A Shapley distance in graphs

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Abstract

A new distance in finite graphs is defined through a game-theoretic approach. This distance arises when solving the problem about the fair cost, for a node in a graph, of attaining access to another node. The distance indicates the level of difficulty in the communication between any pair of nodes, on the understanding that the fewer paths there are between two nodes and the more nodes there are that form those paths, the greater the distance is.

Keywords: game theory, graph, distance, cooperative TU game, Shapley value.

1. Introduction

Myerson [6] studied cooperative games in situations in which there are limitations on the communication among the players. He used the best-known value for cooperative transferable utility (TU) games, the Shapley value, to define and characterize a value for games with communication restrictions. These restrictions were modeled through graphs. Since then, many studies in game theory have been carried out to deal with situations in which there is a cooperative game and a graph that delimits the communication among the players. This has led several game theorists to consider the study of graphs by using game-theoretic tools. The basic idea is the following. Given a graph, whose nodes will be identified with players, we can consider, instead of an exogenous game, a game determined by the graph itself. If the game is properly chosen, we can obtain useful information about the graph by applying a value to the game. Notable examples of this are the studies on centrality in graphs that have been carried out by means of game-theoretic tools. The first of these studies was carried out by Grofman and Owen [4]. They used one of the most important values studied in cooperative game theory, the Banzhaf value, to study power in social networks, and gave other graph-theoretic applications of this value. Later on, Gómez *et al.* [3], following the approaches considered by Myerson [6] and Owen [7], studied centrality in graphs by means

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of the Shapley value of certain conveniently defined games. Suri and Narahari [10] used the Shapley value to create an algorithm to find what they call influential nodes in a graph. Michalak *et al.* [5] developed efficient algorithms to calculate the Shapley value of the game used by Suri and Narahari and other games useful for the study of centrality in graphs. The Shapley value has also been used to study centrality in directed graphs by del Pozo *et al.* [8]. More recently, Gallardo *et al.* [2] used the Shapley value to study and measure power in hierarchical structures. Other examples of the use of game theory in graph problems can be found in [1] and [11].

Suppose that i and k are two nodes in a finite graph such that they are connected in the graph and there is no edge between them. Let us suppose that the nodes in the graph are agents, and that for agent i the capacity to get in contact with k produces a profit. Our goal is to study how costly attaining access to k is for i, on the assumption that i will have to pay a fair toll to the intermediary nodes between i and k in order to contact k. To this end, we will proceed in a similar way as we explained above, that is, a suitable TU-game in the set of nodes will be defined. In principle, one of the components of the Shapley value of this game would provide the solution to our problem. However, considerations about the possibility of using pivotal nodes will emerge. In order to determine whether the use of pivotal nodes is advantageous or not, we will introduce certain function defined on the set of pair of nodes. The problem of determining whether the strategy of using pivotal nodes is useless or profitable will be equivalent to determining whether this function is a distance on the set of nodes or not. The majority of this paper is devoted to proving that this function is indeed a distance. This will be the central theoretical result of the paper. This result will allow us to solve our initial problem about the cost, for a node in a graph, of attaining access to another node.

The paper is organized as follows. In Section 2, several basic definitions concerning cooperative games are recalled. In Section 3, we propose the graph problem that we aim to solve. Moreover, a function on the set of pair of nodes is introduced. In Section 4, we prove that this function is a distance. Finally, in Section 5, the conclusions are drawn.

2. Preliminaries

2.1. Cooperative TU-games

A *transferable utility cooperative game* or *TU-game* is a pair (N, v) where N is a finite and nonempty set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of N are called players, the subsets $E \subseteq N$ are called coalitions and $v(E)$ is the worth of E. For each coalition E, the worth of E can be interpreted as the maximal gain or minimal cost that the players in this coalition can achieve by themselves. A TU-game (N, v) is often identified with the function v. The family of all the games with set of players N is denoted by $\mathcal{G}^N$. A game $v \in \mathcal{G}^N$ is said to be monotonic if $v(E) \leqslant v(F)$ for every $E \subseteq F \subseteq N$. And v is superadditive if $v(E) + v(F) \leqslant v(E \cup F)$ for every $E, F \subseteq N$ with $E \cap F = \emptyset$.

A payoff vector for a game on the set of players N is a vector $x \in \mathbb{R}^N$. A value on $\mathcal{G}^N$ is a function $\psi : \mathcal{G}^N \longrightarrow \mathbb{R}^N$ that assigns a payoff vector to each game. Numerous values have been defined for several families of games in the literature. The *Shapley value*, introduced in [9], is the most important of these values. The Shapley value $Sh\,(v) \in \mathbb{R}^N$ of a game $v \in \mathcal{G}^N$ is a weighted average of the marginal contributions of each player to the coalitions. Let $\Pi(N)$ denote the set of permutations on N. For every $\sigma \in \Pi(N)$ and $i \in N$, let $P_{\sigma,i}$ denote the set of players in N which precede i with respect to permutation σ, that is, $P_{\sigma,i} = \{j \in N : \sigma(j) < \sigma(i)\}$. Then the Shapley value of $v \in \mathcal{G}^N$ is defined as

$$Sh_i\,(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \left(v(P_{\sigma,i} \cup \{i\}) - v(P_{\sigma,i})\right), \quad \text{for all } i \in N.$$

It is easy to check that

$$Sh_i\,(v) = \sum_{\{E \subseteq N : i \in E\}} p_E^N \left(v\,(E) - v\,(E \setminus \{i\})\right), \quad \text{for all } i \in N, \tag{1}$$

where $p_E^N = \dfrac{(|N| - |E|)!\,(|E| - 1)!}{|N|!}$, for every $E \in 2^N \setminus \{\emptyset\}$.

