The Mathematics Teacher's Specialised Knowledge (MTSK) model

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This paper presents the model, *The Mathematics Teacher’s Specialised Knowledge* (MTSK). It acknowledges earlier contributions to understanding and structuring teachers’ knowledge, in particular, the special debt owed to Shulman’s notion of *pedagogical content knowledge* and to Ball and collaborators’ *Mathematical Knowledge for Teaching* (MKT) influential for the specialised nature of one of its sub-domains. Our research with teachers has led us to explore the characteristics of MKT and to refine the descriptors relating to its sub-domains, a task which has underlined the difficulty involved in unambiguously delimiting the boundaries which separate these. As a result, and taking into consideration a broader view of the specialised nature of the teacher's mathematical knowledge, we propose a framework which, whilst respecting the major domains of Content Knowledge and Pedagogical Content Knowledge, regards the specialisation in respect of mathematical knowledge as a property which is inherent to the model and extends across all sub-domains.

Keywords: teachers’ knowledge; specialised knowledge; mathematical knowledge for teaching

Introduction

This paper presents a joint theoretical study into mathematics teachers’ knowledge carried out by the SIDM group of the University of Huelva (Spain). First we consider existing models and discuss design features which could limit their utility. We then move on to review the findings of various studies into the questions we raise, before proposing an alternative model which seeks to circumvent these limitations by focussing on the *specialised* nature of this knowledge. Finally, we give a full explanation of what we believe is the significance of this approach.

One of the group’s areas of research interest concerns mathematics teachers’ knowledge and professional development. In 1999, responding to the request of a group of primary teachers, we formed a working group (called the PIC), which resulted in a reorientation from doing research *into* teachers and their work to doing research *with* teachers, being our guiding principle to advocate a notion of professional development focused on the practitioner, which accords a central role to the teacher's reflections on their practice.

We have explored the conditions for promoting development, change and professional expertise, ensuring throughout that our work is anchored to the collaborative relations with the teachers (both primary and secondary) in the group (PIC) (Muñoz-Catalán, Carrillo, & Climent, 2010).

One of the key findings of our studies, endorsed by the participating teachers, was the need to delve more deeply into the knowledge usable for teaching, and consequently the need for adequate tools or models which would facilitate this analysis and possibly also allow us to make recommendations for teacher training (Carrillo, & Climent, 2011).

The main aim of the PIC collaborative research group, which is currently composed of 12 members including serving pre-school, primary and secondary teachers, trainee teachers and researchers into Mathematics Education, is to identify the kind of learning opportunities created by teachers in the course of their work. The group takes an interpretative approach to exploring mathematics teachers' specialised knowledge with the objective of developing a model that enables this knowledge to be analysed in depth. We briefly describe the chief features of this approach below.

A method frequently used in the PIC is that of analysing recordings of pre-agreed lessons in the group with a view to identifying examples of good practice in mathematics lessons. At one point early in the history of the group, some of the teachers stated their need to improve their mathematical knowledge so as to be able to meet the challenges their work was bringing them. Given that our approach to teacher observation promotes reflection on practice with the teachers, our goal in employing the

MTSK model to analyse teachers' specialised knowledge is one of comprehension and interpretation rather than evaluation. Finally, in order for this analysis to be fertile enough to merit joint discussion and reflection with fellow professionals, the model needs to be able to facilitate levels of reflection beyond a description of subdomains, from which follow our efforts to develop categories of analysis.

We thus redirected the focus of our research onto mathematics teachers' professional knowledge. After using existing models, and coming to understand both their limitations and strengths, we were led to propose a model focusing exclusively on the knowledge specific to the mathematics teacher, that is to the exclusion of areas of professional knowledge held in common with teachers of other subjects (at no point underestimating the importance of these to effective teaching).

In the following section, we discuss the models of teachers' knowledge (chiefly *Mathematical Knowledge for Teaching,* Ball, Thames, & Phelps, 2008) which have been influential in the development of our own, *Mathematics Teacher's Specialised Knowledge* (MTSK), presented afterwards. Finally, we offer some general reflections on the model.

Theoretical background

Among the various attempts to map out teachers' professional knowledge, perhaps the most influential is Shulman's (1986) study, in which he identified three principle domains, *Subject Matter Knowledge (SMK)*, *Pedagogical Content Knowledge (PCK) and Curricular Knowledge (CK)*. Shulman's most significant contribution was the inclusion of PCK and the subject matter to be taught as the defining feature of teachers' knowledge.

Since the publication of Shulman's ground-breaking work, various alternatives for conceptualizing teachers' knowledge have been proposed, each foregrounding different elements and features (e.g. Ma, 1999; Davis, & Simmt, 2006; Schoenfeld, & Kilpatrick, 2008; Rowland, Turner, Thwaites, & Huckstep, 2009; Baumbert, & Kunter, 2013). Of special note is Ball, et al.'s (2008) model, *Mathematical Knowledge for Teaching* (MKT).

