Games with fuzzy permission structure: a conjunctive approach

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Abstract

A cooperative game consists of a set of players and a characteristic function which determines the maximal gain or minimal cost that every subset of players can achieve when they decide to cooperate, regardless of the actions that the other players take. A permission structure over the set of players describes a hierarchical organization where there are players who need permission from certain other players before they are allowed to cooperate with others. Various assumptions can be made about how a permission structure affects the cooperation possibilities. In the conjunctive approach it is assumed that each player needs permission from all his superiors. This paper deals with fuzzy permission structures in the conjunctive approach. In this model, players could depend partially on other players, that is, they may have certain degree of autonomy. First, we define a value for games with fuzzy permission structure that only takes into account the direct relations among players and provide a characterization for this value. Finally, we study a value for games with fuzzy permission structure which takes account of the indirect relations among players.

Keywords: cooperative games, Shapley value, fuzzy sets, fuzzy orders

1. Introduction

In a general way, game theory studies cooperation and conflict models, using mathematical methods. This paper is about cooperative game theory. A cooperative game over a finite set of players is defined as a function establishing the worth of each coalition. Given a cooperative game, the main problem that arises is how to assign to each player a payoff in a reasonable way. In this setting, it is often assumed that all players are socially identical. However, there are situations in which some players have a direct or indirect influence on other players. For instance, there might be a veto relationship between them. Following this idea, Gilles *et al.* [7] modeled situations in which an authority structure is exogenously given and puts some constraints on the behavior of the players in the game. They consider an abstract authority structure in which certain players dominate some other players, in the sense that the superiors have veto power over the activities undertaken by their subordinates. In general, they deal with games with permission structure, consisting of a set of players, a cooperative game and a mapping that assigns to every player a subset of direct

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subordinates. In this respect, the power of a player over a subordinate can be of different kinds. In the conjunctive approach it is assumed that every player needs the permission of all his superiors, that is, every player has complete veto power over the actions undertaken by his inferiors. Bearing this in mind, in order to obtain the payoff of each player they define a new game that takes account the information given by the authority structure.

In the model proposed in [7], the subordinate relationships are total in the sense that the whole capacity for action of a player is controlled by his superiors. Now we propose to study situations in which the dependence between players can be partial, which means that only a part of the resources of a player is controlled by his superiors. This is what we will call games with fuzzy permission structure. The partial dependencies will be represented by a fuzzy relation on the set of players. In the same way as in [7] we will define a new game considering the constraints given by the structure. A critical issue arises at this point: in order to define such new game we need to assign a worth to certain fuzzy coalitions. Fuzzy coalitions were introduced by Aubin [1] to deal with situations where some agents cannot fully participate within a coalition. If we think of certain production games, full participation means to offer all of resources, whereas partial participation means to offer only a fraction of them. So, because of the kind of dependency relationships we intend to deal with, we will be faced with the problem of assigning a gain to a fuzzy coalition. Regarding this matter, different approaches have been developed in literature. In his seminal paper, Aubin proposed an optimal value, also studied in [9]. In [3], Butnariu, assuming that different players should have the same membership grade in order to cooperate, provided a different way to assign a gain to a fuzzy coalition. In [13], Tsurumi *et al.*, by using the Choquet integral, come up with a reasonable method to extend a crisp game to the set of fuzzy coalitions. Following Tsurumi's approach, we will use the Choquet integral to define a new auxiliary game that will combine the information from the original game and from the fuzzy permission structure. This new game will allow us to determine the payoff that each player should receive, derived from certain reasonable basis that we will show in an axiomatic form.

The paper is organized as follows. In Section 2 we recall some basic definitions and results about cooperative games, permission structures, fuzzy sets and the Choquet integral. In section 3, we introduce fuzzy permission structures, which model fuzzy relationships between players. An operator that determines the autonomous part of each coalition in a fuzzy permission structure is also defined. Using this operator, we modify the characteristic function of a game by getting payoffs in accordance with the fuzzy permission structure. Then, we define a value for games on fuzzy permission structures. Finally in section 3, indirect fuzzy relationships are dealt with. In Section 4, we characterize the values introduced in the previous section. A numerical example is given as well. Finally, in Section 5 we give some conclusions.

2. Preliminaries

2.1. Cooperative TU-games

A *transferable utility cooperative game or TU-game* is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of $N = \{1, ..., n\}$ are called *players*, the subsets $E \subseteq N$ *coalitions* and $v(E)$ is the *worth* of E. For each coalition E, the worth of E can be interpreted as the maximal gain or minimal cost that players in this coalition can achieve by themselves against the best offensive threat by the complementary coalition. Often, a TU-game (N, v) is identified with the function v. The family of all the games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^n - 1)$-dimensional real vector space. One basis of this space is the collection $\{u_F : F \subseteq N,\ F \neq \emptyset\}$ where for a coalition $F \subseteq N$, $F \neq \emptyset$, the *unanimity game* u_F is given by

$$u_F(E) = \begin{cases} 1 & \text{if } F \subseteq E, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

These games are monotone games. A game v is *monotone* if for every $F \subseteq E \subseteq N$, it holds that $v(F) \leq v(E)$. A player $i \in N$ is a *null player* in $v \in \mathcal{G}^N$ if $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$. A player i is a *necessary player* in $v \in \mathcal{G}^N$ if $v(E) = 0$ for $E \subseteq N \setminus \{i\}$.

A payoff vector for a game on the set of players N is a vector $x \in \mathbb{R}^N$. A *value* on $\mathcal{G}^N$ is a function $\psi : \mathcal{G}^N \longrightarrow \mathbb{R}^N$ that assigns to each game a payoff vector. Many values have been defined for different families of games in literature. The Shapley value [12] is the most important of them. The *Shapley value* $Sh(v) \in \mathbb{R}^N$ of a game $v \in \mathcal{G}^N$ is a weighted average of the marginal contributions of each player to the coalitions and formally it is defined by

$$Sh_i(v) = \sum_{\{E \subseteq N : i \in E\}} \frac{(n - |E|)!\,(|E| - 1)!}{n!} \left(v(E) - v(E \setminus \{i\})\right), \quad \text{for all } i \in N, \tag{2}$$

where $|E|$ denotes the cardinality of E. The Shapley value is the unique value $\psi : \mathcal{G}^N \longrightarrow \mathbb{R}^N$ that satisfies the following five axioms:

Efficiency: $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in \mathcal{G}^N$.

