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On p-Laplacian reaction-diffusion problems with dynamical boundary conditions in perforated media

María ANGUIANO

Abstract

We study the effect of the p-Laplacian operator in the modelling of the heat equation through a porous medium $\Lambda_\epsilon \subset \mathbb{R}^N$ ($N \geq 2$). The case of $p = 2$ was recently published in (Anguiano, Mediterr. J. Math. 17, 18 (2020)). Using rigorous functional analysis techniques and the properties of Sobolev spaces, we managed to solve additional (nontrivial) difficulties which arise compared to the study for $p = 2$, and we prove a convergence theorem in appropriate functional spaces.

AMS classification numbers: 35K05, 35K58.

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1 Statement of the problem and the results

Homogenization problems in perforated media for the p-Laplacian operator have been considered in the literature over the last decades. The homogenization of the equation

$$-\text{div}\left(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon\right) = f \tag{1}$$

in a periodically perforated domain is considered by Labani and Picard in [36] with Dirichlet boundary conditions. Such a problem is a generalization of the linear problem for Laplace's equation, which corresponds to $p = 2$, and was studied by Cioranescu and Murat in [17]. Donato and Moscariello in [24] study the homogenization of a class of nonlinear elliptic Neumann problems in perforated domains of $\mathbb{R}^N$. As a consequence of [24], we are able in particular to describe the homogenization of (1) with Neumann boundary conditions. This result is a generalization of earlier related works, for instance, Cioranescu and Donato [15] and Cioranescu and Saint Jean Paulin [18]. In [41] Shaposhnikova and Podol'skii study the homogenization of (1) in an ϵ-periodically perforated domain with a nonlinear boundary condition. In [21] Díaz *et al.* consider (1) with a nonlinear perturbed Robin-type boundary condition in an ϵ periodically perforated domain where the size of the particles is smaller than the period ϵ, and the asymptotic behavior of the solution is studied as $\epsilon \to 0$. The closest articles to this one in the literature are [33, 34, 35], where Gómez *et al.* consider the case $2 < p \leq N$, and [22], where Díaz *et al.* study the case $p > N$.

It has been discovered by physicists that, as far as the Allen-Cahn equation is concerned, for certain materials a dynamical interaction with the walls must be taken into account (see Fischer *et al.* [28, 29] for more details). In this sense, in the context of heat equations, dynamical boundary conditions have been rigorously derived in Gal and Shomberg [31] based on first and second thermodynamical principles and their physical interpretation was also given in Goldstein [32]. We point out that these types of boundary conditions are also used for modelling various physical situations including fluid diffusion within a semi-permeable boundary (see Crank [20], Langer [37], and, March and Weaver [39] for more details) or several situations when the hear flow inside the domain is subject to nonlinear heating or cooling on the boundary (see Favini *et al.* [26, 27] for more details).

In the previous literature there is no study for the homogenization of p-Laplacian parabolic models as we consider in this article. Such equations model nonlinear fluid diffusion through a semi-permeable membrane (see

Duvaut and Lions [25, Ch.1]) or nonlinear heat flow with radiation on the boundary causing nonlinear cooling (see Friedman [30, Ch.7, §5]).

Model problem. The heat equation that we study in this paper is the following

$$\partial_t v_\epsilon - \Delta_p\, v_\epsilon + \alpha |v_\epsilon|^{p-2} v_\epsilon = -f_1(v_\epsilon) \quad \text{in } \Lambda_\epsilon \times (0, \bar{T}),$$

where $v_\epsilon = v_\epsilon(x,t)$, $x \in \Lambda_\epsilon$, $t \in (0, \bar{T})$, with $\bar{T} > 0$ and $\alpha > 0$. Assume that $\Lambda_\epsilon \subset \mathbb{R}^N$ ($N \geq 2$) is a fixed bounded domain Λ from which a set T_ϵ of holes has been removed, in particular, Λ_ϵ is a periodically perforated domain with holes of the same size as the period (see [5] for more details on the domain). Here the diffusion is modeled by the p-Laplacian operator $\Delta_p v_\epsilon := \text{div}\left(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon\right)$ with $p \in [2, N]$. On the nonlinear term f_1, we assume that $f_1 \in C(\mathbb{R})$, such that

$$p \leq q_1 < +\infty, \text{ if } p = N \quad \text{and} \quad 2 \leq q_1 \leq \frac{Np}{N-p}, \text{ if } p \in [2, N), \tag{2}$$

$$\eta_1 |s|^{q_1} - \lambda \leq f_1(s)s \leq \eta_2 |s|^{q_1} + \lambda, \quad \text{for all } s \in \mathbb{R}, \tag{3}$$

and

$$(f_1(s_1) - f_1(s_2))(s_1 - s_2) \geq -\beta (s_1 - s_2)^2, \quad \text{for all } s_1, s_2 \in \mathbb{R}, \tag{4}$$

where $\eta_i > 0$, $i = 1, 2$, $\lambda > 0$, and $\beta > 0$.

We consider the following dynamical boundary conditions on the boundary of the holes

$$\partial_{\nu_p} v_\epsilon + \epsilon\, \partial_t v_\epsilon = -\epsilon\, f_2(v_\epsilon) \text{ on } \partial T_\epsilon \times (0, \bar{T}),$$

where T_ϵ is the set of all the holes of this periodic distribution contained in Λ_ϵ (see [5] for more details). The "normal derivative" must be understood as $\partial_{\nu_p} v_\epsilon = |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \cdot \nu$, where ν denotes the outward normal to ∂T_ϵ. This boundary equation is multiplied by ϵ to compensate the growth of the surface by shrinking ϵ, where the value of v_ϵ is assumed to be the trace of the function v_ϵ defined for $x \in \Lambda_\epsilon$. On the nonlinear term f_2, we assume that $f_2 \in C(\mathbb{R})$, such that

$$2 \leq q_2 < +\infty, \text{ if } p = N \quad \text{and} \quad 2 \leq q_2 \leq \frac{(N-1)p}{N-p}, \text{ if } p \in [2, N), \tag{5}$$

$$\eta_1 |s|^{q_2} - \lambda \leq f_2(s)s \leq \eta_2 |s|^{q_2} + \lambda, \quad \text{for all } s \in \mathbb{R}, \tag{6}$$

and

$$(f_2(s_1) - f_2(s_2))(s_1 - s_2) \geq -\beta (s_1 - s_2)^2, \quad \text{for all } s_1, s_2 \in \mathbb{R}. \tag{7}$$

Moreover, we consider Dirichlet boundary condition on the boundary of Λ

$$v_\epsilon = 0, \text{ on } \partial\Lambda \times (0, \bar{T}),$$

and the initial conditions

$$v_\epsilon(x, 0) = v_\epsilon^0(x), \text{ for } x \in \Lambda_\epsilon, \quad v_\epsilon(x, 0) = \phi_\epsilon^0(x), \text{ for } x \in \partial T_\epsilon,$$

where $(v_\epsilon^0, \phi_\epsilon^0)$ satisfies

$$v_\epsilon^0 \in L^2(\Lambda), \quad \phi_\epsilon^0 \in L^2(\partial T_\epsilon), \tag{8}$$

and

$$|v_\epsilon^0|^2_{\Lambda_\epsilon} + \epsilon |\phi_\epsilon^0|^2_{\partial T_\epsilon} \leq K, \tag{9}$$

where $K > 0$ and $|\cdot|_{\Lambda_\epsilon}$ (respectively $|\cdot|_{\partial T_\epsilon}$) is the norm in $L^2(\Lambda_\epsilon)$ (respectively $L^2(\partial T_\epsilon)$). Notice that on ∂T_ϵ we assume that $\phi_\epsilon^0(x)$ is equal to the trace of $v_\epsilon^0(x)$.

