A characterization of the Shapley value for cooperative games with fuzzy characteristic function

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Abstract

The characteristic function of a cooperative game determines the payment that each coalition can obtain when the players in the coalition cooperate. But there are cooperative situations in which the players have only imprecise expectations about the profit that can be achieved by each coalition. These situations are modeled through cooperative games with fuzzy characteristic function, in which the payment of each coalition is a fuzzy quantity. A value for these games assigns to each player in a game a fuzzy quantity that indicates the vaguely expected payoff for the player. There is a Shapley value for games with fuzzy characteristic function, but no characterization of this value has been given in the literature. In this paper a characterization of the Shapley value for games with fuzzy characteristic function is presented.

Keywords: cooperative game; Shapley value; fuzzy set; fuzzy quantity

1. Introduction

A cooperative game is given by a characteristic function that determines the payment that can be obtained by each subset of players. In this setting it is assumed that the players know with precision the profit that can be achieved by each coalition. Nevertheless, in real life this is not always realistic. Sometimes, the payment achievable by a coalition depends on external circumstances which are not completely under control of the players in the coalition. In these situations there are only have vague expectations about the payment that could be obtained by each coalition. Game theorists have considered different approaches to deal with these situations. Charnes and Granot [4] introduced cooperative games in which the payment of each coalition is given by a random variable. Suijs et al. [13] extended this model when they introduced cooperative games in which the players in a coalition may have different kinds of behavior and a collection of random variables is assigned to each coalition. Later on, Timmer [14] introduced stochastic cooperative games without deterministic transfer

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payments. A different approach was considered by Branzei et al. [2] when they handled bankruptcy situations in which the claims are given by intervals of real numbers. This was the origin of cooperative interval games, in which the players only know a lower and a upper bound of the profit that can be obtained by each coalition. Cooperative interval games have multiple applications in economics and operations research (see [3]). The present paper is focused on the approach proposed by Mareš and Vlach [9], who introduced cooperative games with fuzzy characteristic function. In these games the characteristic function assigns to each coalition a fuzzy quantity that establishes imprecise expectations about the payment that the coalition would obtain. Since fuzzy quantities form a family of fuzzy subsets of the set of real numbers, the model introduced by Mareš and Vlach is another application of the theory of fuzzy sets, introduced by Zadeh [17], to cooperative game theory. In principle, a game with fuzzy characteristic function was associated with a crisp game which indicated the modal values of each coalition (i.e., the values with maximum possibility degree). However, the authors extended the model and considered games with fuzzy characteristic function independently of crisp games [10]. As in the case of crisp cooperative games, the main problem that arises when dealing with cooperative games with fuzzy characteristic function is how to share the total profit obtained by the grand coalition. In this regard, it seems plausible to think that the vagueness of the expected outcomes of the coalitions will cause vagueness in the payoffs of the players, even in cases in which the gain of the grand coalition is known with precision but the payments of other coalitions are not known accurately. This means that, for any reasonable solution concept, the payoff of each player in a game should be a fuzzy quantity. Several studies have been carried out in this line of research (see [1], [16]). Mareš [8] introduced different solutions for games with fuzzy characteristic function by considering a fuzzy version of the most usual solutions for deterministic games. In particular, he defined a Shapley value for games with fuzzy characteristic function. He proved that this value satisfies some good properties. However, no characterization has been given in the literature for this value. In this paper a natural characterization of the Shapley value for games with fuzzy characteristic function is presented.

The paper is organized as follows. In section 2 some concepts regarding cooperative games, fuzzy quantities and cooperative games with fuzzy characteristic function are recalled. In section 3 we present a characterization of the Shapley value for games with fuzzy characteristic function.

2. Preliminaries

2.1. Cooperative games

A *cooperative game (with transferable utility)* consists of a finite set of players N and a characteristic function $v : 2^N \to \mathbb{R}$ which satisfies $v(\emptyset) = 0$. The elements of N are called players, and the subsets of N coalitions. Given a coalition E, $v(E)$ is the worth of E, and it is interpreted as the collective payment that the players of E would obtain if they cooperate. Frequently, a cooperative game (N, v) is identified with the function v. The family of games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^{|N|}-1)$-dimensional real vector space. One basis of $\mathcal{G}^N$ is the set $\left\{\delta_E : E \in 2^N \setminus \{\emptyset\}\right\}$ where for a nonempty coalition E the game δ_E is defined by

$$\delta_E(F) = \begin{cases} 1 & \text{if } F = E, \\ 0 & \text{otherwise.} \end{cases}$$

Another basis of $\mathcal{G}^N$ is the set $\left\{u_E : E \in 2^N \setminus \{\emptyset\}\right\}$ where for a nonempty coalition E the unanimity game u_E is defined by

$$u_E(F) = \begin{cases} 1 & \text{if } E \subseteq F, \\ 0 & \text{otherwise.} \end{cases}$$

Every game $v \in \mathcal{G}^N$ can be written as

$$v = \sum_{\{E \in 2^N : E \neq \emptyset\}} \triangle_v(E)\, u_E \tag{1}$$

where $(\triangle_v(E))_{E \subseteq N}$ is the Möbius transform of v on the poset $(2^N, \subseteq)$. The coefficient $\triangle_v(E)$ is called the dividend of the coalition E in the game v and is given by

$$\triangle_v(E) = \sum_{F \subseteq E} (-1)^{|E|-|F|} v(F) \tag{2}$$

for every $E \in 2^N \setminus \{\emptyset\}$.

