Instituto de Ciencias Matemáticas

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PhD Dissertation

Free boundary and turbulence for incompressible viscous fluids

A thesis presented for the degree of Doctor in Mathematics by

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A mis padres, Mariló y Fran

Agradecimientos

"La vida no es la que uno vivió, sino la que uno recuerda y cómo la recuerda para contarla."

— Gabriel García Márquez

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Presentación

Las *ecuaciones de Navier-Stokes* conforman la base de la teoría matemática para la dinámica de fluidos viscosos. Podemos considerar una gran variedad de escenarios que involucran este tipo de fluidos. En particular, los fluidos se pueden clasificar en dos regímenes principales: laminar y turbulento. El *número de Reynolds* es una constante que está íntimamente relacionada con este comportamiento, y asocia las fuerzas de viscosidad que actúan en el fluido (causan fricción entre las partículas), con las fuerzas inerciales (causan la aceleración del fluido). En esta tesis, se estudia la dinámica de fluidos viscosos desde dos perspectivas muy diferentes. Por un lado, estudiamos el caso en el que el número de Reynolds es despreciable, dando lugar al sistema de *Stokes*. Describimos el comportamiento de dos fluidos dos-dimensionales diferentes que evolucionan en el tiempo, y analizamos las propiedades de la interfaz entre ellos. Este problema pertenece a la clasificación de problemas de frontera libre. Por otro lado, consideramos un escenario totalmente diferente, donde el número de Reynolds es grande y se desarrolla turbulencia. Estudiamos el movimiento de un fluido en dos o tres dimensiones, cuya turbulencia es homogénea, isotrópica y *fully-developed.*, a través del *modelo de turbulencia de Kolmogorov*. Esta tesis está dividida en dos partes, cada una de ellas dedicada a uno de los problemas mencionados.

La primera parte de la tesis contiene una introducción y dos capítulos. Está basada en el artículo [32]. En el primer capítulo, presentamos el modelo que describe la dinámica de dos fluidos incompresibles, inmiscibles y viscosos que fluyen en el regimen de Stokes, y están contenidos en una banda dos dimensional que es periódica en la dirección horizontal. Asumimos que los fluidos están sujetos a fuerzas gravitatorias y que tienen distintas densidades. La motivación para estudiar este problema es la falta de resultados en el caso de salto de densidades, que consideran una densidad no integrable y un espacio físico de profundidad infinita. En este caso, la diferencia de densidades induce la dinámica de la interfaz que se genera entre los dos fluidos. Uno de los métodos clásicos para tratar los problemas de interfaz libre es usar la teoría del potencial para construir soluciones explícitas para el sistema. Usando este método, construimos una ecuación de dinámica de contorno para este problema, a través de una versión horizontalmente periódica del *Stokeslet*. Esta técnica genera soluciones explícitas para nuestro escenario, incluso para términos de fuerza más generales al que usamos en nuestro análisis (la fuerza gravitatoria). Además, esta formulación para la velocidad consiste en una ecuación no local y fuertemente no lineal. Como primera estrategia, analizamos el operador lineal que se genera, y probamos que, en el caso estable de las densidades, cuando el fluido más denso está debajo del menos denso, produce un efecto de *weak damping*. Este tipo de comportamiento supone un gran contraste entre este y otros problemas de frontera libre relacionados, cuyos operadores lineales son parabólicos. En nuestro caso, el efecto de *weak damping* sugiere que las soluciones no ganan regularidad, y por lo tanto nuestro problema es de naturaleza hiperbólica. Teniendo esto en cuenta, estudiamos el problema no lineal y probamos existencia y unicidad de soluciones locales en tiempo, cuando la interfaz inicial está descrita por una curva sin auto-intersecciones y con regularidad $C^{1,\gamma}$, con $0 < \gamma < 1$. De acuerdo con el comportamiento esperado, la solución no gana regularidad, es $C^{1,\gamma}$ en espacio. Este resultado de *well-posedness* es cierto independientemente del regimen de las densidades, es decir, el sistema está bien propuesto incluso cuando el fluido más denso se encuentra encima del menos denso. Este fenómeno se debe a la viscosidad de los fluidos.

En el segundo capítulo, estudiamos el comportamiento de las soluciones para tiempos largos, cuando el dato initial es pequeño y está descrito por el grafo de una función. Las técnicas utilizadas aprovechan las propiedades del semi-grupo lineal y el llamado *weak damping*. Con estas técnicas, probamos existencia global en tiempo para el caso estable de las densidades (estabilidad Rayleigh-Taylor). La prueba se basa en estimaciones de energía a priori en espacios de Sobolev y el estudio riguroso de los núcleos que aparecen. También probamos la estabilidad de la interfaz plana, es decir, el decaimiento de la frontera libre al estado estacionario plano. En particular, probamos existencia y unicidad de soluciones globales en tiempo con regularidad Sobolev H^3 y decaimiento polinomial de la interfaz al estado estacionario. Además, podemos extender este resultado a funciones analíticas en espacios de Wiener. Para ello, usamos técnicas de análisis de Fourier en la ecuación de dinámica de contorno y las propiedades del semigrupo lineal en las álgebras de Wiener para obtener existencia global en tiempo y decaimiento exponencial hacia el estado estacionario plano. Finalmente, en el regimen Rayleigh-Taylor inestable de las densidades, construimos una amplia familia de soluciones suaves con crecimiento exponencial en tiempo para intervalos de tiempo arbitrariamente largos, probando que las interfaces pueden crecer exponencialmente en el caso intestable.

La segunda parte de la tesis contiene una introducción y un capítulo. Está basada en el próximo artículo [21]. En este capítulo, se establece la existencia y unicidad de soluciones locales en tiempo para el modelo de Kolmogorov de turbulencia. Este modelo pertenece a los modelos $k-\varepsilon$, y describe la dinámica de un fluido homogéneo, isotrópico con turbulencia *fully-developed*. Generalizamos los resultados previos relajando las hipótesis sobre la energía cinética turbulenta, que puede anularse, para cubrir un mayor rango de casos posibles. En consecuencia, perdemos la parabolicidad del sistema, y se necesita un análisis cuidadoso para encontrar existencia y unicidad de soluciones. Probamos *well-posedness* en espacios de Sobolev críticos H^s para $s > 1 + d/2$, para fluidos de dimensión dos y tres, que ocupan un dominio periódico. Consideramos que la regularidad es fraccionaria, y en consecuencia, nuestro estudio involucra cálculo paradiferencial y la descomposición de Littlewood-Paley, para poder obtener estimaciones de energía a priori.

Abstract

The mathematical bases of the dynamics of viscous fluids are given by the classical Navier-Stokes equations, which model the motion of a viscous incompressible fluid. We can consider a wide variety of scenarios involving these type of fluids. In particular, one can classify the motion of fluids in two general regimes: laminar and turbulent. The Reynolds number is a constant intimately related to this behavior, which associates the viscosity forces acting on the fluid (causing friction between particles), with the inertial forces (causing acceleration of the fluid). In this thesis, we study the dynamics of viscous fluids from two very different perspectives. On the one hand, we study the scenario where the Reynolds number is vanishingly small, giving rise to the Stokes system. We describe the behavior of two different two-dimensional fluids which evolve in time, and we analyze the properties of the interface between them. This problem lies in the class of free boundary problems. On the other hand, we consider a drastically different scenario, where the Reynolds number is large and turbulence is developed. We study the motion of a two or three-dimensional fully developed homogeneous isotropic turbulent fluid, through the Kolmogorov two-equation model of turbulence. This thesis is divided into two parts, each of them devoted to one of the problems.

The first part of the thesis contains an introduction and two chapters. It is based on the submitted paper [32]. In the first chapter, we present the model which describes the dynamics of two incompressible immiscible viscous fluids in the Stokes regime, filling a 2D horizontally periodic strip. We assume that the fluids are subject to the gravity force and they have different densities. This framework is chosen motivated by the lack of results in the density jump setting with an infinitely deep geometry and a non-integrable density. In this scenario, the

density jump induces the dynamics of the free interface arising between the two fluids. One of the classical methods to deal with free boundary problems is to use potential theory to furnish explicit solutions for the system. Using this approach, we derive a contour dynamics formulation for this problem, through a x_1-periodic version of the Stokeslet. This technique yields explicit solutions of the system, even for more general forcing terms than the one used in our analysis (the gravity force). Furthermore, this formulation of the velocity consist of a non-local and strongly non-linear equation. As a first approach, we analyze the linear operator inside the explicit solution, which shows what we call a weak damping effect in the stable stratification of the densities, when the lighter fluid lies above the denser one. This type of operator shows a contrast between this and other related free boundary problems, whose linear operators are of parabolic type. In our case, the weak damping effect suggests that the solutions do not gain regularization in time, hence the nature of the problem is hyperbolic. Having this in mind, we study the full non-linear equation and we show local-in-time well-posedness when the initial interface is described by a curve with no self-intersections and $C^{1+\gamma}$ Hölder regularity, with $0 < \gamma < 1$. According to the expected hyperbolic behavior, the solution does not gain any regularity, it is $C^{1+\gamma}$ in space. This well-posedness result holds regardless of the Rayleigh-Taylor stability of the physical system, i.e., the system is well-posed even when the denser fluid lies above the lighter one. This behavior is due to the viscosity of the fluids.

In the second chapter, we study the long time behavior of solutions when the initial data is small and described by the graph of a function. The techniques used exploit the properties of the linear semi-group and the so-called weak damping effect. With these techniques, we prove the global-in-time existence for the Rayleigh-Taylor stable case of the densities (the lighter fluid lies above the denser fluid). The proof relies on a priori energy estimates on suitable Sobolev spaces and the careful study of the singular kernels appearing. We also prove stability of the flat interface, i.e., the decay of the free interface to the flat steady state. In particular, we prove existence and uniqueness of global interfaces with H^3 regularity and polynomial decay of the interface. Moreover, we can extend this global-in-time existence result to analytic solutions in suitable Wiener spaces. We use Fourier techniques of the contour dynamics equation and the properties of the linear semi-group in Wiener algebras to obtain global-in-time existence and exponential decay to the flat interface. Finally, in the Rayleigh-Taylor unstable regime, we construct a wide family of smooth solutions with exponential in time growth for an arbitrarily large interval of existence, showing that the free boundaries can grow exponentially.

The second part of the thesis contains an introduction and one chapter. It is based on the forthcoming paper [21]. In this chapter, we establish a local

well-posedness result for the Kolmogorov two-equation model of turbulence. This model belongs to the k-ε models, and describes the dynamics of an homogeneous and isotropic fully-developed turbulent flow. We generalize the previous results letting the turbulent kinetic energy vanish, in order to cover a wider range of phenomena. Consequently, we lose the parabolicity of the system, and an accurate analysis is needed to find the existence and uniqueness of solutions. We prove local well-posedness in critical Sobolev spaces H^s for $s > 1 + d/2$, for the cases of two and three dimensional fluids, in a periodic box $\mathbb{T}^d$. We consider fractional regularity, and consequently, our study involves paradifferential calculus, passing through Littlewood-Paley decomposition, in order to have a priori high order energy bounds.

Contents

THE TWO-PHASE STOKES MODEL

Introduction

> *"... the effort expended in trying to answer the following two fundamental questions has not yet attained complete success: Do the equations of hydrodynamics, together with suitable boundary and initial conditions, have a unique solution? How satisfactory is the description of real flows given by the solutions of those equations?"*
>
> — O. A. Ladyzhenskaya

I The dynamics of a viscous fluid

The description of the dynamics of viscous fluids is a classical problem in physics and a very challenging problem from a mathematical point of view.

Let us consider a viscous incompressible fluid of constant density (a viscous liquid) which occupies a region Ω of the d- dimensional space $\mathbb{R}^d$. In general, we are interested in the cases $d = 2, 3$, which are the most common to model physical phenomena (planar and three-dimensional fluids). Under these assumptions, the dynamics of the fluid is described by the following system of partial differential equations:

$$\rho \left(\frac{du}{dt} + u \cdot \nabla u \right) = \mu \Delta u - \nabla p - \rho f, \quad x \in \Omega,\ t \in \mathbb{R}^+, \tag{I.1a}$$

$$\nabla \cdot u = 0, \quad x \in \Omega,\ t \in \mathbb{R}^+. \tag{I.1b}$$

In the previous system, $t \in \mathbb{R}^+$ represents time, $x \in \Omega$ stands for a point in the physical region, and the variables of the system are the velocity vector field $u(x, t)$ and the pressure scalar field $p(x, t)$. The constants ρ, μ denote the density of the fluid and the dynamic viscosity coefficient, respectively, while the function $f(x, t)$ represents the external forces exerted on the fluid.

The mathematical operators inside the equation have the following form in cartesian coordinates:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right)$$

is the gradient operator and

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

is the Laplace operator.

With these definitions in mind, the term in the left-hand side of the equation represents the total acceleration of the fluid particles (the inertial forces). In particular, the non-linear term

$$u \cdot \nabla u = \sum_{i=1}^{d} u_i \frac{\partial u}{\partial x_i}$$

is the convective term and captures the non-linear interactions taking place in the dynamics.

On the right-hand side, the Laplacian of the velocity represents the viscous friction, the gradient of the pressure represents the internal forces acting on the surfaces of fluid volumes, and finally, f captures the external forces (e.g., gravity forces).

The physical law behind equation (I.1a) is the balance of linear momentum, i.e., *Newton's second law*. Equation (I.1b) represents the conservation of mass in the case of a fluid with constant density, which derives into the conservation of volumes. For that reason, this equation is referred to as the *incompressibility condition*.

The system (I.1) is known as the *Navier-Stokes equations*. It was first proposed by Navier [78] and consolidated by Poisson (1831), Saint Venant (1843) and Stokes [84], who settled the bases from a continuum mechanics point of view. In the twentieth century, the advances of Leray [59, 60, 61], Hopf [47], Ladyzhenskaya [56, 57], among others, placed the Navier-Stokes equations as the fundamental system describing the dynamics of an incompressible viscous fluid. Note that if we set the viscosity $\mu = 0$, we recover the well-known *Euler equations* for the dynamics of ideal fluids.

Nowadays, the Navier-Stokes equations constitute one of the main fields of research in mathematical fluid dynamics. It is worth mentioning that, although modern sophisticated techniques have been developed to study this PDE system, there are a wide variety of unsolved questions and problems. From an analytical point of view, the community is mostly interested in the existence of solutions, their uniqueness and their regularity, as well as the possible formation of singularities.

The most celebrated open problem is the well-known millennium problem, which consists in finding global existence of smooth solutions for the Navier-Stokes equations in three dimensions. Such a result holds true in two dimensions, as proved by Leray [61] (see also the contributions to this problem by Ladyzenskaya [57] and Lions & Prodi [64]) and also in the three dimensional case for finite time, according to Leray [59]. Although weak solutions globally exist, as proved by Leray [61] and Hopf [47], it remains an open question whether smooth solutions can be continued for all times (see Robinson & Rodrigo & Sadowski [82] for a review of results).

Notice that the Navier-Stokes system models very general phenomena. In order to tackle a particular phenomenon, a possible approach would be to restrict the model by adding extra assumptions. For instance, one might assume certain intrinsic properties of the fluid, such as a small/large ratio between inertial forces and viscous forces. This particular hypothesis will be our next subject of study.

Let us denote the *characteristic length-scale* and the *characteristic velocity* of a fluid as L and V, respectively. These quantities depend intrinsically on the fluid and the domain where it is flowing. For instance, if we are analyzing the motion of a fluid in a pipe, L represents the diameter of the pipe and V stands for the mean velocity of the fluid. Consider also the *dynamic viscosity* of the fluid μ. Then,

$$\mathrm{Re} = \frac{LV}{\mu} \tag{I.2}$$

is an adimensional quantity and captures the ratio between inertial and viscous forces. This constant is known as the *Reynolds number*. It was introduced by Stokes [85] but named after Reynolds, due to his popular experiments using this parameter (see [81]).

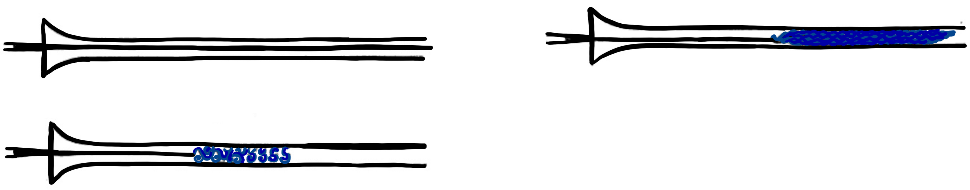

Figure 1: Development of turbulence in an experiment conducted by Reynolds [81], consisting in injecting colored water at different velocities in a glass tube filled with clear water.

The Reynolds number allows us to measure how turbulent is the behavior of a fluid. At small Reynolds numbers, the viscous forces are strong compared to inertial forces, which causes the fluid to flow in a laminar regime. On the contrary, at high Reynolds numbers, the viscosity is not strong enough to prevent the fluid to form eddies and turbulent patterns due to the high acceleration of the particles.

Assuming no external forcing for simplicity, the non-dimensional form of the Navier-Stokes equations becomes

$$\mathrm{Re}\left(\frac{du}{dt} + u \cdot \nabla u\right) = \Delta u - \nabla p. \tag{I.3}$$

II The Stokes law and the *Stokeslet*

In this section, we will focus on the scenario where the Reynolds number Re is small enough to be negligible. This hypothesis added to the non-dimensional form of the Navier-Stokes equations (I.3) gives rise to the Stokes equations

$$\Delta u - \nabla p = 0, \quad x \in \Omega,\ t \in \mathbb{R}^+, \tag{II.1a}$$
$$\nabla \cdot u = 0, \quad x \in \Omega,\ t \in \mathbb{R}^+. \tag{II.1b}$$

There is a vast literature about the study of the Stokes operator and Stokes-based models. For instance we refer to the pioneer works by Ladyzhenskaya [57], Ladyzhenskaya & Solonnikov [56] and Abe & Giga [1].

The existence and uniqueness of weak solutions to the system (II.1) in a bounded domain Ω is due to Odqvist and Lichtenstein in independent works (see [57], Chapter 2). In contrast, we are interested in the special case where the fluid occupies the whole space $\Omega = \mathbb{R}^n$. In this setting, it is possible to construct explicit solutions using the approach of hydrodynamic potentials, which allow us to build, in a similar fashion as for the Laplace equation, a fundamental solution for the Stokes equation, or *Stokeslet* (see [29] for a further discussion).

Let us consider the symmetric tensor field S and the vector field P, defined by the relations

$$S_{ij}(x-y) = \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\right)\Phi(|x-y|), \tag{II.2a}$$

$$P_j(x-y) = -\frac{\partial}{\partial y_j}\Delta\Phi(|x-y|), \tag{II.2b}$$

where $x, y \in \mathbb{R}^n$, δ_{ij} is the Kronecker delta and $\Phi(z)$ is a smooth enough function for $z \neq 0$. Formal computations lead us into

$$\Delta S_{ij}(x-y) + \frac{\partial}{\partial x_i}P_j(x-y) = \delta_{ij}\Delta^2\Phi(|x-y|), \tag{II.3a}$$

$$\frac{\partial}{\partial x_i}S_{ij}(x-y) = 0. \tag{II.3b}$$

Notice that, if we choose Φ to be a fundamental solution for the biharmonic equation in $\mathbb{R}^n$, from (II.3), it follows that the pair S, P defined in (II.2) is a fundamental solution to the Stokes system in $\mathbb{R}^n$.

In the three dimensional case, where $n = 3$, the fundamental solution of the biharmonic equation is

$$\Phi(|x-y|) = -\frac{|x-y|}{8\pi},$$

and the fields S and P are given by

$$S_{ij}(x-y) = -\frac{1}{8\pi}\left(\frac{\delta_{ij}}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right), \tag{II.4a}$$

$$P_j(x-y) = \frac{1}{4\pi}\frac{x_j-y_j}{|x-y|^3}. \tag{II.4b}$$

Similarly, in the planar case, where $n = 2$, we have

$$\Phi(|x-y|) = \frac{1}{8\pi}|x-y|^2\log(|x-y|),$$

$$S_{ij}(x-y) = -\frac{1}{4\pi}\left(\delta_{ij}\log\left(\frac{1}{|x-y|}\right) + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right), \tag{II.5a}$$

$$P_j(x-y) = \frac{1}{2\pi}\frac{x_j-y_j}{|x-y|^2}. \tag{II.5b}$$

The fundamental solution also allows to get explicit solutions for the non-homogeneous Stokes problem

$$\begin{aligned} \Delta u - \nabla p &= f, \quad x \in \mathbb{R}^n, \\ \nabla \cdot u &= 0, \quad x \in \mathbb{R}^n. \end{aligned}$$

One gets from potential theory that the solution for this system is explicitly given by

$$u(x) = \int_{\mathbb{R}^n} S(x-y) \cdot f(y) dy,$$
$$p(x) = -\int_{\mathbb{R}^n} P(x-y) \cdot f(y) dy.$$

Let us briefly discuss an alternative approach to get the previous explicit solution for the non-homogeneous Stokes system in $\mathbb{R}^n$, which may be more clarifying from a physical point of view. We take the stream potential coming from the velocity u, whose existence is assured by (II.1b). It is characterized by the relation

$$u = \nabla \wedge \psi.$$

Applying the curl to the Stokes equation, we find that the stream function solves the elliptic problem

$$-\Delta^2 \psi = \nabla \wedge f.$$

From this point, we can get an explicit formula for the stream function, through the Green function of the biharmonic equation. Therefore, we obtain the formal expression of the velocity u:

$$u = \nabla \wedge (-\Delta^{-2}(\nabla \wedge f)).$$

Similarly, taking the divergence of the Stokes equation, we find the elliptic relation for the pressure (Poisson type)

$$-\Delta p = \nabla \cdot f.$$

It is well-known that we can find an explicit expression for p from the latter elliptic equation.

Let us observe that in order to explicitly solve the Stokes system, we need to know the explicit expressions of the Green functions for some elliptic problems in the physical domain $\mathbb{R}^n$. Once we have understood the difficulties and techniques that are key to find the fundamental solution of the Stokes system in the whole space, our aim is to study the construction of these type of kernels in a domain with a boundary (for instance, see Chapter 4 in [29] for the case of the half-space).

In this part of the thesis, we will study the two-phase non-homogeneous Stokes system in the horizontally periodic planar strip $\mathbb{T} \times \mathbb{R}$. The analysis for this model will strongly rely on the theory of hydrodynamic potentials presented above (we will construct suitable potentials for our scenario) and of Calderón-Zygmund theory for singular kernels (see [9] and [36]).

III The two-phase Stokes problem

III.1 Description of the model

We study the two-phase Stokes problem in the planar strip $\mathbb{T}\times\mathbb{R}$. We consider two different immiscible fluids evolving in the physical domain, and as a consequence, the motion of the fluids will result into the dynamics of the free boundary between them. In other words, we introduce another level of complexity to the problem, since the domain of the fluids becomes a function of time, $\Omega = \Omega(t)$. These type of problems are known as free boundary problems. The methodology consists in analyzing different properties of the free boundary, such as its regularity or the possible formation of singularities, in order to determine certain behavior of the flow. The fact that the domain and the solution of the PDE are coupled makes free boundary problems very technical, since numerous standard PDE techniques do not extend to these problems. Consequently, the study of free interfaces requires the use of a wide toolbox of mathematical analysis techniques, where Fourier analysis and harmonic analysis are fundamental.

Free boundaries appear naturally in many different physical and biological phenomena. The vast variety of interesting questions and scenarios makes it a very active area of research. For instance, they appear in the dynamics of ocean waves, air masses, or in the mixing of two immiscible fluids of different nature. In particular, the motion of a two-phase highly viscous flow is a practical problem in many applications ranging from biology to physics. Furthermore, this is also a first principles step towards the description of viscous water waves. Indeed, although for most of the applications in coastal engineering, water waves are assumed to be inviscid, as already stated by the celebrated applied mathematician Longuet-Higgins [65] (see [37] for more details regarding the mathematical modeling of viscous waves):

> *"For certain applications, however, viscous damping of the waves is important, and it would be highly convenient to have equations and boundary conditions of comparable simplicity as for undamped waves".*

With this physical phenomena in mind, we consider two incompressible, viscous and immiscible fluids filling the 2π-periodic strip in the variable x_1, $\mathbb{T}\times\mathbb{R}$. The curve

$$\Gamma(t) = \{(z_1(\alpha,t), z_2(\alpha,t)); \quad \alpha\in[-\pi,\pi], \quad z(\alpha+2\pi k,t) = (2\pi k,0) + z(\alpha,t)\},$$

is the interface between the fluids in such a way that

$$\mathbb{T}\times\mathbb{R} = \Omega^+(t)\cup\Omega^-(t)\cup\Gamma(t), \quad \Omega^+(t)\cap\Omega^-(t) = \phi, \quad \partial\Omega^\pm(t) = \Gamma(t).$$

Then the upper fluid fills the domain $\Omega^+(t)$, while the lower fluid lies in $\Omega^-(t)$. More precisely, there exists a constant M big enough such that

$$\mathbb{T} \times [M, +\infty) \subset \Omega^+(t) \quad \text{and} \quad \mathbb{T} \times (-\infty, -M] \subset \Omega^-(t).$$

We consider the Reynolds number defined in (I.2) to be vanishingly small, so the fluids flow in the Stokes regime. As a consequence, the dynamics of the fluids is described by the following system of PDEs:

$$-\mu^\pm \Delta u^\pm = -\nabla p^\pm - g\rho^\pm (0,1)^t, \quad x \in \Omega^\pm(t), t \in [0,T], \tag{III.1a}$$
$$\nabla \cdot u^\pm = 0, \quad x \in \Omega^\pm(t), t \in [0,T], \tag{III.1b}$$
$$[\![-pI + \mu((\nabla u + (\nabla u)^T)/2)]\!] \cdot (\partial_\alpha z)^\perp = 0, \quad x \in \Gamma(t), t \in [0,T], \tag{III.1c}$$
$$[\![u]\!] = 0, \quad x \in \Gamma(t), t \in [0,T], \tag{III.1d}$$
$$z_t = u(z,t), \quad t \in [0,T], \tag{III.1e}$$
$$z = z_0, \quad t = 0. \tag{III.1f}$$

Above we have used the notation

$$(\partial_\alpha z)^\perp = (-\partial_\alpha z_2, \partial_\alpha z_1),$$

$$[\![F]\!] = F^+(z(\alpha,t),t) - F^-(z(\alpha,t),t),$$

giving in (III.1c) the continuity of the stress tensor and in (III.1d) the continuity of the velocity due to the viscosity of the fluids. In the previous equations, g refers to the acceleration due to gravity while u, p, μ and ρ denote the velocity vector field, the pressure scalar field, the dynamic viscosity and the density of each fluid, respectively. Then, z describes the internal wave separating both fluids, which moves with the flow. This internal wave is a free boundary and should be recovered from the dynamics of the problem. We assume that these fluids have the same viscosities but different densities, *i.e.*

$$\mu^+ = \mu^-, \quad \rho^+ \neq \rho^-.$$

In what follows, we assume, without loss of generality, that $\mu = g = 1$. We emphasize the form of the density function, which can be expressed in the whole space as a piece-wise defined function with ρ^+ and ρ^- two different constants:

$$\rho(x,t) = \begin{cases} \rho^+, & x \in \Omega^+(t),\ t \in [0,T], \\ \rho^-, & x \in \Omega^-(t),\ t \in [0,T]. \end{cases}$$

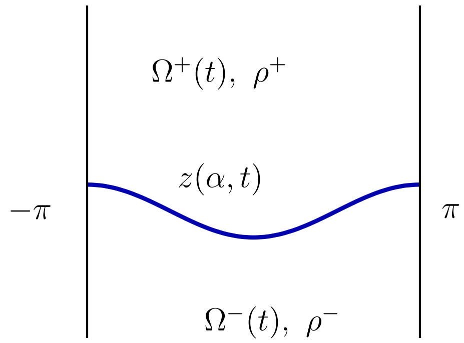

Figure 2: The situation under study.

III.2 State of the art

The dynamics of free boundary viscous waves driven by different forces is a classical and challenging problem, where many scenarios are still unsolved. As discussed in the introduction, the fluid dynamics can be subject to different regimes, such as the Stokes regime, Navier-Stokes regime (see [42]) or a water waves regime with viscosity (see [39]), among others. The fluids are typically driven by gravity and capillarity forces, and their dynamics also depend on the geometry of the problem.

The two-phase Stokes regime driven by capillarity in $\mathbb{R}^2$, where the interface between the two fluids is described by the graph of a function, was first studied by Badea & Duchon [3] in the contour dynamics formulation approach. In this work, the authors prove a global existence and uniqueness result for small initial data in the space of Fourier transforms of bounded measures. This scenario has also been studied by Matioc & Prokert [67, 68], with equal viscosities and viscosity jump, respectively. The authors prove well-posedness in sub-critical Sobolev spaces with arbitrarily large initial data and a criterion for global existence, using the potential theory approach. The linear problem for a graph in the case of a Stokes flow driven by capillarity is

$$f_t^L = -\sigma(-\partial_x^2)^{1/2} f^L, \quad \sigma > 0. \tag{III.2}$$

Taking advantage of the parabolic character of the capillarity problem, they also prove parabolic smoothing of the solution. The same authors have recently proven a similar result for the one-phase problem (i.e., the fluid above behaves like vacuum) as a small viscosity limit [69]. Moreover, Guo & Tice [43] (see also [86]) study the stability of contact lines in fluids. In particular, they consider the dynamics of an incompressible viscous Stokes fluid evolving in a two-dimensional open-top vessel under the influence of gravity and capillary forces. See also [6], where the Stokes system is used to model a de-mixing process of a binary viscous liquid.

Furthermore, the sedimentation of intertialess particles in a viscous flow in 3D is described by a coupled transport-Stokes system subject to gravity. This model was derived by Höfer [44] and Mecherbet [70]. Regarding this problem, Höfer & Schubert [46] and Mecherbet [71] prove in parallel works the existence and uniqueness of solutions with initial density in $L^1 \cap L^\infty$. Further results concerning the transport-Stokes system problem in 2D, are, for instance, [2] and recent work [40], where the author show global well-posedness for compactly supported and $L^1 \cap L^\infty$ initial density and persistence of regularity results. Moreover, in the also recent result [58], the author proves global well-posedness for bounded initial density in the case of bounded domains and the infinite strip $\Omega = \mathbb{R} \times (0, 1)$. See also [45], where Stokes is used to model vortex filaments and [17], where the fractional transport-Stokes is considered. In a very recent result [48], the author uses a Lagrangian approach to show well-posedness of the Stokes transport system with initial L^1 density. A more exhaustive collection of results for the transport-Stokes system can be found in [72].

In this thesis, we will use a contour dynamic approach to write the Stokes free boundary problem as a single non-linear and non-local partial differential equation. Two celebrated problems sharing the linear part of the capillarity-driven Stokes problem are the Peskin and the Muskat problems. Furthermore, both problems can be studied as a non-linear and non-local partial differential equation using a contour dynamics approach similar to the one that we use in this work. On the one hand, the Peskin problem models the dynamics of an elastic filament immersed in a Stokes fluid [77, 35, 63, 15]. On the other hand, the Muskat problem models the dynamics of fluids with different densities flowing in a porous media according to Darcy's law [30, 38]. However, in the problem under consideration in this work, the gravity driven Stokes problem, the dynamics is induced by the gravity force.

The results concerning the gravity driven free boundary Stokes problem are much more scarce when the fluids fill an infinitely deep domain. We start noting that the Stokes gravity interface dynamics when the internal wave is given as the graph of the function $h(x,t)$ verifies the following energy balance

$$\frac{(\rho^- - \rho^+)}{2}\|h(t)\|^2_{L^2(\mathbb{T})} + \int_0^t \|\nabla u\|^2_{L^2(\mathbb{T}\times\mathbb{R})} ds = \frac{(\rho^- - \rho^+)}{2}\|h(0)\|^2_{L^2(\mathbb{T})}.$$

Furthermore, for the Stokes gravity interface dynamics, the linear operator reads

$$(-\partial_x^2)^{1/2} f_t^L = -(\rho^- - \rho^+) f^L,$$

which is not of parabolic type. Instead, we show that this operator has a damping effect without regularization if the denser fluid is below the lighter one: a situation know as the Rayleigh-Taylor stable scenario. We call this a weak damping effect. This is in heavy contrast to the Peskin problem, the Muskat problem and the capillarity-driven Stokes problem where the linear operator is given by a square root of the Laplacian as shown in (III.2). This striking difference makes the gravity driven Stokes problem a hyperbolic problem for the free boundary. For this hyperbolic non-local problem we establish the local existence for $C^{1,\gamma}$ interfaces and global well-posedness and decay for sufficiently small free boundaries in the RT stable case. We provide two global existence theorems. First, we establish global well-posedness of classical solutions in Sobolev spaces and polynomial decay of the L^2 norm towards the flat interface. Secondly, we prove global well-posedness of analytic interfaces in Wiener algebras and exponential decay towards the flat interface. In the two-phase gravity driven Stokes, our result is, to the best of our knowledge, the only one dealing with an infinitely deep domain with non integrable densities, instead our densities are merely in L^∞ with no decay.

The local well-posedness results extends to the Rayleigh-Taylor unstable regime, establishing the existence of local in time $C^{1,\gamma}$ unstable interfaces. Despite being well-posed, in the final result of this work, we prove that interfaces in the unstable regime growth exponentially in some Wiener norms. We prove this result by a careful study of the linear semi-group, which shows exponential growth bounds. We can generalize this behavior to the full non-linear problem, showing exponential growth for smooth small data in Wiener spaces in an arbitrarily large interval of existence. Previous results in this direction are scarce and sometimes are more restricted. In this direction, we have to mention the previous result [41], where Guo, Hallstrom & Spirn showed instability for smooth interfaces in three different physical settings. Namely, vortex sheets with surface tension, the Muskat problem with surface tension and vortex patches. They show that the dynamics of perturbations of the steady states are characterized by the dynamics of their linear semi-groups in a particular time scale related to the size of the perturbation. We would also like to mention the result by Kiselev & Li [49], where the authors show double exponential growth of the curvature of an Euler patch boundary under certain symmetry assumptions and with fixed boundary. We have to emphasize that in our result, we do not need any symmetry or time scale restriction, instead we prove the exponential growth for arbitrary time scales and non-symmetric initial data.

IV Contributions

IV.1 Chapter 1

One of the classical methods to deal with free boundary problems is to exploit potential theory in order to reformulate the problem into a new contour dynamics equation, which will be typically non-local and strongly non-linear. Let us mention that this kind of approach has been extensively and successfully used in other free boundary problems in fluid dynamics to show well-posedness (see [14, 5] for the vortex patch, [18] for water waves, [51, 34, 33] for the SQG sharp-front, [19, 31] for the Muskat problem and [35, 15] for the Peskin problem). Furthermore, it has been applied to prove singularity formation for the water waves, the SQG sharp-front and the Muskat problem [12, 10, 50, 34, 11]. In Chapter 1, we derive the contour dynamics formulation for (III.1) through a new kernel, which we call the x_1-periodic Stokeslet. This approach results in the following equivalent contour dynamics equation for the internal wave $z(\alpha, t)$:

$$z_t(\alpha,t) = (\rho^- - \rho^+) \int_{\mathbb{T}} \mathcal{S}(z(\alpha,t) - z(\beta,t)) \cdot \partial_\beta z^\perp(\beta,t) z_2(\beta,t) d\beta, \tag{IV.1}$$

where the so-called x_1-periodic Stokeslet reads

$$\mathcal{S}(y) = \frac{1}{8\pi} \log\left(2(\cosh(y_2) - \cos(y_1))\right) \cdot I \\ - \frac{y_2}{8\pi(\cosh(y_2) - \cos(y_1))} \begin{pmatrix} -\sinh(y_2) & \sin(y_1) \\ \sin(y_1) & \sinh(y_2) \end{pmatrix}.$$

We take advantage of this contour dynamics formulation (IV.1) to prove the following local in time existence result for the interface with Hölder regularity $C^{1,\gamma}(\mathbb{T})$, in the case when the initial geometry of the interface is a curve $z_0(\alpha)$ satisfying the arc-chord condition (which is measured by the function $\mathcal{F}$ defined in (1.3.1)-(1.3.2)).