In summary, we study in this paper the following problem

$$
\begin{cases}
\partial_t v_\epsilon - \Delta_p \, v_\epsilon + \alpha |v_\epsilon|^{p-2} v_\epsilon = -f_1(v_\epsilon) & \text{in } \Lambda_\epsilon \times (0, \bar{T}), \\
\partial_{\nu_p} v_\epsilon + \epsilon \, \partial_t v_\epsilon = -\epsilon \, f_2(v_\epsilon) & \text{on } \partial T_\epsilon \times (0, \bar{T}), \\
v_\epsilon = 0, & \text{on } \partial \Lambda \times (0, \bar{T}), \\
v_\epsilon(x, 0) = v_\epsilon^0(x), & \text{for } x \in \Lambda_\epsilon, \\
v_\epsilon(x, 0) = \phi_\epsilon^0(x), & \text{for } x \in \partial T_\epsilon,
\end{cases}
\tag{10}
$$

under the assumptions (2)-(9).

In a recent article (see [5]) we addressed the problem (10) with $p = 2$ (for the physical motivation of this model, see, for instance, Timofte [45]). More recently, in [6] we generalize this previous study with a Laplace-Beltrami correction term and in [7] we carried out the first study on the asymptotic behavior of the solution of parabolic models in a thin porous media.

We would like to highlight that analyzing a p-Laplacian problem involves additional (nontrivial) difficulties compared to the study for the Laplacian. Due to the presence of p-Laplacian operator in the domain, the variational formulation of the p-Laplacian reaction-diffusion equation is different that in [5]. We have to work in the space

$$
V_p := \left\{ (u, \gamma(u)) : u \in W^{1,p}(\Lambda_\epsilon), \gamma(u) = 0 \text{ on } \partial \Lambda \right\},
$$

where γ denotes the trace operator

$$
u \in W^{1,p}(\Lambda_\epsilon) \mapsto u|_{\partial \Lambda_\epsilon} \in W^{1 - \frac{1}{p}, p}(\partial \Lambda_\epsilon).
$$

New $W^{1,p}$-estimates are needed to deal with problem (10). To prove these estimates rigorously, we use the Galerkin approximations where we have to introduce a special basis consisting of functions in the space

$$
\mathcal{V}_s := \left\{ (u, \gamma(u)) : u \in W^{s,2}(\Lambda_\epsilon), \gamma(u) = 0 \text{ on } \partial \Lambda \right\}, \quad s \geq \frac{N(p-2)}{2p} + 1,
$$

in the sense of Lions [38, p. 161]. Therefore, thanks to the assumption made on s, we have $\mathcal{V}_s \subset V_p$. We use the so-called energy method introduced by Tartar [44] and considered by many authors (see, for instance, Cioranescu and Donato [15]). Finally, in order to identity the limit equation, it is necessary to use monotonicity arguments. In summary, we prove that the solution of problem (10), properly extended to the whole Λ, converges to the unique solution of a new nonlinear problem, defined all over the domain Λ, given by a new operator and containing extra zero order terms, capturing the effect of the influence of the non-homogeneous dynamical condition imposed on the boundary of T_ϵ.

Theorem 1.1 (Convergence Theorem)**.** *Assume (2)-(4), (6)–(7) and (9). On the nonlinear term f_2, we suppose that $f_2 \in C^1(\mathbb{R})$ and its associated exponent q_2 satisfies*

$$
p \leq q_2 < +\infty \quad \text{if} \;\; p = N \quad \text{and} \quad 2 \leq q_2 \leq \frac{(N-1)p}{N-p} \quad \text{if} \;\; p \in [2, N).
\tag{11}
$$

We suppose that $(v_\epsilon, \phi_\epsilon)$ is the unique solution of (10), with $(v_\epsilon^0, \phi_\epsilon^0) \in V_p$, and where $\phi_\epsilon(t) = \gamma(v_\epsilon(t))$ a.e. $t \in (0, \bar{T}]$. Let $\hat{v}_\epsilon$ be the $W^{1,p}$-extension of v_ϵ to $\Lambda \times (0, \bar{T})$. Then, we have

$$
\hat{v}_\epsilon(t) \to v(t) \quad \text{in } L^p(\Lambda), \quad \text{as } \epsilon \to 0, \quad \forall t \in [0, \bar{T}],
$$

where "$\to$" denotes the strong convergence and v is the unique solution of the problem given by

$$
(\theta^* + \theta_T) \, \partial_t v - \operatorname{div} b(\nabla v) + \theta^* (\alpha |v|^{p-2} v + f_1(v)) + \theta_T f_2(v) = 0, \;\; \text{in } \Lambda \times (0, \bar{T}),
\tag{12}
$$

with Dirichlet boundary condition

$$
v = 0, \;\; \text{on } \partial \Lambda \times (0, \bar{T}),
\tag{13}
$$

3

and initial condition

$$v(x,0) = v_0(x), \text{ for } x \in \Lambda, \tag{14}$$

where $\theta^* = |Z^*|/|Z|$, $\theta_T = |\partial T|/|Z|$, Z is the representative cell in $\mathbb{R}^N$, T is an open subset of Z, $Z^* = Z \setminus \bar{T}$ and $|Z|$ (respectively $|\partial T|$ and $|Z^*|$) denotes the measure of Z (respectively ∂T and Z^*).

For any $\zeta \in \mathbb{R}^N$, if $w(y)$ is the solution of the problem

$$\begin{cases} \displaystyle\int_{Z^*} |\nabla_y w(y)|^{p-2}\nabla_y w(y) \cdot \nabla_y \varphi(y)dy = 0 & \forall \varphi \in \mathbb{H}_{\text{per}}(Z^*), \\ w \in \zeta \cdot y + \mathbb{H}_{\text{per}}(Z^*), \end{cases} \tag{15}$$

then b is defined by

$$b(\zeta) = \frac{1}{|Z|} \int_{Z^*} |\nabla_y w(y)|^{p-2}\nabla_y w(y)dy, \tag{16}$$

where $\mathbb{H}_{\text{per}}(Z^*)$ is the space of functions from $W^{1,p}(Z^*)$ which have the same trace on the opposite faces of Z.

Remark 1.2. *If the diffusion is modeled by the Laplacian operator (i.e. $p = 2$), then the limit problem (12)-(14) is the problem obtained in [5, Theorem 6.1].*

Remark 1.3. *It seems possible to improve the regularity assumed on the functions f_1 and f_2 when they are assumed non-increasing and Hölder continuous (for more details, see the recent book [23] where Díaz et al. present this improvement for the linear diffusion case).*

We organize this work as follows. In the next section, we present some notations, definitions and properties of suitable spaces for the study of (10). Some preliminary results are established in Section 3, some estimates for the solution of (10) are rigorously derived in Section 4, and a convergence result is indicated in Section 5. Finally, in Section 6 we study the limit problem and a conclusion section is established in Section 7.

2 The Sobolev spaces $W^{s,p}(\Lambda_\epsilon)$

In this section we recall the Sobolev spaces $W^{s,p}$, which will be used in this paper (see Adams and Fournier [1], Brézis [13, Chapter 9] and Nečas [40, Chapter 2] for more details about them).

For any positive integer s and $p \geq 1$, we define the Sobolev space $W^{s,p}(\Lambda_\epsilon)$ to be the completion of $C^s(\overline{\Lambda}_\epsilon)$, with respect to the norm

$$||u||_{s,p,\Lambda_\epsilon} = \left(\sum_{0 \leq |\alpha| \leq s} |D^\alpha u|^p_{p,\Lambda_\epsilon} \right)^{1/p}.$$

Observe that $W^{s,p}(\Lambda_\epsilon)$ is a Banach space.

By the Sobolev embedding Theorem (see [1, Chapter 4, p. 99]), we have the embedding

$$W^{s,p}(\Lambda_\epsilon) \subset W^{1,r}(\Lambda_\epsilon), \tag{17}$$

where $s \geq 1$ and $\frac{1}{p} - \frac{s-1}{N} \leq \frac{1}{r} \leq \frac{1}{p}$.