MKT considers SMK and PCK. SMK consists of *Common Content Knowledge* (CCK), *Specialised Content Knowledge* (SCK) and *Horizon Content Knowledge* (HCK); PCK includes *Knowledge of Content and Teaching* (KCT), *Knowledge of Content and Students* (KCS) and *Knowledge of Content and Curriculum* (KCC). CCK is defined as the knowledge which a well-educated adult has of the educational level in question. SCK recognises the specialised nature of the teacher's mathematical knowledge as opposed to the mathematical knowledge required by other professionals that use mathematics. HCK reflects the idea that the teacher should display some awareness of how school mathematics joins up. Ball and Bass (2009) proposed a division of this sub-domain into three components, one relating to topics or themes, another to practice (drawing on the work of Schwab, 1978, and Ball and McDiarmird, 1990), and a third relating to what can be termed mathematics values.

With respect to PCK and its sub-domains, KCS includes the teacher's capacity to foresee what will strike the students as easy, challenging, interesting or motivating. KCT considers the knowledge that guides teachers in negotiating the specifics of the lesson, such as to emphasise or clarify a particular mathematical idea. KCC is comprised of the knowledge employed in determining the direction which the students' learning should take, and the type of content they should learn.

Among the more significant of the model's contributions is its recognition of a type of knowledge exclusive to teachers (SCK), predicated on the idea that teaching requires specialized knowledge that other professions do not. There are certain shortcomings that can be levelled at this model, however. One is the question of whether or not particular elements are indeed exclusive to teachers of mathematics (Flores, Escudero, & Carrillo, 2013), concomitant on the extrinsic nature of the notion of specialisation underpinning the model (Scheiner, Montes, Godino, Carrillo, Pino-Fan, 2017), which could cause difficulties of analysis. Another, connected to the former, is the tendency for sub-domains to overlap when put to analytical use (Silverman, & Thompson, 2008). In this regard, Ball et al. (2008) state that *"by 'mathematical knowledge for teaching', we mean the mathematical knowledge needed to carry out the work of teaching mathematics*" (p.395). Note that the object of analysis in this model is not the mathematical knowledge *used* by teachers to carry out their work, but rather the assessment of the mathematical knowledge *needed* to do so. (e.g. Ball, Hill, & Bass, 2005). Hence, the MKT model, and the work of Rowland et al. (2009), focus their attention on practice as carried out in class, ignoring the knowledge that teachers might bring into play when carrying out any other kind of activity as a teacher.

With our sights set on a fine-grained analysis of the knowledge teachers employ in their work, we propose a conceptualization of mathematics teachers' knowledge in which the different sub-domains of SMK are determined by the mathematical nature of the content. Similarly, and for the purposes of ease of use, the model includes categories of knowledge associated with each of the sub-domains, enabling a detailed analysis of teachers' mathematical knowledge to be undertaken.

The MTSK model

Our starting point is the assumption that in order to carry out their role (including lesson planning, liaising with colleagues, giving lessons and taking time to reflect on them afterwards) the teacher needs specific knowledge. We associate this specificity with mathematics teaching. It includes meanings, the properties and definitions of particular topics, the means of building understanding of the subject, connections between content items, knowledge of teaching mathematics, and characteristics associated with learning mathematics, amongst others. To this extent, we understand that the specificity of the teacher's knowledge in relation to mathematics teaching affects both SMK and PCK together, and as such cannot be considered a sub-domain of either.

Spurred on by a critical analysis of MKT, our principal goal was to construct a model of teachers' knowledge which took holistic account of the specialised nature of teachers' knowledge (that is, permeating all sub-domains within the model). We also sought to ensure that the definitions for each sub-domain were constructed in terms of what the teacher used/needed, without reference to external agencies (other professions), thus avoiding the problems of overlap affecting other models (as noted above).

The MTSK model takes an analytical focus with the aim of gaining insight into the teacher's knowledge, specifically the elements which go to make this knowledge up and the interactions between them. It is, then, preeminently directed towards studying the knowledge, which the teacher puts into *use*. To this effect, we bring to bear domains and sub-domains under the hypothesis that the knowledge in question can be mapped onto these. When we say that a teacher *needs* knowledge pertaining to a particular sub-domain, we are not referring to a predetermined list of contents, rather we mean that the teacher must necessarily have knowledge which can be located in this sub-domain. In this regard, teacher trainers could make use of the MTSK model for organizing the perceived training needs of their trainees.

With these principles in mind, we set about developing a twofold procedure incorporating a *top-down* and *bottom-up* perspective (Grbich, 2013). First, theoretical considerations and early experimentation with this and other models of teacher knowledge provided an initial conceptualization of the MTSK model (*top-down*), leading to a reorganization of content in the mathematical and pedagogical domains. Subsequently, the structure and functionality of the model was carefully scrutinised from the perspective of Grounded Theory (GT) (Charmaz, 2014) (*bottom-up*), to check for any possible new categories or subcategories, or even subdomains, which had not been contemplated by the MTSK model in its original configuration.

