Comment on "A new approach of cooperative interval games: The interval core and Shapley value revisited"

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Abstract

The interval Shapley-like value for cooperative interval games was introduced by W. Han et al. [W. Han, H. Sun, G. Xu, A new approach of cooperative interval games: The interval core and Shapley value revisited, Operations Research Letters 40 (2012) 462–468]. A theorem of characterization of the interval Shapley-like value was provided in that paper. We show that there is an error in the proof of that theorem. We indicate how to avoid this error and complete the proof.

Keywords: Cooperative games; Cooperative interval games; The Shapley value; The interval Shapley-like value

1. Introduction

Cooperative games describe situations in which the cooperation among a set of players gives rise to a profit. Particularly notable are transferable utility games (or TU-games), in which each coalition (subset of players) can obtain a profit that the players in the coalition can freely share. This profit is determined by a real number. But there are situations in which it is not possible to know exactly the worth of each coalition. One of the models developed to deal with these situations is that of cooperative interval games. In this model one knows only a lower bound and an upper bound for the worth of each coalition, and no further probabilistic assumptions can be made. Cooperative interval games were introduced by Branzei et al. [3]. Since then, these games have proved to be a useful tool in the analysis of different problems (see, for instance, [6]). Several solutions for these games have been studied (see [1] and [2]). An important contribution was made by Han et al. [4]. They introduced core-like solutions and Shapley-like values for interval games. In particular, they defined and characterized the so-called interval Shapley-like value. The purpose of this letter is to show that there is an error in [4] in the proof of the theorem of characterization of the

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interval Shapley-like value. We also show how to avoid that error and complete the proof of the theorem.

This letter is organized as follows. In Section 2 the notation and definitions used in [4] are recalled, and some preliminaries concerning TU-games are given. In Section 3 a counter-example is given to show the mistake made in [4]. Moreover, a family of interval games is introduced. In Section 4 we propose some modifications to complete the proof of the theorem of characterization of the interval Shapley-like value.

2. Preliminaries

2.1. TU-games

Let $n \in \mathbb{N}$ and $N = \{1, \ldots, n\}$. A *cooperative transferable utility game* or *TU-game* on N is a function $w : 2^N \to \mathbb{R}$ that satisfies $w(\emptyset) = 0$.

The set of all TU-games on N is denoted by G^N. This set is a $(2^n - 1)$-dimensional real vector space. For each nonempty $T \in 2^N \setminus \{\emptyset\}$ we can consider the game $u_T \in G^N$ defined by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

The game u_T is called the *unanimity game* of T. Shapley [7] proved that the set $\{u_T : T \in 2^N \setminus \{\emptyset\}\}$ is a basis of G^N. Thus, for every game $w \in G^N$ there exist real numbers $\{\Delta_w(T)\}_{T \in 2^N \setminus \{\emptyset\}}$ such that

$$w = \sum_{T \in 2^N \setminus \{\emptyset\}} \Delta_w(T)\, u_T$$

where each coordinate $\Delta_w(T)$ of the game w with respect to the basis of the unanimity games is called *dividend of the coalition* T in the game w (see [5]). The dividends can be obtained recursively:

$$\Delta_w(T) = \begin{cases} w(T) & \text{if } |T| = 1, \\ \\ w(T) - \displaystyle\sum_{\{S \in 2^N \setminus \{\emptyset\} :\, S \subsetneq T\}} \Delta_w(S) & \text{if } |T| > 1. \end{cases}$$

2.2. Interval arithmetic

Let I, J be closed and bounded real intervals with $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$. Let $\alpha \in \mathbb{R}$. Then,

(i) $I \oplus J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$,

(ii) $I \ominus J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$,

(iii) $\alpha I = \begin{cases} [\alpha \underline{I}, \alpha \overline{I}] & \text{if } \alpha \geqslant 0, \\ [\alpha \overline{I}, \alpha \underline{I}] & \text{if } \alpha < 0. \end{cases}$

The set of closed and bounded real intervals will be denoted by $I(\mathbb{R})$.

2.3. Cooperative interval games

Let us recall some of the notations and definitions used in [4].

A *cooperative interval game* is an ordered pair (N, v) where $N = \{1, \ldots, n\}$ with $n \in \mathbb{N}$ and $v : 2^N \to I(\mathbb{R})$ satisfies $v(\emptyset) = [0, 0]$. The set N is called the set of players of (N, v), and v is called the characteristic function of (N, v). For notational simplicity, we will identify a cooperative interval game (N, v) with its characteristic function v. The family of all cooperative interval games with set of players N is denoted by IG^N.