A value on $\mathcal{G}^N$ is a mapping $\psi : \mathcal{G}^N \to \mathbb{R}^N$. If $v \in \mathcal{G}^N$ and $i \in N$ the real number $\psi_i(v)$ is the payoff of the player i in the game v. Multiple values have been defined in the literature. The best-known of them is the *Shapley value* [11], which assigns to each player $i \in N$ in a game $v \in \mathcal{G}^N$ a weighted average of the marginal contributions of i to the coalitions. It is formally defined by

$$\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E \left(v(E) - v(E \setminus \{i\})\right)$$

for every $i \in N$ and every $v \in \mathcal{G}^N$, where

$$p_E = \frac{(|N| - |E|)!\,(|E| - 1)!}{|N|!}$$

for every $E \in 2^N \setminus \{\emptyset\}$.

Some desirable properties for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ are the following:

Efficiency: $\sum_{i\in N} \psi_i (v) = v (N)$ for all $v \in \mathcal{G}^N$.

Additivity: $\psi (v_1 + v_2) = \psi (v_1) + \psi (v_2)$ for all $v_1, v_2 \in \mathcal{G}^N$.

Equal treatment: If $v \in \mathcal{G}^N$, $i, j \in N$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$, then $\psi_i(v) = \psi_j(v)$.

Null player property: A player $i \in N$ is a null player in $v \in \mathcal{G}^N$ if $v (E) = v(E \setminus \{i\})$ for all $E \subseteq N$. If $i \in N$ is a null player in $v \in \mathcal{G}^N$ then $\psi_i (v) = 0$.

Strong monotonicity: If $v, w \in \mathcal{G}^N$, $i \in N$ and $v(E \cup \{i\}) - v(E) \geqslant w(E \cup \{i\}) - w(E)$ for every $E \subseteq N \setminus \{i\}$ then $\psi_i(v) \geqslant \psi_i(w)$.

The properties of efficiency, additivity, equal treatment and null player characterize the Shapley value [11]. Young [15] proved that the Shapley value is characterized also by the properties of efficiency, equal treatment and strong monotonicity.

2.2. Fuzzy quantities

Firstly we recall some definitions regarding fuzzy sets.

Given a set X, a fuzzy subset a of X is defined by its membership function $\mu_a \colon X \to [0, 1]$. For each $x \in X$ the number $\mu_a(x)$ is the degree of membership of x in a. For each $t \in (0, 1]$ the t-cut of a is defined by

$$[a]_t = \{x \in X \colon \mu_a(x) \geqslant t\}$$

Notice that the family of t-cuts determine a. The core of a is defined by

$$core(a) = [a]_1.$$

If a is a fuzzy subset of $\mathbb{R}$, the 0-cut of a is defined by

$$[a]_0 = \overline{\{x \in \mathbb{R} \colon \mu_a(x) > 0\}}.$$

If a, b are fuzzy subsets of X, it is said that a is contained in b, and it is denoted by $a \subseteq b$, if $\mu_a(x) \leqslant \mu_b(x)$ for every $x \in X$.

In this paper we will deal with a particular class of fuzzy subsets of $\mathbb{R}$, the class of fuzzy quantities. The term fuzzy quantity has been used in the literature with slightly different meanings. We will use the concept of fuzzy quantity as defined in [12]. A fuzzy subset a of $\mathbb{R}$ is a fuzzy quantity if it satisfies the following conditions:

i) $core(a) \neq \emptyset$.

ii) $[a]_t$ is a closed and bounded interval for every $t \in [0,1]$.

The set of fuzzy quantities will be denoted by $\mathbb{F}$. If $a \in \mathbb{F}$ and $t \in [0,1]$ we denote

$$a_t^+ = \max[a]_t \quad \text{and} \quad a_t^- = \min[a]_t.$$

In the remainder of this subsection we recall the basics of fuzzy arithmetic (see [5], [6], [7], [12]).

Let $a, b \in \mathbb{F}$.

- The sum $a \oplus b \in \mathbb{F}$ is defined by

  $$\mu_{a\oplus b}(x) = \sup\{\min\{\mu_a(y), \mu_b(z)\} \colon y, z \in \mathbb{R},\ y + z = x\}$$

  for every $x \in \mathbb{R}$. Equivalently,

  $$[a \oplus b]_t = [a_t^- + b_t^-, a_t^+ + b_t^+]$$

  for every $t \in [0,1]$.

- The difference $a \ominus b \in \mathbb{F}$ is defined by

  $$\mu_{a\ominus b}(x) = \sup\{\min\{\mu_a(y), \mu_b(z)\} \colon y, z \in \mathbb{R},\ y - z = x\}$$

  for every $x \in \mathbb{R}$. Equivalently,

  $$[a \ominus b]_t = [a_t^- - b_t^+, a_t^+ - b_t^-]$$

  for every $t \in [0,1]$.

- The product $a \odot b \in \mathbb{F}$ is defined by

  $$\mu_{a\odot b}(x) = \sup\left\{\min\left\{\mu_a(y), \mu_b\left(z\right)\right\} : y, z \in \mathbb{R},\ yz = x\right\}$$

for every $x \in \mathbb{R}$. Equivalently,

$$[a \odot b]_t = \left[\min\{a_t^- b_t^-, a_t^- b_t^+, a_t^+ b_t^-, a_t^+ b_t^+\}, \max\{a_t^- b_t^-, a_t^- b_t^+, a_t^+ b_t^-, a_t^+ b_t^+\}\right]$$

for every $t \in [0, 1]$.

Notice that the set of real numbers can be embedded into $\mathbb{F}$. Indeed, we can identify $p \in \mathbb{R}$ with the fuzzy quantity determined by the following membership function:

$$\mu_p(x) = \begin{cases} 1 & \text{if } x = p, \\ 0 & otherwise. \end{cases}$$

With this identification we have that $\mathbb{R} \subset \mathbb{F}$. Note that the operations $\oplus, \ominus, \odot$ extend, respectively, the sum, subtraction and product of real numbers.

