Cooperative games and coalition cohesion indices: the Choquet-Owen value

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Abstract

In a cooperative game with transferable utility, it is usually assumed that all coalitions are equally feasible. However, if we deal with cooperative games with coalition configuration, only some coalitions are a priori feasible, due to the preferences of the agents. In this paper we propose a generalization of games with coalition configuration. In our model, the feasibility of a coalition is determined by the cohesion of its members and, obviously, this cohesion does not have to be equal for all coalitions. The cohesion of each coalition will be determined by a cohesion index. We introduce the class of games with cohesion index and propose an allocation rule, which is characterized by using reasonable properties. The cohesion idea is not only a concept related to social groups. In software design this concept explains the relationships among all the elements of a module. Our value can be applied in this way as we show in the paper.

Index Terms

game theory, Owen value, coalition configuration, group cohesion, software cohesion

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I. INTRODUCTION

Cooperative game theory studies situations where a set of agents (players) bargain to allocate a common profit obtained from their collaboration. The resulting allocation is given by a vector (payoff vector) whose coordinates are the payoffs assigned to the players. A game on a set of players is modelled by a function (characteristic function) that assigns to each subset of players (coalition) the profit that the members of the coalition can achieve when they decide to cooperate. The Shapley value [1] is one of the most studied solutions for cooperative games. It is a correspondence that assigns a payoff vector to each game based on a set of reasonable conditions (axioms) which allow us to compare this value with alternative allocations. Several variations of the Shapley value have been proposed for situations where some additional information is known. Frequently this additional information is about the interpersonal relationships among the players. This paper focuses on the Owen variation [2]. In that paper, it is considered that there exists a partition of the set of players based on the affinities between them. Owen proposed a Shapley-type solution which takes into account that information in order to get a fair allocation of the profit obtained by the grand coalition. Later on Albizuri *et al.* [3] introduced coalition configurations as a way of modelling exogenous information. A coalition configuration is a family of coalitions, not necessarily disjoint, whose union is the whole set of players. They defined an Owen-type value for these situations, named the configuration Owen value.

The term *group cohesion* (see Beal *et al.* [4]) refers to the cumulative effect of all the factors causing members of a group to stay in it, the ”social glue” that binds a group together while working towards a goal or satisfying the needs of its members. Hence, cohesiveness is a measure of the attraction of the group to its members and their resistance to leaving it. A cohesion index assigns a number between 0 and a maximum level to the groups in order to compare their cohesiveness. A coalition configuration can be seen as a family such that the coalitions in the family have the maximum level of cohesiveness whereas the other coalitions are not cohesive at all. Group cohesion has been studied in different fields, for instance, Sociology and Group Psicology (see Carless *et al.* [5]), sports (see Gammage *et al.* [6]), software design (see Bieman and Ott [7]), ... These studies proposed different measures of cohesiveness and studied the impact of cohesiveness on the performance of a task or project showing the importance of the cohesiveness amongst all the agents involved in the project so as to achieve good results.

In our paper we go a step further in the following sense. Suppose that several departments of a company are involved in a project. Once the project is finished and some benefits/costs arise, we are interested in assigning a portion of them to each worker, taking into account the cohesiveness of the departments. To do this, we will extend the model of transferable utility cooperative games. As an example of application of our model, we will consider, in software design, the cost of design, development and/or maintenance of a program, where each one of its elementary units (modules) is considered as a player. In this setting, a coalition is a set of modules and its cost is given by the sum of its individual costs if there is no connection among them; if the elements are connected the cost will be determined taking into consideration the cohesiveness of the coalition. The final goal is to split the total cost among the elements by using a cohesion measure. We can also take a reverse perspective and model the benefit obtained from creating some elements together.

In this paper, our goal is to provide a framework to deal with those situations. We introduce games with coalition cohesion index. A cohesion index will be given by a function on the family of all coalitions. We also provide and characterize a sharing value for games with coalition index. Our model extends that of games with coalition configuration structure and our value extends the configuration Owen value.

The paper is organized as follows. Section 2 is devoted to some preliminaries on cooperative games, coalition configurations and fuzzy sets. Cohesion indices are introduced in Section 3. In Section 4 we define the Choquet-Owen value for games with coalition cohesion index. This value is characterized in Section 5. Finally, some conclusions are presented.

II. Preliminaries

A. Cooperative games

A *cooperative game* (with transferable utility) is a pair (N, v) where N is a set of cardinality $n \in \mathbb{N}$ and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of N are called *players*, the subsets $S \subseteq N$ *coalitions* and $v(S)$ is the *worth* of S. Let (N, v) be a game. The game is *monotone* if $v(T) \leq v(S)$ for all $T \subseteq S \subseteq N$. A player $i \in N$ is said to be a *null player* for (N, v) if it satisfies $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$ with $i \in S$. Two players $i, j \in N$ are *symmetric* for the game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. If $T \subseteq N$ is a non-empty coalition, the *unanimity game* (N, u_T) is given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. The unanimity games can generate all the characteristic functions in the sense that each game (N, v) satisfies that there exist numbers $\{\alpha_T\}_{\emptyset \neq T \subseteq N} \subset \mathbb{R}$ (named dividends, see [8]) such that

$$v = \sum_{\emptyset \neq T \subseteq N} \alpha_T u_T.$$

A *payoff vector* for the game (N, v) is any $x \in \mathbb{R}^N$ where, for each player $i \in N$, the number x_i represents the payment of i owing to his cooperation possibilities in the game. A *value* for cooperative games assigns to each game (N, v) a payoff vector in $\mathbb{R}^N$. The *Shapley value* [1] of a game (N, v) is defined for any player $i \in N$ as

$$Sh_i(N, v) = \sum_{\{S \subseteq N : i \in S\}} \gamma_n^s \left[v(S) - v(S \setminus \{i\}) \right],$$

where

$$\gamma_n^s = \frac{(n-s)!(s-1)!}{n!} \quad \text{with } s = |S|.$$

This value is the only one satisfying the following conditions:

(S1) *Efficiency.* It holds that $\sum_{i \in N} Sh_i(N, v) = v(N)$ for every game (N, v).

(S2) *Linearity.* For all games $(N, v_1), (N, v_2)$ and for all $a_1, a_2 \in \mathbb{R}$ it holds that

$$Sh(N, a_1 v_1 + a_2 v_2) = a_1 Sh(N, v_1) + a_2 Sh(N, v_2).$$

(S3) *Null player property.* If $i \in N$ is a null player for a game (N, v) then $Sh_i(N, v) = 0$.

(S4) *Equal treatment property.* If i, j are symmetric players for a game (N, v) then $Sh_i(N, v) = Sh_j(N, v)$.

B. Coalition configurations

Let N be a finite set of players in a game situation. Following Albizuri *et al.* [3], a *coalition configuration* of N is a family of non-empty coalitions $\mathcal{C} = \{C_1, ..., C_m\}$ satisfying $\bigcup_{p=1}^m C_p = N$. These coalitions, that we name *groups* in the paper, represent a priori unions of the players caused by common interests or social relationships. A *coalition structure* in the sense of Owen [2] is a particular case of coalition configuration $\mathcal{C}$, whenever $\mathcal{C}$ is a partition of N, that is, for all $p, q \in \{1, \ldots, m\}$ with $p \neq q$ it holds that $C_p \cap C_q = \emptyset$. A *game with coalition configuration* is a triple $(N, \mathcal{C}, v)$ where (N, v) is a cooperative game and $\mathcal{C}$ is a coalition configuration of N.

Since $\bigcup \mathcal{C} = N$ we will denote $(\mathcal{C}, v)$ instead of $(N, \mathcal{C}, v)$.

A *configuration value* assigns a payoff vector to each game with coalition configuration. That is, if ψ is a configuration value then $\psi(\mathcal{C}, v)$ is a payoff vector in $\mathbb{R}^N$ for each game with coalition configuration $(\mathcal{C}, v)$. Particularly the (configuration) Owen value [3] is a configuration value based on the Shapley value. It is defined following a heuristic development in two steps of bargaining: the first one among the groups and the second one into each group. Let $(\mathcal{C}, v)$ be a game with coalition configuration. If $\mathcal{C} = \{C_1, ..., C_m\}$ then we denote $M = \{1, ..., m\}$. The *quotient game* is the cooperative game $(M, v_{\mathcal{C}})$ given by

$$v_{\mathcal{C}}(Q) = v\left(\bigcup_{p \in Q} C_p\right) \qquad \text{for every } Q \subseteq M. \tag{1}$$

Consider $p \in M$. For each $S \subseteq C_p$ we take the coalition configuration

$$\mathcal{C}_p(S) = \{C_1, ..., C_{p-1}, S, C_{p+1}, ..., C_m\}. \tag{2}$$

A game $(C_p, v_p^{\mathcal{C}})$ is defined by

$$v_p^{\mathcal{C}}(S) = Sh_p(M, v_{\mathcal{C}_p(S)}) \qquad \text{for every } S \subseteq C_p. \tag{3}$$

Finally, the *Owen value* of $(\mathcal{C}, v)$ is given by

$$\phi_i(\mathcal{C}, v) = \sum_{\{p \in M \colon i \in C_p\}} Sh_i(C_p, v_p^{\mathcal{C}}) \qquad \text{for every } i \in N. \tag{4}$$

The Owen value satisfies the properties of efficiency (O1), linearity (O2) and null player (O3), which are analogous to (S1), (S2) and (S3) respectively, regardless of the coalition configuration. Albizuri *et al.* [3] introduced three other properties to axiomatize the Owen value: anonymity, coalitional symmetry and merger.

(O4) *Anonymity.* If π is a permutation of N such that $\pi C_p = C_p$ for all $p \in M$ then $\phi_i(\mathcal{C}, \pi v) = \phi_{\pi i}(\mathcal{C}, v)$ for each $i \in N$, with $\pi v(S) = v(\pi S)$ for all $S \subseteq N$.

Anonymity implies that ϕ satisfies a property similar to the *equal treatment property* satisfied by the Shapley value. If $i, j \in N$ are symmetric players for (N, v) and $i \in C_p$ if and only if $j \in C_p$ whenever $C_p \in \mathcal{C}$, then $\phi_i(\mathcal{C}, v) = \phi_j(\mathcal{C}, v)$. To show this it is enough to consider the permutation $\pi(i) = j$, $\pi(j) = i$ and $\pi(k) = k$ for any other player, and apply the anonymity property.