For any $v_1, v_2 \in IG^N$ and $\alpha \in \mathbb{R}$, the interval games $v_1 + v_2, v_1 - v_2, \alpha v_1 \in IG^N$ are defined by

(i) $(v_1 + v_2)(S) = v_1(S) \oplus v_2(S)$ for every $S \in 2^N$,

(ii) $(v_1 - v_2)(S) = v_1(S) \ominus v_2(S)$ for every $S \in 2^N$,

(iii) $(\alpha v_1)(S) = \alpha\, v_1(S)$ for every $S \in 2^N$.

For any closed and bounded interval I, the length of I is denoted by $|I|$. If $v \in IG^N$ then the length game $|v| \in G^N$ is defined by $|v|(S) = |v(S)|$ for every $S \in 2^N$. An interval game $v \in IG^N$ is uncertainty-free if $|v(S)| = 0$ for every $S \in 2^N$. The set of all uncertainty-free interval games in IG^N is denoted by UIG^N. For any $T \in 2^N \setminus \{\emptyset\}$ the interval game $U_T^f \in UIG^N$ is defined by

$$U_T^f(S) = \begin{cases} [1, 1] & \text{if } T \subseteq S, \\ [0, 0] & \text{otherwise.} \end{cases}$$

Two closed and bounded intervals I, J are related by an indifference relationship, denoted by $I \sim J$, if $\frac{\underline{I}+\overline{I}}{2} = \frac{\underline{J}+\overline{J}}{2}$. An interval game $v \in IG^N$ is $[0, 0]$-indifferent if $v(S) \sim [0, 0]$ for every $S \in 2^N$. The set of all $[0, 0]$-indifferent interval games in IG^N is denoted by ZIG^N. For any $T \in 2^N \setminus \{\emptyset\}$ the interval game $U_T^z \in ZIG^N$ is defined by

$$U_T^z(S) = \begin{cases} [-1, 1] & \text{if } T \subseteq S, \\ [0, 0] & \text{otherwise.} \end{cases}$$

2.4. The interval Shapley-like value

An interval value on IG^N is a function $\Psi\colon IG^N \to I(\mathbb{R})^n$. If $v \in IG^N$, the i-th coordinate of the interval vector $\Psi(v)$ represents the payoff of player i in v.

The interval Shapley-like value $\Phi^*\colon IG^N \to I(\mathbb{R})^n$, introduced by Han et al. [4], is defined by

$$\Phi_i^*(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}(v(S \cup \{i\}) \ominus v(S)),$$

for every $v \in IG^N$ and every $i \in N$.

In order to characterize the interval Shapley-like value, Han et al. [4] consider the following properties for a value $\Psi\colon IG^N \to I(\mathbb{R})^n$:

(i) *Indifference Efficiency (IEFF)*: $\sum_{i \in N} \Psi_i(v) \sim v(N)$ for all $v \in IG^N$.

(ii) *Indifference Null Player Property (INP)*: There exists a unique $t \geqslant 0$ such that $\Psi_i(v) = [-t, t]$ for every $v \in IG^N$ and every $i \in N$ null player in v (i.e., $v(S \cup \{i\}) = v(S)$ for all $S \in 2^N$).

(iii) *Symmetry (SYM)*: $\Psi_i(v) = \Psi_j(v)$ for every $v \in IG^N$ and every $i, j \in N$ symmetric players in v (i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$).

(iv) *Additivity (ADD)*: $\Psi_i(v + w) = \Psi_i(v) \oplus \Psi_i(w)$ for every $v, w \in IG^N$ and every $i \in N$.

3. Counter-example

Let $n \in \mathbb{N}$ and $N = \{1, \ldots, n\}$.