This involved various cycles of applying the model to data from a multiplicity of sources, including classroom episodes, in-service training sessions, interviews with mathematics teachers at primary, secondary and university level and trainee mathematics teachers. As categories emerged from the analysis of the data, these were fed back into the cycle for use with subsequent episodes through content analysis (Bardin, 2001). This iterative process helped to refine the conceptual foundations of the model and clarify the definitions of its specific parts.

Below we give examples of this process for the categories in the sub-domain of *Knowledge of Features of Learning Mathematics.*

First we carried out a literature review and explored different types of categories, focussing on the interaction of the student with content, and the characteristics of mathematical content itself as an object of learning. From this were established the categories *strengths and weaknesses in learning mathematics*, *ways pupils interact with mathematical content*, and *students' main interests and expectations in approaching an area of content*. Once the basic framework of categories had been established (*top-down*), they were applied to the previously collected data (see above), maintaining throughout a degree of flexibility to allow for the incorporation of new categories. Once the categories had been empirically tested, they were confirmed and their descriptions carefully composed to reflect what each aimed to encompass.

The final stage of the process analysed transcribed excerpts from the perspective of Grounded Theory (*bottom-up*), at times using the software programme MAXQDA. This confirmed the above categories and identified a new one (*knowledge of theories of mathematical learning associated with a mathematical content area*), which emerged out of statements by teachers which, irrespective of register, were deemed to be indicative of speakers' knowledge, which could be associated to learning theories.

The resultant model we propose is the *Mathematics Teacher's Specialised Knowledge (MTSK)* model (figure 1). With respect to MKT, this model features a reconfiguration of mathematical knowledge, a reinterpretation of pedagogical content knowledge and a new way of conceptualizing the notion of specialization (Scheiner, et al., 2017). Drawing on Shulman (1986), we consider two extensive areas of knowledge.

In the first instance, we consider the knowledge possessed by a mathematics teacher in terms of a scientific discipline within an educational context –the domain of *Mathematical Knowledge* (*MK*). We broaden the idea of *Subject Matter Knowledge* (Shulman, 1986), in that we consider characteristics of mathematics as a scientific discipline, and at the same time recognise a differentiation between *Mathematics per se* and *School Mathematics*[1]. The other domain – *Pedagogical Content Knowledge* (*PCK*) – is comprised of the knowledge relating to mathematical content in terms of teaching-learning.

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[1] Dreher, Lindmeier, Heinze, & Niemand (2018) discuss the gap between school knowledge and academic knowledge, and propose SRCK (school-related content knowledge). Unlike MTSK, which finds room for academic knowledge only insofar as it relates to school knowledge, SRCK focuses on the interrelationship between the two types of knowledge (school and academic) in terms of dimensions, such as the curriculum, which MTSK considers within PCK.

We are also aware that the teacher's classroom practice is deeply influenced by what can be loosely termed a philosophy of mathematics, that is a more or less coherent set of conceptions and beliefs (Thompson, 1992) about mathematics, how it is learnt and how it should be taught, which permeate the teacher's knowledge in each of the sub-domains. Therefore, MTSK also includes beliefs about mathematics and about mathematics teaching and learning. These are represented at the centre of the figure to underline the reciprocity between beliefs and knowledge domains.

Our aim is to construct increasingly precise images by which the teacher's practice can be interpreted in the light of those aspects which most influence it, based on the knowledge underlying this practice.

Next, we present and discuss the content of the various sub-domains into which we organise MTSK, giving examples and details of each, and concluding with the categories of analysis for each one. We do not deal with the beliefs domain in this paper (see Flores, & Carrillo, 2014).

Mathematical knowledge

We understand mathematics as a network of systemic knowledge structured according to its own rules. Having a good understanding of this network – the nodes and connections between them–, the rules and features pertaining to the process of creating mathematical knowledge enables the teacher to teach content in a connected fashion and to validate their own and their students' mathematical conjectures. Thus, we divide the teacher's *mathematical knowledge* into three sub-domains: mathematics content itself (*Knowledge of Topics*), the interlinking systems which bind the subject (*Knowledge of the Structure of Mathematics*), and how one proceeds in mathematics (*Knowledge of Practices in Mathematics*).

One of the difficulties mentioned above was that of differentiating between common knowledge and specialised knowledge in the MKT model. This is especially true in the case of secondary and university teachers, where it becomes increasingly difficult to specify what constitutes common knowledge, as items such as the reasoning underlying procedures and notions, habitually considered specialised knowledge, form a part of what the student is expected to know. The MTSK model seeks to overcome this problem by defining the MK sub-domains in terms of mathematics itself (topics, connections, ways of proceeding), such that inclusion of items is independent of the level the teacher is working at.

