-1}(0,\pi;\mathbb{R}^n)$ and taking into account the expression of the vectors $V_{j,k}$ (see Proposition 2.1), we infer that, for any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\left|M_{j,k}(y^0)\right| \le C_\varepsilon e^{\varepsilon \Re(\lambda_{j,k})} \|y^0\|_{H^{-1}(0,\pi;\mathbb{C}^n)}, \quad \forall k \ge 1, \quad \forall j : 0 \le j \le n-1.$$

Let us take $\varepsilon > 0$ (which will be chosen later). Using the previous estimate together with inequality (34), we get

$$\left\{ \begin{aligned} \|v\|_{L^2(0,T;\mathbb{C})} &\le \sum_{k=1}^{+\infty}\sum_{j=0}^{n-1} C_\varepsilon e^{-\Re(\lambda_{j,k})T} e^{2\varepsilon\Re(\lambda_{j,k})} \|y^0\|_{H^{-1}(0,\pi;\mathbb{C}^n)} \\ &= \sum_{k=1}^{+\infty}\sum_{j=0}^{n-1} C_\varepsilon e^{-\Re(\lambda_{j,k})(T-2\varepsilon)} \|y^0\|_{H^{-1}(0,\pi;\mathbb{C}^n)}. \end{aligned}\right.$$

where C_ε is a positive constant. Taking, for example, $\varepsilon = T/4$, we obtain that the series in the definition of v converges absolutely in $L^2(0,T;\mathbb{C})$. Thus, the previous control v solves the moment problem (32). This proves the null controllability result at time $T > 0$ of system (2).

3.1.2. *Proof of Item 3 of Theorem 1.4.* As in Section 3.1.1, we will apply the moment method to prove the null controllability at time $T > 0$ of system (2) when $\alpha = 0$. In particular, $L^\star = -D^\star \partial_{xx}$ and from Proposition 2.1, we have

$$\sigma(L^\star) = \{\lambda_k := dk^2 : k \in \mathbb{N}^\star\}.$$

As in the previous case, system (2) is null controllable at time $T > 0$ if and only if there exists a control $v \in L^2(0,T)$ such that (29) holds, where $\Phi_{j,k} = e_{n-j} w_k$ (see item b) of Proposition 2.1).

For the initial data $\varphi^T := \Phi_{j,k} = e_{n-j} w_k$, $k \ge 1$, $j : 0 \le j \le n-1$, the solution to the adjoint problem (27) is given by:

$$\varphi_{j,k}(t,x) = e^{-dk^2(T-t)} \sum_{l=0}^{j} (-1)^{j-l} \frac{k^{2(j-l)}}{(j-l)!} (T-t)^{j-l} e_{n-l} w_k(x), \quad (t,x) \in Q_T.$$

From this identity, we infer

$$\left\{ \begin{aligned} &\varphi_{j,k}(0,x) = e^{-dk^2 T} \sum_{l=0}^{j} (-1)^{j-l} \frac{\left(k^2 T\right)^{j-l}}{(j-l)!} e_{n-l} w_k(x), \\ &\partial_x \varphi_{j,k}(t,0) = k\sqrt{\frac{2}{\pi}} e^{-dk^2(T-t)} \sum_{l=0}^{j} (-1)^{j-l} \frac{k^{2(j-l)}}{(j-l)!} (T-t)^{j-l} e_{n-l}, \end{aligned}\right.$$

for any $k \ge 1$ and $j : 0 \le j \le n-1$.

Thus, if we introduce the notation

$$\widetilde{M}_{j,k}(y^0) = -\frac{1}{k}\sqrt{\frac{\pi}{2}} \sum_{l=0}^{j} (-1)^l \frac{\left(k^2 T\right)^{j-l}}{(j-l)!} \langle y^0, e_{n-l} w_k \rangle_{H^{-1}, H_0^1}, \tag{35}$$

for all $(j,k) \in \{0,1,\dots,n-1\} \times \mathbb{N}^\star$, and observing that

$$B^\star D^\star = (0,0,\dots,1,d),$$

the moment problem (29) becomes: find $v \in L^2(0,T)$ such that for all $k \in \mathbb{N}^\star$ one has

$$d\int_0^T v(T-t)e^{-dk^2t}\,dt = e^{-dk^2T}\widetilde{M}_{0,k}(y^0),$$
$$d\frac{k^{2j}}{j!}\int_0^T v(T-t)t^j e^{-dk^2t}\,dt - \frac{k^{2(j-1)}}{(j-1)!}\int_0^T v(T-t)t^{j-1}e^{-dk^2t}\,dt = e^{-dk^2T}\widetilde{M}_{j,k}(y^0),$$
$$\forall j : 1 \le j \le n-1.$$

This linear system can be written in a vectorial form as

$$\mathcal{A}_k X_k = e^{-dk^2T}\widetilde{M}_k(y^0), \tag{36}$$

where

$$\mathcal{A}_k = \begin{pmatrix} d & 0 & 0 & 0 & \cdots & 0 \\ -1 & dk^2 & 0 & 0 & \cdots & 0 \\ 0 & -k^2 & \frac{d}{2}k^4 & 0 & \cdots & 0 \\ 0 & 0 & -\frac{1}{2}k^4 & \frac{d}{6}k^6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{k^{2(n-2)}}{(n-2)!} & d\frac{k^{2(n-1)}}{(n-1)!} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^n),$$

$$X_k = \left(\int_0^T v(T-t)t^j e^{-dk^2t}\,dt\right)_{0\le j\le n-1} \in \mathbb{R}^n,\ \widetilde{M}_k(y^0) = \left(\widetilde{M}_{j,k}(y^0)\right)_{0\le j\le n-1} \in \mathbb{R}^n.$$

System (36) is triangular, then it is equivalent to

$$\begin{cases} \text{Find } v \in L^2(0,T) \text{ such that} \\ \displaystyle\int_0^T v(T-t)t^j e^{-\lambda_k t}\,dt = e^{-dk^2T}M_{j,k}(y^0), \quad \forall k \in \mathbb{N}^\star,\ 0 \le j \le n-1, \end{cases} \tag{37}$$

where the coefficients $M_{j,k}(y^0)$ are given by

$$\left(M_{j,k}(y^0)\right)_{0\le j\le n-1} := \mathcal{A}_k^{-1}\widetilde{M}_k(y^0) \in \mathbb{R}^n. \tag{38}$$

In summarizing, the null controllability property of system (2) at time T is equivalent to the moment problem (37).

Our next step will be to prove that the moment problem (37) admits a solution $v \in L^2(0,T)$. Firstly, we can apply Theorem 2.6, with $\eta = n$, to the real sequence $\Lambda := \{dk^2\}_{k\ge1}$ and deduce the existence of a family $\{q_{j,k}\}_{k\ge1, 0\le j\le n-1} \subset L^2(0,T)$ biorthogonal to $\left\{t^j e^{-\lambda_k t}\right\}_{k\ge1,0\le j\le n-1}$ which satisfies (21). As in the previous case, this fact provides a formal solution to the moment problem (37):

$$v(t) := \sum_{k=1}^{\infty}\sum_{j=0}^{n-1} e^{-dk^2T}M_{j,k}(y^0)q_{j,k}(T-t).$$

On the other hand, the previous series converges absolutely in $L^2(0,T)$. Indeed, taking into account the expression of the coefficients $M_{j,k}(y^0)$ (see (38), (35) and (36)), the estimate (21) and $y^0 \in H^{-1}(0,\pi;\mathbb{R}^n)$, it is not difficult to prove the following property: for any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\left|M_{j,k}(y^0)\right| \|q_{j,k}\|_{L^2(0,T)} \le C_\varepsilon e^{\varepsilon dk^2}\|y^0\|_{H^{-1}(0,\pi;\mathbb{R}^n)}, \quad \forall (j,k) : k \ge 1,\ 0 \le j \le n-1.$$

With this inequality, we can reason as in Section 3.1.1 and prove that v is a solution to the moment problem (37). □

3.2. **Proof of Corollary 1.5.** First of all, observe that if system (2) is null controllable at time T then it is approximately controllable at time T. On the other hand, notice that the approximate controllability at time T for system (2) is equivalent to a Fattorini-Hautus test. More precisely,

Theorem 3.1. *System (2) is approximately controllable at time T if and only if, for every $\lambda \in \mathbb{C}$ and $\Phi \in D(L^\star)$, we have the following property*

$$\left.\begin{array}{ll} L^\star\Phi = \lambda\Phi & \text{in } (0,\pi) \\ B^\star D^\star \partial_x \Phi(0) = 0 & \end{array}\right\} \Longrightarrow \Phi = 0 \text{ in } (0,\pi).$$

For the proof, one just have to apply [33, Theorem 1.1].

