Noncontextual Wirings

Barbara Amaral,[1,2,3] Adán Cabello,[4] Marcelo Terra Cunha,[5] and Leandro Aolita[6,3,7]
[1]*Departamento de Matemática, Universidade Federal de Ouro Preto, 35400-000 Ouro Preto, Minas Gerais, Brazil*
[2]*Departamento de Física e Matemática, CAP—Universidade Federal de São João del-Rei, 36.420-000 Ouro Branco, Minas Gerais, Brazil*
[3]*International Institute of Physics, Federal University of Rio Grande do Norte, 59070-405 Natal, Brazil*
[4]*Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain*
[5]*Departamento de Matemática Aplicada, IMECC-Unicamp, 13084-970 Campinas, São Paulo, Brazil*
[6]*Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, Rio de Janeiro, Rio de Janeiro 21941-972, Brazil*
[7]*ICTP South American Institute for Fundamental Research, Instituto de Física Teórica, UNESP-Universidade Estadual Paulista R. Dr. Bento T. Ferraz 271, Bl. II, São Paulo 01140-070, São Paulo, Brazil*

(Received 6 June 2017; revised manuscript received 5 December 2017; published 29 March 2018)

Contextuality is a fundamental feature of quantum theory necessary for certain models of quantum computation and communication. Serious steps have therefore been taken towards a formal framework for contextuality as an operational resource. However, the main ingredient of a resource theory—a concrete, explicit form of free operations of contextuality—was still missing. Here we provide such a component by introducing noncontextual wirings: a class of contextuality-free operations with a clear operational interpretation and a friendly parametrization. We characterize them completely for general black-box measurement devices with arbitrarily many inputs and outputs. As applications, we show that the relative entropy of contextuality is a contextuality monotone and that maximally contextual boxes that serve as contextuality bits exist for a broad class of scenarios. Our results complete a unified resource-theoretic framework for contextuality and Bell nonlocality.

DOI: 10.1103/PhysRevLett.120.130403

Introduction.—Quantum contextuality refers to the impossibility of explaining the statistical predictions of quantum theory in terms of models where the measurement outcomes reveal preexistent system properties that are independent of the context, i.e., on which other compatible measurements are jointly performed [1,2]. Contextuality can be seen as a generalization of Bell nonlocality [3] to the case where the spacelike separation restriction is removed, so that single systems are included. It thus represents an exotic, intrinsically quantum phenomenon with both fundamental and practical implications. Contextuality has received lots of attention over the last decade. On one hand, it has been experimentally studied in a variety of physical setups [4–8]. On the other one, it is known to be a resource in magic-state [9–12] and measurement-based [13] quantum computing, for random number certification [14], and for several other information-processing tasks in the Bell scenario of spacelike separated measurements [15].

This has motivated considerable interest in resource theories of both contextuality [16–18] and Bell nonlocality [19–21]. Resource theories give powerful frameworks for the formal treatment of a physical property as an operational resource, adequate for its characterization, quantification, and manipulation [22,23]. Their central component is a special class of transformations, called the free operations, that fulfill the essential requirement of mapping every free (i.e., resourceless) object of the theory into a free object. Whereas resource-theoretic approaches for quantum nonlocality are highly developed [19–21,24–28], the operational framework of contextuality as a resource is still less developed. In Refs. [16,17], an abstract characterization of the axiomatic structure of a resource theory of contextuality was done. However, a concrete specification of the free operations of contextuality was not given. Without an explicit parametrization of a physically motivated class of free operations, a resource theory significantly loses applicability. For instance, in Refs. [16,17], an interesting measure of contextuality, called the relative entropy of contextuality, was proposed, but only partial monotonicity under a rather restricted subset of contextuality free operations was shown. Monotonicity (nonincrease under the corresponding free operations) is the fundamental requirement for a function to be a valid quantifier of a resource.

Here, we fill this gap by introducing the class of noncontextual wirings. These are the natural noncontextuality preserving physical operations at hand in the device-independent scenario of black-box measurement devices, where one does not assume any *a priori* knowledge of the state or the observables in question. We derive a friendly analytical expression for generic noncontextual wirings applicable to all nondisturbing boxes, so that both quantum and postquantum boxes are covered. In addition, the framework is versatile in that it allows for transformations between systems with different numbers of inputs and outputs as well as different compatibility constraints. Furthermore, we show that, for the case of Bell tests, the wirings reduce to the canonical free operations of Bell nonlocality [19–21]. Hence, the framework constitutes a unified resource theory for both contextuality and Bell nonlocality in their most general forms. As applications, first we show that an important quantifier called relative entropy of contextuality is monotonic under all noncontextual wirings, a problem left open in Refs. [16,17]. Then, for the broad class of so-called cycle boxes, we show that contextality bits exists in the strongest possible sense: single boxes from which the entire nondisturbing set can be freely obtained with noncontextual wirings.