Knowledge of Topics (KoT)

The term *topic* refers to content items within the definable knowledge areas making up the mathematics syllabus. As a starting point, we referenced the content areas proposed by the NCTM (2000) in its mathematics standards. It is important to note that the topics are specific components within these areas and can vary according to each country's curriculum.

Knowledge of Topics (KoT) describes the *what* and *in what way* the mathematics teacher knows[2] the topics they teach; it implies thoroughgoing knowledge of mathematical content (e.g., concepts, procedures, facts, rules and theorems) and their meanings. It combines the knowledge that the students are expected to learn with a deeper, and maybe more formal and rigorous understanding. Included in this sub-domain of knowledge are: the type of problems the content can be applied to, with their associated contexts and meanings; properties and their underlying principles, definitions and procedures, including connections to items within the same topic; ways of

[2] When writing "the teacher knows" we refer to knowledge a teacher has or may have, with an analytic aim, in which one avoids a prescriptive or evaluative perspective.

representing the contents. For example, a primary teacher's understanding of addition might include being able to attribute to it the significations of combination and change. It might likewise include understanding the foundations of the operation in question, in this case, for example, (formally or informally) recognising addition as a binary operation on a numerical set, with its corresponding properties, such as the fact that the addition acts upon the cardinals and not the set.

We attribute the teacher's knowledge of phenomena or situations organised by a topic to their knowledge of their meanings (Gómez, & Cañadas, 2016). In this way, we consider the teacher's knowledge of models that can be attributed to a topic, these seen as phenomena which can serve to generate mathematical knowledge, amongst them, those which appear in the creation of the concept itself. A simple example is the teacher's knowledge of different contexts associated with the concept of fraction and its meanings. Likewise, included within the teacher's phenomenological knowledge of the topic would be their awareness of its uses and applications.

Another important element within KoT is the knowledge of mathematical properties and their underlying principles, of especial importance to any work on a mathematical item.

In school mathematics, mathematical objects are frequently defined through reference to a series of properties (e.g. an even number is defined as any number that is a multiple of two). The teacher knows definitions and how to choose appropriate sets of properties to characterize mathematical objects (which might include alternatives to the conventional sets). Thus, for example, teachers might ideally know varying definitions of the mathematical object *polygon*, along with the properties brought into play in each. Connected to the notion of definitions, the teacher's knowledge of images and examples of mathematical objects is also included.

Another component of KoT is knowledge of the procedures involved in a topic. This includes knowledge of how to do something (e.g., algorithms, both conventional and alternative), when to do something (the sufficient and necessary conditions to apply an algorithm), why something is done (the principles underlying algorithms), and the characteristics of the resulting object.

One final area which forms part of the teacher's KoT is their knowledge of the different registers in which a topic can be represented [graphic, algebraic, arithmetic, pictographic, through natural language, etc.] (Duval, 1995); as the latter suggests, mathematical vocabulary is also included in KoT.

In summary, KoT (categories in table 1) comprises a thorough-going knowledge of mathematical topics, bringing together knowledge of procedures, definitions and properties, representations and models, as well as contexts, problems and meanings, and to this extent, it recognises the complexity of the mathematical objects that might arise in the classroom.

Knowledge of the Structure of Mathematics (KSM)

The KSM sub-domain describes the teacher's knowledge of connections between mathematical items. There are two different considerations which give rise to connections: temporal considerations, which respond to (non-curricular, but mathematics-related) questions of sequencing, and produce connections associated with an increase in complexity or with simplification; and considerations of demarcation of mathematical objects, which produce inter-conceptual connections.

Our approach to the issue of connections distinguishes between intra-conceptual and inter-conceptual connections. The sub-domain KSM only considers inter-conceptual connections, whilst intra-conceptual connections are considered in KoT (as part of the properties and their foundations) on the grounds that they are located in the proximity of a single concept, and can be regarded as enriched knowledge of a single content item. In the same vein, we exclude from KSM connections to content from other disciplines (included in Ball, and Bass' 2009, HCK with regard to topics); instead, we locate these in KoT, as they correspond to the phenomenology and applications of the topics in question.

On the other hand, KSM acknowledges the temporal connections underlining the generative role of mathematical items in the construction of other items, for which reason they transcend the straightforward curriculum sequence. They, too, can be associated with an increase in complexity or with simplification.

In terms of an increase in complexity, an item is related to later content material, such that the prior, more elementary mathematics is viewed from a subsequent, more advanced vantage point (Klein, 1908) as an aid to future uptake. Conversely, connections associated with simplification recognise the links of the material in question to previous content. In this case, the more advanced mathematics is retrospectively contextualised in the more elementary content on which it builds.

