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# Berge equilibria and the equilibria of the altruistic game

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## Abstract

Berge's notion of equilibrium represents a complementary alternative to the Nash equilibrium when modeling socioeconomic behavior and human interactions. While the notion of Nash equilibrium is based on self-interest, as players seek to maximize their own payoffs given the action of the other players, the idea behind Berge equilibrium is mutual support, as given the action of one of the players, all others select their actions looking for her best interest. However, because of the demanding conditions involved, the existence of Berge equilibria is rarely guaranteed. In this paper, we propose vector-valued normal-form games as a unified framework in which to study and extend the concept of Berge equilibrium. Based on the equilibria of the so-called altruistic game, we introduce new equilibrium concepts which constitute different relaxations of Berge's notion, although they still retain the underlying idea of mutual support. We establish the links between these new equilibria, Nash equilibrium, Berge equilibrium, and other related concepts already existing in the literature. Our approach has the advantage that it permits the incorporation of preference information to identify the equilibria which are consistent with different altruistic attitudes of the players.

**Keywords** Berge equilibria · Nash equilibria · Altruistic equilibria · Preference information

**Mathematics Subject Classification** 91A06 · 91A10 · 90C29

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## 1 Introduction

The initial intuition of the solution concept of Berge equilibrium was included in a book published in 1957 by the mathematician Claude Berge (1957). In this book, the concept of equilibrium of a coalition with respect to another coalition was presented. Applying this concept from an individualistic perspective, Zhukovskiy (1985) formally introduced the so-called Berge–Zhukovskiy equilibrium. Subsequently, Vaisman (1994) and Vaisman and Zhukovskiy (1994) carried out a more detailed study of the Berge equilibrium and its properties, and established a rigorous mathematical description of this concept.

The idea of Berge equilibrium is in a sense opposite to the idea of Nash equilibrium (Nash 1951). While the former can be considered as an implication of the altruistic motivations of the players, in the latter, players have incentives to adopt self-oriented behaviors.

The literature on Berge equilibrium has increased in the last years. The research is mainly devoted to properties of this concept (Colman et al. 2011), existence theorems and procedures of computation (Larbani and Nessah 2008; Deghdak and Florenzano 2011; Musy et al. 2012; Courtois et al. 2017; Enkhbat and Sukhee 2021), as well as relationships between Berge equilibrium and other equilibrium concepts (Pottier and Nessah 2014; Courtois et al. 2015; Crettez 2017). A review of publications on Berge equilibrium can be found in Larbani and Zhukovskii (2017) and in Salukvadze and Zhukovskiy (2020). Applications of the Berge equilibrium in economic situations where players are mutually supportive are studied in Nessah and Larbani (2014), and in Kudryavtsev et al. (2019).

The altruistic behavior of the players has been analyzed from different perspectives providing several notions of equilibrium that can be considered as extensions of the concept of Berge equilibrium. Corley (2015) introduced the dual equilibrium which constitutes a generalization of the Berge equilibrium from pure to mixed strategies and provided sufficient conditions for its existence. Schouten et al. (2019) proposed the unilateral support equilibrium which retains the supportive behavior of the Berge equilibrium, but every player is supported by every other player individually. Crettez and Nessah (2020) established conditions for the existence of unilateral support equilibrium in n-person games. Previously, Colman et al. (2011) had shown that the condition underlying unilateral support equilibria is a necessary condition for an action profile to be a Berge equilibrium. Recently, Safaty and Abdou (2021) have used a tensor approach to provide the extension of pure unilateral support equilibrium to mixed unilateral support equilibrium when the set of strategies of the players are discrete and finite. In Salukvadze and Zhukovskiy (2020, p. 228), by combining the selfishness of Nash equilibrium, the altruism of Berge equilibrium, and the Pareto maximality, the concept of hybrid equilibrium is presented. This concept constitutes a refinement of the Berge equilibrium. They show that, for games with bounded convex and closed strategy sets and continuous payoff functions, the existence of an hybrid equilibrium in mixed strategies is guaranteed.

In this paper, we address the extension of the concept of Berge equilibrium in the general framework of vector-valued games. In a first stage, we consider a joint vector-valued game in which every player takes into account the utility functions of all of them and introduce the concepts of joint equilibrium, and ideal joint equilibrium for scalar games. Joint ideal equilibria combine the properties of Nash equilibria and the idea of mutual support underlying Berge equilibria. Thus, they can be considered a relaxation of the concept of Berge–Nash equilibria (Abalo and Kostreva 2005). Moreover, the hybrid equilibrium (Salukvadze and Zhukovskiy 2020) can be seen as a refinement of the ideal joint equilibrium that includes a condition of Pareto optimality.