Nondisturbing boxes.—We consider a measurement device with N buttons (inputs) and M lights (outputs), with $N, M \in \mathbb{N}$. Not all buttons are *compatible*, i.e., can be pressed jointly. Each subset of compatible buttons defines a *context* [29–31]. Let $\mathcal{X} = \{1, 2, \ldots, N\}$ represent the set of buttons. The contexts can be encoded in an input compatibility hypergraph $\mathcal{I}_{\mathcal{X}} \coloneqq \{\chi_j \subseteq \mathcal{X}\}_{j=1,\ldots,|\mathcal{I}_{\mathcal{X}}|}$, where each *hyperedge* χ_j contains the buttons that can be jointly pressed in context j, with $|\mathcal{I}_{\mathcal{X}}|$ the number of contexts [30,31]. We say that j is a maximal context if, for all $1 \leq j' \leq |\mathcal{I}_{\mathcal{X}}|$, $\chi_j \subseteq \chi_{j'}$ implies $\chi_{j'} = \chi_j$.

Similarly, not all lights can turn on jointly. Let $\mathcal{A} = \{1, 2, \ldots, M\}$ be the set of lights. Then, each kth button has a set $\mathcal{A}^{(k)} \subseteq \mathcal{A}$ of lights associated, one—and only one—of which turns on upon pressing that button. The number of lights on is thus always equal to the number of buttons pressed. Hence, for the lights it is more convenient to work with mutual exclusivity constraints. These can be encoded in an output exclusivity hypergraph $\mathcal{O}_{\mathcal{A}} \coloneqq \{\mathcal{A}^{(k)}\}_{k=1,\ldots,N}$, where $\mathcal{A}^{(k)}$ encodes the exclusivity hyperedge of button $k \in \mathcal{X}$. We denote by $\mathcal{A}^{(\chi)} \coloneqq \bigcup_{k\in\chi} \mathcal{A}^{(k)}$ the subset of lights associated with all the buttons in $\chi \in \mathcal{I}_{\mathcal{X}}$. In turn, note that different buttons may share associated lights. We refer to $\mathcal{X}_{(l)} \coloneqq \{k \in \mathcal{X} : l \in \mathcal{A}^{(k)}\}$ as the subset of buttons associated with light $l \in \mathcal{A}$. We restrict throughout to the case where only incompatible buttons can have common associated lights. That is, for every $l \in \mathcal{A}$, $\{k, k'\} \subseteq \mathcal{X}_{(l)}$ is allowed only if $\{k, k'\} \cap \chi \subset \{k, k'\}$ for all $\chi \in \mathcal{I}_{\mathcal{X}}$.

For any input hypergraph $\mathcal{I}_{\mathcal{X}}$ and output hypergraph $\mathcal{O}_{\mathcal{A}}$, we consider conditional probability distributions

$$\boldsymbol{P}_{\mathcal{A}|\mathcal{X}} \coloneqq \{p_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a}, \chi)\}_{\boldsymbol{a}\in\{0,1\}^M, \chi\in\mathcal{I}_{\mathcal{X}}}. \tag{1}$$

The M-bit string $\boldsymbol{a} \coloneqq (a_1, \cdots a_M) \in \{0, 1\}^M$ represents the state of all M lights: $a_l = 0$ stands for "lth light off" and $a_l = 1$ for "lth light on". Hence, $p_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a}, \chi)$ is the probability of the lights being in state $\boldsymbol{a}$ upon pressing the buttons in the subset χ, which is nonzero only if $\boldsymbol{a}$ assigns the state "on" to one, and only one, of the lights associated with each button in χ. That is, for each $\chi \in \mathcal{I}_{\chi}$, $p_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a}, \chi) \neq 0$ only if $\|\boldsymbol{a}^{(k)}\|_{\mathrm{h}} = 1$, with $\boldsymbol{a}^{(k)} \coloneqq (a_l)_{l\in\mathcal{A}^{(k)}}$ the substring of $\boldsymbol{a}$ of lights associated with button k and $\|\boldsymbol{a}^{(k)}\|_{\mathrm{h}}$ the Hamming norm of (number of ones in) $\boldsymbol{a}^{(k)}$, for all $k \in \chi$. We refer to any such $\boldsymbol{P}_{\mathcal{A}|\mathcal{X}}$ as a box behavior relative to $\mathcal{I}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{A}}$. A specially relevant class is that of nondisturbing behaviors: $\boldsymbol{P}_{\mathcal{A}|\mathcal{X}}$ is nondisturbing if, for all χ, $\chi' \in \mathcal{I}_{\mathcal{X}}$ with $\chi' \subset \chi$,

$$\sum_{a_l : l \notin \mathcal{A}^{(\chi')}} p_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a}, \chi) = p_{\mathcal{A}^{(\chi')}|\mathcal{X}'}(\boldsymbol{a}^{(\chi')}, \chi'), \tag{2}$$

with $\boldsymbol{a}^{(\chi')} \coloneqq (a_l)_{l\in\mathcal{A}^{(\chi')}}$ the substring of $\boldsymbol{a}$ of lights associated with the buttons in χ' (instead of the entire context χ). The nondisturbance condition demands that whenever two contexts have buttons in common the marginal distribution over the common buttons is independent of the context. It is thus the analogue of the no-signaling condition in Bell scenarios [15].