Linearity: $\psi(a_1 v_1 + a_2 v_2) = a_1 \psi(v_1) + a_2 \psi(v_2)$ for all $a_1, a_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathcal{G}^N$.

Null player property: If $i \in N$ is null player in $v \in \mathcal{G}^N$ then $\psi_i(v) = 0$.

Positivity: $\psi(v) \geq 0$ for every monotone game $v \in \mathcal{G}^N$.

Necessary player property: If i is a necessary player in a monotone game $v \in \mathcal{G}^N$, then $\psi_i(v) \geq \psi_j(v)$ for all $j \in N$.

The reader can use [6] or [8] to get more information about cooperative games.

2.2. Permission structures

A *permission structure* on N is represented by a mapping $S : N \to 2^N$ where the players in $S(i)$ are the successors of player $i \in N$ in the permission structure, that is, $S(i)$ contains all the agents that are dominated *directly* by agent i. By $\hat{S}$ is denoted the transitive closure of S, i.e. $j \in \hat{S}(i)$ if and only if there exists a sequence $\{i_p\}_{p=0}^{q}$ such that $i_0 = i$, $i_q = j$ and $i_p \in S(i_{p-1})$ for $1 \leq p \leq q$. Thus, the players in $\hat{S}(i) \setminus S(i)$ are all agents that are dominated *indirectly* by i. The *set of superiors* of i in S is denoted by $\hat{P}_S(i) = \left\{ j \in N : i \in \hat{S}(j) \right\}$. The collection of all permission structures on N is denoted by $\mathcal{S}^N$. Graphically, the family of permission structures on N can be identified with the set of directed graphs on N. The vertex set is N and the pair (i, j) is a link if $j \in S(i)$.

A pair (v, S) where $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$ is called a *game with permission structure over* N. A permission structure limits the possibilities of coalition formation in a TU-game. Several assumptions can be made about how a permission structure affects the cooperation possibilities. In the *conjunctive approach* [7], the authors assume that the players need permission from all their superiors in the permission structure before they are allowed to cooperate. So, the *conjunctive sovereign part* of a coalition E contains those players in E whose superiors are all in E, that is, $A^S(E) = \left\{ i \in E : \hat{P}_S(i) \subseteq E \right\}$.

In order to find reasonable payoff vectors for games with permission structure, it is proposed in [7] to modify the characteristic function $v \in \mathcal{G}^N$ taking account of the limited possibilities of cooperation determined by the permission structure $S \in \mathcal{S}^N$. The conjunctive restriction of v on S is defined as the game $v^S \in \mathcal{G}^N$ given by $v^S(E) = v\left(A^S(E)\right)$ for every coalition $E \subseteq N$. A *value for games with permission structure on* N is an application $\psi : \mathcal{G}^N \times \mathcal{S}^N \to \mathbb{R}^N$. Particularly the *conjunctive permission value* is defined as the value $\phi^{conj} : \mathcal{G}^N \times \mathcal{S}^N \to \mathbb{R}^N$ which assigns to every $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$ the Shapley value of the conjunctive restriction of v on S, that is, $\phi^{conj}(v, S) = Sh\left(v^S\right)$. A set of axioms that uniquely determines the conjunctive permission value for games with permission structure is provided. For a value ψ, these axioms are the following:

Efficiency: $\sum_{i \in N} \psi_i(v, S) = v(N)$, for all $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$.

Additivity: $\psi(v_1 + v_2, S) = \psi(v_1, S) + \psi(v_2, S)$ for all $v_1, v_2 \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$.

Weakly inessential player property: $\psi_i(v, S) = 0$ for every $v \in \mathcal{G}^N$, $S \in \mathcal{S}^N$ and $i \in N$ such that every player $j \in \hat{S}(i) \cup \{i\}$ is a null player in v.

Necessary player property: $\psi_i(v, S) \geq \psi_j(v, S)$ for every monotone $v \in \mathcal{G}^N$, $S \in \mathcal{S}^N$, $j \in N$ and $i \in N$ a necessary player in v.

Structural monotonicity: $\psi_i(v,S) \geq \psi_j(v,S)$ for every monotone $v \in \mathcal{G}^N$, $S \in \mathcal{S}^N$ and $i,j \in N$ such that $j \in S(i)$.

For further information, see [2, 7].

2.3. Fuzzy sets

Fuzzy sets were described by Zadeh [14]. A *fuzzy subset* of N is a mapping $e : N \longrightarrow [0,1]$ where e assigns to $i \in N$ a degree of membership. A fuzzy subset of N is identified with a vector in $[0,1]^N$. Given $e \in [0,1]^N$ the *support* of e is the set $supp\,(e) = \{i \in N : e_i > 0\}$ and the *image* of e is the set $\{e_i : i \in N\} \setminus \{0\}$. We will denote $im\,(e) = \{s_p\}_{p=1}^q$ considering that its elements are written in increasing order, that is, $s_1 < \cdots < s_q$. For all $e, f \in [0,1]^N$ and $i \in N$, internal operations meet and join are introduced. These operations are defined as $(e \cap f)_i = e_i \wedge f_i$, $(e \cup f)_i = e_i \vee f_i$, where $\wedge, \vee$ represent minimum and maximum respectively. The fuzzy sets $e, f \in [0,1]^N$ are called *comonotone* when $(e_i - e_j)(f_i - f_j) \geq 0$ for all $i, j \in N$.

A *fuzzy relation on* N is a fuzzy subset in $N \times N$, so the notions of support, image and comonotonicity can be applied.

In cooperative game theory, Aubin [1] defines a *fuzzy coalition* in N to be a fuzzy subset of N where, for all $i \in N$, the number $e_i \in [0,1]$ is regarded as the degree of participation of player i in e. Every coalition $E \subseteq N$ can be identified with the fuzzy coalition $\mathbf{1}^E \in [0,1]^N$ defined by $\mathbf{1}_i^E = 1$ if $i \in E$ and $\mathbf{1}_i^E = 0$ otherwise.