In Lemma 3.1 (ii) of [4] it is asserted that $\{U_T^z\}_{T \in 2^N \setminus \{\emptyset\}}$ is a basis for ZIG^N. We prefer not to use the term "basis", since ZIG^N is not a linear space. In any case, in [4] it is assumed that for any $v \in ZIG^N$ there exists a set of real numbers $\{\beta_T^z\}_{T \in 2^N \setminus \{\emptyset\}}$ such that v can be written as

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} \beta_T^z U_T^z.$$

This assertion is not correct. Let us see a counter-example. Let $N = \{1, 2\}$. Consider the interval game $v \in ZIG^N$ defined as

$$v(S) = [-1, 1] \quad \text{for every } S \in 2^N \setminus \{\emptyset\} \text{ and } v(\emptyset) = [0, 0]. \tag{1}$$

Suppose that there exists a set of real numbers $\{\beta_T^z\}_{T \in 2^N \setminus \{\emptyset\}}$ such that v can be written as

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} \beta_T^z U_T^z. \tag{2}$$

Then

$$[-1, 1] = v(\{1\}) = \beta_{\{1\}}^z U_{\{1\}}^z(\{1\}) = \beta_{\{1\}}^z [-1, 1].$$

Therefore, $\beta^z_{\{1\}} \in \{-1, 1\}$. Similarly, $\beta^z_{\{2\}} \in \{-1, 1\}$. But then

$$v(\{1,2\}) = \left(\sum_{T \in 2^N \setminus \{\emptyset\}} \beta^z_T U^z_T \right) (\{1,2\}) = [-1,1] \oplus [-1,1] \oplus \beta^z_{\{1,2\}}[-1,1] = [-2,2] \oplus \beta^z_{\{1,2\}}[-1,1].$$

Therefore, we have that

$$[-1,1] = [-2,2] \oplus \beta^z_{\{1,2\}}[-1,1]$$

which is not possible, since there is not any closed and bounded interval I such that

$$[-1,1] = [-2,2] \oplus I.$$

Therefore, v cannot be written as in equation (2).

Notice that Lemma 3.1 is used in [4] within the proof of Theorem 3.1 when it is asserted that for any $v \in IG^N$ there exists a set of real numbers $\{\beta^k_T\}^{k=f,z}_{T \in 2^N \setminus \{\emptyset\}}$ such that

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} (\beta^f_T U^f_T + \beta^z_T U^z_T). \tag{3}$$

That assertion is not correct. It is clear that we can take the game v defined in (1) as a counter-example. In fact, we can describe the games in IG^N which can be written as in equation (3). To this end, we need to recall the concept of totally positive TU-game, which was introduced in [8]

Definition 1 ([8])**.** *A TU-game $w \in G^N$ is called totally positive if $\Delta_w(T) \geqslant 0$ for every $T \in 2^N \setminus \{\emptyset\}$.*

Proposition 1. *Let $v \in IG^N$. The following statements are equivalent:*

1. *There exists a set of real numbers $\{\beta^k_T\}^{k=f,z}_{T \in 2^N \setminus \{\emptyset\}}$ such that*

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} (\beta^f_T U^f_T + \beta^z_T U^z_T).$$

2. *The TU-game $|v| \in G^N$ is totally positive.*

Proof. (1) $\Longrightarrow$ (2). Let $v \in IG^N$ and let $\{\beta^k_T\}^{k=f,z}_{T \in 2^N \setminus \{\emptyset\}}$ be real numbers such that

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} (\beta^f_T U^f_T + \beta^z_T U^z_T).$$

Notice that, for every closed and bounded intervals I, J, and every $\alpha \in \mathbb{R}$, $|I \oplus J| = |I| + |J|$

and $|\alpha I| = |\alpha||I|$. By these properties, it is clear that, for every $S \in 2^N \setminus \{\emptyset\}$,

$$|v(S)| \;=\; \left|\sum_{T\in 2^N\setminus\{\emptyset\}} (\beta_T^f U_T^f(S) \oplus \beta_T^z U_T^z(S))\right| = \sum_{T\in 2^N\setminus\{\emptyset\}} |\beta_T^z||U_T^z(S)| = \sum_{T\in 2^N\setminus\{\emptyset\}} 2|\beta_T^z|u_T(S).$$

Hence, $\Delta_{|v|}(T) = 2|\beta_T^z|$ for every $T \in 2^N \setminus \{\emptyset\}$. Therefore, $|v|$ is totally positive.