However, the concept of Berge equilibrium suggests a more purely altruistic attitude in the sense of desiring to improve the situations of the other players while ignoring one's self-interest. We introduce this circumstance into our framework by means of a new vector-valued game, the altruistic game, in which each player only considers the utilities of the other players and ignores her own utility. The equilibria of this vector-valued game permit the introduction and characterization of the concepts of altruistic and ideal altruistic equilibria. Several relationships between these solution concepts and other concepts proposed in the literature are established. The concept of ideal altruistic equilibrium coincides with the concept of unilateral support equilibrium proposed by Schouten et al. (2019). As a consequence, as proven in Crettez and Nessah (2020), for games with more than two players, the concept of ideal altruistic equilibria is an extension of the concept of strong Berge equilibria (Berge 1957).

The generalizations of the Berge equilibrium proposed with our approach have the advantage that for a wide class of games, they can be characterized as the equilibria of weighted scalar games. If the weights are interpreted as the relative importance that each player assigns to the utilities of the other players, then preference information can be used to identify the equilibria that are in accordance with situations in which the agents exhibit altruistic attitudes that do not necessarily consider all the players equally.

The rest of the paper is organized as follows. In Sect. 2, we introduce notations and definitions. In Sect. 3, the equilibria of a vector-valued game in which every player considers her own utility together with the utilities of the other players are introduced, and the links between these equilibria and the Nash and the Berge equilibria are established. Section 4 is devoted to the concept of altruistic equilibria, in which the players only care about the utilities of the other players. In Sect. 5, we relate altruistic equilibria with the equilibria of weighted games and describe the procedure to include preference information in the identification of ideal altruistic equilibria. Concluding remarks are included in Sect. 6. The Appendix contains the proofs of the results and a detailed analysis of the illustrative examples.

## 2 Preliminaries

The notation of vector inequalities is the following: let $x, y \in \mathrm{IR}^s$, $x > y$ means $x_j > y_j$ for all $j$; $x \geq y$ means $x_j \geq y_j$ for all $j$, with $x \neq y$; and $x \geqq y$ means $x_j \geq y_j$ for all $j$. We denote $\Delta^s = \{y \in \mathrm{IR}^s : y \geq 0, \sum_{j=1}^{s} y_j = 1\}$ and $\Delta_+^s = \{y \in \mathrm{IR}^s : y > 0, \sum_{j=1}^{s} y_j = 1\}$.

A non-cooperative normal-form game is represented by $G = \{(X_i, u_i)_{i \in N}\}$, where $N = \{1, \dots, n\}$ is the set of players, $X_i$ is the set of strategies or actions that player $i \in N$ can adopt and the mapping $u_i : \times_{i \in N} X_i \to \mathrm{IR}$, is the individual utility function of player $i$. An action profile, $x = (x_1, \dots, x_n)$, with $x_i \in X_i$, for a game $G$ can be written as $x = (x_i, x_{-i})$, where $x_i$ is a strategy of player $i$, and $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ stands for the strategy combination of all players except player $i$. We also denote $X_{-i} = \times_{j \neq i} X_j$.

The concept of Nash equilibrium (Nash 1951) is based on the extended assumption in which the players only take care of their own utility, while in a Berge equilibrium (Berge 1957), the players show an altruistic behavior, by seeking to improve the utilities of all other players, in the sense in which for each player $i$, all the other players jointly do their best to maximize the payoff of player $i$.

**Definition 2.1** Let $G = \{(X_i, u_i)_{i \in N}\}$ be a non-cooperative normal-form game.

a) An action profile $x^*$ is a Nash equilibrium for $G$ if for all $i \in N$, $u_i(x^*) \geq u_i(x_i, x^*_{-i})$ for all $x_i \in X_i$.
b) An action profile $x^*$ is a Berge equilibrium[1] for $G$ if for all $i \in N$, $u_i(x^*) \geq u_i(x^*_i, x_{-i})$ for all $x_{-i} \in X_{-i}$.

The set of Nash equilibria and the set of Berge equilibria for the game $G$ are denoted by $N(G)$ and $B(G)$, respectively.

In a Nash equilibrium, if player $i$ deviates from her strategy, she does not improve her payoff. In a Berge equilibrium, given a player $i$, if one or more players other than $i$ deviate from their strategies, then the payoff of the player $i$ is not improved.

In terms of best responses: by denoting, $R^j_{x_i}(x_{-i}) = arg\,max_{x_i} u_j(x_i, x_{-i})$, and by $R^j_{x_{-i}}(x_i) = arg\,max_{x_{-i}} u_j(x_i, x_{-i})$, $x^*$ is a Nash equilibrium if for all $i \in N$, $x^*_i \in R^i_{x_i}(x^*_{-i})$, and $x^*$ is a Berge equilibrium if for all $i \in N$, $x^*_{-i} \in R^i_{x_{-i}}(x^*_i)$. That is, a Nash equilibrium is such that the strategy of every player is the best response to the strategies of all the others, and a Berge equilibrium is an action profile, such that for each $i \in N$, the strategies of all the other players are the best responses to the strategy of player $i$.