2.4. The Choquet integral

The Choquet integral [5] was introduced for capacities. Later on Schmeidler [11] studied this integral for all the set functions. If $v : 2^N \to \mathbb{R}$ is a set function and $e \in [0,1]^N$, then the *Choquet integral* of e with respect to v is

$$\int e\,dv = \sum_{p=1}^{q} (s_p - s_{p-1})\, v\left(\{i \in N : e_i \geq s_p\}\right), \tag{3}$$

where $im\,(e) = \{s_p\}_{p=1}^q$ and $s_0 = 0$. The following properties of the Choquet integral are known:

(C1) $\int \mathbf{1}^E\,dv = v\,(E)$, for all $E \subseteq N$.

(C2) $\int te\,dv = t \int e\,dv$, for $t \in [0,1]$.

(C3) $\int e\,dv \leq \int f\,dv$, when $e \leq f$ and v is monotone.

(C4) $\int e\,d\,(cv) = c \int e\,dv$, for $c \in \mathbb{R}$.

(C5) $\int e\,d\,(v_1 + v_2) = \int e\,dv_1 + \int e\,dv_2$.

(C6) $\int (e + f)\; dv = \int e\,dv + \int f\,dv$, when $e + f \leq \mathbf{1}^N$ and e, f are comonotone.

3. Proposed methodology

In [2], it is assumed that each player needs permission from all his superiors, so the cooperation possibilities depend on the positions of the players in the hierarchical permission structure. Moreover, the dependence among players is total. Now, we are going to introduce a new model in which players could depend on other players but not totally, that is, they may have certain degree of autonomy.

3.1. Fuzzy permission structures. The autonomous operator

Definition 1. *A fuzzy permission structure over N is a reflexive fuzzy relation ρ on N, that is, an application $\rho : N \times N \to [0,1]$ satisfying $\rho(i,i) = 1$ for all $i \in N$. The family of all fuzzy permission structures over N is denoted by $\mathcal{FS}^N$.*

Given $\rho \in \mathcal{FS}^N$ and $i, j \in N$, the number $\rho(i,j) \in [0,1]$ is interpreted as the membership level of player j which needs the permission of i.

In the following, we are going to study games with fuzzy permission structure in a *conjunctive approach*. Therefore, we assume that all the dependency relationships must be respected, that is, player i in a coalition $E \subseteq N$ cannot use the membership level controlled by players out of E, in the sense that if coalition E is formed the degree of participation of player i within E cannot be greater than $1 - \rho(j,i)$ for any player j that is not in E. The following operator determines the degree of autonomy which each player has at his disposal when participating in a coalition.

Definition 2. *Let $\rho \in \mathcal{FS}^N$ be a fuzzy permission structure on N. The autonomous operator associated to ρ is the function $a^\rho : 2^N \to [0,1]^N$ defined, for each $i \in N$, by*

$$a_i^\rho(E) = 1 - \bigvee_{j \in N \setminus E} \rho(j,i), \quad \textit{for all } E \subseteq N. \tag{4}$$

In this definition, it is understood that $a^\rho(N) = \mathbf{1}^N$. For any coalition E, the fuzzy coalition $a^\rho(E)$ is called *autonomous set* of E. Now we show several properties of this operator. First, recall that two fuzzy permission structures $\rho, \rho' \in \mathcal{FS}^N$ are comonotone when for all $i, j, k, l \in N$ it holds that $(\rho(i,j) - \rho(k,l))(\rho'(i,j) - \rho'(k,l)) \geq 0$.

Proposition 1. *Let $\rho, \rho' \in \mathcal{FS}^N$ be fuzzy permission structures and $E, F \subseteq N$. It holds that*

(A1) $a^\rho(E) \leq \mathbf{1}^E$.

(A2) $a^\rho(F) \leq a^\rho(E)$ *if* $F \subseteq E$.

(A3) $a^\rho(E \cap F) = a^\rho(E) \cap a^\rho(F)$.

(A4) $a^{(1-t)\rho + t\rho'}(E) = (1-t)a^\rho(E) + t a^{\rho'}(E)$ *for* $t \in [0,1]$, *where* ρ, ρ' *are comonotone.*

Proof. (A1) and (A2) are evident. By (A2), $a^{\rho}(E\cap F)\leq a^{\rho}(E)\cap a^{\rho}(F)$, for all $E,F\subseteq N$. Now, given $i\in N$, there exists $k\notin E$ (or $k\notin F$) such that $a_i^{\rho}(E\cap F)=1-\rho(k,i)$. Moreover,

$$\rho(k,i)=\bigvee_{j\in N\setminus E\cap F}\rho(j,i)=\bigvee_{j\in N\setminus E}\rho(j,i). \tag{5}$$

So, $a_i^{\rho}(E\cap F)=a_i^{\rho}(E)\geq a_i^{\rho}(E)\wedge a_i^{\rho}(F)$ and it proves (A3). It is easy to check (A4) since

$$(1-t)\bigvee_{j\in N\setminus E}\rho(j,i)+t\bigvee_{j\in N\setminus E}\rho'(j,i)=\bigvee_{j\in N\setminus E}\left[(1-t)\rho(j,i)+t\rho'(j,i)\right],$$

for each $i\in N$ and for all comonotone $\rho,\rho'\in\mathcal{FS}^N$. □

3.2. The restricted game

In order to incorporate the information from a fuzzy permission structure into a game, we will define a new characteristic function. To do this we will follow the idea of Tsurumi *et al.* [13] who used the Choquet integral to extend a game to the set of fuzzy coalitions.

Definition 3. *Let $v\in\mathcal{G}^N$ and $\rho\in\mathcal{FS}^N$. The restricted game of v by the fuzzy permission structure ρ is the game v^{ρ} defined by*

$$v^{\rho}(E)=\int a^{\rho}(E)\,dv,\quad \textit{for all } E\subseteq N. \tag{6}$$

In general, given a fuzzy permission structure, the images of the autonomous sets of different coalitions will be different. The next lemma allows us to write the restricted game using the image set of the fuzzy permission structure. Therefore we can use the same set of levels for all coalitions. For this, we consider, for each $\rho\in\mathcal{FS}^N$, the set

$$A_t^{\rho}(E)=\{i\in E:\rho(j,i)<t\quad\text{for all } j\in N\setminus E\}, \tag{7}$$

where $t\in(0,1]$ and $E\subseteq N$. Note that $A_t^{\rho}(E)\subseteq A_t^{\rho}(F)$ when $E\subseteq F$.