Notice that if $a \in \mathbb{F}$ and $p \in \mathbb{R}$, then

$$\mu_{p \oplus a}(x) = \mu_a(x - p)$$

for every $x \in \mathbb{R}$. Equivalently,

$$[p \oplus a]_t = [p + a_t^-, p + a_t^+]$$

for every $t \in [0, 1]$. And, if $p \in \mathbb{R} \setminus \{0\}$, then

$$\mu_{p \odot a}(x) = \mu_a\left(\frac{x}{p}\right)$$

for every $x \in \mathbb{R}$. Equivalently,

$$[p \odot a]_t = \begin{cases} [pa_t^-, pa_t^+] & \text{if } p > 0, \\ [pa_t^+, pa_t^-] & \text{if } p < 0. \end{cases}$$

for every $t \in [0, 1]$.

Given $a, b \in \mathbb{F}$, it is said that a is greater than or equal to b, which is denoted by $a \geqslant b$, if $a_t^- \geqslant b_t^-$ and $a_t^+ \geqslant b_t^+$ for every $t \in [0, 1]$.

A fuzzy quantity $a \in \mathbb{F}$ is said to be 0-symmetric if $a_t^- = -a_t^+$ for every $t \in [0, 1]$.

Let us recall some basic properties of the arithmetic operations in $\mathbb{F}$. Let $a, b, c, d \in \mathbb{F}$.

a) $a \oplus b = b \oplus a$.

b) $a \odot b = b \odot a$.

c) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

d) $a \odot (b \odot c) = (a \odot b) \odot c$.

e) $a \oplus 0 = a$.

f) $a \odot 1 = a$.

g) $a \odot 0 = 0$.

h) $a \ominus b = a \oplus ((-1) \odot b)$.

The properties above will be used throughout this paper without referring to them. The following properties, although equally simple, are more specific and they will be referred to when applied.

i) If $p \in \mathbb{R}$,

$$p \odot (a \oplus b) = (p \odot a) \oplus (p \odot b), \tag{3}$$

$$p \odot (a \ominus b) = (p \odot a) \ominus (p \odot b). \tag{4}$$

j) If $b, c \geqslant 0$ (or $b, c \leqslant 0$),

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c). \tag{5}$$

k) If $a \subseteq c$ and $b \subseteq d$,

$$a \oplus b \subseteq c \oplus d, \tag{6}$$

$$a \ominus b \subseteq c \ominus d, \tag{7}$$

$$a \odot b \subseteq c \odot d. \tag{8}$$

l) The equation $x \oplus a = b$ either has no solution in $\mathbb{F}$ or has a unique solution in $\mathbb{F}$.

m) If $p \in \mathbb{R}$, then $p \odot (a \ominus a)$ is 0-symmetric.

n) If $a \oplus b \in \mathbb{R}$, then $a, b \in \mathbb{R}$.

2.3. Cooperative games with fuzzy characteristic function

A *cooperative game with fuzzy characteristic function* consists of a finite and nonempty set N and a characteristic function $v : 2^N \to \mathbb{F}$ that satisfies $v(\emptyset) = 0$. The elements of N are called players, and the subsets of N are called coalitions. For each coalition E, the fuzzy quantity $v(E)$ describes the expectations about the collective payment that can be obtained by the players in E when they cooperate. A cooperative game with fuzzy characteristic function (N, v) will be identified with the mapping v. The class of all cooperative games with fuzzy characteristic function and set of players N is denoted by $\mathcal{FG}^N$. Since $\mathbb{R} \subset \mathbb{F}$, we have that $\mathcal{G}^N \subset \mathcal{FG}^N$. If $v, w \in \mathcal{FG}^N$ and $a \in \mathbb{F}$ the games $v \oplus w, a \odot v \in \mathcal{FG}^N$ are defined by

$$\begin{aligned} (v \oplus w)(E) &= v(E) \oplus w(E), \\ (a \odot v)(E) &= a \odot v(E), \end{aligned}$$

for every $E \in 2^N$.

A value on $\mathcal{FG}^N$ is a mapping $\Psi\colon \mathcal{FG}^N \to \mathbb{F}^N$. If $v \in \mathcal{FG}^N$ and $i \in N$ the fuzzy quantity $\Psi_i(v)$ describes the expectations about the payoff of player i in the game v. Mareš [8] introduced the *Shapley value for cooperative games with fuzzy characteristic function*, which is defined by

$$\Phi_i(v) = \bigoplus_{\{E \subseteq N : i \in E\}} p_E \odot (v(E) \ominus v(E \setminus \{i\}))$$

for every $i \in N$ and every $v \in \mathcal{FG}^N$, where the $(p_E)_{E \in 2^N \setminus \{\emptyset\}}$ are the coefficients of the Shapley value on $\mathcal{G}^N$.

3. Characterization

Let us fix a finite and nonempty set N. We introduce some properties that a value $\Psi\colon \mathcal{FG}^N \to \mathbb{F}^N$ may satisfy:

- **ADDITIVITY.** If $v, w \in \mathcal{FG}^N$ then $\Psi(v \oplus w) = \Psi(v) \oplus \Psi(w)$.

- **EQUAL TREATMENT.** If $v \in \mathcal{FG}^N$, $i, j \in N$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$, then $\Psi_i(v) = \Psi_j(v)$.

  Notice that the first two properties considered are natural fuzzy extension of the properties of additivity and equal treatment used to characterize the Shapley value.

- **CENTRAL EFFICIENCY.** For every $v \in \mathcal{FG}^N$ there exists $d_v \in \mathbb{F}$ such that d_v is 0-symmetric and

  $$\bigoplus_{i \in N} \Psi_i(v) = v(N) \oplus d_v.$$

  It would not be reasonable to require that the players' payoffs sum up to $v(N)$. Suppose that $v \in \mathcal{FG}^N \setminus \mathcal{G}^N$ and $v(N) \in \mathbb{R}$. If a value Ψ on $\mathcal{FG}^N$ satisfies efficiency then $\bigoplus_{i \in N} \Psi_i(v) = v(N) \in \mathbb{R}$ and, consequently, by property **n** on page 7, $\Psi_i(v) \in \mathbb{R}$ for every $i \in N$. Hence, this value would ignore the uncertainty in the payments of all the coalitions different from N. Therefore, instead of efficiency, we require a weaker property, central efficiency. This property says, that, for any possibility level $t \in (0,1]$, the t-cut of the sum of the players' payoffs contains the t-cut of $v(N)$ and both intervals have the same center.

