Banzhaf values for cooperative games with fuzzy characteristic function

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ABSTRACT

The Banzhaf value, which determines the power of each agent in a cooperative game, has been used in the literature to analyze fuzzy cooperative situations. In this paper, we propose two Banzhaf values for games with fuzzy characteristic function. In one of them, the players' payoffs are fuzzy quantities. In the other one, the payoffs are real numbers. In each case we provide an axiomatization with reasonable properties.

KEYWORDS

Cooperative game; Banzhaf value; fuzzy set; fuzzy quantity

1. Introduction

In decision situations for committees or centers of distributed control the quantification of the power of each member is an important element to analyze the final position and the different treatment of each of them. Cooperative games are a way to represent these situations and to study the power of their elements in the group. Given a finite set of players, a cooperative game assigns to each subset of these players (coalition) the payment that they could achieve by cooperating. A value is a function that determines, for every cooperative game, a vector which represents a payoff distribution among the players. Values for cooperative games can also be seen as functions which determine the power or influence of the agents in the game based on the power of each coalition in a decision situation. This paper focuses on one of the best known values, the Banzhaf value, introduced by Penrose (1946) and by Banzhaf (1965) for particular situations. Later on Owen (1975) generalized this solution for all cooperative games as the Banzhaf value. The Banzhaf value determines the power of an agent in a cooperative situation according to the expected contribution of this agent to each coalition, considering that the agent is equally likely to join any coalition.

In some cooperative situations, there is only vague information on the formation of coalitions or on their payment. Aubin (1981) studied cooperative games with fuzzy coalitions and Tan et al. (2014) analyzed the Banzhaf value for these games. In the present paper we will focus on situations in which there are only expectations about the payment of the coalitions. Game theorists have introduced different types of cooperative games with can be used to model these situations. For this, they have used

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mathematical tools that allow to deal with uncertainty. Charnes and Granot (1973) made use of Probability Theory and introduced cooperative games in which the coalition payments are random variables with given distribution functions. Various authors have continued this line of research (Suijs et al. 1999; Timmer 2001). Branzei et al. (2003) used real intervals to model cooperative situations in which the players only know a lower and a upper bound of the profit that can be obtained by each coalition. Cooperative interval games have multiple applications in economics and operations research (Branzei et al. 2010). Mareš and Vlach (2001) used another mathematical tool to handle imprecise information: the fuzzy numbers introduced by Zadeh (1965). They defined games with fuzzy characteristic function, in which the payment of a coalition is given by a fuzzy number which establishes the grade of feasibility of each possible profit achievable by the coalition. The present paper is focused on this approach. As with other cooperative games, the main problem that arises when dealing with cooperative games with fuzzy characteristic function is how to share the total profit obtained by the grand coalition. In this regard, it seems reasonable that the imprecision payments of the coalitions should imply imprecision in the players' payoffs. Multiple studies have been carried out in this line of research (Mareš 2001; Borkotokey 2008; Yu and Zhang 2010). The Banzhaf value for these games has been studied by Liang and Li (2019) and Pusillo (2013). In the first half of the present paper we study and characterize a Banzhaf value with fuzzy payoffs for games with fuzzy coalition payments. Both lines of study, games with fuzzy coalitions or with fuzzy payoffs, are recently continued (Borkotokey and Mesiar 2013; Gallardo et al. 2017; Liu et al. 2018).

However, there are situations in which, even if there is vagueness in the profit attainable by the coalitions, precise payoffs for the players are needed. Suppose, for example, a cooperative situation in which the total profit obtained by the grand coalition is known exactly, but there are only expectations about the profit achievable by each proper coalition. Take into account that when a cooperative situation is modeled by a cooperative game, it is supposed that all the players will cooperate and the grand coalition will be formed. This means that the formation of any proper coalition is just a hypothetical scenario and, therefore, it might not be possible to know with precision the profit achievable by each proper coalition. However, the players might need a precise allocation of the total profit. The goal of this paper is to come up with a method for obtaining exact allocations in these situations. We introduce the concept of real value for games with fuzzy characteristic function. By using a function introduced by Yager (1981) with the purpose of ranking fuzzy numbers, we obtain a Banzhaf real value for games with fuzzy characteristic function. We show that this value is characterized by certain nice properties.

The paper is organized as follows. In section 2 some concepts regarding cooperative games, fuzzy quantities and cooperative games with fuzzy characteristic function are recalled. In section 3 we introduce and characterize the Banzhaf value for cooperative games with fuzzy characteristic function. In section 4 we introduce and characterize the real Banzhaf value for cooperative games with fuzzy characteristic function.

2. Preliminaries

2.1. *Cooperative games*

A *cooperative game (with transferable utility)* consists of a finite set of players N and a characteristic function $v : 2^N \to \mathbb{R}$ which satisfies $v(\emptyset) = 0$. The elements of N

are called players, and the subsets of N coalitions. Given a coalition E, $v(E)$ is the worth of E, and it is interpreted as the collective payment that the players of E would obtain if they cooperate. Frequently, a cooperative game (N, v) is identified with the function v. The family of games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^{|N|}-1)$-dimensional real vector space. One basis of $\mathcal{G}^N$ is the set $\left\{\delta_E : E \in 2^N \setminus \{\emptyset\}\right\}$ where for a nonempty coalition E the game δ_E is defined by

$$\delta_E(F) = \begin{cases} 1 & \text{if } F = E, \\ 0 & \text{otherwise.} \end{cases}$$

Another basis of $\mathcal{G}^N$ is the set $\left\{u_E : E \in 2^N \setminus \{\emptyset\}\right\}$ where for a nonempty coalition E the unanimity game u_E is defined by

$$u_E(F) = \begin{cases} 1 & \text{if } E \subseteq F, \\ 0 & \text{otherwise.} \end{cases}$$

Every game $v \in \mathcal{G}^N$ can be written as

$$v = \sum_{\{E \in 2^N : E \neq \emptyset\}} \triangle_v(E)\, u_E \tag{1}$$

where $(\triangle_v(E))_{E \subseteq N}$ is the Möbius transform of v on the poset $(2^N, \subseteq)$. The coefficient $\triangle_v(E)$ is called the dividend of the coalition E in the game v and is given by

$$\triangle_v(E) = \sum_{F \subseteq E} (-1)^{|E|-|F|} v(F) \tag{2}$$

for every $E \in 2^N \setminus \{\emptyset\}$.

A value ψ for cooperative games assigns to each nonempty finite set N and each $v \in \mathcal{G}^N$ a vector $\psi(v) \in \mathbb{R}^N$. Multiple values have been defined in the literature.

The *Banzhaf value* arises from considering that each player is equally likely to join any coalition. Given $v \in \mathcal{G}^N$, the Banzhaf value of v, denoted by $\beta(v)$, is defined by

$$\beta_i(v) = \frac{1}{2^{|N|-1}} \sum_{\{E \subseteq N : i \in E\}} \left(v(E) - v(E \setminus \{i\})\right)$$

for every $i \in N$.

Some properties for a value for cooperative games are the following:

1-efficiency: If $v \in \mathcal{G}^{\{i\}}$ then $\psi_i(v) = v(\{i\})$.

Additivity: $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2)$ for all $v_1, v_2 \in \mathcal{G}^N$.

Equal treatment: If $v \in \mathcal{G}^N$, $i, j \in N$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$, then $\psi_i(v) = \psi_j(v)$.

Null player property: A player $i \in N$ is a null player in $v \in \mathcal{G}^N$ if $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$. If $i \in N$ is a null player in $v \in \mathcal{G}^N$ then $\psi_i(v) = 0$.

Merger: Let $v \in \mathcal{G}^N$ and let i, j be two different players in N. The merger of i and j defines a new player denoted by $\widehat{ij}$. Let $N^{ij} = (N \setminus \{i, j\}) \cup \{\widehat{ij}\}$ and $v^{ij} : 2^{N^{ij}} \to \mathbb{R}$

defined by

$$v^{ij}(E) = \begin{cases} v\,(E) & \text{if } \widehat{ij} \notin E, \\ v\left(\left(E \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right) & \text{if } \widehat{ij} \in E. \end{cases}$$

Then,

$$\psi_i\,(v) + \psi_j\,(v) = \psi_{\widehat{ij}}\left(v^{ij}\right).$$

The properties above characterize the Banzhaf value.

2.2. *Fuzzy quantities*

Firstly we recall some definitions regarding fuzzy sets.

Given a set X, a fuzzy subset a of X is defined by its membership function $\mu_a\colon X \to [0,1]$. For each $x \in X$ the number $\mu_a(x)$ is the degree of membership of x in a. For each $t \in (0,1]$ the t-cut of a is defined by

$$[a]_t = \{x \in X\colon \mu_a(x) \geqslant t\}$$

Notice that the family of t-cuts determine a. The core of a is defined by

$$core(a) = [a]_1.$$

If a is a fuzzy subset of $\mathbb{R}$, the 0-cut of a is defined by

$$[a]_0 = \overline{\{x \in \mathbb{R}\colon \mu_a(x) > 0\}}.$$

If a, b are fuzzy subsets of X, it is said that a is contained in b, and it is denoted by $a \subseteq b$, if $\mu_a(x) \leqslant \mu_b(x)$ for every $x \in X$.

In this paper we will deal with a particular class of fuzzy subsets of $\mathbb{R}$, the class of fuzzy quantities. The term fuzzy quantity has been used in the literature with slightly different meanings. We will use the concept of fuzzy quantity as defined in Stefani et al. (2008). A fuzzy subset a of $\mathbb{R}$ is a fuzzy quantity if it satisfies the following conditions:

i) $core(a) \neq \emptyset$.
ii) $[a]_t$ is a closed and bounded interval for every $t \in [0,1]$.