Lemma 2. *Let $v\in\mathcal{G}^N$, $\rho\in\mathcal{FS}^N$ with $im(\rho)=\{t_k\}_{k=1}^m$ and $t_0=0$. It holds*

$$v^{\rho}(E)=\sum_{k=1}^{m}(t_k-t_{k-1})\,v\left(A_{t_k}^{\rho}(E)\right),\qquad \textit{for all } E\subseteq N. \tag{8}$$

Proof. Let $E\subseteq N$ be a coalition with $im(a^{\rho}(E))=\{s_p\}_{p=1}^q$ and $s_0=0$. For each $p\in\{0,...,q\}$, there exits $k_p\in\{0,...,m\}$ such that $s_p=1-t_{k_p}$. For all $p\in\{1,...,q\}$,

$k_{p-1} \in \{k_p + 1, ..., m\}$. Then, for all k, with $k_p + 1 \leq k \leq k_{p-1}$ it holds

$$\begin{aligned} A^{\rho}_{t_k}(E) &= \left\{ i \in E : \bigvee_{j \in N \setminus E} \rho(j,i) < t_k \right\} = \left\{ i \in E : 1 - \bigvee_{j \in N \setminus E} \rho(j,i) > 1 - t_k \right\} \\ &= \{ i \in N : a^{\rho}_i(E) > 1 - t_k \} = \{ i \in N : a^{\rho}_i(E) \geq s_p \}, \end{aligned} \tag{9}$$

where the last equality is true by the fact that $a^{\rho}_i(E) > 1 - t_k > s_{p-1}$ if and only if $a^{\rho}_i(E) \geq s_p$ for $i \in N$. Using (9) and the definition of the Choquet integral we can write

$$\begin{aligned} \sum_{k=1}^{m} (t_k - t_{k-1}) \, v\left(A^{\rho}_{t_k}(E)\right) &= \sum_{p=1}^{q} \left[\sum_{k=k_p+1}^{k_{p-1}} (t_k - t_{k-1}) \right] v\left(\{ i \in N : a^{\rho}_i(E) \geq s_p \}\right) \\ &= \sum_{p=1}^{q} \left(t_{k_{p-1}} - t_{k_p} \right) v\left(\{ i \in N : a^{\rho}_i(E) \geq s_p \}\right) \\ &= \sum_{p=1}^{q} (s_p - s_{p-1}) \, v\left(\{ i \in N : a^{\rho}_i(E) \geq s_p \}\right) = v^{\rho}(E). \quad \square \end{aligned}$$

3.3. A value for games with fuzzy permission structure

Now, we are going to answer the question of how to divide the gains among the players in a reasonable way when we have a game and a fuzzy permission structure on the set of players. A *value for games with fuzzy permission structure* is a function $\Psi : \mathcal{G}^N \times \mathcal{FS}^N \to \mathbb{R}^N$ that associates to each game $v \in \mathcal{G}^N$ and each fuzzy permission structure $\rho \in \mathcal{FS}^N$ a vector $\Psi(v, \rho) \in \mathbb{R}^N$, where $\Psi_i(v, \rho)$ represents the payoff to player i in (v, ρ). Let us consider a particular value taking into account the direct relations among players.

Definition 4. *The direct conjunctive fuzzy permission value or DCF-value is defined as the application* $\Phi : \mathcal{G}^N \times \mathcal{FS}^N \to \mathbb{R}^N$ *where* $\Phi(v, \rho) = Sh(v^{\rho})$, *for all* $v \in \mathcal{G}^N$ *and* $\rho \in \mathcal{FS}^N$.

3.4. Transitive fuzzy relationships

In the model introduced in [7] the relations of dependence are transitive. In the rest of this section we aim to deal with transitivity in the case of fuzzy relationships. This could be done in different ways. Our approach is based on the concept of transitive fuzzy relation introduced by Zadeh [15].

Let $S \in \mathcal{S}^N$ be a permission structure. We can identify S with the fuzzy permission structure $\rho_S \in \mathcal{FS}^N$ defined, for all $i, j \in N$, by

$$\rho_S(i,j) = \begin{cases} 1 & \text{if } j \in S(i) \cup \{i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

However, in general, given $v \in \mathcal{G}^N$, the conjunctive permission value, $\phi^{conj}(v,S)$ does not coincide with the DCF-value $\Phi(v,\rho_S)$. This is due to the fact that the latter only takes into account the direct relations among players. Zadeh [15] defines a *transitive fuzzy relation* as a mapping $\rho : N \times N \longrightarrow [0,1]$ satisfying $\rho(i,j) \geq \rho(i,k) \wedge \rho(k,j)$ for all $i,j,k \in N$. Now, we introduce a value considering the indirect dependencies among players in a fuzzy permission structure.

Definition 5. *Let $\rho \in \mathcal{FS}^N$. The transitive closure of ρ is a fuzzy permission structure $\widehat{\rho} \in \mathcal{FS}^N$ defined, for all $i,j \in N$, by*

$$\widehat{\rho}(i,j) = \bigvee_{\{i_p\}_{p=0}^{q} \in P_{ij}} \bigwedge_{p=1}^{q} \rho(i_{p-1}, i_p) \tag{11}$$

where $P_{ij} = \left\{ \{i_p\}_{p=0}^{q} \subseteq N : q \in \mathbb{N},\ i_0 = i,\ i_q = j \right\}$.

The fuzzy permission structure $\widehat{\rho}$ is transitive. Moreover, if ρ is transitive then $\widehat{\rho} = \rho$.

Definition 6. *The conjunctive fuzzy permission value or CF-value is defined as the application $\Phi^{conj} : \mathcal{G}^N \times \mathcal{FS}^N \to \mathbb{R}^N$ where $\Phi^{conj}(v,\rho) = \Phi(v,\widehat{\rho})$, for all $v \in \mathcal{G}^N$ and $\rho \in \mathcal{FS}^N$.*

The next proposition shows that the CF-value is an extension of the conjunctive permission value [7].