For general games, there is no link between the existence of Nash equilibria and Berge equilibria. However, for the case of two-person games, it follows from the definitions that $x^*$ is a Berge equilibrium for the game $G = \{(X_i, u_i)_{i=1,2}\}$, if and only if $x^*$ is a Nash equilibrium for the game $G' = \{(X_i, \bar{u}_i)_{i=1,2}\}$, where $\bar{u}_i = u_j$ for $i \neq j$.

[1] This notion of Berge equilibrium was introduced by Zhukovskiy (1985).

A vector-valued normal-form game is represented by $\mathcal{G} = \{(X_i, v_i)_{i\in N}\}$, with $N = \{1, \ldots, n\}$ the set of players and $X_i$ the set of possible strategies of player $i$. The payoff of each player is multidimensional, $v_i : \times_{i\in N} X_i \to \mathrm{IR}^{s_i}$ with $v_i := (v_i^j)_{j=1,\ldots,s_i}$.

The natural extensions of the concept of Nash equilibrium to general vector-valued games were introduced by Shapley (1959) for finite two-person zero-sum games, and by Voorneveld et al. (2000), who proposed the concept of ideal equilibrium. Formally, for general games with vector-valued utilities:

**Definition 2.2** Let $\mathcal{G} = \{(X_i, v_i)_{i\in N}\}$ be a vector-valued game.

(a) An action profile $x^*$ is a weak equilibrium for $\mathcal{G}$ if $\nexists\, i \in N$ with $x_i \in X_i$, such that $v_i(x_i, x^*_{-i}) > v_i(x^*)$.
(b) An action profile $x^*$ is an equilibrium for $\mathcal{G}$ if $\nexists\, i \in N$ with $x_i \in X_i$, such that $v_i(x_i, x^*_{-i}) \geq v_i(x^*)$.
(c) An action profile $x^*$ is an ideal equilibrium for $\mathcal{G}$ if for all $i \in N$, $v_i(x^*) \geqq v_i(x_i, x^*_{-i})$ for all $x_i \in X_i$.

It is straightforward that the set of equilibria is contained in the set of weak equilibria, and the set of ideal equilibria is contained in the set of equilibria. The first two sets of equilibria are often not very different, in fact, under certain convexity conditions they coincide. A sufficient condition for both to coincide is that for all $i \in N$, the sets of strategies $X_i$ are non-empty convex subsets of a finite-dimensional space and the functions $v_i^j$ are strictly concave in $x_i$ for all $j$.

## 3 A joint vector-valued game

Given a non-cooperative normal-form game, $G = \{(X_i, u_i)_{i\in N}\}$, we define a vector-valued normal-form game represented by $\mathcal{J} = \{(X_i, u)_{i\in N}\}$, where all the players consider the same collective vector-valued utility function, $u : \times_{i\in N} X_i \to \mathrm{IR}^n$, $u := (u_j)_{j\in N}$. That is, every player considers her own utility, together with the utilities of the other players. It is assumed that the preferences of each player are monotone in the sense that every player prefers (or weakly prefer) greater utilities for all the players. That is, given $u, \bar{u} \in \mathrm{IR}^n$, such that $u \geqq \bar{u}$, then $u$ would be at least as preferred as $\bar{u}$ by all the players. We call this game the *joint game associated with G*.

The following definitions of equilibria for a normal-form game correspond to the notions of equilibria, weak equilibria, and ideal equilibria of the vector-valued joint game.

**Definition 3.1** Let $G = \{(X_i, u_i)_{i\in N}\}$ be a normal-form non-cooperative game.

a) An action profile $x^*$ is a weak joint equilibrium for the game $G$ if it is a weak equilibrium for the game $\mathcal{J}$, that is, $\nexists\, i \in N$ with $x_i \in X_i$, such that $u_j(x_i, x^*_{-i}) > u_j(x^*)$, for all $j \in N$.
b) An action profile $x^*$ is a joint equilibrium for the game $G$ if it is an equilibrium for the game $\mathcal{J}$, that is, $\nexists\, i \in N$ with $x_i \in X_i$, such that $u_j(x_i, x^*_{-i}) \geq u_j(x^*)$, for all $j \in N$, with at least one strict inequality.
c) An action profile $x^*$ is an ideal joint equilibrium for the game $G$ if it is an ideal equilibrium for the game $\mathcal{J}$, that is, for all $i \in N$, $u_j(x^*) \geq u_j(x_i, x^*_{-i})$ for all $j \in N$ and for all $x_i \in X_i$.