Two groups $C_p, C_q \in \mathcal{C}$ are *symmetric for* $(\mathcal{C}, v)$ if $v(\bigcup_{r \in Q} C_r \cup C_p) = v(\bigcup_{r \in Q} C_r \cup C_q)$ for all $Q \subseteq M \setminus \{p, q\}$.

(O5) *Coalitional symmetry.* For every symmetric groups $C_p, C_q \in \mathcal{C}$ it holds that

$$\sum_{i \in C_p} \phi_i(\mathcal{C}, v) = \sum_{i \in C_q} \phi_i(\mathcal{C}, v).$$

Two players $i, j \in N$ are *double for* $(\mathcal{C}, v)$ when: 1) $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N$ and 2) if there is a coalition $S \subseteq N \setminus \{i, j\}$ with $S \cup \{i\} \in \mathcal{C}$ then $S \cup \{j\}, S \cup \{i, j\} \notin \mathcal{C}$. Double players have the same influence jointly or individually. If i, j are double for $(\mathcal{C}, v)$ then $\mathcal{C}^i_{-j} = (\mathcal{C} \setminus \{C_p \in \mathcal{C} : j \in C_p\}) \cup \{(C_p \setminus \{j\}) \cup \{i\} : C_p \in \mathcal{C}, j \in C_p\}$ is a coalition configuration of $N \setminus \{j\}$. Consider $(\mathcal{C}^i_{-j}, v^i_{-j})$ where

$$v^i_{-j}(S) = \begin{cases} v(S \cup \{j\}) & \text{if } i \in S, \\ v(S) & \text{otherwise.} \end{cases} \tag{5}$$

(O6) *Merger.* If $i, j \in N$ are double then $\phi_k(\mathcal{C}, v) = \phi_k(\mathcal{C}^i_{-j}, v^i_{-j})$ for all $k \in N \setminus \{i, j\}$.

C. Fuzzy sets and the Choquet integral

The functions minimum and maximum will be denoted by $\wedge, \vee$ respectively throughout this paper. Fuzzy sets were described by Zadeh [9]. Let A be a finite set. A *fuzzy subset* of A is a mapping $\rho : A \longrightarrow [0,1]$ where ρ assigns to $a \in A$ a degree of membership. Given a fuzzy set ρ of A, the *support* of ρ is the set $supp\,(\rho) = \{a \in A : \rho(a) > 0\}$ and the *image* of ρ is the set $im\,(\rho) = \{\rho(a) : a \in A\}$. We will denote the image of ρ as $im\,(\rho) = \{\lambda_1 < \cdots < \lambda_q\}$ when we want to consider that its elements are written in increasing order. The fuzzy sets ρ, ρ' of A are called *comonotone* when $(\rho(a) - \rho(b))\,(\rho'(a) - \rho'(b)) \geq 0$ for all $a, b \in A$. For each number $t \in [0,1]$ the t*-level set* is

$$A_t^\rho = \{a \in A : \rho(a) \geq t\}\,. \tag{6}$$

A *set function* over A is any function $f : 2^A \to \mathbb{R}$. Observe that if $f(\emptyset) = 0$ then (A, f) is a game. The Choquet integral [10] was introduced for monotone games over A (capacities). Later on Schmeidler [11] studied this integral for all the set functions. If f is a set function and ρ a fuzzy set of A, then the *Choquet integral* of ρ with respect to f is the continuous operator

$$\int \rho\, df = \sum_{k=1}^{r} (\lambda_k - \lambda_{k-1})\, f\left(A_{\lambda_k}^\rho\right), \tag{7}$$

where $im\,(\rho) = \{\lambda_1 < \cdots < \lambda_r\}$ and $\lambda_0 = 0$. The following properties of the Choquet integral are known:

(C1) For all non-empty $B \subseteq A$, $\int \mathbf{1}^B\, df = f\,(B)$ with $\mathbf{1}^B(a) = 1$ if $a \in B$ and $\mathbf{1}^B(a) = 0$ otherwise.

(C2) If $f(B) = K$ for all non-empty $B \subseteq A$, then $\int \rho\, df = K \bigvee_{a \in A} \rho(a)$.

(C3) $\int t\rho\, df = t \int \rho\, df, \quad$ for $t \in [0,1]\,.$

(C4) $\int \rho \, d\,(a_1 f_1 + a_2 f_2) = a_1 \int \rho \, df_1 + a_2 \int \rho \, df_2$.

(C5) $\int (\rho + \rho')\; df = \int \rho \, df + \int \rho' \, df$, when $\rho(a) + \rho'(a) \leq 1$ and ρ, ρ' are comonotone.

III. Coalition cohesion indices

Let N be a set of agents with $|N| = n \in \mathbb{N}$. Suppose that a social analysis of the cohesiveness for each coalition, taking into consideration all the relevant facets of the situation, has been done and we describe the results of the study by a cohesion index where the most cohesive coalition has level 1. The index of a singleton coalition means the capacity of the agent to keep his positions without the support of the others.

Definition 1. *A coalition cohesion index over* N *is a function* $\rho : 2^N \to [0, 1]$ *such that* $\rho(\emptyset) = 0$ *and there exists* $S \subseteq N$ *with* $\rho(S) = 1$.

This definition corresponds to that of cooperation index, introduced by Amer *et al.* [12], but the sense is totally different. In a cooperation index ρ the number $\rho(S)$ is interpreted as the probability that coalition S is formed, and then the Owen-type value proposed in that case is just an expected payoff; it is not an allocation of the total profit from cooperation.

A coalition cohesion index ρ over N is also a fuzzy set of 2^N (or a fuzzy hypergraph for N, see [13]) with special conditions, thus we can use all the notations and definitions from that context. The coalitions in $supp(\rho)$ are the cohesive coalitions, which are called *groups*, and $im(\rho)$ is the set of the different levels of cohesiveness. Every non-empty family of non-empty coalitions $\mathcal{C} \subset 2^N$ can be identified with the coalition cohesion index $\rho^{\mathcal{C}}$ over N with $\rho^{\mathcal{C}}(S) = 1$ if $S \in \mathcal{C}$ and $\rho^{\mathcal{C}}(S) = 0$ otherwise. That is, the coalitions in the family are well cohesive and the others are not cohesive. In particular, we can identify any coalition configuration $\mathcal{C}$ on N with its corresponding coalition index $\rho^{\mathcal{C}}$. Moreover, if we have a coalition cohesion index ρ over N

with $im(\rho) = \{0, 1\}$ and $\underset{S \in supp(\rho)}{\cup} S = N$, then $supp(\rho)$ is a coalition configuration of N.

Given a coalition cohesion index ρ over N we may find players which do not belong to any cohesive coalition. In our context they are agents which represent a problem in any group, even for themselves. They can be seen as agents that spoil the cohesiveness of any group.

Definition 2. *Let ρ be a coalition cohesion index over N. A buster player in ρ is a player $i \in N$ satisfying $\rho(S) = 0$ for every $S \subseteq N$ with $i \in S$. The set of buster players in ρ is denoted as $B(\rho)$.*

Notice that any coalition configuration $\mathcal{C}$ has $B(\rho^{\mathcal{C}}) = \emptyset$. Besides, any non-empty family of non-empty coalitions $\mathcal{C} \subset 2^N$ is a coalition configuration of $N \setminus B(\rho^{\mathcal{C}})$. Hereinafter we will use the notation $B(\mathcal{C})$ instead of $B(\rho^{\mathcal{C}})$.

Let ρ be a coalition cohesion index over N. For every $t \in [0, 1]$ we denote the t-level set (6) of ρ as $\mathcal{C}_t^{\rho} = \{S \subseteq N : \rho(S) \geq t\}$.

Definition 3. *Let ρ be a coalition cohesion index over the set of agents N. If $a, b \in [0, 1]$ with $a \leq b \neq 0$ then the simplification of ρ to $[a, b]$ is the coalition cohesion index ρ_a^b given by*

$$\rho_a^b(S) = \begin{cases} 0 & \text{if } \rho(S) < a, \\ \dfrac{\rho(S) - a}{b - a} & \text{if } \rho(S) \in [a, b), \\ 1 & \text{if } \rho(S) \geq b. \end{cases}$$

Besides, ρ_0^0 is the coalition cohesion index given by $\rho_0^0(S) = 1$ for all $S \neq \emptyset$ and $\rho_0^0(\emptyset) = 0$.

Remarks. Let ρ be a coalition cohesion index over N and $0 \leq a \leq b \leq 1$. (1) $\rho_0^1 = \rho$. (2) If $\mathcal{C} \subseteq 2^N$ is a non-empty family of non-empty coalitions then $(\rho^{\mathcal{C}})_a^b = \rho^{\mathcal{C}}$ for all $a \leq b$ and $b > 0$. (3) It holds that $supp(\rho_a^a) = \mathcal{C}_a^{\rho}$ when $a > 0$. (4) If $a < b$, ρ_a^b it holds that $supp(\rho_a^b) = \{S \in supp(\rho) \ : \ \rho(S) > a\}$.

The following result proves some facts related to coalition cohesion indices and their simplifications to intervals.

Lemma 1. *Let ρ be a coalition cohesion index over N.*

1) *Let $a_1, a_2, a_3 \in [0,1]$ with $a_1 \leq a_2 \leq a_3$. Then, $\rho_{a_1}^{a_2}$ and $\rho_{a_2}^{a_3}$ are comonotone coalition cohesion indices. It also happens that $(a_2 - a_1)\rho_{a_1}^{a_2}$ and $(a_3 - a_2)\rho_{a_2}^{a_3}$ are comonotone fuzzy sets.*
2) *Let $0 < a_1 \leq \cdots \leq a_d < 1$. Then,*

$$\rho = \sum_{k=1}^{d+1} (a_k - a_{k-1}) \rho_{a_{k-1}}^{a_k}$$

where $a_0 = 0$ and $a_{d+1} = 1$.

3) *Let $0 < a_1 \leq \cdots \leq a_d < 1$ and let f be a set function over the families of coalitions in N. Then,*

$$\int \rho \, df = \sum_{l=1}^{d+1} (a_{l-1} - a_l) \int \rho_{a_{l-1}}^{a_l} \, df,$$

where $a_0 = 0$ and $a_{d+1} = 1$.

Proof. Let ρ be a coalition cohesion index.