Let us use Theorem 3.1 applied to the operator $L = -D\partial_{xx} + A$, combined with Theorem 1.4, in order to deduce that approximate controllability property for system (2) implies the null controllability property. To do this, by contradiction, assume first that $n = 2p+1$, with $p \in \mathbb{N}^\star$, $\alpha < 0$ and $\sqrt{|\alpha|}/d^{n/2} := \mathcal{K} \in \mathbb{N}^\star$. In this case, $\Phi = \Phi_{p,\mathcal{K}}$ (see item *a)* of Proposition 2.1) satisfies $\Phi \not\equiv 0$, $L^\star\Phi = \lambda\Phi$, with $\lambda = \lambda_{p,\mathcal{K}}$, and

$$B^\star D^\star \partial_x \Phi(0) = \mathcal{K}\sqrt{\frac{2}{\pi}} B^\star D^\star V_{p,\mathcal{K}} = -\mathcal{K}\sqrt{\frac{2}{\pi}} |\alpha|^{-\frac{n-2}{n}} k^{2\frac{n-2}{n}} \left(1 - d\,|\alpha|^{-\frac{1}{n}}\,\mathcal{K}^{\frac{2}{n}}\right) = 0.$$

From Theorem 3.1, we deduce that system (2) is not approximately controllable at time T.

On the other hand, assume now that there exists $k, k' \in \mathbb{N}^\star$ and $j, j' \in \{0, ..., n-1\}$ such that $(j,k) \neq (j',k')$ and

$$\lambda := \lambda_{j,k} = \lambda_{j',k'}.$$

The eigenvalue λ is, at least, double (geometric multiplicity) and some associated eigenfunctions are

$$\widetilde{\Phi}_{j,k} = \widetilde{V}_{j,k} w_k \quad \text{and} \quad \widetilde{\Phi}_{j',k'} = \widetilde{V}_{j',k'} w_{k'},$$

where $\widetilde{V}_{j,k} := \left(c^l_{j,k}\right)_{1\le l\le n} \in \mathbb{C}^n$ and the coefficients $c^l_{j,k}$, $1 \le l \le n$, are given in (13) (see Proposition 2.1).

Let us point out that $B^\star D^\star = (0, \cdots, 0, 1, d)$. Thus, we introduce

$$\Phi := k'\left[c^{n-1}_{j',k'} + dc^n_{j',k'}\right]\widetilde{\Phi}_{j,k} - k\left[c^{n-1}_{j,k} + dc^n_{j,k}\right]\widetilde{\Phi}_{j',k'}.$$

It is not difficult to see that Φ is not identically zero and satisfies

$$\left\{\begin{array}{ll} -D^\star\partial_{xx}\Phi + A^\star\Phi = \lambda\Phi & \text{in } (0,\pi) \\ B^\star D^\star \partial_x\Phi(0) = 0. & \end{array}\right.$$

Therefore, Theorem 3.1 leads to the non-approximate controllability of system (2). □

3.3. **Proof of Theorem 1.7.** We will devote this section to prove Theorem 1.7. To this end, let us consider system (2) in the case $n = 2$, with matrices $D, A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ given by (9), with $d \geq 1$ and $\alpha > 0$. In this case, recall Proposition 2.1, the eigenvalues of the operator $L^\star = -D^\star\partial_{xx} + A^\star$ are given by

$$\lambda_{0,k} := dk^2 + \sqrt{\alpha}k, \quad \lambda_{1,k} := dk^2 - \sqrt{\alpha}k, \quad k \geq 1 \tag{39}$$

and $\Phi_{0,k} = V_{0,k} w_k$ and $\Phi_{1,k} = V_{1,k} w_k$, with $V_{j,k} = \widetilde{V}_{j,k}/|\widetilde{V}_{j,k}|$ $(j = 0, 1)$ and

$$\widetilde{V}_{0,k} = \begin{pmatrix} 1 \\ \alpha^{-1/2} k \end{pmatrix}, \quad \widetilde{V}_{1,k} = \begin{pmatrix} 1 \\ -\alpha^{-1/2} k \end{pmatrix}, \tag{40}$$

are eigenvectors associated to $\lambda_{0,k}$ and $\lambda_{1,k}$.

As in Section 3.1.1 and without loss of generality, we are going to assume that $\sigma(L^\star) \subset (0, \infty)$, i.e., $\lambda_{1,k} > 0$ for all $k \in \mathbb{N}^\star$.

Observe that in the case $n = 2$, inequalities (19) and (20) are, in general, not valid when one takes as sequence Λ the real sequence $\Lambda := \{\lambda_{0,k}, \lambda_{1,k}\}_{k\geq 1}$. Indeed, from (39), we deduce the equalities

$$\begin{cases} \lambda_{1,k+m} - \lambda_{0,k} = (2k + m)\left(dm - \sqrt{\alpha}\right), & \forall k, m \in \mathbb{N}^\star, \\ \lambda_{1,k} - \lambda_{1,\ell} = (k - \ell)\left[d(k + \ell) - \sqrt{\alpha}\right], & \forall k, \ell \in \mathbb{N}^\star. \end{cases} \tag{41}$$

Thus, if $\sqrt{\alpha}/d = m \in \mathbb{N}^\star$, we have $\lambda_{0,k} = \lambda_{1,k+m}$, for any $k \geq 1$, and $\lambda_{1,k} = \lambda_{1,m-k}$, for any $k : 1 \leq k \leq m - 1$.

As a consequence, we deduce that the operator $L^\star$ has an infinite number of eigenvalues with geometric multiplicity equal to 2. Therefore, we can follow the arguments of Section 3.2 and conclude that system (2) is neither approximately nor null controllable in $H^{-1}(0, \pi; \mathbb{R}^2)$ at any time $T > 0$. This proves the necessary part of Theorem 1.7.

Let us now assume that $\sqrt{\alpha}/d \notin \mathbb{N}^\star$ and prove that system (2) is null controllable in the space $H^{-1}(0, \pi; \mathbb{R}^2)$ at time $T > 0$. As in Section 3.1.1, this controllability result is equivalent to the moment problem:

$$\begin{cases} \text{Given } y^0 \in H^{-1}(0, \pi; \mathbb{R}^2), \text{ find } v \in L^2(0, T) \ : \ \forall (j, k) \in \{0, 1\} \times \mathbb{N}^\star, \\ B^\star D^\star V_{j,k} \displaystyle\int_0^T v(T - t) e^{-\lambda_{j,k} t}\, dt = -\frac{1}{k}\sqrt{\frac{\pi}{2}} e^{-\lambda_{j,k} T} \langle y^0, \Phi_{j,k} \rangle_{H^{-1}, H_0^1}. \end{cases} \tag{42}$$

We follow the arguments of Section 3.1.1 in order to solve the previous moment problem.

In this case, we can write

$$m - 1 < \frac{\sqrt{\alpha}}{d} < m,$$

where $m \geq 1$ is a positive integer. From (41), it is not difficult to prove that the eigenvalues of $L^\star$ are simple and one has

$$\lambda_{1,k+m-1} < \lambda_{0,k} < \lambda_{1,k+m}, \quad \forall k \geq 1,$$

and

$$\lambda_{1,k+m} - \lambda_{0,k} = (2k + m)\,(dm - \sqrt{\alpha}) \geq 3\,(dm - \sqrt{\alpha}) := \mathcal{C}_0 > 0,$$
$$\lambda_{0,k} - \lambda_{1,k+m-1} = (2k + m - 1)\,[\sqrt{\alpha} - d(m - 1)] \geq 2\,[\sqrt{\alpha} - d(m - 1)] := \mathcal{C}_1 > 0,$$

for any positive integer $k \in \mathbb{N}^\star$. Thus, the sequence $\Lambda = \{\lambda_{0,k}, \lambda_{1,k}\}_{k\geq 1}$ can be rearranged as an increasing sequence of positive real numbers $\Lambda = \{\Lambda_k\}_{k\geq 1}$ as follows:

$$\begin{cases} \{\Lambda_k\}_{1\leq k\leq m} := \{\lambda_{1,k}\}_{1\leq k\leq m}\,, \\ \Lambda_{m+2\ell-1} := \lambda_{0,\ell}, & \forall \ell \geq 1, \\ \Lambda_{m+2\ell} := \lambda_{1,m+\ell}, & \forall \ell \geq 1. \end{cases}$$

Therefore, the increasing sequence Λ satisfies properties (17) and (20) for

$$\rho := \min\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2\} > 0, \quad \mathcal{C}_2 := \frac{1}{m} \min_{1\leq k,\ell\leq m} |\lambda_{1,k} - \lambda_{1,\ell}|\,.$$

On the other hand, as in Section 3.1.1 (see(31)), we can see that the assumption $\sqrt{\alpha}/d \notin \mathbb{N}^\star$ implies that $B^\star D^\star V_{j,k} \neq 0$ for every $k \in \mathbb{N}^\star$ and $j = 0,1$ (see (40)). Therefore, we can apply the arguments of Section 3.1.1 to solve the moment method (42) and obtain the null controllability property of this system in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time $T > 0$. This proves the sufficient part of Theorem 1.7.