Some desirable properties for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ are the following:

Efficiency: $\sum_{i \in N} \psi_i\,(v) = v\,(N)$ for all $v \in \mathcal{G}^N$.

Additivity: $\psi\,(v_1 + v_2) = \psi\,(v_1) + \psi\,(v_2)$ for all $v_1, v_2 \in \mathcal{G}^N$.

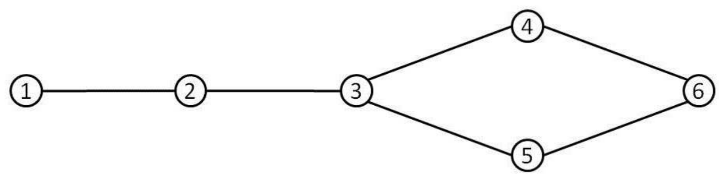
Null player property: If $i \in N$ is a null player in $v \in \mathcal{G}^N$, that is, $v\,(E) = v(E \setminus \{i\})$ for all $E \subseteq N$, then $\psi_i\,(v) = 0$.

Symmetry: If $i, j \in N$ are symmetric with respect to $v \in \mathcal{G}^N$, that is, $v(E \cup \{i\}) = v(E \cup \{j\})$ for every $E \subseteq N \setminus \{i, j\}$, then $\psi_i\,(v) = \psi_j\,(v)$.

The Shapley value is the unique value satisfying efficiency, additivity, null player property and symmetry.

3. Proposed methodology

Throughout this paper, N will denote a finite set. Let $G = (N, L)$ be a graph, where N is the set of nodes and L is the set of edges. Let us suppose that the nodes represent agents that can interact, cooperate and negotiate. Suppose that i, k are two (different) agents in N with $\{i, k\} \notin L$. Suppose that attaining communication with agent k means certain economic profit for agent i. In order to contact k, agent i will need the cooperation of other intermediary agents, who can form a path between i and k. It is reasonable for these intermediaries to demand a part of the profit that i obtains from being in contact with k. Therefore i will have to transfer a percentage of the profit to them. Our goal is to determine the proportion of profit that i will be able to retain, assuming that the profit sharing is done fairly. Notice that, depending on the connection situation of i and k in G, it will be more or less costly for i to get in contact with k. In order to illustrate this, consider the graph represented by the following diagram:



Suppose that 1 obtains a profit from achieving contact with 3, and 3 obtains a profit from achieving contact with 6. In order to contact 3, agent 1 needs the cooperation of 2, in the sense that if agent 2 refuses to cooperate then 1 will not obtain any profit. Therefore, in a fair negotiation between 1 and 2, agent 1 will have to transfer to agent 2 half of the profit that 1 can obtain from attaining access to 3. Agent 3 obtains a profit from achieving contact with 6. In order to receive that profit, agent 3 needs the cooperation of either 4 or 5. Therefore, the position of 3 in negotiation with 4 and 5 is better than the position of 1 in negotiation with 2. Agent 3 will be able to retain more than half of the profit that she/he can obtain from achieving contact with 6.

In order to solve the problem proposed, cooperative game theory will be used. Firstly, we will consider the TU-games which model the cooperative situation described at the beginning of this section.

Definition 1. *Let $G = (N, L)$ be a graph. Let i, k be two (different) nodes in N. We define $A_G^{i,k} : 2^N \to \{0, 1\}$ as*

$$A_G^{i,k}(S) = \begin{cases} 1 & \text{if there exist } j_1, \ldots, j_r \in S \text{ such that } j_1 = i, \{j_q, j_{q+1}\} \in L \\ & \text{for all } q = 1, \ldots, r-1 \text{ and } \{j_r, k\} \in L, \\ & \\ 0 & \text{otherwise,} \end{cases}$$

for any $S \subseteq N$.

Notice that $A_G^{i,k}(S)$ is equal to 1 if and only if $i \in S$ and i can achieve contact with k without leaving S. If we suppose that i obtains a profit equal to 1 from achieving contact with k, then the coalitions which are able to obtain that profit are the coalitions S with $A_G^{i,k}(S) = 1$. Notice that $A_G^{i,k}$ is a $\{0,1\}$-game in N. The Shapley value of this game provides a fair allocation of the profit among i and those agents that can enable i to achieve contact with k. Hence, agent i would finally retain a profit equal to $Sh_i(A_G^{i,k})$. It seems that we have solved the problem proposed at the beginning of this section. However, there is an objection that we explain below.