1) Let $a_1, a_2, a_3 \in [0,1]$ with $a_1 \leq a_2 \leq a_3$. Let $S, T \subseteq N$ be two different non-empty coalitions. If $\rho(S), \rho(T) \geq a_2$ then $\rho_{a_1}^{a_2}(S) - \rho_{a_1}^{a_2}(T) = 0$, and if $\rho(S), \rho(T) \leq a_2$ then $\rho_{a_2}^{a_3}(S) - \rho_{a_2}^{a_3}(T) = 0$. Suppose then $\rho(T) < a_2$ and $\rho(S) > a_2$. We obtain

$$\rho_{a_1}^{a_2}(S) - \rho_{a_1}^{a_2}(T) \geq 1 \wedge \left(1 - \frac{\rho(T) - a_1}{a_2 - a_1}\right) > 0,$$
$$\rho_{a_2}^{a_3}(S) - \rho_{a_2}^{a_3}(T) \geq 1 \wedge \frac{\rho(S) - a_2}{a_3 - a_2} > 0.$$

Therefore $\rho_{a_1}^{a_2}$ and $\rho_{a_2}^{a_3}$ are comonotone coalition cohesion indices.

It is clear that $0 \leq (a_2 - a_1)\rho_{a_1}^{a_2}(S) \leq 1$, $0 \leq (a_3 - a_2)\rho_{a_2}^{a_3}(S) \leq 1$, and

$$(a_2 - a_1)(\rho_{a_1}^{a_2}(S) - \rho_{a_1}^{a_2}(T))(a_3 - a_2)(\rho_{a_2}^{a_3}(S) - \rho_{a_2}^{a_3}(T)) \geq 0$$

for every $S, T \subseteq N$. Thus, $(a_2 - a_1)\rho_{a_1}^{a_2}$ and $(a_3 - a_2)\rho_{a_2}^{a_3}$ are comonote fuzzy sets.

2) Let $S \subseteq N$. Let $k \in \{1, \ldots, d+1\}$ such that $a_{k-1} \leq \rho(S) \leq a_k$. Then,

- $\rho_{a_{l-1}}^{a_l}(S) = 1$, for every $1 \leq l \leq k-1$,
- $\rho_{a_{k-1}}^{a_k}(S) = \frac{\rho(S) - a_{k-1}}{a_k - a_{k-1}}$, and
- $\rho_{a_{l-1}}^{a_l}(S) = 0$, for every $k+1 \leq l \leq d+1$.

Thus,

$$\sum_{l=1}^{d+1}(a_l - a_{l-1})\rho_{a_{l-1}}^{a_l}(S) = \sum_{l=1}^{k-1}(a_l - a_{l-1}) + (a_k - a_{k-1})\frac{\rho(S) - a_{k-1}}{a_k - a_{k-1}} = \rho(S).$$

3) Let $a_1 \leq \cdots \leq a_d \in (0,1)$ and let f be a set function over the families of coalitions. Then,

$$\int \rho \, df = \int (\sum_{k=1}^{d+1}(a_{k-1} - a_k)\rho_{a_{k-1}}^{a_k})df = \sum_{k=1}^{d+1}(a_{k-1} - a_k)\int \rho_{a_{k-1}}^{a_k} df$$

where the last equality follows from (C5), (C3) and the items above. □

IV. THE CHOQUET-OWEN VALUE

We will define a value that extends the Owen value 4 defined in [3] for games with coalition configuration.

Definition 4. *A game with coalition cohesion index is a triple (N, ρ, v) where (N, v) is a game and ρ is a coalition cohesion index over N. A cohesion value is a mapping that assigns a payoff vector to each game with coalition cohesion index.*

Let (N, v) be a game and $\mathcal{C} \subset 2^N$ a non-empty family of non-empty coalitions. If we want to assign a payoff vector to $(N, \mathcal{C}, v)$ by using the Owen value for games with coalition configuration we may need to modify the game. We consider that the participation of the buster players is assured without any demand from them. We can take the game $(N \setminus B(\mathcal{C}), v[\mathcal{C}])$ given by

$$v[\mathcal{C}](S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ v(S \cup B(\mathcal{C})) & \text{otherwise,} \end{cases}$$

for every $S \subseteq N \setminus B(\mathcal{C})$. So we identify $(N, \mathcal{C}, v)$ with the game with coalition configuration $(\mathcal{C}, v[\mathcal{C}])$. Particularly, if $\mathcal{C}$ is a coalition configuration then $B(\mathcal{C}) = \emptyset$ and $v[\mathcal{C}] = v$.

We define the following set functions over the families of coalitions. Let (N, v) be a game and $i \in N$. The *Owen set function* related to (N, v) and player $i \in N$ is given by

$$\phi_i(N, v)(\mathcal{C}) = \begin{cases} \phi_i(\mathcal{C}, v[\mathcal{C}]) & \text{if } i \notin B(\mathcal{C}), \\ 0 & \text{if } i \in B(\mathcal{C}), \end{cases} \tag{8}$$

for every $\mathcal{C} \subseteq 2^N$. We use this family of set functions to extend the Owen value to the family of games with coalition cohesion index.

Definition 5. *The Choquet-Owen value for games with coalition cohesion index is the cohesion value defined for all* (N, ρ, v) *and* $i \in N$ *as*

$$\Phi_i(N, \rho, v) = \int \rho \, d\phi_i(N, v).$$

Remark. The Choquet-Owen value can be expressed using (7) as

$$\Phi_i(N, \rho, v) = \sum_{k=1}^{r} (\lambda_k - \lambda_{k-1}) \phi_i(N, v) \left(\mathcal{C}^{\rho}_{\lambda_k}\right)$$

for each player i in N, where $im(\rho) = \{\lambda_1 < \cdots < \lambda_r\}$ and $\lambda_0 = 0$. Hence, in order to calculate the payoff of a player we need to determine a sequence of payoffs in coalition configurations defined by intervals of cohesiveness and then to weight those payoffs according to the measure of the intervals. Therefore, if $\mathcal{C}$ is a coalition configuration then $\Phi(N, \rho^{\mathcal{C}}, v) = \phi(\mathcal{C}, v)$.

It is possible to follow a heuristic process as in [3] to get the Choquet-Owen value in two steps (see section 2.2) as we show next. Let (N, v) be a game and $\mathcal{C} = \{C_p\}_{p \in M}$, where $M = \{1, ..., m\}$, a non-empty family of non-empty coalitions of N. We extend the quotient game (1) to our context. The *quotient game* of $\mathcal{C}$ in N is $(M, v'_{\mathcal{C}})$ given by $v'_{\mathcal{C}} = v[\mathcal{C}]_{\mathcal{C}}$. In this game each coalition of $\mathcal{C}$ acts as a player and the earnings of a coalition of groups is given by the earnings of the union of all the groups in the coalition jointly with the set of buster players. Notice that the set of buster players is not a player in the quotient game. Moreover, if $p \in M$ and $S \subseteq C_p$ then $v'_{\mathcal{C}_p(S)} = v[\mathcal{C}]_{\mathcal{C}_p(S)}$, following Expression (2).

Let (N, v) be a game and ρ a coalition cohesion index. We take a quotient game for each level $t \in im(\rho)$. Let $im(\rho) = \{\lambda_1 < \cdots < \lambda_r\}$, $\lambda_0 = 0$, $supp(\rho) = \{C_1, ..., C_m\}$ and $M = \{1, ..., m\}$. For each $p \in M$ and $S \subseteq C_p$, we define the *group game* (C_p, w_p^ρ) as

$$w_p^\rho(S) = \sum_{k=1}^{r_p} (\lambda_k - \lambda_{k-1}) Sh_p(M^k, v'_{(\mathcal{C}^\rho_{\lambda_k})_p(S)}),$$

where $\rho(C_p) = \lambda_{r_p}$ and for every $k = 1, \ldots, r_p$ we denote $M^k = \{q \in M : \rho(C_q) \geq \lambda_k\}$ and $(\mathcal{C}^\rho_{\lambda_k})_p(S) = \{C_q \ : q \in M^k\} \setminus \{C_p\} \cup \{S\}$. Now we can follow a procedure similar to that used for the Owen value in order to get the Choquet-Owen value, as the next result states.

Theorem 1. *Let (N, ρ, v) be a game with a coalition cohesion index. Let $i \in N \setminus B(\rho)$ and*

$supp(\rho) = \{C_p\}_{p\in M}$, $M = \{1, ..., m\}$. *The Choquet-Owen value of player i satisfies the equality*

$$\Phi_i(N, \rho, v) = \sum_{\{p\in M:\, i\in C_p\}} Sh_i(C_p, w_p^\rho).$$

Proof. We consider a coalition cohesion index ρ with $im(\rho) = \{\lambda_1 < \cdots < \lambda_r\}$, $\lambda_0 = 0$, $supp(\rho) = \{C_p\}_{p\in M}$, and $M = \{1, ..., m\}$. For each $k = 1, ..., r$ we denote by $M^k = \{p \in M \ :\ \rho(C_p) \geq \lambda_k\}$. Given a player $i \notin B(\rho)$ and $p \in M$ with $i \in C_p$ we will take r_p such that $\rho(C_p) = \lambda_{r_p}$ and

$$r_i = \bigvee_{\{p\in M: i\in C_p\}} r_p.$$

By Definition 5, (7), (4), and (S2) we have

$$\begin{aligned}
\Phi_i(N, \rho, v) = \int \rho\, d\phi_i(N, v) &= \sum_{k=1}^{r_i} (\lambda_k - \lambda_{k-1}) \phi_i(\mathcal{C}^\rho_{\lambda_k}, v[\mathcal{C}^\rho_{\lambda_k}]) \\
&= \sum_{k=1}^{r_i} (\lambda_k - \lambda_{k-1}) \sum_{\{p\in M^k: i\in C_p\}} Sh_i\left(C_p, v[\mathcal{C}^\rho_{\lambda_k}]_p^{\mathcal{C}^\rho_{\lambda_k}}\right) \\
&= \sum_{\{p\in M: i\in C_p\}} Sh_i\left(C_p, \sum_{k=1}^{r_p} (\lambda_k - \lambda_{k-1}) v[\mathcal{C}^\rho_{\lambda_k}]_p^{\mathcal{C}^\rho_{\lambda_k}}\right).
\end{aligned}$$

Finally, using Expression (3) and the fact that $v[\mathcal{C}^\rho_{\lambda_k}]_{(\mathcal{C}^\rho_{\lambda_k})(S)} = v'_{(\mathcal{C}^\rho_{\lambda_k})(S)}$, it suffices to notice that for every $S \subseteq C_p$ it holds that

$$\begin{aligned}
\sum_{k=1}^{r_p} (\lambda_k - \lambda_{k-1}) v[\mathcal{C}^\rho_{\lambda_k}]_p^{\mathcal{C}^\rho_{\lambda_k}}(S) &= \sum_{k=1}^{r_p} (\lambda_k - \lambda_{k-1}) Sh_p(M^k, v[\mathcal{C}^\rho_{\lambda_k}]_{(\mathcal{C}^\rho_{\lambda_k})_p(S)}) \\
&= \sum_{k=1}^{r_p} (\lambda_k - \lambda_{k-1}) Sh_p(M^k, v'_{(\mathcal{C}^\rho_{\lambda_k})_p(S)}) = w_p^\rho(S). \qquad \square
\end{aligned}$$

V. AN AXIOMATIZATION OF THE CHOQUET-OWEN VALUE

Now, in order to characterize the Choquet-Owen value, we will introduce some axioms similar to those considered for the Owen value. Let Ψ be a value for games with coalition cohesion index.