With this, we can at last provide a precise formal definition of the general mathematical objects of the resource theory. Namely, we call every set of input and output hypergraphs $\mathcal{I}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{A}}$, respectively, together with a nondisturbing behavior $\boldsymbol{P}_{\mathcal{A}|\mathcal{X}}$ relative to them, a *box*,

$$\boldsymbol{B} \coloneqq \{\mathcal{I}_{\mathcal{X}}, \mathcal{O}_{\mathcal{A}}, \boldsymbol{P}_{\mathcal{A}|\mathcal{X}}\}. \tag{3}$$

We call the set of all such nondisturbing boxes ND.

In turn, the free objects of the theory, i.e., the resourceless ones, are given by the class $\mathsf{NC} \subset \mathsf{ND}$ of *noncontextual* (NC) boxes, defined by NC box behaviors. A behavior $\boldsymbol{P}_{\mathcal{A}|\mathcal{X}}$ is NC if it admits a NC hidden-variable model, i.e., if, for all $\chi \in \mathcal{I}_{\mathcal{X}}$ and $\boldsymbol{a} \in \{0, 1\}^M$ we have

$$p_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a}, \chi) = \sum_{\lambda} p_{\Lambda}(\lambda) \prod_{l\in\mathcal{A}} D_l(a_l|\chi_{(l)}, \lambda), \tag{4}$$

where Λ is the hidden variable, taking the value λ with probability $p_{\Lambda}(\lambda)$, $\chi_{(l)} \coloneqq \chi \cap \mathcal{X}_{(l)}$ is the single-element subset [32] of χ associated with light l, and $D_l(a_l|\chi_{(l)}, \lambda) \coloneqq \delta(a_l, f_l(\chi_{(l)}, \lambda))$, where $\delta(a_l, f_l(\chi_{(l)}, \lambda))$, with δ the Kronecker delta, is the λth NC deterministic response function for the lth light given the input $\chi_{(l)}$. The function f_l encodes the deterministic assignment of $\chi_{(l)}$ into a_l for the λth global deterministic strategy incorporating the constraints of $\mathcal{O}_{\mathcal{A}}$. That is, it is such that, for all λ, $f_l(\emptyset, \lambda) = 0$ (lth light is off if no associated button is pressed, i.e., if $\chi_{(l)} = \emptyset$) and $f_l(\chi_{(l)}, \lambda) \times f_{l'}(\chi_{(l')}, \lambda) = 0$, whenever $\{l, l'\} \subseteq \mathcal{A}^{(k)}$ for any $k \in \mathcal{X}$ (no mutually exclusive lights simultaneously on). Note that, since f_l depends only on $\chi_{(l)}$ (instead of the entire context χ), D_l can only generate NC behaviors in Eq. (4). In fact, we show in

Sec. IV of the Supplemental Material [33] that, when the contexts are defined by spacelike separated buttons, expression (4) reduces to the usual local hidden-variable models of Bell nonlocality [15]. Any box outside NC is called *contextual*. It is a well-known fact that measurements on quantum states can yield contextual boxes.

Contextuality-free operations.—We consider compositions of the initial box $\boldsymbol{B}$ with a preprocessing box

$$\boldsymbol{B}_{\mathrm{PRE}} \coloneqq \{\mathcal{I}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{B}}, \boldsymbol{P}_{\mathcal{B}|\mathcal{Y}}\} \in \mathsf{NC}, \tag{5}$$

and a $(\boldsymbol{b}, \psi)$-dependent postprocessing box

$$\boldsymbol{B}_{\mathrm{POST}}(\boldsymbol{b}, \psi) \coloneqq \{\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{C}}, \boldsymbol{P}_{\mathcal{C}|\mathcal{Z},\psi,\boldsymbol{b}}\} \in \mathsf{NC}, \tag{6}$$

for all $\boldsymbol{b} \in \{0,1\}^{|\mathcal{B}|}$ and $\psi \in \mathcal{I}_{\mathcal{Y}}$, as shown in Fig. 1. $\mathcal{Y}$ and $\mathcal{B}$ are, respectively, the sets of buttons and lights of $\boldsymbol{B}_{\mathrm{PRE}}$, and $\mathcal{Z}$ and $\mathcal{C}$ those of $\boldsymbol{B}_{\mathrm{POST}}(\boldsymbol{b}, \psi)$. For the composition to be possible, we demand that the set of allowed outputs of $\boldsymbol{B}_{\mathrm{PRE}}$ is a subset of the allowed inputs of $\boldsymbol{B}$, and the same for $\boldsymbol{B}$ with $\boldsymbol{B}_{\mathrm{POST}}$. To this end, we need to introduce the output compatibility hypergraph $\bar{\mathcal{O}}_{\mathcal{A}}$ associated to $\mathcal{O}_{\mathcal{A}}$, given by all subsets $\alpha \subset \mathcal{A}$ of output lights with at most one light per exclusivity hyperedge in $\mathcal{O}_{\mathcal{A}}$: $\bar{\mathcal{O}}_{\mathcal{A}} \coloneqq \{\alpha \subset \mathcal{A} : |\alpha \cap \mathcal{A}^{(k)}| \leq 1, k = 1, \ldots, N\}$, and similarly for $\bar{\mathcal{O}}_{\mathcal{B}}$. That is, $\bar{\mathcal{O}}_{\mathcal{A}}$ and $\bar{\mathcal{O}}_{\mathcal{B}}$ give the compatible combinations of lights on, those not violating any of the constraints in $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$, respectively. Then, we demand that $\bar{\mathcal{O}}_{\mathcal{B}} \subseteq \mathcal{I}_{\mathcal{X}}$ and $\bar{\mathcal{O}}_{\mathcal{A}} \subseteq \mathcal{I}_{\mathcal{Z}}$.