Proposition (Prop 3, Chapter 1. Local existence of solutions in $C^{1,\gamma}(\mathbb{T})$)**.** Let $0 < \gamma < 1$ be a fixed parameter and $z_0(\alpha) \in C^{1,\gamma}(\mathbb{T})$ be the initial data satisfying the arc-chord condition. Then, there is a time $0 < T$ such that there exists a unique local solution

$$z \in C^1((-T,T); C^{1,\gamma}(\mathbb{T}))$$

of (IV.1) satisfying the arc-chord condition on $(-T, T)$.

The proof of this result relies on the Picard Theorem on a suitable Banach space. Note that it is independent of the regime of the densities, being valid in both the stable and unstable case. Further details of the proof are shown in Section 1.3.

In the particular case when the internal wave is described as the graph of a function $h(\alpha, t)$, i.e.,

$$z(\alpha, t) = (\alpha, h(\alpha, t)),$$

we show that the contour dynamics equation is equivalent to

$$
\begin{aligned}
h_t(\alpha,t) = {} & \frac{(\rho^- - \rho^+)}{8\pi}\int_{\mathbb{T}} \log(2(\cosh(h(\alpha,t)-h(\beta,t)) - \cos(\alpha-\beta)))h(\beta) \\
& \times [1 + h_\alpha(\alpha)h_\alpha(\beta)]d\beta \\
& + \frac{(\rho^- - \rho^+)}{8\pi}\int_{\mathbb{T}} \frac{h(\beta)(h(\alpha)-h(\beta))}{\cosh(h(\alpha)-h(\beta)) - \cos(\alpha-\beta)} \\
& \times [(h_\alpha(\alpha)h_\alpha(\beta)-1)\sinh(h(\alpha)-h(\beta))]\, d\beta \\
& + \frac{(\rho^- - \rho^+)}{8\pi}\int_{\mathbb{T}} \frac{h(\beta)(h(\alpha)-h(\beta))}{\cosh(h(\alpha)-h(\beta)) - \cos(\alpha-\beta)} \\
& \times [(h_\alpha(\alpha)+h_\alpha(\beta))\sin(\alpha-\beta)]\, d\beta.
\end{aligned} \tag{IV.2}
$$

The detailed proofs of these formulations are given in Section 1.1.

IV.2 Chapter 2

In order to prove global-in-time existence of solutions in Sobolev spaces in the stable regime of the densities, we study the linear semi-group

$$f_t^L = -(\rho^- - \rho^+)(-\partial_x^2)^{-1/2} f^L,$$

which results from the linearization of (IV.2) around the equilibrium, and prove a decay result (see Section 1.2 for further details). The linear semi-group properties together with a priori estimates in $L^2(\mathbb{T})$ and $H^3(\mathbb{T})$ lead us into one of the main results of this chapter, where we show stability for classical solutions with prescribed small initial data in $H^3(\mathbb{T})$. Although we expect that the optimal regularity is less than H^3, we have chosen this space for the sake of simplicity.

Theorem (Thm 1, Chapter 2. Global existence and decay of solutions for small data)**.** Let $h_0 \in H^3(\mathbb{T})$ be the initial data for (IV.2). Assume that the system is in the RT stable regime, *i.e.*

$$\rho^- - \rho^+ > 0,$$

and take m arbitrarily close to 2, that is, $m = 2 - \varepsilon$ for some $\varepsilon > 0$. There is a $0 < \delta = \delta(\rho^- - \rho^+)$ such that if

$$||h_0||_{H^3(\mathbb{T})} < \delta,$$

there exists a unique global classical solution $h(\alpha, t)$

$$h \in C([0,T]; H^3),$$

of (IV.2) for arbitrary $T > 0$ such that

$$(1+t)^m \|h\|_{L^2} + \|h\|_{H^3} \leq C \|h_0\|_{H^3},$$

for some constant $C > 0$.

Further details of the proof are shown in Section 2.1.

Moreover, we prove global well-posedness in Wiener spaces of analytic functions

$$A^1_\nu = \{u \in L^2(\mathbb{T}) \text{ s.t. } \|u\|_{A^1_\nu} = \sum_{k=-\infty}^{\infty} e^{\nu|k|}|k||\hat{u}(k)| < \infty\}$$

with $\nu \geq 0$, by a careful analysis of equation (IV.2) applying Fourier techniques and taking advantage of the algebra structure of Wiener spaces. This result is stated in the following theorem:

Theorem (Thm 2, Chapter 2. Global existence of solutions in Wiener algebras for small data)**.** Let $h_0 \in A^1_{\nu_0}$ be the initial data for (IV.2) in the RT stable case,

$$\rho^- - \rho^+ > 0.$$

Assume that

$$\nu_0 > 0.$$

There is a $0 < \delta = \delta(\rho^- - \rho^+, \nu_0)$ such that if

$$\|h_0\|_{A^1_{\nu_0}} < \delta,$$

so that

$$\nu_0 - \mathcal{M}(\|h_0\|_{A^1_{\nu_0}}) > 0,$$

for a suitable non negative function $\mathcal{M}(x) \approx x + O(x^2)$, there exists a positive decreasing function $\nu(t) > 0$ with $\nu(0) = \nu_0$ (characterized by (2.2.21)) and a unique global analytic solution $h(\alpha, t)$

$$h \in C([0,T]; A^1_{\nu(t)}),$$

of (IV.2) for arbitrary $T > 0$ satisfying

$$\|h\|_{A^1_{\nu(t)}} \leq \|h_0\|_{A^1_{\nu_0}}.$$

The details of the proof are shown in Section 2.2.2. This result, together with a careful study of the linear semi-group growth and the ideas of the proof of the global existence result in Sobolev spaces, allow us to prove exponential decay of solutions in Wiener algebras. The statement reads as follows:

Theorem (Thm 3, Chapter 2. Global existence and exponential decay of solutions for small data)**.** Let $h_0 \in A^1_{\nu_0}$ be the initial data for (IV.2) in the RT stable case,

$$\rho^- - \rho^+ > 0,$$

fulfilling the hypotheses of Theorem 2. In particular,

$$||h_0||_{A^1_{\nu_0}} < \delta,$$

for a suitable $\delta = \delta(\rho^- - \rho^+, \nu_0)$. Assume that

$$0 < \nu^* \leq \frac{\nu_0}{24}.$$

There is a $0 < \varepsilon = \varepsilon(\nu^*, \rho^- - \rho^+, ||h_0||_{A^1_{\nu_0}})$ such that if

$$||h_0||_{A^0_{\nu^*}} < \varepsilon,$$

there exists a unique global analytic solution $h(\alpha, t)$

$$h \in C([0,T]; A^0_{\nu^*}),$$

of (IV.2) for arbitrary $T > 0$ satisfying

$$e^{\sqrt{\frac{\rho^- - \rho^+}{4}\nu^* t}} ||h||_{A^0} + ||h||_{A^0_{\nu^*}} \leq C \, ||h_0||_{A^0_{\nu^*}},$$

for some constant $C > 0$.

The proof is given in Section 2.2.3.

Finally, using the previous result, we can prove that smooth initial data in the RT unstable scenario can lead to exponential growth in a particular Wiener norm.

Theorem (Thm 4, Chapter 2. Exponential growth of solutions in the RT unstable case for small data)**.** Let $T > 0$ be an arbitrary fixed parameter. Then, it exists a family of smooth initial data

$$g_0 \in A^0_{\nu*},$$

such that

$$g \in C([0,T]; A^0_{\nu*}),$$

is a solution of (IV.2) in the RT unstable regime,

$$\rho^- - \rho^+ < 0,$$

and

$$||g(\tau)||_{A^0_{\nu^*}} \geq \frac{1}{C} e^{\sqrt{\frac{|\rho^+ - \rho^-|}{4}\nu^* \tau}} ||g_0||_{A^0}, \quad \tau \in [0,T]. \tag{IV.3}$$

CHAPTER 1

The free boundary Stokes system

This chapter is based on [32].

1.1 The contour dynamics formulation in $\mathbb{T} \times \mathbb{R}$

In this section we are going to find that system (III.1) is equivalent to a single non-local and nonlinear equation for the internal wave $z(\alpha, t)$. This new equation is called the contour dynamics formulation of (III.1). Although such a formulation for the Stokes problem has been used before [3] in the case of $\mathbb{R}^2$, the fact that we are considering periodic waves makes the form of the Stokeslet different and its computation far from being trivial. Thus, as the closed form of the periodic stream function and the periodic Stokeslet for a given generic force F can be of its own interest, we include some of the details in the next section. Then, we restrict us to the case where the force F reduces to the gravity force to further simplify the integral kernels.

1.1.1 The stream function

Let us consider a generic force F. The purpose of this section is to find a representation formula for the stream function ψ, *i.e.* the function such that

$$u(x,t) = \nabla^{\perp}\psi(x,t) = (-\partial_{x_2}\psi(x,t), \partial_{x_1}\psi(x,t)), \tag{1.1.1}$$

where u is the velocity field given by (III.1). We have

$$\nabla^{\perp} \cdot u(x,t) = \Delta\psi(x,t).$$

Let us consider the Stokes equations written as

$$\begin{aligned} \Delta u - \nabla p = F, &\quad x \in \mathbb{T} \times \mathbb{R} \\ \nabla \cdot u = 0, &\quad x \in \mathbb{T} \times \mathbb{R}. \end{aligned}$$

Using the stream function, we find that

$$\Delta^2\psi = \nabla^{\perp} \cdot F(x).$$

Then, in order to solve the Stokes system of PDEs, we have to first solve the bilaplacian in the strip $\mathbb{T} \times \mathbb{R}$. This latter problem will be solved by finding the Green's function associated to this fourth order elliptic problem K. Namely, we will prove the existence of the stream function by finding

$$\psi(x) = \int_{\mathbb{T}\times\mathbb{R}} K(x-y)\nabla^{\perp} \cdot F(y)dy, \tag{1.1.2}$$

where K is any function satisfying

$$\Delta^2 K = \delta, \ x \text{ in } \mathbb{T} \times \mathbb{R}.$$

As this Green function can be useful in many different problems in viscous fluid dynamics, we give its precise expression and the proof of its construction in the next proposition.

Proposition 1 (Green function of the x_1-periodic bilaplacian)**.** Define

$$K(x) = \sum_{n\in\mathbb{Z}} \beta_n(x_2)e^{inx_1}. \tag{1.1.3}$$

where

$$\beta_0(x_2) = \frac{|x_2|x_2^2}{24\pi}. \tag{1.1.4}$$

$$\beta_n(x_2) = \frac{(|nx_2|+1)e^{-|nx_2|}}{8\pi n^2|n|} \text{ for } n \neq 0. \tag{1.1.5}$$

Then K verifies

$$\Delta^2 K = \delta, \ x \text{ in } \mathbb{T} \times \mathbb{R}.$$

Proof. Above equation can be expressed as the product of two deltas

$$\Delta^2 K(x) = \delta(x_2)\delta(x_1), \quad x_1 \in \mathbb{T}, \quad x_2 \in \mathbb{R}. \tag{1.1.6}$$

The delta function in the periodic setting is given by

$$\delta(x_1) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{inx_1},$$

if it is understood in the distributional sense. Then, it yields

$$\Delta^2 K(x) = \frac{\delta(x_2)}{2\pi}\sum_{n\in\mathbb{Z}} e^{inx_1}. \tag{1.1.7}$$

On the other hand, K must be a $2\pi-$periodic function in x_1 so that we find $K(x)$ given by

$$K(x) = \sum_{n\in\mathbb{Z}} \beta_n(x_2)e^{inx_1}. \tag{1.1.8}$$

Comparing (1.1.7) and (1.1.8), we get an ODE for $\beta_n(x_2)$

$$\frac{\delta(x_2)}{2\pi} = \partial^4_{x_2}\beta_n(x_2) - 2n^2\partial^2_{x_2}\beta_n(x_2) + n^4\beta_n(x_2). \tag{1.1.9}$$

The case $n = 0$ can be solved in a direct manner getting (1.1.4). The case $n \neq 0$ can be solve by applying the Fourier Transform in the x_2 variable

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x_2)e^{-ix_2\xi}dx_2.$$

Then we find that

$$\frac{1}{2\pi} = \xi^4\hat{\beta}_n(\xi) + 2n^2\xi^2\hat{\beta}_n(\xi) + n^4\hat{\beta}_n(\xi). \tag{1.1.10}$$

We apply the inverse Fourier Transform to obtain

$$\beta_n(x_2) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} \frac{1}{\xi^4 + 2n^2\xi^2 + n^4}e^{i\xi x_2}d\xi \tag{1.1.11}$$

Solving the above integral it is possible to get (1.1.5). This concludes the proof of the proposition. □

Let us further simplify the expression for the stream function. Integrating by parts,

$$\begin{aligned}
\psi(x) &= \int_{\mathbb{T}\times\mathbb{R}} K(x-y)(-\partial_{y_2}F_1(y) + \partial_{y_1}F_2(y))dy \\
&= \int_{\mathbb{T}\times\mathbb{R}} -\partial_{x_2}K(x-y)F_1(y) + \partial_{x_1}K(x-y)F_2(y)dy \\
&= \int_{\mathbb{T}\times\mathbb{R}} \Psi(x-y)\cdot F(y)dy,
\end{aligned}$$

with
$$\Psi = (-\partial_{x_2} K, \partial_{x_1} K),$$
where
$$\begin{aligned}\partial_{x_1} K(x) &= \sum_{n\in\mathbb{Z}} in\beta_n(x_2)e^{inx_1} \\ &= \frac{1}{8\pi}\sum_{n\in\mathbb{Z}, n\neq 0}\left(\frac{i|x_2|}{n}e^{-|nx_2|} + \frac{i}{n|n|}e^{-|nx_2|}\right)e^{inx_1} \\ &= -\frac{1}{4\pi}\sum_{n>0}\left(\frac{n|x_2|}{n^2}e^{-n|x_2|} + \frac{1}{n^2}e^{-n|x_2|}\right)\sin(nx_1),\end{aligned}$$
and
$$\begin{aligned}\partial_{x_2} K(x) &= \sum_{n\in\mathbb{Z}} \partial_{x_2}\beta_n(x_2)e^{inx_1} \\ &= \frac{x_2|x_2|}{8\pi} - \frac{x_2}{8\pi}\sum_{n\in\mathbb{Z}, n\neq 0}\frac{1}{|n|}e^{-|nx_2|}e^{inx_1} \\ &= \frac{x_2|x_2|}{8\pi} - \frac{1}{4\pi}\sum_{n>0} nx_2\cos(nx_1)\frac{e^{-n|x_2|}}{n^2}.\end{aligned}$$
We observe that, due to the Fourier series of the kernel K, we have that Ψ is, at least, a function whose Fourier series is absolutely convergent and, as a consequence a $C(\mathbb{T}\times\mathbb{R})$ function. Therefore we can ensure the existence of the continuous function Ψ.

1.1.2 The x_1-periodic Stokeslet

The aim of this section is to use the representation formula for the stream function in order to obtain a representation formula for the velocity u. We recall that
$$\psi(x) = \int_{\mathbb{T}\times\mathbb{R}} \Psi(x-y)\cdot F(y)dy,$$
with
$$\Psi = (-\partial_{x_2} K, \partial_{x_1} K),$$
and F is a generic force. As a consequence,
$$\begin{aligned}u(x) &= \nabla^{\perp}\psi(x) \\ &= \int_{\mathbb{T}\times\mathbb{R}} \mathcal{S}(x-y)\cdot F(y)dy,\end{aligned}$$

where the x_1-periodic kernel is

$$\mathcal{S}=\begin{pmatrix}\partial^2_{x_2}K & -\partial_{x_2}\partial_{x_1}K\\ -\partial_{x_2}\partial_{x_1}K & \partial^2_{x_1}K\end{pmatrix},$$

with

$$\partial^2_{x_1}K(x)=-\frac{1}{4\pi}\sum_{n>0}\left(|x_2|e^{-n|x_2|}+\frac{1}{n}e^{-n|x_2|}\right)\cos(nx_1),$$

$$\partial_{x_1}\partial_{x_2}K(x)=\frac{1}{4\pi}\sum_{n>0}x_2\sin(nx_1)e^{-n|x_2|},$$

and

$$\partial^2_{x_2}K(x)=\frac{|x_2|}{4\pi}-\frac{1}{4\pi}\sum_{n>0}\left(\frac{e^{-n|x_2|}}{n}-|x_2|e^{-n|x_2|}\right)\cos(nx_1).$$

We observe that, even if the kernel Ψ was not explicit, we are able to sum up the Fourier series corresponding to the Stokeslet to find a closed form expression for the kernel $\mathcal{S}$. We compute

$$\begin{aligned}\sum_{n>0}\sin(nx_1)e^{-n|x_2|}&=\sum_{n>0}\frac{e^{n(ix_1-|x_2|)}-e^{n(-ix_1-|x_2|)}}{2i}\\&=\frac{1}{2i}\left(\frac{e^{ix_1-|x_2|}}{1-e^{ix_1-|x_2|}}-\frac{e^{-ix_1-|x_2|}}{1-e^{-ix_1-|x_2|}}\right)\\&=\frac{1}{2}\frac{\sin(x_1)}{\cosh(x_2)-\cos(x_1)}\\&=\frac{1}{2}\partial_{x_1}\log(\cosh(x_2)-\cos(x_1)).\end{aligned}$$

As a consequence

$$\sum_{n>0}\cos(nx_1)\frac{e^{-n|x_2|}}{n}=C(x_2)-\frac{1}{2}\log(\cosh(x_2)-\cos(x_1)).$$

Similarly,

$$
\begin{aligned}
\sum_{n>0} \cos(nx_1)e^{-n|x_2|}|x_2| &= \sum_{n>0} \frac{e^{n(ix_1-|x_2|)} + e^{n(-ix_1-|x_2|)}}{2}|x_2| \\
&= \frac{1}{2}\left(\frac{e^{ix_1-|x_2|}}{1-e^{ix_1-|x_2|}} + \frac{e^{-ix_1-|x_2|}}{1-e^{-ix_1-|x_2|}}\right)|x_2| \\
&= \frac{1}{2}\frac{\cos(x_1)-e^{-|x_2|}}{\cosh(x_2)-\cos(x_1)}|x_2| \\
&= \frac{1}{2}\frac{\cos(x_1)-\cosh(x_2)+\cosh(x_2)-e^{-|x_2|}}{\cosh(x_2)-\cos(x_1)}|x_2| \\
&= -\frac{1}{2}|x_2| + \frac{1}{2}\frac{\sinh(|x_2|)}{\cosh(x_2)-\cos(x_1)}|x_2| \\
&= -\frac{1}{2}|x_2| + \frac{1}{2}\frac{x_2\sinh(x_2)}{\cosh(x_2)-\cos(x_1)}.
\end{aligned}
$$

Thus, we find that

$$
\begin{aligned}
\partial_{x_2}\sum_{n>0}\cos(nx_1)\frac{e^{-n|x_2|}}{n} &= \frac{1}{2}\frac{x_2}{|x_2|} - \frac{1}{2}\frac{\sinh(x_2)}{\cosh(x_2)-\cos(x_1)} \\
&= C'(x_2) - \frac{1}{2}\frac{\sinh(x_2)}{\cosh(x_2)-\cos(x_1)},
\end{aligned}
$$

from where

$$
\sum_{n>0}\cos(nx_1)\frac{e^{-n|x_2|}}{n} = \frac{|x_2|}{2} - \frac{1}{2}\log(\cosh(x_2)-\cos(x_1)) + c.
$$

Choosing $x_2 = 0$ and $x_1 = \pi$, we find

$$
\sum_{n>0}\cos(nx_1)\frac{e^{-n|x_2|}}{n} = \frac{|x_2|}{2} - \frac{1}{2}\log(\cosh(x_2)-\cos(x_1)) - \frac{\log(2)}{2}.
$$

The previous computations lead to

$$
\begin{aligned}
\partial_{y_1}\partial_{y_2}K(y) &= \frac{y_2}{8\pi}\frac{\sin(y_1)}{\cosh(y_2)-\cos(y_1)}, \\
\partial_{y_1}^2 K(y) &= -\frac{y_2}{8\pi}\frac{\sinh(y_2)}{\cosh(y_2)-\cos(y_1)} + \frac{1}{8\pi}\log(2(\cosh(y_2)-\cos(y_1))), \\
\partial_{y_2}^2 K(y) &= \frac{1}{8\pi}\log(2(\cosh(y_2)-\cos(y_1))) + \frac{y_2}{8\pi}\frac{\sinh(y_2)}{\cosh(y_2)-\cos(y_1)}.
\end{aligned}
$$

Collecting all the terms we can write

$$\begin{aligned}\mathcal{S}(y) &= \frac{1}{8\pi}\log\left(2(\cosh(y_2)-\cos(y_1))\right)\cdot I\\ &\quad -\frac{y_2}{8\pi(\cosh(y_2)-\cos(y_1))}\begin{pmatrix} -\sinh(y_2) & \sin(y_1)\\ \sin(y_1) & \sinh(y_2)\end{pmatrix}.\end{aligned}$$

Similarly, we can compute the pressure using the Green function for the Poisson equation in $\mathbb{T}\times\mathbb{R}$, namely

$$\mathcal{P}(x) = -\frac{1}{4\pi}\log\left(\cosh(x_2)-\cos(x_1)\right).$$

which solves

$$-\Delta\mathcal{P} = \delta.$$

Then, we find that

$$\begin{aligned} p(x) &= -\frac{1}{4\pi}\int_{\mathbb{T}\times\mathbb{R}}\log\left(\cosh(x_2-y_2)-\cos(x_1-y_1)\right)\nabla\cdot F(y)dy\\ &= \frac{1}{4\pi}\int_{\mathbb{T}\times\mathbb{R}}\frac{-\sin(x_1-y_1)F_1(y)-\sinh(x_2-y_2)F_2(y)}{\cosh(x_2-y_2)-\cos(x_1-y_1)}dy\\ &= -\frac{1}{4\pi}\int_{\mathbb{T}\times\mathbb{R}}\frac{(\sin(y_1),\sinh(y_2))\cdot F(x-y)}{\cosh(y_2)-\cos(y_1)}dy.\end{aligned}$$

1.1.3 The equation for the free boundary: the case of an arbitrary curve

In this section we are going to focus on the case where the force acting on the fluid F is the gravity force in order to obtain the contour dynamics formulation for the free surface under consideration. Indeed, in our case of two homogeneous fluids separated by an internal wave under the action of gravity, we can write the acting force as

$$F(x,t) = g\rho(x,t)(0,1) = \nabla(\rho(x,t)x_2) \quad \text{for} \quad x\in\Omega^{\pm}(t),$$

with

$$\rho(x,t) = \begin{cases}\rho^{+} & \text{for } x \text{ in } \Omega^{+}(t)\\ \rho^{-} & \text{for } x \text{ in } \Omega^{-}(t)\end{cases}.$$

We can use this specific structure to further simplify the representation formulas in the situation under study. Let us start with the case of the stream function.

We integrate by parts in order to find that

$$
\begin{aligned}
\psi(x) &= \int_{\mathbb{T}\times\mathbb{R}} \Psi(x-y)\cdot\nabla(\rho(y)y_2)dy \\
&= \int_{\Omega^+} \Psi(x-y)\cdot\nabla_y(\rho^+ y_2)dy + \int_{\Omega^-} \Psi(x-y)\cdot\nabla_y(\rho^- y_2)dy \\
&= -\int_{\Omega^+} \nabla_y\cdot\Psi(x-y)\rho^+ y_2 dy - \int_{\Omega^-} \nabla_y\cdot\Psi(x-y)\cdot\rho^- y_2 dy, \\
&\quad - \rho^+\int_{\mathbb{T}} \Psi(x-z(\beta))\cdot(\partial_\beta z(\beta))^\perp z_2(\beta)d\beta \\
&\quad + \rho^-\int_{\mathbb{T}} \Psi(x-z(\beta))\cdot(\partial_\beta z(\beta))^\perp z_2(\beta)d\beta \\
&= (\rho^- - \rho^+)\int_{\mathbb{T}} \Psi(x-z(\beta))\cdot(\partial_\beta z(\beta))^\perp z_2(\beta)d\beta.
\end{aligned}
$$

The previous computation implies that the velocity solves

$$
u(x) = (\rho^- - \rho^+)\int_{\mathbb{T}} \mathcal{S}(x-z(\beta))\cdot z_\beta^\perp(\beta)z_2(\beta)d\beta,
$$

with

$$
\begin{aligned}
\mathcal{S}(y) &= \frac{1}{8\pi}\log\left(2(\cosh(y_2)-\cos(y_1))\right)\cdot I \\
&\quad - \frac{y_2}{8\pi(\cosh(y_2)-\cos(y_1))}\begin{pmatrix} -\sinh(y_2) & \sin(y_1) \\ \sin(y_1) & \sinh(y_2)\end{pmatrix}.
\end{aligned} \tag{1.1.12}
$$

Since the velocity is continuous across the interface we find the contour equation for the case of an arbitrary periodic curve

$$
z_t(\alpha,t) = (\rho^- - \rho^+)\int_{\mathbb{T}} \mathcal{S}(z(\alpha,t)-z(\beta,t))\cdot\partial_\beta z^\perp(\beta,t)z_2(\beta,t)d\beta. \tag{1.1.13}
$$

1.1.4 The equation for the free boundary: the case of a graph

In the previous section we have found the contour equation for the internal wave separating both fluids. However, such a free boundary is not necessarily parameterized as the graph of a function. In fact, the graph parametrization is not maintained by the contour equation (1.1.13). In this section we are going to find the contour equation for the case where the free boundary is parameterized as a graph. In order to do that we start with an initial data given by the graph of certain function and we evolve such initial data with a *modified contour dynamics* *equation*. Due to the smoothness of the evolution, the solution of this modified contour dynamics equation is a graph and, besides this, it is parameterized as a graph. Finally, we show that, due to the definition of our modified evolution, a reparametrization of the graph solves the contour dynamics equation (1.1.13) and as a consequence, the evolving graphs are equivalent to a solution of (1.1.13) after a reparametrization.

Suppose $z(\alpha,t)$ is a solution to the following *modified contour dynamics equation*

$$z_t(\alpha,t)=(\rho^--\rho^+)\int_{\mathbb{T}}\mathcal{S}(z(\alpha,t)-z(\beta,t))\cdot\partial_\beta z^\perp(\beta,t)z_2(\beta,t)d\beta+\lambda(\alpha,t)\partial_\alpha z(\alpha,t),$$

where λ will be defined below. Assuming that the curve z is initially a graph, i.e.,

$$z_1(\alpha,0)=\alpha$$

then, at least for a short time t, we have $\partial_\alpha z_1(\alpha,t')\neq 0$ for every $t'\in[0,t]$. This way, taking

$$\lambda(\alpha,t)=-\frac{(\rho^--\rho^+)}{\partial_\alpha z_1(\alpha,t)}\left(\int_{\mathbb{T}}\mathcal{S}(z(\alpha,t)-z(\beta,t))\cdot\partial_\beta z^\perp(\beta,t)z_2(\beta,t)d\beta\right)\cdot(1,0),$$

we can reparametrize the curve z as the graph of certain function $h(\alpha,t)$.

With this choice of λ, the equation for the graph becomes

$$\begin{aligned}h_t(\alpha,t)=(\rho^--\rho^+)\int_{\mathbb{T}}&\mathcal{S}((\alpha-\beta,h(\alpha,t)-h(\beta,t)))\cdot(-\partial_\alpha h(\beta,t),1)h(\beta,t)d\beta\\&\cdot(-\partial_\alpha h(\alpha,t),1).\end{aligned}\tag{1.1.14}$$

More explicitly,

$$\begin{aligned}h_t(\alpha,t)=&\frac{(\rho^--\rho^+)}{8\pi}\int_{\mathbb{T}}\log(2(\cosh(h(\alpha,t)-h(\beta,t))-\cos(\alpha-\beta)))h(\beta)\\&\times[1+h_\alpha(\alpha)h_\alpha(\beta)]d\beta\\&+\frac{(\rho^--\rho^+)}{8\pi}\int_{\mathbb{T}}\frac{h(\beta)(h(\alpha)-h(\beta))}{\cosh(h(\alpha)-h(\beta))-\cos(\alpha-\beta)}\\&\times[(h_\alpha(\alpha)h_\alpha(\beta)-1)\sinh(h(\alpha)-h(\beta))]\,d\beta\\&+\frac{(\rho^--\rho^+)}{8\pi}\int_{\mathbb{T}}\frac{h(\beta)(h(\alpha)-h(\beta))}{\cosh(h(\alpha)-h(\beta))-\cos(\alpha-\beta)}\\&\times[(h_\alpha(\alpha)+h_\alpha(\beta))\sin(\alpha-\beta)]\,d\beta.\end{aligned}\tag{1.1.15}$$

Then, define a reparametrization of the curve

$$z(\alpha,t)=\tilde{z}(\phi(\alpha,t),t)$$

with new parameter
$$\xi = \phi(\alpha, t).$$
We want to find ϕ such that $\tilde{z}(\xi, t)$ solves

$$\tilde{z}_t(\xi, t) = (\rho^- - \rho^+) \int_{\mathbb{T}} \mathcal{S}(\tilde{z}(\xi, t) - \tilde{z}(\chi, t)) \cdot \partial_\chi \tilde{z}^\perp(\chi, t) \tilde{z}_2(\chi, t) d\chi.$$

Changing variables

$$\chi = \phi(\beta), \quad \partial_\beta z^\perp(\beta, t) = \partial_\chi \tilde{z}^\perp(\chi, t)\phi'(\beta), \quad d\chi = \phi'(\beta) d\beta,$$

$$\begin{aligned}\int_{\mathbb{T}} &\mathcal{S}(\tilde{z}(\xi, t) - z(\beta, t)) \cdot \partial_\beta z^\perp(\beta, t) z_2(\beta, t) d\beta \\ &= \int_{\mathbb{T}} \mathcal{S}(\tilde{z}(\xi, t) - \tilde{z}(\chi, t)) \cdot \partial_\chi \tilde{z}^\perp(\chi, t) \tilde{z}_2(\chi, t) d\chi.\end{aligned}$$

We compute

$$\begin{aligned}z_t(\alpha, t) &= \partial_t \tilde{z}(\phi(\alpha, t), t) \\ &= \phi_t(\alpha, t)\partial_\xi \tilde{z}(\xi, t) + \tilde{z}_t(\xi, t).\end{aligned}$$

Dealing with the non-local part of the contour equation we find

$$\begin{aligned}z_t(\alpha, t) = (\rho^- - \rho^+) &\int_{\mathbb{T}} \mathcal{S}(z(\alpha, t) - z(\beta, t)) \cdot \partial_\beta z^\perp(\beta, t) z_2(\beta, t) d\beta \\ &+ \lambda(\alpha, t)\partial_\alpha z(\alpha, t) \\ = (\rho^- - \rho^+) &\int_{\mathbb{T}} \mathcal{S}(\tilde{z}(\xi, t) - \tilde{z}(\chi, t)) \cdot \partial_\chi \tilde{z}^\perp(\chi, t) \tilde{z}_2(\chi, t) d\chi \\ &+ \lambda(\alpha, t)\phi_\alpha(\alpha, t)\partial_\xi \tilde{z}(\xi, t).\end{aligned}$$

As a consequence, if we chose ϕ solving

$$\phi_t(\alpha, t) = \lambda(z)(\alpha, t)\phi_\alpha(\alpha, t)$$

we obtain that $\tilde{z}$ solves

$$\tilde{z}_t(\xi, t) = (\rho^- - \rho^+) \int_{\mathbb{T}} \mathcal{S}(\tilde{z}(\xi, t) - \tilde{z}(\chi, t)) \cdot \partial_\chi \tilde{z}^\perp(\chi, t) \tilde{z}_2(\chi, t) d\chi.$$

Thus, finding a solution of (1.1.14) is equivalent to finding a solution of (1.1.13).

1.2 The linear operator

Let us linearize around the steady state $z_* = (\alpha, 0)$. Then, we find that the linear problem reads

$$h_t(\alpha,t) = \frac{(\rho^- - \rho^+)}{4}\frac{1}{2\pi}\int_{\mathbb{T}} h(\alpha-\beta,t)\log\left(4\sin^2\left(\frac{\beta}{2}\right)\right)d\beta. \tag{1.2.1}$$

Using the properties of the logarithm, we find the equation

$$h_t(\alpha,t) = -\frac{(\rho^- - \rho^+)}{4}\Lambda^{-1}(h)(\alpha,t) + \frac{(\rho^- - \rho^+)}{4}\frac{\log(4)}{2\pi}\int_{\mathbb{T}} h(\beta,t)d\beta, \tag{1.2.2}$$

where the operator Λ^{-1} is given explicitly as follows

$$\Lambda^{-1}(h)(\alpha,t) = -\frac{1}{2\pi}\int_{\mathbb{T}} h(\alpha-\beta,t)\log\left(\sin^2\left(\frac{\beta}{2}\right)\right)d\beta.$$

Its Fourier coefficients are the following (see [13])

$$\widehat{\Lambda^{-1}}(k) = \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{T}} e^{-ik\beta}\log\left|\sin^2(\beta/2)\right|d\beta = \frac{1}{|k|}, & \text{for } k \neq 0, \\ -\frac{1}{2\pi}\int_{\mathbb{T}} \log\left|\sin^2(\beta/2)\right|d\beta = \log(4), & \text{for } k = 0. \end{cases}$$

Another relevant singular operator is the Hilbert transform in the torus

$$\mathcal{H}(h)(\alpha,t) = \frac{1}{2\pi}\text{PV}\int_{\mathbb{T}} \cot\left(\frac{\beta}{2}\right)h(\alpha-\beta,t)d\beta \tag{1.2.3}$$

which is related to the previous operator via

$$\partial_\alpha\left(\Lambda^{-1}(h)(\alpha,t)\right) = -\mathcal{H}(h)(\alpha,t).$$

Let us emphasize that in the stable case, when $\rho^- - \rho^+ > 0$, the linear operator shows a damping effect without regularization. In this sense, the linear operator is *not* of parabolic type. To simplify the notation we write

$$\overline{\rho} = \frac{(\rho^- - \rho^+)}{4}.$$

Using the previous computations, we have that

$$\hat{h}_t(k,t) = \begin{cases} -\frac{\overline{\rho}}{|k|}\hat{h}(k,t), & \text{for } k \neq 0, \\ 0, & \text{for } k = 0, \end{cases}$$

and it follows that

$$\hat{h}(k,t) = \begin{cases} e^{-\frac{\overline{\rho}}{|k|}t}\hat{h}_0(k), & \text{for } k \neq 0, \\ \hat{h}_0(0), & \text{for } k = 0. \end{cases} \tag{1.2.4}$$

We will use the notation

$$h(x,t) = e^{-\overline{\rho}\Lambda^{-1}t}h_0(x),$$

for the semi-group acting on some zero-mean data h_0. We observe that if we set an initial data with zero mean, i.e., $\hat{h}_0(0,t) = 0$, then, this property is conserved due to (1.2.4). In this section we will focus in studying the time decay of the solution to (1.2.2) in the Rayleigh-Taylor stable case $\overline{\rho} > 0$.