In particular, for $p \geq 1$, we define the Sobolev space $W^{1,p}(\Lambda_\epsilon)$ to be the completion of $C^1(\overline{\Lambda}_\epsilon)$, with respect to the norm

$$||u||_{p,\Lambda_\epsilon} := \left(|u|^p_{p,\Lambda_\epsilon} + |\nabla u|^p_{p,\Lambda_\epsilon} \right)^{1/p},$$

where $|\cdot|_{p,\Lambda_\epsilon}$ is the norm in $L^p(\Lambda_\epsilon)$. We set $H^1(\Lambda_\epsilon) = W^{1,2}(\Lambda_\epsilon)$.

We define

$$p^\star = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p = N. \end{cases}$$

We have the continuous embedding

$$W^{1,p}(\Lambda_\epsilon) \subset \begin{cases} L^r(\Lambda_\epsilon) & \text{if } \quad p < N \quad \text{and} \quad r = p^\star, \\ L^r(\Lambda_\epsilon) & \text{if } \quad p = N \quad \text{and} \quad p \leq r < p^\star. \end{cases} \tag{18}$$

By Rellich-Kondrachov Theorem (see [13, Chapter 9, Theorem 9.16]), we have the compact embedding

$$W^{1,p}(\Lambda_\epsilon) \subset \begin{cases} L^r(\Lambda_\epsilon) & \text{if } \quad p < N \quad \text{and} \quad 1 \leq r < p^\star, \\ L^r(\Lambda_\epsilon) & \text{if } \quad p = N \quad \text{and} \quad p \leq r < p^\star. \end{cases} \tag{19}$$

In particular, we have the compact embedding

$$W^{1,p}(\Lambda_\epsilon) \subset L^2(\Lambda_\epsilon), \quad \forall \, 2 \leq p \leq N. \tag{20}$$

One can define a family of spaces intermediate between L^p and $W^{1,p}$. More precisely for $p \geq 1$ we define the fractional order Sobolev space

$$W^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon) := \left\{ u \in L^p(\partial\Lambda_\epsilon); \frac{|u(x) - u(y)|}{|x-y|^{1-\frac{1}{p}+\frac{N}{p}}} \in L^p(\partial\Lambda_\epsilon \times \partial\Lambda_\epsilon) \right\},$$

equipped with the natural norm. We set $H^{1/2}(\partial\Lambda_\epsilon) = W^{\frac{1}{2},2}(\partial\Lambda_\epsilon)$. These spaces play an important role in the theory of traces.

The trace operator is denoted by γ such that $u \mapsto u|_{\partial\Lambda_\epsilon}$. This operator belongs to $\mathcal{L}(W^{1,p}(\Lambda_\epsilon), W^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon))$. We denote by $||\gamma||$ the norm of γ in this space.

We will use $\|\cdot\|_{p,\partial\Lambda_\epsilon}$ to denote the norm in $W^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon)$, which is given by

$$\|\phi\|_{p,\partial\Lambda_\epsilon} = \inf\{\|u\|_{p,\Lambda_\epsilon} : \gamma(u) = \phi\}.$$

We define

$$p_b^\star = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p = N. \end{cases}$$

We have the continuous embedding

$$W^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon) \subset \begin{cases} L^r(\partial\Lambda_\epsilon) & \text{if } \quad p < N \quad \text{and} \quad r = p_b^\star, \\ L^r(\partial\Lambda_\epsilon) & \text{if } \quad p = N \quad \text{and} \quad 1 \leq r < p_b^\star. \end{cases} \tag{21}$$

By [40, Chaper 2, Theorem 6.2], we have the compact embedding

$$W^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon) := \gamma\left(W^{1,p}(\Lambda_\epsilon)\right) \subset L^r(\partial\Lambda_\epsilon) \quad \text{if } \quad 1 \leq r < p_b^\star. \tag{22}$$

In particular, we have the compact embedding

$$W^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon) \subset L^2(\partial\Lambda_\epsilon) \quad \forall \, 2 \leq p \leq N. \tag{23}$$

Some important notations for reading the paper:

We define a few notations. For Λ_ϵ, we call

$$H_p := L^p(\Lambda_\epsilon) \times L_{\partial\Lambda}^p(\partial\Lambda_\epsilon), \quad \forall p \geq 2,$$

where

$$L_{\partial\Lambda}^p(\partial\Lambda_\epsilon) := \{u \in L^p(\partial\Lambda_\epsilon) : u = 0 \text{ on } \partial\Lambda\}, \quad \forall p \geq 2.$$

On H_p, we consider the norm $|(\cdot,\cdot)|_{H_p}$ given by

$$|(u,\phi)|_{H_p}^p = |u|_{p,\Lambda_\epsilon}^p + \epsilon|\phi|_{p,\partial T_\epsilon}^p, \quad (u,\phi) \in H_p, \quad \forall p \geq 2,$$

where $|\cdot|_{p,\Lambda_\epsilon}$ is the norm in $L^p(\Lambda_\epsilon)$ and $|\cdot|_{p,\partial T_\epsilon}$ is the norm in $L^p(\partial T_\epsilon)$.

In this paper, it is very important the following space

$$V_p := \left\{(u,\gamma(u)) : u \in W_{\partial\Lambda}^{1,p}(\Lambda_\epsilon)\right\}, \quad \forall p \geq 2,$$

where

$$W_{\partial\Lambda}^{1,p}(\Lambda_\epsilon) := \{u \in W^{1,p}(\Lambda_\epsilon) : \gamma(u) = 0 \text{ on } \partial\Lambda\}, \quad \forall p \geq 2.$$

Observe that V_p is a closed vector subspace of $W_{\partial\Lambda}^{1,p}(\Lambda_\epsilon) \times W_{\partial\Lambda}^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon)$, where

$$W_{\partial\Lambda}^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon) := \{u \in W^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon) : u = 0 \text{ on } \partial\Lambda\}, \quad \forall p \geq 2.$$

We endow it with the norm $\|(\cdot,\cdot)\|_{V_p}$ given by

$$\|(u,\gamma(u))\|_{V_p}^p = \|u\|_{p,\Lambda_\epsilon}^p + \|\gamma(u)\|_{p,\partial T_\epsilon}^p, \quad (u,\gamma(u)) \in V_p.$$

Let p', q_1' and q_2' be the conjugate exponents of p, q_1 and q_2, respectively. Taking into account the continuous embeddings (18) and (21), and the assumptions (2)-(5), we have the following useful continuous inclusions

$$V_p \subset W^{1,p}(\Lambda_\epsilon) \subset L^{q_1}(\Lambda_\epsilon) \subset L^2(\Lambda_\epsilon), \quad V_p \subset W^{1-\frac{1}{p},p}(\partial\Lambda_\epsilon) \subset L^{q_2}(\partial\Lambda_\epsilon) \subset L^2(\partial\Lambda_\epsilon), \tag{24}$$

and

$$L^2(\Lambda_\epsilon) \subset L^{q_1'}(\Lambda_\epsilon) \subset \left(W^{1,p}(\Lambda_\epsilon)\right)' \subset V_p', \tag{25}$$

where $\left(W^{1,p}(\Lambda_\epsilon)\right)'$ and V_p' denote the dual of $W^{1,p}(\Lambda_\epsilon)$ and V_p, respectively. Note that $\left(W^{1,p}(\Lambda_\epsilon)\right)'$ is a subspace of $W^{-1,p'}(\Lambda_\epsilon)$, where $W^{-1,p'}(\Lambda_\epsilon)$ denotes the dual of the Sobolev space $W_0^{1,p}(\Lambda_\epsilon) := \overline{\mathcal{D}(\Lambda_\epsilon)}^{W^{1,p}(\Lambda_\epsilon)}$.