Suppose we have a graph $G = (N, L)$, and i, j, k three different nodes in N such that $\{i, k\} \notin L$, i and j are connected in G and that j and k are also connected in G. For simplicity we will suppose that i is not in any minimal path between j and k. We assume that i obtains a profit equal to 1 from achieving contact with k. Since $\{i, k\} \notin L$, agent i will have to negotiate with intermediary agents. We know that, as a result of that negotiation, i will have to cede a part of the profit. But now suppose that, instead of negotiating to obtain access to agent k, agent i considers the following strategy. Agent i assigns a part of the profit to agent j. Agent j then negotiates her/his access to agent k. In this negotiation, agent j will have to cede a part of the quantity that has been received from agent i. Meanwhile, i only has to negotiate for access to j. In this negotiation, agent i will have to cede a part of the quantity that she/he retained. Let us calculate the profit that i will be able to retain if she/he follows the strategy described above. Firstly, an amount equal to α will be allocated to j. Agent j will use this amount to negotiate access to k. We know that, as a result of this negotiation, j will be able to keep $\alpha\, Sh_j(A_G^{j,k})$. This is the final profit obtained by j. On the other hand, i has an amount of $1-\alpha$ to negotiate access to j. As a result of this negotiation, i will be able to keep $(1-\alpha)\, Sh_i(A_G^{i,j})$. It now has to be taken into consideration that agent j is essential in this strategy. Therefore, it is fair that agents i and j obtain the same profit. Hence, it must be $\alpha\, Sh_j(A_G^{j,k}) = (1-\alpha)\, Sh_i(A_G^{i,j})$. We obtain

$$\alpha = \frac{Sh_i(A_G^{i,j})}{Sh_i(A_G^{i,j}) + Sh_j(A_G^{j,k})}.$$

Therefore, the final profit that i will obtain is $(1-\alpha)\, Sh_i(A_G^{i,j})$, which is equal to

$$\frac{Sh_i(A_G^{i,j})\, Sh_j(A_G^{j,k})}{Sh_i(A_G^{i,j}) + Sh_j(A_G^{j,k})}. \tag{2}$$

Of course, the question that arises is whether the strategy followed has been profitable for i. Remember that if agent i negotiates with all the intermediaries for access to k then she/he would receive an amount of

$$Sh_i(A_G^{i,k}). \tag{3}$$

Our goal will be to prove that the strategy of using a pivotal agent in the sense explained above is never advantageous, that is, (3) is greater or equal to (2). This will be done by proving that certain function on N^2 is a distance on N. This function is introduced in the following definition.

Definition 2. *Let $G = (N, L)$ be a graph. We define $d_G^{Sh} : N^2 \to [0, +\infty]$ as*

$$d_G^{Sh}(i,k) = \begin{cases} 0 & \text{if } i = k, \\ \dfrac{1}{Sh_i(A_G^{i,k})} & \text{if } i \neq k \text{ and there is a path in } G \text{ connecting } i \text{ and } k, \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that (2) can be rewritten as

$$\frac{1}{\frac{1}{Sh_i(A_G^{i,j})} + \frac{1}{Sh_j(A_G^{j,k})}},$$

which is equal to

$$= \frac{1}{d_G^{Sh}(i,j) + d_G^{Sh}(j,k)}.$$

On the other hand, (3) is equal to

$$\frac{1}{d_G^{Sh}(i,k)}.$$

Therefore, in order to achieve our goal it is enough to prove that

$$d_G^{Sh}(i,k) \leqslant d_G^{Sh}(i,j) + d_G^{Sh}(j,k). \tag{4}$$

Notice that this expression reminds of the triangle inequality of a distance. In fact, this is the case. In the following section it will be shown that d_G^{Sh} is a distance on N. If, for the moment, we assume that result, we have proved (4) and, consequently, the fact that using a pivotal agent is never advantageous. At this point, one could consider a strategy based on using a sequence of pivotal agents. Each one of these pivotal agents would negotiate for access with the next one, except for the last one, who would negotiate for access to k. It can be checked, in a similar way as we have done in the case of one pivotal agent, that following this strategy will not increase the final profit retained by i. Again, the key point is the fact that d_G^{Sh} is a distance on N. Therefore, we have solved the problem proposed at the beginning of this section. We can assert that if attaining communication with agent k means certain economic profit for agent i, then the proportion of this profit that i can expect to retain is equal to $\frac{1}{d_G^{Sh}(i,k)}$.

The following section is devoted entirely to proving that d_G^{Sh} is a distance on N for every graph $G = (N, L)$.

4. Results

In order to prove that d_G^{Sh} is a distance, we need some previous lemmas and propositions. The first lemma is a well-known result in game theory, but, for the sake of completeness, we provide a proof.

Lemma 1. *Let M be a finite and nonempty set. Then*

$$\sum_{\{D \subseteq M \colon E \subseteq D\}} p_D^M = \frac{1}{|E|}$$

for every $E \in 2^M \setminus \{\emptyset\}$.

Proof. Let M be a finite and nonempty set and let $E \in 2^M \setminus \{\emptyset\}$. Consider $u_E \in \mathcal{G}^M$ defined as

$$u_E(D) = \begin{cases} 1 & \text{if } E \subseteq D, \\ 0 & \text{otherwise,} \end{cases}$$

for every $D \subseteq M$.

Take $i \in E$. On the one hand, from the properties of efficiency, null player and symmetry of the Shapley value, it follows that

$$Sh_i(u_E) = \frac{1}{|E|}.$$

On the other hand, we have

$$Sh_i(u_E) = \sum_{\{D \subseteq M \colon i \in D\}} p_D^M \left(u_E(D) - u_E(D \setminus \{i\}) \right) = \sum_{\{D \subseteq M \colon E \subseteq D\}} p_D^M.$$

$\square$

Lemma 2. *Let M, N be finite, nonempty and disjoint sets. Then*

$$\sum_{\{C, D \subseteq M \colon C \cap D = E\}} p_C^M \, p_{D \cup T}^{M \cup N} + \sum_{\{R, S \subseteq N \colon R \cap S = T\}} p_R^N \, p_{E \cup S}^{M \cup N} = p_E^M \, p_T^N$$

for every $E \in 2^M \setminus \{\emptyset\}$ *and* $T \in 2^N \setminus \{\emptyset\}$.