If $v \in \mathcal{FG}^N$, a player $i \in N$ is said to be a null player in v if $v(E \cup \{i\}) = v(E)$ for every $E \in 2^N$.

- **NULL PLAYER.** If $v \in \mathcal{FG}^N$ and $i \in N$ is a null player in v, then $\Psi_i(v)$ is 0-symmetric.

  Notice that the equality $v(E \cup \{i\}) = v(E)$ does not imply that if coalitions $E \cup \{i\}$ and E were actually formed then the profits obtained (in the game v) by them should be equal. Instead, that equality means that the expectations on the payments achievable by those coalitions are the same. Mathematically, this is expressed by the fact that $v(E \cup \{i\}) = v(E)$ does not imply that $v(E \cup \{i\}) \ominus v(E) = 0$, but only that $v(E \cup \{i\}) \ominus v(E)$ is 0-symmetric. Therefore, we must not require that a null player earn a zero payoff. Instead, we look for values that assign a 0-symmetric fuzzy number to any null player in a game with fuzzy characteristic function.

- **EQUALLY SIGNED MARGINAL CONTRIBUTIONS.** If $v \in \mathcal{FG}^N$, $i \in N$ and $v(E \cup \{i\}) \ominus v(E) \geqslant 0$ (resp. $v(E \cup \{i\}) \ominus v(E) \leqslant 0$) for every $E \subseteq N \setminus \{i\}$, then $\Psi_i(v) \geqslant 0$ (resp. $\Psi_i(v) \leqslant 0$).

  Remember the property of strong monotonicity for a value ψ on $\mathcal{G}^N$ introduced by Young [15]. We could consider a weaker property stating that if $v \in \mathcal{G}^N$, $i \in N$ and $v(E \cup \{i\}) - v(E) \geqslant 0$ (resp. $v(E \cup \{i\}) - v(E) \leqslant 0$) for every $E \subseteq N \setminus \{i\}$, then $\psi_i(v) \geqslant 0$ (resp. $\psi_i(v) \leqslant 0$). If we extend this to the fuzzy case, we obtain the property of equally signed marginal contributions.

- **ZERO SOLUTION.** If $v \in \mathcal{FG}^N$ and $0 \subseteq v(E)$ for every $E \in 2^N$, then $0 \subseteq \Psi_i(v)$ for every $i \in N$.

  The zero solution property says that if it is possible (at the maximum possibility level) that the payments of all the coalitions in a game are equal to zero, then it is possible (at the maximum possibility level) that the payoffs of all the players in the game are equal to zero.

Let us see that Φ satisfies the six properties above.

Theorem 1. *The Shapley value for cooperative games with fuzzy characteristic function satisfies the properties of additivity, equal treatment, central efficiency, null player, equally signed marginal contributions and zero solution.*

Proof.

- *Additivity*

  Let $v, w \in \mathcal{FG}^N$ and let $i \in N$. Then,

$$\Phi_i(v \oplus w) = \bigoplus_{\{E \subseteq N : i \in E\}} p_E \odot ((v \oplus w)(E) \ominus (v \oplus w)(E \setminus \{i\})),$$

  which, by basic arithmetic properties, (3) and (4), is equal to

$$\bigoplus_{\{E \subseteq N : i \in E\}} p_E \odot (v(E) \ominus v(E \setminus \{i\})) \oplus \bigoplus_{\{E \subseteq N : i \in E\}} p_E \odot (w(E) \ominus w(E \setminus \{i\})),$$

  which is, by definition, $\Phi_i(v) \oplus \Phi_i(w)$.

- *Equal treatment*

  Let $v \in \mathcal{FG}^N$ and $i, j \in N$ be such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$. Then,

$$\begin{aligned}
\Phi_i(v) &= \bigoplus_{\{E \subseteq N : i \in E\}} p_E \odot (v(E) \ominus v(E \setminus \{i\})) \\
&= \bigoplus_{\{E \subseteq N : i,j \in E\}} p_E \odot (v(E) \ominus v(E \setminus \{i\})) \\
&\quad \oplus \bigoplus_{\{E \subseteq N : i \in E, j \notin E\}} p_E \odot (v(E) \ominus v(E \setminus \{i\})) \\
&= \bigoplus_{\{E \subseteq N : i,j \in E\}} p_E \odot (v(E) \ominus v(E \setminus \{j\})) \\
&\quad \oplus \bigoplus_{\{E \subseteq N : j \in E, i \notin E\}} p_E \odot (v(E) \ominus v(E \setminus \{j\})) \\
&= \bigoplus_{\{E \subseteq N : j \in E\}} p_E \odot (v(E) \ominus v(E \setminus \{j\})) = \Phi_j(v).
\end{aligned}$$