Let us now prove the last part of Theorem 1.7. So, assume that

$$\frac{\sqrt{\alpha}}{d} = m \in \mathbb{N}^\star. \tag{43}$$

As said before, the operator $L^\star$ has an infinite number of eigenvalues with geometric multiplicity equal to 2 (see (41)) and, in general, the moment problem (42) cannot be solved. Nevertheless, this problem has a solution $v \in L^2(0,T)$ for some initial data $y^0 \in H^{-1}(0,\pi;\mathbb{R}^2)$. Let us see this point.

Firstly, we have $\lambda_{0,k} = \lambda_{1,k+m}$, for any $k \geq 1$, and $\lambda_{1,k} = \lambda_{1,m-k}$, for any $k : 1 \leq k \leq m-1$. Thus, the real sequence $\Lambda := \{\lambda_{0,k}, \lambda_{1,k}\}_{k\geq 1}$ is, in fact, $\Lambda = \{\lambda_{1,k}\}_{k\geq m_0}$, where $m_0 = 1 + \left\lfloor \frac{m-1}{2} \right\rfloor$ ($\lfloor\cdot\rfloor$ is the floor function). From the expression of m (see (43)) and m_0, we can prove that the sequence Λ is increasing and satisfies properties (17) and (20).

Secondly, let us remember that, given $y^0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, there exists $v \in L^2(0,T)$ such that the solution y of (2) satisfies $y(T,\cdot) = 0$ in $(0,\pi)$ if and only if the control v solves the moment problem (42). Using the expressions (40) and (43), this moment problem can be rewritten as

$$\left\{ \begin{array}{l} \text{Given } y^0 \in H^{-1}(0,\pi;\mathbb{R}^2), \text{ find } v \in L^2(0,T) \text{ such that for all } k \in \mathbb{N}^\star, \\ \left(1 + \dfrac{k}{m}\right) \displaystyle\int_0^T v(T-t) e^{-\lambda_{0,k} t}\, dt = -\frac{1}{k}\sqrt{\frac{\pi}{2}} e^{-\lambda_{0,k} T} \left(y_k^0, \widetilde{V}_{0,k}\right)_{\mathbb{R}^2}, \\ \left(1 - \dfrac{k}{m}\right) \displaystyle\int_0^T v(T-t) e^{-\lambda_{1,k} t}\, dt = -\frac{1}{k}\sqrt{\frac{\pi}{2}} e^{-\lambda_{1,k} T} \left(y_k^0, \widetilde{V}_{1,k}\right)_{\mathbb{R}^2}, \end{array} \right. \tag{44}$$

where $y_k^0 = \langle y^0, w_k \rangle_{H^{-1}, H_0^1} \in \mathbb{R}^2$, $k \geq 1$, are the Fourier coefficients of y^0.

For $y^0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, let us consider the conditions

$$\left\{ \begin{array}{ll} \left(y_k^0, \widetilde{V}_{1,k}\right)_{\mathbb{R}^2} = \left(y_{m-k}^0, \widetilde{V}_{1,m-k}\right)_{\mathbb{R}^2} & \forall k : 1 \leq k \leq m_0 - 1, \\ \left(y_k^0, \widetilde{V}_{0,k}\right)_{\mathbb{R}^2} = -\left(y_{k+m}^0, \widetilde{V}_{1,k+m}\right)_{\mathbb{R}^2} & \forall k \geq 1, \\ \left(y_m^0, \widetilde{V}_{1,m}\right)_{\mathbb{R}^2} = 0, & \end{array} \right. \tag{45}$$

and introduce the closed subspace of $H^{-1}(0,\pi;\mathbb{R}^2)$ given by:

$$\mathcal{X} := \left\{ y^0 \in H^{-1}(0,\pi;\mathbb{R}^2) : y^0 \text{ satisfies conditions (45)} \right\}.$$

The set $\mathcal{X}$ is a closed subspace of $H^{-1}(0,\pi;\mathbb{R}^2)$ which has infinite codimension. Indeed, if we consider the closed subspace $\mathcal{Y}$ given by

$$\left\{ e_1 f : f_k = 0,\ 1 \leq k \leq m,\ \{f_k\}_{k\geq 1} \text{ are the Fourier coefficients of } f \in H^{-1}(0,\pi) \right\}$$

then, thanks to conditions (45), one has

$$\mathcal{X} \cap \mathcal{Y} = \{0\}.$$

On the other hand, taking into account that $\dim \mathcal{Y} = \infty$, we deduce that $\mathcal{X}$ has infinite codimension[2].

In order to finish the proof of Theorem 1.7, let us see the property: *"given $y^0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, there exists a control $v \in L^2(0,T)$ such that the solution y of system (2) satisfies $y(T,\cdot) = 0$ in $(0,\pi)$ if and only if $y^0 \in \mathcal{X}$"*, i.e., let us prove that the moment problem (44) has solution if and only if $y^0 \in \mathcal{X}$.

Let us start assuming that $y^0 \in \mathcal{X}$. In this case, from (45) and the expressions of m and the functions $\Phi_{j,k}$ (see (43) and (40)), the moment problem (44) is equivalent to

$$\left\{ \begin{array}{l} \text{Find } v \in L^2(0,T) \text{ such that for all } k \geq m_0,\ k \neq m, \\ \left(1 - \dfrac{k}{m}\right) \displaystyle\int_0^T v(T-t) e^{-\lambda_{1,k} t}\, dt = -\frac{1}{k}\sqrt{\frac{\pi}{2}} e^{-\lambda_{1,k}T} \left(y_k^0, \widetilde{V}_{1,k}\right)_{\mathbb{R}^2}. \end{array} \right.$$

As proved before, the sequence $\Lambda = \{\lambda_{1,k}\}_{k \geq m_0}$ is increasing and satisfies properties (17) and (20). Following the arguments of Section 3.1.1, we deduce that the previous moment problem admits a solution $v \in L^2(0,T)$.

Let us now suppose that $y^0 \notin \mathcal{X}$. In this case, the moment problem is incompatible and does not admit any solution:

1. If $\left(y_m^0, \widetilde{V}_{1,m}\right)_{\mathbb{R}^2} \neq 0$, it is clear that the second equation of (44) has no solution when $k = m$.
2. If for some $k_0 : 1 \leq k_0 \leq m_0 - 1$ (resp., $k_0 \geq 1$) one has $\left(y_{k_0}^0, \widetilde{V}_{1,k_0}\right)_{\mathbb{R}^2} \neq \left(y_{m-k_0}^0, \widetilde{V}_{1,m-k_0}\right)_{\mathbb{R}^2}$ (resp., $\left(y_{k_0}^0, \widetilde{V}_{0,k_0}\right)_{\mathbb{R}^2} \neq -\left(y_{k_0+m}^0, \widetilde{V}_{1,k_0+m}\right)_{\mathbb{R}^2}$), from the equality $\lambda_{1,k_0} = \lambda_{1,m-k_0}$ (resp., $\lambda_{0,k_0} = \lambda_{1,k_0+m}$), it is not difficult to show that the problem (44) is incompatible.

This proves the previous equivalence and ends the proof of Theorem 1.7. □

3.4. **Pointwise controllability.** In this section we will prove the null controllability at time $T > 0$ of system (12) in $L^2(0,\pi;\mathbb{R}^n)$ when $x_0 \in (0,\pi)$ satisfies appropriate properties. To this end, we will follow the same ideas of the proof of Theorem 1.4.

Remark 3.2. Taking into account that $\delta_{x_0} \in H^{-1}(0,\pi)$, we deduce that, for any $y^0 \in L^2(0,\pi;\mathbb{R}^n)$ and $u \in L^2(0,T)$, system (12) admits a unique solution y with regularity

$$y \in L^2(0,T;H_0^1(0,\pi;\mathbb{R}^n)) \cap \mathcal{C}^0([0,T];L^2(0,\pi;\mathbb{R}^n)),$$

and which depends continuously on y^0 and u.