Before stating our results, let us introduce the following Banach scale of spaces

$$H^s_\nu = \{u \in L^2(\mathbb{T}) \text{ s.t. } \sum_{k=-\infty}^{\infty} e^{2\nu|k|}|k|^{2s}|\hat{u}(k)|^2 < \infty\}, \tag{1.2.5}$$

$$A^s_\nu = \{u \in L^2(\mathbb{T}) \text{ s.t. } \sum_{k=-\infty}^{\infty} e^{\nu|k|}|k|^{s}|\hat{u}(k)| < \infty\}. \tag{1.2.6}$$

To simplify notation we write

$$H^s = H^s_0, \quad H_\nu = H^0_\nu$$

together with

$$A^s = A^s_0, \quad A_\nu = A^0_\nu, \quad A = A^0_0.$$

In particular, we have the following result:

Proposition 2 (Decay in L^2 and A)**.** Let us consider equation (1.2.2) in the RT stable case $\overline{\rho} > 0$ with zero mean initial data h_0. Then we have that the solution verifies

$$||h||_{L^2} \leq C\,||h_0||_{H^{s_0}}\,(1+t)^{-s}, \tag{1.2.7}$$

$$||h||_{A} \leq C\,||h_0||_{A^{s_0}}\,(1+t)^{-s}, \tag{1.2.8}$$

where

$$0 < s < s_0 \tag{1.2.9}$$

is arbitrary. Furthermore, with initial analytic regularity, we also have the following exponential decay

$$||h||_{L^2} \leq C\,||h_0||_{H_\nu}\,e^{-\sqrt{\overline{\rho}\nu t}}, \tag{1.2.10}$$

$$||h||_{A} \leq C\,||h_0||_{A_\nu}\,e^{-\sqrt{\overline{\rho}\nu t}}. \tag{1.2.11}$$

Proof. Let us first obtain an algebraic decay rate for Sobolev initial data. In the RT stable case we fix $M > 0 < s_0$ and compute

$$\begin{aligned} ||h||_{L^2}^2 &= \sum_{|k|\leq M, k\neq 0} |\hat{h}(k,t)|^2 + \sum_{|k|>M} |\hat{h}(k,t)|^2 \\ &\leq \sum_{|k|\leq M, k\neq 0} e^{-2\frac{\overline{\rho}}{|k|}t} |\hat{h}_0(k)|^2 + \frac{1}{M^{2s_0}} \sum_{|k|>M} |\hat{h}_0(k)|^2 |k|^{2s_0} \\ &\leq e^{-2\overline{\rho}\frac{1}{M}t} ||h_0||_{L^2}^2 + \frac{1}{M^{2s_0}} ||h_0||_{H^{s_0}}^2 . \end{aligned}$$

The global bound $x^n e^{-x} \leq C(n)$ for $x \geq 0$ provides

$$||h||_{L^2}^2 \leq \frac{C(n)M^n}{\overline{\rho}^n 2^n t^n} ||h_0||_{L^2}^2 + \frac{1}{M^{2s_0}} ||h_0||_{H^{s_0}}^2$$

where n is arbitrary and $t > 1$. Taking M such that

$$C(n)\frac{M^n}{\overline{\rho}^n 2^n t^n} = \frac{1}{M^{2s_0}},$$

it is possible to get

$$||h||_{L^2} \leq C(n,\overline{\rho},s_0) ||h_0||_{H^{s_0}} t^{-\frac{s_0 n}{n+2s_0}} .$$

The uniform bound $\|h\|_{L^2} \leq ||h_0||_{L^2}$ for $t \leq 1$ yields the desired bound

$$||h||_{L^2} \leq C(n,\overline{\rho},s_0) ||h_0||_{H^{s_0}} (1+t)^{-\frac{s_0 n}{n+2s_0}},$$

increasing the constant $C(n,\overline{\rho},s_0)$. Taking n big enough we find

$$s := \frac{ns_0}{n+2s_0} < s_0.$$

A similar approach yields

$$||h||_{A} \leq C(n,\overline{\rho},s_0) ||h_0||_{A^{s_0}} (1+t)^{-\frac{s_0 n}{n+2s_0}} .$$

We compute for a different $M > 0$ the following

$$\begin{aligned} ||h||_{L^2}^2 &= \sum_{|k|\leq M, k\neq 0} |\hat{h}(k,t)|^2 + \sum_{|k|>M} |\hat{h}(k,t)|^2 \\ &\leq e^{-2\frac{\overline{\rho}}{M}t} \sum_{|k|\leq M, k\neq 0} |\hat{h}_0(k)|^2 + e^{-2\nu M} \sum_{|k|>M} |\hat{h}_0(k)|^2 e^{2\nu|k|} \\ &\leq e^{-2\frac{\overline{\rho}}{M}t} ||h_0||_{L^2}^2 + e^{-2\nu M} ||h_0||_{H_\nu}^2 . \end{aligned}$$

Taking

$$\frac{\overline{\rho}}{M}t = \nu M,$$

allows us to conclude that

$$||h||_{L^2} \leq ||h_0||_{H_\nu} e^{-\sqrt{\overline{\rho}\nu t}}.$$

A similar approach provides

$$||h||_{A} \leq ||h_0||_{A_\nu} e^{-\sqrt{\overline{\rho}\nu t}}.$$

□

1.3 Local existence

This section is devoted to prove local in time existence of solutions for the contour dynamics problem in the case of an interface defined by a curve $z(\alpha,t)$ (1.1.13). We will prove that the local existence of smooth initial data is guaranteed regardless of whether the fluids are in a stable/unstable stratification.

As we need to control the arc-chord condition of the interface to avoid self-intersections, we define the function

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{\beta^2}{\cosh(z_2(\alpha,t)-z_2(\alpha-\beta,t)) - \cos(z_1(\alpha,t)-z_1(\alpha-\beta,t))} \quad (1.3.1)$$

for $\alpha, \beta \in (-\pi,\pi)$ and

$$\mathcal{F}(z)(\alpha,0,t) = \frac{2}{|\partial_\alpha z(\alpha,t)|^2}. \quad (1.3.2)$$

Defining

$$||z(t)||_{L^\infty} = \sup_{\alpha\in\mathbb{T}} |z(\alpha,t)|\,,$$

and

$$||\mathcal{F}(z)||_{L^\infty} = \sup_{\alpha,\beta\in\mathbb{T}} |\mathcal{F}(z)|\,,$$

we emphasize that as long as

$$||\mathcal{F}(z)||_{L^\infty} < \infty,$$

self-intersections of the curve are excluded.

Our result is the following:

Proposition 3 (Local existence of solutions in $C^{1,\gamma}(\mathbb{T})$)**.** Let $0<\gamma<1$ be a fixed parameter and $z_0(\alpha)\in C^{1,\gamma}(\mathbb{T})$ be the initial data satisfying

$$||\mathcal{F}(z_0)||_{L^\infty}<\infty.$$

Then, there is a time $0<T$ such that there exists a unique solution

$$z\in C^1((-T,T);C^{1,\gamma}(\mathbb{T})),$$

of (1.1.13) satisfying the arc-chord condition.

Proof. **Sketch of the proof and functional framework;** The proof of this result is based on the Picard Theorem on a suitable Banach Space (see [66] for another application of this method in a different problem). The precise statement of Picard Theorem that we are going to use is:

Lemma 1 (Picard Theorem on a Banach Space)**.** Let $O\subseteq B$ be an open subset of a Banach space B and let $N(X)$ be a nonlinear operator satisfying the following criteria:

- $N(X)$ maps O to B,
- $N(X)$ is locally Lipschitz continuous, i.e., for any $X\in O$ there exists $L>0$ such that
$$\left|\left|N(\widetilde{X})-N(X)\right|\right|_B\leq L\left|\left|\widetilde{X}-X\right|\right|_B \quad \forall \widetilde{X}\in O.$$
Then, for any $X_0\in O$, there exists a time $T>0$ such that the ODE
$$\frac{dX}{dt}=N(X),\quad X|_{t=0}=X_0\in O$$
has a unique local solution $X\in C^1((-T,T);O)$.

Our functional frame will be the Banach space $B=C^{1,\gamma}(\mathbb{T})$, equipped with the norm

$$||z(t)||_{C^{1,\gamma}}=||z(t)||_{L^\infty}+||z_\alpha(t)||_{L^\infty}+|z_\alpha(t)|_\gamma\,,$$

where the $\gamma-$Hölder seminorm is defined as

$$|z(t)|_\gamma=\max_{\alpha\neq\beta}\frac{|z(\alpha,t)-z(\beta,t)|}{|\alpha-\beta|^\gamma}.$$

We define the set $O^M\subset C^{1,\gamma}(\mathbb{T})$ as

$$O^M=\{z\in C^{1,\gamma}(\mathbb{T});\ ||\mathcal{F}(z)||_{L^\infty}<M,\ ||z||_{C^{1,\gamma}}<M\}. \tag{1.3.3}$$

This set is open in $C^{1,\gamma}$ in the appropriate topology. Defining

$$N(z)(\alpha,t) = (\rho^- - \rho^+)\int_{\mathbb{T}} \mathcal{S}(z(\alpha,t)-z(\beta,t))z_\beta^\perp(\beta,t)z_2(\beta,t)d\beta, \tag{1.3.4}$$

equation (1.1.13) can be written as

$$\frac{dz(\alpha,t)}{dt} = N(z)(\alpha,t).$$

Once we have established the functional setting of the theorem, it remains to prove the nonlinear operator satisfies the two hypothesis of the Picard Theorem.

Step 1: $N(z)$ maps O^M to $C^{1,\gamma}(\mathbb{T})$; We need to prove

$$||N(z)||_{C^{1,\gamma}} \leq C(\rho,\gamma,M), \text{ for all } z \in O^M.$$

The proof of this proposition boils down to the following estimate:

$$|N_\alpha(z)|_\gamma \leq C(\rho,\gamma,M). \tag{1.3.5}$$

We write

$$N(z)(\alpha) = \frac{(\rho^- - \rho^+)}{8\pi}\left(N_1(z)(\alpha) - N_2(z)(\alpha)\right)$$

with

$$N_1(z)(\alpha) = \int_{\mathbb{T}} \log\left(2(\cosh(z_2(\alpha)-z_2(\beta)) - \cos(z_1(\alpha)-z_1(\beta)))\right) z_2(\beta) z_\beta^\perp(\beta)d\beta, \tag{1.3.6}$$

$$\begin{aligned} N_2(z)(\alpha) = \int_{\mathbb{T}} &\frac{(z_2(\alpha)-z_2(\beta))z_2(\beta)}{\cosh(z_2(\alpha)-z_2(\beta)) - \cos(z_1(\alpha)-z_1(\beta))} \\ &\times \begin{pmatrix} -\sinh(z_2(\alpha)-z_2(\beta)) & \sin(z_1(\alpha)-z_1(\beta)) \\ \sin(z_1(\alpha)-z_1(\beta)) & \sinh(z_2(\alpha)-z_2(\beta)) \end{pmatrix} z_\alpha^\perp(\beta)d\beta. \end{aligned}$$

For the sake of brevity, we are going to estimate only the term N_1, being N_2 similar. To obtain an expression of $\partial_\alpha N_1(z)(\alpha)$, we differentiate the expression (1.3.6) and we perform the change of variables $\beta = \alpha - \beta'$ for simplicity. We get

$$\begin{aligned} \partial_\alpha N_1(z)(\alpha) = \int_{\mathbb{T}} &\frac{z_\alpha(\alpha)\cdot(\sin(z_{1-}(\alpha)), \sinh(z_{2-}(\alpha)))}{\beta^2} \\ &\times \mathcal{F}(z)(\alpha,\beta) z_2(\alpha-\beta) z_\alpha^\perp(\alpha-\beta)d\beta, \end{aligned} \tag{1.3.7}$$

where

$$z_{i-}(\alpha) = z_i(\alpha) - z_i(\alpha-\beta). \tag{1.3.8}$$

We have the following splitting

$$\partial_\alpha N_1(z)(\alpha+h) - \partial_\alpha N_1(z)(\alpha) = I_1 + I_2 + I_3 + I_4 + I_5, \tag{1.3.9}$$

where

$$I_1 = \int_{\mathbb{T}} \frac{(z_\alpha(\alpha+h) - z_\alpha(\alpha)) \cdot (\sin(z_{1-}(\alpha+h)), \sinh(z_{2-}(\alpha+h)))}{\beta^2} \times \mathcal{F}(z)(\alpha+h,\beta) z_2(\alpha+h-\beta) z_\alpha^\perp(\alpha+h-\beta) d\beta,$$

$$I_2 = \int_{\mathbb{T}} \frac{z_\alpha(\alpha) \cdot (\sin(z_{1-}(\alpha+h)) - \sin(z_{1-}(\alpha)), \sinh(z_{2-}(\alpha+h)) - \sinh(z_{2-}(\alpha)))}{\beta^2} \times \mathcal{F}(z)(\alpha+h,\beta) z_2(\alpha+h-\beta) z_\alpha^\perp(\alpha+h-\beta) d\beta,$$

$$I_3 = \int_{\mathbb{T}} z_\alpha(\alpha) \cdot (\sin(z_{1-}(\alpha)), \sinh(z_{2-}(\alpha))) \times (\mathcal{F}(z)(\alpha+h,\beta) - \mathcal{F}(z)(\alpha,\beta)) \frac{z_2(\alpha+h-\beta) z_\alpha^\perp(\alpha+h-\beta)}{\beta^2} d\beta,$$

$$I_4 = \int_{\mathbb{T}} \mathcal{F}(z)(\alpha,\beta) \frac{z_\alpha(\alpha) \cdot (\sin(z_{1-}(\alpha)), \sinh(z_{2-}(\alpha)))}{\beta^2} \times (z_2(\alpha+h-\beta) - z_2(\alpha-\beta)) z_\alpha^\perp(\alpha-\beta) d\beta,$$

and

$$I_5 = \int_{\mathbb{T}} \mathcal{F}(z)(\alpha,\beta) \frac{z_\alpha(\alpha) \cdot (\sin(z_{1-}(\alpha)), \sinh(z_{2-}(\alpha))))}{\beta^2} z_2(\alpha-\beta) \times (z_\alpha^\perp(\alpha+h-\beta) - z_\alpha^\perp(\alpha-\beta)) d\beta.$$

To prove $|\partial_\alpha N_1(z)|_\gamma \leq C(\gamma, M)$, it suffices to show

$$|I_i| \leq |h|^\gamma C(\gamma, M) \text{ for } i = 1\ldots5. \tag{1.3.10}$$

We are going to estimate only the first term, I_1, being the remainder terms similar. We compute

$$\begin{aligned} I_1 = &\int_{\mathbb{T}} \frac{(z_\alpha(\alpha+h) - z_\alpha(\alpha)) \cdot (\sin(z_{1-}(\alpha+h)), \sinh(z_{2-}(\alpha+h)))}{\beta^2} \\ &\times \mathcal{F}(z)(\alpha+h,\beta) z_2(\alpha+h-\beta) z_\alpha^\perp(\alpha+h-\beta) d\beta \\ = &I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5} + I_{1,6}. \end{aligned}$$

The first of these terms is

$$I_{1,1} = \int_{\mathbb{T}} \frac{(z_\alpha(\alpha+h) - z_\alpha(\alpha)) \cdot [(\sin(z_{1-}(\alpha+h)), \sinh(z_{2-}(\alpha+h))) - z_-(\alpha+h)]}{\beta^2} \\ \times \mathcal{F}(z)(\alpha+h,\beta) z_2(\alpha+h-\beta) z_\alpha^\perp(\alpha+h-\beta) d\beta.$$

Using the integral Mean Value Theorem

$$z_-(\alpha+h) = \int_0^1 z'((\alpha+h)s + (1-s)(\alpha+h-\beta))\beta ds, \tag{1.3.11}$$

which, together with the series expansion of the trigonometric functions sin and sinh, implies

$$|(\sin(z_{1-}(\alpha+h)), \sinh(z_{2-}(\alpha+h))) - z_-(\alpha+h)| \lesssim |\beta|^3 \, ||z_\alpha||_{L^\infty}^3 ,$$

we find that

$$\begin{aligned} |I_{1,1}| &\lesssim |h|^\gamma \, |z_\alpha|_\gamma \, ||\mathcal{F}(z)||_{L^\infty} \, ||z_\alpha||_{L^\infty}^4 \, ||z||_{L^\infty} \int_{\mathbb{T}} |\beta| d\beta \\ &\leq |h|^\gamma C(\gamma, M). \end{aligned}$$

The second term reads

$$I_{1,2} = \int_{\mathbb{T}} \left(\mathcal{F}(z)(\alpha+h,\beta) - \frac{2}{|z_\alpha(\alpha+h)|^2} \right) \frac{(z_\alpha(\alpha+h) - z_\alpha(\alpha)) \cdot z_-(\alpha+h)}{\beta^2} \\ \times z_2(\alpha+h-\beta) z_\alpha^\perp(\alpha+h-\beta) d\beta.$$

In order to estimate this term, we use (1.3.11) and

$$\begin{aligned} &\left| \mathcal{F}(z)(\alpha+h,\beta) - \frac{2}{|z_\alpha(\alpha+h)|^2} \right| \\ &\quad = \left| \frac{\beta^2 |z_\alpha(\alpha+h)|^2 - 2(\cosh(z_{2-}(\alpha+h)) - \cosh(z_{1-}(\alpha+h)))}{|z_\alpha(\alpha+h)|^2 (\cosh z_{2-}(\alpha+h) - \cosh(z_{1-}(\alpha+h)))} \right| \\ &\quad \lesssim \beta^2 \, ||\mathcal{F}(z)||_{L^\infty}^2 \, ||z_\alpha||_{L^\infty}^4 . \end{aligned} \tag{1.3.12}$$

to find that

$$\begin{aligned} |I_{1,2}| &\lesssim |h|^\gamma \, |z_\alpha|_\gamma \, ||\mathcal{F}(z)||_{L^\infty}^2 \, ||z||_{L^\infty}^2 \, ||z_\alpha||_{L^\infty}^5 \\ &\leq |h|^\gamma C(\gamma, M). \end{aligned}$$

Note that thanks to (1.3.2), it holds that

$$\frac{2}{|z_\alpha(\cdot)|^2} = \mathcal{F}(\cdot, \beta, t) \leq M.$$

Then, the third term

$$I_{1,3} = \frac{2}{|z_\alpha(\alpha+h)|^2}\int_{\mathbb{T}} \frac{(z_\alpha(\alpha+h)-z_\alpha(\alpha))\cdot z_-(\alpha+h)}{\beta} \times \frac{z_2(\alpha+h-\beta)-z_2(\alpha+h)}{\beta} z_\alpha^\perp(\alpha+h-\beta)d\beta,$$

can be bounded as

$$|I_{1,3}| \lesssim |h|^\gamma |z_\alpha|_\gamma ||\mathcal{F}(z)||_{L^\infty} ||z_\alpha||^3_{L^\infty} \leq |h|^\gamma C(\gamma, M).$$

Using the previous ideas together with the inequality

$$|z(\alpha+h)-z(\alpha+h-\beta)-z_\alpha(\alpha+h)\beta| \leq |\beta|^{1+\gamma}|z_\alpha|_\gamma, \tag{1.3.13}$$

we have that

$$I_{1,4} = \frac{2z_2(\alpha+h)}{|z_\alpha(\alpha+h)|^2}\int_{\mathbb{T}}(z_\alpha(\alpha+h)-z_\alpha(\alpha)) \times \frac{(z_-(\alpha+h)-\beta z_\alpha(\alpha+h))}{\beta^2} z_\alpha^\perp(\alpha+h-\beta)d\beta$$

verifies

$$|I_{1,4}| \lesssim |h|^\gamma |z_\alpha|^2_\gamma ||\mathcal{F}(z)||_{L^\infty} ||z||_{L^\infty} ||z_\alpha||_{L^\infty} \int_{\mathbb{T}} |\beta|^{1+\gamma-2} d\beta \lesssim |h|^\gamma |z_\alpha|^2_\gamma ||\mathcal{F}(z)||_{L^\infty} ||z||_{L^\infty} ||z_\alpha||_{L^\infty} \leq |h|^\gamma C(\gamma, M).$$

The term

$$I_{1,5} = \frac{2z_2(\alpha+h)(z_\alpha(\alpha+h)-z_\alpha(\alpha))\cdot z_\alpha(\alpha+h)}{|z_\alpha(\alpha+h)|^2} \times \int_{\mathbb{T}} \left(\frac{1}{\beta}-\cot(\beta)\right) z_\alpha^\perp(\alpha+h-\beta)d\beta,$$

can be estimated as follows

$$|I_{1,5}| \lesssim |h|^\gamma |z_\alpha|_\gamma ||\mathcal{F}(z)||_{L^\infty} ||z||_{L^\infty} ||z_\alpha||^2_{L^\infty} \int_{\mathbb{T}} |\beta| d\beta \lesssim |h|^\gamma |z_\alpha|_\gamma ||\mathcal{F}(z)||_{L^\infty} ||z||_{L^\infty} ||z_\alpha||^2_{L^\infty} \leq |h|^\gamma C(\gamma, M).$$

Finally, the last term reads

$$I_{1,6} = \frac{2z_2(\alpha+h)(z_\alpha(\alpha+h)-z_\alpha(\alpha))\cdot z_\alpha(\alpha+h)}{|z_\alpha(\alpha+h)|^2} 2\pi\mathcal{H}(z_\alpha^\perp)(\alpha+h).$$

We observe that the Hilbert transform $\mathcal{H}(\cdot)$ given in (1.2.3) maps continuously C^γ to C^γ. In particular,

$$\left|\left|\mathcal{H}(z_\alpha^\perp)\right|\right|_{L^\infty} \lesssim \left|\left|z_\alpha^\perp\right|\right|_\gamma. \tag{1.3.14}$$

Consequently,

$$\begin{aligned}|I_{1,6}| &\lesssim |h|^\gamma \left|z_\alpha\right|_\gamma ||\mathcal{F}(z)||_{L^\infty} ||z||_{L^\infty} ||z_\alpha||_{L^\infty} ||z_\alpha||_\gamma \\ &\leq |h|^\gamma C(\gamma, M).\end{aligned}$$

Collecting the estimates of the terms $I_{1,1}\ldots I_{1,6}$, we obtain

$$|I_1| \leq |h|^\gamma C(\gamma, M). \tag{1.3.15}$$

Using these ideas we can estimate the remaining terms. This concludes the proof of this step.

Step 2: $N(z)$ is locally Lipschitz continuous; We have to prove that the operator N defined as in (1.3.4) is locally Lipschitz continuous in $C^{1,\gamma}(\mathbb{T})$, i.e., for any $z \in O^M$, there exists a constant $L > 0$ such that

$$||N(z)-N(w))||_{C^{1,\gamma}} \leq L(\rho,\gamma,M)\, ||z-w||_{C^{1,\gamma}} \quad \forall\, w \in O^M. \tag{1.3.16}$$

For the sake of brevity, we are going to show some of the estimates corresponding to

$$|N_\alpha(z)-N_\alpha(w))|_\gamma \leq L(\rho,\gamma,M)\, ||z-w||_{C^{1,\gamma}}.$$

The previous statement is equivalent to prove that

$$\begin{aligned}&|(N_\alpha(z)(\alpha+h)-N_\alpha(w)(\alpha+h))-(N_\alpha(z)(\alpha)-N_\alpha(w)(\alpha))| \\ &\quad\leq L(\rho,\gamma,M)|h|^\gamma\, ||z-w||_{C^{1,\gamma}}.\end{aligned} \tag{1.3.17}$$

As before, we are going to focus on the term N_1. From (1.3.9), we deduce that

$$(\partial_\alpha N_1(z)(\alpha+h)-\partial_\alpha N_1(w)(\alpha+h))-(\partial_\alpha N_1(z)(\alpha)-\partial_\alpha N_1(w)(\alpha)) = \sum_{i=1}^{5}(I_i(z)-I_i(w)).$$

Let us focus on the first term. The remaining terms can be estimated using the same tools and ideas. The first term can be split as follows:

$$I_1(z)-I_1(w) = J_1^1 + J_2^1 + J_3^1 + J_4^1 + J_5^1,$$

where

$$J_1^1 = \int_{\mathbb{T}} \frac{((z-w)_\alpha(\alpha+h)-(z-w)_\alpha(\alpha))\cdot(\sin(z_{1-}(\alpha+h)),\sinh(z_{2-}(\alpha+h)))}{\beta^2} \times \mathcal{F}(z)(\alpha+h,\beta)z_2(\alpha+h-\beta)z_\alpha^\perp(\alpha+h-\beta)d\beta,$$

$$J_2^1 = \int_{\mathbb{T}} \frac{(\sin(z_{1-}(\alpha+h))-\sin(w_{1-}(\alpha+h)),\sinh(z_{2-}(\alpha+h))-\sinh(w_{2-}(\alpha+h)))}{\beta^2} \cdot (w_\alpha(\alpha+h)-w_\alpha(\alpha))\mathcal{F}(z)(\alpha+h,\beta)z_2(\alpha+h-\beta)z_\alpha^\perp(\alpha+h-\beta)d\beta,$$

$$J_3^1 = \int_{\mathbb{T}} \frac{(w_\alpha(\alpha+h)-w_\alpha(\alpha))\cdot(\sin(w_{1-}(\alpha+h)),\sinh(w_{2-}(\alpha+h)))}{\beta^2} \times (\mathcal{F}(z)(\alpha+h,\beta)-\mathcal{F}(w)(\alpha+h,\beta))z_2(\alpha+h-\beta)z_\alpha^\perp(\alpha+h-\beta)d\beta,$$

$$J_4^1 = \int_{\mathbb{T}} \frac{(w_\alpha(\alpha+h)-w_\alpha(\alpha))\cdot(\sin(w_{1-}(\alpha+h)),\sinh(w_{2-}(\alpha+h)))}{\beta^2} \times \mathcal{F}(w)(\alpha+h,\beta)(z_2-w_2)(\alpha+h-\beta)z_\alpha^\perp(\alpha+h-\beta)d\beta,$$

and

$$J_5^1 = \int_{\mathbb{T}} \frac{(w_\alpha(\alpha+h)-w_\alpha(\alpha))\cdot(\sin(w_{1-}(\alpha+h)),\sinh(w_{2-}(\alpha+h)))}{\beta^2} \times \mathcal{F}(w)(\alpha+h,\beta)w_2(\alpha+h-\beta)(z-w)_\alpha^\perp(\alpha+h-\beta)d\beta.$$

The terms J_1^1, J_4^1 and J_5^1 can be estimated similarly to I_1 in Step 1. As a consequence we obtain the estimates

$$|J_i^1| \leq L(\gamma, M)\, ||z-w||_{C^{1,\gamma}} \text{ for } i = 1,4,5.$$

We treat the terms J_2^1, J_3^1 separately, as the difference of the curves is not explicitly present in these terms. We split J_2^1 as follows:

$$\begin{aligned} J_2^1 = &\int_{\mathbb{T}} \frac{(\sin(z_{1-}(\alpha+h))-\sin(w_{1-}(\alpha+h)),\sinh(z_{2-}(\alpha+h))-\sinh(w_{2-}(\alpha+h)))}{\beta^2} \\ &\cdot (w_\alpha(\alpha+h)-w_\alpha(\alpha))\mathcal{F}(z)(\alpha+h,\beta)z_2(\alpha+h-\beta)z_\alpha^\perp(\alpha+h-\beta)d\beta \\ = &J_{2,1}^1 + J_{2,2}^1 + J_{2,3}^1 + J_{2,4}^1 + J_{2,5}^1 + J_{2,6}^1. \end{aligned}$$

We have that

$$
\begin{aligned}
&J^1_{2,1} = \\
&\int_{\mathbb{T}} [(\sin(z_{1-}(\alpha+h)) - \sin(w_{1-}(\alpha+h)), \sinh(z_{2-}(\alpha+h)) - \sinh(w_{2-}(\alpha+h))) \\
&\qquad -(z-w)_-(\alpha+h)] \\
&\qquad \cdot (w_\alpha(\alpha+h) - w_\alpha(\alpha)) \frac{\mathcal{F}(z)(\alpha+h,\beta) z_2(\alpha+h-\beta) z_\alpha^\perp(\alpha+h-\beta)}{\beta^2} d\beta,
\end{aligned}
$$

$$
\begin{aligned}
J^1_{2,2} = &\int_{\mathbb{T}} (z-w)_-(\alpha+h) \cdot (w_\alpha(\alpha+h) - w_\alpha(\alpha)) \left(\mathcal{F}(z)(\alpha+h,\beta) - \frac{2}{|z_\alpha(\alpha+h)|^2}\right) \\
&\times \frac{z_2(\alpha+h-\beta) z_\alpha^\perp(\alpha+h-\beta)}{\beta^2} d\beta,
\end{aligned}
$$

$$
\begin{aligned}
J^1_{2,3} = &\frac{2}{|z_\alpha(\alpha+h)|^2} \int_{\mathbb{T}} (z-w)_-(\alpha+h) \cdot (w_\alpha(\alpha+h) - w_\alpha(\alpha)) \\
&\times \frac{(z_2(\alpha+h-\beta) - z_2(\alpha+h)) z_\alpha^\perp(\alpha+h-\beta)}{\beta^2} d\beta,
\end{aligned}
$$

$$
\begin{aligned}
J^1_{2,4} = &\frac{2z_2(\alpha+h)}{|z_\alpha(\alpha+h)|^2} \int_{\mathbb{T}} ((z-w)_-(\alpha+h) - \beta(z-w)_\alpha(\alpha+h)) \\
&\cdot (w_\alpha(\alpha+h) - w_\alpha(\alpha)) \frac{z_\alpha^\perp(\alpha+h-\beta)}{\beta^2} d\beta,
\end{aligned}
$$

$$
\begin{aligned}
J^1_{2,5} = &\frac{2z_2(\alpha+h)}{|z_\alpha(\alpha+h)|^2} (z-w)_\alpha(\alpha+h) \cdot (w_\alpha(\alpha+h) - w_\alpha(\alpha)) \\
&\times \int_{\mathbb{T}} z_\alpha^\perp(\alpha+h-\beta) \left(\frac{1}{\beta} - \cot(\beta)\right) d\beta,
\end{aligned}
$$

and

$$
J^1_{2,6} = \frac{2z_2(\alpha+h)}{|z_\alpha(\alpha+h)|^2} (z-w)_\alpha(\alpha+h) \cdot (w_\alpha(\alpha+h) - w_\alpha(\alpha)) 2\pi \mathcal{H}(z_\alpha^\perp)(\alpha+h).
$$

To estimate the term $J^1_{2,1}$, we use the integral Mean Value Theorem and the Taylor series expansion of the functions sin and sinh leading us to

$$
\begin{aligned}
&|(\sin(z_{1-}(\alpha+h)) - \sin(w_{1-}(\alpha+h)), \sinh(z_{2-}(\alpha+h)) - \sinh(w_{2-}(\alpha+h))) \\
&\quad - (z-w)_-(\alpha+h)| \\
&\leq C(M)|\beta|^3 \left|\left|(z-w)_\alpha\right|\right|_{L^\infty}.
\end{aligned}
$$

which implies

$$\begin{aligned}\left|J^1_{2,1}\right| &\leq C(M)\,||(z-w)_\alpha||_{L^\infty}\,|h|^\gamma\,|w_\alpha|_\gamma\,||F(z)||_{L^\infty}\,||z||_{L^\infty}\,||z_\alpha||_{L^\infty}\int_{\mathbb{T}}|\beta|d\beta\\ &\leq L(\gamma.M)|h|^\gamma\,||z-w||_{C^{1,\gamma}}\,.\end{aligned}$$

The terms $J^1_{2,2}\dots J^1_{2,6}$ are estimated like the terms $I_{1,2}\dots I_{1,6}$. Note that they contain terms depending on the linear difference of the curves $z-w$, so consequently we obtain estimates of the form:

$$|J^1_{2,i}|\leq L(\gamma,M)|h|^\gamma\,||z-w||_{C^{1,\gamma}}\ \text{ for } i=2\dots 6.$$

We conclude that

$$|J^1_2|\leq L(\gamma,M)|h|^\gamma\,||z-w||_{C^{1,\gamma}}\,.$$

In order to estimate the term J^1_3 we have to use

$$\begin{aligned}&\mathcal{F}(z)(\cdot,\beta)-\mathcal{F}(w)(\cdot,\beta)\\ &\quad=\mathcal{F}(z)(\cdot,\beta)\mathcal{F}(w)(\cdot,\beta)\frac{\cosh(w_{2-}(\cdot))-\cos(w_{1-}(\cdot))-\cosh(z_{2-}(\cdot))+\cos(z_{1-}(\cdot))}{\beta^2},\end{aligned} \tag{1.3.18}$$

which, together with

$$\begin{aligned}&|\cosh(w_{2-}(\alpha+h))-\cos(w_{1-}(\alpha+h))-\cosh(z_{2-}(\alpha+h))+\cos(z_{1-}(\alpha+h))|\\ &\leq C(M)|\beta|^2\,||(z-w)_\alpha||_{L^\infty}\,,\end{aligned}$$

lead to

$$|\mathcal{F}(z)(\alpha+h,\beta)-\mathcal{F}(w)(\alpha+h,\beta)|\leq C(M)\,||(z-w)_\alpha||_{L^\infty}\,.$$

Similarly, we find that

$$\begin{aligned}&\left|\mathcal{F}(z)(\alpha+h,\beta)-\frac{2}{|z_\alpha(\alpha+h)|^2}-\mathcal{F}(w)(\alpha+h,\beta)+\frac{2}{|w_\alpha(\alpha+h)|^2}\right|\\ &\quad\leq C(M)|\beta|^2\,||(z-w)_\alpha||_{L^\infty}\,.\end{aligned}$$

This leads to

$$\left|J^1_3\right|\leq L(\gamma,M)|h|^\gamma\,||z-w||_{C^{1,\gamma}}\,.$$

As a consequence, we have that

$$|I_1(z)-I_1(w)|\leq L(\gamma,M)|h|^\gamma\,||z-w||_{C^{1,\gamma}}\,.$$

We can perform the same type of splitting for the other terms $I_i(z)-I_i(w)$. Then, the desired estimate is obtained using the same ideas. This concludes the proof of the result. □

CHAPTER 2

Stability and instability for small initial data

This chapter is based on [32].