- *Central efficiency*

  Let $v \in \mathcal{FG}^N$. Then,

$$\bigoplus_{i \in N} \Phi_i(v) = \bigoplus_{i \in N} \left(\bigoplus_{\{E \subseteq N : i \in E\}} p_E \odot (v(E) \ominus v(E \setminus \{i\})) \right),$$

which, by basic arithmetic properties, (3), (4) and (5), is equal to

$$\begin{aligned}
&\left(\bigoplus_{E\in 2^N\setminus\{\emptyset\}}\left(\bigoplus_{i\in E} p_E\odot v\left(E\right)\right)\right)\ominus\left(\bigoplus_{E\in 2^N\setminus\{\emptyset\}}\left(\bigoplus_{i\in E} p_E\odot v\left(E\setminus\{i\}\right)\right)\right)\\
=\ &\left(\bigoplus_{E\in 2^N\setminus\{\emptyset\}}\left(\bigoplus_{i\in E} p_E\odot v\left(E\right)\right)\right)\ominus\left(\bigoplus_{E\in 2^N\setminus\{N,\emptyset\}}\left(\bigoplus_{i\in N\setminus E} p_{E\cup\{i\}}\odot v\left(E\right)\right)\right)\\
=\ &\left(\bigoplus_{E\in 2^N\setminus\{\emptyset\}}\left(|E|\frac{(|N|-|E|)!\,(|E|-1)!}{|N|!}\right)\odot v\left(E\right)\right)\\
&\ominus\left(\bigoplus_{E\in 2^N\setminus\{N,\emptyset\}}\left((|N|-|E|)\frac{(|N|-|E|-1)!\,|E|!}{|N|!}\right)\odot v\left(E\right)\right)\\
=\ & v(N)\oplus\left(\bigoplus_{E\in 2^N\setminus\{N,\emptyset\}}\left(\frac{(|N|-|E|)!\,|E|!}{|N|!}\right)\odot\left(v\left(E\right)\ominus v\left(E\right)\right)\right).
\end{aligned}$$

And it is sufficient to notice that

$$d_v=\bigoplus_{E\in 2^N\setminus\{N,\emptyset\}}\left(\frac{(|N|-|E|)!\,|E|!}{|N|!}\right)\odot\left(v\left(E\right)\ominus v\left(E\right)\right)$$

is 0-symmetric, since it is a sum of 0-symmetric fuzzy quantities (recall property **m** on page 7).

- *Null player*

  Let $v\in\mathcal{FG}^N$, $i\in N$ be such that i is a null player in v. Then,

  $$\Phi_i(v)=\bigoplus_{\{E\subseteq N:\,i\in E\}} p_E\odot\left(v\left(E\right)\ominus v\left(E\right)\right),$$

  which is 0-symmetric, since it is a sum of 0-symmetric fuzzy quantities.

- *Equally signed marginal contributions*

  Let $v\in\mathcal{FG}^N$ be such that $v(E\cup\{i\})\ominus v(E)\geqslant 0$ for every $E\subseteq N\setminus\{i\}$. It is clear that $p_E\odot(v(E)\ominus v(E\setminus\{i\})\geqslant 0$ for every $E\in 2^N$ with $i\in E$. If we take into account that if $a,b\in\mathbb{F}$ and $a,b\geqslant 0$ then $a\oplus b\geqslant 0$, we conclude that $\Phi_i(v)\geqslant 0$.

- *Zero solution*

  Let $v\in\mathcal{FG}^N$ be such that $0\subseteq v(E)$ for every $E\in 2^N$. Take $i\in N$. Then, by (6), (7) and (8),

$$0 \subseteq \bigoplus_{\{E \subseteq N:\, i \in E\}} p_E \odot \left(v\left(E\right) \ominus v\left(E \setminus \{i\}\right)\right),$$

that is, $0 \subseteq \Phi_i(v)$.

□

Now we aim to prove that if a value on $\mathcal{FG}^N$ satisfies the six properties stated in the previous theorem then this value is equal to the Shapley value for cooperative games with fuzzy characteristic function. Firstly we need to see two simple lemmas.

Lemma 2. *Let $a, b, c, d \in \mathbb{F}$ be such that*

i) $a, c \geqslant 0$ *(resp. $a, c \leqslant 0$),*

ii) $0 \subseteq a, c$,

iii) b, d *are* 0*-symmetric,*

iv) $a \oplus b = c \oplus d$.

Then, $a = c$ and $b = d$.

Proof. We will prove only the version where $a, c \geqslant 0$.

Let $t \in [0, 1]$. From *i)* and *ii)*, $a_t^- = c_t^- = 0$. By *iii)*, $b_t^- = -b_t^+$ and $c_t^- = -c_t^+$. We have that

$$[a \oplus b]_t = [a_t^- + b_t^-, a_t^+ + b_t^+] = [-b_t^+, a_t^+ + b_t^+] \tag{9}$$

and

$$[c \oplus d]_t = [c_t^- + d_t^-, c_t^+ + d_t^+] = [-d_t^+, c_t^+ + d_t^+]. \tag{10}$$

From (9), (10) and *iv)* we obtain that $b_t^+ = d_t^+$ and $a_t^+ = c_t^+$. It is clear that $[a]_t = [c]_t$ and $[b]_t = [d]_t$. Since these equalities hold for every $t \in [0, 1]$, it follows that $a = c$ and $b = d$.

□

Lemma 3. *Let $a, b \in \mathbb{F}$ be such that*

i) $a \geqslant 0$ *(resp. $a \leqslant 0$),*

ii) $0 \subseteq a$,

iii) $a \subseteq b$,

iv) $b \leqslant a$ *(resp. $b \geqslant a$),* *v) b is 0-symmetric,*

Then, $b = a \ominus a$.

Proof. We will prove only the version where $a \geqslant 0$ and $b \leqslant a$.

Let $x \in [0, +\infty)$. Let us see that $\mu_b(x) \leqslant \mu_a(x)$. Suppose that $\mu_b(x) > \mu_a(x)$. If we take $t \in (\mu_a(x), \mu_b(x))$, then $x \in [b]_t$ and $x \notin [a]_t$. From *i)* and *ii)*, $[a]_t = [0, a_t^+]$. We have that $x \in [0, +\infty)$, $x \in [b_t^-, b_t^+]$ and $x \notin [0, a_t^+]$. It follows that $b_t^+ > a_t^+$, but this contradicts condition *iv)*. We have proved that $\mu_b(x) \leqslant \mu_a(x)$ for every $x \in [0, +\infty)$. By *iii)*, we know that $\mu_b(x) \geqslant \mu_a(x)$ for every $x \in \mathbb{R}$. We conclude that $\mu_b(x) = \mu_a(x)$ for every $x \in [0, +\infty)$. From this and condition *v)* it follows that $[b]_t = [-a_t^+, a_t^+]$ for every $t \in [0, 1]$. And it suffices to notice that, by *i)* and *ii)*, $[a \ominus a]_t = [-a_t^+, a_t^+]$ for every $t \in [0, 1]$.