Let us first describe the approximate controllability result for system (12). One has:

Theorem 3.3. *Let us consider the matrices $D, A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathbb{R}^n$ given by (9), with $d \geq 1$ and $\alpha \in \mathbb{R}$. In addition, assume that $n = 2p+1$ ($p \in \mathbb{N}^\star$) when $\alpha > 0$. Let us also fix $x_0 \in (0,\pi)$. Then,*

1. *If $\alpha \neq 0$, system (12) is approximately controllable at time T if and only if one has:*

$$\left\{ \begin{array}{l} \text{the eigenvalues of } L^\star = -D^\star \partial_{xx} + A^\star \\ \text{has geometric multiplicity equal to } 1; \\ x_0 \neq r\pi, \ \text{with } r \in \mathbb{Q} \cap (0,1). \end{array} \right. \tag{46}$$

[2] $H^{-1}(0,\pi;\mathbb{R}^2)$ is a Hilbert space. Then, consider $P : H^{-1}(0,\pi;\mathbb{R}^2) \to \mathcal{X}$, the orthogonal projection of $H^{-1}(0,\pi;\mathbb{R}^2)$ onto $\mathcal{X}$. Thus, $\mathcal{A} : y \in \mathcal{Y} \mapsto \mathcal{A}(y) = y - Py \in \mathcal{X}^\perp$ is injective.

2. *If $\alpha = 0$, system (12) is approximately controllable at time $T > 0$ if and only if $x_0 \neq r\pi$, with $r \in \mathbb{Q} \cap (0,1)$.*

Proof. As saw in Section 3.2, it is not to difficult to show that system (12) is approximately controllable in $L^2(0,\pi;\mathbb{R}^2)$ at time $T > 0$ if and only if, for every $\lambda \in \mathbb{C}$ and $\Phi \in D(L^\star)$, we have the following property

$$\left.\begin{array}{ll} L^\star\Phi = \lambda\Phi & \text{in } (0,\pi) \\ B^\star\Phi(x_0) = 0 & \end{array}\right\} \Longrightarrow \Phi = 0 \text{ in } (0,\pi). \tag{47}$$

We will do the proof for $\alpha \neq 0$. The case $\alpha = 0$ can be obtained following the same argument.

Let us first see that the conditions in (46) are necessary for the approximate controllability of system (12). We can argue as in Section 3.2 and prove that the first condition in (46) is necessary. On the other hand, if $x_0 = r\pi$, with $r = m/\ell \in \mathbb{Q} \cap (0,1)$ and $m, \ell \in \mathbb{N}^\star$, we deduce that $\lambda = \lambda_{0,\ell} \in \mathbb{R}$ and $\Phi := \Phi_{0,\ell} = V_{0,\ell} w_\ell$ (see Proposition 2.1) satisfy $L^\star\Phi = \lambda\Phi$,

$$B^\star\Phi(x_0) = \sqrt{\frac{2}{\pi}} B^\star V_{0,\ell} \sin(\ell x_0) = \sqrt{\frac{2}{\pi}} B^\star D^\star V_{0,\ell} \sin(m\pi) = 0,$$

and $\Phi \not\equiv 0$ in $(0,\pi)$. Therefore, from (47) we deduce that system (12) cannot be approximately controllable at time $T > 0$.

Let us see that conditions in (46) imply the approximate controllability of system (12) at time $T > 0$ i.e., the Fattorini-Hautus test (47). Indeed, first, if $\Phi \in D(L^\star)$ satisfies $L^\star\Phi = \lambda\Phi$ in $(0,\pi)$, for $\lambda \in \mathbb{C}$, we deduce $\lambda = \lambda_{j,k}$ and $\Phi = C\Phi_{j,k}$, with $k \geq 1$, $0 \leq j \leq n-1$ and $C \in \mathbb{C}$. Secondly, taking into account the expression of $\Phi_{j,k}$ (see Proposition 2.1), condition $B^\star\Phi(x_0) = 0$ can be also written as

$$0 = B^\star\Phi(x_0) = \begin{cases} S_{j,k}\left(\alpha^{-\frac{n-1}{n}} k^{2\frac{(l-1)}{n}} e^{-\frac{2\pi j}{n}(n-1)i}\right)\sin(kx_0), & \text{if } \alpha > 0, \\ S_{j,k}\left(|\alpha|^{-\frac{n-1}{n}} k^{2\frac{(n-1)}{n}} e^{-\frac{(2j+1)\pi}{n}(n-1)i}\right)\sin(kx_0), & \text{if } \alpha < 0. \end{cases}$$

where $S_{j,k} = \frac{C}{\left|\left(c^l_{j,k}\right)_{1\leq l\leq n}\right|}\sqrt{\frac{2}{\pi}}$. In both cases, assumptions (46) imply $C = 0$ and, therefore, $\Phi \equiv 0$. Then, one has (47) and the approximate controllability of system (12) at time $T > 0$. This ends the proof. □

Remark 3.4. Taking into account Proposition 2.7 (see Remark 2.8), we deduce that, when $\alpha \neq 0$ is under the assumptions of Theorem 3.3, the eigenvalues of $L^\star$ satisfy $m(\lambda_{j,k}) = 1$ for any $k \geq k_1$ and any $j : 0 \leq j \leq n-1$. Thus, if $x_0 \in (0,\pi)$ is such that $x_0/\pi \notin \mathbb{Q}$, system (12) is approximately controllable in $L^2(0,\pi;\mathbb{R}^2)$ at time $T > 0$ apart from a finite dimensional space of $L^2(0,\pi;\mathbb{R}^2)$.

On the other hand, the proof of Theorem 3.3 is still valid when $\alpha > 0$ and $n = 2p$, with $p \in \mathbb{N}^\star$, that is to say, system (12) is approximately controllable at time T if and only if (46) holds. Nevertheless, in this case, we can have an infinite number of eigenvalues λ of $L^\star$ such that $\mu(\lambda) \geq 2$ (this is the case when $n = 2$ and p and α satisfy (43), see Section 3.3). So, we cannot conclude that, if $x_0 \in (0,\pi)$ is such that $x_0/\pi \notin \mathbb{Q}$, system (12) is approximately controllable in $L^2(0,\pi;\mathbb{R}^2)$ at time $T > 0$ apart from a finite dimensional space of $L^2(0,\pi;\mathbb{R}^2)$.

Let us now study the null controllability of system (12). The result reads as follows:

Theorem 3.5. *Under conditions of Theorem 3.3, let us assume that (46) holds when $\alpha \neq 0$. In addition, assume that $x_0 = \vartheta\pi$, with $\vartheta \in (0,1)$ an irrational number, and consider*

$$T_\vartheta = \limsup_{k\to\infty} \frac{-\log|\sin(k\vartheta\pi)|}{dk^2} \in [0,\infty].$$

Then:

1. *System (12) is null controllable in $L^2(0,\pi;\mathbb{C}^n)$ at any time $T > T_\vartheta$.*
2. *System (12) is not null controllable in $L^2(0,\pi;\mathbb{C}^n)$ for $T < T_\vartheta$.*

Proof. Again, we will do the proof of the result when $\alpha \neq 0$. The case $\alpha = 0$ can be obtained from a similar argument (see Section 3.1.2). So, assume that (46) holds and take $x_0 = \vartheta\pi$ where $\vartheta \in (0,1)$ is an irrational number.

Let us first prove the first item in Theorem 3.5. To this end, assume $T_\vartheta \in [0,\infty)$ and take $T > T_\vartheta$. As in Section 3.1.1, the null controllability at time $T > 0$ of system (12) is equivalent to the following moment problem: given $y^0 \in L^2(0,\pi;\mathbb{C}^n)$, find a control $v \in L^2(0,\pi;\mathbb{C})$ such that

$$\sqrt{\frac{2}{\pi}} \sin(k\vartheta\pi)\,(B^\star V_{j,k})^\star \int_0^T v(T-t)e^{-\lambda_{j,k}^\star t}\,dt = -e^{-\lambda_{j,k}^\star T}(y^0,\Phi_{j,k})_{L^2},$$

for any $k \in \mathbb{N}^\star$ and any $j : 0 \le j \le n-1$. From (46) we deduce that $\mu(\lambda_{j,k}) = 1$ and $\sin(k\vartheta\pi) \neq 0$. Also, it is easy to check that $B^\star V_{j,k} \neq 0$. Thus, the previous moment problem is equivalent to

$$\begin{cases} \forall y^0 \in L^2(0,\pi;\mathbb{C}^n), \exists v \in L^2(0,\pi;\mathbb{C}) \ : \ \forall (j,k) \in \{0\dots,n-1\} \times \mathbb{N}^\star, \\ \displaystyle\int_0^T v(T-t)e^{-\lambda_{j,k}^\star t}\,dt = e^{-\lambda_{j,k}^\star T}\widetilde{M}_{j,k}(y^0), \end{cases} \tag{48}$$

where

$$\widetilde{M}_{j,k}(y^0) := -\frac{1}{\sin(k\vartheta\pi)}\sqrt{\frac{\pi}{2}}\frac{(y^0,\Phi_{j,k})_{L^2}}{V_{j,k}^\star B}.$$

Again, a formal solution to the previous moment problem is

$$v(t) = \sum_{k=1}^\infty \sum_{j=0}^{n-1} e^{-\lambda_{j,k}^\star T}\widetilde{M}_{j,k}(y^0)q_{j,k}(T-t),$$

where $\{q_{j,k}\}_{k\ge1,0\le j\le n-1}$ is a biorthogonal family to $\{p_{j,k}\}_{k\ge1,0\le j\le n-1}$ (see (33)) in $L^2(0,T;\mathbb{C})$ satisfying (34) (see Theorem 2.4).