Moreover, we allow $\boldsymbol{P}_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi}$ to have only a restricted dependence on $(\boldsymbol{b}, \psi)$, in such a way that each output light of the postprocessing box is causally influenced only by the inputs and outputs of the preprocessing box that are

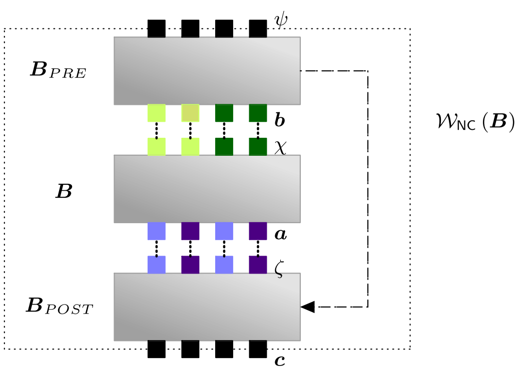

FIG. 1. A noncontextual wiring $\mathcal{W}_{\mathsf{NC}}$ with respect to pre- and postprocessing boxes $\boldsymbol{B}_{\mathrm{PRE}}$ and $\boldsymbol{B}_{\mathrm{POST}}$, respectively, mapping an initial box $\boldsymbol{B}$ into a final box $\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})$. The buttons and lights of $\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})$ are given by the buttons of $\boldsymbol{B}_{\mathrm{PRE}}$ and the lights of $\boldsymbol{B}_{\mathrm{POST}}$, respectively. Only the lights (buttons) of $\boldsymbol{B}$ of the same color can be on (pressed) at the same time. The behavior of $\boldsymbol{B}_{\mathrm{POST}}$ is causally influenced by $\boldsymbol{B}_{\mathrm{PRE}}$, but in a restricted way such that the statistics of each output light of $\boldsymbol{B}_{\mathrm{POST}}$ depends only on the buttons and lights of $\boldsymbol{B}_{\mathrm{PRE}}$ that are associated with it (see text). As a result, if $\boldsymbol{B}$ is noncontextual so is $\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})$.

associated with it. That is, we demand that, for all $\boldsymbol{b} \in \{0,1\}^{|\mathcal{B}|}$, $\boldsymbol{c} \in \{0,1\}^{|\mathcal{C}|}$, $\psi \in \mathcal{I}_{\mathcal{Y}}$, and $\zeta \in \mathcal{I}_{\mathcal{Z}}$,

$$p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi}(\boldsymbol{c}, \zeta) = \sum_{\phi} p_{\Phi}(\phi) \prod_{n \in \mathcal{C}} D_n(c_n|\zeta_{(n)}, \chi_{[n]}^{(\boldsymbol{b})}, \psi_{[n]}, \phi), \tag{7}$$

with $D_n(c_n|\zeta_{(n)}, \chi_{[n]}^{(\boldsymbol{b})}, \psi_{[n]}, \phi)$ defined analogously to $D_l(a_l|\chi_{(l)}, \lambda)$ in Eq. (4). Similarly to $\chi_{(l)}$ there, $\zeta_{(n)}$ is the single-element subset of ζ associated with light $n \in \mathcal{C}$. In turn, we now introduce the short-hand notations $\chi_{[n]}^{(\boldsymbol{b})}$ and $\psi_{[n]} \coloneqq \psi_{(\chi_{[n]}^{(\boldsymbol{b})})}$ [34]. The subset $\chi_{[n]}^{(\boldsymbol{b})}$ is composed of the single button in $\chi_{(\zeta_{(n)})}$ directly wired to some light on in $\boldsymbol{b}$, whereas $\psi_{[n]}$ is the single-button subset of ψ associated to the light directly wired to the button of $\chi_{[n]}^{(\boldsymbol{b})}$. These subsets are all well defined through the hypergraphs $\mathcal{I}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{A}}$, independently of the specific behavior $\boldsymbol{P}_{\mathcal{A}|\mathcal{X}}$ in question, as shown in Sec. I of the Supplemental Material [33]. This is crucial for the composition not to create contextuality.

With this, we are now in a good position to introduce the free operations of contextuality.