2.1 Stability of classical solutions in the stable case

In this section we assume that the system is in the Rayleigh-Taylor stable case where the lighter fluid lies above the denser fluid, namely

$$\rho^- - \rho^+ > 0.$$

Then we prove the global existence and decay to equilibrium for small enough initial data

$$z(\alpha, 0) = (\alpha, h(\alpha, 0))$$

given as a graph in the Sobolev space H^3. In particular, the result we prove is the following:

Theorem 1 (Global existence and decay of solutions for small data)**.** Let $h_0 \in H^3(\mathbb{T})$ be the initial data for (1.1.15) in the RT stable case,

$$\rho^- - \rho^+ > 0,$$

and take m arbitrarily close to 2, that is, $m = 2 - \varepsilon$ for some $\varepsilon > 0$. There is a $0 < \delta = \delta(\rho^- - \rho^+)$ such that if

$$||h_0||_{H^3(\mathbb{T})} < \delta$$

there exists a unique global classical solution $h(\alpha, t)$

$$h \in C([0, T]; H^3)$$

of (1.1.15) for arbitrary $T > 0$ satisfying

$$(1 + t)^m \, ||h||_{L^2} + ||h||_{H^3} \leq C \, ||h_0||_{H^3}$$

for some constant $C > 0$.

Proof. **The linear part:** First we observe that, without lost of generality, we can consider the initial data $h_0(\alpha)$ having zero mean. Due to the fact that the equation for the case of a graph can be written as

$$\partial_t h(\alpha, t) d\alpha = \partial_\alpha \psi(\alpha, h(\alpha, t))$$

where ψ is the stream function, we have that the solution $h(\alpha, t)$ satisfies

$$\int_{\mathbb{T}} h(\alpha, t) d\alpha = 0 \text{ for every } t > 0.$$

In this setting, equation (1.1.15) can be written as

$$h_t(\alpha, t) = -\overline{\rho}\Lambda^{-1}(h)(\alpha, t) + \frac{\overline{\rho}}{2\pi}(I_1(\alpha, t) + I_2(\alpha, t) + I_3(\alpha, t) + I_4(\alpha, t)), \quad (2.1.1)$$

where

$$\overline{\rho} = \rho^- - \rho^+,$$

$$I_1(\alpha, t) = \int_{\mathbb{T}} \log\left(1 + \frac{\sinh^2\left(\frac{h_-(\alpha,t)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)}\right) h(\alpha - \beta, t) d\beta,$$

$$I_2(\alpha, t) = \int_{\mathbb{T}} \log\left(\sinh^2\left(\frac{h_-(\alpha, t)}{2}\right) + \sin^2\left(\frac{\beta}{2}\right)\right) h(\alpha - \beta, t) h_\alpha(\alpha, t) h_\alpha(\alpha - \beta, t) d\beta,$$

$$I_3(\alpha,t) = \int_{\mathbb{T}} \frac{h(\alpha-\beta,t)h_-(\alpha,t)}{2\sinh^2\left(\frac{h_-(\alpha,t)}{2}\right)+2\sin^2\left(\frac{\beta}{2}\right)} \times \left[(h_\alpha(\alpha,t)h_\alpha(\alpha-\beta,t)-1)\sinh(h_-(\alpha,t))\right] d\beta,$$

$$I_4(\alpha,t) = \int_{\mathbb{T}} \frac{h(\alpha-\beta,t)h_-(\alpha,t)}{2\sinh^2\left(\frac{h_-(\alpha,t)}{2}\right)+2\sin^2\left(\frac{\beta}{2}\right)} \left[(h_\alpha(\alpha,t)+h_\alpha(\alpha-\beta,t))\sin(\beta)\right] d\beta.$$

Above we use the notation $h_-(\alpha) = h(\alpha)-h(\alpha-\beta)$ as in previous sections. Define

$$|||h||| = \sup_{t\in[0,T]} \left((1+t)^m \left|\left|h\right|\right|_{L^2} + \left|\left|h\right|\right|_{H^3}\right). \tag{2.1.2}$$

The proof follows the ideas in [16]. Namely, our goal now is to obtain an inequality of the form

$$|||h||| \leq \mathcal{M}_0 + F(|||h|||),$$

where F is a smooth $O(x^k)$ function with $k>1$.

The L^2 estimate: In this subsection we prove a L^2 estimate of the interface $h(\alpha,t)$ by using Duhamel principle. Using Duhamel formula we obtain that

$$h(\alpha,t) = e^{-\overline{\rho}\Lambda^{-1}t}h_0(\alpha) + \frac{\overline{\rho}}{2\pi}\int_0^t e^{-\overline{\rho}\Lambda^{-1}(t-s)}(I_1(\alpha,s)+I_2(\alpha,s)+I_3(\alpha,s)+I_4(\alpha,s))ds.$$

Then, by (1.2.7),

$$\left|\left|h\right|\right|_{L^2} \lesssim (1+t)^{-m}\left|\left|h_0\right|\right|_{H^2} + \frac{\overline{\rho}}{2\pi}\int_0^t (1+t-s)^{-m}\left|\left|(I_1(s)+I_2(s)+I_3(s)+I_4(s))\right|\right|_{H^2} ds \tag{2.1.3}$$

with $m<2$ to be chosen later.

We first estimate each of the terms $I_1(\alpha,s)\dots I_4(\alpha,s)$ in H^2. Secondly, we use Gagliardo-Niremberg interpolation inequality, in order to get estimates in our norm, defined by (2.1.2). Some basic inequalities we will use are:

$$\left|\left|\partial_\alpha^k h(t)\right|\right|_{L^\infty} \lesssim \left|\left|h(t)\right|\right|_{H^3}^{(1+2k)/6}\left|\left|h(t)\right|\right|_{L^2}^{(5-2k)/6} \lesssim (1+t)^{-m\frac{(5-2k)}{6}}|||h|||,\ 0\leq k\leq 2,$$

$$\left|\left|h(t)\right|\right|_{H^k} \lesssim \left|\left|h(t)\right|\right|_{H^3}^{k/3}\left|\left|h(t)\right|\right|_{L^2}^{(3-k)/3} \lesssim (1+t)^{-m\frac{(3-k)}{3}}|||h|||,\ 0\leq k\leq 3.$$

For the sake of simplicity, we will drop the time dependence in the notation when it does not cause any ambiguity. We start by estimating I_1 in H^2. We have that

$$\left|\left|I_1\right|\right|_{H^2} \lesssim K_1^1 + K_2^1 + K_3^1,$$

where

$$K_1^1 = \left\| \int_{\mathbb{T}} \log \left(1 + \frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)} \right) \partial_\alpha^2 h(\alpha - \beta) d\beta \right\|_{L^2},$$

$$K_2^1 = \left\| \int_{\mathbb{T}} \partial_\alpha \log \left(1 + \frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)} \right) h_\alpha(\alpha - \beta) d\beta \right\|_{L^2},$$

and

$$K_3^1 = \left\| \int_{\mathbb{T}} \partial_\alpha^2 \log \left(1 + \frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)} \right) h(\alpha - \beta) d\beta \right\|_{L^2}.$$

First of all,

$$K_1^1 \lesssim \sup_{\alpha,\beta\in\mathbb{T}} \left| \log \left(1 + \frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)} \right) \right| ||h||_{H^2}.$$

Using the integral Mean Value Theorem in (1.3.11), we can estimate the finite differences in h as

$$\left| \partial_\alpha^k \frac{h_-(\alpha)}{\beta} \right| \lesssim \left|\left| \partial_\alpha^{k+1} h \right|\right|_{L^\infty}.$$

This fact together with

$$\frac{\sinh(x)}{x} \leq \cosh(x), \quad \frac{\beta}{2\sin\left(\frac{\beta}{2}\right)} \leq \frac{\pi}{2},$$

imply

$$\begin{aligned} \left| \log \left(1 + \frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)} \right) \right| &\leq \left| \frac{\sinh\left(\frac{h_-(\alpha)}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \right|^2, \\ &\leq \left| \frac{\sinh\left(\frac{h_-(\alpha)}{2}\right)}{\frac{h_-(\alpha)}{2}} \right|^2 \left| \frac{h_-(\alpha)}{\beta} \right|^2 \left| \frac{\frac{\beta}{2}}{\sin\left(\frac{\beta}{2}\right)} \right|^2 \\ &\lesssim \cosh^2(||h||_{L^\infty}) \, ||h_\alpha||_{L^\infty}^2. \end{aligned}$$

The previous computations yield

$$K_1^1 \lesssim \cosh^2(||h||_{L^\infty}) \, ||h_\alpha||_{L^\infty}^2 \, ||h||_{H^2}. \tag{2.1.4}$$

Similarly,

$$K_2^1 \lesssim \cosh(2\,||h||_{L^\infty}) \, ||h_\alpha||_{L^\infty}^2 \, ||h||_{H^2} \tag{2.1.5}$$

and

$$K_3^1 \lesssim \cosh^2(2\,||h||_{L^\infty})\,||h||_{L^\infty} \\ \times \left(||h||_{H^3}\,||h_\alpha||_{L^\infty} + ||h||_{H^2}\left|\left|\partial_\alpha^2 h\right|\right|_{L^\infty} + ||h||_{H^2}\left|\left|\partial_\alpha^2 h\right|\right|_{L^\infty}||h_\alpha||_{L^\infty}^2\right). \quad (2.1.6)$$

Now, we apply Gagliardo-Nirenberg interpolation inequalities to (2.1.4), (2.1.5) and (2.1.6), which will produce powers of the energy multiplied by factors of the type $(1+s)^{-m\mu}$ for some $\mu > 0$. As a consequence, we find the estimate

$$||I_1||_{H^2} \lesssim (1+s)^{-m\frac{4}{3}}\cosh^2(2\,||h||_{L^\infty})\left(|||h|||^3 + |||h|||^5\right). \quad (2.1.7)$$

Now, let us estimate the second term I_2 using the same techniques:

$$||I_2||_{H^2} \lesssim (1+s)^{-m\frac{4}{3}}\left(1+\cosh^2(2\,||h||_{L^\infty})\right)\left(|||h|||^3 + |||h|||^5 + |||h|||^7\right). \quad (2.1.8)$$

The term I_3 is similarly estimated. In this way we find that

$$||I_3||_{H^2} \lesssim (1+s)^{-m\frac{4}{3}}\cosh^3(4\,||h||_{L^\infty})\left(|||h|||^3 + |||h|||^5 + |||h|||^7 + |||h|||^9\right). \quad (2.1.9)$$

Finally,

$$||I_4||_{H^2} \lesssim (1+s)^{-m\frac{4}{3}}\left(1+\cosh^2(2\,||h||_{L^\infty})\right)\left(|||h|||^3 + |||h|||^5 + |||h|||^7\right). \quad (2.1.10)$$

In order to close the estimate in L^2, we need integrability in time in (2.1.3). Note that the norm in B does not depend on time. For that purpose, we use Lemma 2.4 in [23], which establishes that, for $m, \eta > 0$,

$$\int_0^t (1+t-s)^{-m}(1+s)^{-1-\eta} \leq C(m,\eta)(1+t)^{-m}. \quad (2.1.11)$$

Introducing estimates (2.1.7), (2.1.8), (2.1.9) and (2.1.10) in (2.1.3) and using (2.1.11), we obtain that, for any $m > \frac{3}{4}$, it holds that

$$||h||_{L^2} \leq (1+t)^{-m}\,||h_0||_{H^2} + C(1+t)^{-m}\mathcal{B}_1(|||h|||)\mathcal{P}_1(|||h|||), \quad (2.1.12)$$

where $\mathcal{P}_1$ is a polynomial with monomials at least cubic and

$$\mathcal{B}_1(x) = 1 + \cosh^3(4x).$$

The H^3 estimate: We compute that

$$\frac{1}{2}\frac{d}{dt}\,||h(t)||_{H^3}^2 = -\overline{\rho}\,||h(t)||_{H^{2.5}}^2 \\ + \frac{\overline{\rho}}{2\pi}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha,t)\partial_\alpha^3\left(I_1(\alpha,t) + I_2(\alpha,t) + I_3(\alpha,t) + I_4(\alpha,t)\right)d\alpha.$$

Then,

$$\frac{d}{dt}\left|\left|h\right|\right|_{H^3}^2 \lesssim \frac{\overline{\rho}}{2\pi}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\partial_\alpha^3\left(I_1(\alpha)+I_2(\alpha)+I_3(\alpha)+I_4(\alpha)\right)d\alpha.$$

The aim is to estimate each of the four terms in H^3. The strategy to bound most of the terms is similar to the one in [20].

For the sake of brevity we will estimate in detail the term corresponding to the first integral. Denote

$$\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\partial_\alpha^3 I_1(\alpha)d\alpha = R_1^1 + R_2^1 + R_3^1 + R_4^1,$$

where

$$R_1^1 = \int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\left(\int_{\mathbb{T}} \log\left(1+\frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)}\right)\partial_\alpha^3 h(\alpha-\beta)d\beta\right)d\alpha$$

$$R_2^1 = 3\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\left(\int_{\mathbb{T}} \partial_\alpha\log\left(1+\frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)}\right)\partial_\alpha^2 h(\alpha-\beta)d\beta\right)d\alpha,$$

$$R_3^1 = 3\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\left(\int_{\mathbb{T}} \partial_\alpha^2\log\left(1+\frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)}\right)\partial_\alpha h(\alpha-\beta)d\beta\right)d\alpha,$$

and

$$R_4^1 = \int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\left(\int_{\mathbb{T}} \partial_\alpha^3\log\left(1+\frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)}\right)h(\alpha-\beta)d\beta\right)d\alpha.$$

Similarly to the estimates performed for the L^2 estimate of the interface, we find that

$$\begin{aligned} R_1^1 &\lesssim \left|\left|h\right|\right|_{H^3}^2 \cosh^2(\left|\left|h\right|\right|_{L^\infty})\left|\left|h_\alpha\right|\right|_{L^\infty}^2 \\ &\lesssim (1+t)^{-m}\cosh^2(\left|\left|\left|h\right|\right|\right|)\left|\left|\left|h\right|\right|\right|^4, \end{aligned}$$

$$\begin{aligned} R_2^1 &\lesssim \left|\left|h\right|\right|_{H^3}\left|\left|h\right|\right|_{H^2}\cosh(\left|\left|h\right|\right|_{L^\infty})\left|\left|\partial_\alpha^2 h\right|\right|_{L^\infty}\left|\left|h_\alpha\right|\right|_{L^\infty} \\ &\lesssim (1+t)^{-m}\cosh(\left|\left|\left|h\right|\right|\right|)\left|\left|\left|h\right|\right|\right|^4, \end{aligned}$$

$$R_3^1 \lesssim (1+t)^{-m}\cosh^2(|||h|||)\left(|||h|||^4+|||h|||^6\right).$$

The term R_4^1 is more singular. We split it into parts by computing the derivative

$$\partial_\alpha^3 \log\left(1+\frac{\sinh^2(h_-(\alpha)/2)}{\sin^2(\beta/2)}\right)$$

explicitly:

$$R_4^1 = R_{4,1}^1 + R_{4,2}^1 + R_{4,3}^1 + R_{4,4}^1 + R_{4,5}^1 + R_{4,6}^1 + R_{4,7}^1 + R_{4,8}^1,$$

with

$$R_{4,1}^1 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\frac{(\partial_\alpha^3 h)_-(\alpha)}{2}\frac{\sinh(h_-(\alpha))}{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)}h(\alpha-\beta)d\beta d\alpha$$

$$R_{4,2}^1 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)3\frac{(\partial_\alpha^2 h)_-(\alpha)(h_\alpha)_-(\alpha)}{2}\frac{\cosh(h_-(\alpha))}{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)}h(\alpha-\beta)d\beta d\alpha$$

$$R_{4,3}^1 = -\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\frac{(\partial_\alpha^2 h)_-(\alpha)(h_\alpha)_-(\alpha)}{4}\frac{\sinh^2(h_-(\alpha))}{\left(\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)\right)^2}h(\alpha-\beta)d\beta d\alpha$$

$$R_{4,4}^1 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\frac{((h_\alpha)_-(\alpha))^3}{2}\frac{\sinh(h_-(\alpha))}{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)}h(\alpha-\beta)d\beta d\alpha$$

$$R_{4,5}^1 = -\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\frac{((h_\alpha)_-(\alpha))^3}{4}\frac{\cosh(h_-(\alpha))\sinh(h_-(\alpha))}{\left(\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)\right)^2}h(\alpha-\beta)d\beta d\alpha$$

$$R_{4,6}^1 = -\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)(h_\alpha)_-(\alpha)(\partial_\alpha^2 h)_-(\alpha)\frac{\sinh^2(h_-(\alpha))}{\left(\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)\right)^2}h(\alpha-\beta)d\beta d\alpha$$

$$R_{4,7}^1 = -\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\frac{((h_\alpha)_-(\alpha))^3}{2}\frac{\sinh(2h_-(\alpha))}{\left(\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)\right)^2}h(\alpha-\beta)d\beta d\alpha$$

$$R_{4,8}^1 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\frac{((h_\alpha)_-(\alpha))^3}{2}\frac{\sinh^3(h_-(\alpha))}{\left(\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)\right)^3}h(\alpha-\beta)d\beta d\alpha$$

We focus on the most singular term $R^1_{4,1}$. We split $R^1_{4,1}$ into parts, in order to localize the singularity:

$$R^1_{4,1} = R^1_{4,1,1} + R^1_{4,1,2} + R^1_{4,1,3} + R^1_{4,1,4} + R^1_{4,1,5} + R^1_{4,1,6},$$

where

$$\begin{aligned} R^1_{4,1,1} &= \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\frac{(\partial_\alpha^3 h)_-(\alpha)}{2}\frac{\sinh(h_-(\alpha)) - h_-(\alpha)}{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)} h(\alpha-\beta)d\beta d\alpha \\ &\lesssim ||h||^2_{H^3}\cosh(2\,||h||_{L^\infty})\,||h_\alpha||^2_{L^\infty}\,||h||^2_{L^\infty}\,. \end{aligned}$$

Similarly, using

$$\left(\frac{\beta^2}{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)} - \frac{4}{1+|h_\alpha(\alpha)|^2}\right) \lesssim ||h_\alpha||_{L^\infty}\,||h_{\alpha\alpha}||_{L^\infty}\,\beta,$$

we find that

$$\begin{aligned} R^1_{4,1,2} &= \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 h(\alpha)\frac{(\partial_\alpha^3 h)_-(\alpha)}{2}\frac{h_-(\alpha)}{\beta^2} \\ &\quad\times\left(\frac{\beta^2}{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)} - \frac{4}{1+|h_\alpha(\alpha)|^2}\right) h(\alpha-\beta)d\beta d\alpha \\ &\lesssim ||h_{\alpha\alpha}||_{L^\infty}\,||h||^2_{H^3}\,||h_\alpha||^2_{L^\infty}\,||h||_{L^\infty}\,. \end{aligned}$$

We compute

$$\begin{aligned} R^1_{4,1,3} &= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{4}{1+|h_\alpha(\alpha)|^2}\partial_\alpha^3 h(\alpha)\frac{(\partial_\alpha^3 h)_-(\alpha)}{2}\frac{h_-(\alpha)-\beta h_\alpha(\alpha)}{\beta^2} h(\alpha-\beta)d\beta d\alpha \\ &\lesssim ||h||^2_{H^3}\,||h||_{\dot{C}^2}\,||h||_{L^\infty}\,. \end{aligned}$$

Furthermore, we can continue as follows

$$\begin{aligned} R^1_{4,1,4} &= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{4}{1+|h_\alpha(\alpha)|^2}\partial_\alpha^3 h(\alpha)\frac{\partial_\alpha^3 h(\alpha)}{2} h_\alpha(\alpha)h(\alpha-\beta)\left(\frac{1}{\beta}-\cot(\beta)\right)d\beta d\alpha \\ &\quad+\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{4}{1+|h_\alpha(\alpha)|^2}\partial_\alpha^3 h(\alpha)\frac{\partial_\alpha^3 h(\alpha-\beta)}{2} h_\alpha(\alpha)\frac{h(\alpha-\beta)-h(\alpha)}{\beta}d\beta d\alpha \\ &\lesssim ||h||^2_{H^3}\,||h_\alpha||_{L^\infty}\,||h||_{L^\infty} + ||h||^2_{H^3}\,||h_\alpha||^2_{L^\infty} \end{aligned}$$

and

$$\begin{aligned}R^1_{4,1,5} &= \int_{\mathbb{T}} 2\pi \frac{4}{1+|h_\alpha(\alpha)|^2}\partial^3_\alpha h(\alpha)\frac{\partial^3_\alpha h(\alpha)}{2} h_\alpha(\alpha)\mathcal{H}(h)(\alpha)d\alpha \\ &\quad + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{4}{1+|h_\alpha(\alpha)|^2}\partial^3_\alpha h(\alpha)\frac{\partial^3_\alpha h(\alpha-\beta)}{2} h_\alpha(\alpha)h(\alpha)\left(\frac{1}{\beta}-\cot(\beta)\right)d\beta d\alpha \\ &\lesssim ||h||^2_{H^3}||h_\alpha||_{L^\infty}||h||_{H^1} + ||h||^2_{H^3}||h_\alpha||_{L^\infty}||h||_{L^\infty}.\end{aligned}$$

Finally, we have that

$$\begin{aligned}R^1_{4,1,6} &= \int_{\mathbb{T}} 2\pi \frac{4}{1+|h_\alpha(\alpha)|^2}\partial^3_\alpha h(\alpha)\frac{\mathcal{H}(\partial^3_\alpha h)(\alpha)}{2} h_\alpha(\alpha)h(\alpha)d\alpha \\ &\lesssim ||h||^2_{H^3}||h_\alpha||_{L^\infty}||h||_{L^\infty}.\end{aligned}$$

Consequently,

$$R^1_4 \lesssim (1+t)^{-m}\mathcal{B}(||h||_{L^\infty})\,|||h|||\,\mathcal{P}(|||h|||),$$

and hence

$$\int_{\mathbb{T}} \partial^3_\alpha h(\alpha)\partial^3_\alpha I_1(\alpha)d\alpha \lesssim (1+t)^{-m}\mathcal{B}(||h||_{L^\infty})\,|||h|||\,\mathcal{P}(|||h|||).$$

This estimate provides the condition $m>1$ in order to have integrability in time. The terms corresponding to the remaining integrals I_2, I_3 and I_4 can be estimated following the previous procedure, giving rise to

$$\frac{d}{dt}||h||^2_{H^3} \leq (1+t)^{-m\frac{2}{3}}C(\rho)\mathcal{B}_2(|||h|||)\mathcal{P}_2(|||h|||), \tag{2.1.13}$$

where $\mathcal{P}_2$ is a polynomial with monomials with degree at least four and

$$\mathcal{B}_2(x) = 1+\cosh^3(4x).$$

The final estimate (2.1.13) requires the choice of m to be $m>3/2$ (to assure the further integrability in time). Nevertheless, the optimal choice of m will be arbitrarily close to 2, in order to gain rate of decay. Equipped with this regularity and using standard energy arguments (see [33] section 6 for details), we conclude the continuity in time of h in H^3.

Global existence: Collecting estimates (2.1.12) and the integrated (in time) version of (2.1.13), we find

$$|||h||| \leq 2\,||h_0||_{H^3} + C(\rho)\mathcal{B}(|||h|||)\mathcal{P}(|||h|||), \tag{2.1.14}$$

where $\mathcal{P}$ is a polynomial with monomials of degree at least 3, and

$$\mathcal{B}(x) = 1+\cosh^3(4x).$$

Let the initial data be such that

$$||h_0||_{H^3} < \delta.$$

Now we prove that, if δ is small enough,

$$|||h||| < 4\delta.$$

We argue by contradiction. Assume that the solution reaches the value $|||h||| = 4\delta$ at certain time $t = T$. Then, by (2.1.14),

$$4\delta \leq 2\delta + C(\rho)\mathcal{B}(4\delta)\mathcal{P}(4\delta).$$

Using the smallness conditions of δ and the fact that $\mathcal{B}$ is a monotonic increasing function, we observe

$$4\delta < 2\delta + C\mathcal{B}(4\delta)16\delta^2, \quad \text{and therefore} \quad 2\delta < C\mathcal{B}(1)16\delta^2,$$

which is a contradiction if

$$\delta < \min\{(C\mathcal{B}(1)8)^{-1}, 4^{-1}\}.$$

Consequently,

$$|||h||| < 4\delta.$$

Finally, we can choose a big enough constant such that

$$C\,||h_0||_{H^3} \geq \delta,$$

thus

$$4\delta \leq 4C\,||h_0||_{H^3}\,.$$

Therefore, abusing notation for an arbitrary constant C, we have shown that as long as the initial data is small enough, we can find a classical solution to (1.1.15) such that

$$|||h||| = \sup_{t\in[0,T]} \left((1+t)^m\,||h||_{L^2} + ||h||_{H^3}\right) \leq C\,||h_0||_{H^3} \tag{2.1.15}$$

for every $T > 0$. □

2.2 Stability of analytic solutions in the stable case

This section is devoted to prove global well-posedness and exponential decay of analytic solutions to (1.1.15). For that purpose, we will exploit the algebra structure of Wiener spaces A^s_ν (see (1.2.6) in Section 1.2 for the definition of these spaces). We assume the RT stable scenario, where

$$\overline{\rho} = \frac{\rho^- - \rho^+}{4} > 0.$$

In this case, we will prove global existence and exponential decay of solutions for small enough initial data, given as a graph in a suitable Wiener space

$$h_0 \in A^1_{\nu_0}, \text{ with } \nu_0 > 0.$$

We recall that we can assume that the initial data has zero mean without loss of generality, and this property is propagated thanks to the existence of a stream function ψ. In particular, we prove the following result:

Theorem 2 (Global existence of solutions in Wiener algebras for small data)**.** Let $h_0 \in A^1_{\nu_0}(\mathbb{T})$ be the initial data for (1.1.15) in the RT stable case,

$$\rho^- - \rho^+ > 0.$$

Assume that

$$\nu_0 > 0.$$

There is a $0 < \delta = \delta(\rho^- - \rho^+, \nu_0)$ such that if

$$||h_0||_{A^1_{\nu_0}} < \delta$$

so that

$$\nu_0 - \mathcal{M}(||h_0||_{A^1_{\nu_0}}) > 0$$

for a suitable non negative function $\mathcal{M}(x) \approx x + O(x^2)$, there exists a positive decreasing function $\nu(t) > 0$ with $\nu(0) = \nu_0$ (characterized by (2.2.21)) and a unique global analytic solution $h(\alpha, t)$

$$h \in C([0,T]; A^1_{\nu(t)})$$

of (1.1.15) for arbitrary $T > 0$ satisfying

$$||h||_{A^1_{\nu(t)}} \leq ||h_0||_{A^1_{\nu_0}}.$$

The details of the proof are shown in Subsection 2.2.2. The control of the analyticity of solutions is similar in spirit as in Foias & Temam [27]. Moreover, we can prove that, assuming the hypotheses of the previous theorem and certain extra conditions, we can get global existence and exponential decay of solutions in $A^0_{\nu^*}$, for a suitable $\nu^* < \nu_0$. This result is captured in the following statement:

Theorem 3 (Global existence and exponential decay of solutions for small data)**.** Let $h_0 \in A^1_{\nu_0}$ be the initial data for (1.1.15) in the RT stable case,

$$\rho^- - \rho^+ > 0.$$

fulfilling the hypotheses of Theorem 2. In particular,

$$||h_0||_{A^1_{\nu_0}} < \delta$$

for a suitable $\delta = \delta(\rho^- - \rho^+, \nu_0)$. Assume that

$$0 < \nu^* \leq \frac{\nu_0}{24}.$$

There is a $0 < \varepsilon = \varepsilon(\nu^*, \rho^- - \rho^+, ||h_0||_{A^1_{\nu_0}})$ such that if

$$||h_0||_{A^0_{\nu^*}} < \varepsilon$$

there exists a unique global analytic solution $h(\alpha, t)$

$$h \in C([0,T]; A^0_{\nu^*})$$

of (1.1.15) for arbitrary $T > 0$ satisfying

$$e^{\sqrt{((\rho^- - \rho^+)/4)\nu^* t}}\, ||h||_{A^0} + ||h||_{A^0_{\nu^*}} \leq C\, ||h_0||_{A^0_{\nu^*}}$$

for some constant $C > 0$.

The details of the proof are shown in Subsection 2.2.3.

2.2.1 Fourier analysis

In this subsection, we study the Fourier side of equation (1.1.15), in order to exploit its structure in the Wiener Space context. We will use the equivalent expression given in (2.1.1).

First of all, we recall some properties of Wiener spaces. The fundamental property of Wiener spaces is their algebra structure:

$$||f_1 \cdot f_2 \dots f_n||_{A^s_\nu} \leq \prod_{i=1}^{n} ||f_i||_{A^s_\nu}.$$

We can also interpolate Wiener algebras using Hölder inequality in the following general way:

$$\begin{aligned}||h(t)||_{A^s_{\nu(t)}} &= \sum_{k\in\mathbb{Z}} |k|^{\alpha s+(1-\alpha)s}|\hat{h}(k)|e^{(\beta\nu(t)+(1-\beta)\nu(t))|k|}\\ &\leq ||h(t)||^{1/p}_{A^{\alpha sp}_{\beta\nu(t)p}}\,||h(t)||^{1/q}_{A^{(1-\alpha)sq}_{(1-\beta)\nu(t)q}},\end{aligned}$$

with $p,q\geq 1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $0\leq\alpha,\beta\leq 1$.

Note that nonlinearities in the original equation will result in convolutions in the Fourier side. In the following, we will represent the convolution of n copies of f ($n-1$ convolutions), as

$$*^n f(k) = (f*\underbrace{\ldots}_{n-1}*f)(k).$$

Also note that translations in the original variable h will result in multipliers in the Fourier side. In this sense,

$$\widehat{h(\cdot-\beta)}(k,\beta) = \hat{h}(k)e^{-ik\beta} \text{ and } \widehat{h_-}(k,\beta) = \hat{h}(k)m(k,\beta),$$

with

$$m(k,\beta) = (1-e^{-ik\beta}) = -ik\beta\int_0^1 e^{-ik\beta(1-s)}ds.$$

Now, we take the $k-$th Fourier coefficient of equation (2.1.1):

$$\hat{h}_t(k,t) = -\overline{\rho}\frac{\hat{h}(k,t)}{|k|} + \frac{\overline{\rho}}{2\pi}\left(\hat{I}_1(k,t)+\hat{I}_2(k,t)+\hat{I}_3(k,t)+\hat{I}_4(k,t)\right). \tag{2.2.1}$$

In the following, we will drop the time dependence in the notation when it does not cause any ambiguity. Moreover, we consider that h is small enough in order to represent functions $\sinh(x)$ and $\log(1+x)$ by their Taylor series, and $\frac{1}{1+x^2}$ as a geometric sum. Define

$$\mathcal{T}_1(\alpha,\beta) = \log\left(1+\frac{\sinh^2\left(\frac{h_-(\alpha)}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sin\left(\frac{\beta}{2}\right)^{2n} n}\left(\sum_{j=0}^{\infty}\frac{(h_-(\alpha))^{2j+1}}{2^{2j+1}(2j+1)!}\right)^{2n}$$

and

$$\begin{aligned}\mathcal{T}_2(\alpha,\beta) &= \frac{h(\alpha-\beta)h_-(\alpha)}{2\sinh^2\left(\frac{h_-(\alpha)}{2}\right)+2\sin^2\left(\frac{\beta}{2}\right)}\\ &= \frac{1}{2\sin^2\left(\frac{\beta}{2}\right)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sinh\left(\frac{h_-(\alpha)}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\right)^{2n} h(\alpha-\beta)h_-(\alpha).\end{aligned}$$

Then, the nonlinear terms in (1.1.15) can be expressed in the following way:

$$I_1(\alpha) = \int_{\mathbb{T}} \mathcal{T}_1(\alpha,\beta) h(\alpha-\beta) d\beta,$$

$$I_2(\alpha) = \int_{\mathbb{T}} \left(\mathcal{T}_1(\alpha,\beta) + \log\left(\sin^2\left(\frac{\beta}{2}\right)\right)\right) h(\alpha-\beta) h_\alpha(\alpha) h_\alpha(\alpha-\beta) d\beta,$$

$$I_3(\alpha) = \int_{\mathbb{T}} \mathcal{T}_2(\alpha,\beta) \left[(h_\alpha(\alpha) h_\alpha(\alpha-\beta) - 1) \sinh(h_-(\alpha)) \right] d\beta,$$

$$I_4(\alpha) = \int_{\mathbb{T}} \mathcal{T}_2(\alpha,\beta) \left[(h_\alpha(\alpha) + h_\alpha(\alpha-\beta)) \sin(\beta) \right] d\beta.$$

Therefore, their Fourier coefficients are

$$\hat{I}_1(k) = \int_{\mathbb{T}} \widehat{\mathcal{T}_1}(k,\beta) * e^{-ik\beta} \hat{h}(k) d\beta, \tag{2.2.2}$$

$$\begin{aligned} \hat{I}_2(k) = & \int_{\mathbb{T}} \left(\widehat{\mathcal{T}_1}(k,\beta) \right) * \hat{h}(k) e^{-ik\beta} * (ik\hat{h}(k)) * (e^{-ik\beta} ik\hat{h}(k)) d\beta \\ & + \int_{\mathbb{T}} \log\left(\sin^2\left(\frac{\beta}{2}\right)\right) \hat{h}(k) e^{-ik\beta} * (ik\hat{h}(k)) * (e^{-ik\beta} ik\hat{h}(k)) d\beta. \end{aligned} \tag{2.2.3}$$

$$\begin{aligned} \hat{I}_3(k) = & \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} \int_{\mathbb{T}} \widehat{\mathcal{T}_2}(k,\beta) * (ik\hat{h}(k) * (ike^{-ik\beta}\hat{h}(k)) * *^{2j+1} \left(m(k,\beta)\hat{h}(k) \right) d\beta \\ & - \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} \int_{\mathbb{T}} \widehat{\mathcal{T}_2}(k,\beta) * *^{2j+1} \left(m(k,\beta)\hat{h}(k) \right) d\beta. \end{aligned} \tag{2.2.4}$$

$$\hat{I}_4(k) = \int_{\mathbb{T}} \sin(\beta) \widehat{\mathcal{T}_2}(k,\beta) * \left[(ik\hat{h}(k)) + (ike^{-ik\beta}\hat{h}(k)) \right] d\beta. \tag{2.2.5}$$

Taking the $k-$th Fourier coefficient, we find

$$\begin{aligned}
\widehat{\mathcal{T}_1}(k,\beta) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} *^{2n}\left(\sum_{j=0}^{\infty}\frac{(h_-(\alpha))^{2j+1}}{\sin\left(\frac{\beta}{2}\right)2^{2j+1}(2j+1)!}\right)^{\widehat{}} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sin^{2n}\left(\frac{\beta}{2}\right)} *^{2n}\left(\sum_{j=0}^{\infty}\frac{*^{2j+1}(m(k,\beta)\hat{h}(k))}{2^{2j+1}(2j+1)!}\right) \\
&= \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{n+1}}{n\sin^{2n}\left(\frac{\beta}{2}\right)2^{2j+1}(2j+1)!} *^{2(j+n)+1}\left[m(k,\beta)\hat{h}(k)\right] \\
&= \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{n+1}}{n\sin^{2n}\left(\frac{\beta}{2}\right)2^{2j+1}(2j+1)!} \\
&\quad\times\sum_{k,k_1,k_2,\dots k_{2(j+n)}\neq 0}\left(m(k_{2(j+n)})m(k-k_1)\prod_{l=1}^{2(j+n)-1}m(k_l-k_{l+1})\right) \\
&\quad\times\left(\hat{h}(k_{2(j+n)})\hat{h}(k-k_1)\prod_{l=1}^{2(j+n)-1}\hat{h}(k_l-k_{l+1})\right). \qquad (2.2.6)
\end{aligned}$$

Note that

$$\begin{aligned}
&\left(m(k_{2(j+n)})m(k-k_1)\prod_{l=1}^{2(j+n)-1}m(k_l-k_{l+1})\right) \\
&= \int_0^1 ds\int_0^1 ds_1\dots\int_0^1 ds_{2(j+n)}(-i)^{2(j+n)+1}\left(k_{2(j+n)}(k-k_1)\prod_{l=1}^{2(j+n)-1}(k_l-k_{l+1})\right) \\
&\quad\times\beta^{2n+2j+1}\left(e^{-ik_{2(j+n)}\beta(1-s)}e^{-i(k-k_1)\beta(1-s)}\prod_{l=1}^{2(j+n)-1}e^{-i(k_l-k_{l+1})\beta(1-s_l)}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}\widehat{\mathcal{T}}_2(k,\beta) &= \left(\frac{1}{2\sin^2\left(\frac{\beta}{2}\right)}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^n}{(2j+1)!2^{2j+1}\sin^{2n}\left(\frac{\beta}{2}\right)} *^{2(n+j)+1}\left[m(k)\hat{h}(k)\right]\right) \\ &\quad * (e^{-ik\beta}\hat{h}(k)) * (m(k,\beta)\hat{h}(k)) \\ &= \left[\frac{1}{2\sin^2\left(\frac{\beta}{2}\right)}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^n}{(2j+1)!2^{2j+1}\sin^{2n}\left(\frac{\beta}{2}\right)}\right. \\ &\quad \times \sum_{k_1,k_2,\dots k_{2(j+n)}\in\mathbb{Z}}\left(m(k_{2(j+n)})m(k-k_1)\prod_{l=1}^{2(j+n)-1} m(k_l-k_{l+1})\right) \\ &\quad \left.\times\left(\hat{h}(k_{2(j+n)})\hat{h}(k-k_1)\prod_{l=1}^{2(j+n)-1}\hat{h}(k_l-k_{l+1})\right)\right] * (e^{-ik\beta}\hat{h}(k)) * (m(k,\beta)\hat{h}(k)).\end{aligned} \tag{2.2.7}$$

2.2.2 Proof of Theorem 2

In this section, we prove Theorem 2 exploiting the structure of nonlinear terms (2.2.2)-(2.2.5).