Proposition 3. *If $S \in \mathcal{S}^N$ then $\Phi^{conj}(v,\rho_S) = \phi^{conj}(v,S)$ for each game $v \in \mathcal{G}^N$.*

Proof. It is easy to check that $A_1^{\widehat{\rho_S}}(E) = A^S(E)$ for all $E \subseteq N$ and thus $v^{\widehat{\rho_S}} = v^S$. Therefore,

$$\Phi^{conj}(v,\rho_S) = \Phi(v,\widehat{\rho_S}) = Sh\left(v^{\widehat{\rho_S}}\right) = \phi^{conj}(v,S). \quad \square \tag{12}$$

4. Results

Firstly we will characterize the values introduced in the previous section. After that, a practical example is provided.

4.1. A characterization of the DCF-value

Let $\Psi : \mathcal{G}^N \times \mathcal{FS}^N \to \mathbb{R}^N$ be a value for games with fuzzy permission structure. We consider the following properties:

Efficiency. If $v \in \mathcal{G}^N$ and $\rho \in \mathcal{FS}^N$ then

$$\sum_{i \in N} \Psi_i(v,\rho) = v(N). \tag{13}$$

Additivity. For all $v_1, v_2 \in \mathcal{G}^N$ and $\rho \in \mathcal{FS}^N$, it holds

$$\Psi\left(v_1 + v_2, \rho\right) = \Psi\left(v_1, \rho\right) + \Psi\left(v_2, \rho\right). \tag{14}$$

Inessential player. A player $i \in N$ is an *inessential player* in $(v, \rho) \in \mathcal{G}^N \times \mathcal{FS}^N$ if every player $j \in N$ with $\rho(i, j) > 0$ is a null player in v. If i is inessential player in (v, ρ) then $\Psi_i(v, \rho) = 0$.

Veto power over a necessary player. If i is a necessary player in a monotone $v \in \mathcal{G}^N$ and $\rho \in \mathcal{FS}^N$ is a fuzzy permission structure, then $\Psi_j(v, \rho) \geq \Psi_k(v, \rho)$ for each $j \in N$ with $\rho(j, i) = 1$ and for all $k \in N$.

Comonotonicity. Let $\rho, \rho' \in \mathcal{FS}^N$ be comonotone fuzzy permission structures. For all $t \in [0, 1]$, it holds that

$$\Psi\left(v, t\rho + (1 - t)\rho'\right) = t\Psi(v, \rho) + (1 - t)\Psi\left(v, \rho'\right). \tag{15}$$

The next result provides a characterization of the DCF-value.

Theorem 4. *A value $\Psi : \mathcal{G}^N \times \mathcal{FS}^N \to \mathbb{R}^N$ is equal to the DCF-value if and only if it satisfies the properties of additivity, efficiency, inessential player, veto power over a necessary player and comonotonicity.*

Proof. Firstly we will prove that Φ satisfies the properties in the theorem.

Additivity follows directly from the linearity of the Shapley value and (C5).

Efficiency is clear from the fact that the Shapley value is efficient and (C1). Given $(v, \rho) \in \mathcal{G}^N \times \mathcal{FS}^N$,

$$\sum_{i \in N} \Phi_i(v, \rho) = \sum_{i \in N} \phi_i\left(v^\rho\right) = v^\rho(N) = \int a^\rho(N)\, dv = \int \mathbf{1}^N\, dv = v(N). \tag{16}$$

In order to prove that the DCF-value satisfies the property of inessential player, let $i \in N$ be an inessential player for (v, ρ). We must check that $\Phi_i(v, \rho) = 0$. Taking into consideration the null player property of the Shapley value, it suffices to show that player i is a null player in v^ρ. If $im(\rho) = \{t_k\}_{k=1}^m$, $t_0 = 0$ and $E \subseteq N$ is a coalition such that $i \in E$, (8) implies that

$$v^\rho(E) \quad = \quad \sum_{k=1}^m \left(t_k - t_{k-1}\right) v\left(A_{t_k}^\rho(E)\right)$$

and

$$v^{\rho}(E\setminus\{i\}) \;=\; \sum_{k=1}^{m}(t_k-t_{k-1})\,v\left(A^{\rho}_{t_k}(E\setminus\{i\})\right).$$

Clearly, $A^{\rho}_{t_k}(E\setminus\{i\})\subseteq A^{\rho}_{t_k}(E)$ for $k\in\{1,...,m\}$. If $j\in A^{\rho}_{t_k}(E)\setminus A^{\rho}_{t_k}(E\setminus\{i\})$ then $\rho(i,j)\geq t_k>0$ and therefore, j is a null player in v. So, $v\left(A^{\rho}_{t_k}(E)\right)=v\left(A^{\rho}_{t_k}(E\setminus\{i\})\right)$.

Now, let $i\in N$ be a necessary player in a monotone $v\in\mathcal{G}^N$ and $j\in N$ a player with $\rho(j,i)=1$. Then, $a^{\rho}_i(E)=0$ for all $E\subset N$ with $j\notin E$. If $im(a^{\rho}(E))=\{s_p\}_{p=1}^{q}$ then $i\notin\{k\in N: a^{\rho}_k(E)\geq s_p\}$ for all p and so, $v(\{k\in N: a^{\rho}_k(E)\geq s_p\})=0$. Hence $v^{\rho}(E)=0$ and j is a necessary player in v^{ρ}. Properties (A2) and (C3) imply v^{ρ} is also monotone. As the Shapley value satisfies the necessary player property then, for all $k\in N$,