Taking into account the compact embeddings (20) and (23), we have the compact embedding

$$V_p \subset H_2 \quad \forall 2 \leq p \leq N. \tag{26}$$

Finally, for $s \geq 1$, we consider the space

$$\mathcal{V}_s := \left\{(u,\gamma(u)) : u \in W_{\partial\Lambda}^{s,2}(\Lambda_\epsilon)\right\}.$$

We note that $\mathcal{V}_s$ is a closed vector subspace of $W_{\partial\Lambda}^{s,2}(\Lambda_\epsilon) \times W_{\partial\Lambda}^{s-\frac{1}{2},2}(\partial\Lambda_\epsilon)$.

Observe that by (17), we have that $W^{s,2}(\Lambda_\epsilon) \subset W^{1,2}(\Lambda_\epsilon)$ and by Rellich's Theorem (see [40, Chapter 1, Theorem 1.4]), we obtain the compact embedding

$$W^{1,2}(\Lambda_\epsilon) \subset L^2(\Lambda_\epsilon),$$

and by [40, Chapter 1, Exercise 1.2], we have the compact embedding

$$W^{s-\frac{1}{2},2}(\partial\Lambda_\epsilon) := \gamma_0\left(W^{s,2}(\Lambda_\epsilon)\right) \subset L^2(\partial\Lambda_\epsilon).$$

Thus, we can deduce the compact embedding

$$\mathcal{V}_s \subset H_2. \tag{27}$$

3 Preliminary results

In the latter we will need the following results:

Remark 3.1 (Additional conditions on the nonlinear terms). *Observe that we have the following conditions on the nonlinear terms:*

$$|f_1(s)| \le K\left(1 + |s|^{q_1-1}\right), \quad |f_2(s)| \le K\left(1 + |s|^{q_2-1}\right), \quad \text{for all } s \in \mathbb{R}, \quad K > 0. \tag{28}$$

Theorem 3.2 (Existence and uniqueness of solution of (10)). *Assume (2)–(8), then there is a unique solution $(v_\epsilon, \phi_\epsilon)$ of the problem (10) such that, for all $\bar{T} > 0$, $v_\epsilon \in C([0,\bar{T}]; L^2(\Lambda_\epsilon)) \cup L^p(0,\bar{T}; W^{1,p}(\Lambda_\epsilon))$, $\phi_\epsilon \in C([0,\bar{T}]; L^2_{\partial\Lambda}(\partial\Lambda_\epsilon)) \cup L^p(0,\bar{T}; W^{1-\frac{1}{p},p}_{\partial\Lambda}(\partial\Lambda_\epsilon))$ where $\gamma(v_\epsilon(t)) = \phi_\epsilon(t)$ a.e. $t \in (0,\bar{T}]$.*

We have that $(v_\epsilon, \phi_\epsilon)$ satisfies

$$d_t(v_\epsilon(t), w)_{\Lambda_\epsilon} + \epsilon\, d_t(\phi_\epsilon(t), \gamma(w))_{\partial T_\epsilon} + (|\nabla v_\epsilon(t)|^{p-2}\nabla v_\epsilon(t), \nabla w)_{\Lambda_\epsilon} + \alpha(|v_\epsilon(t)|^{p-2}v_\epsilon(t), w)_{\Lambda_\epsilon}$$
$$+(f_1(v_\epsilon(t)), w)_{\Lambda_\epsilon} + \epsilon\,(f_2(\phi_\epsilon(t)), \gamma(w))_{\partial T_\epsilon} = 0, \quad \forall w \in W^{1,p}_{\partial\Lambda}(\Lambda_\epsilon), \tag{29}$$

in $\mathcal{D}'(0,\bar{T})$, with the initial condition

$$v_\epsilon(0) = v_\epsilon^0, \quad \text{and} \quad \phi_\epsilon(0) = \phi_\epsilon^0, \tag{30}$$

where $(\cdot,\cdot)_{\Lambda_\epsilon}$ is the inner product in $L^2(\Lambda_\epsilon)$ or $(L^2(\Lambda_\epsilon))^N$ and the duality product between $L^{r'}(\Lambda_\epsilon)$ and $L^r(\Lambda_\epsilon)$ if $r \ne 2$, and $(\cdot,\cdot)_{\partial T_\epsilon}$ is the inner product in $L^2(\partial T_\epsilon)$ and the duality product between $L^{r'}(\partial T_\epsilon)$ and $L^r(\partial T_\epsilon)$ if $r \ne 2$.

Moreover $(v_\epsilon, \phi_\epsilon)$ satisfies the energy equality

$$\frac{1}{2}d_t\left(|(v_\epsilon(t), \phi_\epsilon(t))|^2_{H_2}\right) + |\nabla v_\epsilon(t)|^p_{p,\Lambda_\epsilon} + \alpha|v_\epsilon(t)|^p_{p,\Lambda_\epsilon}$$
$$+(f_1(v_\epsilon(t)), v_\epsilon(t))_{\Lambda_\epsilon} + \epsilon\,(f_2(\phi_\epsilon(t)), \phi_\epsilon(t))_{\partial T_\epsilon} = 0, \quad a.e.\ t \in (0,\bar{T}). \tag{31}$$

Proof. It is based on the theory of monotonicity of Lions [38]. We will show that operator $B_p : V_p \to V_p'$, given by

$$\langle B_p((w, \gamma(w))), (u, \gamma(u))\rangle := (|\nabla w|^{p-2}\nabla w, \nabla u)_{\Lambda_\epsilon} + \alpha(|w|^{p-2}w, u)_{\Lambda_\epsilon}, \quad \forall w, u \in W^{1,p}_{\partial\Lambda}(\Lambda_\epsilon), \tag{32}$$

is coercive, that is $\dfrac{\langle B_p((w, \gamma(w)), (w, \gamma(w)))\rangle}{\|(w, \gamma(w))\|_{V_p}} \to +\infty$ when $\|(w, \gamma(w))\|_{V_p} \to +\infty$.

We have

$$
\begin{aligned}
\langle B_p((w, \gamma(w)), (w, \gamma(w)))\rangle \quad &\ge \quad \min\{1, \alpha\}\|w\|^p_{p,\Lambda_\epsilon} \tag{33}\\
&= \quad \frac{1}{1 + \|\gamma\|^p}\min\{1, \alpha\}\|w\|^p_{p,\Lambda_\epsilon} + \frac{\|\gamma\|^p}{1 + \|\gamma\|^p}\min\{1, \alpha\}\|w\|^p_{p,\Lambda_\epsilon}\\
&\ge \quad \frac{1}{1 + \|\gamma\|^p}\min\{1, \alpha\}\|(w, \gamma(w))\|^p_{V_p}, \quad \forall w \in W^{1,p}_{\partial\Lambda}(\Lambda_\epsilon),
\end{aligned}
$$

so B_p is coercive. By [38, Ch.2,Th.1.4], we have that (29)–(30) has a unique solution and satisfies the energy equality.
$\qquad\square$

Remark 3.3 (Energy inequality). *By (31) and using (3) and (6), we obtain the energy inequality*

$$d_t\left(|(v_\epsilon(t), \phi_\epsilon(t))|^2_{H_2}\right) + 2|\nabla v_\epsilon(t)|^p_{p,\Lambda_\epsilon} + 2\alpha|v_\epsilon(t)|^p_{p,\Lambda_\epsilon} + 2\eta_1|v_\epsilon(t)|^{q_1}_{q_1,\Lambda_\epsilon} \tag{34}$$
$$+2\eta_1\epsilon\,|\phi_\epsilon(t)|^{q_2}_{q_2,\partial T_\epsilon} \le 2\lambda\left(|\Lambda_\epsilon| + \epsilon\,|\partial T_\epsilon|\right),$$

where $|\Lambda_\epsilon|$ (respectively $|\partial T_\epsilon|$) denotes the measure of Λ_ϵ (respectively ∂T_ϵ).