Proof. Consider the evolution of the norm

$$\frac{d}{dt}\left|\left|h(t)\right|\right|_{A^0_{\nu(t)}} = \nu'(t)\left|\left|h(t)\right|\right|_{A^1_{\nu(t)}} + \sum_{k\neq 0}\frac{\hat{h}_t(k,t)\overline{\hat{h}(k,t)} + \hat{h}(k,t)\overline{\hat{h}_t(k,t)}}{2|\hat{h}(k,t)|}e^{\nu(t)|k|}.$$

From (2.2.1), we find that

$$\frac{d}{dt}\left|\left|h(t)\right|\right|_{A^0_{\nu(t)}} \leq \nu'(t)\left|\left|h(t)\right|\right|_{A^1_{\nu(t)}} - \overline{\rho}\left|\left|h\right|\right|_{A^{-1}_{\nu(t)}} \tag{2.2.8}$$

$$\begin{aligned}&\quad + \frac{\overline{\rho}}{2\pi}\sum_{k\neq 0}\sum_{i=1}^{4}\frac{\hat{I}_i(k,t)\overline{\hat{h}(k,t)} + \hat{h}(k,t)\overline{\hat{I}_i(k,t)}}{2|\hat{h}(k,t)|}e^{\nu(t)|k|} \\ &\leq \nu'(t)\left|\left|h(t)\right|\right|_{A^1_{\nu(t)}} - \overline{\rho}\left|\left|h\right|\right|_{A^{-1}_{\nu(t)}} + \frac{\overline{\rho}}{2\pi}\sum_{k\neq 0}\sum_{i=1}^{4}|\hat{I}_i(k,t)|e^{\nu(t)|k|} \\ &\leq \nu'(t)\left|\left|h(t)\right|\right|_{A^1_{\nu(t)}} - \overline{\rho}\left|\left|h\right|\right|_{A^{-1}_{\nu(t)}} + \frac{\overline{\rho}}{2\pi}\sum_{i=1}^{4}\left|\left|I_i\right|\right|_{A^0_{\nu(t)}}.\end{aligned} \tag{2.2.9}$$

Let us estimate the norms $||I_i||_{A^0_{\nu(t)}}$. Note that

$$|k| \leq |k-k_1| + |k_{2(n+j)}|\sum_{m=1}^{2(n+j)-1}|k_m - k_{m+1}|,$$

thus

$$e^{i\nu(t)|k|} \leq e^{i\nu(t)|k-k_1|} e^{i\nu(t)|k_{2n+2j}|} \prod_{m=1}^{2(n+j)-1} e^{i\nu(t)|k_m - k_{m+1}|}.$$

Therefore, using (2.2.6) and

$$\sum_{i=1}^{\infty} \frac{x^{2n}}{n} = -\log(1-x^2) = \log\left(\frac{1}{1-x^2}\right)$$

for x small enough, it holds

$$\begin{aligned}
||\mathcal{T}_1||_{A^0_{\nu(t)}} &\leq \sum_{k\neq 0}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{\beta^{2j+2n+1}}{n\sin^{2n}\left(\frac{\beta}{2}\right)2^{2j+1}(2j+1)!} \left|*^{2(j+n)+1}\left[ik\hat{h}(k)\right]\right| e^{\nu(t)|k|} \\
&\leq \sum_{k\neq 0}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{\pi^{2n}\beta^{2j+1}}{n2^{2j+1}(2j+1)!} \left|*^{2(j+n)+1}\left[ik\hat{h}(k)\right]\right| e^{\nu(t)|k|} \\
&\leq \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{\pi^{2n+2j+1}}{n2^{2j+1}(2j+1)!} ||h||^{2(j+n)+1}_{A^1_{\nu(t)}} \\
&\leq \sinh\left(\frac{\pi}{2}||h||_{A^1_{\nu(t)}}\right)\log\left(\frac{1}{1-\pi^2||h||^2_{A^1_{\nu(t)}}}\right) \\
&\leq \mathcal{G}_1\left(||h||_{A^1_{\nu(t)}}\right),
\end{aligned} \tag{2.2.10}$$

where we defined

$$\mathcal{G}_1\left(||h||_{A^1_{\nu(t)}}\right) = \sinh\left(\frac{\pi}{2}||h||_{A^1_{\nu(t)}}\right)\log\left(\frac{1}{1-\pi^2||h||^2_{A^1_{\nu(t)}}}\right).$$

From (2.2.2) and (2.2.10), we get the estimates

$$\begin{aligned}
||I_1||_{A^0_{\nu(t)}} &\leq \int_{\mathbb{T}} ||\mathcal{T}_1||_{A^0_{\nu(t)}} ||h||_{A^0_{\nu(t)}}\, d\beta \\
&\leq 2\pi\mathcal{G}_1\left(||h||_{A^1_{\nu(t)}}\right)||h||_{A^0_{\nu(t)}}
\end{aligned} \tag{2.2.11}$$

and

$$\begin{aligned}
||I_2||_{A^0_{\nu(t)}} &\leq \int_{\mathbb{T}}\left(||\mathcal{T}_1||_{A^0_{\nu(t)}} + \log\left(\sin^2\left(\frac{\beta}{2}\right)\right)\right)||h||^2_{A^1_{\nu(t)}}||h||_{A^0_{\nu(t)}}\, d\beta \\
&\leq 2\pi\mathcal{G}_1\left(||h||_{A^1_{\nu(t)}}\right)||h||_{A^0_{\nu(t)}}||h||^2_{A^1_{\nu(t)}} + 4\pi\log(2)||h||_{A^0_{\nu(t)}}||h||^2_{A^1_{\nu(t)}}.
\end{aligned} \tag{2.2.12}$$

From (2.2.3), (2.2.7) and
$$\sum_{i=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$$
for x small enough, we get

$$\begin{aligned}
||\mathcal{T}_2||_{A^0_{\nu(t)}} &\leq \sum_{k\neq 0} \frac{1}{2\sin^2\left(\frac{\beta}{2}\right)} \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{\beta^{2n+2j+2}}{(2j+1)!2^{2j+1}\sin^{2n}\left(\frac{\beta}{2}\right)} \\
&\quad \times \left| *^{2(n+j)+2} \left[ik\hat{h}(k)\right] * \hat{h}(k)\right| e^{i\nu(t)|k|} \\
&\leq \frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{\pi^{2(n+j)+2}}{(2j+1)!2^{2j+1}} ||h||^{2(n+j)+2}_{A^1_{\nu(t)}} ||h||_{A^0_{\nu(t)}} \\
&\leq \frac{\pi}{2}\sinh\left(\frac{\pi}{2}||h||_{A^1_{\nu(t)}}\right) \frac{1}{1-\pi^2||h||^2_{A^1_{\nu(t)}}} ||h||_{A^1_{\nu(t)}} ||h||_{A^0_{\nu(t)}} \\
&\leq \mathcal{G}_2(||h||_{A^1_{\nu(t)}}) ||h||_{A^0_{\nu(t)}}, \qquad (2.2.13)
\end{aligned}$$

where we defined
$$\mathcal{G}_2(||h||_{A^1_{\nu(t)}}) = \frac{\pi}{2}\sinh\left(\frac{\pi}{2}||h||_{A^1_{\nu(t)}}\right) \frac{1}{1-\pi^2||h||^2_{A^1_{\nu(t)}}} ||h||_{A^1_{\nu(t)}}.$$

From (2.2.4), (2.2.5) and (2.2.13), we get the estimates

$$\begin{aligned}
||I_3||_{A^0_{\nu(t)}} &\leq \int_{\mathbb{T}} ||\mathcal{T}_2||_{A^0_{\nu(t)}} \left(||h||^2_{A^1_{\nu(t)}}+1\right)\sinh\left(\pi||h||_{A^1_{\nu(t)}}\right) d\beta \\
&\leq 2\pi\mathcal{G}_2(||h||_{A^1_{\nu(t)}}) ||h||_{A^0_{\nu(t)}} \left(||h||^2_{A^1_{\nu(t)}}+1\right)\sinh\left(\pi||h||_{A^1_{\nu(t)}}\right) \qquad (2.2.14)
\end{aligned}$$

and

$$\begin{aligned}
||I_4||_{A^0_{\nu(t)}} &\leq \int_{\mathbb{T}} 2|\sin(\beta)|\, ||\mathcal{T}_2||_{A^0_{\nu(t)}} ||h||_{A^1_{\nu(t)}} d\beta \\
&\leq 8\mathcal{G}_2(||h||_{A^1_{\nu(t)}}) ||h||_{A^0_{\nu(t)}} ||h||_{A^1_{\nu(t)}}. \qquad (2.2.15)
\end{aligned}$$

Joining the estimates of terms $||I_i||_{A^0_{\nu(t)}}$ in (2.2.9),

$$\begin{aligned}
\frac{d}{dt}||h(t)||_{A^0_{\nu(t)}} &\leq \nu'(t)||h(t)||_{A^1_{\nu(t)}} - \overline{\rho}\,||h||_{A^{-1}_{\nu(t)}} \\
&\quad + C_1(\overline{\rho})\,||h||_{A^1_{\nu(t)}} ||h||_{A^0_{\nu(t)}} \mathcal{J}_1(||h||_{A^1_{\nu(t)}}), \qquad (2.2.16)
\end{aligned}$$

where $\mathcal{J}_1$ is an increasing function and $C(\overline{\rho})$ depends linearly on the density jump. In fact,
$$\mathcal{J}_1(x) \approx x + O(x^2) \qquad (2.2.17)$$

The evolution of the $A^1_{\nu(t)}$ norm can be expressed, similarly, as

$$\frac{d}{dt}\,||h(t)||_{A^1_{\nu(t)}} \le \nu'(t)\,||h(t)||_{A^2_{\nu(t)}} - \overline{\rho}\,||h||_{A^0_{\nu(t)}} + \frac{\overline{\rho}}{2\pi}\sum_{i=1}^{4}||I_i||_{A^1_{\nu(t)}}\,.$$

We have

$$\begin{aligned}
||\mathcal{T}_1||_{A^1_{\nu(t)}} &\le \sum_{k\neq 0}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{\pi^{2n}\beta^{2j+1}|k|}{n2^{2j+1}(2j+1)!}\left|*^{2(j+n)+1}\left[ik\hat{h}(k)\right]\right|e^{\nu(t)|k|}\\
&\le \sum_{k\neq 0}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}(2(j+n)+1)\frac{\pi^{2n+2j+1}}{n2^{2j+1}(2j+1)!}\\
&\quad\times\left|*^{2(j+n)}\left[ik\hat{h}(k)\right]*\left[|k|^2 i\hat{h}(k)\right]\right|e^{\nu(t)|k|}\\
&\le \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}(2(j+n)+1)\frac{\pi^{2n+2j+1}}{n2^{2j+1}(2j+1)!}\,||h||^{2(j+n)}_{A^1_{\nu(t)}}\,||h||_{A^2_{\nu(t)}}\\
&\le ||h||_{A^2_{\nu(t)}}\frac{\pi}{2}\cosh\left(\frac{\pi}{2}\,||h||_{A^1_{\nu(t)}}\right)\log\left(\frac{1}{1-\pi^2\,||h||^2_{A^1_{\nu(t)}}}\right)\\
&\quad+||h||_{A^2_{\nu(t)}}\sinh\left(\frac{\pi}{2}\,||h||_{A^1_{\nu(t)}}\right)\frac{2\pi^2\,||h||_{A^1_{\nu(t)}}}{1-\pi^2\,||h||^2_{A^1_{\nu(t)}}}\\
&\le \mathcal{G}_1'(||h||_{A^1_{\nu(t)}})\,||h||_{A^2_{\nu(t)}}
\end{aligned}\tag{2.2.18}$$

From (2.2.18), we get

$$\begin{aligned}
||I_1||_{A^1_{\nu(t)}} &\le \int_{\mathbb{T}}\left|\left|\mathcal{T}_1(k,\beta)*e^{-ik\beta}\hat{h}(k)\right|\right|_{A^1_{\nu(t)}}d\beta\\
&\le \int_{\mathbb{T}}\left(||\mathcal{T}_1||_{A^1_{\nu(t)}}\,||h||_{A^0_{\nu(t)}}+||\mathcal{T}_1||_{A^0_{\nu(t)}}\,||h||_{A^1_{\nu(t)}}\right)d\beta\\
&\le 2\pi\mathcal{G}_1'(||h||_{A^1_{\nu(t)}})\,||h||_{A^2_{\nu(t)}}\,||h||_{A^0_{\nu(t)}}+2\pi\mathcal{G}_1(||h||_{A^1_{\nu(t)}})\,||h||_{A^1_{\nu(t)}}\\
&\le C\,||h||_{A^0_{\nu(t)}}\,||h||_{A^2_{\nu(t)}}\,F_1\left(||h||_{A^1_{\nu(t)}}\right)
\end{aligned}$$

and

$$\begin{aligned}||I_2||_{A^1_{\nu(t)}} &\leq \int_{\mathbb{T}} \left|\left|\widehat{\mathcal{T}}_1(k,\beta) * \hat{h}(k)e^{-ik\beta} * (ik\hat{h}(k)) * (e^{-ik\beta}ik\hat{h}(k))\right|\right|_{A^1_{\nu(t)}} d\beta \\ &\quad + \int_{\mathbb{T}} \left|\log\left(\sin^2\left(\frac{\beta}{2}\right)\right)\right| \left|\left|\hat{h}(k)e^{-ik\beta} * (ik\hat{h}(k)) * (e^{-ik\beta}ik\hat{h}(k))\right|\right|_{A^1_{\nu(t)}} d\beta \\ &\leq ||\mathcal{T}_1||_{A^1_{\nu(t)}} ||h||_{A^0_{\nu(t)}} ||h||^2_{A^1_{\nu(t)}} + ||\mathcal{T}_1||_{A^0_{\nu(t)}} ||h||^3_{A^1_{\nu(t)}} \\ &\quad + 2\, ||\mathcal{T}_1||_{A^0_{\nu(t)}} ||h||_{A^0_{\nu(t)}} ||h||_{A^1_{\nu(t)}} ||h||_{A^2_{\nu(t)}} \\ &\leq C\, ||h||_{A^0_{\nu(t)}} ||h||_{A^2_{\nu(t)}} F_2\left(||h||_{A^1_{\nu(t)}}\right).\end{aligned}$$

Furthermore,

$$||\mathcal{T}_2||_{A^1_{\nu(t)}} \leq \mathcal{G}_2'(||h||_{A^1_{\nu(t)}})\, ||h||_{A^0_{\nu(t)}} ||h||_{A^2_{\nu(t)}} + \mathcal{G}_2(||h||_{A^1_{\nu(t)}})\, ||h||_{A^1_{\nu(t)}}. \tag{2.2.19}$$

From (2.2.19), we get

$$||I_3||_{A^1_{\nu(t)}} \leq C\, ||h||_{A^0_{\nu(t)}} ||h||_{A^2_{\nu(t)}} F_3\left(||h||_{A^1_{\nu(t)}}\right).$$

and

$$||I_4||_{A^1_{\nu(t)}} \leq C\, ||h||_{A^0_{\nu(t)}} ||h||_{A^2_{\nu(t)}} F_4\left(||h||_{A^1_{\nu(t)}}\right),$$

where the functions $F_1, \ldots F_4$ are increasing functions.

Joining all the estimates of terms $||I_i||_{A^1_{\nu(t)}}$ together, we get

$$\begin{aligned}\frac{d}{dt} ||h(t)||_{A^1_{\nu(t)}} &\leq \nu'(t)\, ||h(t)||_{A^2_{\nu(t)}} - \overline{\rho}\, ||h||_{A^0_{\nu(t)}} \\ &\quad + C_2(\overline{\rho})\, ||h||_{A^0_{\nu(t)}} ||h||_{A^2_{\nu(t)}} \mathcal{J}_2\left(||h||_{A^1_{\nu(t)}}\right),\end{aligned} \tag{2.2.20}$$

where $\mathcal{J}_2$ is, again, an increasing function, and $C_2(\overline{\rho})$ depends linearly on the density jump.

Control of norms for all time. Define

$$\nu'(t) = -\left(\max_{i=1,2}\{C_i(\overline{\rho})\mathcal{J}_i\left(||h||_{A^1_{\nu(t)}}\right)\} + \varepsilon\right) ||h||_{A^0_{\nu(t)}}. \tag{2.2.21}$$

for $\varepsilon > 0$. Then, by (2.2.16), (2.2.20) and (2.2.21)

$$\frac{d}{dt} ||h(t)||_{A^0_{\nu_0}} < 0,$$

$$\frac{d}{dt}\,||h(t)||_{A^1_{\nu_0}} < 0,$$

for every $t \geq 0$. Consequently

$$||h(t)||_{A^0_{\nu(t)}} \leq ||h_0||_{A^0_{\nu_0}} \text{ and } ||h(t)||_{A^1_{\nu(t)}} \leq ||h_0||_{A^1_{\nu_0}} \tag{2.2.22}$$

for every $t \geq 0$. In addition, from (2.2.20), taking into account the damping term, we have

$$||h(t)||_{A^1_{\nu(t)}} + \overline{\rho} \int_0^t ||h(s)||_{A^0_{\nu(s)}}\, ds \leq ||h_0||_{A^1_{\nu_0}}\,. \tag{2.2.23}$$

Consequently, the solution is controlled by the initial data for all time $t \geq 0$.

Control of the analyticity band. Using (2.2.23) and the monotonic increasing character of functions $\mathcal{J}_i$,

$$\begin{aligned}
\nu(t) &= \nu_0 + \int_0^t \nu'(t)dt \\
&\geq \nu_0 - \left(\max_{i=1,2}\{C_i(\overline{\rho})\mathcal{J}_i\left(||h_0||_{A^1_{\nu_0}}\right)\} + \varepsilon\right)\int_0^t ||h(s)||_{A^0_{\nu(s)}}\, ds \\
&\geq \nu_0 - \frac{1}{\overline{\rho}}\left(\max_{i=1,2}\{C_i(\overline{\rho})\mathcal{J}_i\left(||h_0||_{A^1_{\nu_0}}\right)\} + \varepsilon\right)||h_0||_{A^1_{\nu_0}} \\
&> 0
\end{aligned}$$

for small enough data $||h_0||_{A^1_{\nu_0}}$. This ensures that the analyticity bound does not collapse at any time $t \geq 0$.

In particular, we can choose small enough initial data such that

$$0 < \frac{\nu_0}{2} \leq \nu(t) \leq \nu_0. \tag{2.2.24}$$

The estimates (2.2.22) and (2.2.24) give us the global existence of solutions in $A^1_{\nu(t)}$. □

2.2.3 Proof of Theorem 3

In this section, we prove Theorem 3. The proof relies on the proof of Theorem 2 and a similar strategy to the one used in the proof of Theorem 1: we exploit the decay properties of the semi-group in Wiener algebras and combine them with a priori estimates.

Proof. Before starting the argument of the proof, let us establish the following decay lemma:

Lemma 2. Let $a, b > 0$. Then,

$$\int_0^t e^{-\sqrt{a(t-s)}} e^{-\sqrt{bs}} ds \leq c(\min\{a,b\}) e^{-\sqrt{\frac{\min\{a,b\}}{2} t}}.$$

Proof. Define $m = \min\{a, b\}$. Then,

$$\begin{aligned}
\int_0^t e^{-\sqrt{a(t-s)}} e^{-\sqrt{bs}} ds &= \int_0^{\frac{t}{2}} e^{-\sqrt{a(t-s)}} e^{-\sqrt{bs}} ds + \int_{\frac{t}{2}}^t e^{-\sqrt{a(t-s)}} e^{-\sqrt{bs}} ds \\
&\leq \int_0^{\frac{t}{2}} e^{-\sqrt{m(t-s)}} e^{-\sqrt{ms}} ds \quad + \int_{\frac{t}{2}}^t e^{-\sqrt{m(t-s)}} e^{-\sqrt{ms}} ds \\
&\leq e^{-\sqrt{m\frac{t}{2}}} \int_0^{\frac{t}{2}} e^{-\sqrt{ms}} ds + e^{-\sqrt{m\frac{t}{2}}} \int_{\frac{t}{2}}^t e^{-\sqrt{m(t-s)}} ds \\
&\leq e^{-\sqrt{m\frac{t}{2}}} \frac{4}{m} \left(1 - e^{-\sqrt{mt}} \left(1 + \sqrt{mt}\right)\right) \\
&\leq c(m) e^{-\sqrt{m\frac{t}{2}}}.
\end{aligned}$$

□

Let $h \in C([0,T]; A^1_{\nu(t)})$ be a solution of (1.1.15) in the RT stable case constructed under the hypotheses of Theorem 2, with small enough initial data

$$||h_0||_{A^1_{\nu_0}} < \delta.$$

Define ν^* in such a way that

$$0 < 12\nu^* \leq \frac{\nu_0}{2} \leq \nu(t) \leq \nu_0,$$

and define the norm

$$|||h|||_{\nu^*} = \sup_{t\in[0,T]} \left(e^{\sqrt{\rho\nu^* t}} ||h||_{A^0} + ||h||_{A^0_{\nu^*}} \right). \tag{2.2.25}$$

The aim is to produce uniform bounds of (2.2.25), only depending on the size of the initial data h_0.

Using the estimate (1.2.11) in Proposition 2 for $\nu = 2\nu^*$ and Duhamel formula, we get

$$||h||_{A^0} \lesssim e^{-\sqrt{\bar{\rho}2\nu^* t}} ||h_0||_{A^0_{2\nu^*}} + \frac{\bar{\rho}}{2\pi} \int_0^t e^{-\sqrt{\bar{\rho}2\nu^*(t-s)}} ||I_1 + I_2 + I_3 + I_4||_{A^0_{2\nu^*}} ds.$$

Using $x \leq c(\nu^*)e^{2\nu^* x}$ and Hölder inequality, we can easily derive the estimates

$$\begin{aligned} ||h||_{A^1_{2\nu^*}} &= \sum_{k\neq 0} |k| e^{2\nu^*|k|} |\hat{h}(k)| \\ &\leq c(\nu^*)\, ||h||_{A^0_{3\nu^*}} \\ &\leq c(\nu^*)\, ||h||^{3/4}_{A^0}\, ||h||^{1/4}_{A^0_{12\nu^*}} \end{aligned}$$

and

$$||h||_{A^0_{2\nu^*}} \leq c(\nu^*)\, ||h||^{1/2}_{A^0}\, ||h||^{1/2}_{A^0_{4\nu^*}} .$$

Since $12\nu^* \leq \frac{\nu_0}{2} \leq \nu(t)$, we have the uniform estimates

$$||h||_{A^0_{4\nu^*}} \leq ||h||_{A^0_{12\nu^*}} \leq ||h||_{A^0_{\nu(t)}} \leq ||h_0||_{A^0_{\nu_0}}$$

and

$$||h||_{A^1_{2\nu^*}} \leq ||h||_{A^1_{\nu(t)}} \leq ||h_0||_{A^1_{\nu_0}} ,$$

so that, by (2.2.11),(2.2.12), (2.2.14) and (2.2.15) in a band of width $2\nu^*$ and the estimates above,

$$\begin{aligned} ||I_1 + I_2 + I_3 + I_4||_{A^0_{2\nu^*}} &\leq c(\overline{\rho})\, ||h||_{A^1_{2\nu^*}}\, ||h||_{A^0_{2\nu^*}}\, \mathcal{J}_1(||h||_{A^1_{2\nu^*}}) \\ &\leq c(\overline{\rho})\, ||h||^2_{A^1_{2\nu^*}}\, ||h||_{A^0_{2\nu^*}}\, F(||h||_{A^1_{2\nu^*}}) \\ &\leq c(\overline{\rho})\, ||h||^2_{A^0}\, ||h||_{A^0_{\nu(t)}}\, F(||h||_{A^1_{\nu(t)}}) \\ &\leq c(\overline{\rho}, ||h_0||_{A^1_{\nu_0}}) e^{-2\sqrt{\overline{\rho}\nu^* s}}\, |||h|||^2_{\nu^*} , \end{aligned}$$

where the first inequality comes from (2.2.17).

Consequently, applying Lemma 2,

$$\begin{aligned} ||h||_{A^0} &\leq e^{-\sqrt{\overline{\rho}2\nu^* t}}\, ||h_0||_{A^0_{2\nu^*}} + c(\nu^*, \overline{\rho}, ||h_0||_{A^1_{\nu_0}})\, |||h|||^2_{\nu^*} \int_0^t e^{-\sqrt{\overline{\rho}2\nu^*(t-s)}} e^{-\sqrt{\overline{\rho}4\nu^* s}} ds \\ &\leq e^{-\sqrt{\overline{\rho}\nu^* t}}\, ||h_0||_{A^0_{2\nu^*}} + c(\nu^*, \overline{\rho}, ||h_0||_{A^1_{\nu_0}})\, |||h|||^2_{\nu^*}\, e^{-\sqrt{\overline{\rho}\nu^* t}}. \end{aligned}$$

Thus,

$$e^{\sqrt{\overline{\rho}\nu^* t}}\, ||h||_{A^0} \leq ||h_0||_{A^0_{2\nu^*}} + c(\nu^*, \overline{\rho}, ||h_0||_{A^1_{\nu_0}})\, |||h|||^2_{\nu^*} . \tag{2.2.26}$$

Furthermore, by Hölder inequality, we can similarly derive the estimates

$$\begin{aligned} ||h||_{A^1_{\nu^*}} &= \sum_{k\neq 0} |k| e^{\nu^*|k|} |\hat{h}(k)| \\ &\leq c(\nu^*)\, ||h||_{A^0_{2\nu^*}} \\ &\leq c(\nu^*)\, ||h||^{3/4}_{A^0}\, ||h||^{1/4}_{A^0_{8\nu^*}} \end{aligned}$$

and

$$||h||_{A^0_{\nu^*}} \leq c(\nu^*)\, ||h||^{1/2}_{A^0}\, ||h||^{1/2}_{A^0_{2\nu^*}}\,.$$

By (2.2.16) in a band of width ν^* and (2.2.17), it holds that

$$\begin{aligned}\frac{d}{dt}\, ||h||_{A^0_{\nu^*}} &\leq C_1(\overline{\rho})\, ||h||^2_{A^1_{\nu^*}}\, ||h||_{A^0_{\nu^*}}\, F(||h||_{A^1_{\nu^*}}) \\ &\leq c(\nu^*, \overline{\rho}, ||h_0||_{A^1_{\nu_0}}) e^{-2\sqrt{\overline{\rho}\nu^* t}}\, |||h|||^2_{\nu^*}\,.\end{aligned}$$

Integrating from 0 to t, we get

$$\begin{aligned}||h||_{A^0_{\nu^*}} &\leq ||h_0||_{A^0_{\nu^*}} + c(\nu^*, \overline{\rho}, ||h_0||_{A^1_{\nu_0}})\, |||h|||^2_{\nu^*} \int_0^t e^{-\sqrt{\overline{\rho}4\nu^* s}} ds \\ &\leq ||h_0||_{A^0_{\nu^*}} + c(\nu^*, \overline{\rho}, ||h_0||_{A^1_{\nu_0}})\, |||h|||^2_{\nu^*}\,. \end{aligned} \tag{2.2.27}$$

Joining the estimates (2.2.26) and (2.2.27), we find

$$|||h|||_{\nu^*} \leq ||h_0||_{A^0_{\nu^*}} + c(\nu^*, \overline{\rho}, ||h_0||_{A^1_{\nu_0}})\, |||h|||^2_{\nu^*}\,.$$

Assuming the condition

$$||h_0||_{A^0_{\nu^*}} < \varepsilon,$$

for $\varepsilon > 0$ small enough, we can reproduce the argument in the proof of Theorem 1 and get

$$|||h|||_{\nu^*} = \sup_{t\in[0,T]} \left(e^{\sqrt{\overline{\rho}\nu^* t}}\, ||h||_{A^0} + ||h||_{A^0_{\nu^*}} \right) \leq C\, ||h_0||_{A^0_{\nu^*}} \tag{2.2.28}$$

for every $T > 0$. This gives us the global existence of analytic interfaces in $A^0_{\nu^*}$ and the exponential decay of the A^0 norm in the stable regime.

□

2.3 Exponential growth of solutions in the unstable case

In this section, we prove that in the unstable regime of the densities, where the denser fluid lies above the less dense, smooth solutions grow exponentially in certain Wiener norms. We collect this result in the following theorem:

Theorem 4 (Exponential growth of solutions in the RT unstable case for small data)**.** Let $T > 0$ be an arbitrary fixed parameter. Then, it exists a family of smooth initial data

$$g_0 \in A^0_{\nu*}$$

such that

$$g \in C([0,T]; A^0_{\nu*})$$

is a solution of (1.1.15) in the RT unstable regime,

$$\rho^- - \rho^+ < 0,$$

and

$$||g(\tau)||_{A^0_{\nu^*}} \geq \frac{1}{C} e^{\sqrt{(|\rho^- - \rho^+|/4)\nu^* \tau}} ||g_0||_{A^0} \quad \tau \in [0,T]. \tag{2.3.1}$$

Proof. Let $T > 0$ arbitrary. By Theorem 3, there exists a solution for every $\tau \in [0,T]$

$$h \in C([0,\tau]; A^0_{\nu^*})$$

in the RT stable case starting from a small enough initial data h_0,

$$||h_0||_{A^0_{\nu^*}} < \varepsilon.$$

Setting a family of different initial data h_0, we can find a family of different solutions $h(\alpha,t)$. Furthermore, each solution satisfies

$$e^{\sqrt{|\overline{\rho}|\nu^* t}} ||h(t)||_{A^0} \leq C\, ||h_0||_{A^0_{\nu^*}} \quad t \in [0,\tau] \tag{2.3.2}$$

where $C > 0$.

Now, define the function

$$g(\alpha,t) = h(\alpha, \tau - t).$$

It is clear from the definition that

$$g \in C([0,\tau]; A^0_{\nu^*}).$$

Moreover, we find that

$$\begin{aligned} g_t(\alpha,t) &= -(h_t)(\alpha,\tau - t) \\ &= |\overline{\rho}|\Lambda^{-1}(g)(\alpha,t) - \frac{|\overline{\rho}|}{2\pi}\left(I_1(g)(\alpha,t) + I_2(g)(\alpha,t) + I_3(g)(\alpha,t) + I_4(g)(\alpha,t)\right), \end{aligned}$$

i.e., $g(\alpha,t)$ solves (1.1.15) (equivalently (2.1.1)) in the unstable RT scenario. Note that

$$g_0(\alpha) = h(\alpha,\tau) \text{ and } g(\alpha,\tau) = h_0(\alpha).$$

Then, by (2.3.2), it holds that

$$||g(\tau)||_{A^0_{\nu^*}} \geq \frac{1}{C} e^{\sqrt{|\overline{\rho}|\nu^* \tau}} ||g_0||_{A^0} \quad \tau \in [0, T]. \tag{2.3.3}$$

The previous equation shows exponential growth of smooth analytic solutions of the unstable RT scenario for small initial data, in a time interval which is arbitrarily large.

□

THE KOLMOGOROV MODEL OF TURBULENCE

Introduction

"When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first."

— W. Heisenberg

IV Turbulence in fluids

There is not a completely satisfactory and precise definition of turbulence. However, one can have an intuitive notion of what turbulence is by numerous phenomena occurring in oceanography, meteorology, astrophysics, etc. For instance, we cannot predict accurately the behavior of ocean currents, and we observe eddies in the atmospheres of the Earth, Jupiter and Saturn, or in the galaxy.

Following the discussion in [62], we can agree on some desired properties for a turbulent flow: the motion is unpredictable, since small uncertainty in the data can lead to bifurcation or drastically different behavior due to the nonlinear terms; transported quantities are mixed much more faster than in diffusion processes; the motion covers a wide range of spacial wave lengths.

If we return to the discussion in Section I, we find that, contrary to what happens in a laminar flow, in a turbulent motion, the Reynolds number must be large. For instance, one can measure the Reynolds number as the product of the characteristic length L associated to large eddies and the characteristic fluctuating velocity u, divided by the viscosity forces. In turbulent fluids, this ratio is large. This is in connection with the existence and uniqueness of global in time solutions for the Navier-Stokes equations. Such solutions have not been found for all ranges of Reynolds numbers, and one can justify this gap in the theory with the appearance of turbulence, which might cause a loss of regularity and loss of continuous dependence on the initial data.

Another relevant factor that characterizes turbulent flow is vorticity. There are numerous experiments supporting that turbulent flows are rotational, i.e., they have non-zero vorticity $\nabla \times u \neq 0$. Even in uniform irrotational fluids, vorticity can be produced by the interaction with boundaries or obstacles; for example, the contact with an obstacle produces a zero velocity condition which induces vorticity.