Efficiency. For all (N, ρ, v) game with coalition cohesion index it holds that

$$\sum_{i \in N} \Psi_i(N, \rho, v) = v(N).$$

We consider that buster players should have null payoff.

Buster player property. If $i \in N$ is a buster player for the coalition cohesion index ρ then $\Psi_i(N, \rho, v) = 0$ for any game (N, v).

Linearity. For all games $(N, \rho, v_1), (N, \rho, v_2)$ and numbers $a_1, a_2 \in \mathbb{R}$ it holds that

$$\Psi(N, \rho, a_1 v_1 + a_2 v_2) = a_1 \Psi(N, \rho, v_1) + a_2 \Psi(N, \rho, v_2).$$

The maximal cohesion degree of a player in a coalition cohesion index ρ over N is defined as

$$\rho_i = \bigvee_{\{S \subseteq N : i \in S\}} \rho(S).$$

A game with a coalition cohesion index (N, ρ, v) is *a-null* with $a \in (0, 1]$ if $v(S) = 0$, for every $S \subseteq N$ with $\bigvee_{i \in S} \rho_i < a$. In particular, $v(S) = 0$ for every $S \subseteq B(\rho)$. Given a null player in (N, v), his payoff in an $a-$null game (N, v, ρ) should be proportional to his payoff in the simplification to the interval $[a, 1]$.

Fuzzy null player property. If $i \in N$ is a null player in (N, v) and (N, v, ρ) is an a-null game with coalition cohesion index, then

$$\Psi_i(N, \rho, v) = (1 - a) \Psi_i(N, \rho_a^1, v).$$

If two symmetric players are also symmetric for the index from certain cohesiveness level, then the difference between the payoffs obtained in a game with a coalition cohesion index should be proportional to the difference of the payoffs in the simplification to the interval $[0, a]$.

Fuzzy equal treatment property. Let i, j be two symmetric players for the game (N, v) and ρ a coalition cohesion index. If there exists $a \in [0, 1)$ such that for every coalition S with $\rho(S) > a$ it holds that $i \in S$ if and only if $j \in S$, then

$$\Psi_i(N, \rho, v) - \Psi_j(N, \rho, v) = a \left[\Psi_i(N, \rho_0^a, v) - \Psi_j(N, \rho_0^a, v)\right].$$

The concept of symmetry for players can be extended to groups in the context of coalition cohesion indices. Let (N, ρ, v) be a game with a coalition cohesion index. Two coalitions $S, T \in supp(\rho)$ are *separable symmetric groups* if:

- $v(S \cup R) = v(T \cup R)$ for all $R \subseteq N \setminus (S \cup T)$ and,
- $S \cap R = T \cap R = S \cap T = \emptyset$ for all $R \in supp(\rho) \setminus \{S, T\}$.

Separable symmetric groups property. Let $S, T \subseteq N$ be separable symmetric groups for (N, ρ, v) with $0 < \rho(T) \leq \rho(S)$. It holds

$$\sum_{i \in S} [\Psi_i(N, \rho, v) - [\rho(S) - \rho(T)]\Psi_i(N, \rho_{\rho(T)}^{\rho(S)}, v)] = \sum_{j \in T} \Psi_j(N, \rho, v).$$

Let $a \in [0, 1)$. Two players $i, j \in N$ are *a-double for* (N, ρ, v) if:

- there are coalitions $S, T \subseteq N$ (they can be equal) with $\rho(S), \rho(T) > a$ and $i \in S, j \in T$,
- $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalition $S \subseteq N$ and
- if $S \subseteq N \setminus \{i, j\}$ with $\rho(S \cup \{i\}) > a$ then $\rho(S \cup \{j\}), \rho(S \cup \{i, j\}) \leq a$.

If i, j are a-double then ρ_{-j}^i is another coalition cohesion index over $N \setminus \{j\}$ defined for all

$S \subseteq N \setminus \{j\}$ as

$$\rho^i_{-j}(S) = \begin{cases} \rho(S) \vee \rho((S \setminus \{i\}) \cup \{j\}) & \text{if } i \in S, \\ \rho(S) & \text{otherwise.} \end{cases}$$

We also consider the game $(N \setminus \{j\}, v^i_{-j})$ defined in Expression 5.

Merger of players. Let i, j be a-double for (N, ρ, v). It holds that

$$\Psi_k(N, \rho, v) = a\Psi_k(N, \rho^a_0, v) + (1 - a)\Psi_k(N \setminus \{j\}, (\rho^1_a)^i_{-j}, v^i_{-j})$$

for every $k \in N \setminus \{i, j\}$.

Observe what happens with these axioms when we take $\mathcal{C}$ as a coalition configuration. Efficiency, linearity, fuzzy null player property and merger of players (taking into account that $im(\rho) = \{0, 1\}$ in that case) coincide with the respective axioms in [3]. Moreover, the fuzzy equal treatment property and the separable symmetric groups property coincides with the equal treatment property and the group symmetry property of the Owen value defined in [2], respectively.

Theorem 2. *The Choquet-Owen value satisfies efficiency, linearity, fuzzy null player property, buster player property, fuzzy equal treatment property, separable symmetric groups and merger of players.*

Proof. We consider each of the axioms mentioned in the theorem.

BUSTER PLAYER PROPERTY. Let (N, ρ, v) be a game with coalition cohesion index. If $i \in B(\rho)$ then $i \in B(\mathcal{C}^\rho_t)$ for all $t \in im(\rho)$. Hence, (8) implies that $\phi_i(N, v)(\mathcal{C}^\rho_t) = 0$ for all $t \in im(\rho)$ and, by definition of the Choquet integral (7), $\Phi_i(N, \rho, v) = 0$.

EFFICIENCY. We know that the Owen value ϕ is efficient (O1) for coalition configurations.

For each family $\mathcal{C} \subseteq 2^N$ we get

$$\sum_{i\in N} \phi_i(N,v)(\mathcal{C}) = \sum_{i\in N\setminus B(\mathcal{C})} \phi_i\left(\mathcal{C}, v[\mathcal{C}]\right) = v[\mathcal{C}](N \setminus B(\mathcal{C})) = v(N).$$

Let (N, ρ, v) be a game with coalition cohesion index. Using (C2) and (C4) we obtain

$$\sum_{i\in N} \Phi_i(N,\rho,v) = \sum_{i\in N} \int \rho \, d\phi_i(N,v) = \int \rho \, d \sum_{i\in N} \phi_i(N,v) = v(N).$$

LINEARITY. The linearity of Φ follows from the linearity of the Owen value (O2) over coalition configurations. Let $(N, v_1), (N, v_2)$ be two games and $a_1, a_2 \in \mathbb{R}$. For each family of coalitions $\mathcal{C}$ we get the equality $(a_1v_1 + a_2v_2)[\mathcal{C}] = a_1v_1[\mathcal{C}] + a_2v_2[\mathcal{C}]$ and, then, for every player $i \in N \setminus B(\mathcal{C})$

$$\begin{aligned}\phi_i(N, a_1v_1 + a_2v_2)(\mathcal{C}) &= \phi_i(\mathcal{C}, (a_1v_1 + a_2v_2)[\mathcal{C}]) = a_1\phi_i(\mathcal{C}, v_1[\mathcal{C}]) + a_2\phi_i(\mathcal{C}, v_2[\mathcal{C}]) \\ &= a_1\phi_i(N, v_1)(\mathcal{C}) + a_2\phi_i(N, v_2)(\mathcal{C}).\end{aligned}$$

If $i \in B(\mathcal{C})$ then $\phi_i(N, a_1v_1 + a_2v_2)(\mathcal{C}) = 0 = a_1\phi_i(N, v_1)(\mathcal{C}) + a_2\phi_i(N, v_2)(\mathcal{C})$. So, using again (C4), for a coalition cohesion index ρ over N.

$$\Phi_i(N, \rho, a_1v_1 + a_2v_2) = a_1 \int \rho \, d\phi_i(N, v_1) + a_2 \int \rho \, d\phi_i(N, v_2) = a_1\Phi_i(N, \rho, v_1) + a_2\Phi_i(N, \rho, v_2).$$

FUZZY NULL PLAYER PROPERTY. Let (N, ρ, v) be an a-null game with coalition cohesion index and $i \in N$ a null player for (N, v). If $i \in B(\rho) = B(\rho_0^a) \subseteq B(\rho_a^1)$ we have the equalities $\Phi_i(N, \rho, v) = \Phi_i(N, \rho_0^a, v) = 0 = \Phi_i(N, \rho_a^1, v)$ because Φ satisfies the buster player property. Suppose $i \notin B(\rho)$. We will calculate $\Phi_i(N, \rho_0^a, v)$. Firstly, $B(\rho_0^a) = B(\rho)$ and (N, ρ_0^a, v) is also a-null. Let $t \in im(\rho_0^a)$ with $t < a$. Then, $B(\rho_0^a) \subseteq B(\rho_t^a)$, $\rho_0^a(B(\rho_t^a)) < t$ and $v(B(\rho_t^a)) = 0$.