The set of fuzzy quantities will be denoted by $\mathbb{F}$. If $a \in \mathbb{F}$ and $t \in [0,1]$ we denote

$$a_t^+ = \max[a]_t \quad \text{and} \quad a_t^- = \min[a]_t.$$

In the remainder of this subsection we recall the basics of fuzzy arithmetic (Stefani et al. 2008; Dubois and Prade 1978,b; Kaufmann and Gupta 1991).

Let $a, b \in \mathbb{F}$.

- The sum $a \oplus b \in \mathbb{F}$ is defined by

$$\mu_{a\oplus b}(x) = \sup\{\min\{\mu_a(y), \mu_b(z)\} \colon y, z \in \mathbb{R},\ y + z = x\}$$

  for every $x \in \mathbb{R}$. Equivalently,

$$[a \oplus b]_t = [a_t^- + b_t^-, a_t^+ + b_t^+]$$

  for every $t \in [0, 1]$.
- The difference $a \ominus b \in \mathbb{F}$ is defined by

$$\mu_{a\ominus b}(x) = \sup\{\min\{\mu_a(y), \mu_b(z)\} \colon y, z \in \mathbb{R},\ y - z = x\}$$

  for every $x \in \mathbb{R}$. Equivalently,

$$[a \ominus b]_t = [a_t^- - b_t^+, a_t^+ - b_t^-]$$

  for every $t \in [0, 1]$.
- The product $a \odot b \in \mathbb{F}$ is defined by

$$\mu_{a\odot b}(x) = \sup\left\{\min\left\{\mu_a(y), \mu_b\left(z\right)\right\} : y, z \in \mathbb{R},\ yz = x\right\}$$

  for every $x \in \mathbb{R}$. Equivalently,

$$\begin{aligned}[a \odot b]_t = \quad & \big[\min\{a_t^- b_t^-, a_t^- b_t^+, a_t^+ b_t^-, a_t^+ b_t^+\}\ , \\ & \max\{a_t^- b_t^-, a_t^- b_t^+, a_t^+ b_t^-, a_t^+ b_t^+\}\big]\end{aligned}$$

  for every $t \in [0, 1]$.

Notice that the set of real numbers can be embedded into $\mathbb{F}$. Indeed, we can identify $p \in \mathbb{R}$ with the fuzzy quantity determined by the following membership function:

$$\mu_p(x) = \begin{cases} 1 & \text{if } x = p, \\ 0 & \text{otherwise.} \end{cases}$$

With this identification we have that $\mathbb{R} \subset \mathbb{F}$. Note that the operations $\oplus, \ominus, \odot$ extend, respectively, the sum, subtraction and product of real numbers. Moreover, also the bounded closed intervals of real numbers are in $\mathbb{F}$. If $[p, q]$ is a real interval then we identify it with the fuzzy quantity $[p, q]$ with membership fucntion:

$$\mu_{[p,q]}(x) = \begin{cases} 1 & \text{if } x \in [p, q], \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $a \in \mathbb{F}$ and $p \in \mathbb{R}$, then

$$\mu_{p\oplus a}(x) = \mu_a(x - p)$$

for every $x \in \mathbb{R}$. Equivalently,

$$[p \oplus a]_t = [p + a_t^-, p + a_t^+]$$

for every $t \in [0,1]$. And, if $p \in \mathbb{R} \setminus \{0\}$, then

$$\mu_{p \odot a}(x) = \mu_a\left(\frac{x}{p}\right)$$

for every $x \in \mathbb{R}$. Equivalently,

$$[p \odot a]_t = \begin{cases} [pa_t^-, pa_t^+] & \text{if } p > 0, \\ [pa_t^+, pa_t^-] & \text{if } p < 0. \end{cases}$$

for every $t \in [0,1]$.

Given $a, b \in \mathbb{F}$, it is said that a is greater than or equal to b, which is denoted by $a \geqslant b$, if $a_t^- \geqslant b_t^-$ and $a_t^+ \geqslant b_t^+$ for every $t \in [0,1]$.

A fuzzy quantity $a \in \mathbb{F}$ is said to be 0-symmetric if $a_t^- = -a_t^+$ for every $t \in [0,1]$.

Let us recall some basic properties of the arithmetic operations in $\mathbb{F}$. Let $a, b, c, d \in \mathbb{F}$.

a) $a \oplus b = b \oplus a$.
b) $a \odot b = b \odot a$.
c) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
d) $a \odot (b \odot c) = (a \odot b) \odot c$.
e) $a \oplus 0 = a$.
f) $a \odot 1 = a$.
g) $a \odot 0 = 0$.
h) $a \ominus b = a \oplus ((-1) \odot b)$.

The properties above will be used throughout this paper without referring to them. The following properties, although equally simple, are more specific and they will be referred to when applied.

i) If $p \in \mathbb{R}$,

$$p \odot (a \oplus b) = (p \odot a) \oplus (p \odot b), \tag{3}$$
$$p \odot (a \ominus b) = (p \odot a) \ominus (p \odot b). \tag{4}$$

j) If $b, c \geqslant 0$ (or $b, c \leqslant 0$),

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c). \tag{5}$$

k) If $a \subseteq c$ and $b \subseteq d$,

$$a \oplus b \subseteq c \oplus d, \tag{6}$$
$$a \ominus b \subseteq c \ominus d, \tag{7}$$
$$a \odot b \subseteq c \odot d. \tag{8}$$

l) The equation $x \oplus a = b$ either has no solution in $\mathbb{F}$ or has a unique solution in $\mathbb{F}$.
m) If $p \in \mathbb{R}$, then $p \odot (a \ominus a)$ is 0-symmetric.
n) If $a \oplus b \in \mathbb{R}$, then $a, b \in \mathbb{R}$.

The best known and most employed metric in $\mathbb{F}$ is the supremum distance. Let us introduce it. Let A and B be nonempty bounded subsets of $\mathbb{R}$. Then,

$$d^*(A,B) = \sup\{\inf\{|x-y| \colon y \in B\} \colon x \in A\}.$$

The Hausdorff distance between A and B is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}.$$

If $a, b \in \mathbb{F}$ the supremum distance between a and b is defined as

$$d_\infty(a,b) = \sup\{d_H([a]_t,[b]_t) \colon t \in [0,1]\}.$$

2.3. *Cooperative games with fuzzy characteristic function*

A *cooperative game with fuzzy characteristic function* consists of a nonempty finite set N and a characteristic function $v : 2^N \to \mathbb{F}$ that satisfies $v(\emptyset) = 0$. The elements of N are called players, and the subsets of N are called coalitions. For each coalition E, the fuzzy quantity $v(E)$ describes the expectations about the collective payment that can be obtained by the players in E when they cooperate. A cooperative game with fuzzy characteristic function (N,v) will be identified with the mapping v. The class of all cooperative games with fuzzy characteristic function and set of players N is denoted by $\mathcal{FG}^N$. Since $\mathbb{R} \subset \mathbb{F}$, we have that $\mathcal{G}^N \subset \mathcal{FG}^N$. If $v, w \in \mathcal{FG}^N$ and $a \in \mathbb{F}$ the games $v \oplus w, a \odot v \in \mathcal{FG}^N$ are defined by

$$\begin{aligned}(v \oplus w)(E) &= v(E) \oplus w(E),\\ (a \odot v)(E) &= a \odot v(E),\end{aligned}$$

for every $E \in 2^N$.

3. The Banzhaf value for cooperative games with fuzzy characteristic function

A value for cooperative games with fuzzy characteristic function assigns to each nonempty finite set N, $v \in \mathcal{FG}^N$ and $i \in N$ a fuzzy quantity $\Psi_i(v)$.

Definition 3.1. The Banzhaf value for cooperative games with fuzzy characteristic function is defined by

$$B_i(v) = \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N \colon i \in E\}} (v(E) \ominus v(E \setminus \{i\})) \right)$$

for every nonempty finite set N, $v \in \mathcal{FG}^N$ and $i \in N$.

We introduce some properties that a value Ψ for games with fuzzy characteristic function may satisfy:
1-EFFICIENCY. If $v \in \mathcal{FG}^{\{i\}}$ then $\Psi_i(v) = v(\{i\})$.
ADDITIVITY. If $v, w \in \mathcal{FG}^N$ then $\Psi(v \oplus w) = \Psi(v) \oplus \Psi(w)$.

EQUAL TREATMENT. If $v \in \mathcal{FG}^N$, $i, j \in N$ and

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

for every $S \subseteq N \setminus \{i, j\}$, then $\Psi_i(v) = \Psi_j(v)$.
NULL PLAYER. If $v \in \mathcal{FG}^N$, a player $i \in N$ is said to be a null player in v if $v(E \cup \{i\}) = v(E)$ for every $E \in 2^N$. If $v \in \mathcal{FG}^N$ and $i \in N$ is a null player in v, then $\Psi_i(v)$ is 0-symmetric.
MERGER. Let $v \in \mathcal{FG}^N$ and let i, j be two different players in N. The merger of i and j defines a new player denoted by $\widehat{ij}$. Let $N^{ij} = (N \setminus \{i, j\}) \cup \{\widehat{ij}\}$ and $v^{ij} : 2^{N^{ij}} \to \mathbb{F}$ defined by

$$v^{ij}(E) = \begin{cases} v(E) & \text{if } \widehat{ij} \notin E, \\ v\left(\left(E \setminus \{\widehat{ij}\}\right) \cup \{i, j\}\right) & \text{if } \widehat{ij} \in E. \end{cases}$$

Then, there exists a 0-symmetric fuzzy quantity d (which depends on v, i and j) such that

$$\Psi_i(v) \oplus \Psi_j(v) = \Psi_{\widehat{ij}}\left(v^{ij}\right) \oplus d.$$

Weber (1988) introduced monotonicity as an axiom for values over classic games. We proposed now a similar axiom for games with fuzzy characteristic functions.
EQUALLY SIGNED MARGINAL CONTRIBUTIONS. If $v \in \mathcal{FG}^N$, $i \in N$ and $v(E \cup \{i\}) \ominus v(E) \geqslant 0$ (resp. $v(E \cup \{i\}) \ominus v(E) \leqslant 0$) for every $E \subseteq N \setminus \{i\}$, then $\Psi_i(v) \geqslant 0$ (resp. $\Psi_i(v) \leqslant 0$).
ZERO SOLUTION. If $v \in \mathcal{FG}^N$ and $0 \subseteq v(E)$ for every $E \in 2^N$, then $0 \subseteq \Psi_i(v)$ for every $i \in N$.