Remark 3.4 (Estimates for the measures of Λ_ϵ and ∂T_ϵ). *Taking into account the number of holes, we can deduce (see [6, Section 4] for more details)*

$$|\partial T_\epsilon| \leq \frac{K}{\epsilon}, \quad K > 0.$$

And since $|\Lambda_\epsilon| \leq |\Lambda|$, we have that

$$|\Lambda_\epsilon| + \epsilon |\partial T_\epsilon| \leq K, \quad K > 0. \tag{35}$$

4 Some estimates for the solution of (10)

If we denote

$$\mathcal{F}_1(s) := \int_0^s f_1(r)dr \quad \text{and} \quad \mathcal{F}_2(s) := \int_0^s f_2(r)dr,$$

we can deduce that

$$\widetilde{\eta}_1 |s|^{q_1} - \widetilde{\lambda} \leq \mathcal{F}_1(s) \leq \widetilde{\eta}_2 |s|^{q_1} + \widetilde{\lambda} \quad \forall s \in \mathbb{R}, \tag{36}$$

and

$$\widetilde{\eta}_1 |s|^{q_2} - \widetilde{\lambda} \leq \mathcal{F}_2(s) \leq \widetilde{\eta}_2 |s|^{q_2} + \widetilde{\lambda} \quad \forall s \in \mathbb{R}. \tag{37}$$

with $\widetilde{\eta}_i, \widetilde{\lambda} > 0$, $i = 1, 2$.

Lemma 4.1. *We suppose (2)–(7) and (9). There is a constant K independent of ϵ, such that the solution $(v_\epsilon, \phi_\epsilon)$ of the problem (10) satisfies*

$$\|v_\epsilon\|_{p,\Lambda_\epsilon,\bar{T}} \leq K, \quad \sup_{t \in [0,\bar{T}]} \|v_\epsilon(t)\|_{p,\Lambda_\epsilon} \leq K, \tag{38}$$

for any initial condition $(v_\epsilon^0, \phi_\epsilon^0) \in V_p$, and where $\|\cdot\|_{p,\Lambda_\epsilon,\bar{T}}$ is the norm in $L^p(0, \bar{T}; W^{1,p}(\Lambda_\epsilon))$.

Proof. Using (35) in (34), we obtain

$$d_t \left(|(v_\epsilon(t), \phi_\epsilon(t))|_{H_2}^2 \right) + 2|\nabla v_\epsilon(t)|_{p,\Lambda_\epsilon}^p + 2\alpha |v_\epsilon(t)|_{p,\Lambda_\epsilon}^p + 2\eta_1 |v_\epsilon(t)|_{q_1,\Lambda_\epsilon}^{q_1} + 2\eta_1 \epsilon |\phi_\epsilon(t)|_{q_2,\partial T_\epsilon}^{q_2} \leq K. \tag{39}$$

From (32)-(33), in particular, we can deduce

$$d_t \left(|(v_\epsilon(t), \phi_\epsilon(t))|_{H_2}^2 \right) + \frac{2 \min\{1, \alpha\}}{1 + \|\gamma\|^p} \|(v_\epsilon(t), \gamma(v_\epsilon(t)))\|_{V_p}^p \leq K. \tag{40}$$

We integrate between 0 and t and taking into account (9), in particular, we have the first estimate in (38).

In order to obtain the second estimate in (38), we have to take the inner product in the problem (10) with v_ϵ'. To do this, we need that $v_\epsilon' \in L^p(0, \bar{T}; W_{\partial\Lambda}^{1,p}(\Lambda_\epsilon)) \cap L^{q_1}(0, \bar{T}; L^{q_1}(\Lambda_\epsilon))$ with $\gamma(v_\epsilon') \in L^{q_2}(0, \bar{T}; L_{\partial\Lambda}^{q_2}(\partial\Lambda_\epsilon))$. As we do not have it for our solution, we have to use the Galerkin method with the properties of B_p given by (32).

At first, we introduce a special basis consisting of functions $(u_j, \gamma(u_j)) \in \mathcal{V}_s$ with $s \geq \frac{N(p-2)}{2p} + 1$ in the sense of [38, Chapter 2, Remark 1.6, p. 161]. Therefore, thanks to the assumption made on s, taking into account (17), we have $\mathcal{V}_s \subset V_p$. The scalar product in H_2 generates on $\mathcal{V}_s \subset H_2$ the bilinear functional $((w, \gamma(w)), (u, \gamma(u)))_{H_2}$ which can be represented in the form

$$((w, \gamma(w)), (u, \gamma(u)))_{H_2} = \langle L((w, \gamma(w))), (u, \gamma(u)) \rangle_{\mathcal{V}_s},$$

where L is a self-adjoint operator. The compact embedding (27) implies the compactness of the operator L. Hence, L has a complete system of eigenvectors $\{(u_j, \gamma(u_j)) : j \geq 1\}$. These vectors are orthonormal in H_2 and orthogonal in $\mathcal{V}_s$. Observe that $span\{(u_j, \gamma(u_j)) : j \geq 1\}$ is dense in V_p.

We use the Galerkin approximation of the solution of (10) given by

$$(v_{\epsilon,n}(t), \gamma(v_{\epsilon,n}(t))) = \sum_{j=1}^{n} \sigma_{\epsilon nj}(t)(u_j, \gamma(u_j)), \quad n \geq 1 \tag{41}$$

such that

$$d_t((v_{\epsilon,n}(t), \gamma(v_{\epsilon,n}(t))), (u_j, \gamma(u_j)))_{H_2} + \langle B_p((v_{\epsilon,n}(t), \gamma(v_{\epsilon,n}(t)))), (u_j, \gamma(u_j)) \rangle$$
$$+(f_1(v_{\epsilon,n}(t)), u_j)_{\Lambda_\epsilon} + \epsilon(f_2(\gamma(v_{\epsilon,n}(t))), \gamma(u_j))_{\partial T_\epsilon} = 0, \quad j = 1, \dots, n, \tag{42}$$

$$(v_{\epsilon,n}(0), \gamma(v_{\epsilon,n}(0))) = (v_{\epsilon,n}^0, \gamma(v_{\epsilon,n}^0)), \tag{43}$$

and

$$\sigma_{\epsilon nj}(t) = (v_{\epsilon,n}(t), u_j)_{\Lambda_\epsilon} + (\gamma(v_{\epsilon,n}(t)), \gamma(u_j))_{\partial T_\epsilon}.$$

As $(v_\epsilon^0, \phi_\epsilon^0) \in V_p$, there is $(v_{\epsilon,n}^0, \gamma(v_{\epsilon,n}^0)) \in span\{(u_j, \gamma(u_j)) : 1 \leq j \leq n\}$, such that

$$||(v_{\epsilon,n}^0, \gamma(v_{\epsilon,n}^0))||_{V_p} \leq K, \quad K > 0. \tag{44}$$

We multiply by $\sigma'_{\epsilon nj}$ in (42), we sum from $j = 1$ to n, and we obtain

$$|(v'_{\epsilon,n}(t), \gamma(v'_{\epsilon,n}(t)))|_{H_2}^2 + \frac{1}{p} d_t(\langle B_p((v_{\epsilon,n}(t), \gamma(v_{\epsilon,n}(t)))), (v_{\epsilon,n}(t), \gamma(v_{\epsilon,n}(t))) \rangle)$$
$$+(f_1(v_{\epsilon,n}(t)), v'_{\epsilon,n}(t))_{\Lambda_\epsilon} + \epsilon(f_2(\gamma(v_{\epsilon,n}(t))), \gamma(v'_{\epsilon,n}(t)))_{\partial T_\epsilon} = 0. \tag{45}$$