Definition 1: Noncontextual wirings.—We define the noncontextual wiring with respect to the pre- and post-processing boxes described above, as the linear map $\mathcal{W}_{\mathsf{NC}}$ that takes any initial box $\boldsymbol{B} \in \mathsf{ND}$, given by Eq. (3), into a final box $\boldsymbol{B}_{\mathrm{f}} \coloneqq \mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})$ with $N_{\mathrm{f}} \coloneqq |\mathcal{Y}|$ buttons and $M_{\mathrm{f}} \coloneqq |\mathcal{C}|$ lights, with

$$\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B}) \coloneqq \{\mathcal{I}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{C}}, \boldsymbol{P}_{\mathcal{C}|\mathcal{Y}}\}, \tag{8}$$

where $\boldsymbol{P}_{\mathcal{C}|\mathcal{Y}}$ is the final behavior, given by

$$p_{\mathcal{C}|\mathcal{Y}}(\boldsymbol{c}, \psi) = \sum_{\substack{\boldsymbol{a} \in \{0,1\}^{|\mathcal{A}|} \\ \boldsymbol{b} \in \{0,1\}^{|\mathcal{B}|}}} p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi}(\mathbf{c}, \zeta^{(\boldsymbol{a})}) p_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a}, \chi^{(\boldsymbol{b})}) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b}, \psi), \tag{9}$$

for all $\boldsymbol{c} \in \{0,1\}^{|\mathcal{C}|}$ and $\psi \in \mathcal{I}_{\mathcal{Y}}$. We denote the class of all such wirings by NCW.

Self-consistency of the theory requires that NCW satisfies the following property, proven in Sec. II of the Supplemental Material [33].

Lemma 1: Nondisturbance preservation.—The class of boxes ND is closed under all wirings in NCW.

In addition, to give valid free operations, NCW must fulfill the following requirement, proven in Sec. III of the Supplemental Material [33].

Theorem 1: Noncontextuality preservation.—The class of boxes NC is closed under all wirings in NCW.

Intuitively, this is connected to the fact that the composition of any three independent noncontextual boxes yields a final box that is also noncontextual (with three independent noncontextual hidden variables). NCW is, however, more powerful than such compositions because the pre- and postprocessing boxes here are not independent. Still, the restriction of Eq. (7) enables noncontextuality preservation (see Sec. III of the Supplemental Material). Finally, in Sec. IV of the Supplemental Material [33], we show that, for spacelike separated measurements, NCW reduces to local operations assisted by shared randomness, the canonical free operations of Bell nonlocality [19–21].

Contextuality monotones.—In Ref. [16], a measure of contextuality called the relative entropy of contextuality, R_C, was introduced. For an arbitrary box $\boldsymbol{B} \in \mathsf{ND}$,

$$R_C(\boldsymbol{B}) \coloneqq \min_{\boldsymbol{B}^* \in \mathsf{NC}} S(\boldsymbol{B} \| \boldsymbol{B}^*). \tag{10}$$

$S(\boldsymbol{B}\|\boldsymbol{B}^*)$ is the relative entropy of $\boldsymbol{B}$ with respect to $\boldsymbol{B}^*$ (see Sec. IV of the Supplemental Material [33]), which measures the distinguishability of $\boldsymbol{B}$ from $\boldsymbol{B}^*$ in a broad class of scenarios [21]. Hence, $R_C(\boldsymbol{B})$ quantifies the distinguishability of $\boldsymbol{B}$ from its closest (with respect to S) noncontextual box $\boldsymbol{B}^*$, providing a direct generalization to contextuality of the statistical strength of Bell nonlocality proofs [35].

The essential requirement for a function to be a valid measure of a resource is that it is monotonic (i.e., nonincreasing) under the corresponding free operations. In Ref. [16], the authors show, for quantum boxes, monotonicity of R_C under probabilistic mixtures of independent channels on each quantum observable (each context). This corresponds to a restricted subset of NCW [36]. Here, we show monotonicity of R_C under the whole class NCW and for all boxes $\boldsymbol{B} \in \mathsf{ND}$.

Lemma 2: Monotonicity of R_C.—Let $\boldsymbol{B} \in \mathsf{ND}$. Then, $R_C[\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})] \leq R_C(\boldsymbol{B})$ for all $\mathcal{W}_{\mathsf{NC}} \in \mathsf{NCW}$.

The proof (given in Sec. V of the Supplemental Material [33]) relies explicitly on the parametrization of NCW in Eq. (9).

Interestingly, also, another measure of contextuality, the contextual fraction $C(\boldsymbol{B})$ [29,37], was recently shown to be monotonic under some specific classes of contextuality-free operations [18]. A straightforward calculation (see Sec. VI of the Supplemental Material [33]) shows that $C(\boldsymbol{B})$ is also monotonic under the NCW class.

Lemma 3: Monotonicity of C.—Let $\boldsymbol{B} \in \mathsf{ND}$. Then, $C[\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})] \leq C(\boldsymbol{B})$ for all $\mathcal{W}_{\mathsf{NC}} \in \mathsf{NCW}$.