Let us see that B satisfies the seven properties above.

Theorem 3.2. *The Banzhaf value for cooperative games with fuzzy characteristic function satisfies the properties of* 1*-efficiency, additivity, equal treatment, null player, merger, equally signed marginal contributions and zero solution.*

Proof. 1-EFFICIENCY. It can be easily checked.
ADDITIVITY. Let $v, w \in \mathcal{FG}^N$ and let $i \in N$. Then, $B_i(v \oplus w)$ is

$$\frac{1}{2^{|N|-1}} \odot \bigoplus_{\{E \subseteq N : i \in E\}} (v \oplus w)(E) \ominus (v \oplus w)(E \setminus \{i\}), \tag{9}$$

which, by basic arithmetic properties, (3) and (4), is equal to

$$\begin{aligned} & \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N : i \in E\}} v(E) \ominus v(E \setminus \{i\}) \right) \\ \oplus \; & \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N : i \in E\}} w(E) \ominus w(E \setminus \{i\}) \right), \end{aligned} \tag{10}$$

which is, by definition, $B_i(v) \oplus B_i(w)$.
EQUAL TREATMENT. Let $v \in \mathcal{FG}^N$ and $i, j \in N$ be such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$. Then,

$$\begin{aligned}
B_i(v) &= \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N \colon i \in E\}} v\left(E\right) \ominus v\left(E \setminus \{i\}\right) \right) \\
&= \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N \colon i \in E,\, j \in E\}} v\left(E\right) \ominus v\left(E \setminus \{i\}\right) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N \colon i \in E,\, j \notin E\}} v\left(E\right) \ominus v\left(E \setminus \{i\}\right) \right) \\
&= \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N \colon j \in E,\, i \in E\}} v\left(E\right) \ominus v\left(E \setminus \{j\}\right) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N \colon j \in E,\, i \notin E\}} v\left(E\right) \ominus v\left(E \setminus \{j\}\right) \right) \\
&= \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N \colon j \in E\}} v\left(E\right) \ominus v\left(E \setminus \{j\}\right) \right) \\
&= B_j(v).
\end{aligned} \tag{11}$$

NULL PLAYER. Let $v \in \mathcal{FG}^N$, $i \in N$ be such that i is a null player in v. Then,

$$B_i(v) = \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N \colon i \in E\}} \left(v\left(E\right) \ominus v\left(E\right)\right) \right),$$

which is 0-symmetric, since it is the multiplication of a real number and a 0-symmetric fuzzy quantity (take into account that the addition of 0-symmetric fuzzy quantities is 0-symmetric).
MERGER. Let $v \in \mathcal{FG}^N$ and let i, j be two different players in N. Then $B_i(v) \oplus B_j(v)$

is equal to

$$
\begin{aligned}
&= \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N : i \in E\}} v(E) \ominus v(E \setminus \{i\}) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N : j \in E\}} v(E) \ominus v(E \setminus \{j\}) \right) \\
&= \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N : i \in E, j \in E\}} v(E) \ominus v(E \setminus \{i\}) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N : i \in E, j \notin E\}} v(E) \ominus v(E \setminus \{i\}) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N : i \in E, j \in E\}} v(E) \ominus v(E \setminus \{j\}) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N : i \notin E, j \in E\}} v(E) \ominus v(E \setminus \{j\}) \right)
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{2^{|N|-2}} \odot \left(\bigoplus_{\{E \subseteq N:\, i \in E,\, j \in E\}} v(E) \ominus v(E \setminus \{i,j\}) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N:\, i \in E,\, j \notin E\}} v(E) \ominus v(E) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N:\, i \notin E,\, j \in E\}} v(E) \ominus v(E) \right) \\
&= \frac{1}{2^{|N^{ij}|-1}} \odot \left(\bigoplus_{\left\{H \subseteq N^{ij}:\, \widehat{ij} \in H\right\}} v^{ij}(H) \ominus v^{ij}\left(H \setminus \left\{\widehat{ij}\right\}\right) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N:\, i \in E,\, j \notin E\}} v(E) \ominus v(E) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N:\, i \notin E,\, j \in E\}} v(E) \ominus v(E) \right) \\
&= B_{\widehat{ij}}\left(v^{ij}\right) \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N:\, i \in E,\, j \notin E\}} v(E) \ominus v(E) \right) \\
&\quad \oplus \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N:\, i \notin E,\, j \in E\}} v(E) \ominus v(E) \right).
\end{aligned}$$

Equally signed marginal contributions. It easily follows from the definition of B and the fact that if $a, b \in \mathbb{F}$ and $a, b \geqslant 0$ then $a \oplus b \geqslant 0$ and $a \odot b \geqslant 0$.

Zero solution. Let $v \in \mathcal{FG}^N$ be such that $0 \subseteq v(E)$ for every $E \in 2^N$. Take $i \in N$. Then, by (6), (7) and (8),

$$0 \subseteq \frac{1}{2^{|N|-1}} \odot \left(\bigoplus_{\{E \subseteq N:\, i \in E\}} v(E) \ominus v(E \setminus \{i\}) \right),$$

that is, $0 \subseteq B_i(v)$. □

Now we aim to prove that if a value for games with fuzzy characteristic function satisfies the seven properties stated in the previous theorem then this value is equal to the Banzhaf value for cooperative games with fuzzy characteristic function. Firstly we need to see two simple lemmas.

Lemma 3.3. *Let $a, b, c, d \in \mathbb{F}$ be such that*

i) $a, c \geqslant 0$ *(resp.* $a, c \leqslant 0$*),*
ii) $0 \subseteq a, c$*,*

iii) b, d *are* 0*-symmetric,*
iv) $a \oplus b = c \oplus d$.

Then, $a = c$ *and* $b = d$.

Proof. We will prove only the version where $a, c \geqslant 0$.

Let $t \in [0,1]$. From *i)* and *ii)*, $a_t^- = c_t^- = 0$. By *iii)*, $b_t^- = -b_t^+$ and $c_t^- = -c_t^+$. We have that

$$[a \oplus b]_t = [a_t^- + b_t^-, a_t^+ + b_t^+] = [-b_t^+, a_t^+ + b_t^+] \tag{12}$$

and

$$[c \oplus d]_t = [c_t^- + d_t^-, c_t^+ + d_t^+] = [-d_t^+, c_t^+ + d_t^+]. \tag{13}$$

From (12), (13) and *iv)* we obtain that $b_t^+ = d_t^+$ and $a_t^+ = c_t^+$. It is clear that $[a]_t = [c]_t$ and $[b]_t = [d]_t$. Since these equalities hold for every $t \in [0,1]$, it follows that $a = c$ and $b = d$. □

Lemma 3.4. *Let* $a, b \in \mathbb{F}$ *be such that*

i) $a \geqslant 0$ *(resp.* $a \leqslant 0$*),*
ii) $0 \subseteq a$*,*
iii) $a \subseteq b$*,*
iv) $b \leqslant a$ *(resp.* $b \geqslant a$*),*
v) b *is* 0*-symmetric,*

Then, $b = a \ominus a$.

Proof. We will prove only the version where $a \geqslant 0$ and $b \leqslant a$.

Let $x \in [0, +\infty)$. Let us see that $\mu_b(x) \leqslant \mu_a(x)$. Suppose that $\mu_b(x) > \mu_a(x)$. If we take $t \in (\mu_a(x), \mu_b(x))$, then $x \in [b]_t$ and $x \notin [a]_t$. From *i)* and *ii)*, $[a]_t = [0, a_t^+]$. We have that $x \in [0, +\infty)$, $x \in [b_t^-, b_t^+]$ and $x \notin [0, a_t^+]$. It follows that $b_t^+ > a_t^+$, but this contradicts condition *iv)*. We have proved that $\mu_b(x) \leqslant \mu_a(x)$ for every $x \in [0, +\infty)$. By *iii)*, we know that $\mu_b(x) \geqslant \mu_a(x)$ for every $x \in \mathbb{R}$. We conclude that $\mu_b(x) = \mu_a(x)$ for every $x \in [0, +\infty)$. From this and condition *v)* it follows that $[b]_t = [-a_t^+, a_t^+]$ for every $t \in [0,1]$. And it suffices to notice that, by *i)* and *ii)*, $[a \ominus a]_t = [-a_t^+, a_t^+]$ for every $t \in [0,1]$. □

Now we are in conditions to complete the characterization of the Banzhaf value for cooperatives games with fuzzy characteristic function.

Theorem 3.5. *If a value* Ψ *for games with fuzzy characteristic function satisfies the properties of* 1*-efficiency, additivity, equal treatment, null player, merger, equally signed marginal contributions and zero solution, then* Ψ *is equal to the Banzhaf value for cooperative games with fuzzy characteristic function.*

Proof. Suppose that Ψ satisfies the properties stated in the theorem. Our goal is to prove that $\Psi = B$. The proof will be done in several steps. In each step it will be shown that $\Psi(v) = B(v)$ for every v in a certain class of games.

STEP 1. We aim to prove that

$$\Psi(v) = B(v) \tag{14}$$

for every nonempty finite set N and every $v \in \mathcal{G}^N$.