Now, we integrate between 0 and t, using (33) and (35)-(37), we can deduce

$$\int_0^t |(v'_{\epsilon,n}(s), \gamma(v'_{\epsilon,n}(s)))|_{H_2}^2 ds + \frac{\min\{1,\alpha\}}{1 + \|\gamma\|^p} \frac{1}{p} ||(v_{\epsilon,n}(t), \gamma(v_{\epsilon,n}(t)))||_{V_p}^p$$
$$+\widetilde{\eta}_1 \left(|v_{\epsilon,n}(t)|_{q_1,\Lambda_\epsilon}^{q_1} + \epsilon|\gamma(v_{\epsilon,n}(t))|_{q_2,\partial T_\epsilon}^{q_2} \right) \leq \frac{\max\{1,\alpha\}}{p} ||(v_{\epsilon,n}^0, \gamma(v_{\epsilon,n}^0))||_{V_p}^p$$
$$+\widetilde{\eta}_2 \left(|v_{\epsilon,n}^0|_{q_1\Lambda_\epsilon}^{q_1} + \epsilon|\gamma(v_{\epsilon,n}^0)|_{q_2,\partial T_\epsilon}^{q_2} \right) + 2\widetilde{\lambda}K,$$

for all $t \in (0, \bar{T})$. In order to estimate the right hand side of the last inequality we use (24) together with (44) and $\epsilon \ll 1$. In particular, we can deduce

$$\int_0^t |(v'_{\epsilon,n}(s), \gamma(v'_{\epsilon,n}(s)))|_{H_2}^2 ds + \frac{\min\{1,\alpha\}}{1 + \|\gamma\|^p} \frac{1}{p} ||(v_{\epsilon,n}(t), \gamma(v_{\epsilon,n}(t)))||_{V_p}^p \leq K, \tag{46}$$

for all $t \in (0, \bar{T})$.

Now, if we argue as in the proof of [5, Lemma 4.5], we can deduce

$$\sup_{t \in [0,\bar{T}]} ||(v_\epsilon(t), \gamma(v_\epsilon(t)))||_{V_p} \leq K,$$

and, in particular, the second estimate in (38) is proved. $\qquad\square$

Now, we use the following extension result given by Donato and Moscariello [24, Lemma 2.4]:

Let $\hat{v}_\epsilon \in W_0^{1,p}(\Lambda)$ be a $W^{1,p}$-extension of v_ϵ, that satisfies the following condition

$$|\nabla \hat{v}_\epsilon|_{p,\Lambda} \leq K|\nabla v_\epsilon|_{p,\Lambda_\epsilon}, \quad K > 0. \tag{47}$$

Corollary 4.2. *We suppose the assumptions in Lemma 4.1. Then, there are a constant K independend on ϵ and a $W^{1,p}$-extension $\hat{v}_\epsilon$ of the solution v_ϵ of (10) into $\Lambda \times (0, \bar{T})$, such that*

$$\|\hat{v}_\epsilon\|_{p,\Lambda,\bar{T}} \leq K, \quad \sup_{t \in [0,\bar{T}]} \|\hat{v}_\epsilon(t)\|_{p,\Lambda} \leq K, \tag{48}$$

where $||\cdot||_{p,\Lambda,\bar{T}}$ is the norm in $L^p(0, \bar{T}; W^{1,p}(\Lambda))$ and $||\cdot||_{p,\Lambda}$ is the norm in $W^{1,p}(\Lambda)$.

9

5 Convergence result

Proposition 5.1. *We suppose the assumptions in Lemma 4.1. Then, there is a function $v \in L^p(0, \bar{T}; W_0^{1,p}(\Lambda))$ such that, at least after extraction of a subsequence, we obtain*

$$\hat{v}_\epsilon(t) \rightharpoonup v(t) \qquad in \;\; W_0^{1,p}(\Lambda), \quad \forall t \in [0, \bar{T}], \quad \forall \bar{T} > 0, \tag{49}$$

$$\hat{v}_\epsilon(t) \to v(t) \qquad in \;\; L^p(\Lambda), \quad \forall t \in [0, \bar{T}], \quad \forall \bar{T} > 0, \tag{50}$$

$$|\hat{v}_\epsilon(t)|^{p-2}\hat{v}_\epsilon(t) \to |v(t)|^{p-2}v(t) \qquad in \;\; L^{p'}(\Lambda), \quad \forall t \in [0, \bar{T}], \quad \forall \bar{T} > 0, \tag{51}$$

$$f_1(\hat{v}_\epsilon(t)) \to f_1(v(t)) \qquad in \;\; L^{q_1'}(\Lambda), \quad \forall t \in [0, \bar{T}], \quad \forall \bar{T} > 0, \tag{52}$$

where "$\rightharpoonup$" denotes the weak convergence and "$\to$" denotes the strong convergence.

Moreover, there is a function $\rho \in L^{p'}(0, \bar{T}; L^{p'}(\Lambda))$ such that, at least after extraction of a subsequence, we obtain

$$\hat{\rho}_\epsilon \rightharpoonup \rho \qquad in \;\; L^{p'}(0, \bar{T}; L^{p'}(\Lambda)), \quad \forall \bar{T} > 0, \tag{53}$$

where $\hat{\rho}_\epsilon$ is given by

$$\hat{\rho}_\epsilon = \begin{cases} \rho_\epsilon & in \; \Lambda_\epsilon \times (0, \bar{T}), \\ 0 & in \; (\Lambda \setminus \overline{\Lambda_\epsilon}) \times (0, \bar{T}), \end{cases} \tag{54}$$

with $\rho_\epsilon := |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon$.

Finally, we suppose that $f_2 \in C^1(\mathbb{R})$. Then, we obtain

$$f_2(\hat{v}_\epsilon(t)) \to f_2(v(t)) \qquad in \;\; L^{\bar{r}}(\Lambda), \quad \forall t \in [0, \bar{T}], \quad \forall \bar{T} > 0, \tag{55}$$

$$f_2(\hat{v}_\epsilon(t)) \rightharpoonup f_2(v(t)) \qquad in \;\; W_0^{1,\bar{r}}(\Lambda), \quad \forall t \in [0, \bar{T}], \quad \forall \bar{T} > 0, \tag{56}$$

where $\bar{r} > 1$ is given by

$$\bar{r} \in (1, p) \quad if \; p = N, \quad \bar{r} = \frac{Np}{(N-p)(q_2-2) + N} \quad if \; p < N,$$

with q_2 satisfying (11).

Proof. If we argue as in the proof of [5, Proposition 5.1], we obtain (49)-(50) and (52)-(53).

On the other hand, observe that

$$||w|^{p-2}w| \leq C(1 + |w|^{p-1}).$$

Using [19, Theorem 2.4] with $G(x, w) = |w|^{p-2}w$, $t = p'$ and $r = p$, we can deduce that $w \in L^p(\Lambda) \mapsto |w|^{p-2}w \in L^{p'}(\Lambda)$ is continuous in the strong topologies. And, using (50), we can deduce (51).

For the nonlinear term f_2, we separate the cases $p < N$ and $p = N$. We argue as in the proof of [6, Proposition1] in order to obtain (55)-(56).

$\square$

6 Study of the limit problem

Let $w \in \mathcal{D}(\Lambda)$ be a test function. We multiply (10) by w and integrating by parts, we can deduce, in $\mathcal{D}'(0, \bar{T})$, the following variational formulation

$$d_t \left(\int_\Lambda \omega_{\Lambda_\epsilon} \hat{v}_\epsilon(t) w dx \right) + \epsilon \, d_t \left(\int_{\partial T_\epsilon} \gamma(v_\epsilon(t)) w d\sigma(x) \right) + \int_\Lambda \hat{\rho}_\epsilon \cdot \nabla w dx$$

$$+ \alpha \int_\Lambda \omega_{\Lambda_\epsilon} |\hat{v}_\epsilon(t)|^{p-2}\hat{v}_\epsilon(t) w dx + \int_\Lambda \omega_{\Lambda_\epsilon} f_1(\hat{v}_\epsilon(t)) w dx + \epsilon \int_{\partial T_\epsilon} f_2(\gamma(v_\epsilon(t))) w d\sigma(x) = 0,$$

10

where $\omega_{\Lambda_\epsilon}$ is the characteristic function of the domain Λ_ϵ. Observe that the main difference with [5, Theorem 6.1] is the presence of the term $\int_\Lambda \omega_{\Lambda_\epsilon} |\hat{v}_\epsilon(t)|^{p-2} \hat{v}_\epsilon(t) w dx$.