As an example of connections involving an increase in complexity, we can take the case of a preschool teacher comparing objects in terms of size and making a connection with the idea of scale. This connection enables him or her not only to orient the comparison as a necessary step on the way to the notion of magnitude, invoking the logical process of classification, but also to highlight the invariability of the shape and other attributes (with the exception of size), and even, in certain cases, to refer to the (approximate) ratio of proportionality.

An example of connections involving simplification can be found in the simplification of algebraic expressions when they are compared to expressions using natural numbers. This is the case with one of our teachers, who, noting that his pupils were having difficulties with the algebraic expression in Figure 2 below (the result of calculating the second derivative of a function), suggested that they associate it with the simpler expression $\frac{3+\frac{1}{4}}{7}$, thus establishing a link between the two based on the syntactic-algebraic similarity between them.

$$\frac{-\sqrt{(x^2+1)^3}+\frac{(x-1)-3(x^2+1)^2}{\sqrt{(x^2+1)^3}}}{(x^2+1)^3}$$

Figure 2. Algebraic expression obtained by calculating the second derivative of a function.

With respect to inter-conceptual connections, one type, which we call auxiliary connections, concerns the necessary participation of an item in larger processes. An example is the use of equations as an auxiliary element in calculating the roots of a function. The other type, transverse connections, result when different content items have features in common such as is the case with the concepts of limit, derivative, local and global continuity, and integration, which are all connected by the underlying notion of infinity as one of mathematics' big ideas. Auxiliary connections and connections associated with simplification are not made explicit in other models of mathematics teachers' knowledge (categories in table 2).

Knowledge of Practices in Mathematics (KPM)

The term *practice* has been used in various ways by researchers. Perhaps the most frequent is to refer to actions occurring in the teaching-learning process, whereby the expressions *(mathematics) teaching practice*, emphasising the role of the teacher (NCTM, 2014), and *(mathematics) classroom practice*, englobing both teacher and students individually and in interaction (e.g. Franke, Kazemi, & Battey, 2007), enjoy widespread usage.

Another use occurs in the phrase practices in mathematics, in which the object of said practice is mathematics itself. Here the focus is on the workings of mathematics rather than the process of teaching it. We define it as any mathematical activity carried out systematically, which represents a pillar of mathematical creation and which conforms to a logical basis from which rules can be extracted. Amongst many other things, the mathematics teacher's knowledge about this practice includes knowing about demonstrating, justifying, defining, making deductions and inductions, giving examples, and understanding the role of counterexamples. It also includes an understanding of the logic underpinning each of these practices – in short, it concerns what can be termed a syntactic knowledge (Schwab, 1978) of mathematics.

These features mean that Knowledge of Practices in Mathematics is closely related to the notion of mathematical metaknowledge (defined by methods, structures and organization of mathematical knowledge, Robert, & Robinet 1996), and knowledge about mathematics (e.g. Ball, & McDiarmid 1990).

In the MTSK model, Knowledge of Practices in Mathematics focuses specifically on means of production and mathematical functioning, leaving aside structuring and organization as part of Knowledge of the Structure of Mathematics.

The presentation of this sub-domain based on its descriptors follows. A table with the categories is not supplied, as the descriptors' grouping into categories is under study.

KPM can be either general or specific to a topic. General KPM includes knowledge about how mathematics is developed beyond any particular concept (e.g., knowing the meaning of necessary and sufficient conditions). It is the knowledge deployed in performing general mathematical tasks, for example, the type of proof for testing the truth-value of a proposition, along with knowledge of how such a demonstration can be applied, and the different characteristics of definitions (Mamona-Downs, & Downs, 2016). In this respect, the various argumentation practices available (see Stylianides, Bieda, & Morselli, 2016, for a full review of the literature) represent one of the central planks of KPM. In like manner, the sub-domain also encompasses teachers' knowledge of heuristic aids to problem solving and of theory-building practices such as *"seeing connections, sensing structure, and abstracting commonalities*" (Bass, 2017, p. 230). Specific KPM is a particular instance of general KPM associated with the peculiarities of the topic in question, for example, the use of induction to prove a certain property is associated with the manner of proceeding with numerable infinite sets. Likewise we can find here the knowledge concerning the application of heuristic strategies to specific topics, such as the choice of appropriate subsets of the natural numbers (usually applying divisibility criteria) to tackle the proof of a property of this set, which corresponds to the general heuristic dividing the problem into cases.

To this effect, this component is fundamental not only to the teacher's awareness of mathematical reasoning (in general and with respect to specific topics), KPM is also about knowing how to explore and generate new knowledge in mathematics and gives substance to teachers' knowledge, permitting them to manage the mathematical reasoning brought into play by their pupils by accepting, refuting, or refining this as necessary.