If $\mathbf{0}$ denotes the game that assigns zero to all coalitions $E \in 2^N$, it is clear, from the property of equally signed marginal contributions, that $\Psi_i(\mathbf{0}) = 0$ for every $i \in N$. Let $v \in \mathcal{G}^N$. By additivity,

$$\Psi_i(v) \oplus \Psi_i(-v) = \Psi_i(\mathbf{0}) = 0, \tag{15}$$

for every $i \in N$. From (15) and property $\mathbf{n}$ on page 6 it follows that $\Psi_i(v) \in \mathbb{R}$ for every $i \in N$. Therefore, the restriction of Ψ to the family of games with real characteristic function, denoted by $\Psi_{|\mathcal{G}}$, is a value for cooperative crisp games. Taking into account that Ψ satisfies the properties of 1-efficiency, additivity, equal treatment, null player and merger and the fact that $\Psi(v) \in \mathbb{R}^N$ for every nonempty finite set N and every $v \in \mathcal{G}^N$, it can easily be verified that $\Psi_{|\mathcal{G}}$ satisfies the properties (for values on crisp games) of 1-efficiency, additivity, equal treatment, null player and merger. Since these properties characterize the Banzhaf value, we conclude that $\Psi_{|\mathcal{G}} = \beta$. By the same reasoning, $B_{|\mathcal{G}} = \beta$. We have proved (14).

STEP 2. Our goal is to prove that

$$\Psi(a \odot u_E) = B(a \odot u_E) \tag{16}$$

for every nonempty finite set N, every $E \in 2^N \setminus \{\emptyset\}$ and every $a \in \mathbb{F}$ with $a \geqslant 0$ and $0 \subseteq a$.

Let $a \in \mathbb{F}$ be such that $a \geqslant 0$ and $0 \subseteq a$. Firstly we will prove that

$$\Psi_i(a \cdot u_E) = \frac{1}{2^{|E|-1}} \odot a \tag{17}$$

for every nonempty finite set N, every $E \in 2^N \setminus \{\emptyset\}$ and every $i \in E$. Let us prove (17) by induction on $|N|$.

Base case. $|N| = 1$. We have that $E = N$. Let $N = \{i\}$. By the property of 1-efficiency, $\Psi_i(a \odot u_{\{i\}}) = a$. Therefore, (17) holds.

Inductive step. Let N be a finite set with $|N| > 1$ and let $E \in 2^N \setminus \{\emptyset\}$. Take $i \in E$. Since $|N| > 1$ we can take $j \in N \setminus \{i\}$. We distinguish two cases:

- $j \in E$. From the property of merger it follows that there exists a 0-symmetric fuzzy quantity $d \in \mathbb{F}$ such that

$$\Psi_i(a \odot u_E) \oplus \Psi_j(a \odot u_E) = \Psi_{\widehat{ij}}(a \odot u_{(E\setminus\{i,j\})\cup\{\widehat{ij}\}}) \oplus d. \tag{18}$$

By induction hypothesis,

$$\Psi_{\widehat{ij}}(a \odot u_{(E \setminus \{i,j\}) \cup \{\widehat{ij}\}}) = \frac{1}{2^{|E|-2}} \odot a. \tag{19}$$

By equal treatment,

$$\Psi_i(a \odot u_E) = \Psi_j(a \odot u_E). \tag{20}$$

From (18), (19) and (20) we obtain that

$$2 \odot \Psi_i(a \odot u_E) = \frac{1}{2^{|E|-2}} \odot a \oplus d.$$

If $i \in E$ and $F \subseteq N \setminus \{i\}$, then $(a \odot u_E)(F \cup \{i\}) \ominus (a \odot u_E)(F) \geqslant 0$. By the property of equally signed marginal contributions, it follows that $\Psi_i(a \odot u_E) \geqslant 0$. Since $0 \subseteq (a \odot u_E)(F)$ for every $F \in 2^N$, we obtain, by the property of zero solution, that $0 \subseteq \Psi_i(a \odot u_E)$ for every $i \in N$. Therefore, the following conditions hold:

i) $2 \odot \Psi_i(a \odot u_E) \geqslant 0$, $\frac{1}{2^{|E|-2}} \odot a \geqslant 0$,
ii) $0 \subseteq 2 \odot \Psi_i(a \odot u_E)$, $0 \subseteq \frac{1}{2^{|E|-2}} \odot a$,
iii) 0 and d are 0-symmetric,
iv) $2 \odot \Psi_i(a \odot u_E) \oplus 0 = \frac{1}{2^{|E|-2}} \odot a \oplus d.$.

By applying Lemma 3.3 we obtain that $2 \odot \Psi_i(a \odot u_E) = \frac{1}{2^{|E|-2}} \odot a$, whence it easily follows that $\Psi_i(a \odot u_E) = \frac{1}{2^{|E|-1}} \odot a$.

- $j \in N \setminus E$. From the property of merger it follows that there exists a 0-symmetric fuzzy quantity $d \in \mathbb{F}$ such that

$$\Psi_i(a \odot u_E) \oplus \Psi_j(a \odot u_E) = \Psi_{\widehat{ij}}(a \odot u_{(E \setminus \{i\}) \cup \{\widehat{ij}\}}) \oplus d. \tag{21}$$

By induction hypothesis,

$$\Psi_{\widehat{ij}}(a \odot u_{(E \setminus \{i\}) \cup \{\widehat{ij}\}}) = \frac{1}{2^{|E|-1}} \odot a. \tag{22}$$

From (21) and (22) we obtain that

$$\Psi_i(a \odot u_E) \oplus \Psi_j(a \odot u_E) = \frac{1}{2^{|E|-1}} \odot a \oplus d.$$

If $i \in E$ and $F \subseteq N \setminus \{i\}$, then $(a \odot u_E)(F \cup \{i\}) \ominus (a \odot u_E)(F) \geqslant 0$. By the property of equally signed marginal contributions, it follows that $\Psi_i(a \odot u_E) \geqslant 0$. Since $0 \subseteq (a \odot u_E)(F)$ for every $F \in 2^N$, we obtain, by the property of zero solution, that $0 \subseteq \Psi_i(a \odot u_E)$ for every $i \in N$. Moreover, by the null player property, $\Psi_j(a \odot u_E)$ is 0-symmetric. Therefore, the following conditions hold:

i) $\Psi_i(a \odot u_E) \geqslant 0$, $\frac{1}{2^{|E|-1}} \odot a \geqslant 0$,
ii) $0 \subseteq \Psi_i(a \odot u_E)$, $0 \subseteq \frac{1}{2^{|E|-1}} \odot a$,
iii) $\Psi_j(a \odot u_E)$ and d are 0-symmetric,
iv) $\Psi_i(a \odot u_E) \oplus \Psi_j(a \odot u_E) = \frac{1}{2^{|E|-1}} \odot a \oplus d$.

By applying Lemma 3.3 we obtain that $\Psi_i(a \odot u_E) = \frac{1}{2^{|E|-1}} \odot a$.

Therefore, we have proved (17).

Now our goal is to show that

$$\Psi_j(a \cdot u_E) = \frac{1}{2^{|E|}} \odot (a \ominus a) \tag{23}$$

for every nonempty finite set N, every $E \in 2^N \setminus \{N, \emptyset\}$ and every $j \in N \setminus E$.

Let N be a nonempty finite set, $E \in 2^N \setminus \{N, \emptyset\}$ and $j \in N \setminus E$. By (17) we know that

$$\Psi_j(a \odot u_{E\cup\{j\}}) = \frac{1}{2^{|E|}} \odot a. \tag{24}$$

Let $w \in \mathcal{FG}^N$ defined by

$$w(F) = \begin{cases} a & \text{if } E \subseteq F \text{ and } j \notin F, \\ 0 & \text{otherwise,} \end{cases}$$

for every $F \in 2^N$. We have that $a \odot u_E = (a \odot u_{E\cup\{j\}}) \oplus w$. By additivity,

$$\Psi_j(a \odot u_E) = \Psi_j(a \odot u_{E\cup\{j\}}) \oplus \Psi_j(w). \tag{25}$$

Since $0 \subseteq a$, we have that $0 \subseteq w(F)$ for every $F \in 2^N$. By the property of zero solution,

$$0 \subseteq \Psi_j(w). \tag{26}$$

From (6), (24), (25) and (26),

$$\frac{1}{2^{|E|}} \odot a \subseteq \Psi_j(a \odot u_E). \tag{27}$$

Note that $w(F \cup \{j\}) \ominus w(F) \leqslant 0$ for every $F \subseteq N \setminus \{j\}$. By the property of equally signed marginal contributions,

$$\Psi_j(w) \leqslant 0. \tag{28}$$

From (24), (25) and (28) it easily follows that

$$\Psi_j(a \odot u_E) \leqslant \frac{1}{2^{|E|}} \odot a. \tag{29}$$

Notice that j is a null player in $a \odot u_E$. By the property of null player, $\Psi_j(a \odot u_E)$ is 0-symmetric. From this fact together with $a \geqslant 0$, $0 \subseteq a$, (27) and (29) we obtain that the following conditions hold:

i) $\frac{1}{2^{|E|}} \odot a \geqslant 0$,

ii) $0 \subseteq \frac{1}{2^{|E|}} \odot a$,

iii) $\frac{1}{2^{|E|}} \odot a \subseteq \Psi_j(a \odot u_E)$,

iv) $\Psi_j(a \odot u_E) \leqslant \dfrac{1}{2^{|E|}} \odot a$,

v) $\Psi_j(a \odot u_E)$ is 0-symmetric.

By Lemma 3.4 and (4),

$$\Psi_j(a \odot u_E) = \frac{1}{2^{|E|}} \odot (a \ominus a). \tag{30}$$

From (17) and (30),

$$\Psi_i(a \odot u_E) = \begin{cases} \dfrac{1}{2^{|E|-1}} \odot a & \text{if } i \in E, \\ \\ \dfrac{1}{2^{|E|}} \odot (a \ominus a) & \text{if } i \in N \setminus E. \end{cases}$$

Since we have used only the properties stated in the theorem, we have proved (16).