We will denote $WJ(G)$ to the set of weak joint equilibria, $J(G)$ to the set of joint equilibria, and $IJ(G)$ to the set of ideal joint equilibria. It is straightforward that $IJ(G) \subseteq J(G) \subseteq WJ(G)$.

Note that the condition for an action profile to be a joint equilibrium is very undemanding. It suffices that one of the players cannot deviate improving the utility of at least one of the players (including her own) without making worse the utility of another one. The set of joint equilibria is typically a wide set. On the other hand, the condition for the ideal joint equilibrium is quite restrictive. It entails that no deviation of any of the players makes any of them better-off.

The following links between the equilibria for the scalar game with any number of players and the joint equilibria defined hold. The Appendix contains the proofs of these results.

**Proposition 3.2** *Let $G = \{(X_i, u_i)_{i \in N}\}$ be a non-cooperative normal-form game*:

(a) *If $x^*$ is a Nash equilibrium of G, then $x^*$ is a weak joint equilibrium.*
(b) *If $x^*$ is a Berge equilibrium of G, then $x^*$ is a weak joint equilibrium.*
(c) *If $x^*$ is an ideal joint equilibrium, then it is a Nash equilibrium of G.*
(d) *If $x^*$ is a Nash equilibrium and a Berge equilibrium of G, then $x^*$ is an ideal joint equilibrium.*

It follows from the definitions that for the case of games with two players, $x^*$ is an ideal joint equilibrium if and only if $x^*$ is both a Nash and a Berge equilibrium of $G$. However, for games with more than two players, an ideal joint equilibrium is not necessarily a Berge equilibrium, as will be shown in Example 2.

The concept of joint ideal equilibria encompasses the notion of Nash equilibrium and the idea of mutual support underlying Berge equilibrium. It constitutes a relaxation of the concept of Berge–Nash equilibria introduced by Abalo and Kostreva (2005). On the other hand, as a consequence of the fourth point in the above proposition, the notion of hybrid equilibrium proposed by Salukvadze and Zhukovskiy (2020) is a refinement of the notion of ideal joint equilibrium that includes a condition of Pareto optimality.

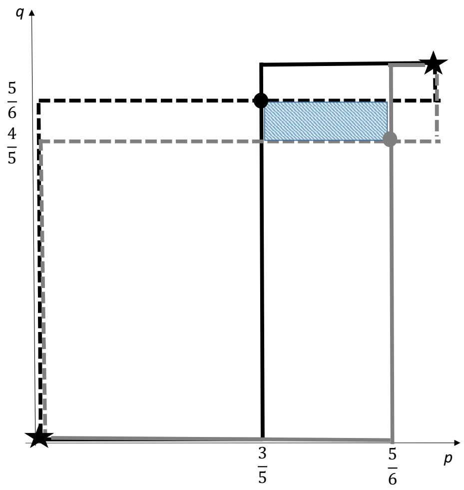


**Fig. 1** Berge, Nash, and joint equilibria of the bimatrix game

***Example 1*** A two-person game. Consider a bimatrix game where each of the two players has two pure strategies, represented as

$$\begin{array}{cc} & \begin{array}{cc} s_2^1 & s_2^2 \end{array} \\ \begin{array}{c} s_1^1 \\ s_2^1 \end{array} & \begin{pmatrix} (2,1) & (0,0) \\ (1,0) & (4,5) \end{pmatrix} \end{array}.$$

With this representation of the game, e.g., when Player 1 plays $s_2^1$ and Player 2 plays $s_1^2$, Player 1 receives a payoff of 1 and Player 2 a payoff of 0.

The payoff functions of the players in the mixed extension of the bimatrix game are

$$u_1(p,q) = (5q-4)p + 4 - 3q, \quad u_2(p,q) = (6p-5)q - 5p + 5,$$

where $p$ represents the probability with which Player 1 plays $s_1^1$ and $q$ the probability with which Player 2 plays $s_1^2$. Accordingly, $1-p$ and $1-q$ are the probabilities with which they choose $s_2^1$ and $s_2^2$, respectively.

This game has two Nash equilibria in pure strategies: $(p,q) = (1,1)$ and $(p,q) = (0,0)$. That is, the first pure strategies of both players, $\left(s_1^1, s_1^2\right)$, and the second pure strategies of both players, $\left(s_2^1, s_2^2\right)$. The game has also two Berge equilibria in pure strategies which coincide with the Nash equilibria in pure strategies. In addition, the game has one Nash equilibrium in mixed strategies: $(p,q) = \left(\frac{5}{6}, \frac{4}{5}\right)$. There is also a Berge equilibrium in mixed strategies: $(p,q) = \left(\frac{3}{5}, \frac{5}{6}\right)$.