However, it is also interesting to study turbulence which evolves without imposed constraints. In this case, we consider the turbulence to be *fully-developed*. In real phenomena, we can only find fully-developed turbulence at the small scales, in the case where the Reynolds number is large enough so that the viscosity does not influence the dynamics of the small scales. This is the case of the atmosphere of the Earth, Jupiter and Saturn. Nevertheless, when studying a homogeneous turbulent flow where no boundaries or external forcing is considered, for theoretical purposes, we may assume the turbulence to be fully-developed even in the large scales. This simplification will be useful to derive models based on statistical assumptions. We will also consider the turbulence to be *homogeneous* and *isotropic*. A turbulence is called *homogeneous* if the velocity $u(x, t)$ is statistically independent of the point of space x, that is, it is statistically invariant under translations; it is *isotropic* if the means and moments of the velocity are invariant under rotations.

In the celebrated *Kolmogorov's 1941 theory*, Kolmogorov and Obukhov established the foundations of the mathematical theory of turbulence (see [52] and the Appendix in [83] for an English translation). Kolmogorov's theory is based on three principal hypotheses, coming from experimental observations and dimensional arguments:

- *"At sufficiently large Reynolds numbers, the small-scale turbulent motions ($l << L$) are statistically isotropic".* Here, L refers to the scale of the large eddies, and it is stated that the directional biases of the large scales are lost as the energy is transferred to small scales.

- *"In every turbulent flow at sufficiently high Reynolds number, the statistics of the small scale motions ($l << L_E$) have a universal form that is uniquely determined by ε and ν".* In this statement, L_E is the length scale which determines the difference between large scale anisotropic eddies and small scale isotropic eddies, ε refers to the energy dissipation rate and ν is the kinematic viscosity.

- *"In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale l in the range $L >> l >> \nu$ have a universal form that is uniquely determined by ε independent of ν."*

Kolmogorov's theory has proved to work well in very high Reynolds numbers, although some important simplifications are made which may not be true in practice. For instance, it is assumed that turbulence at large scales is completely random, while in practice we might find large scale coherent structures. This fact is key when it comes to develop the mathematical models. Randomness is introduced and we work with stochastic averages.

This section is based on [28, 80, 62], where the reader can find a deep discussion on turbulence and the related $k - \varepsilon$ models.

V The Kolmogorov two-equation model of turbulence

V.1 Derivation of the model

One of the principal hypothesis to derive the statistical mathematical models for turbulence is to assume that one can decompose the incompressible velocity field into its mean part $\langle u \rangle$ and its oscillations $\tilde{u}$:

$$u = \langle u \rangle + \tilde{u}.$$

There is not a unique choice of the mean $\langle u \rangle$, although the natural choice in the case of Navier Stokes equations with random initial data seems to be the *statistical average*. If we inject this decomposition into the Navier-Stokes equations, we obtain the so-called *Reynolds averaged Navier-Stokes equations* (RANS). We find a cascade of differential equations for $\langle u \rangle$, $\tilde{u}$ and their high order correlations. The main inconvenience about these models is that further information about the correlations is needed, since the models are not closed. For example, if we search for an equation for the averaged velocity $\langle u \rangle$, one can check that it satisfies the so-called *Reynolds equation*

$$\partial_t \langle u \rangle + \langle u \rangle \nabla \langle u \rangle + \nabla \langle p \rangle + \nabla \cdot \langle \tilde{u} \otimes \tilde{u} \rangle - \nu \Delta \langle u \rangle = 0, \quad \nabla \cdot \langle u \rangle = 0, \tag{V.1}$$

where the extra term $\langle \widetilde{u} \otimes \widetilde{u} \rangle$, called *Reynolds Stress tensor* appears. Then, if one tries to use the equation for $\widetilde{u}$ to find the correlation $\langle \widetilde{u} \otimes \widetilde{u} \rangle$, a higher order correlation is involved.

One possibility would be to determine only one closure hypothesis, and a reasonable assumption is choosing the Reynolds stress tensor as a function of ∇u. This is the case of the Smagorinsky turbulence model, and the k-model proposed by Prandtl [79] for boundary layers. For many applications, this may be too simple and a more general form of the Reynolds tensor is needed. This will be the case of the k-ε models, introduced by Launder and Spalding (see [83]), which assume that the stress tensor is a function of ∇u, and also of the *average turbulent kinetic energy* k and of the *rate of dissipation of turbulent energy* by the small eddies ε. In this way, the mean velocity satisfies the Reynolds equation (V.1) together with the Reynolds hypothesis

$$\nabla \cdot \langle \widetilde{u} \otimes \widetilde{u} \rangle \sim -\nabla \cdot \left(\frac{k^2}{\varepsilon} (\nabla \langle u \rangle + \nabla \langle u \rangle^T) \right).$$

The scalars k and ε are defined as

$$k = \frac{1}{2} |\widetilde{u}|^2 \quad \text{and} \quad \varepsilon = \frac{\nu}{2} \langle |\nabla \widetilde{u} + \nabla \widetilde{u}^T|^2 \rangle.$$

One can find a relation between k and ε through the scalar function ω, the so-called *mean frequency of the turbulent fluctuations*:

$$\varepsilon = k\,\omega.$$

In this way, we can equivalently define k-ω models. The function

$$L = \frac{\sqrt{k}}{\omega}$$

is known as the external length scale and represents the size of the largest eddies. It will play an important role for the study of the turbulence.

With this brief introduction in mind, we introduce the PDE system for the Kolmogorov two-equation model of turbulence derived in [52] (see also [83] for the English version), which will be our subject of study in this part of the thesis:

$$\begin{cases} \partial_t u \,+\, (u \cdot \nabla) u \,+\, \nabla p \,-\, \nu \operatorname{div} \left(\dfrac{k}{\omega} \mathbb{D} u \right) \,=\, 0 \\ \partial_t \omega \,+\, u \cdot \nabla \omega \,-\, \alpha_1 \operatorname{div} \left(\dfrac{k}{\omega} \nabla \omega \right) \,=\, -\, \alpha_2\, \omega^2 \\ \partial_t k \,+\, u \cdot \nabla k \,-\, \alpha_3 \operatorname{div} \left(\dfrac{k}{\omega} \nabla k \right) \,=\, -\, k\, \omega \,+\, \alpha_4 \, \dfrac{k}{\omega} \left| \mathbb{D} u \right|^2 \\ \operatorname{div} u \,=\, 0\,. \end{cases} \tag{V.2}$$

In the model, the function $\nu \frac{k}{\omega}$ denotes the kinematic eddy viscosity, while the diffusion coefficients for ω and k are $\alpha_1 \frac{k}{\omega}$ and $\alpha_3 \frac{k}{\omega}$, respectively.

The three unknowns u, ω and k are functions of time and space variables $(t,x) \in \mathbb{R}_+ \times \Omega$, where Ω is a domain in $\mathbb{R}^d$, with $d = 2,3$. The vector field $u \in \mathbb{R}^d$ represents the average velocity of the fluid. It is assumed to be incompressible, whence the last equation in (V.2). Moreover, p is the mean pressure of the fluid. The other two unknowns ω and k are positive scalar functions and they are related to the small scale quantities used to describe the turbulent fluid motion. For simplicity, we assume that no external force is acting on the fluid.

All the parameters $\nu, \alpha_1 \ldots \alpha_4$ are physical adimensional parameters which are strictly positive numbers: we have

$$\nu > 0\,, \qquad \alpha_1\,,\ \alpha_2\,,\ \alpha_3\,,\ \alpha_4\ >\ 0\,.$$

Finally, the symbol $\mathbb{D}$ appearing in the first and third equations stands for the symmetric part of the gradient of u:

$$\mathbb{D}u\ :=\ \frac{1}{2}\Big(Du\,+\,\nabla u\Big)\,,$$

where we have denoted by Du the Jacobian matrix of u and by ∇u its transpose matrix.

We will consider the physical domain to be a periodic box of dimension d in order to avoid incompatibilities with the size of the domain or the interaction with boundaries, since the object of study is the fully-developed turbulence.

V.2 State of the art

The previous works devoted to the mathematical analysis of the system (V.2) are mostly from the last decade.

In [75] (see also [74]), Mielke and Naumann prove the existence of global in time weak solutions to (V.2) in a periodic three dimensional box. Their analysis strongly relies on the condition $\frac{k}{\omega} > 0$, that is, the eddy viscosity is strictly positive, which makes the system fully parabolic. In particular, they assume that the mean turbulent kinetic energy and the mean turbulent frequency are initially away from zero, meaning that

$$0 < \omega_{0,*} \le \omega_0 \quad \text{ and } \quad 0 < k_{0,*} \le k_0.$$

The positivity is preserved by the flow, so that for all times $t > 0$ the previous lower bounds remain valid (see the discussion in Subsection 3.2.1). Let us mention the result by Mielke [73], where the author considers a toy model for the Kolmogorov two-equation model of turbulence.

In the works [54] and [55], Kosewski and Kubica prove local in time well-posedness and global in time well-posedness for small initial data, respectively. Again, the positivity condition for the quantities k and ω is crucial for their analysis. In a very recent result, Kosewski [53] generalizes the local well-posedness result to fractional Sobolev regularity.

The only result dealing with the possible vanishing of k_0 is due to Bulíček and Málek. In [8], they study system (V.2) in a three-dimensional bounded domain with a smooth enough boundary and mixed boundary conditions, so that the turbulent occurs at the Dirichlet part of the boundary. They construct global in time weak solutions though a Galerkin approximation, by means of a finite energy E which relates the velocity u and the turbulent kinetic energy k, so that they reformulate the system in terms of the variables u, ω and E. Eventually, it is possible to obtain a solution to the original system from the reformulated one in a suitable class of weak solutions. Let us note that, rather than being away from zero, the vanishing of the initial mean turbulent kinetic energy is assumed to be controlled by the following logarithmic bound:

$$\log k_0 \in L^1(\Omega).$$

This assumption also propagates for all times $t > 0$.

With these results in mind, it would be interesting for completeness, to extend the local in time well-posedness to the case where k is allowed to vanish. Citing from [75]: *"It would be desirable to develop an existence theory without this condition, because this would allow us to study how the support of k, which is may be called the turbulent region, invades the non-turbulent region where $k \equiv 0$".* According to the previous cite, it could be an approach towards the understanding of a setting where there is a transition from turbulent to non-turbulent flow.

VI Contributions

VI.1 Chapter 3

In this part of the thesis, we continue the study of [25], devoted to the one-dimensional reduction of system (V.2). In that paper, the authors proved well-posedness and blow-up in finite time of solutions at H^m level of regularity, for any integer $m \geq 2$. The proof of the well-posedness result is based on a priori energy estimates and the construction of solutions via a Galerkin process together with a convergence argument. The blow-up mechanism is similar in spirit to that of the Burgers equation, and the proof relies on the possible vanishing of the turbulent kinetic energy k. In [26], the same authors proved a different blow-up mechanism for a toy model of the 1-D reduction of (V.2).

While it is not clear at present whether or not the blow-up results may be extended to higher dimensions, so to the true Kolmogorov model (V.2), in the present paper we generalize the local well-posedness result of [25] in two aspects: first of all, we extend it to the physically relevant situation of two and three-dimensional flows; in addition, we prove well-posedness in optimal Sobolev spaces $H^s(\mathbb{T}^d)$, with $s > 1 + d/2$ and $d = 2, 3$. Here, "optimal" means in terms of hyperbolic regularity, the only one we can hope to propagate, owing to the degeneracy of the system for $k \approx 0$.

The generalization to higher dimensions involves some extra technical difficulties in the computations of a priori estimates, although the spirit of the proof is the same one as in the one-dimensional model. However, the generalization to non-integer regularity requires different techniques with respect to the integer regularity case. In particular, our strategy involves paradifferential calculus, passing though Littlewood-Paley decomposition, in order to tackle the commutator estimates appearing in the high order energy bounds, which are crucial to prove the existence of solutions.

Our local well-posedness result reads as follows:

Theorem 5. Let $s > 1 + d/2$ and $\Omega = \mathbb{T}^d$. Define (u_0, ω_0, k_0) as a triplet of functions under the following hypotheses:

1. $u_0, \omega_0 \in H^s(\Omega), \quad \operatorname{div} u_0 = 0$;
2. there exist two constants such that $0 < \omega_* \le \omega_0 \le \omega^*$;
3. $k_0 \ge 0$ is such that $\beta_0 := \sqrt{k_0} \in H^s(\Omega)$.

Then, there exists a time $T > 0$ such that the system (V.2) equipped with the initial data (u_0, ω_0, k_0), admits a unique solution (u, ω, k) on $[0, T] \times \Omega$ such that

1. the non-negativity of ω and k is propagated, i.e., for any $(t, x) \in [0, T] \times \Omega$, $\omega(t, x) > 0$ and $k(t, x) \ge 0$;
2. the functions u, ω and $\sqrt{k}$ belong to the space
$$L^\infty([0,T]; H^s(\Omega)) \cap \bigcap_{l<s} C([0,T]; H^l(\Omega));$$
3. the gradient of the pressure ∇p belongs to the space
$$L^2([0,T]; H^{s-1}(\Omega)) \cap \bigcap_{l<s} C([0,T]; H^{l-2});$$
4. the functions $\sqrt{\frac{k}{\omega}}\mathbb{D}u$, $\sqrt{\frac{k}{\omega}}\nabla\omega$ and $\sqrt{\frac{k}{\omega}}\nabla(\sqrt{k})$ belong to the space $L^2([0,T]; H^s(\Omega))$;

5. the solution (u, ω, k) is unique in the class

$$\begin{aligned}\mathbb{X}_T(\Omega) := \{(u, \omega, k) \;\Big|\; & \omega, \omega^{-1}, k \in L^\infty([0,T] \times \Omega), \quad \omega > 0, \quad k \geq 0, \\ & u, \omega, \sqrt{k} \in C([0,T]; L^2(\Omega)), \\ & \nabla u, \nabla \omega, \nabla(\sqrt{k}) \in L^\infty([0,T] \times \Omega)\}.\end{aligned}$$

Furthermore, we have the following blow-up criterion:

Theorem 6. We denote $T^* > 0$ as the lifespan of the solution and $\lfloor s \rfloor$ the floor of s.

Then, the solution blows-up at time $T^* < +\infty$ if and only if

$$\int_0^{T^*} \mathcal{F}(s)\ \mathrm{d}s = +\infty, \tag{VI.1}$$

where

$$\begin{aligned}\mathcal{F}(t) = & \left(1 + ||\omega, \beta||_{L^\infty}^2\right)\left(1 + ||\nabla\omega||_{L^\infty}^{2\lfloor s \rfloor}\right) ||\nabla u, \nabla \omega, \nabla \beta||_{L^\infty}^2 \\ & + (1 + ||\beta||_{L^\infty})\left(1 + ||\nabla \omega||_{L^\infty}^{\lfloor s \rfloor}\right) \\ & \times \left(\left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right\|_{L^\infty} + \left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\nabla\omega\right)\right\|_{L^\infty} + \left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\nabla\beta\right)\right\|_{L^\infty}\right).\end{aligned}$$

CHAPTER 3

Local well-posedness in critical Sobolev spaces

This chapter is based on the forthcoming paper [21].

3.1 Littlewood-Paley theory on the torus

We present a summary of some fundamental elements of Littlewood-Paley theory and use them to derive some useful inequalities. We refer to Chapter 2 of [4] for details on the construction in the $\mathbb{R}^d$ setting and to [22] for the adaptation to the case of a d-dimensional periodic box $\mathbb{T}_a^d$, where $a \in \mathbb{R}^d$ (this means that the domain is periodic in space with period equal to $2\pi a_j$ with respect to the j-th component for any $1 \leq j \leq d$).

For simplicity of presentation, we focus here on the case in which all a_j are equal to 1. We denote by $\left|\mathbb{T}^d\right| = \mathcal{L}\left(\mathbb{T}^d\right)$ the Lebesgue measure of the box $\mathbb{T}^d$.

First of all, let us recall that, for a tempered distribution $u \in \mathcal{S}'(\mathbb{T}^d)$, we denote by $\mathcal{F}u = \left(\widehat{u}_k\right)_{k\in\mathbb{Z}^d}$ its Fourier series, so that we have

$$u(x) = \sum_{k\in\mathbb{Z}^d} \widehat{u}_k \, e^{ik\cdot x} , \qquad \text{with} \qquad \widehat{u}_k := \frac{1}{\left|\mathbb{T}^d\right|} \int_{\mathbb{T}^d} u(x)\, e^{-ik\cdot x}\, dx \,.$$

Next, we introduce the so called *Littlewood-Paley decomposition*, based on a non-homogeneous dyadic partition of unity with respect to the Fourier variable. We fix a smooth scalar function φ such that $0 \leq \varphi \leq 1$, φ is even and supported in the ring $\left\{r \in \mathbb{R} \,\middle|\, 5/6 \leq |r| \leq 12/5\right\}$, and such that

$$\forall\, r \in \mathbb{R} \setminus \{0\} , \qquad \sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\, r\right) = 1 \,.$$

Let us define $|D| := (-\Delta)^{1/2}$ as the Fourier multiplier[1] of symbol $|k|$, for $k \in \mathbb{Z}^d$. The dyadic blocks $(\Delta_j)_{j\in\mathbb{Z}}$ are then defined by

$$\forall\, j \in \mathbb{Z} , \qquad \Delta_j u := \varphi(2^{-j}|D|)u = \sum_{k\in\mathbb{Z}^d} \varphi(2^{-j}|k|)\, \widehat{u}_k \, e^{ik\cdot x} \,.$$

Notice that, because we are working on a compactly supported set, one has that eventually, $\Delta_j \equiv 0$ for $j < 0$ negative enough (depending on the size of $\mathbb{T}^d_a$). In addition, one has the following Littlewood-Paley decomposition in $\mathcal{S}'(\mathbb{T}^d)$:

$$\forall\, u \in \mathcal{S}'(\mathbb{T}^d) , \qquad u = \widehat{u}_0 + \sum_{j\in\mathbb{Z}} \Delta_j u \qquad \text{in} \qquad \mathcal{S}'(\mathbb{T}^d) \,.$$

In the decomposition above, $\widehat{u}_0$ stands for the mean value of u on $\mathbb{T}^d$, i.e.,

$$\widehat{u}_0 = \overline{u} = \frac{1}{\left|\mathbb{T}^d\right|} \int_{\mathbb{T}^d} u(x)\, dx \,.$$

It is relevant to note that the Fourier multipliers Δ_j are linear operators which are bounded on L^p for any $p \in [1, +\infty]$, and additionally, their norms are *independent* of j and p.

Now, let us present a version of the classical *Bernstein inequalities* adapted to our functional framework (see Chapter 2 of [4] for a general statement of this result).

[1]Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}(f\, \mathcal{F}u)$.

Lemma 3. There exists a universal constant $C > 0$, only depending on the size of the torus $\mathbb{T}^d$ and on the support of the function φ defined above, such that for any $j \in \mathbb{Z}$, for any $m \in \mathbb{N}$, for any couple (p, q) such that $1 \leq p \leq q \leq +\infty$, and for any smooth enough $u \in \mathcal{S}'(\mathbb{T}^d)$, it holds

$$\left\|\Delta_j u\right\|_{L^q} \leq C\, 2^{jd\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|\Delta_j u\right\|_{L^p}$$

and

$$C^{-m-1}\, 2^{-jm} \left\|\Delta_j u\right\|_{L^p} \leq \left\|D^m \Delta_j u\right\|_{L^p} \leq C^{m+1}\, 2^{jm} \left\|\Delta_j u\right\|_{L^p}.$$

By use of Bernstein inequalities, we prove the following Poincaré-Wirtinger type inequality:

$$\forall\, f \in W^{1,\infty}(\mathbb{T}^d) \ \text{ with } \ \overline{f} = 0\,, \qquad \|f\|_{L^\infty} \lesssim \|f\|_{L^2}^{2/(d+2)}\, \|\nabla f\|_{L^\infty}^{d/(d+2)}\,. \tag{3.1.1}$$

The proof relies on an optimization procedure for the dyadic partition of f and the systematic use of Bernstein inequalities. In particular, for $N \in \mathbb{N}$ to be fixed later, we can estimate

$$\begin{aligned}
\|f\|_{L^\infty}^2 &\leq \sum_{j<0} \|\Delta_j f\|_{L^\infty}^2 + \sum_{j=0}^{N} \|\Delta_j f\|_{L^\infty}^2 + \sum_{j\geq N+1} \|\Delta_j f\|_{L^\infty}^2 \\
&\lesssim \sum_{j<0} 2^{jd}\, \|\Delta_j f\|_{L^2}^2 + \sum_{j=0}^{N} 2^{jd}\, \|\Delta_j f\|_{L^2}^2 + \sum_{j\geq N+1} 2^{-2j} 2^{2j}\, \|\Delta_j f\|_{L^\infty}^2 \\
&\lesssim \left(1 + 2^{Nd}\right) \|f\|_{L^2}^2 + 2^{-2N}\, \|\nabla f\|_{L^\infty}^2.
\end{aligned}$$

Now, we can choose N such that

$$2^{Nd}\, \|f\|_{L^2}^2 \approx 2^{-2N}\, \|\nabla f\|_{L^\infty}^2 \qquad \Longrightarrow \qquad 2^N \approx \left(\frac{\|\nabla f\|_{L^\infty}}{\|f\|_{L^2}}\right)^{2/(d+2)}.$$

Inequality (3.1.1) follows immediately from the previous choice of N.

It is well known that Sobolev spaces $H^s(\mathbb{T}^d)$ for $s \in \mathbb{R}$ are characterized in terms of Littlewood-Paley decomposition (see Section 2.7 of [4]) through the following norm equivalence:

$$\|u\|_{H^s}^2 \sim |\widehat{u}_0|^2 + \sum_{j\in\mathbb{Z}} 2^{2sj}\, \|\Delta_j u\|_{L^2}^2\,. \tag{3.1.2}$$

This characterization involves the low order term $|\widehat{u}_0|^2$, which indeed can be substituted by the square of the L^2 norm, by noticing that $|\widehat{u}_0|^2 \leq ||u||_{L^2}^2$. Thus

$$||u||_{H^s}^2 \sim ||u||_{L^2}^2 + \sum_{j\in\mathbb{Z}} 2^{2sj}\, \|\Delta_j u\|_{L^2}^2. \tag{3.1.3}$$

The high order term is equivalent to the homogeneous Sobolev norm of regularity s

$$\|u\|_{\dot{H}^s}^2 \sim \sum_{j\in\mathbb{Z}} 2^{2sj} \|\Delta_j u\|_{L^2}^2 .$$

In the next section, we will present a priori energy estimates involving the low order and high order terms. In particular, the high order a priori estimates will involve some commutators; the structure described in the following lemma will be present through the estimates.

Lemma 4. Let $s > 0$ and $d = 2, 3$. Let α be a scalar function and u a vector field, both defined over $\mathbb{T}^d$. There exists a constant $c = c(s, d)$ such that

$$\left(\sum_{j\in\mathbb{Z}} 2^{2js} \left|\left|[\Delta_j, u] \cdot \nabla\alpha\right|\right|_{L^2}^2\right)^{1/2} \leq c \left(||\nabla u||_{L^\infty} ||\alpha||_{H^s} + ||\nabla\alpha||_{L^\infty} ||\nabla u||_{H^{s-1}}\right). \tag{3.1.4}$$

In addition, if $\operatorname{div} u = 0$, we have

$$[\Delta_j, u] \cdot \nabla\alpha = \operatorname{div}\left([\Delta_j, u]\, \alpha\right),$$

thus

$$\left(\sum_{j\in\mathbb{Z}} 2^{2js} \left|\left|\operatorname{div}\left([\Delta_j, u]\, \alpha\right)\right|\right|_{L^2}^2\right)^{1/2} \leq c \left(||\nabla u||_{L^\infty} ||\alpha||_{H^s} + ||\nabla\alpha||_{L^\infty} ||\nabla u||_{H^{s-1}}\right). \tag{3.1.5}$$

Proof. Estimates (3.1.4) and (3.1.5) are particular cases of Lemma 2.100 in [4], Chapter 2. We adapt the proof in the previous reference to the geometry of the torus. For that purpose, we show that the estimates do not depend on the mean of u and α. We write

$$u = \widetilde{u} + \hat{u}_0, \quad \alpha = \widetilde{\alpha} + \hat{\alpha}_0,$$

where $\widetilde{u}$ and $\widetilde{\alpha}$ are zero mean functions. We notice that

$$[\Delta_j, u]\cdot\nabla\alpha = [\Delta_j, \widetilde{u} + \hat{u}_0,]\cdot\nabla\left(\widetilde{\alpha} + \hat{\alpha}_0\right) = [\Delta_j, \hat{u}_0]\cdot\nabla\widetilde{\alpha} + [\Delta_j, \widetilde{u}]\cdot\nabla\widetilde{\alpha} = [\Delta_j, \widetilde{u}]\cdot\nabla\widetilde{\alpha}.$$

Therefore, we can assume that functions u and α have zero mean and the proof is straightforward adapted to the torus $\mathbb{T}^d$, since now we can work with the non-homogeneous dyadic decomposition as in the $\mathbb{R}^d$ case. □

Furthermore, the next lemma, which involves commutators estimates, will turn out to be fundamental to close the estimates.

Lemma 5. Let $s > 1+d/2$. Let α and f be scalar functions and u a d-dimensional vector field, all of them defined over $\mathbb{T}^d$, such that they belong to $H^s(\mathbb{T}^d)$.

Assume that

$$\sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^3} \alpha^2 \left|\Delta_j \nabla f\right|^2 \, \mathrm{d}x < +\infty .$$

Then, the product $\alpha \nabla f$ belongs to $H^s(\mathbb{T}^d)$. In addition, one has the following inequalities, stating the "equivalence of norms" up to lower order terms:

$$\begin{aligned} \|\alpha \nabla f\|_{H^s}^2 \lesssim \; & \sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^d} \alpha^2 \left|\Delta_j \nabla f\right|^2 \, \mathrm{d}x \\ & + \left(||\nabla f||_{L^\infty} ||\alpha||_{H^s} + ||\nabla \alpha||_{L^\infty} ||f||_{H^s}\right)^2 , \end{aligned}$$

$$\sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^d} \alpha^2 \left|\Delta_j \nabla f\right|^2 \, \mathrm{d}x \lesssim \|\alpha \nabla f\|_{H^s}^2 + \left(||\nabla f||_{L^\infty} ||\alpha||_{H^s} + ||\nabla \alpha||_{L^\infty} ||f||_{H^s}\right)^2 .$$

Furthermore, if

$$\sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^3} \alpha^2 \left|\Delta_j \mathbb{D} u\right|^2 \, \mathrm{d}x < +\infty ,$$

then, the product $\alpha \, \mathbb{D}u$ belongs to $H^s(\mathbb{T}^d)$. Similarly, it holds

$$\begin{aligned} \|\alpha \, \mathbb{D}u\|_{H^s}^2 \lesssim \; & \sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^d} \alpha^2 \left|\Delta_j \mathbb{D} u\right|^2 \, \mathrm{d}x + \\ & + \left(||\nabla u||_{L^\infty} ||\alpha||_{H^s} + ||\nabla \alpha||_{L^\infty} ||u||_{H^s}\right)^2 , \end{aligned}$$

$$\sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^d} \alpha^2 \left|\Delta_j \nabla u\right|^2 \, \mathrm{d}x \lesssim \|\alpha \, \mathbb{D}u\|_{H^s}^2 + \left(||\nabla u||_{L^\infty} ||\alpha||_{H^s} + ||\nabla \alpha||_{L^\infty} ||u||_{H^s}\right)^2 .$$

Proof. The proof of the previous proposition is based on the dyadic characterization of Sobolev spaces, the equivalence of norms (3.1.2) and the application of Lemma 4. We estimate

$$\begin{aligned} \|\alpha \nabla f\|_{H^s}^2 \lesssim \; & \left(\int_{\mathbb{T}^d} \alpha \left|\nabla f\right| \mathrm{d}x \right)^2 + \sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^d} \alpha^2 \left|\Delta_j \nabla f\right|^2 \, \mathrm{d}x \\ & + \sum_{j\in\mathbb{Z}} 2^{2js} \, ||[\Delta_j, \alpha] \, \nabla f||_{L^2}^2 \\ \lesssim \; & ||\nabla f||_{L^\infty}^2 \, ||\alpha||_{L^2}^2 + \sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^d} \alpha^2 \left|\Delta_j \nabla f\right|^2 \, \mathrm{d}x \\ & + \left(||\nabla f||_{L^\infty} ||\alpha||_{H^s} + ||\nabla \alpha||_{L^\infty} ||f||_{H^s}\right)^2 \\ \lesssim \; & \sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^d} \alpha^2 \left|\Delta_j \nabla f\right|^2 \, \mathrm{d}x + \left(||\nabla f||_{L^\infty} ||\alpha||_{H^s} + ||\nabla \alpha||_{L^\infty} ||f||_{H^s}\right)^2 . \end{aligned}$$

On the other hand,

$$\sum_{j\in\mathbb{Z}} 2^{2js} \int_{\mathbb{T}^d} \alpha^2 \left|\Delta_j \nabla f\right|^2 \, \mathrm{d}x \lesssim \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left|[\alpha, \Delta_j]\, \nabla f\right|\right|_{L^2}^2 + \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left|\Delta_j(\alpha \nabla f)\right|\right|_{L^2}^2$$
$$\lesssim \left(||\nabla f||_{L^\infty} \, ||\alpha||_{H^s} + ||\nabla \alpha||_{L^\infty} \, ||f||_{H^s}\right)^2 + \|\alpha \, \nabla f\|_{H^s}^2 .$$

The second part of the lemma is straightforward taking into account the decomposition

$$\mathbb{D}v = \frac{(\nabla v)^T + \nabla v}{2}.$$

□

Finally, we give the statement of a technical result for the control of Sobolev norms involving products.

Lemma 6. Given $s > 0$, the space $L^\infty(\mathbb{T}^d) \cap H^s(\mathbb{T}^d)$ is an algebra, and a constant C exists such that

$$||uv||_{H^s} \leq C(s) \left(||u||_{L^\infty} \, ||v||_{H^s} + ||u||_{H^s} \, ||v||_{L^\infty}\right).$$

This result is a particular case of Corollary 2.86 of [4] in the $\mathbb{R}^d$ setting. The proof can be easily adapted to the configuration of the torus using the Littlewood-Paley decomposition in this setting.

3.2 *A priori* estimates

We are now in the position of showing *a priori* estimates for system (V.2). As in [25], their derivation is based on a two-step procedure: first of all, we are going to bound the energy norm (more precisely, suitable L^p norms) of the solution $\big(u, \omega, k\big)$; next, we will derive higher order estimates for the dyadic blocks $\Delta_j\big(u, \omega, k\big)$. All together, those bounds will imply the sought control of the Sobolev norm H^s of the solution.

We point out that, in order to carry out the higher order estimates, it will be fundamental to resort to the formulation of the system on the new unknowns $\big(u, \omega, \beta\big)$, where $\beta = \sqrt{k}$.

3.2.1 Estimates coming from the energy

We can derive pointwise lower and upper bounds for the functions ω and k following the ODE analysis as in Section 3.1 of [25]. We define the quantities

$$\omega_* := \min_{x\in\Omega} \omega_0(x), \quad \omega^* := \max_{x\in\Omega} \omega_0(x), \quad k_* := \min_{x\in\Omega} k_0(x).$$

Note that, due to the H^s regularity of the solutions for $s > 1 + \frac{d}{2}$, we find that k and ω are Lipschitz continuous, thus differentiable almost everywhere by Rademacher theorem. This fact together with the hypotheses of Theorem 5 allows us to get the following bounds:

$\forall\ (t, x) \in \mathbb{R}_+ \times \Omega$,

$$0 < \omega_{\min}(t) = \frac{\omega_*}{\omega_* \alpha_2 t + 1} \leq \omega(t, x) \leq \frac{\omega^*}{\omega^* \alpha_2 t + 1} = \omega^{\max}(t) \leq \omega^* \tag{3.2.1}$$

and

$$k(t, x) \geq \frac{k_*}{(\omega^* \alpha_2 + 1)^{1/\alpha_2}} = k_{\min}(t) \geq 0. \tag{3.2.2}$$

Now, we perform estimates coming from the energy identities derived from the PDE system. First of all, we multiply the equation for u by u itself and integrate in Ω. We then perform an integration by parts, taking into account the divergence free property for the velocity u, to find the following energy estimate:

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx + \nu \int_\Omega \frac{k}{\omega} |\mathbb{D} u|^2 dx = 0.$$

Integrating in time, we find that

$$||u||^2_{L^2} + 2\nu \int_0^t \int_\Omega \frac{k}{\omega} |\mathbb{D} u|^2 \, \mathrm{d}x\mathrm{d}s \leq ||u_0||^2_{L^2}. \tag{3.2.3}$$

We perform the same computations to the equation for ω, getting

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\omega|^2 \, \mathrm{d}x + \alpha_1 \int_\Omega \frac{k}{\omega} |\nabla \omega|^2 dx + \alpha_2 \int_\Omega \omega^3 \, \mathrm{d}x = 0.$$

Integrating in time, we have

$$||\omega||^2_{L^2} + 2\alpha_1 \int_0^t \int_\Omega \frac{k}{\omega} |\nabla \omega|^2 \, \mathrm{d}x\mathrm{d}s + 2\alpha_2 \int_0^t \int_\Omega \omega^3 \, \mathrm{d}x\mathrm{d}s \leq ||\omega_0||^2_{L^2}. \tag{3.2.4}$$

Let us turn to the equation for k. We cannot expect to get estimates in L^2 for this variable, due to the presence of a term in the right-hand side of the third equation in (V.2) which is merely in L^1. We perform an estimate in L^1, getting

$$\frac{d}{dt} \int_\Omega k \, \mathrm{d}x + \int_\Omega k\omega \, \mathrm{d}x = \alpha_4 \int_\Omega \frac{k}{\omega} |\mathbb{D} u|^2 \, \mathrm{d}x.$$

We then integrate in time and note that the right-hand side is uniformly bounded by (3.2.3). Thus,

$$||k||_{L^1} + \int_0^t \int_\Omega k\omega \, \mathrm{d}x \leq \frac{\alpha_4}{2\nu} ||u_0||^2_{L^2} + ||k_0||_{L^1}. \tag{3.2.5}$$

The discussion in [25] suggests to work with the good unknown $\beta := \sqrt{k}$. Note that the previous estimate would be translated into L^2 control of the variable β,

$$||\beta||_{L^2}^2 + \int_0^t \int_\Omega \beta^2 \omega \, \mathrm{d}x \leq \frac{\alpha_4}{2\nu} ||u_0||_{L^2}^2 + ||\beta_0||_{L^2}^2 . \tag{3.2.6}$$

Finally, let us note that, applying divergence to the equation for u, the pressure satisfies the following elliptic problem:

$$-\Delta p = \mathrm{div} \left(\mathrm{div} \left(\frac{k}{\omega} \mathbb{D}u \right) - u \cdot \nabla u \right) . \tag{3.2.7}$$

We get the following a priori L^2 estimate for the gradient of the pressure:

$$||\nabla p||_{L^2} \leq \left\| \mathrm{div} \left(\frac{k}{\omega} \mathbb{D}u \right) - u \cdot \nabla u \right\|_{L^2} \tag{3.2.8}$$

3.2.2 Reformulation of the system and localization

In this section, we reformulate the PDE system (V.2) in the new variables (u, ω, β). To derive a new equation for β, we multiply the third equation in (V.2) by $1/(2\sqrt{k})$ and straightforward computations provide

$$\partial_t \beta \, + \, u \cdot \nabla \beta \, - \, \alpha_3 \, \mathrm{div} \left(\frac{\beta^2}{\omega} \nabla \beta \right) \, = \, -\frac{\beta \, \omega}{2} \, + \, \frac{\alpha_4}{2} \frac{\beta}{\omega} \left| \mathbb{D}u \right|^2 + \alpha_3 \frac{\beta}{\omega} \left| \nabla \beta \right|^2 .$$

We recall the complete PDE system in the new variables

$$\begin{cases} \partial_t u \, + \, (u \cdot \nabla) u \, + \, \nabla p \, - \, \nu \, \mathrm{div} \left(\dfrac{\beta^2}{\omega} \, \mathbb{D}u \right) \, = \, 0 \\ \partial_t \omega \, + \, u \cdot \nabla \omega \, - \, \alpha_1 \, \mathrm{div} \left(\dfrac{\beta^2}{\omega} \, \nabla \omega \right) \, = \, - \, \alpha_2 \, \omega^2 \\ \partial_t \beta \, + \, u \cdot \nabla \beta \, - \, \alpha_3 \, \mathrm{div} \left(\dfrac{\beta^2}{\omega} \, \nabla \beta \right) \, = \, -\dfrac{\beta \, \omega}{2} \, + \, \dfrac{\alpha_4}{2} \, \dfrac{\beta}{\omega} \left| \mathbb{D}u \right|^2 \, + \, \alpha_3 \, \dfrac{\beta}{\omega} \, \left| \nabla \beta \right|^2 \\ \mathrm{div} \, u \, = \, 0 \, . \end{cases} \tag{3.2.9}$$

We will perform the high order Sobolev estimates on this new system. We note that we need to control the possible vanishing of $\beta := \sqrt{k} \geq 0$. For the moment, we will assume that the new system is well-defined and perform formal computations.