Step 3. We must see that

$$\Psi(a \odot u_E) = B(a \odot u_E) \tag{31}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$ with $a \leqslant 0$ and $0 \subseteq a$.

The proof is similar to that of (16). The only difference lies in the versions used of Lemma 3.3, Lemma 3.4 and the property of equally signed marginal contributions.

Step 4. Let us prove that

$$\Psi(a \odot u_E) = B(a \odot u_E) \tag{32}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$.

Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$. Take $z \in \mathit{core}(a)$. Let $b, c \in \mathbb{F}$ defined by

$$\mu_b(x) = \begin{cases} \mu_a(z+x) & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases}$$

$$\mu_c(x) = \begin{cases} 0 & \text{if } x > 0, \\ \mu_a(z+x) & \text{if } x \leqslant 0. \end{cases}$$

Notice that $b \geqslant 0$, $c \leqslant 0$ and $0 \subseteq b, c$. It can easily be verified that $[b]_t = [0, a_t^+ - z]$ and $[c]_t = [a_t^- - z, 0]$ for every $t \in [0,1]$. It follows that $a = z \oplus b \oplus c$. Hence, $a \odot u_E = (z\, u_E) \oplus (b \odot u_E) \oplus (c \odot u_E)$. By additivity, (14), (16) and (31) we obtain that

$$\begin{aligned} \Psi(a \odot u_E) &= \Psi(zu_E) \oplus \Psi(b \odot u_E) \oplus \Psi(c \odot u_E) \\ &= B(zu_E) \oplus B(b \odot u_E) \oplus B(c \odot u_E) \\ &= B(a \odot u_E). \end{aligned}$$

We have proved (32).

STEP 5. Our goal is to prove that

$$\Psi(a \odot \delta_E) = B(a \odot \delta_E) \tag{33}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$.

Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$. By (1) and (2),

$$\delta_E = \sum_{\{F \in 2^N :\, E \subseteq F\}} (-1)^{|F|-|E|} u_F,$$

whence

$$\begin{aligned} \delta_E \quad &+ \sum_{\{F \in 2^N :\, E \subseteq F\}} u_F \\ &= \sum_{\{F \in 2^N :\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} 2u_F, \end{aligned} \tag{34}$$

that is,

$$\begin{aligned} \delta_E(H) \quad &+ \sum_{\{F \in 2^N :\, E \subseteq F\}} u_F(H) \\ &= \sum_{\{F \in 2^N :\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} 2u_F(H), \end{aligned} \tag{35}$$

for every $H \subseteq N$. As the fuzzy arithmetic coincides with the crisp arithmetic over the real numbers (seeing them as fuzzy numbers),

$$\begin{aligned} \delta_E(H) \quad &\oplus \bigoplus_{\{F \in 2^N :\, E \subseteq F\}} u_F(H) \\ &= \bigoplus_{\{F \in 2^N :\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} 2u_F(H), \end{aligned} \tag{36}$$

If we multiply by a and apply (5) we obtain

$$\begin{aligned} (a \odot \delta_E)(H) \quad &\oplus \bigoplus_{\{F \in 2^N :\, E \subseteq F\}} (a \odot u_F)(H) \\ &= \bigoplus_{\{F \in 2^N :\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} ((2 \odot a) \odot u_F)(H), \end{aligned} \tag{37}$$

for every $H \subseteq N$. Hence,

$$\begin{aligned}(a \odot \delta_E) &\oplus \bigoplus_{\{F \in 2^N :\, E \subseteq F\}} (a \odot u_F) \\ &= \bigoplus_{\{F \in 2^N :\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} (2 \odot a) \odot u_F.\end{aligned} \tag{38}$$

which, by additivity, leads to

$$\begin{aligned}\Psi_i(a \odot \delta_E) &\oplus \bigoplus_{\{F \in 2^N \setminus \{\emptyset\} :\, E \subseteq F\}} \Psi_i(a \odot u_F) \\ &= \bigoplus_{\{F \in 2^N \setminus \{\emptyset\} :\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} \Psi_i((2 \odot a) \odot u_F)\end{aligned} \tag{39}$$

and

$$\begin{aligned}B_i(a \odot \delta_E) &\oplus \bigoplus_{\{F \in 2^N \setminus \{\emptyset\} :\, E \subseteq F\}} B_i(a \odot u_F) \\ &= \bigoplus_{\{F \in 2^N \setminus \{\emptyset\} :\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} B_i((2 \odot a) \odot u_F)\end{aligned} \tag{40}$$

for every $i \in N$. From (32), (39), (40) and property **1** on page 6 it is concluded that $\Psi_i(a \odot \delta_E) = B_i(a \odot \delta_E)$ for every $i \in N$. We have proved (33).

Step 6. We aim to prove that

$$\Psi(v) = B(v)$$

for every $v \in \mathcal{FG}^N$.

Let $v \in \mathcal{FG}^N$. Notice that

$$v = \bigoplus_{E \in 2^N \setminus \{\emptyset\}} (v(E) \odot \delta_E).$$

By additivity and (33),

$$\begin{aligned}\Psi(v) &= \bigoplus_{E \in 2^N \setminus \{\emptyset\}} \Psi(v(E) \odot \delta_E) \\ &= \bigoplus_{E \in 2^N \setminus \{\emptyset\}} B(v(E) \odot \delta_E) = B(v),\end{aligned}$$

which completes the proof. □

Remark. Observe that the first five axioms form a classic axiomatization of the Banzhaf value when we apply them only over the set of crisp games, those classic games with real characteristic function. The last axiom really is a trivial condition for crisp games. If $0 \subseteq v(E)$ for every coalition and $v(E) \in \mathbb{R}$ then $v(E) = 0$. So, in the crisp case the null player axiom implies that the payoff is zero for all the players. This axiom is necessary in the fuzzy case on the problem of the difference of fuzzy numbers. The sixth axiom is perhaps more surprising because in the crisp case this axiom says the same but it is not necessary. We will see that the equally signed marginal contributions axiom cannot remove by the others in the fuzzy case. Suppose $v \in \mathcal{FG}^N$, we denote by $a^v \in \mathbb{F}$ the interval $a^v = [-(v(N)_1^+ - v(N)_1^-), (v(N)_1^+ - v(N)_1^-)]$. Fuzzy quantity a^v is 0-symmetric. We propose this another solution for fuzzy games. If $v \in \mathcal{FG}^N$ and $i \in N$ then

$$D_i(v) = B_i(v) \oplus [(|N| - 1) \odot a^v] .$$

Notice that $D = B$ on classic games because $a^v = 0$. It is easy to test that D satisfies 1-efficiency, additivity, null player, equal treatment, merger and zero solution but D does not satisfy equally signed marginal contributions.

4. The real Banzhaf value for cooperative games with fuzzy characteristic function

A real value for cooperative games with fuzzy characteristic function assigns to each nonempty finite set N and $v \in \mathcal{FG}^N$ a vector $(\Theta_i(v))_{i\in N} \in \mathbb{R}^N$.

We aim to define a real value for cooperative games with fuzzy characteristic function with nice properties. To this end, we will use the function $M\colon \mathbb{F} \to \mathbb{R}$ introduced by Yager (1981) for ordering fuzzy numbers. Given $a \in \mathbb{F}$,

$$M(a) = \frac{1}{2}\int_0^1 a_t^+ dt + \frac{1}{2}\int_0^1 a_t^- dt.$$

Definition 4.1. The real Banzhaf value for cooperative games with fuzzy characteristic function is defined by

$$\mathfrak{B}_i(v) = M(B_i(v))$$

for every nonempty finite set N, $v \in \mathcal{FG}^N$ and $i \in N$.

Remark. It is easy to check that $M(a \oplus b) = M(a) + M(b)$, $M(a \ominus b) = M(a) - M(b)$ and $M(p \odot a) = pM(a)$ for every $a, b \in \mathbb{F}$ and every $p \in \mathbb{F}$. From these equalities and the definitions of B and $\mathfrak{B}$ it easily follows that

$$\mathfrak{B}_i(v) = \beta_i(M \circ v)$$

for every nonempty finite set N, $v \in \mathcal{FG}^N$ and $i \in N$.

Example. A company is advertising its products on three websites a, b, c. The managers want to keep this advertising, but they aim to reallocate the amounts spent on these websites, according to the effectiveness of the company's advertising on each one of them. To this end we propose to calculate an index quantifying the interest of each web for the company. Let $N = \{a, b, c\}$. Suppose we know, for each customer that purchased a product from the company, which websites she/he visited (within N). Table 1 shows these data, where $s(E)$ is the number of buyers who visited the pages in E (all of them and only them) before purchase. For instance, $s(\{a, b\})$ means the number of buyers who visited both pages, a and b, before purchase. Notice that we do not know if they bought as a result of their visit to a, b or both.

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	N
$s(E)$	250	150	50	50	100	150	250

Table 1. Number of buyers according to the websites they visited.