Secondly, as i is null player,

$$v[\mathcal{C}_t^{\rho_0^a}](S\cup\{i\}) = v(S\cup B(\mathcal{C}_t^{\rho_0^a})\cup\{i\}) = v(S\cup B(\mathcal{C}_t^{\rho_0^a})) = v[\mathcal{C}_t^{\rho_0^a}](S)$$

for every $S\subseteq N\setminus B(\mathcal{C}_t^{\rho_0^a})$. Then, i is also a null player in $(\mathcal{C}_t^{\rho_0^a}, v[\mathcal{C}_t^{\rho_0^a}])$ for every $t\in im(\rho_0^a)$ with $t<a$. Since the Owen value satisfies the null player property we obtain that $\phi_i(N,v)(\mathcal{C}_t^{\rho_0^a})=0$. Using (7) we conclude $\Phi_i(N,\rho_0^a,v)=0$. Finally, since $\rho = a\rho_0^a + (1-a)\rho_a^1$ and using Lemma 1 item 3 we get $\Phi_i(N,\rho,v) = (1-a)\Phi_i(N,\rho_a^1,v)$.

FUZZY EQUAL TREATMENT PROPERTY. Let (N,ρ,v) be a game with a coalition index. Let i,j be symmetric players for the game (N,v) and $a\in[0,1)$ such that if $\rho(S)>a$, then $i\in S$ iff $j\in S$. If $i,j\in B(\rho)$, the result is clearly true because $i,j\in B(\rho_0^a)$. Let $i,j\notin B(\rho)$. Using that $\rho = a\rho_0^a + (1-a)\rho_a^1$ and using Lemma 1 item 3, we have

$$\Phi_i(N,\rho,v) = a\Phi_i(N,\rho_0^a,v) + (1-a)\Phi_i(N,\rho_a^1,v),$$

$$\Phi_j(N,\rho,v) = a\Phi_j(N,\rho_0^a,v) + (1-a)\Phi_j(N,\rho_a^1,v).$$

Consider any $t\in im(\rho_a^1)$. We have $\mathcal{C}_t^{\rho_a^1} = \mathcal{C}_{t(1-a)+a}^{\rho}$, because $\rho_a^1(S)\geq t$ iff $\dfrac{\rho(S)-a}{1-a}\geq t$. Since i and j are symmetric players in $(\mathcal{C}_t^{\rho_a^1}, v[\mathcal{C}_t^{\rho_a^1}])$, if $\rho_a^1(S)\geq t$ we have $i\in S$ iff $j\in S$, and ϕ satisfies equal treatment of symmetric players, then $\phi_i(N,v)(\mathcal{C}_t^{\rho_a^1}) = \phi_j(N,v)(\mathcal{C}_t^{\rho_a^1})$. Hence, by (7), $\Phi_i(N,\rho_a^1,v) = \Phi_j(N,\rho_a^1,v)$ and

$$\Phi_i(N,\rho,v) - \Phi_j(N\rho,v) = a\left[\Phi_i(N,\rho_0^a,v) - \Phi_j(N,\rho_0^a,v)\right].$$

SEPARABLE SYMMETRIC GROUPS. Let $C_p, C_q\in\mathcal{C}$ with $\mathcal{C} = \{C_{p'}\}_{p'=1}^m \subseteq 2^N$ a non-empty family of non-empty coalitions and $M=\{1,...,m\}$. Suppose C_p, C_q separable symmetric coalitions for $(N,\mathcal{C},v)$, that is C_p, C_q are disjoint and disjoint with the rest of the groups.

We prove that p, q are symmetric players for $v[\mathcal{C}]_{\mathcal{C}}$. For each $Q \subseteq M \setminus \{p,q\}$ we have

$$v[\mathcal{C}]_{\mathcal{C}}(Q \cup \{p\}) = v\left(\bigcup_{p' \in Q} C_{p'} \cup C_p \cup B(\mathcal{C})\right) = v\left(\bigcup_{p' \in Q} C_{p'} \cup C_q \cup B(\mathcal{C})\right) = v[\mathcal{C}]_{\mathcal{C}}(Q \cup \{q\}).$$

From the facts that the Shapley value is efficient (S1) and satisfies the equal treatment property (S4), and using Theorem 2, we obtain

$$\begin{aligned}\sum_{i \in C_p} \Phi_i(N, \mathcal{C}, v) &= \sum_{i \in C_p} Sh_i(C_p, v_p^{\mathcal{C}}) = v_p^{\mathcal{C}}(C_p) = Sh_p(M, v'_{\mathcal{C}}) \\ &= Sh_q(M, v'_{\mathcal{C}}) = \sum_{j \in C_q} \Phi_i(N, \mathcal{C}, v).\end{aligned}$$

Now, we take S, T separable symmetric groups for (N, ρ, v) with $0 < \rho(T) \leq \rho(S)$. Using Lemma 1 item 3, we get for each $i \in S$,

$$\Phi_i(N, \rho, v) = \rho(T)\Phi_i(N, \rho_0^{\rho(T)}, v) + (\rho(S) - \rho(T))\Phi_i(N, \rho_{\rho(T)}^{\rho(S)}, v) + (1 - \rho(S))\Phi_i(N, \rho_{\rho(S)}^1, v).$$

Since i is a buster player for $\rho_{\rho(S)}^1$ and the Choquet-Owen value satisfies the buster player property, $\Phi_i(N, \rho_{\rho(S)}^1, v) = 0$. If $j \in T$ by the same reasoning we obtain

$$\Phi_j(N, \rho, v) = \rho(T)\Phi_j(N, \rho_0^{\rho(T)}, v) + (1 - \rho(T))\Phi_j(N, \rho_{\rho(T)}^1, v) = \rho(T)\Phi_j(N, \rho_0^{\rho(T)}, v),$$

because j is a buster player for $\rho_{\rho(T)}^1$. As S, T are separable symmetric groups for $\mathcal{C}_t^{\rho_0^{\rho(T)}}$, for all $t \in im(\rho_0^{\rho(T)})$, it holds

$$\sum_{i \in S} \Phi_i(N, \rho_0^{\rho(T)}, v) = \sum_{j \in T} \Phi_j(N, \rho_0^{\rho(T)}, v).$$

MERGER OF PLAYERS. Suppose i, j are double players (0-double) for $(N, \mathcal{C}, v)$ with $\mathcal{C} = \{C_p\}_{p=1}^m \subseteq 2^N$ a non-empty family of non-empty coalitions and $M = \{1, ..., m\}$. We denote

$Q_i = \{p \in M : i \in C_p\}$. Consider the new family of $2^{N\setminus\{j\}}$

$$\mathcal{C}^i_{-j} = (\rho^{\mathcal{C}})^i_{-j} = \{C^*_p\}^m_{p=1} = \{C_p : p \in M \setminus Q_j\} \cup \{(C_p \setminus \{j\}) \cup \{i\} : p \in Q_j\}.$$

Observe that $|\mathcal{C}^i_{-j}| = |\mathcal{C}|$, because i, j are double players, and then M also represent the set of subindices of the groups in $\mathcal{C}^i_{-j}$. Moreover, $B(\mathcal{C}^i_{-j}) = B(\mathcal{C})$. Let $k \in N \setminus \{i, j\}$. If $k \in B(\mathcal{C})$ then $\phi_k(N, v)(\mathcal{C}) = 0 = \phi_k(N \setminus \{j\}, v^i_{-j})(\mathcal{C}^i_{-j})$. It is easy to check that $v[\mathcal{C}]^i_{-j} = v^i_{-j}[\mathcal{C}^i_{-j}]$. If $k \notin B(\mathcal{C})$, we get

$$\phi_k(N, v)(\mathcal{C}) = \phi_k(\mathcal{C}^i_{-j}, v[\mathcal{C}]^i_{-j}) = \phi_k(\mathcal{C}^i_{-j}, v^i_{-j}[\mathcal{C}^i_{-j}]) = \phi_k(N \setminus \{j\}, v^i_{-j})(\mathcal{C}^i_{-j})$$

because the Owen value satisfies merger (O6). Applying (6), we obtain $\Phi_k(N, \mathcal{C}, v) = \Phi_k(N \setminus \{j\}, \mathcal{C}^i_{-j}, v^i_{-j})$, for every $k \in N \setminus \{i, j\}$.

Now consider ρ a coalition cohesion index and i, j a pair of $a-$double players for some $a \in (0, 1)$. By Lemma 1 item 3, we obtain for (N, ρ, v) and $k \in N \setminus \{i, j\}$

$$\Phi_k(N, \rho, v) = a\Phi_k(N, \rho^a_0, v) + (1 - a)\Phi_k(N, \rho^1_a, v).$$

Let $t \in im(\rho^1_a)$. We will see that i, j are a-double players for $(N, \mathcal{C}^{\rho^1_a}_t, v)$. For each $S \subseteq N \setminus \{i, j\}$ with $S \cup \{i\} \in \mathcal{C}^{\rho^1_a}_t$ then $\rho^1_a(S \cup \{i\}) \geq t$ iff $\rho(S \cup \{i\}) > a$. Hence $\rho(S \cup \{j\}), \rho(S \cup \{i, j\}) \leq a$ implies $\rho^1_a(S \cup \{j\}) = \rho^1_a(S \cup \{i, j\}) = 0 < t$ and $S \cup \{j\}, S \cup \{i, j\} \notin \mathcal{C}^{\rho^1_a}_t$. We have the equality $\phi_k(N, v)(\mathcal{C}^{\rho^1_a}_t) = \phi_k(N \setminus \{j\}, v^i_{-j})((\mathcal{C}^{\rho^1_a}_t)^i_{-j})$ for every $k \in N \setminus \{i, j\}$, because i, j are double players in the game $(\mathcal{C}^{\rho^1_a}_t, v[\mathcal{C}^{\rho^1_a}_t)])$ and (O6). But $(\mathcal{C}^{\rho^1_a}_t)^i_{-j} = \mathcal{C}^{(\rho^1_a)^i_{-j}}_t$ as we show next. Let $C \in (\mathcal{C}^{\rho^1_a}_t)^i_{-j}$ with $i \in C$ (otherwise the result is trivial). By the construction of $(\mathcal{C}^{\rho^1_a}_t)^i_{-j}$, we have $\rho^1_a(C) \geq t$ or $\rho^1_a(C \setminus \{i\} \cup \{j\}) \geq t$. Equivalently, $(\rho^1_a)_{-j,i}(C) \geq t$ iff $C \in \mathcal{C}^{(\rho^1_a)^i_{-j}}_t$. Then, $\Phi_k(N, \rho^1_a, v) = \Phi_k(N \setminus \{j\}, (\rho^1_a)^i_{-j}, v^i_{-j})$ for every $k \in N \setminus \{i, j\}$ by (7). $\square$

In the following theorem we prove that the Choquet-Owen value is the only one satisfying the seven axioms above.