In order to tackle high order Sobolev estimates, we will localize the equation by taking the operators Δ_j in the Littlewood-Paley decomposition as shown in Section 3.1. Some commutators will arise from this localization procedure.

We first deal with the equation for u. We apply the operator Δ_j to the equation and as a result, we get

$$\begin{aligned}&\partial_t \Delta_j u \,+\, (u\cdot\nabla)\Delta_j u \,+\, \nabla\Delta_j p \,-\, \nu \text{ div}\left(\frac{\beta^2}{\omega}\,\mathbb{D}\Delta_j u\right)\\ &= -[\Delta_j, u]\nabla u + \nu \text{ div}\left(\left[\Delta_j, \frac{\beta^2}{\omega}\right]\mathbb{D}u\right).\end{aligned}$$

Analogous computations based on the equations for ω and β give

$$\begin{aligned}&\partial_t \Delta_j \omega \,+\, (u\cdot\nabla)\Delta_j \omega \,-\, \alpha_1 \text{ div}\left(\frac{\beta^2}{\omega}\,\nabla\Delta_j \omega\right)\\ &= -\,[\Delta_j, u]\cdot\nabla\omega + \alpha_1 \text{ div}\left(\left[\Delta_j, \frac{\beta^2}{\omega}\right]\nabla\omega\right) -\, \alpha_2\,\Delta_j\omega^2\end{aligned}$$

and

$$\begin{aligned}\partial_t \Delta_j \beta \,+\, (u\cdot\nabla)\Delta_j \beta \,-\, \alpha_3 \text{ div}\left(\frac{\beta^2}{\omega}\,\nabla\Delta_j \beta\right)&\\ = -\,[\Delta_j, u]\cdot\nabla\beta + \alpha_3 \text{ div}\left(\left[\Delta_j, \frac{\beta^2}{\omega}\right]\nabla\beta\right)&\\ -\,\frac{\Delta_j(\beta\,\omega)}{2} \,+\, \frac{\alpha_4}{2}\,\Delta_j\left(\frac{\beta}{\omega}\left|\mathbb{D}u\right|^2\right) + \alpha_3\Delta_j\left(\frac{\beta}{\omega}\,|\nabla\beta|^2\right)&.\end{aligned}$$

3.2.3 Estimates for the localized system

In the following estimates, we generally use the notation $f \lesssim g$ to denote

$$f \le c(\nu, \alpha_1, \alpha_2, \alpha_3, \alpha_4, d, s, \omega_*, \omega^*)g,$$

i.e., where the constant c is a harmless constant depending on the fixed parameters of the system, the dimension of the space $\mathbb{T}^d$, the regularity of the solution s and the lower and upper bound of the variable ω, which are ω_* and ω^*, respectively.

Note that, when it does not cause any ambiguity, we will drop the time dependence from the notation through the estimates for the sake of simplicity. For the same reason, in general, when doing the a priori estimates, we denote the scalar product of two vectors simply as $u\,v := u\cdot v$.

In the following sections, we search for energy estimates for the homogeneous part of the Sobolev norm, which in terms of the Littlewood-Paley decomposition translates into

$$||u||_{\dot H^s} \sim \sum_{j\in\mathbb{Z}} 2^{2js}\,||\Delta_j u||_{L^2}^2\,.$$

High order estimates for u

We test the localized equation for u against $\Delta_j u$ and integrate in Ω, resulting

$$\frac{1}{2}\frac{d}{dt}\left|\left|\Delta_j u\right|\right|_{L^2}^2 + \nu\int_\Omega \frac{\beta^2}{\omega}\left|\mathbb{D}\Delta_j u\right|^2 \, \mathrm{d}x = -\int_\Omega \left([\Delta_j, u]\nabla u\right)\Delta_j u \, \mathrm{d}x + \nu\int_\Omega \mathrm{div}\left(\left[\Delta_j, \frac{\beta^2}{\omega}\right]\mathbb{D}u\right)\Delta_j u \, \mathrm{d}x.$$

We multiply the equation by 2^{2js} and sum over the integers $j \in \mathbb{Z}$ to get the formal identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}} 2^{2js}\left|\left|\Delta_j u\right|\right|_{L^2}^2 + \nu\sum_{j\in\mathbb{Z}} 2^{2js}\left|\left|\frac{\beta}{\sqrt{\omega}}\mathbb{D}(\Delta_j u)\right|\right|_{L^2}^2 \\ = -\sum_{j\in\mathbb{Z}} 2^{2js}\int_\Omega \left([\Delta_j, u]\nabla u\right)\Delta_j u \, \mathrm{d}x + \nu\sum_{j\in\mathbb{Z}} 2^{2js}\int_\Omega \mathrm{div}\left(\left[\Delta_j, \frac{\beta^2}{\omega}\right]\mathbb{D}u\right)\Delta_j u \, \mathrm{d}x.\end{aligned} \tag{3.2.10}$$

Let us observe that the terms on the left side of the identity are composed by the derivative of the homogeneous Sobolev norm of order s and a viscosity term which will be crucial to close our estimates (it will be fundamental to apply Lemma 5).

We now focus on the right hand side of the equation. The first commutator is estimated using Cauchy-Schwarz inequality and Lemma 4 as

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{2js}\int_\Omega \left([\Delta_j, u]\nabla u\right)\Delta_j u \, \mathrm{d}x &\lesssim \sum_{j\in\mathbb{Z}} 2^{2js}\left|\left|[\Delta_j, u]\,\nabla u\right|\right|_{L^2}\left|\left|\Delta_j u\right|\right|_{L^2} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{2js}\left|\left|[\Delta_j, u]\,\nabla u\right|\right|_{L^2}^2\right)^{1/2}\left(\sum_{j\in\mathbb{Z}} 2^{2js}\left|\left|\Delta_j u\right|\right|_{L^2}^2\right)^{1/2} \\ &\lesssim \left|\left|\nabla u\right|\right|_{L^\infty}\left|\left|u\right|\right|_{H^s}^2 .\end{aligned}$$

The second commutator requires a further decomposition. Notice that, if we estimate it directly using Lemma 4, we lose derivatives and the estimate cannot be closed. For this reason, we must find an accurate decomposition which allow us to exploit the viscosity term on the left hand side of the equation. The key for this decomposition is to pull out a factor of $\beta/\sqrt{\omega}$ in order to find a suitable structure to apply Lemma 5.

We have

$$\begin{aligned}
&\nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div}\left(\left[\Delta_j, \frac{\beta^2}{\omega}\right]\mathbb{D}u\right)\Delta_j u \,\mathrm{d}x \\
&\quad = \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div}\left(\frac{\beta}{\sqrt{\omega}}\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\mathbb{D}u + \left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right)\Delta_j u \,\mathrm{d}x \\
&\quad = T_1 + \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div}\left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right)\Delta_j u \,\mathrm{d}x
\end{aligned}$$

The first term in the divergence (T1) has a good structure to be estimated using Lemma 5. Note that, after integrating by parts, we find a suitable commutator structure multiplied by $\frac{\beta}{\sqrt{\omega}}\mathbb{D}(\Delta_j u)$. The latter structure is present in the localized equation as a viscous term, hence the idea is to absorb it with the left-hand side of the equation. The remaining term needs further decomposition. Straightforward computations give

$$\begin{aligned}
&\nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div}\left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right)\Delta_j u \,\mathrm{d}x \\
&\quad = \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\operatorname{div}\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right) + \left[\Delta_j, \nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right]\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right)\Delta_j u \,\mathrm{d}x \\
&\quad = T_2 + \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right]\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right)\Delta_j u \,\mathrm{d}x
\end{aligned}$$

At this point, we can estimate (T2) using analogous techniques as in the previous term, but we cannot close an estimate for the last part without losing derivatives. For this reason, we decompose

$$\begin{aligned}
&\nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right]\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right)\Delta_j u \,\mathrm{d}x \\
&\quad = \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right]\mathbb{D}u + \nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\left[\frac{\beta}{\sqrt{\omega}}, \Delta_j\right]\mathbb{D}u\right)\Delta_j u \,\mathrm{d}x \\
&\quad = T_3 + T_4,
\end{aligned}$$

where now, we have suitable structures to apply the commutator estimates.

In conclusion, we have

$$\nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div}\left(\left[\Delta_j, \frac{\beta^2}{\omega}\right]\mathbb{D}u\right)\Delta_j u \,\mathrm{d}x = T_1 + T_2 + T_3 + T_4, \tag{3.2.11}$$

where

$$T_1 = \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div}\left(\frac{\beta}{\sqrt{\omega}}\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\mathbb{D}u\right)\Delta_j u \,\mathrm{d}x,$$

$$T_2 = \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\operatorname{div}\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right)\Delta_j u \,\mathrm{d}x,$$

$$T_3 = \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right]\mathbb{D}u\right)\Delta_j u \,\mathrm{d}x$$

and

$$T_4 = \nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\left(\left[\frac{\beta}{\sqrt{\omega}}, \Delta_j\right]\mathbb{D}u\right)\Delta_j u \,\mathrm{d}x.$$

We find that, integrating by parts,

$$T_1 = -\nu \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\frac{\beta}{\sqrt{\omega}}\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\mathbb{D}u\right) : \mathbb{D}(\Delta_j u)\,\mathrm{d}x.$$

Then, by C-S, the weighted Young inequality and Lemma 4,

$$\begin{aligned}
T_1 &\lesssim \sum_{j\in\mathbb{Z}} 2^{2js} \left\|\frac{\beta}{\sqrt{\omega}}\mathbb{D}(\Delta_j u)\right\|_{L^2}\left\|\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\mathbb{D}u\right\|_{L^2}\\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{2js}\left(\delta\left\|\frac{\beta}{\sqrt{\omega}}\mathbb{D}(\Delta_j u)\right\|_{L^2}^2 + C(\delta)\left\|\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\mathbb{D}u\right\|_{L^2}^2\right)\\
&\lesssim \delta\sum_{j\in\mathbb{Z}} 2^{2js}\left\|\frac{\beta}{\sqrt{\omega}}\mathbb{D}(\Delta_j u)\right\|_{L^2}^2\\
&\quad + C(\delta)\left(\left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right\|_{L^\infty}||u||_{H^s} + ||\nabla u||_{L^\infty}\left\|\frac{\beta}{\sqrt{\omega}}\right\|_{H^s}\right)^2.
\end{aligned}$$

Notice that, at this point, we can choose $\delta > 0$ small enough in order to absorb the first term by the left hand side of (3.2.10).

The next term is estimated using C-S, weighted Young inequality and Lemma

5, as

$$
\begin{aligned}
T_2 &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \left[\Delta_j, \frac{\beta}{\sqrt{\omega}} \right] \operatorname{div} \left(\frac{\beta}{\sqrt{\omega}} \mathbb{D}u \right) \right|\right|_{L^2}^2 \right)^{1/2} \left(\sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \Delta_j u \right|\right|_{L^2}^2 \right)^{1/2} \\
&\lesssim \left(\left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}u \right|\right|_{H^s} \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty} + \left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s} \left|\left| \nabla \left(\mathbb{D}u \frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty} \right) ||u||_{H^s} \\
&\lesssim \delta \left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}u \right|\right|_{H^s}^2 + C(\delta) \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty}^2 ||u||_{H^s}^2 \\
&\quad + \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \mathbb{D}u \right) \right|\right|_{L^\infty} \left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s} ||u||_{H^s} \\
&\lesssim \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right|\right|_{L^2}^2 \\
&\quad + C(\delta) \left(||\nabla u||_{L^\infty}^2 \left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s}^2 + \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty}^2 ||u||_{H^s}^2 \right) \\
&\quad + \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \mathbb{D}u \right) \right|\right|_{L^\infty} \left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s} ||u||_{H^s} .
\end{aligned}
$$

Again, the first term can by absorbed by the viscosity term in (3.2.10) for a small enough $\delta > 0$.

T_3 and T_4 are estimated using C-S and Lemma 4,

$$
T_3 \lesssim \left(\left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right) \right|\right|_{L^\infty} ||u||_{H^s} + \left|\left| \frac{\beta}{\sqrt{\omega}} \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{H^s} ||\nabla u||_{L^\infty} \right) ||u||_{H^s} ,
$$

$$
T_4 \lesssim \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty} ||u||_{H^s} \left(\left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty} ||u||_{H^s} + \left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s} ||\nabla u||_{L^\infty} \right) .
$$

It is relevant to note that the variable $\frac{\beta}{\sqrt{\omega}}$ plays a role through all the estimates. This structure is present in most of the estimates and we can consider $\frac{\beta}{\sqrt{\omega}}$ to be a natural variable for the system. Indeed, this "variable" can be estimated in the regular spaces as

$$
\begin{aligned}
\left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s} &\lesssim (1+t)^{1/2} ||\beta||_{H^s} + (1+t)^{\lfloor s \rfloor + 3/2} (1 + ||\nabla \omega||_{L^\infty}^{\lfloor s \rfloor}) ||\beta||_{L^\infty} ||\omega||_{H^s} , \\
\left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{L^\infty} &\lesssim (1+t)^{1/2} ||\beta||_{L^\infty} , \\
\left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty} &\lesssim (1+t)^{1/2} ||\nabla \beta||_{L^\infty} + (1+t)^{3/2} ||\nabla \omega||_{L^\infty} ||\beta||_{L^\infty} .
\end{aligned}
$$

Above, we have used the estimate

$$\begin{aligned}\left|\left|\frac{1}{\sqrt{\omega}}\right|\right|_{H^s} &= \left|\left|\frac{1}{\sqrt{\omega}}\right|\right|_{L^2} + \left|\left|\frac{\nabla\omega}{\omega^{3/2}}\right|\right|_{\dot{H}^{s-1}} \\ &\lesssim (1+t)^{3/2}\left|\left|\omega\right|\right|_{H^s} + \left|\left|\nabla\omega\right|\right|_{L^\infty}\left|\left|\frac{1}{\omega^{3/2}}\right|\right|_{\dot{H}^{s-1}} \\ &\lesssim (1+t)^{3/2}\left|\left|\omega\right|\right|_{H^s} + \left|\left|\nabla\omega\right|\right|_{L^\infty}\left|\left|\frac{1}{\omega^{3/2}}\right|\right|_{\dot{H}^{\lfloor s\rfloor}} \\ &\lesssim (1+t)^{3/2}\left|\left|\omega\right|\right|_{H^s} + (1+t)^{\lfloor s\rfloor+3/2}(1+\left|\left|\nabla\omega\right|\right|)^{\lfloor s\rfloor}\left|\left|\omega\right|\right|_{H^s} \\ &\lesssim (1+t)^{\lfloor s\rfloor+3/2}(1+\left|\left|\nabla\omega\right|\right|)^{\lfloor s\rfloor}\left|\left|\omega\right|\right|_{H^s},\end{aligned}$$

where $\lfloor s\rfloor$ is the floor of s.

Additionally, we need to control some other terms depending on $\frac{\beta}{\sqrt{\omega}}$ arising in T_3. Firstly, using analogous techniques as in the previous estimates,

$$\begin{aligned}\left|\left|\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\frac{\beta}{\sqrt{\omega}}\right|\right|_{H^s} &= \left|\left|\frac{1}{\sqrt{\omega}}\frac{\beta}{\sqrt{\omega}}\nabla\beta - \frac{1}{2\omega}\frac{\beta}{\sqrt{\omega}}\nabla\omega\right|\right|_{H^s} \\ &\lesssim \left|\left|\frac{1}{\sqrt{\omega}}\right|\right|_{L^\infty}\left|\left|\frac{\beta}{\sqrt{\omega}}\nabla\beta\right|\right|_{H^s} + \left|\left|\frac{1}{2\omega}\right|\right|_{L^\infty}\left|\left|\frac{\beta}{\sqrt{\omega}}\nabla\omega\right|\right|_{H^s} \\ &\quad + \left|\left|\frac{1}{\sqrt{\omega}}\right|\right|_{H^s}\left|\left|\frac{\beta}{\sqrt{\omega}}\nabla\beta\right|\right|_{L^\infty} + \left|\left|\frac{1}{2\omega}\right|\right|_{H^s}\left|\left|\frac{\beta}{\sqrt{\omega}}\nabla\omega\right|\right|_{L^\infty} \\ &\lesssim \delta\sum_{j\in\mathbb{Z}}2^{2js}\left(\left|\left|\frac{\beta}{\sqrt{\omega}}\nabla\Delta_j\beta\right|\right|_{L^2}^2 + \left|\left|\frac{\beta}{\sqrt{\omega}}\nabla\Delta_j\omega\right|\right|_{L^2}^2\right) \\ &\quad + (1+t)^{\lfloor s\rfloor+5/2}\left(1+\left|\left|\beta\right|\right|_{L^\infty}^2 + \left|\left|\nabla\omega\right|\right|_{L^\infty}^2 + \left|\left|\nabla\beta\right|\right|_{L^\infty}^2\right) \\ &\qquad \times\left(1+\left|\left|\nabla\omega\right|\right|_{L^\infty}^{\lfloor s\rfloor}\right)\left(\left|\left|\omega\right|\right|_{H^s} + \left|\left|\beta\right|\right|_{H^s}\right).\end{aligned}$$

We observe that, in order for this norm to be controlled, it will be fundamental to have a viscosity term equivalent to the one for the localized equation of u. Again, we will choose $\delta > 0$ small enough in order to absorb the first term with the help of the localized equations for ω and β.

Secondly, we bound

$$\begin{aligned}\left|\left|\nabla\left(\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\frac{\beta}{\sqrt{\omega}}\right)\right|\right|_{L^\infty} &\lesssim (1+t)^3\left(1+\left|\left|\beta\right|\right|_{L^\infty}^2\right)\left(1+\left|\left|\nabla\omega\right|\right|_{L^\infty}^2 + \left|\left|\nabla\beta\right|\right|_{L^\infty}^2\right) \\ &\quad + (1+t)^{3/2}(1+\left|\left|\beta\right|\right|_{L^\infty})\left(\left|\left|\nabla\left(\frac{\beta}{\sqrt{\omega}}\nabla\beta\right)\right|\right|_{L^\infty} + \left|\left|\nabla\left(\frac{\beta}{\sqrt{\omega}}\nabla\omega\right)\right|\right|_{L^\infty}\right).\end{aligned}$$

Collecting all the estimates, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}} 2^{2js}\,||\Delta_j u||^2_{L^2} \,+\, (\nu\,-c\,\delta)\sum_{j\in\mathbb{Z}} 2^{2js}\left|\left|\frac{\beta}{\sqrt{\omega}}\mathbb{D}(\Delta_j u)\right|\right|^2_{L^2}\\
&\lesssim \delta\sum_{j\in\mathbb{Z}} 2^{2js}\left(\left|\left|\frac{\beta}{\sqrt{\omega}}\nabla(\Delta_j\omega)\right|\right|^2_{L^2}+\left|\left|\frac{\beta}{\sqrt{\omega}}\nabla(\Delta_j\beta)\right|\right|^2_{L^2}\right)\\
&+(1+t)^{2\lfloor s\rfloor+3}\left(1+||\omega||^2_{L^\infty}+||\beta||^2_{L^\infty}\right)\left(1+||\nabla u||^2_{L^\infty}+||\nabla\omega||^2_{L^\infty}+||\nabla\beta||^2_{L^\infty}\right)\\
&\quad\times\left(1+||\nabla\omega||^{2\lfloor s\rfloor}_{L^\infty}\right)\left(||u||^2_{H^s}+||\omega||^2_{H^s}+||\beta||^2_{H^s}\right)\\
&+(1+t)^{\lfloor s\rfloor+3/2}\left(1+||\beta||_{L^\infty}\right)\\
&\quad\times\left(\left|\left|\nabla\left(\frac{\beta}{\sqrt{\omega}}\mathbb{D}u\right)\right|\right|_{L^\infty}+\left|\left|\nabla\left(\frac{\beta}{\sqrt{\omega}}\nabla\omega\right)\right|\right|_{L^\infty}+\left|\left|\nabla\left(\frac{\beta}{\sqrt{\omega}}\nabla\beta\right)\right|\right|_{L^\infty}\right)\\
&\quad\times\left(1+||\nabla\omega||^{\lfloor s\rfloor}_{L^\infty}\right)\left(||u||^2_{H^s}+||\omega||^2_{H^s}+||\beta||^2_{H^s}\right).
\end{aligned}$$

High order estimates for ω

We test the localized equation for ω against $\Delta_j\omega$, and integrate in Ω, getting

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\,||\Delta_j\omega||^2_{L^2}+\alpha_1\int_\Omega\frac{\beta^2}{\omega}|\nabla(\Delta_j\omega)|^2\,\mathrm{d}x &= -\int_\Omega\left([\Delta_j,u]\,\nabla\omega\right)\Delta_j\omega\,\mathrm{d}x\\
&+\alpha_1\int_\Omega \mathrm{div}\left(\left[\Delta_j,\frac{\beta^2}{\omega}\right]\nabla\omega\right)\Delta_j\omega\,\mathrm{d}x-\,\alpha_2\int_\Omega\Delta_j(\omega^2)\Delta_j\omega\,\mathrm{d}x.
\end{aligned}\tag{3.2.12}$$

Then, we multiply by 2^{2js} and sum over the integers $j\in\mathbb{Z}$ in the same fashion as in the previous estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}} 2^{2js}\,||\Delta_j\omega||^2_{L^2}\,+\,\alpha_1\sum_{j\in\mathbb{Z}} 2^{2js}\left|\left|\frac{\beta}{\sqrt{\omega}}\nabla(\Delta_j\omega)\right|\right|^2_{L^2}=\\
&\quad-\sum_{j\in\mathbb{Z}} 2^{2js}\int_\Omega\left([\Delta_j,u]\,\nabla\omega\right)\Delta_j\omega\,\mathrm{d}x+\alpha_1\sum_{j\in\mathbb{Z}} 2^{2js}\int_\Omega\mathrm{div}\left(\left[\Delta_j,\frac{\beta^2}{\omega}\right]\nabla\omega\right)\Delta_j\omega\,\mathrm{d}x\\
&\quad-\,\alpha_2\sum_{j\in\mathbb{Z}} 2^{2js}\int_\Omega\Delta_j(\omega^2)\Delta_j\omega\,\mathrm{d}x.
\end{aligned}$$

We observe that, in the left hand side of the equation, a viscosity term arises as in the equation for u. In addition, we must control the three terms appearing on the right-hand side of the equation. The third term is a para-product which can be controlled using C-S and Lemma 6 as

$$\left|-\alpha_2\sum_{j\in\mathbb{Z}} 2^{2js}\int_\Omega\Delta_j(\omega^2)\Delta_j\omega\,\mathrm{d}x\right|\lesssim ||\omega||^2_{H^s}\,||\omega||_{L^\infty}.$$

Moreover, the first term is bounded using C-S and Lemma 4 as

$$\left| -\sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left([\Delta_j, u]\, \nabla\omega \right) \Delta_j\omega \,\mathrm{d}x \right| \lesssim ||\omega||_{H^s} \left(||\nabla u||_{L^\infty} ||\omega||_{H^s} + ||\nabla\omega||_{L^\infty} ||u||_{H^s} \right).$$

Finally, the second term requires again a decomposition of the commutator.

$$\alpha_1 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \mathrm{div} \left(\left[\Delta_j, \frac{\beta^2}{\omega} \right] \nabla\omega \right) \Delta_j\omega \,\mathrm{d}x = R_1 + R_2 + R_3 + R_4,$$

where

$$R_1 = \alpha_1 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \mathrm{div} \left(\frac{\beta}{\sqrt{\omega}} \left[\Delta_j, \frac{\beta}{\sqrt{\omega}} \right] \nabla\omega \right) \Delta_j\omega \,\mathrm{d}x,$$

$$R_2 = \alpha_1 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}} \right] \mathrm{div} \left(\frac{\beta}{\sqrt{\omega}} \nabla\omega \right) \right) \Delta_j\omega \,\mathrm{d}x,$$

$$R_3 = \alpha_1 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}} \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right] \nabla\omega \right) \Delta_j\omega \,\mathrm{d}x$$

and

$$R_4 = \alpha_1 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \left(\left[\frac{\beta}{\sqrt{\omega}}, \Delta_j \right] \nabla\omega \right) \Delta_j\omega \,\mathrm{d}x.$$

Notice that these commutators are very similar to the ones arising from the decomposition (3.2.11), hence we omit some details of the computations.

Firstly,

$$\begin{aligned} R_1 \lesssim\ & \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j\omega) \right|\right|_{L^2}^2 \\ & + C(\delta) \left(\left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty} ||\omega||_{H^s} + ||\nabla\omega||_{L^\infty} \left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s} \right)^2. \end{aligned}$$

This time, the first term can be absorbed by the viscosity term in (3.2.12).

We have similar estimates for the rest of the terms:

$$\begin{aligned} R_2 \lesssim\ & \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j\omega) \right|\right|_{L^2}^2 \\ & + C(\delta) \left(||\nabla\omega||_{L^\infty}^2 \left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s}^2 + \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right|\right|_{L^\infty}^2 ||\omega||_{H^s}^2 \right) \\ & + \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \nabla\omega \right) \right|\right|_{L^\infty} \left|\left| \frac{\beta}{\sqrt{\omega}} \right|\right|_{H^s} ||\omega||_{H^s}, \end{aligned}$$

$$R_3 \lesssim \left(\left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right) \right\|_{L^\infty} ||\omega||_{H^s} + \left\| \frac{\beta}{\sqrt{\omega}} \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right\|_{H^s} ||\nabla\omega||_{L^\infty} \right) ||\omega||_{H^s}$$

and

$$R_4 \lesssim \left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right\|_{L^\infty} ||\omega||_{H^s} \left(\left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \right) \right\|_{L^\infty} ||\omega||_{H^s} + \left\| \frac{\beta}{\sqrt{\omega}} \right\|_{H^s} ||\nabla\omega||_{L^\infty} \right).$$

All the norms arising are analogous or have been estimated in the previous subsection.

Collecting all the estimates, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}} 2^{2js} ||\Delta_j\omega||_{L^2}^2 + (\nu - c\,\delta) \sum_{j\in\mathbb{Z}} 2^{2js} \left\| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j\omega) \right\|_{L^2}^2 \\
&\lesssim \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left\| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j\beta) \right\|_{L^2}^2 \\
&+ (1+t)^{2\lfloor s\rfloor+3} \left(1 + ||\omega||_{L^\infty}^2 + ||\beta||_{L^\infty}^2\right) \left(1 + ||\nabla u||_{L^\infty}^2 + ||\nabla\omega||_{L^\infty}^2 + ||\nabla\beta||_{L^\infty}^2\right) \\
&\quad \times \left(1 + ||\nabla\omega||_{L^\infty}^{2\lfloor s\rfloor}\right) \left(||u||_{H^s}^2 + ||\omega||_{H^s}^2 + ||\beta||_{H^s}^2\right) \\
&+ (1+t)^{\lfloor s\rfloor+3/2} \left(1 + ||\beta||_{L^\infty}\right) \left(\left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \nabla\omega \right) \right\|_{L^\infty} + \left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \nabla\beta \right) \right\|_{L^\infty} \right) \\
&\quad \times \left(1 + ||\nabla\omega||_{L^\infty}^{\lfloor s\rfloor}\right) \left(||u||_{H^s}^2 + ||\omega||_{H^s}^2 + ||\beta||_{H^s}^2\right).
\end{aligned}$$

High order estimates for β

We test the localized equation for β against $\Delta_j\beta$ and integrate in Ω, getting

$$\begin{aligned}
&\frac{d}{dt}\sum_{j\in\mathbb{Z}} 2^{2js} ||\Delta_j\beta||_{L^2}^2 + \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \left\| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j\beta) \right\|_{L^2}^2 \\
&\qquad = -\sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left([\Delta_j, u]\,\nabla\beta\right) \Delta_j\beta \,\mathrm{d}x \\
&\quad + \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div} \left(\left[\Delta_j, \frac{\beta^2}{\omega}\right] \nabla\beta \right) \Delta_j\beta \,\mathrm{d}x - \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \frac{\Delta_j(\beta\,\omega)}{2} \Delta_j\beta \,\mathrm{d}x \\
&\quad + \frac{\alpha_4}{2} \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \Delta_j \left(\frac{\beta}{\omega} \left|\mathbb{D}u\right|^2 \right) \Delta_j\beta \,\mathrm{d}x + \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \Delta_j \left(\frac{\beta}{\omega} |\nabla\beta|^2 \right) \Delta_j\beta \,\mathrm{d}x.
\end{aligned}$$

This time, we have some additional terms appearing in the right hand side. The first term is controlled using C-S and Lemma 4 as

$$\left| -\sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left([\Delta_j, u]\,\nabla\beta\right) \Delta_j\beta \,\mathrm{d}x \right| \lesssim \left(||\nabla u||_{L^\infty} ||\beta||_{H^s} + ||\nabla\beta||_{L^\infty} ||u||_{H^s}\right) ||\beta||_{H^s}.$$

The second term is decomposed analogously as in the previous estimates for u and ω. We decompose it as

$$\alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div}\left(\left[\Delta_j, \frac{\beta^2}{\omega}\right]\nabla\beta\right)\Delta_j\beta \,\mathrm{d}x = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 = \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \operatorname{div}\left(\frac{\beta}{\sqrt{\omega}}\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\nabla\beta\right)\Delta_j\beta \,\mathrm{d}x,$$

$$S_2 = \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\right]\operatorname{div}\left(\frac{\beta}{\sqrt{\omega}}\nabla\beta\right)\right)\Delta_j\beta \,\mathrm{d}x,$$

$$S_3 = \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\sqrt{\omega}}\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right]\nabla\beta\right)\Delta_j\beta \,\mathrm{d}x$$

and

$$S_4 = \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\left(\left[\frac{\beta}{\sqrt{\omega}}, \Delta_j\right]\nabla\beta\right)\Delta_j\beta \,\mathrm{d}x.$$

They are estimated as

$$\begin{aligned} S_1 \lesssim{}& \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left\|\frac{\beta}{\sqrt{\omega}}\nabla\Delta_j\beta\right\|_{L^2}^2 \\ &+ C(\delta)\left(\left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right\|_{L^\infty}||\beta||_{H^s} + ||\nabla\beta||_{L^\infty}\left\|\frac{\beta}{\sqrt{\omega}}\right\|_{H^s}\right)^2, \end{aligned}$$

$$\begin{aligned} S_2 \lesssim{}& \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left\|\frac{\beta}{\sqrt{\omega}}\nabla(\Delta_j\beta)\right\|_{L^2}^2 \\ &+ C(\delta)\left(||\nabla\beta||_{L^\infty}^2\left\|\frac{\beta}{\sqrt{\omega}}\right\|_{H^s}^2 + \left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right\|_{L^\infty}^2||\beta||_{H^s}^2\right) \\ &+ \left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\nabla\beta\right)\right\|_{L^\infty}\left\|\frac{\beta}{\sqrt{\omega}}\right\|_{H^s}||\beta||_{H^s}, \end{aligned}$$

$$S_3 \lesssim \left(\left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right)\right\|_{L^\infty}||\beta||_{H^s} + \left\|\frac{\beta}{\sqrt{\omega}}\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right\|_{H^s}||\nabla\beta||_{L^\infty}\right)||\beta||_{H^s}$$

and

$$S_4 \lesssim \left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right\|_{L^\infty}||\beta||_{H^s}\left(\left\|\nabla\left(\frac{\beta}{\sqrt{\omega}}\right)\right\|_{L^\infty}||\beta||_{H^s} + \left\|\frac{\beta}{\sqrt{\omega}}\right\|_{H^s}||\nabla\beta||_{L^\infty}\right).$$

The third term is estimated using C-S and Lemma 6 as

$$\left| -\sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \frac{\Delta_j(\beta\,\omega)}{2} \Delta_j\beta \,\mathrm{d}x \right| \lesssim \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left|\Delta_j(\beta\omega)\right|\right|_{L^2} \left|\left|\Delta_j\beta\right|\right|_{L^2}$$

$$\lesssim \left|\left|\beta\right|\right|_{H^s} \left(\left|\left|\beta\right|\right|_{L^\infty} \left|\left|\omega\right|\right|_{H^s} + \left|\left|\omega\right|\right|_{L^\infty} \left|\left|\beta\right|\right|_{H^s}\right).$$

The fourth term also needs a further decomposition in order to close the estimate. We decompose it as

$$\frac{\alpha_4}{2} \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \Delta_j \left(\frac{\beta}{\omega} \left|\mathbb{D}u\right|^2 \right) \Delta_j\beta \,\mathrm{d}x = S_5 + S_6,$$

with

$$S_5 = \frac{\alpha_4}{2} \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \frac{\beta}{\omega} \mathbb{D}u : \Delta_j(\mathbb{D}u) \Delta_j\beta \,\mathrm{d}x$$

and

$$S_6 = \frac{\alpha_4}{2} \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\omega}\mathbb{D}u \right] : \mathbb{D}u \right) \Delta_j\beta \,\mathrm{d}x.$$