Proof. Let M, N be finite, nonempty and disjoint sets. The equality stated in the lemma will be proved by strong induction on $k = |M| + |N| - |E| - |T|$.

1. Base case. If $k = 0$, that is, $E = M$ and $T = N$, we have

$$p_M^M \, p_{M\cup N}^{M\cup N} + p_N^N \, p_{M\cup N}^{M\cup N} = \frac{1}{|M|}\frac{1}{|M|+|N|} + \frac{1}{|N|}\frac{1}{|M|+|N|} = \frac{1}{|M|}\frac{1}{|N|} = p_M^M \, p_N^N.$$

2. Inductive step. Take $E \in 2^M \setminus \{\emptyset\}$ and $T \in 2^N \setminus \{\emptyset\}$. By induction hypothesis we know that

$$\sum_{\{C,D\subseteq M:\, C\cap D=A\}} p_C^M \, p_{D\cup Z}^{M\cup N} + \sum_{\{R,S\subseteq N:\, R\cap S=Z\}} p_R^N \, p_{A\cup S}^{M\cup N} = p_A^M \, p_Z^N \tag{5}$$

for every $A \in 2^M \setminus \{\emptyset\}$ and $Z \in 2^N \setminus \{\emptyset\}$ with $E \subseteq A$, $T \subseteq Z$ and $(A, Z) \neq (E, T)$. On the one hand, we have

$$\begin{aligned}
& \sum_{\{A\subseteq M,\, Z\subseteq N:\, E\subseteq A,\, T\subseteq Z\}} \left(\sum_{\{C,D\subseteq M:\, C\cap D=A\}} p_C^M \, p_{D\cup Z}^{M\cup N} + \sum_{\{R,S\subseteq N:\, R\cap S=Z\}} p_R^N \, p_{A\cup S}^{M\cup N} \right) \\
= & \sum_{\{A\subseteq M,\, Z\subseteq N:\, E\subseteq A,\, T\subseteq Z\}} \sum_{\{C,D\subseteq M:\, C\cap D=A\}} p_C^M \, p_{D\cup Z}^{M\cup N} \\
& + \sum_{\{A\subseteq M,\, Z\subseteq N:\, E\subseteq A,\, T\subseteq Z\}} \sum_{\{R,S\subseteq N:\, R\cap S=Z\}} p_R^N \, p_{A\cup S}^{M\cup N} \\
= & \sum_{\{Z\subseteq N:\, T\subseteq Z\}} \sum_{\{C,D\subseteq M:\, E\subseteq C\cap D\}} p_C^M \, p_{D\cup Z}^{M\cup N} \\
& + \sum_{\{A\subseteq M:\, E\subseteq A\}} \sum_{\{R,S\subseteq N:\, T\subseteq R\cap S\}} p_R^N \, p_{A\cup S}^{M\cup N} \\
= & \left(\sum_{\{C\subseteq M:\, E\subseteq C\}} p_C^M \right) \left(\sum_{\{D\subseteq M,\, Z\subseteq N:\, E\subseteq D,\, T\subseteq Z\}} p_{D\cup Z}^{M\cup N} \right) \\
& + \left(\sum_{\{R\subseteq N:\, T\subseteq R\}} p_R^N \right) \left(\sum_{\{A\subseteq M,\, S\subseteq N:\, E\subseteq A,\, T\subseteq S\}} p_{A\cup S}^{M\cup N} \right) \\
= & \frac{1}{|E|}\frac{1}{|E|+|T|} + \frac{1}{|T|}\frac{1}{|E|+|T|} = \frac{1}{|E|}\frac{1}{|T|}
\end{aligned} \tag{6}$$

where we have used Lemma 1.

And, on the other hand,

$$\sum_{\{A\subseteq M,\, Z\subseteq N\colon E\subseteq A,\, T\subseteq Z\}} p_A^M\, p_Z^N = \left(\sum_{\{A\subseteq M\colon E\subseteq A\}} p_A^M\right)\left(\sum_{\{Z\subseteq N\colon T\subseteq Z\}} p_Z^N\right) = \frac{1}{|E|}\frac{1}{|T|} \tag{7}$$

where we have used again Lemma 1.

From (5), (6) and (7) we obtain the equality in the lemma.

□

Cooperative games that only take the values 0 and 1 are called $\{0,1\}$-games. Given a finite and nonempty set N, a $\{0,1\}$-game $v \in \mathcal{G}^N$ is said to be a simple game if it is monotonic and $v(N) = 1$. We need to deal with simple superadditive games. Let $\mathcal{SG}^N$ denote the set of all simple superadditive games on N.

Notice that if $v \in \mathcal{SG}^N$, $i \in N$ and $v(N \setminus \{i\}) = 0$ then

$$Sh_i(v) = \sum_{\{E\subseteq N\colon E\neq\emptyset\}} p_E^N\, v\,(E)\,.$$

We will use this equality in the following propositions.