Let us check that the previous series is absolutely convergent in $L^2(0,T;\mathbb{C})$. Indeed, from Proposition 2.1, we get

$$dk^2 - |\alpha|^{1/n}k^{2-\frac{2}{n}} \le \Re(\lambda_{j,k}^\star) \le dk^2 + |\alpha|^{1/n}k^{2-\frac{2}{n}}, \quad \forall k \ge 1,\ j : 0 \le j \le n-1,$$

On the other hand, using the expression of T_ϑ, we deduce that for any $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\frac{1}{|\sin(k\vartheta\pi)|} \le C_\varepsilon e^{(T_\vartheta+\varepsilon)dk^2}, \quad \forall k \ge 1.$$

Finally, repeating the arguments in Section 3.1.1, we also have that, for any $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\left|\frac{(y^0,\Phi_{j,k})_{L^2}}{V_{j,k}^\star B}\right| \le C_\varepsilon e^{\varepsilon dk^2}\|y^0\|_{L^2(0,\pi;\mathbb{C}^n)}, \quad \forall k \ge 1,\ j : 0 \le j \le n-1.$$

Therefore, since $\{q_{j,k}\}_{k\geq 1, 0\leq j\leq n-1}$ satisfies (34), we have

$$\left\{\begin{aligned} \|v\|_{L^2(0,T;\mathbb{C})} &\leq \sum_{k=1}^{\infty}\sum_{j=0}^{n-1} C_\varepsilon e^{-\Re(\lambda_{j,k})T} e^{(T_\vartheta+\varepsilon)dk^2} e^{\varepsilon dk^2} \|y^0\|_{L^2(0,\pi;\mathbb{C}^n)} e^{\varepsilon\Re(\lambda_{j,k})} \\ &\leq \sum_{k=1}^{\infty}\sum_{j=0}^{n-1} C_\varepsilon e^{-dk^2 T} e^{(T+\varepsilon)|\alpha|^{1/n}k^{2-\frac{2}{n}}} e^{(T_\vartheta+\varepsilon)dk^2} e^{2\varepsilon dk^2} \|y^0\|_{L^2(0,\pi;\mathbb{C}^n)} \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{n} C_\varepsilon e^{-dk^2(T-T_\vartheta-3\varepsilon)} e^{(T+\varepsilon)|\alpha|^{1/n}k^{2-\frac{2}{n}}} \|y^0\|_{L^2(0,\pi;\mathbb{C}^n)}. \end{aligned}\right.$$

where C_ε is a new positive constant. It is clear that, taking $\varepsilon = (T-T_\vartheta)/6$, the previous series converges absolutely. This proves that $v \in L^2(0,T;\mathbb{C})$ and we have constructed a solution of the moment problem (48). This shows the first item in Theorem 3.5.

Let us now assume that $T_\vartheta \in (0,\infty]$ and consider $0 < T < T_\vartheta$. The objective is to prove that system (12) is not null controllable in $L^2(0,\pi;\mathbb{C}^n)$ at time T. To this end, we will use the following result:

Theorem 3.6. *Let us consider the matrices $D, A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathbb{R}^n$ given by (9), with $d \geq 1$ and $\alpha \in \mathbb{R}$. Then, system (12) is null controllable in $L^2(0,\pi;\mathbb{C}^n)$ at time $T > 0$ if and only if there exists a constant $C_T > 0$ such that*

$$\|\varphi(0,\cdot)\|^2_{L^2(0,\pi;\mathbb{C}^n)} \leq C_T \int_0^T |B^\star \varphi(t,x_0)|^2 \, dt, \quad \forall \varphi^T \in L^2(0,\pi;\mathbb{C}^n), \tag{49}$$

where $\varphi \in L^2(0,T;H_0^1(0,\pi;\mathbb{C}^n)) \cap \mathcal{C}^0([0,T];L^2(0,\pi;\mathbb{C}^n))$ is the solution of the adjoint problem (27) associated to $\varphi^T \in L^2(0,\pi;\mathbb{C}^n)$.

For a proof of this result, see [35] and [36].

Let us see that the observability inequality (49) fails when $T < T_\vartheta$ and, therefore, system (12) is not null controllable in $L^2(0,\pi;\mathbb{C}^n)$ at time T. By contradiction, assume that, for a positive constant C_T, inequality (49) holds. In particular, if we take

$$\varphi^T(x) = V_{0,k} w_k(x) = \sqrt{\frac{2}{\pi}} V_{0,k} \sin(kx), \quad x \in (0,\pi),$$

(see Proposition 2.1), the corresponding solution of system (12) is given by

$$\varphi(t,x) = \sqrt{\frac{2}{\pi}} e^{-\lambda_{0,k}(T-t)} V_{0,k} \sin(kx), \quad (t,x) \in Q_T,$$

and inequality (49) becomes

$$|V_{0,k}|^2 e^{-2\Re(\lambda_{0,k})T} \leq \frac{2}{\pi} C_T |B^\star V_{0,k}|^2 \frac{1}{2\Re(\lambda_{0,k})} \left(1 - e^{-2\Re(\lambda_{0,k})T}\right) |\sin(k\vartheta\pi)|^2, \tag{50}$$

for all $k \geq 1$.

On the other hand, let us take $\varepsilon > 0$ such that $T + \varepsilon < T_\vartheta$. From the definition of T_ϑ we deduce the existence of a subsequence $\{k_n\}_{n\geq 1}$ such that

$$T + \varepsilon < \frac{-\log|\sin(k_n\vartheta\pi)|}{dk_n^2}, \quad \forall n \geq 1,$$

or, equivalently,

$$|\sin(k_n\vartheta\pi)|^2 < e^{-2dk_n^2(T+\varepsilon)}, \quad \forall n \geq 1.$$

Combining the previous inequality and (50) written for the subsequence $\{k_n\}_{n\geq 1}$, we get

$$|V_{0,k_n}|^2 \, e^{-2\Re(\lambda_{0,k_n})T} \leq \frac{2}{\pi} C_T \, |B^\star V_{0,k_n}|^2 \, \frac{1}{2\Re(\lambda_{0,k_n})} \left(1 - e^{-2\Re(\lambda_{0,k_n})T}\right) e^{-2dk_n^2(T+\varepsilon)},$$

for all $n \geq 1$, that is to say,

$$0 < \frac{\pi}{2} C_T^{-1} \leq \mathcal{A}_n, \quad \forall n \geq 1,$$

where

$$\mathcal{A}_n := \frac{|B^\star V_{0,k_n}|^2}{|V_{0,k_n}|^2} \frac{1}{2\Re(\lambda_{0,k_n})} \left(1 - e^{-2\Re(\lambda_{0,k_n})T}\right) e^{-2dk_n^2\varepsilon + 2[\Re(\lambda_{0,k_n}) - dk_n^2]T} \quad \forall n \geq 1.$$

Finally, taking into account the expressions of $V_{0,k}$ and $\lambda_{0,k}$ (see Proposition 2.1) and the inequality

$$\Re(\lambda_{0,k}) \leq dk^2 + |\alpha|^{1/n} k^{2-\frac{2}{n}}, \quad \forall k \geq 1,$$

we obtain that $\lim\limits_{n\to+\infty} \mathcal{A}_n = 0$.

This provides a contradiction and the proof of item 2 of Theorem 3.5. □

Remark 3.7. To the authors' knowledge, the first pointwise null controllability result for a parabolic equation was proved in [16] for the one dimensional heat equation. Similar results were obtained for coupled parabolic systems in [8].