We use $\vartheta \in C_c^1([0, \bar{T}])$ with $\vartheta(\bar{T}) = 0$ and $\vartheta(0) \neq 0$. We multiply by ϑ and we integrate between 0 and $\bar{T}$, in order to obtain

$$-\vartheta(0) \left(\int_\Lambda \omega_{\Lambda_\epsilon} \hat{v}_\epsilon(0) w dx \right) - \int_0^{\bar{T}} \frac{d}{dt} \vartheta(t) \left(\int_\Lambda \omega_{\Lambda_\epsilon} \hat{v}_\epsilon(t) w dx \right) dt$$

$$-\epsilon\vartheta(0) \left(\int_{\partial T_\epsilon} \gamma(v_\epsilon(0)) w d\sigma(x) \right) - \epsilon \int_0^{\bar{T}} \frac{d}{dt} \vartheta(t) \left(\int_{\partial T_\epsilon} \gamma(v_\epsilon(t)) w d\sigma(x) \right) dt$$

$$+ \int_0^{\bar{T}} \vartheta(t) \int_\Lambda \hat{\rho}_\epsilon \cdot \nabla w dx dt + \alpha \int_0^{\bar{T}} \vartheta(t) \int_\Lambda \omega_{\Lambda_\epsilon} |\hat{v}_\epsilon(t)|^{p-2} \hat{v}_\epsilon(t) w dx dt \qquad (57)$$

$$+ \int_0^{\bar{T}} \vartheta(t) \int_\Lambda \omega_{\Lambda_\epsilon} f_1(\hat{v}_\epsilon(t)) w dx dt + \epsilon \int_0^{\bar{T}} \vartheta(t) \int_{\partial T_\epsilon} f_2(\gamma(v_\epsilon(t))) w d\sigma(x) dt = 0.$$

We separately analyze the special term $\int_0^{\bar{T}} \vartheta(t) \int_\Lambda \omega_{\Lambda_\epsilon} |\hat{v}_\epsilon(t)|^{p-2} \hat{v}_\epsilon(t) w dx dt$.

Taking into account (51), using [16, Theorem 2.6] and Lebesgue's Dominated Convergence Theorem, we can deduce

$$\int_0^{\bar{T}} \vartheta(t) \int_\Lambda \omega_{\Lambda_\epsilon} |\hat{v}_\epsilon(t)|^{p-2} \hat{v}_\epsilon(t) w dx dt \to \theta^* \int_0^{\bar{T}} \vartheta(t) \int_\Lambda |v(t)|^{p-2} v(t) w dx dt, \quad \text{if} \quad \epsilon \to 0,$$

where $\theta^* = |Z^*|/|Z|$.

For the other terms we reason as in the proof of [5, Theorem 6.1]. Then, as $\epsilon \to 0$ in (57), we have

$$-\vartheta(0) (\theta^* + \theta_T) \left(\int_\Lambda v(0) w dx \right) - (\theta^* + \theta_T) \int_0^{\bar{T}} \frac{d}{dt} \vartheta(t) \left(\int_\Lambda v(t) w dx \right) dt$$

$$+ \int_0^{\bar{T}} \vartheta(t) \int_\Lambda \rho \cdot \nabla w dx dt + \alpha \theta^* \int_0^{\bar{T}} \vartheta(t) \int_\Lambda |v(t)|^{p-2} v(t) w dx dt \qquad (58)$$

$$+ \theta^* \int_0^{\bar{T}} \vartheta(t) \int_\Lambda f_1(v(t)) w dx dt + \theta_T \int_0^{\bar{T}} \vartheta(t) \int_\Lambda f_2(v(t)) w dx dt = 0,$$

where $\theta_T = |\partial T|/|Z|$.

We observe that the function ρ satisfies

$$(\theta^* + \theta_T) \partial_t v - \text{div} \rho + \theta^* \left(\alpha |v|^{p-2} v + f_1(v) \right) + \theta_T f_2(v) = 0, \quad \text{in} \quad \Lambda \times (0, \bar{T}). \qquad (59)$$

Following the proof of [24, Theorem 3.1], we can deduce that

$$\rho = b(\nabla v) \quad \text{a.e. in } \Lambda \times (0, \bar{T}), \qquad (60)$$

where b is defined by (16).

We observe that v satisfies (12) using (59) and (60). The boundary condition (13) is obviously satisfied. Moreover, taking into account (60) in (58), we obtain exactly the variational formulation of the limit problem (12)-(14), so we get the initial condition (14). A weak solution v of (12)-(14) satisfies $v \in C([0,\bar{T}];L^2(\Lambda)) \cup L^p(0,\bar{T};W_0^{1,p}(\Lambda))$, for all $\bar{T} > 0$, and

$$(\theta^* + \theta_T)\, d_t(v(t),w)_\Lambda + (b\,(\nabla v(t))\,, \nabla w)_\Lambda + \theta^*\alpha(|v(t)|^{p-2}v(t),w)_\Lambda \tag{61}$$
$$+\theta^*(f_1(v(t)),w)_\Lambda + \theta_T(f_2(v(t)),w)_\Lambda = 0, \quad \forall w \in W_0^{1,p}(\Lambda),$$

in $\mathcal{D}'(0,\bar{T})$ and with the initial condition

$$v(0) = v_0, \tag{62}$$

where $(\cdot,\cdot)_\Lambda$ is the inner product in $L^2(\Lambda)$ or $(L^2(\Lambda))^N$ and the duality product between $L^{r'}(\Lambda)$ and $L^r(\Lambda)$ if $r \neq 2$.

The existence and uniqueness of solution of (12)-(14) is based on the theory of monotonicity of Lions [38]. We give a sketch of a proof.

We take into account the structure properties of b, in particular, from [24, Lemmas 2.10-2.13]), for any $\zeta \in \mathbb{R}^N$, we have

$$|b(\zeta)| \leq c(1 + |\zeta|)^{p-1}, \tag{63}$$

where $c > 0$, and for $\zeta_1, \zeta_2 \in \mathbb{R}^N$, we have

$$(b(\zeta_1) - b(\zeta_2), \zeta_1 - \zeta_2) \geq \kappa|\zeta_1 - \zeta_2|^p, \quad \text{if } p \geq 2, \tag{64}$$

where $\kappa > 0$.

On the space $W_0^{1,p}(\Lambda)$ we define the nonlinear monotone operator $B_p : W_0^{1,p}(\Lambda) \to (W_0^{1,p}(\Lambda))'$, given by

$$\langle B_p(w), u \rangle := C_1(b(\nabla w), \nabla u)_\Lambda + C_1\theta^*\alpha(|w|^{p-2}w, u)_\Lambda, \quad \forall w, u \in W_0^{1,p}(\Lambda), \tag{65}$$

where $C_1 = (\theta^* + \theta_T)^{-1}$.

Taking into account (64) with $\zeta_1 = \nabla w$ and $\zeta_2 = 0$, we can deduce

$$\langle B_p(w), w \rangle = C_1(b(\nabla w), \nabla w)_\Lambda + C_1\theta^*\alpha|w|_{p,\Lambda}^p \tag{66}$$
$$\geq C_1\kappa|\nabla w|_{p,\Lambda}^p + C_1\int_\Lambda b(0)\nabla w dx + C_1\theta^*\alpha|w|_{p,\Lambda}^p, \quad \forall w \in W_0^{1,p}(\Lambda),$$

where $|\cdot|_{p,\Lambda}$ is the norm in $L^p(\Lambda)$.