Proposition 3. *Let M, N be finite, nonempty and disjoint sets. Let $u \in \mathcal{SG}^M$ and $v \in \mathcal{SG}^N$. Let $i \in M$, $j \in N$ be such that $u(M \setminus \{i\}) = v(N \setminus \{j\}) = 0$. Let us define $w \in \mathcal{SG}^{M\cup N}$ as $w(E \cup T) = u(E) \wedge v(T)$ for every $E \subseteq M$ and every $T \subseteq N$. Then,*

$$\frac{1}{Sh_i(u)} + \frac{1}{Sh_j(v)} \geqslant \frac{1}{Sh_i(w)}.$$

Proof. We will prove the equivalent equality

$$Sh_i(u)Sh_j(v) \leqslant Sh_i(u)Sh_i(w) + Sh_j(v)Sh_i(w).$$

We have

$$\begin{aligned} Sh_i(u)Sh_j(v) &= \left(\sum_{\{E\subseteq M\colon E\neq\emptyset\}} p_E^M\, u(E)\right)\left(\sum_{\{T\subseteq N\colon T\neq\emptyset\}} p_T^N\, v(T)\right) \\ &= \sum_{\{E\subseteq M,\, T\subseteq N\colon E,T\neq\emptyset\}} p_E^M\, p_T^N\, u(E)\, v(T) \end{aligned}$$

which, by Lemma 2, is equal to

$$
\begin{aligned}
& \sum_{\{E\subseteq M,\, T\subseteq N:\, E,T\neq\emptyset\}} \; \sum_{\{C,D\subseteq M:\, C\cap D=E\}} p_C^M \, p_{D\cup T}^{M\cup N} \, u(E)\, v(T) \\
& + \sum_{\{E\subseteq M,\, T\subseteq N:\, E,T\neq\emptyset\}} \; \sum_{\{R,S\subseteq N:\, R\cap S=T\}} p_R^N \, p_{E\cup S}^{M\cup N} \, u(E)\, v(T) \\
= \; & \sum_{\{C,D\subseteq M,\, T\subseteq N:\, T\neq\emptyset,\, C\cap D\neq\emptyset\}} p_C^M \, p_{D\cup T}^{M\cup N} \, u(C\cap D)\, v(T) \\
& + \sum_{\{E\subseteq M,\, R,S\subseteq N:\, E\neq\emptyset,\, R\cap S\neq\emptyset\}} p_R^N \, p_{E\cup S}^{M\cup N} \, u(E)\, v(R\cap S) \\
= \; & \sum_{\{C,D\subseteq M,\, T\subseteq N:\, C,D,T\neq\emptyset\}} p_C^M \, p_{D\cup T}^{M\cup N} \, u(C\cap D)\, v(T) \\
& + \sum_{\{E\subseteq M,\, R,S\subseteq N:\, E,R,S\neq\emptyset\}} p_R^N \, p_{E\cup S}^{M\cup N} \, u(E)\, v(R\cap S) \\
\leqslant \; & \sum_{\{C,D\subseteq M,\, T\subseteq N:\, C,D,T\neq\emptyset\}} p_C^M \, p_{D\cup T}^{M\cup N} \, u(C)\, u(D)\, v(T) \\
& + \sum_{\{E\subseteq M,\, R,S\subseteq N:\, E,R,S\neq\emptyset\}} p_R^N \, p_{E\cup S}^{M\cup N} \, u(E)\, v(R)\, v(S) \\
= \; & \sum_{\{C,D\subseteq M,\, T\subseteq N:\, C\neq\emptyset,\, D\cup T\neq\emptyset\}} p_C^M \, p_{D\cup T}^{M\cup N} \, u(C)\, u(D)\, v(T) \\
& + \sum_{\{E\subseteq M,\, R,S\subseteq N:\, R\neq\emptyset,\, E\cup S\neq\emptyset\}} p_R^N \, p_{E\cup S}^{M\cup N} \, u(E)\, v(R)\, v(S) \\
= \; & \sum_{\{C,D\subseteq M,\, T\subseteq N:\, C\neq\emptyset,\, D\cup T\neq\emptyset\}} p_C^M \, p_{D\cup T}^{M\cup N} \, u(C)\, w(D\cup T) \\
& + \sum_{\{E\subseteq M,\, R,S\subseteq N:\, R\neq\emptyset,\, E\cup S\neq\emptyset\}} p_R^N \, p_{E\cup S}^{M\cup N} \, v(R)\, w(E\cup S) \\
= \; & \left(\sum_{\{C\subseteq M:\, C\neq\emptyset\}} p_C^M u(C) \right) \left(\sum_{\{D\subseteq M,\, T\subseteq N:\, D\cup T\neq\emptyset\}} p_{D\cup T}^{M\cup N} \, w(D\cup T) \right) \\
& + \left(\sum_{\{R\subseteq N:\, R\neq\emptyset\}} p_R^N \, v(R) \right) \left(\sum_{\{E\subseteq M,\, S\subseteq N:\, E\cup S\neq\emptyset\}} p_{E\cup S}^{M\cup N} \, w(E\cup S) \right) \\
= \; & Sh_i(u) Sh_i(w) + Sh_j(v) Sh_i(w).
\end{aligned}
$$

$\square$

Proposition 4. *Let N be a finite nonempty set. Let $u, v \in \mathcal{SG}^N$. Let $i, j \in N$ be such that $u(N \setminus \{i\}) = v(N \setminus \{j\}) = 0$. Then,*

$$\frac{1}{Sh_i(u)} + \frac{1}{Sh_j(v)} \geqslant \frac{1}{Sh_i(u \wedge v)}.$$

Proof. We can suppose that $N = \{1, \ldots, |N|\}$.

