No-hidden-variables proof for two spin-$\frac{1}{2}$ particles preselected and postselected in unentangled states

Adán Cabello*
Departamento de Física Teórica, Universidad Complutense, 28040 Madrid, Spain
and Departamento de Física Aplicada, Universidad de Sevilla, 41012 Sevilla, Spain

(Received 4 November 1996)

It is a well-known fact that all the statistical predictions of quantum mechanics on the state of any physical system represented by a two-dimensional Hilbert space can always be duplicated by a noncontextual hidden-variables model. In this paper, I show that, in some cases, when we consider an additional independent (unentangled) two-dimensional system, the quantum description of the resulting composite system cannot be reproduced using noncontextual hidden variables. In particular, a no-hidden-variables proof is presented for two individual spin-$\frac{1}{2}$ particles preselected in an uncorrelated state $|A\rangle\otimes|B\rangle$ and postselected in another uncorrelated state $|a\rangle\otimes|B\rangle$, $|B\rangle$ being the same state for the second particle in both preselection and postselection. [S1050-2947(97)02006-4]

PACS number(s): 03.65.Bz

Noncontextual hidden-variables (NCHV) models that are capable of reproducing all statistical predictions of quantum mechanics (QM) for physical systems described by *two-dimensional* Hilbert spaces do exist [1–5]. Examples of such systems, also known as two-state systems or ''qubits'' [6], include a single spin-$\frac{1}{2}$ particle without translational motion, the polarization of a photon, the relative phase and intensity of a single photon in two arms of an interferometer, or an arbitrary superposition of two atomic states. On the contrary, for physical systems described by Hilbert spaces of dimension greater than 2, a fundamental theorem proved by Gleason [7], Bell [1], and Kochen and Specker [3] excludes NCHV alternatives to QM. On the other hand, Bell's theorem [8] prohibits local hidden variables (a particular type of NCHV) for *composite* systems of two (or more) parts (usually two-state systems) initially prepared in an *entangled* state. However, for systems composed of several uncorrelated (unentangled) two-dimensional subsystems, one might think that NCHV descriptions are possible. In this paper, I show that even in such a case, some quantum inferences cannot be duplicated using a NCHV theory. For this purpose, I present a simple no-go proof for an individual system of two spin-$\frac{1}{2}$ particles preselected and postselected [9] in uncorrelated states. The argument contains both measured (i.e., actual) and nonmeasured (i.e., hypothetical) values of the composite system. The former are the results of separate measurements on each particle in the preselection or postselection processes. The latter are hypothetical values of the whole system that are assumed to be determined (in a NCHV theory) in the time interval between the preselection and postselection, invoking one of the following criteria: (a) they can be predicted with certainty after the preselection; (b) they can be *retrodicted* [9,10] with certainty before the postselection; or (c) they must verify the *sum rule* [11] for the results of any measurement of an orthogonal resolution of the identity. The joint use of the preselection and postselection and of these three criteria to infer the values of some properties of the system is legitimate in the context of a NCHV theory in which quantum observables are assumed to have preexisting values revealed by the act of measurement (although of course not in QM itself).

The proof runs as follows. Consider the following experiment: a single spin-$\frac{1}{2}$ particle is prepared at time $t_1<t$ in the state $|A\rangle$ (for instance, in the eigenstate of the spin component in the z direction with eigenvalue $+1$), and at time $t_2>t$, a measurement is performed and the system is found in a different state $|a\rangle$. At time t we have a quantum system both preselected in the state $|A\rangle$ and postselected in the state $|a\rangle$ [9]. Whatever $|A\rangle$ and $|a\rangle$, there exists a trivial NCHV description (compatible with QM) for this individual preselected and postselected system [12]. Now consider a second spin-$\frac{1}{2}$ particle independently prepared at time $t_1<t$ in the state $|B\rangle$; at time $t_2>t$, a measurement confirms that the second particle is still in the state $|B\rangle$ (for simplicity's sake we suppose the free Hamiltonian in t to be zero). Now let us see the quantum description of the composite system. We shall use greek letters for the states of the composite system and latin letters for the states of each particle. At time $t_1<t$, the system is prepared in the uncorrelated quantum state