Theorem 3. *The Choquet-Owen value is the only cohesion value satisfying efficiency, linearity, fuzzy null player property, buster player property, fuzzy equal treatment property, separable symmetric groups property and merger of players.*

Proof. Let Ψ be a cohesion value satisfying the axioms in the theorem. Using linearity, it is enough to prove the result for unanimity games. Let ρ be a coalition cohesion index over N with $supp(\rho) = \{C_p\}_{p=1}^{m}$ and $M = \{1, ..., m\}$. It is clear that $\Psi_i(N, \rho, u_T) = \Phi_i(N, \rho, u_T)$, for every $i \in B(\rho)$. It remains to prove the result for non-buster players. We define

$$L(\rho) = \bigcup_{\{p,q \in M : p \neq q\}} C_p \cap C_q$$

and $l(\rho) = |L(\rho)|$. We prove uniqueness by induction on $l(\rho)$.

BASE CASE. Let $l(\rho) = 0$. Then, $C_p \cap C_q = \emptyset$, for every $p, q \in M$. Let us consider $T \subseteq N$ and the unanimity game (N, u_T). We prove the result for non-buster players by a second induction on the size of $|im(\rho)|$. Let $M_T = \{p \in M : C_p \cap T \neq \emptyset\}$. Notice that if $M_T = \emptyset$, then $T \subseteq B(\rho)$; otherwise, $T \setminus B(\rho) \neq \emptyset$.

BASE SUBCASE. If $|im(\rho)| = 1$ then $\mathcal{C}$ is a non-empty family of non-empty coalitions and $\rho(C_p) = 1$, for every $p \in M$. We consider two cases.

- $M_T = \emptyset$. Then, (N, u_T) is not a-null game for any $a \in (0, 1]$ because $u_T(T) = 1$ and $\bigvee_{i \in T} \rho_i = 0 < a$. Every pair of players $i, j \notin B(\rho)$ are symmetric in the game (N, u_T). Let $p \in M$. Then, $\rho(C_p) = 1 > a = 0$ and applying the fuzzy equal treatment property, we get

$$\Psi_i(N, \rho, u_T) = \Psi_j(N, \rho, u_T)$$

for every $i, j \in C_p$. In addition, every pair of coalitions $C_p,\ C_q \in supp(\rho)$ are separable symmetric groups with $\rho(C_p) = \rho(C_q) = 1$. Then,

$$|C_p|\Psi_i(N, \rho, u_T) = \sum_{l \in C_p} \Psi_l(N, \rho, u_T) = \sum_{l \in C_q} \Psi_l(N, \rho, u_T) = |C_q|\Psi_j(N, \rho, u_T)$$

with $i \in C_p$ and $j \in C_q$. Finally, if we apply the efficiency property, we obtain

$$\Psi_i(N, \rho, u_T) = \frac{1}{m}\frac{1}{|C_p|} = \Phi_i(N, \rho, u_T)$$

for every $i \in C_p$ and $p \in M$.

- $M_T \neq \emptyset$. Let $p \in M_T$ and $i_0 \in T \cap C_p$. Then, $\rho_{i_0} = 1$ and for every coalition $S \subseteq N$ with $\bigvee_{i \in S} \rho_i < 1$, we have $i_0 \notin S$ and $u_T(S) = 0$. Hence, (N, u_T) is an $1-$null game. Let $i \notin T$. Then, $\Psi_i(N, \rho, u_T) = 0$ because i is a null player in (N, u_T), (N, u_T) is an $1-$null game and the value Ψ satisfies the fuzzy null player property. Then, $\rho(C_p) = 1$ and every pair of players $i, j \in T \cap C_p$ are symmetric players. If we apply the fuzzy equal treatment property to the value Ψ, we obtain

$$\Psi_i(N, \rho, u_T) = \Psi_j(N, \rho, u_T)$$

for every $i, j \in T \cap C_p$ and $p \in M_T$. If $M_T = \{p\}$, then

$$\Psi_i(N, \rho, u_T) = \frac{1}{|T \cap C_p|} = \Phi_i(N, \rho, u_T)$$

for every $i \in T \cap C_p$. In other case, let $p, q \in M_T$. Then, C_p and C_q are separable symmetric groups. Using the separable symmetric groups property, we get

$$|T \cap C_p|\Psi_i(N, \rho, u_T) = \sum_{l \in C_p} \Psi_l(N, \rho, u_T) = \sum_{l \in C_q} \Psi_l(N, \rho, u_T) = |T \cap C_q|\Psi_j(N, \rho, u_T).$$

Finally, if we apply the efficiency property, we obtain

$$\Psi_i(N,\rho,u_T) = \frac{1}{m}\frac{1}{|T \cap C_p|}$$

for every $i \in T \cap C_p$.

GENERAL SUBCASE. Suppose that the result is true when $|im(\rho)| < K$. Let ρ be a coalition cohesion index with $|im(\rho)| = K > 1$. We also consider two cases.

- $M_T = \emptyset$. Then, (N, u_T) is not a-null game for any $a \in (0,1]$ because $u_T(T) = 1$ and $\bigvee_{i\in T} \rho_i = 0 < a$. Every pair of players $i, j \notin B(\rho)$ are symmetric in the game (N, u_T). Let $p \in M$. Applying the fuzzy equal treatment property, for every $i, j \in C_p$ we get $\Psi_i(N,\rho,u_T) = \Psi_j(N,\rho,u_T)$ because $\rho(C_p) > 0$. In addition, every pair of coalitions $C_p,\ C_q \in supp(\rho)$ are separable symmetric groups. Then, if $\rho(C_p) \leq \rho(C_q)$ we obtain

$$\begin{aligned}
\textstyle\sum_{l\in C_q} \Psi_l(N,\rho,u_T) - \sum_{l\in C_p} \Psi_l(N,\rho,u_T) &= |C_q|\Psi_i(N,\rho,u_T) - |C_p|\Psi_j(N,\rho,u_T) \\
&= (\rho(C_q) - \rho(C_p)) \textstyle\sum_{l\in C_q} \Psi_l(N, \rho^{\rho(C_q)}_{\rho(C_p)}, u_T) \\
&= (\rho(C_q) - \rho(C_p)) \textstyle\sum_{l\in C_q} \Phi_l(N, \rho^{\rho(C_q)}_{\rho(C_p)}, u_T) \\
&= |C_q|\Phi_i(N,\rho,u_T) - |C_p|\Phi_j(N,\rho,u_T)
\end{aligned} \tag{9}$$

with $i \in C_q$ and $j \in C_p$, where the last equality follows by the induction hypothesis. Let $p_0 \in M$ be such that $\rho(C_q) \geq \rho(C_{p_0})$ for every $q \in M$. Finally, if we apply the efficiency property and (9) to p_0 and $q \in M$, we obtain

$$1 = m|C_{p_0}|(\Psi_j(N,\rho,u_T) - \Phi_j(N,\rho,u_T)) + 1$$

for every $j \in C_{p_0}$. Then, $\Psi_j(N,\rho,u_T) = \Phi_j(N,\rho,u_T)$ for every $j \in C_{p_0}$. Finally, using Equality (9), we obtain $\Psi_i(N,\rho,u_T) = \Phi_i(N,\rho,u_T)$, for every $i \in C_q$ and $q \in M$.

- $M_T \neq \emptyset$. Let $p \in M_T$ and $i_0 \in T \cap C_p$. Then, (N, ρ, u_T) is a ρ_{i_0}-null game because for every $S \subseteq N$ with $\bigvee_{i \in S} \rho_i < \rho_{i_0}$ we have $i_0 \notin S$ and $u_T(S) = 0$. Thus, using the fuzzy null player property, we have $\Psi_i(N, \rho, u_T) = (1 - \rho_{i_0})\Psi_i(N, \rho^1_{\rho_{i_0}}, u_T)$, for every $i \notin T$. Furthermore, applying the induction hypothesis to $(N, \rho^1_{\rho_{i_0}}, u_T)$, we obtain

$$\Psi_i(N, \rho, u_T) = (1 - \rho_{i_0})\Psi_i(N, \rho^1_{\rho_{i_0}}, u_T) = (1 - \rho_{i_0})\Phi_i(N, \rho^1_{\rho_{i_0}}, u_T) = \Phi_i(N, \rho, u_T),$$

for every $i \notin T$. If $M_T = \{p\}$, then taking into account that $i, j \in T \cap C_p$ are symmetric players and $\rho(C_p) > 0$, then $\Psi_i(N, \rho, u_T) = \Psi_j(N, \rho, u_T)$. Using the efficiency property, we obtain

$$\Psi_i(N, \rho, u_T) = \frac{1}{|T \cap C_p|} = \Phi_i(N, \rho, u_T).$$

Now consider the case $|M_T| > 1$. Besides, every pair of players $i, j \in T$ are symmetric players in (N, u_T). Then, $\Psi_i(N, \rho, u_T) = \Psi_j(N, \rho, u_T)$ for every $i, j \in T \cap C_p$, for every $p \in M_T$ because Ψ satisfies the fuzzy equal treatment property. Let $p, q \in M_T$ with $\rho(C_p) \leq \rho(C_q)$. Then, C_p and C_q are separable symmetric groups and, since Ψ satisfies the separable symmetric groups property, we have

$$\begin{aligned}
\textstyle\sum_{l \in C_q} \Psi_l(N, \rho, u_T) - \sum_{l \in C_p} \Psi_l(N, \rho, u_T) &= |C_q|\Psi_i(N, \rho, u_T) - |C_p|\Psi_j(N, \rho, u_T) \\
&= (\rho(C_q) - \rho(C_p)) \textstyle\sum_{l \in C_q} \Psi_l(N, \rho^{\rho(C_q)}_{\rho(C_p)}, u_T) \\
&= (\rho(C_q) - \rho(C_p)) \textstyle\sum_{l \in C_q} \Phi_l(N, \rho^{\rho(C_q)}_{\rho(C_p)}, u_T) \\
&= |C_q|\Phi_i(N, \rho, u_T) - |C_p|\Phi_j(N, \rho, u_T)
\end{aligned} \tag{10}$$

with $i \in C_q$ and $j \in C_p$, where the second-to-last equality follows by the induction hypothesis. Let $p_0 \in M$ be such that $\rho(C_q) \geq \rho(C_{p_0})$ for every $q \in M$. Finally, if we apply the efficiency property and Equality (10) to p_0 and $q \in M$, we obtain

$$1 = m|C_{p_0}|(\Psi_j(N,\rho,u_T) - \Phi_j(N,\rho,u_T)) + 1$$

for every $j \in C_{p_0}$. Then, $\Psi_j(N,\rho,u_T) = \Phi_j(N,\rho,u_T)$ for every $j \in C_{p_0}$. Finally, using Equality (10), we obtain $\Psi_i(N,\rho,u_T) = \Phi_i(N,\rho,u_T)$, for every $i \in C_q$ and $q \in M$.