Now we define a cooperative game (N, v) with fuzzy characteristic function. For each coalition E, $v(E)$ is the number of buyers for whose purchase it was essential to visit all the websites in E. Notice that the buyers who visited only the websites in E are among them, $\sum_{F \subseteq E} s(F) \leqslant v(E)$. But also some customers accounted for $s(F)$, with $F \cap E \neq \emptyset$, $F \setminus E \neq \emptyset$, could have purchased based on their visit to websites in E exclusively. Therefore, we do not know $v(E)$ exactly, but we can consider a description by means of fuzzy numbers in the following way:

$$\mu_{v(a)}(x) = \begin{cases} \dfrac{650 - x}{400} & \text{if } x \in [250, 650] \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{v(b)}(x) = \begin{cases} \dfrac{600 - x}{450} & \text{if } x \in [150, 600] \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{v(c)}(x) = \begin{cases} \dfrac{450 - x}{400} & \text{if } x \in [50, 450] \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{v(a,b)}(x) = \begin{cases} \dfrac{950 - x}{500} & \text{if } x \in [450, 950] \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{v(a,c)}(x) = \begin{cases} \dfrac{850 - x}{450} & \text{if } x \in [400, 850] \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{v(b,c)}(x) = \begin{cases} \dfrac{750 - x}{400} & \text{if } x \in [350, 750] \\ 0 & \text{otherwise,} \end{cases}$$

$$v(N) = 1000.$$

We calculate the real Banzhaf value of v by using the remark above. Firstly we determine the crisp game $M \circ v$. For instance, if $E = \{a\}$ then the cuts are $[250, 650 - 400t]$ for each $t \in [0,1]$, whence

$$(M \circ v)(a) = \int_0^1 \frac{1}{2}(250 + 650 - 400t)\; dt = 400.$$

E	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	N
$M \circ v$ (E)	350	237.5	150	575	512.5	450	1000

Table 2. Game $M \circ v$.

We obtain that the real Banzhaf value is

$$\mathfrak{B}(v) = (400, 312.5, 237.5).$$

The company will spend on advertising on websites a, b and c proportionally to this index.

In the remainder of this section, our goal will be to characterize this value.

Below we introduce some properties that a real value for cooperative games with fuzzy characteristic function may satisfy:

1-EFFICIENCY FOR SYMMETRIC PROFIT. If $v \in \mathcal{FG}^{\{i\}}$ and $v(\{i\})$ is a symmetric fuzzy quantity (i.e., there exists $p \in \mathbb{R}$ such that $v(\{i\}) \ominus p$ is 0-symmetric) then

$$\Theta_i(v) = p.$$

ADDITIVITY. If $v, w \in \mathcal{FG}^N$ then $\Theta(v \oplus w) = \Theta(v) + \Theta(w)$.

EQUAL TREATMENT. If $v \in \mathcal{FG}^N$, $i, j \in N$ and $v(E \cup \{i\}) = v(E \cup \{j\})$ for every $E \subseteq N \setminus \{i,j\}$, then $\Theta_i(v) = \Theta_j(v)$.

NULL PLAYER. If $v \in \mathcal{FG}^N$ and $i \in N$ is a null player in v, then $\Theta_i(v) = 0$.

MERGER. Let $v \in \mathcal{FG}^N$ and let i, j be two different players in N.

Then,

$$\Theta_i(v) + \Theta_j(v) = \Theta_{\widehat{ij}}\left(v^{ij}\right).$$

Let us consider the topology on $\mathbb{F}$ induced by the metric d_∞. Let us endow $\mathbb{F}^{2^N \setminus \{\emptyset\}}$ with the product topology. Since $\mathcal{FG}^N$ can be identified with the set $\mathbb{F}^{2^N \setminus \{\emptyset\}}$, we have endowed $\mathcal{FG}^N$ with a topology. Now we can state the following property.

CONTINUITY. The restriction of Θ to $\mathcal{FG}^N$ is a continuous mapping.

COMONOTONICITY. Let $v, w \in \mathcal{FG}^N$ be such that $core(v(E)) \cap core(w(E)) \neq \emptyset$ for every $E \in 2^N$. Let $\alpha \in (0,1)$. Consider $h \in \mathcal{FG}^N$ defined by

$$\mu_{h(E)}(x) = \alpha\mu_{v(E)}(x) + (1-\alpha)\mu_{w(E)}(x)$$

for every $E \in 2^N$ and every $x \in \mathbb{R}$. Then,

$$\Theta(h) = \alpha\Theta(v) + (1-\alpha)\Theta(w).$$

Comonotonicity can be understood in the following sense. Considering one particular situation, we can define different games with fuzzy payoffs to represent that situation with slight differences (so, usually the cores have non-empty intersection) depending on the analysis of estimations. The axiom says that the payoff vector of a weighted average of these games is the weighted average of the payoff vectors of them.

Let us see that $\mathfrak{B}$ satisfies the seven properties above.

Theorem 4.2. *The real Banzhaf value for cooperative games with fuzzy characteristic function satisfies the properties of* 1*-efficiency for symmetric profit, additivity, equal treatment, null player, merger, continuity and comonotonicity.*

Proof. 1-EFFICIENCY FOR SYMMETRIC PROFIT. Let $v \in \mathcal{FG}^{\{i\}}$ with $v(\{i\})$ symmetric. Since B satisfies 1-efficiency, $B_i(v) = v(\{i\})$. Therefore, $\mathfrak{B}_i(v) = M(v(\{i\}))$. Let $p \in \mathbb{R}$ be such that $v(\{i\}) \ominus p$ is 0-symmetric. It is clear that

$$\frac{1}{2}v(N)_t^+ + \frac{1}{2}v(N)_t^- = p \qquad \text{for every } t \in [0,1].$$

We have that

$$\begin{aligned} \mathfrak{B}_i(v) &= M(v(\{i\})) \\ &= \frac{1}{2}\int_0^1 v(\{i\})_t^+ dt + \frac{1}{2}\int_0^1 v(\{i\})_t^- dt \\ &= p. \end{aligned}$$

ADDITIVITY. Let $v, w \in \mathcal{FG}^N$. Let $E \in 2^N$ and $i \in N$. We have

$$\begin{aligned} \mathfrak{B}_i(v \oplus w) &= M(B_i(v \oplus w)) = M(B_i(v) + B_i(w)) \\ &= M(B_i(v)) + M(B_i(w)) = \mathfrak{B}_i(v) + \mathfrak{B}_i(w), \end{aligned}$$

where we have used the additivity of B and M.

EQUAL TREATMENT. Let $v \in \mathcal{FG}^N$ and $i, j \in N$ be such that $v(E \cup \{i\}) = v(E \cup \{j\})$ for every $E \subseteq N \setminus \{i, j\}$. Since B satisfies the property of equal treatment, $B_i(v) = B_j(v)$. Consequently, $\mathfrak{B}_i(v) = M(B_i(v)) = M(B_j(v)) = \mathfrak{B}_j(v)$.

NULL PLAYER. Let $v \in \mathcal{FG}^N$, $i \in N$ be such that i is a null player in v. Since B satisfies the property of null player, $B_i(v) = 0$. Therefore, $\mathfrak{B}_i(v) = M(B_i(v)) = M(0) = 0$.

MERGER. Let $v \in \mathcal{FG}^N$ and let i, j be two different players in N. Since B satisfies the merger property, there exists a 0-symmetric fuzzy quantity d such that

$$B_i(v) \oplus B_j(v) = B_{\widehat{ij}}\left(v^{ij}\right) \oplus d.$$

If we apply M on both sides we obtain

$$M(B_i(v)) + M(B_j(v)) = M(B_{\widehat{ij}}\left(v^{ij}\right)),$$

where we have applied that M is additive and that $M(d) = 0$. Therefore, we conclude that $\mathfrak{B}_i(v) + \mathfrak{B}_j(v) = \mathfrak{B}_{\widehat{ij}}\left(v^{ij}\right)$.

CONTINUITY. Let N be a nonempty finite set. In Remark 1 we saw that $\mathfrak{B}(v) = \beta(M \circ v)$ for every $v \in \mathcal{FG}^N$. Notice that the restriction of β to $\mathcal{G}^N$ is linear, and, consequently, continuous. Therefore, it is clear that in order to prove that the restriction of $\mathfrak{B}$ to $\mathcal{FG}^N$ is continuous, it suffices to show that the function $M\colon \mathbb{F} \to \mathbb{R}$ is continuous. Let $M^+, M^-\colon \mathbb{F} \to \mathbb{R}$ defined by

$$M^+(a) = \int_0^1 a_t^+ dt,$$

$$M^-(a) = \int_0^1 a_t^- dt,$$

for every $a \in \mathbb{F}$. Since $M = \frac{1}{2}(M^+ + M^-)$ it is enough to prove that M^+ and M^- are continuous. Let us prove that M^+ is continuous (the reasoning for M^- is analogous). Let $a \in \mathbb{F}$ and let $\epsilon > 0$. Let $b \in \mathbb{F}^N$ be such that $d_\infty(a,b) < \epsilon$. We have that

$$\left|M^+(a) - M^+(b)\right| \leqslant \int_0^1 \left|a_t^+ - b_t^+\right| dt. \tag{41}$$

Take $t_0 \in [0,1]$. Let us prove that $\left|a_{t_0}^+ - b_{t_0}^+\right| \leqslant d_H([a]_{t_0}, [b]_{t_0})$. Suppose that $a_{t_0}^+ \geqslant b_{t_0}^+$ (the case $a_{t_0}^+ < b_{t_0}^+$) is analogous). We have that

$$\begin{aligned}\left|a_{t_0}^+ - b_{t_0}^+\right| &= a_{t_0}^+ - b_{t_0}^+ = \min\{|a_{t_0}^+ - y|\colon y \in [b]_{t_0}\}\\ &\leqslant \max\{\min\{|x - y|\colon y \in [b]_{t_0}\}\colon x \in [a]_{t_0}\}\\ &= d^*([a]_{t_0}, [b]_{t_0}) \leqslant d_H([a]_{t_0}, [b]_{t_0}).\end{aligned}$$

We have proved that

$$\left|a_t^+ - b_t^+\right| \leqslant d_H([a]_t, [b]_t) \tag{42}$$

for every $t \in [0,1]$.