$$|\psi_1\rangle=|A\rangle\otimes|B\rangle, \tag{1}$$

and at time $t_2>t$, a measurement is performed and the system is found in the uncorrelated state

$$|\psi_2\rangle=|a\rangle\otimes|B\rangle. \tag{2}$$

Therefore, at time t we have an individual system both preselected in the state $|\psi_1\rangle$ and postselected in the state $|\psi_2\rangle$. $|A\rangle$ and $|a\rangle$ are two different spin states for the first particle and $|B\rangle$ is the same state for the second particle in both preselection and postselection. In particular, for our argument we suppose

$$|a\rangle=\tfrac{1}{3}(|A\rangle-\sqrt{8}|A^{\perp}\rangle), \tag{3}$$

*Electronic address: adancab@eucmax.sim.ucm.es

$$|a^{\perp}\rangle = \tfrac{1}{3}(\sqrt{8}|A\rangle + |A^{\perp}\rangle), \tag{4}$$

where $\{|A\rangle,|A^{\perp}\rangle\}$ and $\{|a\rangle,|a^{\perp}\rangle\}$ are two orthonormal bases for the states of the first particle. With election (3) the probability of postselecting $|\psi_2\rangle$ when preselecting $|\psi_1\rangle$ is

$$|\langle\psi_2|\psi_1\rangle|^2 = \tfrac{1}{9}. \tag{5}$$

Consider the three physical quantities represented by the projection operators $P_{\alpha} = |\alpha\rangle\langle\alpha|$, $P_{\beta+} = |\beta_+\rangle\langle\beta_+|$, and $P_{\beta-} = |\beta_-\rangle\langle\beta_-|$, where

$$|\alpha\rangle = |A^{\perp}\rangle \otimes |B^{\perp}\rangle, \tag{6}$$

$$|\beta_{\pm}\rangle = \tfrac{1}{2}(|A\rangle \otimes |B^{\perp}\rangle \pm \sqrt{3}|A^{\perp}\rangle \otimes |B\rangle). \tag{7}$$

The state $|\psi_1\rangle$ is an eigenstate of P_{α}, $P_{\beta+}$, and $P_{\beta-}$ with eigenvalue zero; therefore, a measurement of any of these projectors will give with certainty the value zero. Since we can predict with certainty the result of measuring P_{α}, $P_{\beta+}$, and $P_{\beta-}$ at time t, then, following [13], in a NCHV theory at time t there exist three *elements of reality* [14] corresponding to the three physical quantities P_{α}, $P_{\beta+}$, and $P_{\beta-}$ and having a value equal to the predicted measurement result, zero in all three cases. We will designate these elements of reality as

$$v[P_{\alpha}(t)] = v[P_{\beta+}(t)] = v[P_{\beta-}(t)] = 0. \tag{8}$$

Consider now the physical quantities represented by the projectors $P_{\gamma+} = |\gamma_+\rangle\langle\gamma_+|$, $P_{\gamma-} = |\gamma_-\rangle\langle\gamma_-|$, where

$$|\gamma_{\pm}\rangle = \frac{1}{2\sqrt{3}}(\sqrt{8}|A\rangle \otimes |B\rangle + |A^{\perp}\rangle \otimes |B\rangle \mp \sqrt{3}|A\rangle \otimes |B^{\perp}\rangle) \tag{9}$$

or, in the basis (3) and (4),

$$|\gamma_{\pm}\rangle = \tfrac{1}{6}[3\sqrt{3}|a^{\perp}\rangle \otimes |B\rangle \mp (|a\rangle \otimes |B^{\perp}\rangle + \sqrt{8}|a^{\perp}\rangle \otimes |B^{\perp}\rangle)]. \tag{10}$$