GENERAL CASE. Suppose true the uniqueness when $l(\rho) < H$, regardless of the size of N. Consider ρ with $l(\rho) = H > 0$. Then, there are $p, q \in M$ with $p \neq q$ such that $C_p \cap C_q \neq \emptyset$. Let $i \in C_p \cap C_q$. We introduce, following [3], a new player $j \notin N$ and extend the coalitional cohesion index ρ over $N \cup \{j\}$ as follows:

$$\hat{\rho}(S) = \begin{cases} \rho(C_p) & \text{if } S = (C_p \setminus \{i\}) \cup \{j\}, \\ \rho(S) & \text{if } j \notin S \text{ and } S \neq C_p, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\hat{\rho}(C_p) = 0$ and, hence, $l(\hat{\rho}) < H$. We distinguish two cases.

- $i \notin T$. We extend the game u_T and the coalitional cohesion index to $N \cup \{j\}$ as follows: $w(S) = u_T(S)$ if $j \notin S$, and $w(S) = u_T((S \setminus \{j\}) \cup \{i\})$ for every $S \subseteq N \cup \{j\}$. Notice that $(N \cup \{j\}, w) = (N \cup \{j\}, u_T)$. On the other hand i, j are 0-double players for $(N \cup \{j\}, \hat{\rho}, u_T)$ because $i \in C_q$, $j \in (C_p \setminus \{i\}) \cup \{j\}$ with $\hat{\rho}(C_q) > 0$, $\hat{\rho}((C_p \setminus \{i\}) \cup \{j\}) > 0$, $u_T(S \cup \{i\}) = u_T(S \cup \{j\})$, for every $S \subset N \setminus \{j\}$, and if $S \subseteq N \setminus \{i\}$ such that $\hat{\rho}(S \cup \{i\}) > 0$ then $\hat{\rho}(S \cup \{j\}), \hat{\rho}(S \cup \{i,j\}) \leq 0$. Then, we define the coalitional cohesion index $\hat{\rho}^i_{-j}$ for every $S \subseteq N$ as

$$\hat{\rho}^i_{-j}(S) = \begin{cases} \hat{\rho}(S) \vee \hat{\rho}((S \setminus \{i\}) \cup j) = \rho(S) & \text{if } i \in S, \\ \hat{\rho}(S) = \rho(S) & \text{otherwise,} \end{cases}$$

and the game (N, w^i_{-j}) for every $S \subseteq N$ as

$$w^i_{-j}(S) = \begin{cases} w(S \cup \{j\}) = u_T(S \cup \{j\}) = u_T(S) & \text{if } i \in S, \\ w(S) = u_T(S) & \text{otherwise.} \end{cases}$$

Notice that $\hat{\rho}^i_{-j} = \rho$ and $(N, w^i_{-j}) = (N, u_T)$. Since Ψ satisfies the merger players property we have

$$\Psi_k(N \cup \{j\}, \hat{\rho}, u_T) = \Psi_k(N, \hat{\rho}^i_{-j}, u_T) = \Psi_k(N, \rho, u_T), \tag{11}$$

for every $k \in N \setminus \{i\}$. Using the induction hypothesis and the fact that Φ also satisfies the merger players property, we have

$$\Psi_k(N \cup \{j\}, \hat{\rho}, u_T) = \Phi_k(N \cup \{j\}, \hat{\rho}, u_T) = \Phi_k(N, \hat{\rho}^i_{-j}, u_T) = \Phi_k(N, \rho, u_T), \tag{12}$$

for every $k \in N \setminus \{i\}$. Combining (11) and (12), we obtain $\Psi_k(N, \rho, v) = \Phi_k(N, \rho, u_T)$, for every $k \in N \setminus \{i\}$. Finally, using the efficiency property, we obtain $\Psi_i(N, \rho, u_T) = \Phi_i(N, \rho, u_T)$.

- $i \in T$. We take the game $(N \cup \{j\}, u_{(T \setminus \{i\}) \cup \{j\}})$. Players i, j are 0-double players for $(N \cup \{j\}, \hat{\rho}, u_{(T \setminus \{i\}) \cup \{j\}})$ because $i \in C_q$, $j \in (C_p \setminus \{i\}) \cup \{j\}$ with $\hat{\rho}(C_q) > 0$, $\hat{\rho}((C_p \setminus \{i\}) \cup \{j\}) > 0$, $u_{(T \setminus \{i\}) \cup \{j\}}(S \cup \{i\}) = u_{(T \setminus \{i\}) \cup \{j\}}(S \cup \{j\})$, for every $S \subset N \setminus \{j\}$, and if $S \subseteq N \setminus \{i\}$ such that $\hat{\rho}(S \cup \{i\}) > 0$ then $\hat{\rho}(S \cup \{j\}), \hat{\rho}(S \cup \{i, j\}) \leq 0$. Then, we define the coalitional cohesion index $\hat{\rho}^i_{-j}$ for every $S \subseteq N$ as

$$\hat{\rho}^i_{-j}(S) = \begin{cases} \hat{\rho}(S) \vee \hat{\rho}(S \setminus \{i\} \cup j) = \rho(S) & \text{if } i \in S, \\ \hat{\rho}(S) = \rho(S) & \text{otherwise.} \end{cases}$$

Since i, j are 0-double players, we define the game $(N, (u_{(T \setminus \{i\}) \cup \{j\}})^i_{-j})$ for every $S \subseteq N$

as

$$(u_{(T\setminus\{i\})\cup\{j\}})^i_{-j}(S) = \begin{cases} u_{(T\setminus\{i\})\cup\{j\}}(S\cup\{j\}) = u_T(S) & \text{if } i \in S, \\ u_{(T\setminus\{i\})\cup\{j\}}(S) = 0 = u_T(S) & \text{otherwise.} \end{cases}$$

Notice that $\hat{\rho}^i_{-j} = \rho$ and $(N, (u_{T\setminus\{i\}\cup\{j\}})^i_{-j}) = (N, u_T)$. Since Ψ satisfies the merger players property we have

$$\Psi_k(N\cup\{j\}, \hat{\rho}, u_{T\setminus\{i\}\cup\{j\}}) = \Psi_k(N, \hat{\rho}^i_{-j}, u_T) = \Psi_k(N, \rho, u_T), \tag{13}$$

for every $k \in N \setminus \{i\}$. Using the induction hypothesis and the fact that Φ also satisfies the merger players property, we have

$$\Psi_k(N\cup\{j\}, \hat{\rho}, u_{T\setminus\{i\}\cup\{j\}}) = \Phi_k(N\cup\{j\}, \hat{\rho}, u_{T\setminus\{i\}\cup\{j\}}) = \Phi_k(N, \hat{\rho}^i_{-j}, u_T) = \Phi_k(N, \rho, u_T), \tag{14}$$

for every $k \in N \setminus \{i\}$. Combining (13) and (14), we obtain $\Psi_k(N, \rho, v) = \Phi_k(N, \rho, u_T)$, for every $k \in N \setminus \{i\}$. Finally, using the efficiency property, we obtain $\Psi_i(N, \rho, u_T) = \Phi_i(N, \rho, u_T)$. □

VI. Example

Let us consider a slight modification of the example proposed on Figure 4 in [7]. First of all we recall some notions about cohesion abstractions in computer science (see [7], for more details). Roughly speaking, a *slice abstraction* of a procedure is the set of data slices of the procedure, where a data slice is a sequence of data tokens. In [7] it is defined the *Weak Functional Cohesion* as the ratio among the number of *glue tokens* and the total number of data tokens in a procedure,

$$WFC(p) = \frac{|\text{gluetokens}(p)|}{|\text{data-tokens}(p)|}$$

where p is a procedure and a *glue token* is a data token that belongs to at least two data slices of a procedure. Take the three-slice abstraction of a program with *glue* and *super-glue*[1] tokens, depicted in the Table I. Each line is a data-token, each column is a data slice, and a "|" represents that the data-token belongs to the corresponding data slice.

	S_1	S_2	S_3
Super-glue:	$\vert$	$\vert$	$\vert$
	$\vert$		
		$\vert$	
			$\vert$
Super-glue:	$\vert$	$\vert$	$\vert$
		$\vert$	
Glue:	$\vert$	$\vert$	
Glue:		$\vert$	$\vert$
	$\vert$		
		$\vert$	
			$\vert$

TABLE I
THREE-SLICE ABSTRACTION OF A PROGRAM.