In the joint game associated with this bimatrix game, the collective vector-valued payoff function of each player consists of $u := (u_1, u_2)$. The set of weak equilibria of the joint game includes the Nash equilibria as well as the Berge equilibria of the original game. Moreover, the ideal joint equilibria are at the same time Berge and Nash equilibria. Figure 1 shows these equilibria.

The set of ideal joint equilibria is $IJ(G) = \{(0,0)\} \cup \{(1,1)\}$. The set of weak joint equilibria is $WJ(G) = \{(0,0)\} \cup \{(p,q) \in \mathrm{IR}^2 : \frac{3}{5} \leq p \leq \frac{5}{6}, \frac{4}{5} \leq q \leq \frac{5}{6}\} \cup \{(1,1)\}$.□

## 4 The altruistic game

In this section, we introduce a vector-valued game which aims to capture the nature of altruism in the sense of being ready to give without compensation. With this altruistic attitude, each player cares about the welfare of the other players, but each one ignores her individual interest and does not expect a reciprocal attitude from the others.

Given the scalar game $G = \{(X_i, u_i)_{i\in N}\}$, we consider a vector-valued normal-form game, $\mathcal{A} = \{(X_i, v_i)_{i\in N}\}$, where the vector-valued utility function of player $i \in N$ is $v_i : \times_{i\in N} X_i \to \mathrm{IR}^{n-1}$, $v_i := (u_j)_{j\in N\setminus i}$. That is, every player only considers the utilities of the other players, ignoring her own utility. We call this game the *altruistic game associated with G*.

The concepts of weak equilibrium, equilibrium, and ideal equilibrium of the altruistic vector-valued game, expressed in terms of the individual utility functions of the players in the scalar game, allow the definitions of new equilibria concepts for the scalar game $G$.

**Definition 4.1** Let $G = \{(X_i, u_i)_{i\in N}\}$ be a non-cooperative normal-form game.

a) An action profile $x^*$ is a weak altruistic equilibrium for the game $G$ if it is a weak equilibrium for the game $\mathcal{A}$, that is, $\nexists\, i \in N$ with $x_i \in X_i$, such that $u_j(x_i, x^*_{-i}) > u_j(x^*)$, for all $j \neq i$.
b) An action profile $x^*$ is an altruistic equilibrium for the game $G$ if it is an equilibrium for the game $\mathcal{A}$, that is, $\nexists\, i \in N$ with $x_i \in X_i$, such that $u_j(x_i, x^*_{-i}) \geq u_j(x^*)$, for all $j \neq i$ with at least one strict inequality.
c) An action profile $x^*$ is an ideal altruistic equilibrium for the game $G$ if it is an ideal equilibrium for the game $\mathcal{A}$, that is, for all $i \in N$, and for all $j \neq i$, $u_j(x^*_i, x^*_{-i}) \geq u_j(x_i, x^*_{-i})$ for all $x_i \in X_i$.

We denote by $WA(G)$ the set of weak altruistic equilibria of $G$, by $A(G)$ the set of altruistic equilibria of $G$ and by $IA(G)$ the set of ideal altruistic equilibria of $G$. It is straightforward that $IA(G) \subseteq A(G) \subseteq WA(G)$.

For two-person games, the two last definitions coincide. Moreover, they coincide with the definition of Berge equilibrium. The following results hold for games with any number of players. These results are proved in the Appendix.

**Proposition 4.2** *Let* $G = \{(X_i, u_i)_{i\in N}\}$ *be a non-cooperative normal-form game*

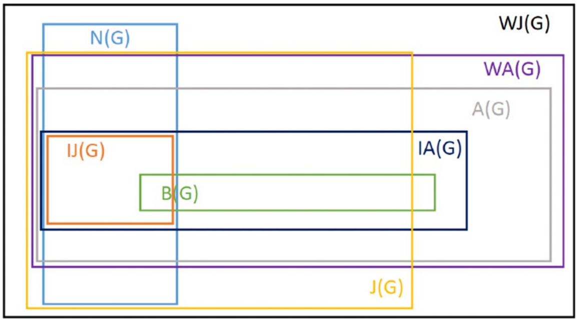


**Fig. 2** The sets of equilibria of a game $G$

*a*) *If* $x^*$ *is an altruistic equilibria of* $G$, *then* $x^*$ *is a weak joint equilibrium.*
*b*) $x^*$ *is an ideal joint equilibrium if and only if* $x^*$ *is a Nash equilibrium and an ideal altruistic equilibrium.*
*c*) *If* $x^*$ *is a Berge equilibrium of* $G$, *then* $x^*$ *is an ideal altruistic equilibrium.*

Figure 2 illustrates the relationships between the concepts of equilibria defined.