Since $|\psi_2\rangle$ is an eigenstate of $P_{\gamma+}$ and $P_{\gamma-}$ with zero eigenvalues, then we can infer (retrodict [10]), with certainty, the result of measuring $P_{\gamma+}$ and $P_{\gamma-}$ at time t; therefore, following an extended definition for elements of reality proposed by Vaidman [10] (consisting of the change of ''predict'' to ''infer'' in Redhead's sufficient condition for elements of reality [13]), at the time t, there exist two more elements of reality corresponding to these physical quantities and having a value equal to the inferred measurement result; that is,

$$v[P_{\gamma+}(t)] = v[P_{\gamma-}(t)] = 0. \tag{11}$$

Finally, consider the physical quantities $P_{\delta+} = |\delta_+\rangle\langle\delta_+|$, $P_{\delta-} = |\delta_-\rangle\langle\delta_-|$, where

$$|\delta_{\pm}\rangle = \frac{1}{2\sqrt{3}}[\sqrt{6}|A\rangle \otimes |B^{\perp}\rangle \pm (2|A\rangle \otimes |B\rangle - \sqrt{2}|A^{\perp}\rangle \otimes |B\rangle)]. \tag{12}$$

The propositions $P_{\alpha}, P_{\beta+}, P_{\gamma+}, P_{\delta+}$ form a set of compatible observables for the composite system and, therefore, on any individual quantum system, we can measure them jointly without mutual disturbance. In addition, they are mutually orthogonal projectors and provide a resolution of the identity

$$P_{\alpha} + P_{\beta+} + P_{\gamma+} + P_{\delta+} = I. \tag{13}$$

Therefore, in any joint measurement of $P_{\alpha}, P_{\beta+}, P_{\gamma+}, P_{\delta+}$ *in any state*, the results must be one 1 and three zeros. This allows us to use a particular case of the *sum rule* [11]: at time t the values in a NCHV theory must satisfy

$$v[P_{\alpha}(t)] + v[P_{\beta+}(t)] + v[P_{\gamma+}(t)] + v[P_{\delta+}(t)] = 1. \tag{14}$$

Since in our preselected and postselected individual system $v[P_{\alpha}(t)] = v[P_{\beta+}(t)] = v[P_{\gamma+}(t)] = 0$, we are forced to conclude that $v[P_{\delta+}(t)] = 1$. Similarly, since $P_{\alpha}, P_{\beta-}, P_{\gamma-}, P_{\delta-}$ form another set of compatible observables and a resolution of the identity, a completely analogous reasoning leads us to conclude that $v[P_{\delta-}(t)] = 1$. But $P_{\delta+}$ and $P_{\delta-}$ are commutative and orthogonal projections representing compatible and mutually exclusive physical propositions, so the results of any joint measurement of $P_{\delta+}$ and $P_{\delta-}$ can never *both* be 1. So we have reached a contradiction between QM and NCHV for a system preselected and postselected in uncorrelated states.

The reason why NCHV models compatible with QM are impossible for this composite system (although they exist for each particle) is because the dimension of the whole quantum system is 4 and, therefore, the Gleason-Bell-Kochen-Specker theorem applies. In particular, in our argument, NCHV theory must assign definite values to some propositions that cannot be measured by *local* measurements on each particle but only by nonlocal measurements on both particles (in our example, these propositions are $P_{\beta+}$, $P_{\gamma+}, P_{\delta+}, P_{\beta-}, P_{\gamma-}, P_{\delta-}$). The particular election of propositions involved in the argument has been made in order to achieve the maximum probability (5) for the preselection and postselection process, preserving the relations of orthogonality among states and projectors necessary for the proof.

The same structure of orthogonality relations is behind Hardy's proof of Bell's theorem [15] and also appears in some recent proofs of the Gleason-Bell-Kochen-Specker theorem [16,17]. In Hardy's example, an individual system is preselected in an entangled state $|\eta_1\rangle$, which is orthogonal to three unentangled states

$$|\hat{\alpha}\rangle = |A\rangle \otimes |B\rangle, \tag{15}$$

$$|\hat{\beta}_+\rangle = |a\rangle \otimes |B^{\perp}\rangle, \tag{16}$$

$$|\hat{\beta}_-\rangle = |A^{\perp}\rangle \otimes |b\rangle, \tag{17}$$

where $\{|A\rangle,|A^{\perp}\rangle\}$ and $\{|a\rangle,|a^{\perp}\rangle\}$ are two orthonormal bases for the states of the first particle and $\{|B\rangle,|B^{\perp}\rangle\}$, and $\{|b\rangle, |b^{\perp}\rangle\}$ are two orthonormal bases for the states of the second particle. The system is also postselected in the unentangled state