Contextuality bits.—The operational framework developed allows us to study contextuality interconversions. A natural question is whether there exists a box from which all boxes, for fixed input and output hypergraphs, can be obtained for free (i.e., through noncontextual wirings). This is intimately connected to quantification: such a superior box can be taken as a unit of contextuality, or contextuality bit, yielding a natural and unambiguous (measure-independent) definition of maximally contextual boxes. Here we answer that question affirmatively for a broad class given by the so-called N-cycle boxes (see Fig. 2). A N-cycle box has as many maximal contexts as buttons (N),

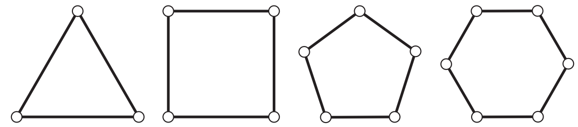

FIG. 2. N-cycle graphs C_N for $N = 3, 4, 5$, and 6 buttons. A N-cycle box is such that the union of all hyperedges in $\mathcal{I}_{\mathcal{X}}$ equals C_N and each input button has its own pair of output lights. For even N, the class is also intimately connected to the well-known chained inequalities of Bell nonlocality [38]. It includes the Clauser-Horne-Shimony-Holt (CHSH) scenario [39], where the inputs define the square C_4, and the Klyachko-Can-Binicioğlu-Shumovsky one [40], where the inputs form the pentagon C_5. For any $N \geq 3$, there exist contextuality bits, i.e., maximally contextual N-cycle boxes from which all other N-cycle boxes can be obtained for free (see text).

each kth maximal context consists of two buttons (k and $k+1$), each kth button belongs to two maximal contexts (χ_k and χ_{k-1}) and has two associated output lights, the $(2k-1)$th and the $(2k)$th lights, so that $M = 2N$. Modulo N is implicitly assumed for the labels of buttons, contexts, and lights. These boxes admit 2^{N-1} contextuality bits:

Lemma 4: Existence of contextuality bits.—For any $N \geq 3$, all N-cycle boxes in ND can be freely obtained from an N-cycle box with behavior $\boldsymbol{P}^{(\boldsymbol{\gamma})}_{\mathcal{A}|\mathcal{X}}$ of components

$$p^{(\boldsymbol{\gamma})}_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a},\chi) \coloneqq \begin{cases} \frac{1}{2}, & \text{if } \chi = \{k, k+1\} \text{ and } a_{2k-s} = a_{2(k+1)-s+\gamma_k}, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

for all $s \in \{0,1\}$ and $k \in \mathcal{X}$, with $\boldsymbol{\gamma} \coloneqq (\gamma_1, \ldots, \gamma_N)$, such that $\gamma_k = 0$ or 1 and $\|\boldsymbol{\gamma}\|_{\mathrm{h}}$ is an odd integer.

Equation (11) describes any of the 2^{N-1} contextual N-cycle behaviors extremal in ND, derived (in a different notation) and shown to be equivalent under noncontextual relabelings of outputs in Ref. [41]. The proof of the lemma, given in Sec. VII of the Supplemental Material [33], consists then of showing that any convex mixing of such relabelings is in NCW. For the particular case $N = 4$ (the CHSH scenario), the behaviors in Eq. (11) become equivalent to the no-signaling extremal box of [42], known to generate all no-signaling boxes under local wirings assisted by shared randomness [19–21]. Lemma 5 thus generalizes this fact to arbitrary $N \geq 3$ and noncontextual wirings. Finally, it is important to mention that, for even N, the buttons can be split into two disjoint subsets of $N/2$ incompatible buttons each, and the lights can be reduced from $2N$ to only 4 (one mutually exclusive pair per subset of buttons), as in the chained inequalities [38]. This is an alternative representation of the same physical box. Our formalism is totally versatile in this sense, as it can directly deal with any chosen representation of a box.

Final discussion.—Recent investigations suggest that contextuality may be a key resource for quantum advantages in various information-processing tasks [9–14]. Here we take a step forward towards contextuality as an operational resource by introducing and characterizing noncontextual wirings. In contrast to more abstract approaches [16,17], noncontextual wirings have a clear operational interpretation and admit a friendly analytical parametrization. This is useful to classify, quantify, and manipulate contextuality as a formal resource. For instance, the question of monotonicity of contextuality was until recently unclear. While in Refs. [16–18] monotonocity of the relative entropy of contextually and of the contextual fraction is proven under some specific operations, here we have settled the problem of monotonicity under all noncontextual wirings for both contextually measures. Furthermore, we have also shown that maximally contextual single boxes that serve as contextuality bits exist for all cycle boxes, which encompass important Bell scenarios [39,40] and play a crucial role in contextuality theory [43–48]. This result can also be extended to boxes with more outputs [24]. Interesting questions are, e.g., what the simplest box admitting inequivalent (not freely interconvertible) classes of contextuality is and what the simplest one allowing for contextuality distillation. Finally, we have shown that, for Bell scenarios, noncontextual wirings reduce to the usual free operations of Bell nonlocality [19–21], which is interesting in itself. Hence, our findings yield a main missing ingredient for a complete, unified resource theory of contextuality and Bell nonlocality.