As a conclusion, the concept of ideal altruistic equilibria is a relaxation of the concept of Berge equilibrium. It entails that a unilateral deviation of one of the players will not benefit any of the other players. Ideal altruistic equilibria coincide with unilateral support equilibria, as defined in Schouten et al. (2019). They prove this fact in their Theorem 3.1. Their definition relies on the games obtained with all the different permutations of the other players' payoffs that each player can consider. Colman et al. (2011) had previously shown that a necessary condition for an action profile to be a Berge equilibrium is that it be a Nash equilibrium in all games obtained with these permutations. Thus, the concept of unilateral support equilibria extends the concept of Berge equilibrium. Moreover, Theorem 4.2. in Schouten et al. (2019) straightforwardly follows from our definitions: $N(G) \cap IA(G) = IJ(G)$.

On the other hand, the concept of altruistic equilibrium constitutes one step further in the relaxation of the concept of Berge equilibria, which still captures the underlying idea of mutual support. In an altruistic equilibrium, the deviation of one of the players does not benefit another player without making worse the payoff of another one.

Note that, even in the case in which an ideal joint equilibrium exists, it is not necessarily Pareto optimal, as can be seen in Example 2. Therefore, being a joint equilibrium or an altruistic equilibrium does not guarantee being Pareto optimal. A refinement of the ideal joint equilibrium including a condition of Pareto optimality has been introduced by Salukvadze and Zhukovskiy (2020): the so-called hybrid equilibrium.

In Examples 2 and 3, we show that the inclusions in Proposition 4.2 can be strict. The Appendix contains a detailed analysis of these examples.

***Example 2*** A three-person game. Consider the three-person game where each player has two pure strategies. The mixed extension of the game is represented by

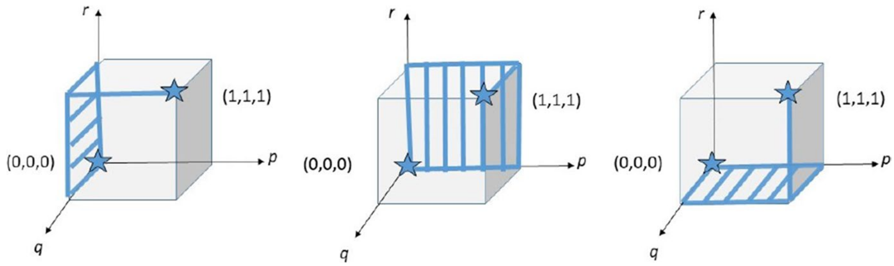


**Fig. 3** Individual best responses and Nash equilibria of the three-person game

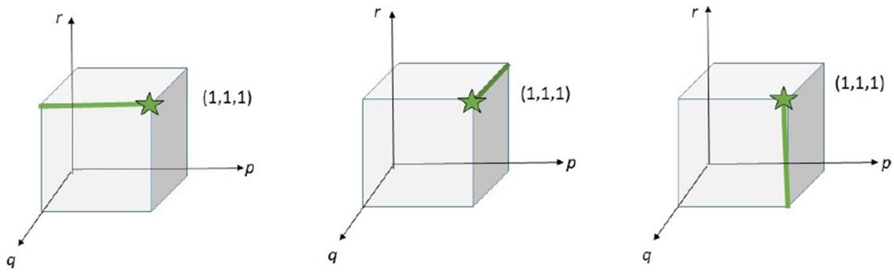


**Fig. 4** Best responses and Berge equilibrium of the three-person game

$$
s_1^3 : \begin{array}{c} \\ s_1^1 \\ s_2^1 \end{array} \begin{array}{cc} s_2^1 & s_2^2 \\ \multicolumn{2}{c}{\begin{pmatrix} (4,4,4) & (1,4,1) \\ (4,1,1) & (2,2,1) \end{pmatrix}} \end{array} \quad s_2^3 : \begin{array}{c} \\ s_1^1 \\ s_2^1 \end{array} \begin{array}{cc} s_2^1 & s_2^2 \\ \multicolumn{2}{c}{\begin{pmatrix} (1,1,4) & (1,2,2) \\ (2,1,2) & (3,3,3) \end{pmatrix}} \end{array} .
$$

We denote by $p \in [0, 1]$ the probability with which Player 1 selects $s_1^1$, by $q \in [0, 1]$ the probability with which Player 2 selects $s_1^2$, and by $r \in [0, 1]$ the probability with which Player 3 selects $s_1^3$. Accordingly, $1 - p$, $1 - q$, and $1 - r$ are the probabilities with which they choose $s_2^1$, $s_2^2$, and $s_2^3$, respectively. With this representation of the game, e.g., when Player 1 plays $s_2^1$, Player 2 plays $s_1^2$, and Player 3 $s_1^3$, then players 1, 2, and 3 receive a payoff of 4, 1, and 1 respectively.