$$|\eta_2\rangle = |a\rangle \otimes |b\rangle, \tag{18}$$

which is orthogonal to

$$|\hat{\gamma}_{+}\rangle = |a^{\perp}\rangle \otimes |B^{\perp}\rangle, \tag{19}$$

$$|\hat{\gamma}_{-}\rangle = |A^{\perp}\rangle \otimes |b^{\perp}\rangle. \tag{20}$$

Considering also the states

$$|\hat{\delta}_{+}\rangle = |A^{\perp}\rangle \otimes |B\rangle, \tag{21}$$

$$|\hat{\delta}_{-}\rangle = |A\rangle \otimes |B^{\perp}\rangle, \tag{22}$$

we have two orthogonal resolutions of the identity: $\{P_{\hat{\alpha}}, P_{\hat{\beta}+}, P_{\hat{\gamma}+}, P_{\hat{\delta}+}\}$ and $\{P_{\hat{\alpha}}, P_{\hat{\beta}-}, P_{\hat{\gamma}-}, P_{\hat{\delta}-}\}$. Therefore we have the same relations of orthogonality as in the previous example. The connection between these and Hardy's proof is explained in [17]. For Hardy's example the maximum probability for the preselection and postselection process is [15]

$$|\langle \eta_2 | \eta_1 \rangle|^2 = \left(\frac{\sqrt{5}-1}{2} \right)^5, \tag{23}$$

which is smaller than Eq. (5). On the other hand, in Hardy's example all states (except the preselected) are unentangled, so it also works as a no-local-hidden-variables proof.

All along in this paper it has been assumed that every projector on a Hilbert space represents a physical proposition; i.e., that there exists an experimental setup for measuring it. Several results suggest that there is no problem in designing such setups, since any discrete unitary operator admits an experimental realization in terms of optical devices [18] or generalized Stern-Gerlach experiments [19]. Therefore, each of the quantum inferences used in the argument (predictions, retrodictions, and the sum rule for an orthogonal resolution of the identity) can be experimentally tested (although not all of them on the same individual system).

In summary, a contradiction between QM and NCHV models can be found, even for a system composed of two uncorrelated parts, each of them described by two-dimensional Hilbert spaces.

I would like to acknowledge Guillermo García Alcaine and Asher Peres for their many helpful comments.

[1] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[2] D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 453 (1966).
[3] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).
[4] J. F. Clauser, Am. J. Phys. **39**, 1095 (1971).
[5] F. Selleri, *Quantum Paradoxes and Physical Reality* (Kluwer, Dordrecht, 1990), p. 48.
[6] B. Schumacher, Phys. Rev. A **51**, 2738 (1995).
[7] A. M. Gleason, J. Math. Mech. **6**, 885 (1957).
[8] J. S. Bell, Physics **1**, 195 (1964).
[9] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).
[10] L. Vaidman, Phys. Rev. Lett. **70**, 3369 (1993).
[11] M. L. G. Redhead, *Incompleteness, Nonlocality, and Realism* (Oxford University Press, New York, 1987), p. 121.
[12] M. L. G. Redhead, *Incompleteness, Nonlocality, and Realism* (Ref. [11]), p. 124.
[13] M. L. G. Redhead, *Incompleteness, Nonlocality, and Realism* (Ref. [11]), p. 72.
[14] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[15] L. Hardy, Phys. Rev. Lett. **68**, 2981 (1992).
[16] A. Cabello and G. García Alcaine, J. Phys. A **29**, 1025 (1996).
[17] A. Cabello, J. M. Estebaranz, and G. García Alcaine, Phys. Lett. A **212**, 183 (1996).
[18] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).
[19] A. R. Swift and R. Wright, J. Math. Phys. **21**, 77 (1980).