Pedagogical Content Knowledge (PCK)

Since the notion was introduced (Shulman, 1986), a great deal of research into PCK has been done, due in large part to its definition as the teacher's knowledge specific to teaching content, and its widely accepted status as the necessary foundation for effective teaching.

The MTSK model recognises the importance of knowledge of mathematical content in terms of teaching and learning. It is the area of teachers' knowledge which most closely concerns classroom practice. However, we consider that PCK represents only part of the knowledge set for teaching, and needs to be complemented by MK. Operating together, they inform and guide the decisions and actions the teacher must take in the course of their teaching.

In our view, the specific focus of PCK is related to mathematics itself. More than being about the intersection between mathematical and general pedagogical knowledge, it is a specific type of knowledge of pedagogy which derives chiefly from mathematics. Hence, we do not include in this sub-domain general pedagogical knowledge applied to mathematical contexts, but rather only that knowledge in which the mathematical content determines the teaching and learning which takes place. It is in this domain that the research literature in mathematics education has a major role as a source of knowledge for teachers.

Like other researchers before us, we specify two sub-domains in PCK, concerned with teaching and learning (Ball et al. 2008), which have been denominated *Knowledge of Mathematics Teaching* (KMT) and *Knowledge of Features of Learning Mathematics* (KFLM), respectively. The contents of these sub-domains, and how these differ from previous models, are described in detail below. The third sub-domain, *Knowledge of Mathematics Learning Standards* (KMLS), also reflects our agreement with Ball et al. (2008) on the importance of the teacher being aware of the curriculum specifications at any particular level. Nevertheless, we see no reason to limit this knowledge to the curriculum. It is an area of knowledge which enables the teacher to be critical and reflective in considering what the student should learn, and what focus should be taken, at any particular level, or period of development.

Knowledge of Features of Learning Mathematics (KFLM)

This sub-domain encompasses knowledge associated with features inherent to learning mathematics, placing the focus on mathematical content (as the object of learning) rather than on the learner. The main sources of teachers' knowledge within this sub-domain tend to be their own experience built up over time along with research results in Mathematics Education.

KFLM refers to the need for the teacher to be aware of how students think and construct knowledge when tackling mathematical activities and tasks. It includes understanding the process pupils must go through to get to grips with different content items, and the features peculiar to each item which might offer learning advantages or, conversely, present difficulties. As such, the sub-domain takes account of the teacher's knowledge about their students' manner of reasoning and proceeding in mathematics (in particular, their errors, areas of difficulty and misconceptions), which informs his or her interpretation of their output (e.g. Fernández, Callejo, & Márquez, 2014, in the case of quotitive division).

KFLM incorporates knowledge of learning styles and different ways of perceiving the traits inherent in certain content. Along the same lines, the sub-domain includes theories, both personal and institutionalised, of students' cognitive development with respect to both mathematics in general and specific content, such as (in general terms) the APOS theory of learning (Arnon et al., 2014), or (with respect to specific content) grounded knowledge about learning calculus.

More specifically, the sub-domain includes awareness of where students have difficulties, and conversely where they show strengths, both in general and with respect to specific content. For example, a teacher might know that learners tend to mistake 'prove' for 'exemplify', or that they often use what they are setting out to demonstrate as an argument in the demonstration itself. Or in terms of a specific area, a primary teacher might be aware, for example, that pupils tend to be more familiar with situations involving sharing items out equally than with those involving grouping items together, and so use the former as a way into the topic of division, rather than the latter, where the association with division is weaker.

The range of knowledge comprising KFLM also includes the procedures and strategies –whether conventional or unconventional – that students use to do mathematics, as well as the terminology used to talk about specific contents, in short, the different ways in which pupils interact with mathematical content.

The final element of KFLM concerns the emotional aspects of learning mathematics (Hannula, 2006). At one extreme, this involves awareness of, for example, mathematics anxiety (Maloney, Schaeffer, & Beilock, 2013), but it includes, too, such everyday things as what motivates the students, their interests and expectations of mathematics (both in general and in terms of specific areas), and manifests itself, for example, in the choice of registers of representation when setting problems for a particular topic (see table 3).

Knowledge of Mathematics Teaching (KMT)

As in the case of KFLM, this sub-domain concerns knowledge intrinsically bound up with content, to the exclusion of aspects of general pedagogical knowledge. In common with the other sub-domains in PCK, this knowledge might be based on theories drawn from the research literature into mathematics education, or on teachers' personal experience and reflection on their practice.

In general terms, the sub-domain concerns theoretical knowledge (both personal and institutional) specific to mathematics teaching, such as Brousseau's theory of didactical situations (Brousseau, 1986), which can be applied to the design of learning opportunities.