By (41) and (42),

$$\begin{aligned}\left|M^+(a) - M^+(b)\right| &\leqslant \int_0^1 d_H([a]_t, [b]_t)dt\\ &\leqslant \int_0^1 d_\infty(a,b)dt = d_\infty(a,b) < \epsilon\end{aligned}$$

COMONOTONICITY. Let $v, w \in \mathcal{FG}^N$ be such that $core(v(F)) \cap core(w(F)) \neq \emptyset$ for every $F \in 2^N$. Let $\alpha \in (0,1)$. Consider $h \in \mathcal{FG}^N$ defined by

$$\mu_{h(F)}(x) = \alpha\mu_{v(F)}(x) + (1-\alpha)\mu_{w(F)}(x) \tag{43}$$

for every $F \in 2^N$ and every $x \in \mathbb{R}$. We aim to prove that $\mathfrak{B}(h) = \alpha\mathfrak{B}(v) + (1-\alpha)\mathfrak{B}(w)$. Taking into account Remark 1 and the linearity of β, it suffices to prove that

$$M \circ h = \alpha(M \circ v) + (1-\alpha)(M \circ w). \tag{44}$$

Let $E \in 2^N$. Take $p \in core(v(E)) \cap core(w(E))$. Let λ denote the Lebesgue measure. We have that $M(h(E))$ is equal to

$$\begin{aligned}
& \frac{1}{2}\int_0^1 h(E)_t^+ dt + \frac{1}{2}\int_0^1 h(E)_t^- dt \\
= \; & \frac{1}{2}\int_0^1 (p + \lambda(\{x \geqslant p \colon \mu_{h(E)}(x) \geqslant t\}))dt \\
& + \frac{1}{2}\int_0^1 (p - \lambda(\{x \leqslant p \colon \mu_{h(E)}(x) \geqslant t\}))dt \\
= \; & p + \frac{1}{2}\lambda(\{(x,t) \in [p, +\infty) \times [0,1] \colon \mu_{h(E)}(x) \geqslant t\}) \\
& - \frac{1}{2}\lambda(\{(x,t) \in [-\infty, p) \times [0,1] \colon \mu_{h(E)}(x) \geqslant t\}) \\
= \; & p + \frac{1}{2}\int_p^{+\infty} \lambda(\{t \in [0,1] \colon \mu_{h(E)}(x) \geqslant t\})dx \\
& - \frac{1}{2}\int_{-\infty}^p \lambda(\{t \in [0,1] \colon \mu_{h(E)}(x) \geqslant t\})dx \\
= \; & p + \frac{1}{2}\int_p^{+\infty} \mu_{h(E)}(x)dx - \frac{1}{2}\int_{-\infty}^p \mu_{h(E)}(x)dx
\end{aligned} \tag{45}$$

where we have applied Fubini's theorem. Similarly, we can prove that

$$M(v(E)) = p + \frac{1}{2}\int_p^{+\infty} \mu_{v(E)}(x)dx - \frac{1}{2}\int_{-\infty}^p \mu_{v(E)}(x)dx \tag{46}$$

and

$$M(w(E)) = p + \frac{1}{2}\int_p^{+\infty} \mu_{w(E)}(x)dx - \frac{1}{2}\int_{-\infty}^p \mu_{w(E)}(x)dx. \tag{47}$$

From (43), (45), (46) and (47) we obtain (44). □

Now we aim to prove that if a value for cooperative games with fuzzy characteristic function satisfies the seven properties stated in the previous theorem then this value is equal to the real Banzhaf value for cooperative games with fuzzy characteristic function.

Theorem 4.3. *If a real value for cooperative games with fuzzy characteristic function satisfies the properties of* 1*-efficiency for symmetric profit, additivity, equal treatment, null player, merger, continuity and comonotonicity, then* Θ *is equal to the real Banzhaf value for cooperative games with fuzzy characteristic function.*

Proof. Suppose that a real value Θ for cooperative games with fuzzy characteristic function satisfies the properties stated in the theorem. Our goal is to prove that $\Theta = \mathfrak{B}$. The proof will be done in several steps. In each step it will be shown that $\Theta(v) = (v)$ for every v in a certain class of games.

STEP 1. We aim to prove that

$$\Theta(a \odot u_E) = \mathfrak{B}(a \odot u_E). \tag{48}$$

for every nonempty finite set N, every $E \in 2^N \setminus \{\emptyset\}$ and every $a \in \mathbb{F}$ with $|\{\mu_a(z)\colon z \in \mathbb{R}\}| = 2$.

Let us prove (48) by induction on $|N|$.

Base case. $|N| = 1$. We have that $E = N$. Let $N = \{i\}$. Since $|\{\mu_a(z)\colon z \in \mathbb{R}\}| = 2$, there exist $x, y \in \mathbb{R}$ with $x \leqslant y$ such that

$$\mu_a(z) = \begin{cases} 1 & \text{if } z \in [x, y], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x, y]. \end{cases}$$

Notice that a is symmetric, since $a \ominus \frac{x+y}{2}$ is 0-symmetric. By the property of 1-efficiency for symmetric profit, $\Theta_i(a \odot u_{\{i\}}) = \frac{x+y}{2}$. We conclude that $\Theta(a \odot u_{\{i\}}) = \mathfrak{B}(a \odot u_{\{i\}})$.

Inductive step. Let N be a finite set with $|N| > 1$ and let $E \in 2^N \setminus \{\emptyset\}$. Take $i \in E$. Since $|N| > 1$ we can take $j \in N \setminus \{i\}$. We distinguish two cases:

- $j \in E$. From the property of merger it follows that

$$\Theta_i(a \odot u_E) + \Theta_j(a \odot u_E) = \Theta_{\widehat{ij}}(a \odot u_{(E \setminus \{i,j\}) \cup \{\widehat{ij}\}}). \tag{49}$$

  By the property of equal treatment,

$$\Theta_i(a \odot u_E) = \Theta_j(a \odot u_E). \tag{50}$$

  By (49) and (50),

$$\Theta_i(a \odot u_E) = \frac{1}{2}\Theta_{\widehat{ij}}(a \odot u_{(E \setminus \{i,j\}) \cup \{\widehat{ij}\}}). \tag{51}$$

  Similarly,

$$\mathfrak{B}_i(a \odot u_E) = \frac{1}{2}\mathfrak{B}_{\widehat{ij}}(a \odot u_{(E \setminus \{i,j\}) \cup \{\widehat{ij}\}}). \tag{52}$$

  By induction hypothesis,

$$\Theta_{\widehat{ij}}(a \odot u_{(E \setminus \{i,j\}) \cup \{\widehat{ij}\}}) = \mathfrak{B}_{\widehat{ij}}(a \odot u_{(E \setminus \{i,j\}) \cup \{\widehat{ij}\}}). \tag{53}$$

  By (51), (52) and (53),

$$\Theta_i(a \odot u_E) = \mathfrak{B}_i(a \odot u_E).$$

- $j \in N \setminus E$. From the property of merger it follows that

$$\Theta_i(a \odot u_E) + \Theta_j(a \odot u_E) = \Theta_{\widehat{ij}}(a \odot u_{(E\setminus\{i,j\})\cup\{\widehat{ij}\}}). \tag{54}$$

By the null player property,

$$\Theta_j(a \odot u_E) = 0. \tag{55}$$

By (54) and (55),

$$\Theta_i(a \odot u_E) = \Theta_{\widehat{ij}}(a \odot u_{(E\setminus\{i,j\})\cup\{\widehat{ij}\}}). \tag{56}$$

Similarly,

$$\mathfrak{B}_i(a \odot u_E) = \mathfrak{B}_{\widehat{ij}}(a \odot u_{(E\setminus\{i,j\})\cup\{\widehat{ij}\}}). \tag{57}$$

By induction hypothesis,

$$\Theta_{\widehat{ij}}(a \odot u_{(E\setminus\{i,j\})\cup\{\widehat{ij}\}}) = \mathfrak{B}_{\widehat{ij}}(a \odot u_{(E\setminus\{i,j\})\cup\{\widehat{ij}\}}). \tag{58}$$

By (56), (57) and (58),

$$\Theta_i(a \odot u_E) = \mathfrak{B}_i(a \odot u_E).$$

Therefore, we conclude that

$$\Theta_i(a \odot u_E) = \mathfrak{B}_i(a \odot u_E) \quad \text{for every } i \in E. \tag{59}$$

Moreover, by the null player property,

$$\Theta_i(a \odot u_E) = \mathfrak{B}_i(a \odot u_E) = 0 \quad \text{for every } i \in N \setminus E. \tag{60}$$

From (59) and (60) it follows that $\Theta(a \odot u_E) = \mathfrak{B}(a \odot u_E)$.