4. Parabolic systems with non-diagonalizable diffusion matrix and non-constant coefficients.

This section is devoted to prove Theorem 1.9. The negative part relies on the Fattorini-Hautus test applied to the operator $L_0 := -D\partial_{xx} + qA_0$. On the other hand, the positive part relies on the *algebraic resolvability* (see [27]).

Proof of Theorem 1.9, item a). First of all, notice that, from Theorem 3.1, system (10) is approximately controllable at time T if and only if the following property for the adjoint operator $L_0^\star$ holds:

For every $\lambda \in \mathbb{C}$ and $(\psi, \varphi) \in H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)$, it holds

$$\left.\begin{array}{lll} L_0^\star \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \lambda \begin{pmatrix} \psi \\ \varphi \end{pmatrix} & \text{in} & (0,\pi) \\ B^\star \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = 0 & \text{in} & \omega \end{array}\right\} \Longrightarrow \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in} \quad (0,\pi). \tag{51}$$

The idea is to construct a potential $q \in \mathcal{C}^\infty([0,\pi])$ such that property (51) does not holds. To this end, consider $\omega := (5\pi/12, 7\pi/12)$ and let us construct three functions $\varphi,\ \psi \in H^2(0,\pi) \cap H_0^1(0,\pi)$ and $q \in \mathcal{C}^\infty([0,\pi])$ satisfying

$$\left\{\begin{array}{ll} -\partial_{xx}\psi + q(\cdot)\psi = 36\psi & \text{in } (0,\pi), \\ -\partial_{xx}\varphi - \partial_{xx}\psi = 36\varphi & \text{in } (0,\pi), \\ \varphi = 0 & \text{in } \omega, \\ \psi \not\equiv 0, \ \varphi \not\equiv 0 & \text{in } (0,\pi). \end{array}\right. \tag{52}$$

The strategy will be to construct a suitable function ψ as a perturbation of $\sin(2x)$. With such a function, we define q and φ as $(52)_1$ and $(52)_2$, respectively, and we check that (φ, ψ) satisfies (52).

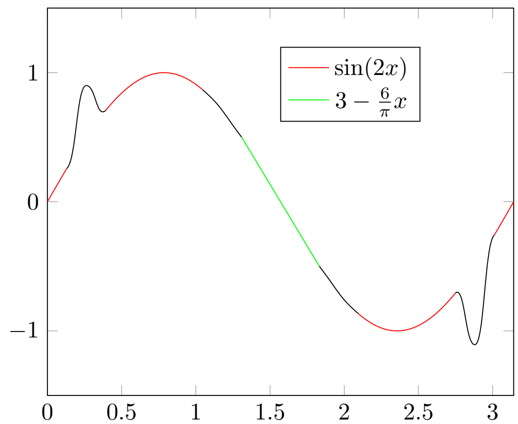

FIGURE 3. Example of a function ψ in $[0,\pi]$

Consider ψ a function of $\mathcal{C}^\infty([0,\pi])\cap H_0^1(0,\pi)$ satisfying

$$\begin{cases} \psi(x)=\sin(2x)+C_1\theta_1(x)-C_2\theta_2(x) & \forall x\in\left[0,\dfrac{\pi}{3}\right]\cup\left[\dfrac{2\pi}{3},\pi\right], \\ \psi(x)=-\dfrac{6}{\pi}x+3 & \forall x\in\overline{\omega}=\left[\dfrac{5\pi}{12},\dfrac{7\pi}{12}\right], \\ |\psi(x)-\sin(2x)|<\varepsilon & \forall x\in\left[\dfrac{\pi}{3},\dfrac{5\pi}{12}\right]\cup\left[\dfrac{7\pi}{12},\dfrac{2\pi}{3}\right], \end{cases} \tag{53}$$

where θ_1 and θ_2 are two nontrivial nonnegative functions of $\mathcal{C}^\infty([0,\pi])$ satisfying

$$\begin{cases} \operatorname{supp}\theta_1\subset\left(\dfrac{\pi}{24},\dfrac{3\pi}{24}\right), \\ \operatorname{supp}\theta_2\subset\left(\dfrac{21\pi}{24},\dfrac{23\pi}{24}\right), \end{cases} \tag{54}$$

$\varepsilon>0$ is small enough and C_1 and C_2 are two positive constants to be determined. The graph of ψ is given in Figure 3.

Notice that the function $\varphi\in\mathcal{C}^\infty([0,\pi])$, defined by

$$\varphi(x):=\left[\tau-\frac{1}{6}\int_0^x\cos(6y)\psi_{yy}(y)\,dy\right]\sin(6x)+\left[\frac{1}{6}\int_0^x\sin(6y)\psi_{yy}(y)\,dy\right]\cos(6x),$$

for all $x\in[0,\pi]$, satisfies the second equation of (52), where $\tau\in\mathbb{R}$ is a constant to be fixed later.

Let us now verify the boundary conditions and $(52)_3$ for φ. Let us first prove that C_1, in (53), and τ can be chosen such that $\varphi\equiv 0$ in ω. As $\psi(x)=-\frac{6}{\pi}x+3$ for $x\in\omega$ and ψ coincides with $\sin(2x)$ in a neighborhood of 0, we have

$$\begin{cases} -\dfrac{1}{6}\displaystyle\int_0^{\frac{5\pi}{12}}\cos(6y)\psi_{yy}(y)\,dy=6\int_0^{\frac{5\pi}{12}}\cos(6y)\psi(y)\,dy-\dfrac{1}{6}, \\ \dfrac{1}{6}\displaystyle\int_0^{\frac{5\pi}{12}}\sin(6y)\psi_{yy}(y)\,dy=-6\int_0^{\frac{5\pi}{12}}\sin(6y)\psi(y)dy-\dfrac{1}{\pi}, \end{cases} \tag{55}$$

and for all $x \in \omega$:

$$\varphi(x) = \left[\tau - \frac{1}{6} + 6\int_0^{\frac{5\pi}{12}} \cos(6y)\psi(y)\,dy\right]\sin(6x) - \left[\frac{1}{\pi} + 6\int_0^{\frac{5\pi}{12}} \sin(6y)\psi(y)\,dy\right]\cos(6x).$$

Since

$$\frac{1}{\pi} + 6\int_0^{\frac{5\pi}{12}} \sin(6y)\sin(2y)\,dy = \frac{1}{\pi} - \frac{3\sqrt{3}}{16} < 0,$$

thanks to (53)$_3$ (for ε small enough), one can choose C_1 (recalling that $\sin(6x) > 0$ in the interval $(\pi/24, 3\pi/24)$) such that

$$\frac{1}{\pi} + 6\int_0^{\frac{5\pi}{12}} \sin(6y)\psi(y)\,dy = 0. \tag{56}$$

In this way, for

$$\tau := \frac{1}{6} - 6\int_0^{\frac{5\pi}{12}} \cos(6y)\psi(y)\,dy,$$

we obtain $\varphi = 0$ in ω.

Let us now verify the boundary conditions for φ. Notice that $\varphi(0) = 0$, by definition. A suitable choice of C_2 will give us $\varphi(\pi) = 0$. Indeed, using that ψ is an affine function in ω and from equalities (55) and (56), we have

$$\varphi(\pi) = \frac{1}{6}\int_0^{\pi} \sin(6y)\psi_{yy}(y)\,dy = \frac{1}{6}\int_0^{\frac{5\pi}{12}} \sin(6y)\psi_{yy}(y)\,dy + \frac{1}{6}\int_{\frac{7\pi}{12}}^{\pi} \sin(6y)\psi_{yy}(y)\,dy$$
$$= \frac{1}{6}\int_{\frac{7\pi}{12}}^{\pi} \sin(6y)\psi_{yy}(y)\,dy = -\frac{1}{\pi} - 6\int_{\frac{7\pi}{12}}^{\pi} \sin(6y)\psi(y)\,dy.$$

Since

$$-\frac{1}{\pi} - 6\int_{\frac{7\pi}{12}}^{\pi} \sin(6y)\sin(2y)\,dy = -\frac{1}{\pi} + \frac{3\sqrt{3}}{16} > 0$$

again thanks to (53)$_3$ (for ε small enough) one can choose C_2 (recalling that $\sin(6x) < 0$ in $(21\pi/24, 23\pi/24)$) such that

$$-\frac{1}{\pi} - 6\int_{\frac{7\pi}{12}}^{\pi} \sin(6y)\psi(y)\,dy = 0$$

and then $\varphi(\pi) = 0$.