□

Now we are in conditions to complete the characterization of the Shapley value for cooperatives games with fuzzy characteristic function.

Theorem 4. *If a value Ψ on $\mathcal{FG}^N$ satisfies the properties of additivity, equal treatment, central efficiency, null player, equally signed marginal contributions and zero solution, then Ψ is equal to the Shapley value for cooperative games with fuzzy characteristic function.*

Proof. Suppose that $\Psi\colon \mathcal{FG}^N \to \mathbb{F}^N$ satisfies the properties stated in the theorem. Our goal is to prove that $\Psi = \Phi$. The proof will be done in several steps. In each step it will be shown that $\Psi(v) = \Phi(v)$ for every v in a certain class of games in $\mathcal{FG}^N$.

Step 1 We aim to prove that

$$\Psi(v) = \Phi(v) \qquad \text{for every } v \in \mathcal{G}^N. \tag{11}$$

If $\mathbf{0}$ denotes the game that assigns zero to all coalitions $E \in 2^N$, it is clear, from the property of equally signed marginal contributions, that $\Psi_i(\mathbf{0}) = 0$ for every $i \in N$. Let $v \in \mathcal{G}^N$. By additivity,

$$\Psi_i(v) \oplus \Psi_i(-v) = \Psi_i(\mathbf{0}) = 0, \tag{12}$$

for every $i \in N$. From (12) and property **n** on page 7 it follows that $\Psi_i(v) \in \mathbb{R}$ for every $i \in N$. Therefore, the restriction of Ψ to $\mathcal{G}^N$, denoted by $\Psi_{|\mathcal{G}^N}$, is a value on $\mathcal{G}^N$. Taking into account that Ψ satisfies the properties of additivity, central efficiency, null player and equal treatment for values on $\mathcal{FG}^N$ and the fact that $\Psi(v) \in \mathbb{R}^N$ for every $v \in \mathcal{G}^N$, it can easily be verified that $\Psi_{|\mathcal{G}^N}$ satisfies the properties of additivity, efficiency, null player and equal treatment for values on $\mathcal{G}^N$. Since these properties characterize the Shapley value ϕ on $\mathcal{G}^N$, we conclude that $\Psi_{|\mathcal{G}^N} = \phi$. By the same reasoning, $\Phi_{|\mathcal{G}^N} = \phi$. We have proved (11).

Step 2 Our goal is to prove that

$$\Psi(a \odot u_E) = \Phi(a \odot u_E) \tag{13}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$ with $a \geqslant 0$ and $0 \subseteq a$.

Let $a \in \mathbb{F}$ be such that $a \geqslant 0$ and $0 \subseteq a$. Let $E \in 2^N \setminus \{\emptyset\}$.

By the property of equal treatment, there exist $b, c \in \mathbb{F}$ such that

$$\Psi_i(a \odot u_E) = \begin{cases} b & \text{if } i \in E, \\ c & \text{if } i \in N \setminus E. \end{cases} \tag{14}$$

If $i \in E$ and $F \subseteq N \setminus \{i\}$, then $(a \odot u_E)(F \cup \{i\}) \ominus (a \odot u_E)(F) \geqslant 0$. By the property of equally signed marginal contributions, it follows that $b \geqslant 0$. Since $0 \subseteq (a \odot u_E)(F)$ for every $F \in 2^N$, we obtain, by the property of zero solution, that $0 \subseteq \Psi_i(a \odot u_E)$ for every $i \in N$. In particular, $0 \subseteq b$. Since the players in $N \setminus E$ are null players in $a \odot u_E$, we obtain, by the property of null player, that c is 0-symmetric. From (14),

$$\bigoplus_{i \in N} \Psi_i(a \odot u_E) = (|E| \odot b) \oplus (|N \setminus E| \odot c), \tag{15}$$

By the property of central efficiency, there exists $d \in \mathbb{F}$ such that d is 0-symmetric and

$$\bigoplus_{i \in N} \Psi_i(a \odot u_E) = a \oplus d. \tag{16}$$

From (15) and (16), $(|E| \odot b) \oplus (|N \setminus E| \odot c) = a \oplus d$.

Therefore, the following conditions hold:

i) $|E| \odot b \geqslant 0$, $a \geqslant 0$,

ii) $0 \subseteq |E| \odot b$, $0 \subseteq a$,

iii) $|N \setminus E| \odot c$ and d are 0-symmetric,

iv) $(|E| \odot b) \oplus (|N \setminus E| \odot c) = a \oplus d$.

By applying Lemma 2 we obtain that $|E| \odot b = a$, whence it easily follows that

$b = \dfrac{1}{|E|} \odot a$. We have proved that

$$\Psi_i(a \odot u_E) = \frac{1}{|E|} \odot a \quad \text{for every } i \in E. \tag{17}$$

Notice that if we consider the case $E = N$ we have calculated $\Psi(a \odot u_N)$ and, since we have used only the properties in the theorem, it is clear that $\Psi(a \odot u_N) = \Phi(a \odot u_N)$. Suppose now that $E \neq N$. Take $j \in N \setminus E$. Similarly to (17), we have that

$$\Psi_i(a \odot u_{E\cup\{j\}}) = \frac{1}{|E|+1} \odot a \quad \text{for every } i \in E \cup \{j\}.$$