For any $H \subseteq N$ we consider $u_H, v_H, w_H \in \mathcal{SG}^{N \cup (-H)}$ defined as

$$\begin{aligned} u_H(E) &= u(E \cap N), \\ v_H(E) &= v((E \cap (N \setminus H)) \cup ((-E) \cap H)), \\ w_H(E) &= u_H(E) \wedge v_H(E), \end{aligned}$$

for every $E \subseteq N \cup (-H)$.

We aim to prove that, for every $H \subsetneqq N$ and $l \in N \setminus H$,

$$Sh_i(w_{H \cup \{l\}}) \leqslant Sh_i(w_H). \tag{8}$$

Let $H \subsetneqq N$ and $l \in N \setminus H$. Let us consider the game $w'_H \in \mathcal{SG}^{N \cup (-H) \cup \{-l\}}$ defined as $w'_H(E) = w_H(E \setminus \{-l\})$ for every $E \subseteq N \cup (-H) \cup \{-l\}$. It can be easily verified that $Sh_i(w_H) = Sh_i(w'_H)$. Therefore, in order to prove (8), we can show that

$$Sh_i(w_{H \cup \{l\}}) \leqslant Sh_i(w'_H). \tag{9}$$

It can easily be proved that, for every $E \subseteq N \cup (-H) \cup \{-l\}$,

$$w_{H \cup \{l\}}(E) = \begin{cases} w_H(E) & \text{if } l, -l \notin E, \\ u_H(E) \wedge v_H(E \setminus \{l\}) & \text{if } l \in E \text{ and } -l \notin E, \\ u_H(E \setminus \{-l\}) \wedge v_H((E \setminus \{-l\}) \cup \{l\}) & \text{if } l \notin E \text{ and } -l \in E, \\ w_H(E \setminus \{-l\}) & \text{if } l, -l \in E. \end{cases} \tag{10}$$

We have

$$\begin{aligned}
Sh_i(w_{H\cup\{l\}}) &= \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, E\neq\emptyset\}} p_E^{N\cup(-H)\cup\{-l\}} w_{H\cup\{l\}}(E) \\
&= \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\notin E,\, E\neq\emptyset\}} p_E^{N\cup(-H)\cup\{-l\}} w_{H\cup\{l\}}(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\in E\}} p_E^{N\cup(-H)\cup\{-l\}} w_{H\cup\{l\}}(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l\in E,-l\notin E\}} p_E^{N\cup(-H)\cup\{-l\}} w_{H\cup\{l\}}(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l\notin E,-l\in E\}} p_E^{N\cup(-H)\cup\{-l\}} w_{H\cup\{l\}}(E) \\
&= \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\notin E,\, E\neq\emptyset\}} p_E^{N\cup(-H)\cup\{-l\}} w_H(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\in E\}} p_E^{N\cup(-H)\cup\{-l\}} w_H(E\setminus\{-l\}) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l\in E,-l\notin E\}} p_E^{N\cup(-H)\cup\{-l\}} w_{H\cup\{l\}}(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l\in E,-l\notin E\}} p_{(E\setminus\{l\})\cup\{-l\}}^{N\cup(-H)\cup\{-l\}} w_{H\cup\{l\}}((E\setminus\{l\})\cup\{-l\}) \\
&= \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\notin E,\, E\neq\emptyset\}} p_E^{N\cup(-H)\cup\{-l\}} w'_H(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\in E\}} p_E^{N\cup(-H)\cup\{-l\}} w'_H(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l\in E,-l\notin E\}} p_E^{N\cup(-H)\cup\{-l\}} \left(w_{H\cup\{l\}}(E) + w_{H\cup\{l\}}((E\setminus\{l\})\cup\{-l\})\right).
\end{aligned} \tag{11}$$

Now we will prove that, for every $E \subseteq N \cup (-H) \cup \{-l\}$ with $l \in E$ and $-l \notin E$, the following holds:

$$w_{H\cup\{l\}}(E) + w_{H\cup\{l\}}((E\setminus\{l\})\cup\{-l\}) \leqslant w'_H(E) + w'_H((E\setminus\{l\})\cup\{-l\}). \tag{12}$$

Let $E \subseteq N \cup (-H) \cup \{-l\}$ with $l \in E$ and $-l \notin E$. We consider the following three cases:

a) If $w_{H\cup\{l\}}(E) = 1$ and $w_{H\cup\{l\}}((E \setminus \{l\}) \cup \{-l\}) = 0$, then

$$\begin{aligned} 1 &= w_{H\cup\{l\}}(E) = u_H(E) \wedge v_H(E \setminus \{l\}) \\ &\leqslant u_H(E) \wedge v_H(E) = w_H(E) = w_H(E \setminus \{-l\}) = w'_H(E). \end{aligned}$$

Therefore, $w'_H(E) = 1$, from which we can derive (12).

b) If $w_{H\cup\{l\}}(E) = 0$ and $w_{H\cup\{l\}}((E \setminus \{l\}) \cup \{-l\}) = 1$, then

$$\begin{aligned} 1 &= w_{H\cup\{l\}}((E \setminus \{l\}) \cup \{-l\}) = u_H(E \setminus \{l\}) \wedge v_H(E) \\ &\leqslant u_H(E) \wedge v_H(E) = w_H(E) = w_H(E \setminus \{-l\}) = w'_H(E). \end{aligned}$$