The present work was initiated during the workshop "Quantum Correlations, Contextuality, and All That… Again" at the International Institute of Physics (IIP), Natal, Brazil. The participants of the workshop as well as the hospitality of IIP are gratefully acknowledged. We thank D. Cavalcanti, C. Duarte, M. Pusey, R. Soares Barbosa, S. Mansfield and an anonymous Referee for fruitful discussions. B. A. thanks the Instituto de Matemática Pura e Aplicada (IMPA) for the hospitality at Rio de Janeiro, Brazil. B. A., M. T. C., and L. A. acknowledge financial support from the Brazilian ministries MEC and MCTIC and agencies CNPq, CAPES, FAEPEX, FAPERJ, FAPESP, and INCT-IQ. A. C. acknowledges support from Project No. FIS2014-60843-P, "Advanced Quantum Information" (MINECO, Spain), with FEDER funds, the FQXi Large Grant "The Observer Observed: A Bayesian Route to the Reconstruction of Quantum Theory," and the project "Photonic Quantum Information" (Knut and Alice Wallenberg Foundation, Sweden).

[1] E. P. Specker, Die Logik nicht gleichzeitig entscheidbarer Aussagen, Dialectica **14**, 239 (1960); The logic of non-simultaneously decidable propositions, arXiv:1103.4537.

[2] S. Kochen and E. P. Specker, The problem of hidden variables in quantum mechanics, J. Math. Mech. **17**, 59 (1967).

[3] J. S. Bell, On the problem of hidden variables in quantum mechanics, Rev. Mod. Phys. **38**, 447 (1966).

[4] Y. Hasegawa, R. Loidl, G. Badurek, M. Baron, and H. Rauch, Quantum Contextuality in a Single-Neutron Optical Experiment, Phys. Rev. Lett. **97**, 230401 (2006).

[5] G. Kirchmair, F. Zähringer, R. Gerritsma, M. Kleinmann, O. Gühne, A. Cabello, R. Blatt, and C. F. Roos, State-independent experimental test of quantum contextuality, Nature (London) **460**, 494 (2009).

[6] E. Amselem, M. Rådmark, M. Bourennane, and A. Cabello, State-Independent Quantum Contextuality with Single Photons, Phys. Rev. Lett. **103**, 160405 (2009).

[7] R. Łapkiewicz, P. Li, C. Schaeff, N. Langford, S. Ramelow, M. Wieśniak, and A. Zeilinger, Experimental non-classicality of an indivisible quantum system, Nature (London) **474**, 490 (2011).

[8] G. Borges, M. Carvalho, P.-L. de Assis, J. Ferraz, M. Araújo, A. Cabello, M. Terra Cunha, and S. Pádua, Quantum contextuality in a Young-type interference experiment, Phys. Rev. A **89**, 052106 (2014).

[9] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, Negative quasi-probability as a resource for quantum computation, New J. Phys. **14**, 113011 (2012).

[10] M. Howard, J. J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the 'magic' for quantum computation, Nature (London) **510**, 351 (2014).

[11] N. Delfosse, P. A. Guerin, J. Bian, and R. Raussendorf, Wigner Function Negativity and Contextuality in Quantum Computation on Rebits, Phys. Rev. X **5**, 021003 (2015).

[12] J. Bermejo-Vega, N. Delfosse, D. E. Browne, C. Okay, and R. Raussendorf, Contextuality as a Resource for Models of Quantum Computation with Qubits, Phys. Rev. Lett. **119**, 120505 (2017).

[13] R. Raussendorf, Contextuality in measurement-based quantum computation, Phys. Rev. A **88**, 022322 (2013).

[14] M. Um, Mark, X. Zhang, J. Zhang, Y. Wang, S. Yangchao, D. Deng, L. Duan, and K. Kim, Experimental certification of random numbers via quantum contextuality, Sci. Rep. **3**, 1627 (2013).

[15] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014).

[16] A. Grudka, K. Horodecki, M. Horodecki, P. Horodecki, R. Horodecki, P. Joshi, W. Kłobus, and A. Wójcik, Quantifying Contextuality, Phys. Rev. Lett. **112**, 120401 (2014).

[17] K. Horodecki, A. Grudka, P. Joshi, W. Kłobus, and J. Łodyga, Axiomatic approach to contextuality and nonlocality, Phys. Rev. A **92**, 032104 (2015).

[18] S. Abramsky, R. Soares Barbosa, and S. Mansfield, The Contextual Fraction as a Measure of Contextuality, Phys. Rev. Lett. **119**, 050504 (2017).

[19] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, Operational Framework for Nonlocality, Phys. Rev. Lett. **109**, 070401 (2012).

[20] J. I. de Vicente, On nonlocality as a resource theory and nonlocality measures, J. Phys. A **47**, 424017 (2014).