The payoff functions for the players in the mixed extension of the game are

$$
\begin{aligned}
u_1(p, q, r) &= 3 - 2p - q - r + pq + pr + 3qr, \\
u_2(p, q, r) &= 3 - p - 2q - r + pq + 3pr + qr, \\
u_3(p, q, r) &= 3 - p - q - 2r + 3pq + pr + qr.
\end{aligned}
$$

*a) Nash equilibria.* The best response of each player to the actions of the other two players is illustrated in Fig. 3. The game has two Nash equilibria: $(p, q, r) = (1, 1, 1)$ and $(p, q, r) = (0, 0, 0)$, that is, the first pure strategies of each player $\left(s_1^1, s_1^2, s_1^3\right)$ and the second pure strategies of each player $\left(s_2^1, s_2^2, s_2^3\right)$.

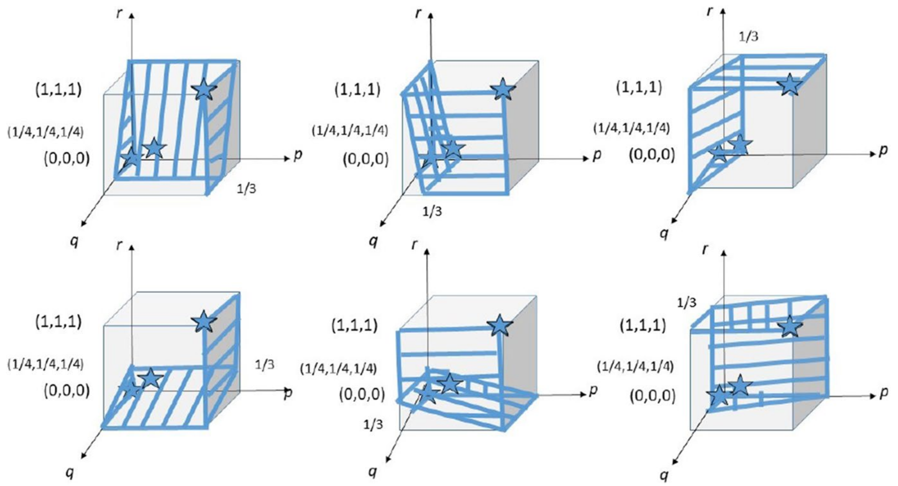


**Fig. 5** Best responses in the altruistic game and ideal altruistic equilibria

*b) Berge equilibria.* The best responses of two of the players to the actions of the other player, as represented in Fig. 4 are: $R^1_{q,r}(p) = (1,1)$ for all $p \in [0,1]$, $R^2_{p,r}(q) = (1,1)$ for all $q \in [0,1]$, and $R^3_{p,q}(r) = (1,1)$ for all $r \in [0,1]$.

Only one Berge equilibrium exists which coincides with one of the Nash equilibria of the game: $(p,q,r) = (1,1,1)$. This action profile is thus an ideal joint equilibria, and therefore, it is also an ideal altruistic equilibria.

*c) Altruistic equilibria.* Other ideal altruistic equilibria exist. It is shown in the Appendix that $(0,0,0) \in IA(G)$. Since (0, 0, 0) is also a Nash equilibrium, then $(0,0,0) \in IJ(G)$. Note that (0, 0, 0) is not a Berge equilibrium, nor is it Pareto optimal. It is also shown that $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \in IA(G)$. Note also that $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is an ideal altruistic equilibrium, that is not a Nash equilibrium nor a Berge equilibrium. The whole set of weak altruistic equilibria is also analyzed in the Appendix, where we also prove that in this game, no more altruistic equilibria exist. Hence, only three weak altruistic equilibria exist which coincide with the three ideal altruistic equilibria

$$IA(G) = A(G) = WA(G) = \left\{(1,1,1), (0,0,0), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right\}.$$

Figure 5 represents the corresponding best responses. The ideal altruistic equilibria are indicated with stars.