Hence, $w'_H(E) = 1$, and we conclude (12).

c) If $w_{H\cup\{l\}}(E) = 1$ and $w_{H\cup\{l\}}((E \setminus \{l\}) \cup \{-l\}) = 1$, then

$$\begin{aligned} 1 &= w_{H\cup\{l\}}(E) = u_H(E) \wedge v_H(E \setminus \{l\}), \\ 1 &= w_{H\cup\{l\}}((E \setminus \{l\}) \cup \{-l\}) = u_H(E \setminus \{l\}) \wedge v_H(E). \end{aligned}$$

From these equalities we obtain $u_H(E \setminus \{l\}) = v_H(E \setminus \{l\}) = 1$. Therefore,

$$\begin{aligned} w'_H((E \setminus \{l\}) \cup \{-l\}) &= w_H(E \setminus \{l\}) = u_H(E \setminus \{l\}) \wedge v_H(E \setminus \{l\}) = 1, \\ w'_H(E) &= w_H(E \setminus \{-l\}) = w_H(E) = u_H(E) \wedge v_H(E) = 1. \end{aligned}$$

Hence, (12) holds.

From (11) and (12) it follows that

$$
\begin{aligned}
Sh_i(w_{H\cup\{l\}}) &\leqslant \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\notin E,\, E\neq\emptyset\}} p_E^{N\cup(-H)\cup\{-l\}} w'_H(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\in E\}} p_E^{N\cup(-H)\cup\{-l\}} w'_H(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l\in E,-l\notin E\}} p_E^{N\cup(-H)\cup\{-l\}} \left(w'_H(E) + w'_H((E\setminus\{l\})\cup\{-l\})\right) \\
&= \sum_{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\notin E,\, E\neq\emptyset\}} p_E^{N\cup(-H)\cup\{-l\}} w'_H(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l,-l\in E\}} p_E^{N\cup(-H)\cup\{-l\}} w'_H(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l\in E,-l\notin E\}} p_E^{N\cup(-H)\cup\{-l\}} w'_H(E) \\
&\quad + \sum_{\{E\subseteq N\cup(-H)\cup\{-l\}:\, l\notin E,-l\in E\}} p_E^{N\cup(-H)\cup\{-l\}} w'_H(E) \\
&= Sh_i(w'_H). \qquad (13)
\end{aligned}
$$

We have proved (9) and, consequently, (8) is also proved.

Notice that $u_\emptyset = u$ and $v_\emptyset = v$. Therefore, $w_\emptyset = u \wedge v$. From (8) we can derive that $Sh_i(w_N) \leqslant Sh_i(w_\emptyset)$. Hence,

$$Sh_i(w_N) \leqslant Sh_i(u \wedge v) \qquad (14)$$

and, therefore

$$\frac{1}{Sh_i(w_N)} \geqslant \frac{1}{Sh_i(u \wedge v)}. \qquad (15)$$

Consider $v' \in \mathcal{SG}^{-N}$ defined as $v'(T) = v(-T)$ for every $T \subseteq -N$. Notice that $w_N(F \cup T) = u(F) \wedge v'(T)$ for every $F \subseteq N$ and $T \subseteq -N$. From Proposition 3, we obtain

$$\frac{1}{Sh_i(u)} + \frac{1}{Sh_{-j}(v')} \geqslant \frac{1}{Sh_i(w_N)}.$$

Taking into consideration that $Sh_{-j}(v') = Sh_j(v)$, we have

$$\frac{1}{Sh_i(u)} + \frac{1}{Sh_j(v)} \geqslant \frac{1}{Sh_i(w_N)},$$

and this, together with (15), leads to

$$\frac{1}{Sh_i(u)} + \frac{1}{Sh_j(v)} \geqslant \frac{1}{Sh_i(u \wedge v)}.$$

$\square$

Theorem 5. *Let $G = (N, L)$ be a graph. Then, d_G^{Sh} is a distance on N.*

Proof. The only non-trivial part is the proof of the triangular inequality. Let $i, j, k \in N$. We must prove that

$$d_G^{Sh}(i,k) \leqslant d_G^{Sh}(i,j) + d_G^{Sh}(j,k).$$

We can suppose that $d_G^{Sh}(i,j) < +\infty$ and $d_G^{Sh}(j,k) < +\infty$.

Notice that $A_G^{i,k}(S) \geqslant A_G^{i,j}(S) \wedge A_G^{j,k}(S)$ for every $S \subseteq N$. We can easily derive that

$$Sh_i(A_G^{i,k}) \geqslant Sh_i(A_G^{i,j} \wedge A_G^{j,k})$$

which leads to

$$\frac{1}{Sh_i(A_G^{i,k})} \leqslant \frac{1}{Sh_i(A_G^{i,j} \wedge A_G^{j,k})}. \tag{16}$$

From Proposition 4, we know that

$$\frac{1}{Sh_i(A_G^{i,j} \wedge A_G^{j,k})} \leqslant \frac{1}{Sh_i(A_G^{i,j})} + \frac{1}{Sh_j(A_G^{j,k})}. \tag{17}$$

From (16) and (17) we obtain

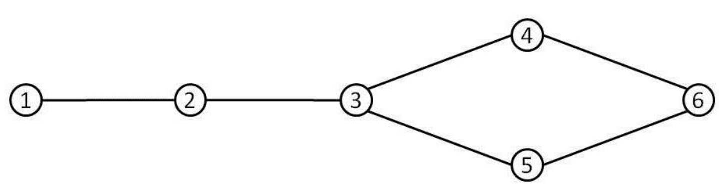
$$\frac{1}{Sh_i(A_G^{i,k})} \leqslant \frac{1}{Sh_i(A_G^{i,j})} + \frac{1}{Sh_j(A_G^{j,k})},$$

which completes the proof.