(2) $\Longrightarrow$ (1). Let $v \in IG^N$ be such that $|v|$ is totally positive. Consider $w \in G^N$ defined by

$$w(S) = \frac{\underline{v(S)} + \overline{v(S)}}{2} \qquad \text{for every } S \in 2^N.$$

For any $S \in 2^N \setminus \{\emptyset\}$, we have that

$$\begin{aligned}
v(S) &= [w(S), w(S)] \oplus \frac{1}{2}[-|v(S)|, |v(S)|] \\
&= \left[\sum_{T\in 2^N\setminus\{\emptyset\}} \Delta_w(T)u_T(S), \sum_{T\in 2^N\setminus\{\emptyset\}} \Delta_w(T)u_T(S)\right] \\
&\oplus \frac{1}{2}\left[-\sum_{T\in 2^N\setminus\{\emptyset\}} \Delta_{|v|}(T)u_T(S), \sum_{T\in 2^N\setminus\{\emptyset\}} \Delta_{|v|}(T)u_T(S)\right]
\end{aligned}$$

which, taking into account that $\Delta_{|v|}(T) \geqslant 0$ for every $T \in 2^N \setminus \{\emptyset\}$, is equal to

$$\begin{aligned}
&\sum_{T\in 2^N\setminus\{\emptyset\}} [\Delta_w(T)u_T(S), \Delta_w(T)u_T(S)] \oplus \sum_{T\in 2^N\setminus\{\emptyset\}} \left[-\frac{1}{2}\Delta_{|v|}(T)u_T(S), \frac{1}{2}\Delta_{|v|}(T)u_T(S)\right] \\
= &\sum_{T\in 2^N\setminus\{\emptyset\}} \Delta_w(T)U_T^f(S) \oplus \sum_{T\in 2^N\setminus\{\emptyset\}} \frac{1}{2}\Delta_{|v|}(T)U_T^z(S).
\end{aligned}$$

Therefore,

$$v = \sum_{T\in 2^N\setminus\{\emptyset\}} \left(\Delta_w(T)U_T^f + \frac{1}{2}\Delta_{|v|}(T)U_T^z\right).$$

$\square$

Example 1. *Let $N = \{1, 2, 3\}$. Let v be the interval game in IG^N defined by*

$$\begin{gathered}
v(\{1\}) = [-1, 3], \qquad v(\{2\}) = [-1, 1], \qquad v(\{3\}) = [2, 2], \\
v(\{1, 2\}) = [1, 7], \qquad v(\{1, 3\}) = [1, 9], \qquad v(\{2, 3\}) = [0, 4], \quad v(N) = [2, 14].
\end{gathered}$$

Then,

$$\begin{array}{cccc} & |v|(\{1\}) = 4, & |v|(\{2\}) = 2, & |v|(\{3\}) = 0, \\ |v|(\{1,2\}) = 6, & |v|(\{1,3\}) = 8, & |v|(\{2,3\}) = 4, & |v|(N) = 12, \end{array}$$

from which we easily obtain

$$\begin{array}{cccc} & \Delta_{|v|}(\{1\}) = 4, & \Delta_{|v|}(\{2\}) = 2, & \Delta_{|v|}(\{3\}) = 0, \\ \Delta_{|v|}(\{1,2\}) = 0, & \Delta_{|v|}(\{1,3\}) = 4, & \Delta_{|v|}(\{2,3\}) = 2, & \Delta_{|v|}(N) = 0. \end{array}$$

Notice that $|v|$ is totally positive. Therefore, by Proposition 1, v can be written as in equation (3).
Consider $w \in G^N$ defined by $w(S) = \frac{\underline{v}(S)+\overline{v}(S)}{2}$ for every $S \in 2^N$. We have that

$$\begin{array}{cccc} & w(\{1\}) = 1, & w(\{2\}) = 0, & w(\{3\}) = 2, \\ w(\{1,2\}) = 4, & w(\{1,3\}) = 5, & w(\{2,3\}) = 2, & w(N) = 8. \end{array}$$

It follows that

$$\begin{array}{cccc} & \Delta_w(\{1\}) = 1, & \Delta_w(\{2\}) = 0, & \Delta_w(\{3\}) = 2, \\ \Delta_w(\{1,2\}) = 3, & \Delta_w(\{1,3\}) = 2, & \Delta_w(\{2,3\}) = 0, & \Delta_w(N) = 0. \end{array}$$