In particular,

$$\Psi_j(a \odot u_{E\cup\{j\}}) = \frac{1}{|E|+1} \odot a. \tag{18}$$

Let $w \in \mathcal{FG}^N$ defined by

$$w(F) = \begin{cases} a & \text{if } E \subseteq F \text{ and } j \notin F, \\ 0 & \text{otherwise,} \end{cases}$$

for every $F \in 2^N$. We have that $a \odot u_E = (a \odot u_{E\cup\{j\}}) \oplus w$. By additivity,

$$\Psi_j(a \odot u_E) = \Psi_j(a \odot u_{E\cup\{j\}}) \oplus \Psi_j(w). \tag{19}$$

Since $0 \subseteq a$, we have that $0 \subseteq w(F)$ for every $F \in 2^N$. By the property of zero solution,

$$0 \subseteq \Psi_j(w). \tag{20}$$

From (6), (18), (19) and (20),

$$\frac{1}{|E|+1} \odot a \subseteq \Psi_j(a \odot u_E). \tag{21}$$

Note that $w(F \cup \{j\}) \ominus w(F) \leqslant 0$ for every $F \subseteq N \setminus \{j\}$. By the property of equally signed marginal contributions,

$$\Psi_j(w) \leqslant 0. \tag{22}$$

From (18), (19) and (22) it easily follows that

$$\Psi_j(a \odot u_E) \leqslant \frac{1}{|E|+1} \odot a. \tag{23}$$

Notice that j is a null player in $a \odot u_E$. By the property of null player, $\Psi_j(a \odot u_E)$ is 0-symmetric. From this fact together with $a \geqslant 0$, $0 \subseteq a$, (21) and (23) we obtain that the following conditions hold:

i) $\frac{1}{|E|+1} \odot a \geqslant 0$,

ii) $0 \subseteq \frac{1}{|E|+1} \odot a$,

iii) $\frac{1}{|E|+1} \odot a \subseteq \Psi_j(a \odot u_E)$,

iv) $\Psi_j(a \odot u_E) \leqslant \frac{1}{|E|+1} \odot a$,

v) $\Psi_j(a \odot u_E)$ is 0-symmetric.

By Lemma 3 and (4),

$$\Psi_j(a \odot u_E) = \frac{1}{|E|+1} \odot (a \ominus a). \tag{24}$$

From (14), (17) and (24),

$$\Psi_i(a \odot u_E) = \begin{cases} \frac{1}{|E|} \odot a & \text{if } i \in E, \\ \\ \frac{1}{|E|+1} \odot (a \ominus a) & \text{if } i \in N \setminus E. \end{cases}$$

Since we have used only the properties stated in the theorem, we have proved (13).

Step 3 We must see that

$$\Psi(a \odot u_E) = \Phi(a \odot u_E) \tag{25}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$ with $a \leqslant 0$ and $0 \subseteq a$.

The proof is similar to that of (13). The only difference lies in the versions used of Lemma 2, Lemma 3 and the property of equally signed marginal contributions.

Step 4 Let us prove that

$$\Psi(a \odot u_E) = \Phi(a \odot u_E) \tag{26}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$.

Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$. Take $z \in core(a)$. Let $b, c \in \mathbb{F}$ defined by

$$\mu_b(x) = \begin{cases} \mu_a(z+x) & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases}$$

$$\mu_c(x) = \begin{cases} 0 & \text{if } x > 0, \\ \mu_a(z+x) & \text{if } x \leqslant 0. \end{cases}$$

Notice that $b \geqslant 0$, $c \leqslant 0$ and $0 \subseteq b, c$. It can easily be verified that $[b]_t = [0, a_t^+ - z]$ and $[c]_t = [a_t^- - z, 0]$ for every $t \in [0,1]$. It follows that $a = z \oplus b \oplus c$. Hence, $a \odot u_E = (z\, u_E) \oplus (b \odot u_E) \oplus (c \odot u_E)$. By additivity, (11), (13) and (25) we obtain that

$$\begin{aligned} \Psi(a \odot u_E) &= \Psi(zu_E) \oplus \Psi(b \odot u_E) \oplus \Psi(c \odot u_E) \\ &= \Phi(zu_E) \oplus \Phi(b \odot u_E) \oplus \Phi(c \odot u_E) = \Phi(a \odot u_E). \end{aligned}$$

We have proved (26).

Step 5 Our goal is to prove that

$$\Psi(a \odot \delta_E) = \Phi(a \odot \delta_E) \tag{27}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$.

Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$. By (1) and (2),

$$\delta_E = \sum_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F\}} (-1)^{|F|-|E|} u_F,$$

whence

$$\delta_E + \sum_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F\}} u_F = \sum_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} 2u_F,$$

that is,

$$\delta_E(H) + \sum_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F\}} u_F(H) = \sum_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} 2u_F(H),$$

for every $H \subseteq N$. If we multiply by a and apply (5) we obtain

$$(a \odot \delta_E)(H) \oplus \bigoplus_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F\}} (a \odot u_F)(H) = \bigoplus_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} ((2 \odot a) \odot u_F)(H),$$

for every $H \subseteq N$. Hence,

$$(a \odot \delta_E) \oplus \bigoplus_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F\}} (a \odot u_F) = \bigoplus_{\{F \in 2^N \setminus \{\emptyset\} \colon E \subseteq F,\, |F|-|E| \in 2\mathbb{Z}\}} ((2 \odot a) \odot u_F).$$

which, by additivity, leads to

$$\Psi_i(a\odot\delta_E)\oplus\bigoplus_{\{F\in 2^N\setminus\{\emptyset\}\colon E\subseteq F\}}\Psi_i(a\odot u_F)=\bigoplus_{\{F\in 2^N\setminus\{\emptyset\}\colon E\subseteq F,|F|-|E|\in 2\mathbb{Z}\}}\Psi_i((2\odot a)\odot u_F) \quad (28)$$

and

$$\Phi_i(a\odot\delta_E)\oplus\bigoplus_{\{F\in 2^N\setminus\{\emptyset\}\colon E\subseteq F\}}\Phi_i(a\odot u_F)=\bigoplus_{\{F\in 2^N\setminus\{\emptyset\}\colon E\subseteq F,|F|-|E|\in 2\mathbb{Z}\}}\Phi_i((2\odot a)\odot u_F) \quad (29)$$

for every $i\in N$. From (26), (28), (29) and property **1** on page 7 it is concluded that $\Psi_i(a\odot\delta_E)=\Phi_i(a\odot\delta_E)$ for every $i\in N$. We have proved (27).