$\square$

Finally, we give an example that illustrates the different natures of the Shapley distance and the best-known distance in graphs, the geodesic distance.

Example. Let us take the graph G considered in Section 3:



Notice that the geodesic distance between 1 and 3 is equal to the geodesic distance between 3 and 6. Let us see that, for the Shapley distance d_G^{Sh}, 3 and 6 are closer than 1 and 3.

In order to calculate $d_G^{Sh}(1,3)$, notice that 3, 4, 5 and 6 are null players in $A_G^{1,3}$. This implies that $Sh_1(A_G^{1,3}) = Sh_1({A_G^{1,3}}_{|\{1,2\}})$, where ${A_G^{1,3}}_{|\{1,2\}}$ denotes the restriction of $A_G^{1,3}$ to $2^{\{1,2\}}$. We have

$$A_G^{1,3}(\{1\}) = A_G^{1,3}(\{2\}) = 0,\ A_G^{1,3}(\{1,2\}) = 1,$$

from which we obtain $Sh_1({A_G^{1,3}}_{|\{1,2\}}) = \frac{1}{2}$. Thus, $Sh_1(A_G^{1,3}) = \frac{1}{2}$. Therefore, $d_G^{Sh}(1,3) = 2$.

In order to calculate $d_G^{Sh}(3,6)$, notice that 1, 2 and 6 are null players in $A_G^{3,6}$. This implies that $Sh_3(A_G^{3,6}) = Sh_3({A_G^{3,6}}_{|\{3,4,5\}})$, where ${A_G^{3,6}}_{|\{3,4,5\}}$ denotes the restriction of $A_G^{3,6}$ to $2^{\{3,4,5\}}$. We have

$$A_G^{3,6}(\{3\}) = A_G^{3,6}(\{4\}) = A_G^{3,6}(\{5\}) = A_G^{3,6}(\{4,5\}) = 0,$$

$$A_G^{3,6}(\{3,4\}) = A_G^{3,6}(\{3,5\}) = A_G^{3,6}(\{3,4,5\}) = 1,$$

from which we can easily obtain that $Sh_3({A_G^{3,6}}_{|\{3,4,5\}}) = \frac{2}{3}$. Hence, $Sh_3(A_G^{3,6}) = \frac{2}{3}$. Therefore, $d_G^{Sh}(3,6) = \frac{3}{2}$.

5. Conclusions

Given a graph, we have identified the nodes with agents, and supposed that each agent can obtain a profit from achieving contact with another agent. However, if there is no edge between these two agents, then a percentage of that profit must be given to other intermediary agents. Our goal was to calculate, for a node in a graph, the cost of attaining access to another node. This has been done by proving that certain function is a distance on the set of nodes. This new distance in graphs indicates the level of difficulty in the communication between any pair of nodes.

References

[1] D. Cheng, Y. Wang and T. Liu, *A survey on potential evolutionary game and its applications*, Journal of Control and Decision 2 (2015), 26–45.

[2] J.M. Gallardo, N. Jiménez and A. Jiménez-Losada, *A Shapley measure of power in hierarchies*, Information Sciences 372 (2016), 98–110.

[3] D. Gómez, E. González-Arangüena, C. Manuel, G. Owen, M. del Pozo and J. Tejada, *Centrality and power in social networks: a game theoretic approach*, Mathematical Social Sciences 46 (2003), 27–54.

[4] B. Grofman and G. Owen, *A game theoretic approach to measuring degree of centrality in social networks*, Social Networks 4 (1982), 213–224.

[5] T.P. Michalak, K.V. Aadithya, P.L. Szczepański, B. Ravindran and N.R. Jennings, *Efficient computation of the Shapley value for game-theoretic network centrality*, Journal of Artificial Intelligence Research 46 (2013), 607–650.

[6] R.B. Myerson, *Graphs and cooperation in games*, Mathematics of Operations Research 2 (1977), 225–229.

[7] G. Owen, *Values of graph-restricted games*, SIAM Journal on Algebraic and Discrete Methods 7 (1986), 210–220.

[8] M. del Pozo, C. Manuel, E. González-Arangüena and G. Owen, *Centrality in directed social networks. A game theoretic approach*, Social Networks 33 (2011), 191–200.

[9] L.S. Shapley, *A value for n-person games*, Contributions to the Theory of Games (H.W. Kuhn and A.W. Tucker eds.), vol. 2, Princeton, New Jersey, (1953), 307–317.

[10] N. Suri and Y. Narahari, *A Shapley value based approach to discover influential nodes in social networks*, IEEE Transactions on Automatic Science and Engineering 99 (2010), 1–18.

[11] P. Yi, Y. Zhang and Y. Hong, *Potential game design for a class of distributed optimisation problems*, Journal of Control and Decision 1 (2014), 166–179.