$$\Phi_j(v,\rho)=Sh_j(v^{\rho})\geq Sh_k(v^{\rho})=\Phi_k(v,\rho).$$

To check that the DCF-value satisfies the property of comonotonicity, let $\rho,\rho'\in\mathcal{FS}^N$ be comonotone fuzzy permission structures. First, we show that the fuzzy sets $a^{\rho}(E)$ and $a^{\rho'}(E)$ are comonotone for all $E\subseteq N$. Indeed, if $i,j\in N$, it holds $a^{\rho}_i(E)>a^{\rho}_j(E)$ if and only if $\bigvee_{k\in N\setminus E}\rho(k,j)>\bigvee_{k\in N\setminus E}\rho(k,i)$. The last inequality is equivalent to writing that for all $k\in N\setminus E$ there exists $k'\in N\setminus E$ such that $\rho(k',j)>\rho(k,i)$. Moreover, since ρ,ρ' are comonotone, for $k\in N\setminus E$, there exists $k'\in N\setminus E$ such that $\rho'(k',j)\geq\rho'(k,i)$ or equivalently $a^{\rho'}_i(E)\geq a^{\rho'}_j(E)$. Clearly, if $t\in[0,1]$, then $ta^{\rho}(E)$ and $(1-t)a^{\rho'}(E)$ are also comonotone and $ta^{\rho}(E)+(1-t)\,a^{\rho'}(E)\leq\mathbf{1}^N$. Then, by (A4), (C6) and (C2), for $v\in\mathcal{G}^N$ it holds that

$$\begin{aligned}v^{t\rho+(1-t)\rho'}(E) &= \int a^{t\rho+(1-t)\rho'}(E)\,dv=\int\left(ta^{\rho}(E)+(1-t)\,a^{\rho'}(E)\right)dv\\ &= t\int a^{\rho}(E)\,dv+(1-t)\int a^{\rho'}(E)\,dv=tv^{\rho}(E)+(1-t)\,v^{\rho'}(E).\end{aligned}\tag{17}$$

Finally, from (17) and the linearity of the Shapley value it follows that

$$\begin{aligned}\Phi(v,t\rho+(1-t)\,\rho') &= Sh\left(v^{t\rho+(1-t)\rho'}\right)=Sh\left(tv^{\rho}+(1-t)\,v^{\rho'}\right)\\ &= tSh(v^{\rho})+(1-t)\,Sh\left(v^{\rho'}\right)=t\Phi(v,\rho)+(1-t)\,\Phi(v,\rho').\end{aligned}$$

It remains to prove uniqueness. For this, let us consider a value Ψ satisfying the five properties and we show that $\Psi=\Phi$.

First, let $\rho\in\mathcal{FS}^N$ be a fuzzy permission value with $im(\rho)=\{1\}$ and $F\in 2^N\setminus\{\emptyset\}$. Let $L=\{k\in N:\rho(k,j)=1\text{ for some }j\in F\}$ and $\alpha\geq 0$. If $i\notin L$ then i is inessential player in $(\alpha u_F, \rho)$. The property of inessential player implies that $\Psi_i(\alpha u_F, \rho) = 0$. If $i \in L$, then there exists $j \in F$ such that $\rho(i,j) = 1$. From the property the veto power over a necessary player there is $c \geq 0$ with $\Psi_i(\alpha u_F, \rho) = c$. From efficiency it follows that

$$\sum_{i \in N} \Psi_i(\alpha u_F, \rho) = c|L| = \alpha, \tag{18}$$

and thus $\Psi(\alpha u_F, \rho) = \Phi(\alpha u_F, \rho)$. If $\alpha = 0$ notice that all players are inessential in $(0, \rho)$. Applying the property of inessential player to both values, we obtain $\Psi(0, \rho) = \Phi(0, \rho) = 0$. From this and the property of additivity, if $\alpha < 0$ then we can derive

$$\Psi(\alpha u_F, \rho) = -\Psi(-\alpha u_F, \rho) = -\Phi(-\alpha u_F, \rho) = \Phi(\alpha u_F, \rho). \tag{19}$$

Since $v \in \mathcal{G}^N$ can be written as $v = \sum_{\{F \subseteq N,\, F \neq \emptyset\}} \alpha_F u_F$, it is clear that $\Psi(v, \rho) = \Phi(v, \rho)$ for all $(v, \rho) \in \mathcal{G}^N \times \mathcal{FS}^N$.

Following a recurrence argument, we assume that the equality $\Psi(v, \rho) = \Phi(v, \rho)$ is true for all $(v, \rho) \in \mathcal{G}^N \times \mathcal{FS}^N$ with $|im(\rho)| = m - 1$ and consider $\rho \in \mathcal{FS}^N$ with $|im(\rho)| = m$. Let $\rho_1, \rho_R \in \mathcal{FS}^N$ be the fuzzy permission structures defined, for all $i, j \in N$, by

$$\rho_1(i,j) = \begin{cases} 1 & \text{if } \rho(i,j) = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \rho_R(i,j) = \begin{cases} 1 & \text{if } \rho(i,j) = 1, \\ \dfrac{\rho(i,j)}{t_R} & \text{otherwise,} \end{cases} \tag{20}$$

where $t_R = \vee(im(\rho) \setminus \{1\})$. It is clear that $|im(\rho_1)| = 1$ and $|im(\rho_R)| = m - 1$. Note that $\rho = (1 - t_R)\rho_1 + t_R \rho_R$. Moreover, $(1 - t_R)\rho_1$ and $t_R \rho_R$ are comonotone. From the property of comonotonicity and the induction hypothesis we obtain

$$\Psi(v, \rho) = (1 - t_R)\Psi(v, \rho_1) + t_R \Psi(v, \rho_R) = (1 - t_R)\Phi(v, \rho_1) + t_R \Phi(v, \rho_R) = \Phi(v, \rho). \tag{21}$$

□

4.2. A characterization of the CF-value

Next we present some properties for $\Psi : \mathcal{G}^N \times \mathcal{FS}^N \to \mathbb{R}^N$ that uniquely determine the CF-value. We will consider the properties of additivity, efficiency, comonotonicity and the following ones:

Transitive inessential player. A player $i \in N$ is a *transitive inessential player* in $(v, \rho) \in \mathcal{G}^N \times \mathcal{FS}^N$ if j is a null player in v for all $j \in N$ such that there exists a sequence $\{i_p\}_{p=0}^q$ with $i_0 = i$, $i_q = j$ and $\rho(i_{p-1}, i_p) > 0$ for all $p = 1, \ldots, q$. If i is a transitive inessential player in (v, ρ) then $\Psi_i(v, \rho) = 0$.