Let us assume that the data slices S_1, S_2, and S_3 can be implemented individually, forming groups of two data slices, or jointly. Considering all the possibilities and calculating their corresponding weak functional cohesions, we can define a coalition cohesion index where, for each $T \subseteq N = \{1, 2, 3\}$, $\rho(T)$ represents the weak functional cohesion of the procedure formed by the data slices in $\{S_t : t \in T\}$:

$$\rho(i) = 1 \text{ for all } i \in \{1, 2, 3\}, \quad \rho(1, 3) = \frac{1}{4}, \quad \rho(1, 2) = \rho(2, 3) = \frac{1}{3}, \quad \rho(1, 2, 3) = \frac{4}{11}.$$

Now, consider the following TU game (N, v) where, for each $T \subseteq N$, $v(T)$ represents the benefit

[1] A *super-glue* token is a data token that belongs to all the data slices.

of implementing the data slices in $\{S_t : t \in T\}$:

T	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(T)$	0	1	1	1	2	2	3	4

We want to share the total benefit of implementing all the data slices among each of them but taking into account their contributions to the cohesion of the procedure. In order to obtain this we compute the Choquet-Owen value. For this example, $im(\rho) = \{\frac{1}{4} < \frac{1}{3} < \frac{4}{11} < 1\}$,

$$\mathcal{C} = supp(\rho) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\},$$

and $M = \{1, 2, \ldots, 7\}$. For each level $\lambda_k \in im(\rho)$, the set M^k is given by:

$$M^1 = M; \quad M^2 = \{1,2,3,4,6,7\}; \quad M^3 = \{1,2,3,7\} \text{ and } M^4 = \{1,2,3\}$$

Since ρ has not buster players, we have $v[\mathcal{C}] = v$ and hence, for all $p \in M$ and all $S \subseteq C_p$, $v'_{\mathcal{C}_p(S)} = v[\mathcal{C}]_{\mathcal{C}_p(S)} = v_{\mathcal{C}_p(S)}$. Besides, for all $\lambda_k \in im(\rho)$, $p \in M$, and all $S \subseteq N$, we have that $v'_{(\mathcal{C}^\rho_{\lambda_k})_p(S)}(T) - v'_{(\mathcal{C}^\rho_{\lambda_k})_p(S)}(T \setminus \{i\}) = v_{(\mathcal{C}^\rho_{\lambda_k})_p(S)}(T) - v_{(\mathcal{C}^\rho_{\lambda_k})_p(S)}(T \setminus \{i\}) = 0$ for all $T \subseteq M$ such that $|T| \geq 5$. We do not compute the quotient games, but we illustrate the computation of the Choquet Owen value by computing the coalitional Owen value in an example. In particular, we calculate $Sh_p(M^k, v'_{(\mathcal{C}^\rho_{\lambda_k})_p(S)})$ for $p = 4$, $k = 2$, and for each $\emptyset \neq S \subseteq C_4 = \{1, 2\}$. For these cases, we have

$$(\mathcal{C}^\rho_{\lambda_2})_4(\{1\}) = \{\{1\}, \{2\}, \{3\}, \{1\}, \{2,3\}, \{1,2,3\}\} \text{ and}$$

$$Sh_4(M^2, v'_{(\mathcal{C}^\rho_{\lambda_2})_4(\{1\})}) = \frac{0!\ 5!}{6!}1 + \frac{1!\ 4!}{6!}3 + \frac{2!\ 3!}{6!}3 + \frac{3!\ 2!}{6!}1 = \frac{40}{120}$$

$$(\mathcal{C}^\rho_{\lambda_2})_4(\{2\}) = \{\{1\}, \{2\}, \{3\}, \{2\}, \{2,3\}, \{1,2,3\}\} \text{ and}$$

$$Sh_4(M^2, v'_{(\mathcal{C}^{\rho}_{\lambda_2})_4(\{2\})}) = \frac{0!\ 5!}{6!}1 + \frac{1!\ 4!}{6!}3 + \frac{2!\ 3!}{6!}2 + \frac{3!\ 2!}{6!}0 = \frac{36}{120}$$

$$(\mathcal{C}^{\rho}_{\lambda_2})_4(\{1,2\}) = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,2,3\}\} \text{ and}$$

$$Sh_4(M^2, v'_{(\mathcal{C}^{\rho}_{\lambda_2})_4(\{1,2\})}) = \frac{0!\ 5!}{6!}2 + \frac{1!\ 4!}{6!}6 + \frac{2!\ 3!}{6!}5 + \frac{3!\ 2!}{6!}1 = \frac{76}{120}$$

The characteristic function of each group game (C_p, w^{ρ}_p) are depicted in Table II.

T	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
(C_1, w^{ρ}_1)	0	$\frac{1175}{1584}$						
(C_2, w^{ρ}_2)	0		$\frac{4289}{3960}$					
(C_3, w^{ρ}_3)	0			$\frac{2172}{1980}$				
(C_4, w^{ρ}_4)	0	$\frac{65}{720}$	$\frac{78}{720}$		$\frac{143}{720}$			
(C_5, w^{ρ}_5)	0	$\frac{45}{720}$		$\frac{60}{720}$		$\frac{105}{720}$		
(C_6, w^{ρ}_6)	0		$\frac{78}{720}$	$\frac{88}{720}$			$\frac{208}{720}$	
(C_7, w^{ρ}_7)	0	$\frac{835}{7920}$	$\frac{1018}{7920}$	$\frac{1128}{7920}$	$\frac{1853}{7920}$	$\frac{1963}{7920}$	$\frac{2688}{7920}$	$\frac{3523}{7920}$

TABLE II
CHARACTERISTIC FUNCTION OF EACH GROUP GAME.

Table III shows the amount that the Shapley value assigns to each player in each group game.

	1	2	3
(C_1, w^{ρ}_1)	$\frac{1175}{1584}$		
(C_2, w^{ρ}_2)		$\frac{4289}{3960}$	
(C_3, w^{ρ}_3)			$\frac{2172}{1980}$
(C_4, w^{ρ}_4)	$\frac{65}{720}$	$\frac{78}{720}$	
(C_5, w^{ρ}_5)	$\frac{45}{720}$		$\frac{60}{720}$
(C_6, w^{ρ}_6)		$\frac{99}{720}$	$\frac{109}{720}$
(C_7, w^{ρ}_7)	$\frac{835}{7920}$	$\frac{1289}{7920}$	$\frac{1399}{7920}$

TABLE III
SHAPLEY VALUE FOR EACH GROUP GAME.

Then, the Choquet-Owen value is obtained by adding up each column in Table III

$$\Phi(N,\rho,v) = \left(1, \frac{179}{120}, \frac{181}{120}\right).$$

Player 2 and player 3 are symmetric in the (N, v) but the additional information given by the cohesion function ρ breaks such symmetry. The Choquet-Owen value assigns a different value to each player. By contrast the Shapley value of the game (N, v), $Sh(N, v) = (1, 1.5, 1.5)$ assigns the same value to player 2 and player 3.

VII. Conclusions

We can see the model of games with coalition configuration as a class of games with some coalitions having maximal degree of cohesiveness, understanding this value as 1. In this paper we extend this model to situations where the cohesiveness degree of a coalition is not necessarily maximal. We present the model and also provide a value that can be applied to share the total gain taking into account the cohesiveness degree of all the coalitions. This value is inspired by the Owen value. We show by an example how this value can be used in software design. In the future we are going to explore the application of any other value or semivalue to this context. Additionally, we can study set-solutions like the core in this setting.

There are some other contexts where our model can be applied. For instance, situations where also uncertainty on payoffs is present (Alparsalan-Gök *et al.* [14], Gao *et al* [15], Palanci *et al.* [16], Branzei *et al.* [17], Shen and Gao [18]) or additional information like a network is provided (Fujimoto [19]). In this last case, since a network can be represented as a coalition cohesion index (1 if the coalition is connected and zero, otherwise), following a similar procedure we can define globally efficient values by contrast to locally efficient values as the Myerson value (Myerson [20]). On the other way around, a coalition cohesion index defines a fuzzy hypergraph that can be seen as a conference situation of Myerson (Myerson [21]). The study of the relationship between both models might deserve some attention in the future.

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REFERENCES

[1] Shapley L.S. (1953). A value for n-person games. In: Kuhn H.W., Tucker A.W. (eds.), Contributions to the theory of games II, Princeton University Press, New Jersey: 307-317.

[2] Owen G. (1977). Values of games with a priori unions. In: Henn R., Moeschlin O. (eds.), Essays in Mathematical Economics and Game Theory, Berlin, Springer-Verlag: 76-88.

[3] Albizuri M.J., Aurrecoechea J. and Zarzuelo J.M. (2005). Configuration values: extensions of the coalitional Owen value. Games and Economic Behavior 57: 1-17.

[4] Beal D.J., Cohen R., Burke, M.J. and McLendon C.L. (2003). Cohesion and perfomance in groups: a meta-analytic clarification of construct relation. Journal of Applied Psichology, 88: 989-1004.

[5] Carless S.A. and De Paola C. (2000). The measurement of cohesion in work teams. Small Group Research 31: 71-88.

[6] Gammage K.L., Carron A.V. and Estabrooks, P.A. (2001). Team cohesion and individual productivity. The influence of the norm for productivity and the identifiability of individual effort. Small Groups Research 32: 3-18.

[7] Bieman J.M. and Ott L.M. (1994). Measuring functional cohesion. IEEE Transactions on Software Engineering 20: 644-657.

[8] Harsanyi J.C. (1963). A simplified bargaining model for the n-person cooperative game. International Economic Review 4: 194-220.

[9] Zadeh L.A. (1965). Fuzzy sets. Information and Control 8: 338-353.

[10] Choquet G. (1953), Theory of Capacities. Annales de l'Institut Fourier vol. 5: 131-295.

[11] Schmeidler D. (1986). Integral representation without additivity. Proceedings of the American Mathematical Society 97: 255-261.

[12] Amer R. and Carreras F. (2001). Power, cooperation indices and coalition structures. In: Holler M.J. and Owen G. (eds.), Power Indices and Coalition Formation, Kluwer Academic Publishers, Boston/Dordrecht/London: 153-173.

[13] Mordeson J.N. and Nair P.S. (2000). Fuzzy graphs and fuzzy hypergraphs, Studies in Fuzziness and Soft Computing 46: Physica-Verlag Heidelberg.

[14] Alparslan-Gök S.Z., Branzei R., and Tijs S. (2010) The interval Shapley value: an axiomatization. Central European Journal of Operations Research18: 131-140.

[15] Gao, J., Zhang, Q. and Shen, P. (2011). Coalitional game with fuzzy payoffs and credibilistic Shapley value. Iran Journal of Fuzzy Systems 8: 107-117.

[16] Palanci, O., Alparslan-Gök, Ergün, S. and Weber, G.W. (2014). Cooperative grey games and the grey Shapley value. Optimization: A Journal of Mathematical Programming and Operations Research, In Press.

[17] Branzei, R., Alparslan-Gök, and Branzei, O. (2011). Cooperative games under interval uncertainty: on the convexity of the interval undominated cores. Central European Journal of Operations Research 19: 523-532.

[18] Shen, P. and Gao, J. (2011). Coalitional games with credibilistic core. Soft Computing, 15: 781-786.

[19] Fujimoto, K. (2014). Cooperative game as non-additive measure. In: Torra, V., Narukawa, Y. and Sugeno, M. (eds), Studies in Fuzziness and Soft Computing 310, Springer International Publishing, Switzerland.

[20] Myerson, R.B. (1977). Graphs and cooperation in games. Mathematics of Operations Research 2: 225-229.

[21] Myerson, R.B. (1980). Conference structures and fair allocation rules. International Journal of game Theory 9: 169-182.