[21] R. Gallego and L. Aolita, Nonlocality free wirings and the distinguishability between Bell boxes, Phys. Rev. A **95**, 032118 (2017).

[22] F. G. S. L. Brandão and G. Gour, Reversible Framework for Quantum Resource Theories, Phys. Rev. Lett. **115**, 070503 (2015); Erratum, Phys. Rev. Lett. **115**, 199901 (2015).
[23] B. Coecke, T. Fritz, and R. W. Spekkens, A mathematical theory of resources, Inf. Comput. **250**, 59 (2016).
[24] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Nonlocal correlations as an information-theoretic resource, Phys. Rev. A **71**, 022101 (2005).
[25] J. Allcock, N. Brunner, N. Linden, S. Popescu, P. Skrzypczyk, and T. Vértesi, Closed sets of nonlocal correlations, Phys. Rev. A **80**, 062107 (2009).
[26] P. Joshi, M. Horodecki, R. Horodecki, A. Grudka, K. Horodecki, and P. Horodecki, No-broadcasting of non-signaling boxes via operations which transform local boxes into local ones, Quantum Inf. Comput. **13**, 567 (2013).
[27] B. Lang, T. Vértesi, and M. Navascués, Closed sets of correlations: Answers from the zoo, J. Phys. A **47**, 424029 (2014).
[28] R. Gallego and L. Aolita, Resource Theory of Steering, Phys. Rev. X **5**, 041008 (2015).
[29] S. Abramsky and A. Brandenburger, The sheaf-theoretic structure of non-locality and contextuality, New J. Phys. **13**, 113036 (2011).
[30] A. Cabello, S. Severini, and A. Winter, Graph-Theoretic Approach to Quantum Correlations, Phys. Rev. Lett. **112**, 040401 (2014).
[31] A. Acín, T. Fritz, A. Leverrier, and A. B. Sainz, A combinatorial approach to nonlocality and contextuality, Commun. Math. Phys. **334**, 533 (2015).
[32] Since $\chi \in \mathcal{I}_{\mathcal{X}}$ is a valid context, it has at most one button associated to each light. Hence, $|\chi_{(l)}| = 1$.
[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.120.130403 for technical details.
[34] Subindices in round brackets refer to association with lights or buttons immediately below (children) and supraindices in round brackets to association with buttons or lights immediately above (parents). In turn, subindices in square brackets refer to association with lights or buttons at lower levels (descendants) in the box composition of Fig. 1 and supraindices in square brackets to association with buttons or lights at higher levels (ancestors).
[35] W. van Dam, R. D. Gill, and P. D. Grünwald, The statistical strength of nonlocality proofs, IEEE Trans. Inf. Theory **51**, 2812 (2005).
[36] In fact, in Ref. [16], restricted monotonicity of another related quantity, the uniform relative entropy of contextuality, was shown. However, one can show [21] that the uniform variant is not monotonous under general wirings in NCW, even for trivial (identity) postprocessing boxes.
[37] E. Amselem, L. E. Danielsen, A. J. López-Tarrida, J. R. Portillo, M. Bourennane, and A. Cabello, Experimental Fully Contextual Correlations, Phys. Rev. Lett. **108**, 200405 (2012).
[38] S. L. Braunstein and C. M. Caves, Wringing out better Bell inequalities, Ann. Phys. (N.Y.) **202**, 22 (1990).
[39] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).
[40] A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Simple Test for Hidden Variables in Spin-1 Systems, Phys. Rev. Lett. **101**, 020403 (2008).
[41] M. Araújo, M. T. Quintino, C. Budroni, M. Terra Cunha, and A. Cabello, All noncontextuality inequalities for the n-cycle scenario, Phys. Rev. A **88**, 022118 (2013).
[42] S. Popescu and D. Rohrlich, Quantum nonlocality as an axiom, Found. Phys. **24**, 379 (1994).
[43] Y.-C. Liang, R. W. Spekkens, and H. M. Wiseman, Specker's parable of the overprotective seer: A road to contextuality, nonlocality and complementarity, Phys. Rep. **506**, 1 (2011).
[44] A. Cabello, P. Badziąg, M. Terra Cunha, and M. Bourennane, Simple Hardy-Like Proof of Quantum Contextuality, Phys. Rev. Lett. **111**, 180404 (2013).
[45] S. Mansfield, Ph.D. thesis, University of Oxford, Oxford, England, 2013.
[46] B. Amaral, M. Terra Cunha, and A. Cabello, Exclusivity principle forbids sets of correlations larger than the quantum set, Phys. Rev. A **89**, 030101(R) (2014).
[47] E. N. Dzhafarov, J. V. Kujala, and V. H. Cervantes, *Contextuality-by-Default: A Brief Overview of Ideas, Concepts, and Terminology*, International Symposium on Quantum Interaction (Springer, New York, 2015).
[48] J. V. Kujala, E. N. Dzhafarov, and J.-Å. Larsson, Necessary and Sufficient Conditions for an Extended Noncontextuality in a Broad Class of Quantum Mechanical Systems, Phys. Rev. Lett. **115**, 150401 (2015).