Finally, to verify the first equality in (52), we define $q \in \mathcal{C}^\infty([0,\pi])$ by

$$q := \frac{\partial_{xx}\psi + 36\psi}{\psi}, \tag{57}$$

with ψ given in (53). Taking into account that ψ is null only at points 0, $\pi/2$ and π, we infer the existence of neighborhoods of 0 and π in which ψ is equal to $\sin(2x)$ and a neighborhood of $\pi/2$ in which ψ is equal to $-\frac{6}{\pi}x + 3$. Therefore, we have that q is equal to 32 in the neighborhoods of 0 and π and equal to 36 in the neighborhood of $\pi/2$. Therefore, the function q is bounded and item a) in Theorem 1.9 is proved. □

Remark 4.1. The previous proof provides an argument to construct functions $\varphi,\ \psi \in H^2(0,\pi) \cap H_0^1(0,\pi)$ and $q \in \mathcal{C}^\infty([0,\pi])$ satisfying (52). In fact, this construction is valid for any functions $\theta_1, \theta_2 \in \mathcal{C}^\infty([0,\pi])$ satisfying (54). We will use this construction in the proof of Theorem 1.9, item b).

Let us now prove item b) of Theorem 1.9. To do that, we will use the following result, whose proof is given below, after the proof of this theorem.

Theorem 4.2. *Let $T > 0$, $\omega \subset (0,\pi)$, a nonempty open set, and $q \in \mathcal{C}^\infty([0,\pi])$ such that q is not constant on an open subset $\omega_1 \subset \omega$. Then, system (10) is null controllable (then, approximately controllable) at time T.*

Proof of Theorem 1.9, item b). Let us take $\omega = (\pi/24, 3\pi/24)$ and the functions $\theta_1, \theta_2 \in \mathcal{C}^\infty([0,\pi])$ satisfying (54) and $\theta_1(x) = e^x$ for any $x \in \omega_1$, with ω_1 an open interval such that $\omega_1 \subset\subset (\pi/24, 3\pi/24)$. With the previous choice, let us consider the function $q \in \mathcal{C}^\infty([0,\pi])$ given in (57), with ψ given in (53). It is clear that the functions $\varphi,\ \psi \in H^2(0,\pi) \cap H^1_0(0,\pi)$ and $q \in \mathcal{C}^\infty([0,\pi])$ satisfy (52) and

$$q(x) = 37 - 5\frac{\sin(2x)}{\sin(2x) + C_1 e^x}, \quad \forall x \in \omega_1.$$

Since q is not constant on ω_1, we can apply Theorem 4.2 and guarantee that system (10) is null controllable at time $T > 0$. This ends the proof of item *b*). □

Let us now return to Theorem 4.2 and establish its proof. This relies on the algebraic resolvability for differential operators, named as *fictitius control method*.

Remark 4.3. The idea of algebraic resolvability for differential operators can be found in [27, Section 2.3.8]. In the context of control theory, the Gromov algebraic resolvability was widely used, for instance, in [24] for parabolic systems with one control force, in [15] for a Navier Stokes control system, in [3] for first-order quasi-linear hyperbolic systems and in [18, 19] for zero and first-order coupled linear parabolic systems with a reduced number of distributed controls.

Proof of Theorem 4.2. Since q is not constant on the open set $\omega_1 \subset\subset \omega$, there exist a constant $C > 0$ and a new open subset $\widehat{\omega} \subset \omega_1$ such that

$$|\partial_x q| > C > 0 \text{ and } |q| > C > 0 \text{ on } \widehat{\omega}. \tag{58}$$

Now, let us reduce the proof of the null controllability of system (10) is null controllable at time T to the resolution of two problems:

- *Analytic problem:* find $(\widehat{y}, \widehat{u})$, with
$$\widehat{y} \in L^2(0,T; H^1_0(0,\pi;\mathbb{R}^2)) \cap \mathcal{C}^0([0,T]; L^2(0,\pi;\mathbb{R}^2)))$$
and $\widehat{u} \in \mathcal{C}^k(Q_T;\mathbb{R}^2)$ (k is a positive integer that will be fixed later) such that:
$$\begin{cases} \partial_t \widehat{y}_1 = \partial_{xx}\widehat{y}_1 + \partial_{xx}\widehat{y}_2 - q(\cdot)\widehat{y}_1 + \widehat{u}_1 & \text{in } Q_T, \\ \partial_t \widehat{y}_2 = \partial_{xx}\widehat{y}_2 + \widehat{u}_2 & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } (0,T)\times\{0,\pi\}, \\ \widehat{y}(0,\cdot) = y^0,\ y(T,\cdot) = 0 & \text{in } (0,\pi), \\ \operatorname{supp}\widehat{u} \subset (0,T)\times\widehat{\omega}, \\ \widehat{\omega} \subset\subset \omega. \end{cases} \tag{59}$$
- *Algebraic problem:* find (z,v) in
$$L^2(0,T; H^1_0(0,\pi;\mathbb{R}^2)) \cap \mathcal{C}^0([0,T]; L^2(0,\pi;\mathbb{R}^2))) \times L^2(Q_T)$$
such that:
$$\begin{cases} \partial_t z_1 = \partial_{xx} z_1 + \partial_{xx} z_2 - q(\cdot) z_1 + \widehat{u}_1 & \text{in } (0,T)\times\omega, \\ \partial_t z_2 = \partial_{xx} z_2 + v + \widehat{u}_2 & \text{in } (0,T)\times\omega, \\ \operatorname{supp}(z,v) \subset\subset (0,T)\times\omega. \end{cases} \tag{60}$$

If we are able to solve the analytic and algebraic problems, then $(\widehat{y}-z,-v)$ is a solution to the null controllability problem for system (10).

The next task will be to solve the analytic and algebraic problems.

The resolution of the analytic problem (59) is standard and can be established taking into account that $q\in\mathcal{C}^\infty([0,\pi])$ and thanks to the local regularity of parabolic equations (see [12] and [24] where the local regularity is used to construct regular controls). Indeed, in a first step, thanks to the regularizing effect of parabolic systems with regular coefficients, one can restrict to the case of regular initial data. Secondly, by Carleman inequalities, one can control system (59) to zero with L^2-controls. Finally, thanks to the local parabolic regularity, one can construct regular controls $\widehat{u}=(\widehat{u}_1,\widehat{u}_2)$ which solve (59).

Now, let us present a resolution of the algebraic problem. System (60) can be rewritten as follow

$$\mathcal{L}(z,v)=\widehat{u},$$

where

$$\mathcal{L}(z,v)=\begin{pmatrix}\partial_t z_1-\partial_{xx}z_1-\partial_{xx}z_2+q(\cdot)z_1\\ \partial_t z_2-\partial_{xx}z_2-v\end{pmatrix}.$$

Let us search a differential operator $\mathcal{M}$ of order $m\geq 1$ (to be determined) with $\mathcal{C}^\infty$ coefficients such that

$$\mathcal{L}\circ\mathcal{M}=Id, \tag{61}$$

thus $(z,v):=\mathcal{M}(\widehat{u})$ will be a solution to (60) (in particular, notice that, since $\mathcal{M}$ is a differential operator, $\operatorname{supp}(z,v)\subset\operatorname{supp}\widehat{u}\subset(0,T)\times\widehat{\omega}$ and by taking $k\geq m+1$, one deduces the regularity of (z,v) in the algebraic problem). In the analytic problem (59), we search for $\widehat{u}\in\mathcal{C}^k(Q_T;\mathbb{R}^2)$ $(k\geq m+1)$ in order to apply the differential operator $\mathcal{M}$. The formal adjoint to (61) is given by

$$(\mathcal{M}^\star\circ\mathcal{L}^\star)\,\psi=\psi, \tag{62}$$

where

$$\mathcal{L}^\star(\psi)=\begin{pmatrix}\mathcal{L}_1^\star\psi\\ \mathcal{L}_2^\star\psi\\ \mathcal{L}_3^\star\psi\end{pmatrix}=\begin{pmatrix}-\partial_t\psi_1-\partial_{xx}\psi_1+q(\cdot)\psi_1\\ -\partial_t\psi_2-\partial_{xx}\psi_2-\partial_{xx}\psi_1\\ -\psi_2\end{pmatrix}.$$

To build the differential operator $\mathcal{M}^\star$ (then we find $\mathcal{M}$), the goal is to apply some differential operator to the components of $\mathcal{L}^\star\Psi$ to obtain ψ. On the other hand, observe that it is sufficient to find $\mathcal{M}^\star$ (and therefore $\mathcal{M}$) over $(0,T)\times\widehat{\omega}$ because, as mentioned before, to solve the algebraic problem, we need to find a differential operator $\mathcal{M}$ that will be applied over functions supported in $(0,T)\times\widehat{\omega}$. The advantage is that in the subset $(0,T)\times\widehat{\omega}$ the potential q satisfies (58) and we can perform manipulations with the multiplicative inverse of q and q_x.