Transitive veto power over a necessary player. If i is a necessary player in a monotone game $v \in \mathcal{G}^N$ then $\Psi_j(v,\rho) \geq \Psi_k(v,\rho)$ for all $k \in N$ for each $j \in N$ such that there exists a sequence $\{i_p\}_{p=0}^q$ with $i_0 = j$, $i_q = i$ and $\rho(i_{p-1}, i_p) = 1$ for all $p = 1, \ldots, q$.

Theorem 5. *A value $\Psi : \mathcal{G}^N \times \mathcal{FS}^N \to \mathbb{R}^N$ is equal to the CF-value if and only if it satisfies the properties of additivity, efficiency, transitive inessential player, transitive veto power over a necessary player and comonotonicity.*

Proof. Firstly we will prove that Φ^{conj} satisfies the five properties.

Additivity and efficiency of Φ^{conj} are evident.

The properties of transitive inessential player and transitive veto power over a necessary player are easy to show. If $i \in N$ is a transitive inessential player in (v,ρ), then $i \in N$ is an inessential player in $(v,\widehat{\rho})$ and hence $\Phi_i(v,\widehat{\rho}) = 0 = \Phi_i^{conj}(v,\rho)$. On the other hand, if $i \in N$ is a necessary player in a monotone game v and $j \in N$ is a player such that there exists a sequence $\{i_p\}_{p=0}^q$ with $i_0 = j$, $i_q = i$ and $\rho(i_{p-1}, i_p) = 1, 1 \leq p \leq q$, then it holds that $\widehat{\rho}(j,i) = 1$ and hence, $\Phi_i^{conj}(v,\rho) = \Phi_i(v,\widehat{\rho}) \geq \Phi_k(v,\widehat{\rho}) = \Phi_k^{conj}(v,\rho)$ for all $k \in N$.

In order to prove that Φ^{conj} satisfies the property of comonotonicity, let $\rho, \rho' \in \mathcal{FS}^N$ be comonotone fuzzy permission structures and $t \in [0,1]$. If $\tau = t\rho + (1-t)\rho'$ then we show that $\widehat{\tau} = t\widehat{\rho} + (1-t)\widehat{\rho'}$, for $t \in [0,1]$. Indeed, given $i, j \in N$, it holds that

$$\begin{aligned}
\widehat{\tau}(i,j) &= \bigvee_{\{i_p\}_{p=0}^q \in P_{ij}} \bigwedge_{p=1}^{q} (t\rho + (1-t)\rho')(i_{p-1}, i_p) \\
&= t \bigvee_{\{i_p\}_{p=0}^q \in P_{ij}} \bigwedge_{p=1}^{q} \rho(i_{p-1}, i_p) + (1-t) \bigvee_{\{i_p\}_{p=0}^q \in P_{ij}} \bigwedge_{p=1}^{q} \rho'(i_{p-1}, i_p) \\
&= t\widehat{\rho}(i,j) + (1-t)\widehat{\rho'}(i,j). \qquad (22)
\end{aligned}$$

These equalities are true because ρ and ρ' are comonotone and therefore the maximum is obtained for the same sequence $\{i_p\}_{p=0}^q$ and the corresponding minimum is attained at the same (i_{p-1}, i_p). Moreover, it is easy to check that $\widehat{\rho}$ and $\widehat{\rho'}$ are also comonotone. From (22) and the fact that Φ satisfies the property of comonotonicity, it follows that

$$\begin{aligned}
\Phi^{conj}(v, t\rho + (1-t)\rho') &= \Phi(v,\widehat{\tau}) = \Phi\left(v, t\widehat{\rho} + (1-t)\widehat{\rho'}\right) \\
&= t\Phi(v,\widehat{\rho}) + (1-t)\Phi\left(v,\widehat{\rho'}\right) = t\Phi^{conj}(v,\rho) + (1-t)\Phi^{conj}(v,\rho').
\end{aligned}$$

Now suppose that $\Psi : \mathcal{G}^N \times \mathcal{FS}^N \to R^N$ satisfies the five properties.

Let $\rho \in \mathcal{FS}^N$ be a fuzzy permission structure with $im(\rho) = \{1\}$ and $F \in 2^N \setminus \{\emptyset\}$. If we consider $L = \{k \in N : \widehat{\rho}(k,j) = 1 \text{ for some } j \in F\}$, then by repeating the reasoning in the proof of Theorem 4 we obtain that $\Psi(v,\rho) = \Phi^{conj}(v,\rho)$ for all $(v,\rho) \in \mathcal{G}^N \times \mathcal{FS}^N$ with $im(\rho) = \{1\}$. If we assume that the equality $\Psi(v,\rho) = \Phi^{conj}(v,\rho)$ is true for all $v \in \mathcal{G}^N$ and $\rho \in \mathcal{FS}^N$ with $|im(\rho)| = m-1$, and apply the induction hypothesis to the comonotone fuzzy permission structures $\rho_1, \rho_R \in \mathcal{FS}^N$ defined, for all $i, j \in N$, by

$$\rho_1(i,j) = \begin{cases} 1 & \text{if } \rho(i,j) = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \rho_R(i,j) = \begin{cases} 1 & \text{if } \rho(i,j) = 1, \\ \dfrac{\rho(i,j)}{t_R} & \text{otherwise,} \end{cases} \tag{23}$$

where $t_R = \vee(im(\rho) \setminus \{1\})$, we obtain, by using the property of comonotonicity, that $\Psi(v,\rho) = \Phi^{conj}(v,\rho)$ for all $(v,\rho) \in \mathcal{G}^N \times \mathcal{FS}^N$. □

4.3. Example

Consider three factories 1, 2 and 3. Each one of them can produce, let us say, one million units of a certain electronic component. For any factory i, the component manufactured by i can be sold at a profit of i euros each. Besides, any two of the factories, i and j, can decide to assemble their components (one unit for one unit) and produce a new component that can be sold at a profit of $2(i+j)$ euros each. Finally, the three factories can agree to work together and produce one million units of another more sophisticated component that can be sold at a profit of 20 euros each. This situation can be modeled by a classical cooperative game.