Firstly, we use weighted Young inequality to estimate S_5:

$$\begin{aligned} S_5 &\lesssim \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left|\Delta_j\beta\right|\right|_{L^2} \left|\left| \frac{\mathbb{D}u}{\sqrt{\omega}} \right|\right|_{L^\infty} \left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right|\right|_{L^2} \\ &\lesssim C(\delta) \left|\left| \frac{\mathbb{D}u}{\sqrt{\omega}} \right|\right|_{L^\infty}^2 \left|\left|\beta\right|\right|_{H^s}^2 + \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right|\right|_{L^2}^2 \\ &\lesssim \frac{C(\delta)}{\omega_{\min}} \left|\left|\nabla u\right|\right|_{L^\infty}^2 \left|\left|\beta\right|\right|_{H^s}^2 + \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right|\right|_{L^2}^2 \\ &\lesssim C(\delta)(1+t) \left|\left|\nabla u\right|\right|_{L^\infty}^2 \left|\left|\beta\right|\right|_{H^s}^2 + \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right|\right|_{L^2}^2. \end{aligned}$$

The term S_6 is estimated by using C-S and Lemma 4. Afterwards, me must decompose the term $\left|\left|\frac{\beta}{\omega}\mathbb{D}u\right|\right|_{H^s}$ in a suitable way to be able to apply Lemma 5 and close the estimate.

$$
\begin{aligned}
S_6 &\lesssim ||\beta||_{H^s} \left(\left|\left| \nabla \left(\frac{\beta}{\omega} \mathbb{D}u \right) \right|\right|_{L^\infty} ||u||_{H^s} + ||\nabla u||_{L^\infty} \left|\left| \frac{\beta}{\omega} \mathbb{D}u \right|\right|_{H^s} \right) \\
&\lesssim \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right|\right|_{L^2}^2 \\
&+ (1+t)^{\lfloor s \rfloor + 2} \left(1 + ||\omega||_{L^\infty}^2 + ||\beta||_{L^\infty}^2 \right) \left(1 + ||\nabla u||_{L^\infty}^2 + ||\nabla \omega||_{L^\infty}^2 + ||\nabla \beta||_{L^\infty}^2 \right) \\
&\quad \times \left(1 + ||\nabla \omega||_{L^\infty}^{\lfloor s \rfloor} \right) \left(||u||_{H^s}^2 + ||\omega||_{H^s}^2 + ||\beta||_{H^s}^2 \right) \\
&+ (1+t)^{1/2} \left|\left| \nabla \left(\frac{\beta}{\sqrt{\omega}} \mathbb{D}u \right) \right|\right|_{L^\infty} \left(||u||_{H^s}^2 + ||\beta||_{H^s}^2 \right).
\end{aligned}
$$

Finally, the fifth term is decomposed as follows:

$$
\alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \Delta_j \left(\frac{\beta}{\omega} |\nabla \beta|^2 \right) \Delta_j \beta \, \mathrm{d}x = S_7 + S_8,
$$

with

$$
S_7 = \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \frac{\beta}{\omega} \nabla(\Delta_j \beta) \nabla \beta \Delta_j \beta \, \mathrm{d}x
$$

and

$$
S_8 = \alpha_3 \sum_{j\in\mathbb{Z}} 2^{2js} \int_\Omega \left(\left[\Delta_j, \frac{\beta}{\omega} \nabla \beta \right] \nabla \beta \right) \Delta_j \beta \, \mathrm{d}x.
$$

They are estimated in a similar fashion as S_5 and S_6:

$$
\begin{aligned}
S_7 &\lesssim \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\nabla \beta}{\sqrt{\omega}} \right|\right|_{L^\infty} ||\Delta_j \beta||_{L^2} \left|\left| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j \beta) \right|\right|_{L^2} \\
&\lesssim \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j \beta) \right|\right|_{L^2}^2 + C(\delta) \frac{1}{\omega_{\min}} ||\nabla \beta||_{L^\infty}^2 ||\beta||_{H^s}^2 \\
&\lesssim \delta \sum_{j\in\mathbb{Z}} 2^{2js} \left|\left| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j \beta) \right|\right|_{L^2}^2 + C(\delta)(1+t) ||\nabla \beta||_{L^\infty}^2 ||\beta||_{H^s}^2 .
\end{aligned}
$$

and

$$
\begin{aligned}
S_8 &\lesssim ||\beta||_{H^s} \left(\left\| \nabla \left(\frac{\beta}{\omega} \nabla \beta \right) \right\|_{L^\infty} ||\beta||_{H^s} + ||\nabla \beta||_{L^\infty} \left\| \frac{\beta}{\omega} \nabla \beta \right\|_{H^s} \right) \\
&\lesssim \delta \sum_{j \in \mathbb{Z}} 2^{2js} \left\| \frac{\beta}{\sqrt{\omega}} \nabla (\Delta_j \beta) \right\|_{L^2}^2 \\
&\quad + (1+t)^{\lfloor s \rfloor + 2} \left(1 + ||\omega||_{L^\infty}^2 + ||\beta||_{L^\infty}^2 \right) \left(1 + ||\nabla \omega||_{L^\infty}^2 + ||\nabla \beta||_{L^\infty}^2 \right) \\
&\qquad \times \left(1 + ||\nabla \omega||_{L^\infty}^{\lfloor s \rfloor} \right) \left(||\omega||_{H^s}^2 + ||\beta||_{H^s}^2 \right) \\
&\quad + (1+t)^{1/2} \left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \mathbb{D} u \right) \right\|_{L^\infty} ||\beta||_{H^s}^2 .
\end{aligned}
$$

Collecting all the estimates, we find

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \sum_{j \in \mathbb{Z}} 2^{2js} ||\Delta_j \beta||_{L^2}^2 + (\nu - c\,\delta) \sum_{j \in \mathbb{Z}} 2^{2js} \left\| \frac{\beta}{\sqrt{\omega}} \nabla (\Delta_j \beta) \right\|_{L^2}^2 \\
&\lesssim \delta \sum_{j \in \mathbb{Z}} 2^{2js} \left(\left\| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right\|_{L^2}^2 + \left\| \frac{\beta}{\sqrt{\omega}} \nabla (\Delta_j \omega) \right\|_{L^2}^2 \right) \\
&+ (1+t)^{2\lfloor s \rfloor + 3} \left(1 + ||\omega||_{L^\infty}^2 + ||\beta||_{L^\infty}^2 \right) \\
&\quad \times \left(1 + ||\nabla u||_{L^\infty}^2 + ||\nabla \omega||_{L^\infty}^2 + ||\nabla \beta||_{L^\infty}^2 \right) \\
&\quad \times \left(1 + ||\nabla \omega||_{L^\infty}^{2\lfloor s \rfloor} \right) \left(||u||_{H^s}^2 + ||\omega||_{H^s}^2 + ||\beta||_{H^s}^2 \right) \\
&+ (1+t)^{\lfloor s \rfloor + 3/2} \left(1 + ||\beta||_{L^\infty} \right) \\
&\quad \times \left(\left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \mathbb{D} u \right) \right\|_{L^\infty} + \left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \nabla \omega \right) \right\|_{L^\infty} + \left\| \nabla \left(\frac{\beta}{\sqrt{\omega}} \nabla \beta \right) \right\|_{L^\infty} \right) \\
&\quad \times \left(1 + ||\nabla \omega||_{L^\infty}^{\lfloor s \rfloor} \right) \left(||u||_{H^s}^2 + ||\omega||_{H^s}^2 + ||\beta||_{H^s}^2 \right) .
\end{aligned}
$$

In the next section, we will join together the estimates for u, ω and β to find uniform energy bounds for the triplet (u, ω, β).

3.2.4 Closing the estimates

We define the energy of the system as

$$
E(t) = ||u(t)||_{H^s}^2 + ||\omega(t)||_{H^s}^2 + ||\beta(t)||_{H^s}^2 . \tag{3.2.13}
$$

Due to the characterization of norms in H^s established in (3.1.3)

$$
E(t) = E_0(t) + E_h(t),
$$

where

$$E_0(t) = ||u(t)||_{L^2}^2 + ||\omega(t)||_{L^2}^2 + ||\beta(t)||_{L^2}^2$$

and

$$E_h(t) = \sum_{j\in\mathbb{Z}} 2^{2js} \left(||\Delta_j u(t)||_{L^2}^2 + ||\Delta_j \omega(t)||_{L^2}^2 + ||\Delta_j \beta(t)||_{L^2}^2 \right).$$

First of all, low order estimates given by (3.2.3), (3.2.4) and (3.2.6) show

$$\forall\, t > 0, \quad E_0(t) \leq c_1\, E_0(0). \tag{3.2.14}$$

On the other hand, the collection of all the high order estimates in Section 3.2.3 derives into

$$\begin{aligned} &\frac{d}{dt} E_h(t) + \sum_{j\in\mathbb{Z}} 2^{2js} \left(\left|\left| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right|\right|_{L^2}^2 + \left|\left| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j \omega) \right|\right|_{L^2}^2 + \left|\left| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j \beta) \right|\right|_{L^2}^2 \right) \\ &\quad \leq c_2\, (1+t)^{2\lfloor s \rfloor + 3}\, F(t)\, (c_1 E_0(0) + E_h(t)), \end{aligned}$$

where

$$\begin{aligned} F(t) = &\left(1 + ||\beta||_{L^\infty}^2\right) \left(1 + ||\nabla\omega||_{L^\infty}^{2\lfloor s \rfloor}\right) \left(1 + ||\nabla u, \nabla\omega, \nabla\beta||_{L^\infty}^2\right) \\ &+ (1 + ||\beta||_{L^\infty}) \left(1 + ||\nabla\omega||_{L^\infty}^{\lfloor s \rfloor}\right) \\ &\times \left(\left|\left| \nabla\left(\frac{\beta}{\sqrt{\omega}} \mathbb{D} u\right) \right|\right|_{L^\infty} + \left|\left| \nabla\left(\frac{\beta}{\sqrt{\omega}} \nabla\omega\right) \right|\right|_{L^\infty} + \left|\left| \nabla\left(\frac{\beta}{\sqrt{\omega}} \nabla\beta\right) \right|\right|_{L^\infty} \right). \end{aligned}$$

We deduce from Grönwall lemma that

$$\forall\, t > 0, \quad E_h(t) + c_1 E_0(0) \leq (E_h(0) + c_1 E_0(0)) \exp\left(c_2 \int_0^t (1+s)^{2\lfloor s \rfloor + 3} F(s)\ \mathrm{d}s \right). \tag{3.2.15}$$

From the definition of the energy in (3.2.13) and the previous inequalities, it holds

$$\forall\, t > 0, \quad E(t) \leq E(0) \exp\left(c \int_0^t (1+s)^{2\lfloor s \rfloor + 3} F(s)\ \mathrm{d}s \right). \tag{3.2.16}$$

We define the time instant $T > 0$ as

$$T := \sup \left\{ t > 0 \,\middle|\, \int_0^t (1+s)^{2\lfloor s \rfloor + 3} F(s) ds \leq 2\log 2 \right\},$$

so that it holds

$$\sup_{t\in[0,T]} E(t) \leq C E(0) \tag{3.2.17}$$

for some constant $C = C(\nu, \alpha_1, \dots, \alpha_4, d, s, \omega_*, \omega^*) > 0$.

Note that the uniform estimates of the energy also provide

$$\int_0^t \sum_{j\in\mathbb{Z}} 2^{2js} \left(\left\| \frac{\beta}{\sqrt{\omega}} \mathbb{D}(\Delta_j u) \right\|_{L^2}^2 + \left\| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j \omega) \right\|_{L^2}^2 + \left\| \frac{\beta}{\sqrt{\omega}} \nabla(\Delta_j \beta) \right\|_{L^2}^2 \right) \mathrm{d}s \lesssim E(0),$$

for $t \in [0, T]$.

At this point, we can develop a blow-up criterion analogously as in [25]. We have that, due to the previous estimates, if the blow-up instant of time is $T^* < \infty$ and

$$\int_0^{T^*} (1+s)^{2\lfloor s\rfloor+3} F(s) ds < \infty,$$

then $E(T^*) < \infty$. Consequently, the solution cannot blow up at time T^* and can be continued into a solution of (V.2) with the same regularity, if uniqueness is granted (this will be proven later, as well as the rigorous existence of the solution). Therefore, the finity of the previous integral provides us with a continuation criterion for our local solution.

Now, we will refine the continuation criterion. Firstly, it is straightforward that one can remove the factor $(1+t)^{2\lfloor s\rfloor+3}$ when T^* is finite. In addition, we can estimate, using (3.1.1),

$$\begin{aligned}\int_0^{T^*} ||\beta||_{L^\infty}^2 \, dt &\lesssim \int_0^{T^*} \left(\widehat{\beta_0}^2 + \|\beta\|_{L^2}^{4/(d+2)} \, \|\nabla\beta\|_{L^\infty}^{2d/(d+2)} \right) \mathrm{d}s \\ &\lesssim ||\beta_0||_{L^2}^2 \, T^* + (1 + ||u_0, \beta_0||_{L^2}) \int_0^{T^*} ||\nabla\beta||_{L^\infty}^{2d/(d+2)} \, \mathrm{d}s.\end{aligned}$$

With this in mind, the equivalent continuation criterion (VI.1) is established.

Furthermore, we can get uniform energy estimates in terms of the initial energy. Owing to the regularity $s > 1 + d/2$ and Sobolev embeddings,

$$\frac{d}{dt} E_h(t) \le C(1 + E(0))(1+t)^{4\lfloor s\rfloor+6}(1 + E_h(t))^{\lfloor s\rfloor+3}$$

with $C > 0$ a uniform constant. Integrating from 0 to t, we get

$$-\frac{1}{(1+E_h(t))^{\lfloor s\rfloor+2}} + \frac{1}{(1+E_h(0))^{\lfloor s\rfloor+2}} \le C(1+E(0)) \left[(1+t)^{4\lfloor s\rfloor+7} - 1 \right],$$

thus

$$E_h(t) \le \frac{1 + E(0)}{\left(1 - C\left[(1+t)^{4\lfloor s\rfloor+7} - 1\right](1+E(0))^{\lfloor s\rfloor+3}\right)^{1/(\lfloor s\rfloor+2)}} \tag{3.2.18}$$

for some constant $C = C(\nu, \alpha_1, \dots, \alpha_4, d, s, \omega_*, \omega^*) > 0$.

3.3 Local existence and uniqueness of solutions

In order to proof Theorem 5, we follow the strategy in [25] for the 1-D model. In the previous reference, the proofs are discussed in detail, so when it is appropriate, we will omit some details and refer the reader to the latter reference. The proof begins by the construction of approximated solutions to system (V.2). Then, we will find a real solution by a convergence argument. Finally, uniqueness of solutions follows from a stability estimate for the system (3.2.9) and the equivalence between the latter system and the original one.

3.3.1 Local existence of solutions

We will deal with sequences of solutions, denoted $(f_\varepsilon)_\varepsilon$. When a sequence is bounded in a normed space X, we will write $(f_\varepsilon)_\varepsilon \subset X$. For simplicity, we will sometimes use the notation $L^p_T(X) := L^p([0,T];X)$.

We begin by removing the degeneration created by the possible vanishing of turbulent kinetic energy k by lifting the initial data: for $0 < \varepsilon < 1$, we define

$$k_{0,\varepsilon} := \left(\sqrt{k_0} + \varepsilon\right)^2.$$

From the initial regularity $\sqrt{k_0} \in H^s(\Omega)$, it is easy to see that

$$(k_{0,\varepsilon})_\varepsilon \in H^s(\Omega), \quad k_{0,\varepsilon} > 0.$$

At this point, for any fixed $0 < \varepsilon < 1$, we can solve the original system (V.2) with respect to the initial data

$$(u_0, \omega_0, k_{0,\varepsilon}) \in H^s(\Omega) \times H^s(\Omega) \times H^s(\Omega),$$

by Theorem 1 in [53]. The solutions $(u_{0,\varepsilon}, \omega_{0,\varepsilon}, k_{0,\varepsilon})$ are constructed via a Galerkin method, hence they are smooth and the a priori estimates are justified for these solutions. Furthermore, some uniform-in-ε properties hold.

The solutions are defined on the common time interval $[0,T]$ thanks to (3.2.18), with

$$T := \inf_{\varepsilon \in (0,1)} T_\varepsilon > 0.$$

Additionally, pointwise bounds described in (3.2.1) and (3.2.2) imply

$$(\omega_\varepsilon)_\varepsilon \subset L^\infty([0,T] \times \Omega), \quad 0 < \omega_\varepsilon(t,x) \le \omega^*$$

and

$$k_\varepsilon(t,x) > 0$$

for all $(t,x) \in [0,T] \times \Omega$. Furthermore, the uniform estimate (3.2.17) gives

$$(u_\varepsilon, \omega_\varepsilon, \sqrt{k_\varepsilon})_\varepsilon \subset L^\infty_T(H^s) \times L^\infty_T(H^s) \times L^\infty_T(H^s)$$

and it is direct in our setting that then, $(k_\varepsilon)_\varepsilon \subset L^\infty_T(H^s)$. We also obtain from the uniform estimates that

$$\left(\sqrt{\frac{k_\varepsilon}{\omega_\varepsilon}}\mathbb{D}u_\varepsilon\right)_\varepsilon, \quad \left(\sqrt{\frac{k_\varepsilon}{\omega_\varepsilon}}\nabla\omega_\varepsilon\right)_\varepsilon, \quad \left(\sqrt{\frac{k_\varepsilon}{\omega_\varepsilon}}\nabla(\sqrt{k_\varepsilon})\right)_\varepsilon \quad \subset \; L^2_T(H^s). \tag{3.3.1}$$

By Banach-Alaoglu theorem, we can extract a weak star convergent subsequence in $L^\infty_T(H^s)$. We will prove that the limit of the mentioned subsequence (u, ω, k) is indeed the solution to our original system (V.2), by a compactness argument.

We recall the equation for u_ε,

$$\partial_t u_\varepsilon \,+\, (u_\varepsilon \cdot \nabla)u_\varepsilon \,+\, \nabla p_\varepsilon \,-\, \nu \,\mathrm{div}\left(\frac{k_\varepsilon}{\omega_\varepsilon}\,\mathbb{D}u_\varepsilon\right) \,=\, 0.$$

Taking the divergence operator to the equation we find the elliptic PDE for the pressure as in (3.2.7). Therefore,

$$\nabla p_\varepsilon = \nabla(-\Delta)^{-1}\,\mathrm{div}\left((u_\varepsilon \cdot \nabla)u_\varepsilon \,-\, \nu\,\mathrm{div}\left(\frac{k_\varepsilon}{\omega_\varepsilon}\,\mathbb{D}u_\varepsilon\right)\right),$$

so, due to the L^2 estimate (3.2.8), the regularity of u_ε and (3.3.1), we find

$$\nabla p \in L^2_T(H^{s-1}),$$

thus

$$(\partial_t u_\varepsilon)_\varepsilon \subset L^2_T(H^{s-1})$$

and

$$(u_\varepsilon)_\varepsilon \subset L^\infty_T(H^s) \cap W^{1,2}_T(H^{s-1}).$$

Ascoli-Arzelà theorem gives the compact inclusion $(u_\varepsilon)_\varepsilon \subset C_T(H^{s-1})$. Then, by interpolation of the previous space and $L^\infty_T(H^s)$, the strong convergence

$$u_\varepsilon \to u \quad \text{in } C_T(H^l) \quad \text{for } 0 \le l < s$$

holds for some subsequence (we abuse notation to avoid relabeling). Equipped with this regularity, we can assure pointwise convergence of u_ε and its first order derivatives

$$u_\varepsilon \to u \quad \text{and} \quad Du_\varepsilon \to Du \quad \text{almost everywhere in } [0,T] \times \Omega. \tag{3.3.2}$$

An analogous analysis can be made for the sequences $(\omega_\varepsilon)_\varepsilon$ and $(k_\varepsilon)_\varepsilon$, finding pointwise convergence for them and their first order derivatives. We conclude that the gradient of the pressure, recovered from the velocity, has regularity

$$\nabla p \in C_T(H^{s-2}).$$

With this analysis, we conclude that the triplet (u, ω, k) is a solution to the original system (V.2). It is not difficult to conclude that the triplet $(u, \omega, \sqrt{k})$ solves (3.2.9) with the same regularity properties.

3.3.2 Uniqueness of solutions

First of all, we can prove that it is equivalent to find a weak solution to the original system (V.2) and to the modified system (3.2.9) in the following functional framework:

$$\mathbb{X}_T(\Omega) := \{(u, \omega, k) \;\Big|\; \omega, \omega^{-1}, k \in L^\infty([0,T] \times \Omega), \quad \omega > 0, \quad k \geq 0, \\ u, \omega, \sqrt{k} \in C([0,T]; L^2(\Omega)), \quad \nabla u, \nabla \omega, \nabla(\sqrt{k}) \in L^\infty([0,T] \times \Omega)\}.$$

The proof follows from Lemma 4.1 in [25].

Secondly, we can state a stability estimate in L^2 for strong solutions to (3.2.9). Once we prove it, uniqueness of solutions to the original system (V.2) follows as a consequence of the equivalence between the two systems.

Theorem 7. We assume that the triplets (u_1, ω_1, β_1) and (u_2, ω_2, β_2) are solutions to (3.2.9). Assume also that, for some time $T > 0$ and $j = 1, 2$, it holds that

$$(u_j, \omega_j, \beta_j) \in \{(u, \omega, \beta) \;\Big|\; \omega, \omega^{-1}, \beta \in L^\infty([0,T]; L^\infty(\Omega)), \quad \omega > 0, \quad \beta \geq 0, \\ \nabla u, \nabla \omega, \nabla \beta \in L^\infty([0,T]; L^\infty(\Omega))\}.$$

Define the difference of the solutions as

$$U := u_1 - u_2, \quad \Sigma := \omega_1 - \omega_2, \quad B := \beta_1 - \beta_2,$$

and assume that $U, \Sigma, B \in C([0,T]; L^2(\Omega))$. Define the energy norm

$$\mathbb{E}(t) := ||U(t)||_{L^2}^2 + ||\Sigma(t)||_{L^2}^2 + ||B(t)||_{L^2}^2 .$$

Then, there exists a constant $C = C(\nu, \alpha_1, \ldots, \alpha_4)$ and a function $\Theta \in L^1([0,T])$ such that, for all $t \in [0,T]$, the following stability estimate holds

$$\mathbb{E}(t) \leq \mathbb{E}(0) \exp\left(C \int_0^t \Theta(\tau) d\tau\right).$$

Proof. The proof follows the same strategy as Theorem 4.2 in [25]. First of all, we find that U solves the symmetric equations

$$\partial_t U + u_1 \nabla U + \nabla(p_1 - p_2) - \nu \operatorname{div}\left(\frac{k_1}{\omega_1}\mathbb{D}U\right) = -U\nabla u_2 + \nu \operatorname{div}(P\,\mathbb{D}u_2) \quad (3.3.3)$$

and

$$\partial_t U + u_2 \nabla U + \nabla(p_2 - p_1) - \nu \operatorname{div}\left(\frac{k_2}{\omega_2}\mathbb{D}U\right) = -U\nabla u_1 + \nu \operatorname{div}(P\,\mathbb{D}u_1) \quad (3.3.4)$$

where P is defined as

$$P := \frac{\beta_1^2}{\omega_1} - \frac{\beta_2^2}{\omega_2} = -\frac{\beta_1^2}{\omega_1\omega_2}\Sigma + \frac{1}{\omega_2}(\beta_1 - \beta_2)B$$

and p_j stands for the pressure associated to the velocity u_j. We perform energy estimates for (3.3.3) and (3.3.4), integrate by parts when it is useful and sum the two symmetric estimates to get

$$\frac{d}{dt}\|U\|_{L^2}^2 + \nu\int_\Omega \left(\frac{\beta_1^2}{\omega_1} + \frac{\beta_2^2}{\omega_2}\right)|\mathbb{D}U|^2\,\mathrm{d}x \le (\|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty})\|U\|_{L^2}^2$$
$$+\nu\left|\int_\Omega \frac{\beta_1^2}{\omega_1\omega_2}\Sigma\,(\mathbb{D}u_1 + \mathbb{D}u_2)\,\mathbb{D}U\,\mathrm{d}x\right| + \nu\left|\int_\Omega \frac{1}{\omega_2}(\beta_1+\beta_2)B\,(\mathbb{D}u_1 + \mathbb{D}u_2)\,\mathbb{D}U\,\mathrm{d}x\right|.$$

We estimate

$$\left|\int_\Omega \frac{\beta_1^2}{\omega_1\omega_2}\Sigma\,(\mathbb{D}u_1 + \mathbb{D}u_2)\,\mathbb{D}U\,\mathrm{d}x\right|$$
$$\le (\|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty})\left\|\frac{\beta_1}{\omega_2\sqrt{\omega_1}}\right\|_{L^\infty}\|\Sigma\|_{L^2}\left\|\frac{\beta_1}{\sqrt{\omega_1}}\mathbb{D}U\right\|_{L^2}$$
$$\le \delta\int_\Omega \frac{\beta_1^2}{\omega_1}|\mathbb{D}U|^2\,\mathrm{d}x + C(\delta)\left(\|\nabla u_1\|_{L^\infty}^2 + \|\nabla u_2\|_{L^\infty}^2\right)\left\|\frac{\beta_1}{\omega_2\sqrt{\omega_1}}\right\|_{L^\infty}^2\|\Sigma\|_{L^2}^2.$$

Moreover, for $j = 1, 2$, it holds

$$\left|\int_\Omega \frac{\beta_j}{\omega_2}B\,(\mathbb{D}u_1 + \mathbb{D}u_2)\,\mathbb{D}U\,\mathrm{d}x\right|$$
$$\le (\|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty})\left\|\frac{\sqrt{\omega_j}}{\omega_2}\right\|_{L^\infty}\|B\|_{L^2}\left\|\frac{\sqrt{\beta_j}}{\sqrt{\omega_j}}\mathbb{D}U\right\|_{L^2}$$
$$\le \delta\int_\Omega \frac{\beta_j^2}{\omega_j}|\mathbb{D}U|^2\,\mathrm{d}x + C(\delta)\left(\|\nabla u_1\|_{L^\infty}^2 + \|\nabla u_2\|_{L^\infty}^2\right)\left\|\frac{\sqrt{\omega_j}}{\omega_2}\right\|_{L^\infty}^2\|B\|_{L^2}^2.$$

Consequently, up to the choice of $\delta > 0$ which will be chosen later to be small enough, we get the estimate

$$\frac{d}{dt}\,||U||^2_{L^2} + (\nu - c\delta)\int_\Omega \left(\frac{\beta_1^2}{\omega_1} + \frac{\beta_2^2}{\omega_2}\right)|\mathbb{D}U|^2 \,\mathrm{d}x \lesssim \Theta_1(t)\mathbb{E}(t), \tag{3.3.5}$$

up to a multiplicative constant only depending on the parameter ν. The functional $\Theta_1(t)$ is defined as

$$\Theta_1(t) := ||\nabla u_1, \nabla u_2||_{L^\infty} + ||\nabla u_1, \nabla u_2||^2_{L^\infty}\left(\left\|\frac{\beta_1}{\omega_2\sqrt{\omega_1}}\right\|^2_{L^\infty} + \left\|\frac{\sqrt{\omega_1}}{\omega_2}\right\|^2_{L^\infty} + \left\|\frac{1}{\omega_2}\right\|_{L^\infty}\right).$$

We move on to the equation for Σ, in order to get a similar L^2 estimate. It is easy to check that Σ solves the symmetric equations

$$\partial_t\Sigma + u_1\nabla\Sigma - \alpha_1 \,\mathrm{div}\left(\frac{k_1}{\omega_1}\nabla\Sigma\right) + \alpha_2(\omega_1+\omega_2)\Sigma = -U\nabla\omega_2 + \alpha_1\,\mathrm{div}\,(P\,\nabla\omega_2)$$

and

$$\partial_t\Sigma + u_2\nabla\Sigma - \alpha_1 \,\mathrm{div}\left(\frac{k_2}{\omega_2}\nabla\Sigma\right) + \alpha_2(\omega_1+\omega_2)\Sigma = -U\nabla\omega_1 + \alpha_1\,\mathrm{div}\,(P\,\nabla\omega_1)\,.$$

At this point, we can perform similar computations to find

$$\frac{d}{dt}\,||\Sigma||^2_{L^2} + (\alpha_1 - c\delta)\int_\Omega\left(\frac{\beta_1^2}{\omega_1} + \frac{\beta_2^2}{\omega_2}\right)|\nabla\Sigma|^2\,\mathrm{d}x + \alpha_2\int_\Omega(\omega_1+\omega_2)|\Sigma|^2\,\mathrm{d}x \lesssim \Theta_2(t)\mathbb{E}(t), \tag{3.3.6}$$

where the multiplicative constant depends only on α_1 and

$$\Theta_2(t) := ||\nabla\omega_1, \nabla\omega_2||_{L^\infty} + ||\nabla\omega_1, \nabla\omega_2||^2_{L^\infty}\left(\left\|\frac{\beta_1}{\omega_2\sqrt{\omega_1}}\right\|^2_{L^\infty} + \left\|\frac{\sqrt{\omega_1}}{\omega_2}\right\|^2_{L^\infty} + \left\|\frac{1}{\omega_2}\right\|_{L^\infty}\right).$$

Finally, we need to find a L^2 estimate for B. We define

$$\widetilde{P} := \frac{\beta_1}{\omega_1} - \frac{\beta_2}{\omega_2} = -\frac{\beta_1}{\omega_1\omega_2}\Sigma + \frac{1}{\omega_2}B$$

to control the extra terms appearing in the equation for β_1, β_2. We find the two symmetric equations for B:

$$\begin{aligned}\partial_t B + u_1\nabla B - \alpha_3\,\mathrm{div}\left(\frac{\beta_1^2}{\omega_1}\nabla B\right) + \frac{1}{2}\omega_1 B = {} & -U\nabla\beta_2 + \alpha_3\,\mathrm{div}\,(P\,\nabla\beta_2) - \frac{1}{2}\Sigma\beta_2 \\ & + \frac{\alpha_4}{2}\frac{\beta_1}{\omega_1}\mathbb{D}U : (\mathbb{D}u_1 + \mathbb{D}u_2) + \frac{\alpha_4}{2}\widetilde{P}|\mathbb{D}u_2|^2 \\ & + \alpha_3\frac{\beta_1}{\omega_1}\nabla B : (\nabla\beta_1 + \nabla\beta_2) + \alpha_3\widetilde{P}|\nabla\beta_2|^2\end{aligned}$$

and

$$\partial_t B + u_2 \nabla B - \alpha_3 \operatorname{div}\left(\frac{\beta_2^2}{\omega_2}\nabla B\right) + \frac{1}{2}\omega_2 B = -U\nabla\beta_1 + \alpha_3 \operatorname{div}(P\,\nabla\beta_1) - \frac{1}{2}\Sigma\beta_1 + \frac{\alpha_4}{2}\frac{\beta_2}{\omega_2}\mathbb{D}U : (\mathbb{D}u_1 + \mathbb{D}u_2) + \frac{\alpha_4}{2}\widetilde{P}|\mathbb{D}u_1|^2 + \alpha_3\frac{\beta_2}{\omega_2}\nabla B : (\nabla\beta_1 + \nabla\beta_2) + \alpha_3\widetilde{P}|\nabla\beta_1|^2.$$

We notice that the first two terms in the right hand side are analogous to the terms appearing in the previous equations. The third term appears in the energy estimate as a contribution of

$$\frac{1}{2}\left|\int_\Omega (\beta_1+\beta_2)\Sigma B\,\mathrm{d}x\right| \lesssim (||\beta_1||_{L^\infty} + ||\beta_2||_{L^\infty})\mathbb{E}(t).$$

The fourth term can be estimated as

$$\left|\int_\Omega \frac{\alpha_4}{2}\frac{\beta_j}{\omega_j}\mathbb{D}U : (\mathbb{D}u_1 + \mathbb{D}u_2)B\,\mathrm{d}x\right| \lesssim \delta\int_\Omega \frac{\beta_j^2}{\omega_j}|\mathbb{D}U|^2\,\mathrm{d}x + C(\delta)\left|\left|\frac{1}{\omega_j}\right|\right|_{L^\infty} ||\nabla u_1, \nabla u_2||_{L^\infty}^2\, ||B||_{L^2}^2\,.$$

The sixth term is similarly estimated as

$$\left|\int_\Omega \alpha_3\frac{\beta_j}{\omega_j}\nabla B : (\nabla\beta_1 + \nabla\beta_2)B\,\mathrm{d}x\right| \lesssim \delta\int_\Omega \frac{\beta_j^2}{\omega_j}|\nabla B|^2\,\mathrm{d}x + C(\delta)\left|\left|\frac{1}{\omega_j}\right|\right|_{L^\infty} ||\nabla \beta_1, \nabla \beta_2||_{L^\infty}^2\, ||B||_{L^2}^2\,.$$

Finally, we can easily control the terms where $\widetilde{P}$ appears. We have

$$\left|\int_\Omega \left(\frac{\alpha_4}{2}\widetilde{P}|\mathbb{D}u_j|^2 + \alpha_3\widetilde{P}|\nabla\beta_j|^2\right)B\,\mathrm{d}x\right| \lesssim ||\nabla u_j, \nabla\beta_j||_{L^\infty}^2\left(\left|\left|\frac{\beta_1}{\omega_1\omega_2}\right|\right|_{L^\infty} + \left|\left|\frac{1}{\omega_2}\right|\right|_{L^\infty}\right)\mathbb{E}(t).$$

Gathering the previous estimates, we extract the energy bound

$$\frac{d}{dt}||B||_{L^2}^2 + (\alpha_3 - c\delta)\int_\Omega\left(\frac{\beta_1^2}{\omega_1} + \frac{\beta_2^2}{\omega_2}\right)|\nabla B|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega(\omega_1+\omega_2)|B|^2\,\mathrm{d}x \lesssim C(\delta)\Theta_3(t)\mathbb{E}(t) + \delta\int_\Omega\left(\frac{\beta_1^2}{\omega_1} + \frac{\beta_2^2}{\omega_2}\right)|\mathbb{D}U|^2\,\mathrm{d}x, \tag{3.3.7}$$

up to a multiplicative constant depending only on α_3, α_4, where

$$\begin{aligned}\Theta_3(t) := &||\nabla u_1, \nabla u_2, \nabla\beta_1, \nabla\beta_2||_{L^\infty} + ||\beta_1, \beta_2||_{L^\infty}\\ &+ ||\nabla u_1, \nabla u_2, \nabla\beta_1, \nabla\beta_2||^2_{L^\infty}\\ &\times\left(\left|\left|\frac{\beta_1}{\sqrt{\omega_1}\omega_2}\right|\right|^2_{L^\infty} + \left|\left|\frac{\sqrt{\omega_1}}{\omega_2}\right|\right|^2_{L^\infty} + \left|\left|\frac{1}{\omega_1}, \frac{1}{\omega_2}\right|\right|_{L^\infty} + \left|\left|\frac{\beta_1}{\omega_1\omega_2}\right|\right|_{L^\infty}\right).\end{aligned}$$

Summing the estimates (3.3.5), (3.3.6) and (3.3.7) and fixing the value of $\delta > 0$ small enough to absorb the extra terms with the left hand side of the inequalities, we find the stability estimate

$$\frac{d}{dt}\mathbb{E}(t) \lesssim (\Theta_1(t) + \Theta_2(t) + \Theta_3(t))\mathbb{E}(t),$$

where the multiplicative constant is such that $C = C(\nu, \alpha_1, \ldots, \alpha_4)$. The application of Grönwall lemma concludes the proof.

□

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