Step 2. Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$ be such that $|\{\mu_a(z) \colon z \in \mathbb{R}\}| = 3$. We aim to prove that

$$\Theta(a \odot u_E) = \mathfrak{B}(a \odot u_E). \tag{61}$$

It is clear that there exist $l \in (0,1)$ and $x, y, r, s \in \mathbb{R}$ with $x \leqslant r \leqslant s \leqslant y$ and $s - r < y - x$ such that

$$\mu_a(z) = \begin{cases} 1 & \text{if } z \in [r,s], \\ l & \text{if } z \in [x,y] \setminus [r,s], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x,y]. \end{cases}$$

Let $b, c \in \mathbb{F}$ defined by

$$\mu_b(z) = \begin{cases} 1 & \text{if } z \in [x, y], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x, y], \end{cases}$$

$$\mu_c(z) = \begin{cases} 1 & \text{if } z \in [r, s], \\ 0 & \text{if } z \in \mathbb{R} \setminus [r, s]. \end{cases}$$

Notice that

$$\mu_a(z) = l\mu_b(z) + (1 - l)\mu_c(z)$$

for every $z \in \mathbb{R}$. Therefore,

$$\mu_{(a \odot u_E)(F)}(z) = l\mu_{(b \odot u_E)(F)}(z) + (1 - l)\mu_{(c \odot u_E)(F)}(z)$$

for every $F \in 2^N$ and every $z \in \mathbb{R}$. Moreover,

$$core((b \odot u_E)(F)) \cap core((c \odot u_E)(F)) \neq \emptyset$$

for every $F \in 2^N$. By the property of comonotonicity and (48),

$$\begin{aligned} \Theta(a \odot u_E) &= l\Theta(b \odot u_E) + (1 - l)\Theta(c \odot u_E) \\ &= l\mathfrak{B}(b \odot u_E) + (1 - l)\mathfrak{B}(c \odot u_E) \\ &= \mathfrak{B}(a \odot u_E). \end{aligned}$$

STEP 3. Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$ be such that $\{\mu_a(z) \colon z \in \mathbb{R}\}$ is a finite set. We aim to prove that

$$\Theta(a \odot u_E) = \mathfrak{B}(a \odot u_E). \tag{62}$$

It is clear that there exist $l_1, \ldots, l_{n-1} \in (0, 1)$ with $l_1 < \ldots < l_{n-1} = 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$ with $x_1 \leqslant \ldots \leqslant x_n \leqslant y_n \leqslant \ldots y_1$ such that

$$\mu_a(z) = \begin{cases} 1 & \text{if } z \in [x_n, y_n] \\ l_i & \text{if } z \in [x_i, y_i] \setminus [x_{i+1}, y_{i+1}], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x_1, y_1]. \end{cases}$$

Consider $b_1, \ldots, b_n \in \mathbb{F}$ defined as

$$\mu_{b_n}(z) = \begin{cases} 1 & \text{if } z \in [x_n, y_n], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x_n, y_n], \end{cases}$$

$$\mu_{b_i}(z) = \begin{cases} 1 & \text{if } z = 0, \\ l_i & \text{if } z \in [x_i - x_{i+1}, y_i - y_{i+1}] \setminus \{0\}, \\ 0 & \text{if } z \in \mathbb{R} \setminus [x_i - x_{i+1}, y_i - y_{i+1}]. \end{cases}$$

for $i = 1, \ldots, n-2$. It can be easily verified that

$$a = \bigoplus_{i=1}^{n} b_i.$$

Therefore, it is clear that

$$a \odot u_E = \bigoplus_{i=1}^{n} b_i \odot u_E.$$

By additivity, (61) and (48),

$$\begin{aligned}
\Theta(a \odot u_E) &= \Theta(\bigoplus_{i=1}^{n} b_i \odot u_E) = \sum_{i=1}^{n} \Theta(b_i \odot u_E) \\
&= \sum_{i=1}^{n} \mathfrak{B}(b_i \odot u_E) = \mathfrak{B}(\bigoplus_{i=1}^{n} b_i \odot u_E) \\
&= \mathfrak{B}(a \odot u_E).
\end{aligned}$$

Step 4. Our goal is to prove that

$$\Theta(a \odot u_E) = \mathfrak{B}(a \odot u_E) \tag{63}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$.

Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$. Since we have already proved (62) we can suppose that $\{\mu_a(z) \colon z \in \mathbb{R}\}$ is not finite. By continuity and (62), in order to prove (63) it is enough to show that there are games with the form $b \odot u_E$, where $\{\mu_b(z) \colon z \in \mathbb{R}\}$ is finite, arbitrarily close to the game $a \odot u_E$. To this end, it suffices to see that we can find fuzzy quantities b, with $\{\mu_b(z) \colon z \in \mathbb{R}\}$ finite, arbitrarily close to a.

Let $\epsilon > 0$. Let $[a]_0 = [r, s]$ and let $x_1, \ldots, x_n \in \mathbb{R}$ such that $r = x_1 < \ldots < x_n = s$, $x_i - x_{i-1} < \epsilon$ for every $i = 2, \ldots, n$ and $x_k \in core(a)$ for some $k \in \{1, \ldots, n\}$. Consider $b \in \mathbb{F}$ defined by

$$\mu_b(z) = \begin{cases} \mu_a(x_i) & \text{if } z \in [x_i, x_{i+1}) \text{ with } i < k, \\ 1 & \text{if } z = x_k, \\ \mu_a(x_i) & \text{if } z \in (x_{i-1}, x_i] \text{ with } i > k, \\ 0 & \text{if } z \in \mathbb{R} \setminus [r, s]. \end{cases}$$

Let us see that $d_\infty(a, b) \leqslant \epsilon$. To this end, it suffices to prove that $d_H([a]_t, [b]_t) < \epsilon$ for every $t \in (0, 1]$. Let $t \in (0, 1]$. Let $[a]_t = [p, q]$. It is clear that $p \leqslant x_k \leqslant q$. Let $h \in \{1, \ldots, k-1\}$ and $l \in \{k+1, \ldots, n\}$ be such that $p \in (x_h, x_{h+1}]$ and $q \in [x_{l-1}, x_l)$. We have that

$$[x_{h+1}, x_{l-1}] \subseteq [a]_t \subset (x_h, x_l) \tag{64}$$

It can be easily verified that

$$\mu_b(x_h) = \mu_a(x_h) < t,$$

$$\mu_b(x_{h+1}) = \mu_a(x_{h+1}) \geqslant t,$$

$$\mu_b(x_{l-1}) = \mu_a(x_{l-1}) \geqslant t,$$

$$\mu_b(x_l) = \mu_a(x_l) < t.$$

Hence, $[x_{h+1}, x_{l-1}] \subseteq [b]_t \subset (x_h, x_l)$. From these inclusions, (64) and the inequalities $x_{h+1} - x_h < \epsilon$ and $x_l - x_{l-1} < \epsilon$ it easily follows that $d_H([a]_t, [b]_t) < \epsilon$.

STEP 5. Our goal is to prove that

$$\Theta(a \odot \delta_E) = \mathfrak{B}(a \odot \delta_E) \tag{65}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$.

Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$. By (1) and (2),

$$\delta_E = \sum_{\{F \in 2^N \setminus \{\emptyset\} :\ E \subseteq F\}} (-1)^{|F|-|E|} u_F,$$

whence

$$\begin{aligned} \delta_E \quad + & \sum_{\{F \in 2^N :\ E \subseteq F\}} u_F \\ = & \sum_{\{F \in 2^N :\ E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} 2u_F, \end{aligned}$$

that is,

$$\begin{aligned} \delta_E(H) \quad + & \sum_{\{F \in 2^N\} :\ E \subseteq F\}} u_F(H) \\ = & \sum_{\{F \in 2^N\} :\ E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} 2u_F(H), \end{aligned}$$

for every $H \subseteq N$. If we multiply by a and apply (5) we obtain

$$\begin{aligned} & (a \odot \delta_E)(H) \oplus \bigoplus_{\{F \in 2^N :\ E \subseteq F\}} (a \odot u_F)(H) \\ = & \bigoplus_{\{F \in 2^N :\ E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} ((2 \odot a) \odot u_F)(H), \end{aligned}$$

for every $H \subseteq N$. Hence,

$$
\begin{aligned}
& (a \odot \delta_E) \oplus \bigoplus_{\{F \in 2^N \,:\, E \subseteq F\}} (a \odot u_F) \\
= & \bigoplus_{\{F \in 2^N\} \,:\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} ((2 \odot a) \odot u_F).
\end{aligned}
$$

which, by additivity, leads to

$$
\begin{aligned}
& \Theta_i(a \odot \delta_E) + \sum_{\{F \in 2^N \,:\, E \subseteq F\}} \Theta_i(a \odot u_F) \\
= & \sum_{\{F \in 2^N \,:\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} \Theta_i((2 \odot a) \odot u_F)
\end{aligned}
\tag{66}
$$

and

$$
\begin{aligned}
& \mathfrak{B}_i(a \odot \delta_E) + \sum_{\{F \in 2^N \,:\, E \subseteq F\}} \mathfrak{B}_i(a \odot u_F) \\
= & \sum_{\{F \in 2^N \,:\, E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} \mathfrak{B}_i((2 \odot a) \odot u_F)
\end{aligned}
\tag{67}
$$

for every $i \in N$. From (63), (66) and (67), it is concluded that $\Theta_i(a \odot \delta_E) = \mathfrak{B}_i(a \odot \delta_E)$ for every $i \in N$. We have proved (65).

STEP 6. We aim to prove that

$$\Theta(v) = \mathfrak{B}(v)$$

for every $v \in \mathcal{FG}^N$.

Let $v \in \mathcal{FG}^N$. Notice that

$$v = \bigoplus_{E \in 2^N \setminus \{\emptyset\}} (v(E) \odot \delta_E).$$

By additivity and (65),

$$
\begin{aligned}
\Theta(v) &= \sum_{E \in 2^N \setminus \{\emptyset\}} \Theta(v(E) \odot \delta_E) \\
&= \sum_{E \in 2^N \setminus \{\emptyset\}} \mathfrak{B}(v(E) \odot \delta_E) = \mathfrak{B}(v),
\end{aligned}
$$

which completes the proof. □

Remark. Observe that the first five axioms form again (as in the case of the fuzzy solution in the before section) the classic axiomatization of the Banzhaf value that we commented in preliminaries, when we apply them only over the set of crisp games. Continuity (as Aubin (1981) showed) is a normal condition to extent discrete functions to the continuum. But we also need comonotonicity although we have additivity. Let

$m\colon \mathbb{F} \to \mathbb{F}$ be defined as $m(a) = \dfrac{1}{2}(a_1^+ + a_1^-)$ for every $a \in \mathbb{F}$. For each $v \in \mathcal{FG}^N$ we define $v_m \in \mathcal{G}^N$ as $v_m = m \circ v$. Let

$$\mathcal{D}(v) = \beta(v_m)$$

for every $v \in \mathcal{FG}^N$. It is easy to check that $\mathcal{D}$ satisfies 1-efficiency for symmetric profit, additivity, equal treatment, null player, merger and continuity, but $\mathcal{D}$ does not satisfy comonotonicity.

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