In terms of specific content, it involves awareness of the potential of activities, strategies and techniques for teaching specific mathematical content, along with any potential limitations and obstacles which might arise. Also included is knowledge of resources and teaching materials, including textbooks, manipulatives, technological resources, interactive whiteboards, and so on. It should be noted that this knowledge goes beyond mere awareness of these resources and how they are used, to encompass critical evaluation of how they can enhance teaching a particular item, and the limitations involved. In the case of a geoboard, for example, this might include being aware that if a rectangular geoboard is used for classifying triangles, then equilateral triangles cannot be obtained.

Finally, there is knowledge of different ways of representing specific content (whether through metaphors, situations or explanations). As an example we can take the metaphor of *borrowing* as an aid to comprehending the American method of subtraction (and at a deeper level, potential difficulties and alternative explanations such as *regrouping* – Ma, 1999).

This kind of knowledge stems from many sources – research publications, curriculum specifications, and the teacher's own classroom experience and formative legacy (see table 4).

Knowledge of Mathematics Learning Standards (KMLS)

By learning standard we mean any instrument designed to measure students' level of ability in understanding, constructing and using mathematics, and which can be applied at any specific stage of schooling. The notions underpinning this measure can be constructed by the teacher drawing on various sources, chief amongst which, and typically demarcating their work, are the curriculum specifications (Santos, & Cai, 2016). Other sources might include non-official curriculum documents (for example NCTM 2000, or curriculum specifications from other countries) and research literature. A simple example of a learning standard is the learning objectives for third year primary pupils with regard to classifying flat shapes.

Also located within KMLS is the knowledge of the mathematical contents to be taught at any particular level. This knowledge is acquired by the teacher from the relevant curriculum specifications or by abstracting the specific abilities which need to be worked on at any particular moment. For example, the NCTM (2000) standards state that students should acquire the ability to explore similarity and congruence between third and fifth grade. The knowledge brought to bear by the teacher in deciding what topics to *use* in developing this ability pertains to KMLS.

Also of relevance to this sub-domain is the question of sequencing topics. The demands placed upon the pupils in terms of the knowledge and skills required for any particular task leads the teacher to locate topics both retrospectively, in terms of previously acquired knowledge, and prospectively, according to the knowledge that will need to be acquired to tackle later topics. Hence, it may be that (like in Spain) multiplication is conceptually glossed as *the number of times* in grades 1 and 2, but in grades 3 and 4 is treated as *abbreviated addition*, in rectangular format and with combinatory problems. This kind of knowledge – here involving the sequencing of conceptual and procedural levels of multiplication – represents a typical entry for this category.

In summary (see table 5), this sub-domain includes the teacher's knowledge of everything the student should, or is able to, achieve at a particular level, in combination with what the student has previously studied and the specifications for subsequent levels.

Final remarks

In this paper we have described an analytical model designed to be used as a tool for approaching the complexity of teachers' knowledge. Its contribution to the discipline lies in its refining of the different facets of knowledge deployed by mathematics teachers in the course of their work. The analytical perspective is built on a refocused approach to the notion of specialization, in which our new conceptualisation respects Shulman's (1986) original dichotomy and refines the content of PCK and (especially) SMK in a manner which is intrinsic to the discipline itself (and considers a third domain, beliefs, in interaction with these).

MTSK considers only the specialised components of mathematics teachers' knowledge, that is, their knowledge of mathematics as the object of teaching and learning. Consequently, MTSK has no interest in other types of knowledge shared with teachers of other subjects (such as general pedagogical knowledge), nor in knowing whether some elements of knowledge are shared with other professionals who use mathematics (for example, the knowledge of derivatives, which is of particular interest to engineers). The key point for MTSK is that the element of knowledge in question is significant to the mathematics teacher and that it is mathematics which conditions said knowledge. Thus, for example, the model does not include awareness of how group dynamics might be exploited, but does an understanding of the features of a geoboard, as it concerns the types of triangles that can be represented. In this way, each of the sub-domains comprising MTSK emerge from mathematics itself, or from those components of teaching and learning which are specific to it.

Hence, with respect to the domain of mathematical knowledge, MTSK concerns itself only with phenomena involving mathematical concepts and procedures. These might include different registers of representation, the processes by which they are defined and transformed, the mathematical structure or structures in which such concepts are embedded and connected to other mathematical concepts, and syntax. In like fashion, the domain of pedagogical content knowledge is founded on situations in which mathematics is seen as the object of teaching and learning.

MTSK is not limited to providing a snapshot of the knowledge a teacher deployed in a particular instance of their teaching, but also allows us to reflect on other kinds of knowledge which might have led to different outcomes in that situation. In addition to opening up discussion of the kind of knowledge that best serves education, this aspect of the model provides us with concrete teaching experiences that can be used in teacher training programmes. Indeed, our teacher-training syllabus is now structured around the sub-domains of the model. In a recent review of the university curriculum, the Mathematics Education Department followed the structure of the model to ensure that all the mathematics that a primary should know was included in the syllabus, and to organise course content according to the MTSK organisers. By this means it is intended that future teachers are equipped to construct as many elements of specialized knowledge as possible.