Now we use the last equation obtained in the proof of Proposition 1:

$$\begin{aligned} v &= \sum_{T \in 2^N \setminus \{\emptyset\}} \left(\Delta_w(T) U_T^f + \frac{1}{2} \Delta_{|v|}(T) U_T^z \right) \\ &= U_{\{1\}}^f + 2U_{\{3\}}^f + 3U_{\{1,2\}}^f + 2U_{\{1,3\}}^f + 2U_{\{1\}}^z + U_{\{2\}}^z + 2U_{\{1,3\}}^z + U_{\{2,3\}}^z. \end{aligned}$$

4. Proposed correction

Recall that Lemma 3.1 is used in [4] to prove Theorem 3.1, and this theorem is used to prove Theorem 5.1. We have seen that Lemma 3.1 and Theorem 3.1 are not correct. Nevertheless, the result stated in Theorem 5.1 is true. Our goal is to give a proof of this. To this end, we need to introduce a family of interval games.

Definition 2. *An interval game $v \in IG^N$ is called size totally positive if the TU-game $|v| \in G^N$ is totally positive.*

The set of all size totally positive games in IG^N will be denoted by $STPIG^N$.

Proposition 2. *Let $\Psi\colon IG^N \to I(\mathbb{R})^n$ be an interval value. If Ψ satisfies IEFF, INP, SYM and ADD on IG^N then $\Psi_i(v) \sim \Phi_i^*(v)$ for any $v \in STPIG^N$ and $i \in N$.*

Proof. Take $v \in STPIG^N$. By Proposition 1 there exists a set of real numbers $\{\beta_T^k\}_{T\in 2^N\setminus\{\emptyset\}}^{k=f,z}$ such that

$$v = \sum_{T\in 2^N\setminus\{\emptyset\}} (\beta_T^f U_T^f + \beta_T^z U_T^z).$$

Now proceed as in the proof of Theorem 5.1 of [4]. $\square$

Now we are in a position to prove the result stated in Theorem 5.1 of [4].

Theorem 3. *Let $\Psi\colon IG^N \to I(\mathbb{R})^n$ be an interval value. If Ψ satisfies IEFF, INP, SYM and ADD on IG^N then $\Psi_i(v) \sim \Phi_i^*(v)$ for any $v \in IG^N$ and $i \in N$.*

Proof. Let $v \in IG^N$. Consider $g, h \in IG^N$ defined by

$$g = \frac{1}{2} \sum_{T\in 2^N\setminus\{\emptyset\}} \Delta_{|v|}(T) U_T^z \quad \text{and} \quad h = v + g.$$

By Proposition 1 we know that $g \in STPIG^N$. Furthermore, for every $S \in 2^N \setminus \{\emptyset\}$ we have that

$$|g(S)| = \left| \frac{1}{2} \sum_{T\in 2^N\setminus\{\emptyset\}} \Delta_{|v|}(T) U_T^z(S) \right| = \frac{1}{2} \sum_{T\in 2^N\setminus\{\emptyset\}} |\Delta_{|v|}(T)||U_T^z(S)| = \sum_{T\in 2^N\setminus\{\emptyset\}} |\Delta_{|v|}(T)| u_T(S).$$

Therefore,

$$\Delta_{|g|}(T) = |\Delta_{|v|}(T)| \quad \text{for every } T \in 2^N \setminus \{\emptyset\}. \tag{4}$$

From $|h| = |v| + |g|$ and the linearity of the dividends we obtain that

$$\Delta_{|h|}(T) = \Delta_{|v|}(T) + \Delta_{|g|}(T) \quad \text{for every } T \in 2^N \setminus \{\emptyset\}. \tag{5}$$

From (4) and (5) it follows that

$$\Delta_{|h|}(T) = \Delta_{|v|}(T) + |\Delta_{|v|}(T)| \geqslant 0 \quad \text{for every } T \in 2^N \setminus \{\emptyset\}.$$

Hence, $h \in STPIG^N$.
Let $i \in N$. Since $g, h \in STPIG^N$ we know, by Proposition 2, that

$$\Psi_i(g) \sim \Phi_i^*(g) \quad \text{and} \quad \Psi_i(h) \sim \Phi_i^*(h). \tag{6}$$

By ADD we have that

$$\Psi_i(h) = \Psi_i(v) \oplus \Psi_i(g) \quad \text{and} \quad \Phi_i^*(h) = \Phi_i^*(v) \oplus \Phi_i^*(g). \tag{7}$$

From (6) and (7) it is clear that $\Psi_i(v) \sim \Phi_i^*(v)$. $\square$

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