On the other hand, using (63) with $\zeta = 0$ and Young's inequality, we can deduce

$$\left|\int_\Lambda b(0)\nabla w dx\right| \leq \int_\Lambda c|\nabla w| dx \leq \frac{2}{p\kappa}\frac{c^{p'}}{p'}|\Lambda| + \frac{\kappa}{2}|\nabla w|_{p,\Lambda}^p, \tag{67}$$

where $|\Lambda|$ denotes the measure of Λ and p' is the conjugate exponent of p.

Then, taking into account (67) in (66), we obtain

$$\langle B_p(w), w \rangle + C_1\frac{2}{p\kappa}\frac{c^{p'}}{p'}|\Lambda| \geq C_1\frac{\kappa}{2}|\nabla w|_{p,\Lambda}^p + C_1\theta^*\alpha|w|_{p,\Lambda}^p$$
$$\geq \min\{C_1\frac{\kappa}{2}, C_1\theta^*\alpha\}||w||_{p,\Lambda}^p, \quad \forall w \in W_0^{1,p}(\Lambda),$$

so B_p is coercive.

Now, we consider the following spaces and operators

$$V_1 = W_0^{1,p}(\Lambda), \quad V_2 = L^{q_1}(\Lambda), \quad V_3 = L^{q_2}(\Lambda),$$

$$B_1(v) = B_p, \quad B_2(v) = C_1\theta^* f_1(w), \quad B_3(v) = C_1\theta_T f_2(w).$$

We observe that from (28), we can deduce that $B_i : V_i \to V_i'$ for $i = 2, 3$.

Taking into account the continuous embedding (18) for Λ, and the assumptions (2) and (11), we have the following useful continuous inclusions

$$W_0^{1,p}(\Lambda) \subset W^{1,p}(\Lambda) \subset L^{q_1}(\Lambda) \subset L^2(\Lambda), \quad W_0^{1,p}(\Lambda) \subset W^{1,p}(\Lambda) \subset L^{q_2}(\Lambda) \subset L^2(\Lambda), \tag{68}$$

and

$$L^2(\Lambda) \subset L^{q_1'}(\Lambda) \subset \left(W^{1,p}(\Lambda)\right)' \subset \left(W_0^{1,p}(\Lambda)\right)', \quad L^2(\Lambda) \subset L^{q_2'}(\Lambda) \subset \left(W^{1,p}(\Lambda)\right)' \subset \left(W_0^{1,p}(\Lambda)\right)', \tag{69}$$

where $\left(W^{1,p}(\Lambda)\right)'$ and $\left(W_0^{1,p}(\Lambda)\right)'$ denote the dual of $W^{1,p}(\Lambda)$ and $W_0^{1,p}(\Lambda)$, respectively.

Finally, if we apply [38, Ch.2,Th.1.4], we have that (61)–(62) has a unique solution. As v is uniquely determined, the whole sequence $\hat{v}_\epsilon$ converges to v and this completes the proof of Theorem 1.1.

7 Conclusions

In this paper, we consider a parabolic model in a perforated media $\Lambda_\epsilon \subset \mathbb{R}^N$ ($N \geq 2$) with periodically distributed holes of size ϵ. The p-Laplacian operator appears in wide range of scientific fields, for instance in fluid dynamics (e.g. flow in a porous media), nonlinear elasticity, glaciology and image restoration. In this sense, in $\Lambda_\epsilon \times (0, \bar{T})$, with $\bar{T} > 0$, we consider the p-Laplace heat equation

$$\partial_t v_\epsilon - \Delta_p v_\epsilon + \alpha |v_\epsilon|^{p-2} v_\epsilon = -f_1(v_\epsilon),$$

where

$$\Delta_p v_\epsilon := \text{div}\left(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon\right),$$

and f_1 is the derivative of a potential function $\mathcal{F}_1$, that is, $\mathcal{F}_1(s) = \int_0^s f_1(r)dr$.

The usual boundary conditions considered in the literature are Dirichlet or Neumann. With this standard boundary condition, if we consider the following energy functional

$$\mathcal{E}_{\Lambda_\epsilon}(v_\epsilon(t)) := \int_{\Lambda_\epsilon} \left(\frac{1}{p}|\nabla v_\epsilon(t)|^p + \frac{\alpha}{p}|v_\epsilon(t)|^p + \mathcal{F}_1(v_\epsilon(t))\right) dx, \tag{70}$$

and taking into account that

$$d_t \left(\int_{\Lambda_\epsilon} \mathcal{F}_1(v_\epsilon(t))dx\right) = (f_1(v_\epsilon(t)), v_\epsilon'(t))_{\Lambda_\epsilon},$$

then, we observe that this energy functional is decreasing.

In this paper, instead of Dirichlet or Neumann boundary condition, we consider a boundary condition, which depends on the time, on the boundary of the holes. In this sense, we add to (70) the following energy functional

$$\mathcal{E}_{\partial T_\epsilon}(v_\epsilon(t)) := \int_{\partial T_\epsilon} \mathcal{F}_2(v_\epsilon(t))dx, \tag{71}$$

where T_ϵ is the set of all the holes contained in a bounded open set $\Lambda \subset \mathbb{R}^N$ and $\mathcal{F}_2$ is a nonlinear function such that $\mathcal{F}_2 = \int_0^s f_2(r)dr$. Then, we obtain a total energy functional

$$\mathcal{E}(v_\epsilon(t)) = \mathcal{E}_{\Lambda_\epsilon}(v_\epsilon(t)) + \mathcal{E}_{\partial T_\epsilon}(v_\epsilon(t)),$$

which, using integration by parts, is decreasing for all time $t \geq 0$ if we consider the following dynamic boundary condition on the boundary of the holes

$$\partial_{\nu_p} v_\epsilon + \partial_t v_\epsilon = -f_2(v_\epsilon),$$

13

where $\partial_{\nu_p} v_\epsilon = |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \cdot \nu$, with ν the outward normal to ∂T_ϵ, and Dirichlet boundary condition on the boundary of Λ.

We extend the results of [5] obtained for $p = 2$ to the case $2 \leq p \leq N$. The main result of this paper (Theorem 1.1) could be summarized by the following expansion for v_ϵ

$$\hat{v}_\epsilon \sim v,$$

where $\hat{v}_\epsilon$ is the $W^{1,p}$-extension of v_ϵ to Λ and v is the solution of a parabolic model coming the homogenization in the porous media.

Using the present study as a starting point, various improvements can be proposed. The first one is the generalization of the asymptotic study to a other types of nonlinear diffusion (and not only p-Laplacian operator). For instance, it is very interesting if the operators $\Delta_p v_\epsilon$ and $\partial_{\nu_p} v_\epsilon$ from (10) are replaced by the operators $\text{div}\,(a\,(|\nabla v_\epsilon|)\,\nabla v_\epsilon)$ and $b(x)a\,(|\nabla v_\epsilon|)\,\nabla v_\epsilon \cdot \nu$, where $b \in L^\infty(\partial T_\epsilon)$, $b \geq b_0 > 0$ and $a \in C^1\left(\mathbb{R}^N, \mathbb{R}\right)$ is a monotone nondecreasing function such that there are two positive constants c_1, c_2 such that

$$|a(y)| \leq c_1(1 + |y|^{p-2}), \quad a(y)|y|^2 \geq c_2|y|^p, \quad \forall y \in \mathbb{R}^N.$$

Another possible way is to study this parabolic model in a thin porous media (see, for instance, [2, 8, 10, 12, 42, 43] for more details on the importance of this type of domains). Finally, another problem could be to consider a porous media containing a thin fissure. This type of domain is very interesting because it models cracks in geological strata (see, for instance, [3, 4, 9, 11] for more details). Mathematical models of such domains include several small parameters, one is connected to the domain height or the width of a thin fissure and the others to the microstructure. This approach could be very interesting.

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