Step 6 We aim to prove that

$$\Psi(v)=\Phi(v)$$

for every $v\in\mathcal{FG}^N$.

Let $v\in\mathcal{FG}^N$. Notice that

$$v=\bigoplus_{E\in 2^N\setminus\{\emptyset\}}(v(E)\odot\delta_E).$$

By additivity and (27),

$$\Psi(v)=\bigoplus_{E\in 2^N\setminus\{\emptyset\}}\Psi(v(E)\odot\delta_E)=\bigoplus_{E\in 2^N\setminus\{\emptyset\}}\Phi(v(E)\odot\delta_E)=\Phi(v),$$

which completes the proof.

□

Logical independence

Finally, we will show that the properties used in the characterization of Φ are independent. We need to introduce a notation. If $y,z\in\mathbb{R}$ and $y\leqslant z$ then the fuzzy number $[y,z]\in\mathbb{F}$ is defined by

$$\mu_{[y,z]}(x)=\begin{cases}1 & \text{if } y\leqslant x\leqslant z,\\ 0 & \text{otherwise.}\end{cases}$$

We require that $|N|\geqslant 2$.

(i) Fix $j\in N$. Let $p\in\mathbb{F}^N$ defined by

$$p_i=\begin{cases}[0,2] & \text{if } i=j,\\ 0 & \text{if } i\in N\setminus\{j\}.\end{cases}$$

Let $\Xi\colon \mathcal{FG}^N \to \mathbb{F}^N$ defined by

$$\Xi(v) = \begin{cases} \Phi(v) & \text{if } v \in \mathcal{FG}^N \setminus \{u_{\{j\}}\}, \\ p & \text{if } v = u_{\{j\}}. \end{cases}$$

Then, Ξ satisfies equal treatment, central efficiency, null player, equally signed marginal contributions and zero solution and it does not satisfy additivity.

(ii) Let $n = |N|$ and $N = \{i_1, \dots, i_n\}$. Consider $\Gamma\colon \mathcal{FG}^N \to \mathbb{F}^N$ defined by

$$\Gamma_{i_k}(v) = \begin{cases} v(\{i_1\}) & \text{if } k = 1, \\ v(\{i_1, \dots, i_k\}) \ominus v(\{i_1, \dots, i_{k-1}\}) & \text{if } k \in \{2, \dots, n\}, \end{cases}$$

for every $v \in \mathcal{FG}^N$.

Then, Γ satisfies additivity, central efficiency, null player, equally signed marginal contributions and zero solution and it does not satisfy equal treatment.

(iii) Let $\Psi\colon \mathcal{FG}^N \to \mathbb{F}^N$ defined by

$$\Psi_i(v) = 0$$

for every $v \in \mathcal{FG}^N$ and every $i \in N$.

Then, Ψ satisfies additivity, equal treatment, null player, equally signed marginal contributions and zero solution and it does not satisfy central efficiency.

(iv) Let $f, g\colon \mathbb{F} \to \mathbb{F}$ defined by

$$\mu_{f(a)}(x) = \begin{cases} \mu_a(x) & \text{if } x \leqslant a_1^+, \\ 0 & \text{if } x > a_1^+, \end{cases}$$

$$\mu_{g(a)}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \mu_a(a_1^+ + x) & \text{if } x \geqslant 0, \end{cases}$$

for every $a \in \mathbb{F}$.

Let $\Omega\colon \mathcal{FG}^N \to \mathbb{F}^N$ defined by

$$\Omega_i(v) = \Phi_i(f \circ v) \oplus \frac{1}{|N|} \odot g(v(N)),$$

for every $v \in \mathcal{FG}^N$ and every $i \in N$.

Then, Ω satisfies additivity, equal treatment, central efficiency, equally signed marginal contributions and zero solution and it does not satisfy null player.

(v) Let $\Lambda\colon \mathcal{FG}^N \to \mathbb{F}^N$ defined by

$$\Lambda_i(v) = \Phi_i(v) \oplus [v(N)_1^- - v(N)_1^+, v(N)_1^+ - v(N)_1^-],$$

for every $v \in \mathcal{FG}^N$ and every $i \in N$.

Then, Λ satisfies additivity, equal treatment, central efficiency, null player and zero solution and it does not satisfy equally signed marginal contributions.

(vi) Let $h\colon \mathbb{F} \to \mathbb{F}$ defined by

$$\mu_{h(a)}(x) = \begin{cases} \mu_a\left(x + \dfrac{a_1^+ - a_1^-}{2}\right) & \text{if } x \geqslant \dfrac{a_1^+ + a_1^-}{2}, \\ \\ \mu_a\left(x - \dfrac{a_1^+ - a_1^-}{2}\right) & \text{if } x < \dfrac{a_1^+ + a_1^-}{2}, \end{cases}$$

for every $a \in \mathbb{F}$.

Let $\Theta\colon \mathcal{FG}^N \to \mathbb{F}^N$ defined by

$$\Theta_i(v) = \Phi_i(h \circ v) \oplus \left[\frac{v(N)_1^- - v(N)_1^+}{2|N|}, \frac{v(N)_1^+ - v(N)_1^-}{2|N|}\right],$$

for every $v \in \mathcal{FG}^N$ and every $i \in N$.

Then, Θ satisfies additivity, equal treatment, central efficiency, null player, equally signed marginal contributions and it does not satisfy zero solution.

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