By way of example, the scheme for dealing with plane shapes is given below:

- Conceptual aspects of plane shapes, such as the concept of polygon, their main features and their definition (KoT – definitions, properties and foundations).
- Classification of plane shapes (KoT – procedures).

- Classifications appropriate to different stages in the primary syllabus (KMLS - expected level of conceptual or procedural development).
- Representations of plane shapes, mathematical notation and the appropriate geometrical vocabulary (KoT – representations).
- Geometrical vocabulary as an indicator of learning (KFLM - ways pupils interact with mathematical content).
- Aspects of mathematics practice (KPM), such as definitions and their construction, the role of examples and counterexamples in constructing definitions, deductive and inductive reasoning, hypothesis forming, formally checking and demonstrating (associated at primary level with KFLM - ways pupils interact with mathematical content)
- Learning theories (KFLM), such as the image and definition of a geometric concept, or the Van Hiele levels for describing geometric learning.
- The Van Hiele teaching phases (KMT – education theories).
- Common errors associated with learning about plane shapes, such as the relationship between area and perimeter (the larger the area, the larger the perimeter, and viceversa) (KFLM - Strengths and weaknesses in learning mathematics).
- Learning resources, such as dot grid paper and its associated geometry, and the treatment of geometry in textbooks (KMT – resources).
- Plane shapes in everyday situations (KoT - Phenomenology and applications).
- Reflections about other non-Euclidean geometries, such as projective and analytical geometry (KSM - connections based on increased complexity).

The potential value of an analytical model such as MTSK lies in its contribution to helping those involved in the discipline to achieve a fine-grained analysis of such aspects as ought to be highlighted. In this sense, the conceptualization and categorization of MTSK, whilst recognising the holistic nature of teachers' knowledge, allows attention to be given to specific aspects which require study. In the same spirit, the process of developing the categories is open to both the incorporation of new elements emerging from further studies, and to more finely defined categories which meet specific contextual aspects. At the moment, several studies are being carried out into specific classroom-based mathematical practices with a view to refining the definitions of categories in the KPM subdomain. One such project currently in progress concerns how teachers can promote the construction of definitions of geometric objects (such as polygons). Consideration of the kind of knowledge displayed by the teachers in these sessions has led us to draw up indicators for the practice of defining within KPM and other subdomains connected with the notion of definition as an object of mathematical teaching and learning.

We assume that the teacher's knowledge is a resource they draw on in the course of their day-to-day professional life (Schoenfeld, 2010). In this respect, analysis of the knowledge thus deployed (e.g. in evaluation, planning, giving examples, responding to students, and reflecting on one's practice) will enable a better appreciation of the complex pattern of relations between elements of knowledge across the various sub-domains. We also intend to explore the affective domain (currently only partly represented in the MTSK model within beliefs) and its interconnections with teachers' specialised knowledge, thus addressing one of the limitations levelled at studies which seek to account for teachers' practice through careful description of their knowledge (Neubrand, 2018).

Regarding uses of the model, beyond the purely analytical, the SIDM group has used it to analyse teachers' knowledge in the contexts of Primary, Secondary and University Education, and for considering different mathematical concepts (fractions, probability, geometry, infinity, linear algebra and functions). Based on the results of these studies, we propose to design assessment instruments for measuring teachers' mathematical knowledge of specific topics (as in Martinovic, & Manizade, 2018). Future projects include an examination of the content of primary teacher training courses in respect of the width and variety of knowledge these aim to tackle. In addition, we are developing studies focusing on the relations between different sub-domains. As an analytical tool, MTSK enables mathematics teachers' knowledge to be studied in great detail, but it must be remembered that this knowledge is not comprised of isolated items, but rather a complex network of relations. Our current studies, supported by evidence of these relations, are leading us to a deeper understanding of this complexity.

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Procedures	How to do something?
	When to do something?
	Why something is done this way?
	Characteristics of the result
Definitions, properties and foundations	
Registers of representation	
Phenomenology and applications	

Table1: Categories of Knowledge of Topics

Connections based on simplification
Connections based on increased complexity
Auxiliary connections
Transverse connections

Table 2: Categories of Knowledge of the Structure of Mathematics

Theories of mathematical learning
Strengths and weaknesses in learning mathematics
Ways pupils interact with mathematical content
Emotional aspects of learning mathematics

Table 3: Categories of Knowledge of Features of Learning Mathematics

Theories of mathematics teaching
Teaching resources (physical and digital)
Strategies, techniques, tasks, and examples

Table 4: Categories of Knowledge of Mathematics Teaching

Expected learning outcomes
Expected level of conceptual or procedural development
Sequencing of topics

Table 5: Categories of Knowledge of Mathematics Learning Standards

Figure 1. The *Mathematics Teacher's Specialised Knowledge* model