Let us introduce the operator $\mathcal{M}_1^\star:=(0\quad -Id\quad \partial_t+\partial_{xx})$. Then, we obtain

$$\mathcal{M}_1^\star\circ\mathcal{L}^\star\psi=\partial_{xx}\psi_1.$$

Taking $\mathcal{M}_2^\star:=-\mathcal{M}_1^\star-(Id\quad 0\quad 0)$, we deduce

$$\mathcal{M}_2^\star\circ\mathcal{L}^\star\psi=\partial_t\psi_1-q(\cdot)\psi_1.$$

Using (58), we define $\mathcal{M}_3^\star:=(-2\partial_x q)^{-1}[(-\partial_t+qId)\circ\mathcal{M}_1^\star+\partial_{xx}\circ\mathcal{M}_2^\star]$. Hence, we have

$$\mathcal{M}_3^\star\circ\mathcal{L}^\star\psi=\partial_x\psi_1+\frac{\partial_{xx}q}{2\partial_x q}\psi_1.$$

Setting $\mathcal{M}_4^\star := \partial_x \circ \mathcal{M}_2^\star - \partial_t \circ \mathcal{M}_3^\star$, it holds

$$\mathcal{M}_4^\star \circ \mathcal{L}^\star \psi = -q\partial_x\psi_1 - \partial_x q\psi_1 - \frac{\partial_{xx}q}{2\partial_x q}\partial_t\psi_1.$$

Again, using (58), we consider $\mathcal{M}_5^\star := -(q)^{-1}[\mathcal{M}_4^\star + \frac{\partial_{xx}q}{2\partial_x q}\mathcal{M}_2^\star] - \mathcal{M}_3^\star$. Then, we obtain

$$\mathcal{M}_5^\star \circ \mathcal{L}^\star \psi = \frac{\partial_x q(\cdot)}{q(\cdot)}\psi_1.$$

Finally, if we take

$$\mathcal{M}^\star := \begin{pmatrix} \frac{q}{\partial_x q}\mathcal{M}_5^\star \\ \mathcal{M}_6^\star \end{pmatrix},$$

where $\mathcal{M}_6^\star := (0 \quad 0 \quad -Id)$, we obtain (62).

Notice that $\mathcal{M}^*$, then $\mathcal{M}$, is a differential operator of order 5. Therefore, by taking $m = 5$ and $k \geq 6$, we have solved the algebraic problem (60). This ends the proof of Theorem 4.2. □

Acknowledgments. The second and third authors have been partially supported by Grant PID2020–114976GB–I00, funded by MCIN/AEI/10.13039/501100011033. The third author was partially supported by Grant IJC2018–037863-I funded by MCIN/AEI/10.13039/501100011033.

REFERENCES

[1] S. Avdonin, J. Park and L. de Teresa, The Kalman condition for the boundary controllability of coupled 1-D wave equations, *Evol. Equ. Control Theory*, **9** (1), (2020), 255-273.

[2] F. Alabau-Boussouira, A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems, *SIAM J. Control Optim.*, **42** (3), (2003), 871-906.

[3] F. Alabau-Boussouira, J.-M. Coron and G. Olive, Internal controllability of first order quasi-linear hyperbolic systems with a reduced number of controls, *SIAM J. Control Optim.*, **55** (1), (2017), 300-323.

[4] F. Ammar Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems, *Differ. Equ. Appl.*, **1** (3), (2009), 427-457.

[5] F. Ammar Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems, *J. Evol. Equ.*, **9** (2), (2009), 267-291.

[6] F. Ammar Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials, *J. Math. Pures Appl.*, (9), **96** (6), (2011), 555-590.

[7] F. Ammar Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, Recent results on the controllability of linear coupled parabolic problems: a survey, *Math. Control Relat. Fields*, **1** (3), (2011), 267-306.

[8] F. Ammar Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences, *J. Funct. Anal.*, **267** (7), (2014), 2077-2151.

[9] F. Ammar Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, Minimal time of controllability of two parabolic equations with disjoint control and coupling domains, *C. R. Math. Acad. Sci. Paris*, **352** (5), (2014), 391-396.

[10] F. Ammar Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, New phenomena for the null controllability of parabolic systems: minimal time and geometrical dependence, *J. Math. Anal. Appl.*, **444** (2), (2016), 1071-1113.

[11] A. Benabdallah, F. Boyer and M. Morancey, A block moment method to handle spectral condensation phenomenon in parabolic control problems, *Annales Henri Lebesgue*, **3**, (2020), 717-793.

[12] O. Bodart, M. González-Burgos and R. Pérez-García, Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity, *Comm. Partial Differential Equations*, **29** (7-8), (2004), 1017-1050.
[13] F. Boyer and G. Olive, Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients, *Math. Control Relat. Fields*, **4** (3), (2014), 263-287.
[14] J.-M. Coron and Nguyen, H.-M.: Optimal time for the controllability of linear hyperbolic systems in one-dimensional space, *SIAM J. Control Optim.*, **57** (2), (2019), 1127-1156.
[15] J.-M. Coron and P. Lissy, Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components, *Invent. Math.*, **198** (3), (2014), 833-880.
[16] S. Dolecki, Observability for the one-dimensional heat equation, *Studia Math.*, **48**, (1973), 291-305.
[17] M. Duprez, Controllability of a 2×2 parabolic system by one force with space-dependent coupling term of order one, *ESAIM Control Optim. Calc. Var.*, **23** (4), (2017), 1473-1498.
[18] M. Duprez and P. Lissy, Indirect controllability of some linear parabolic systems of m equations with $m-1$ controls involving coupling terms of zero or first order, *J. Math. Pures Appl.*, (9), **106** (5), (2016), 905-934.
[19] M. Duprez and P. Lissy, Positive and negative results on the internal controllability of parabolic equations coupled by zero and first order terms, *J. Evol. Equ.*, **18** (2), (2017), 659-680.
[20] H. O. Fattorini and D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, *Arch. Rational Mech. Anal.*, **43**, (1971), 272-292.
[21] E. Fernández-Cara, M. González-Burgos and L. de Teresa, Boundary controllability of parabolic coupled equations, *J. Funct. Anal.*, **259** (7), (2010), 1720-1758.
[22] E. Fernández-Cara, M. González-Burgos and L. de Teresa, Controllability of linear and semilinear non-diagonalizable parabolic systems, *ESAIM Control Optim. Calc. Var.*, **21** (4), (2015), 1178-1204.
[23] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series, 34. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.
[24] M. González-Burgos and R. Pérez-García, Controllability results for some nonlinear coupled parabolic systems by one control force, *Asymptot. Anal.*, **46** (2), (2006), 123-162.
[25] M. González-Burgos and G. R. Sousa-Neto, Boundary controllability of a one-dimensional phase-field system with one control force, *J. Differential Equations*, **269** (5), (2020), 4286-4331.
[26] M. González-Burgos and L. de Teresa, Controllability results for cascade systems of m coupled parabolic PDEs by one control force, *Port. Math.*, **67** (1), (2010), 91-113.
[27] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9. Springer-Verlag, Berlin, 1986.
[28] R. E. Kalman, P. L. Falb and M. A. Arbib, Topics in mathematical system theory, McGraw-Hill Book Co., New York, 1969.
[29] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur, *Comm. Partial Differential Equations*, **20** (1-2), (1995), 335-356.
[30] T. Liard and Lissy, A Kalman rank condition for the indirect controllability of coupled systems of linear operator groups, *Math. Control Signals Systems*, **29** (2), (2017), Art. 9, 35 pp..
[31] J.-L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Avant propos de P. Lelong. Dunod, Paris; Gauthier-Villars, Paris, 1968.
[32] P. Lissy and E. Zuazua, Internal observability for coupled systems of linear partial differential equations, *SIAM J. Control Optim.*, **57** (2), (2019), 832-853.
[33] G. Olive, Boundary approximate controllability of some linear parabolic systems, *Evol. Equ. Control Theory*, **3** (1), (2014), 167-189.
[34] J. R. Shackell, Overconvergence of Dirichlet series with complex exponents, *J. Analyse Math.*, **22**, (1969), 135-170.
[35] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2009.
[36] J. Zabczyk, *Mathematical Control Theory: An Introduction*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992.

Received xxxx 20xx; revised xxxx 20xx; early access xxxx 20xx.