Now imagine the following. Factory 2 sues factory 1 for violation of patent rights and factory 3 sues 2 for the same reason (suppose that both patent infringements are not related). So, in the end, the situation is this: factory 1 cannot use or sell the component they make without permission from factory 2, and the latter cannot use or sell their component without permission from factory 3. Notice that this scenario cannot be modeled with the conjunctive structures introduced in [7], since there is no transitivity in the dependency relationships. Although our model can be applied to this situation, we prefer to focus on proper fuzzy relationships. So we will consider the following sequence of events. The patent problem is difficult to clear up and factory 3 is not confident of winning a possible lawsuit. They are also afraid of the cost of litigation. So they decide to make a proposal to factory 2, according to which the latter would not be able to make use of a 50% of the components they make without the permission from factory 3. The managers of factory 2 are aware that they have probably infringed a patent, so they accept. Then, a 50% of the operating capacity of player 2 ends up being dependent on the permission from player 3. Imagine that something similar happens between 1 and 2. But in this case we will suppose that there are more doubts regarding the possible infringement, so they finally sign an exclusivity agreement that only affects a 30% of the production of factory 1.

Now, we will apply our model to this last scenario. Let $N = \{1,2,3\}$ be the set of players (factories). The characteristic function indicating the profit (in millions) of each coalition without taking into account the dependency relationships is given by

$$\begin{array}{c} v(\{1\}) = 1, \qquad v(\{2\}) = 2, \qquad v(\{3\}) = 3, \\ v(\{1,2\}) = 6, \qquad v(\{1,3\}) = 8, \qquad v(\{2,3\}) = 10, \quad v(N) = 20. \end{array} \tag{24}$$

The dependency relationships are the result of the agreements and can be represented by the fuzzy permission structure ρ given by the matrix $[\rho(i,j)]_3$

$$\rho = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}. \tag{25}$$

The autonomous operator (4) associated to ρ determines the fraction of resources that each player can use within a coalition,

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$a^{\rho}(E)$	$(0.7,0,0)$	$(0,0.5,0)$	$(0,0,1)$	$(1,0.5,0)$	$(0.7,0,1)$	$(0,1,1)$	$(1,1,1)$

Table 1: The autonomous sets.

The Choquet integral with respect to v of the autonomous set of a coalition is the best profit for that coalition. Hence we determine the restricted game (6):

$$\begin{array}{c} v^{\rho}(\{1\}) = 0.7\,v(\{1\}) = 0.7, \qquad v^{\rho}(\{2\}) = 0.5\,v(\{2\}) = 1, \qquad v^{\rho}(\{3\}) = v(\{3\}) = 3, \\ v^{\rho}(\{1,2\}) = 0.5\,v(\{1,2\}) + 0.5\,v(\{1\}) = 3.5, \qquad v^{\rho}(\{2,3\}) = v(\{2,3\}) = 10, \\ v^{\rho}(\{1,3\}) = 0.7\,v(\{1,3\}) + 0.3\,v(\{3\}) = 6.5, \qquad v^{\rho}(N) = v(N) = 20, \end{array} \tag{26}$$

Finally, a payoff vector for the factories is $\Phi(v,\rho) = (4.5667, 6.4667, 8.9667)$.

Now, we suppose that both patent infringements are related. In that case we apply the value introduced in Definition 6. The transitive closure (11) of ρ is given by the matrix

$$\hat{\rho} = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0.3 & 0.5 & 1 \end{bmatrix}. \tag{27}$$

We apply formula (8) to obtain the restricted game. We use the sets in the following table:

t_k	$A_{t_k}^{\hat{\rho}}(\{1\})$	$A_{t_k}^{\hat{\rho}}(\{2\})$	$A_{t_k}^{\hat{\rho}}(\{3\})$	$A_{t_k}^{\hat{\rho}}(\{1,2\})$	$A_{t_k}^{\hat{\rho}}(\{1,3\})$	$A_{t_k}^{\hat{\rho}}(\{2,3\})$	$A_{t_k}^{\hat{\rho}}(N)$
0.3	$\emptyset$	$\emptyset$	$\{3\}$	$\emptyset$	$\{3\}$	$\{2,3\}$	$\{1,2,3\}$
0.5	$\{1\}$	$\emptyset$	$\{3\}$	$\{1\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$

Table 2: Set $A_{t_k}^{\hat{\rho}}(E)$ for $E \subseteq N$.

The restricted game of v by the fuzzy permission structure $\hat{\rho}$ is,

$$\begin{gathered} v^{\hat{\rho}}(\{1\}) = 0.7, \qquad v^{\hat{\rho}}(\{2\}) = 1, \qquad v^{\hat{\rho}}(\{3\}) = 3, \\ v^{\hat{\rho}}(\{1,2\}) = 3.2, \qquad v^{\hat{\rho}}(\{1,3\}) = 6.5, \qquad v^{\hat{\rho}}(\{2,3\}) = 10, \quad v^{\hat{\rho}}(N) = 20, \end{gathered} \tag{28}$$

and a payoff vector for the factories is $\Phi^{conj}(v,\rho) = (4.5167, 6.4167, 9.0667)$.

5. Conclusions and remarks

We have defined and characterized a value for games with fuzzy permission structure. This value extends the conjunctive value introduced in [7] in several ways. Firstly, it deals with fuzzy relationships, allowing us to give reasonable payoffs in situations in which there are players that depend partially on other players. In addition, this new value is applicable not only to hierarchies, but to a wide range of dependency structures. Moreover, in this new model dependency relations are not necessarily transitive.

We have also studied a value for games with fuzzy permission structure applicable to situations in which indirect relations must be considered. A characterization of this value is given as well.

Our model can be applied whenever we have a cooperative TU-game and a collection of pairwise dependency relationships.

Different approaches could be considered for further research. For instance, we can think of situations in which the dependence relationships do not have a pairwise pattern, that is, situations in which a player may depend on a coalition but not necessarily on any player of this coalition. Apart from that, we could consider different operations to deal with fuzzy relationships. In this respect, our definition of the autonomous set of a coalition is inspired by the standard intersection and union of fuzzy sets introduced by Zadeh [14]. Depending on the nature of the relationships among the players, it might be convenient to consider alternative fuzzy set operations, like those given by Butnariu and Klement [4] or others. Moreover, in order to deal with transitive relationships we have considered the concept of transitive relationship introduced by Zadeh [15], that makes use of the minimum t-norm.

But in some situations it would be more appropriate to consider the transitivity with respect to the product t-norm. This approach remains to be studied.

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