The set of weak joint equilibria of the three-person game can be described as

$$\begin{aligned} WJ(G) =& \{(0,0,0)\} \cup \{(p,q,r) : -1 + q + 3r \geq 0, -1 + 3q + r \geq 0, -1 + p + 3r \geq 0, \\ & -1 + 3p + r \geq 0, -1 + p + 3q \geq 0, -1 + 3p + q \geq 0, p, q, r \in [0,1]\}. \end{aligned}$$

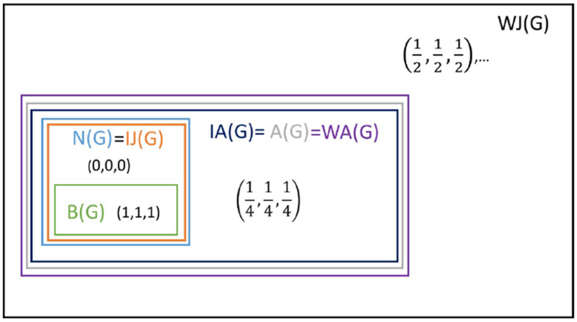


**Fig. 6** Nash, Berge, and ideal altruistic equilibria in Example 2

The set of weak altruistic equilibria is a proper subset of the set of weak joint equilibria. For instance, $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \left(1,1,\frac{3}{4}\right) \in WJ(G)\backslash WA(G)$.

Figure 6 shows the inclusions between the different sets of equilibria for this game.

$\square$

***Example 3*** Consider now the three-person game where each player has two pure strategies, represented by

$$\begin{array}{ccc} & & \begin{array}{cc} s_2^1 & s_2^2 \end{array} \\ s_1^3 : & \begin{array}{c} s_1^1 \\ s_2^1 \end{array} & \begin{pmatrix} (4,4,1) & (1,4,0) \\ (4,1,0) & (2,2,0) \end{pmatrix} \end{array} \quad \begin{array}{ccc} & & \begin{array}{cc} s_2^1 & s_2^2 \end{array} \\ s_2^3 : & \begin{array}{c} s_1^1 \\ s_2^1 \end{array} & \begin{pmatrix} (1,1,0) & (1,2,0) \\ (2,1,0) & (3,3,1) \end{pmatrix} \end{array}.$$

The payoff functions are

$$\begin{aligned} u_1(p,q,r) &= 3 - 2p - q - r + pq + pr + 3qr, \\ u_2(p,q,r) &= 3 - p - 2q - r + pq + 3pr + qr, \\ u_3(p,q,r) &= 1 - p - q - r + pq + pr + qr. \end{aligned}$$

As shown in the Appendix, the game has two Nash equilibria: $(p,q,r) = (1,1,1)$ and $(p,q,r) = (0,0,0)$. And only one Berge equilibrium exists which coincides with one of the Nash equilibria $(p,q,r) = (1,1,1)$. It is also shown that this action profile is an ideal altruistic equilibrium, and, since it is a Nash equilibrium, therefore, it is also an ideal joint equilibrium.

In this case, the ideal altruistic equilibria coincide with the two Nash equilibria, since they correspond to any best response of each player to each of the utility functions of the other players. It can be shown that other altruistic equilibria exist that are not ideal altruistic equilibria. For instance, $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ is a weak altruistic equilibria. That is, the set of ideal altruistic equilibria is a proper subset of the set of altruistic equilibria. $\square$

## 5 Altruistic equilibria and weighted games

The approach we have followed to address the extensions of Berge equilibria, based on non-cooperative vector-valued games, has the advantage that it permits us to establish the links between these concepts and the equilibria of certain weighted scalar games.

From the operative point of view, both altruistic equilibria and ideal altruistic equilibria can be characterized as the equilibria of weighted scalar games. Moreover, if we assume that the preferences of the players can be represented by additive value functions, the corresponding weights can be interpreted as the relative importance that the players assign to the other players. This representation adds flexibility when trying to identify altruistic equilibria, opening the possibility to include partial preference information on these weights.

Bade (2005) established the relationship between the set of equilibria of a vector-valued game and the set of equilibria of the weighted games with positive weights and with non-negative weights. In addition, Wang (1993) stated that the equilibria of weighted games with non-negative weights are weak equilibria for the game with vector-valued utilities and, under concavity assumptions, Mármol et al. (2017) characterized the sets of weak equilibria and of equilibria of the vector-valued game by means of the equilibria of the weighted games. On the other hand, in Voorneveld et al. (2000), a characterization of the ideal equilibria of the vector-valued game in terms of weighted games is also presented. The application of these results for the altruistic game allows us to establish the following results regarding altruistic equilibria.

Given the normal-form game $G = \{(X_i, u_i)_{i\in N}\}$, for each player $i \in N$, consider an $(n-1)$-dimensional vector of weights $\lambda^i \in \Delta^{n-1}$, where for $j \neq i$, $\lambda^i_j$ represents the weight that player $i$ attaches to player $j$. Denote by $\Gamma^{n-1} = \times_{i\in N}\Delta^{n-1}$ and $\Gamma^{n-1}_+ = \times_{i\in N}\Delta^{n-1}_+$. For each $\lambda = (\lambda^i)_{i\in N} \in \Gamma^{n-1}$, consider the scalar game $G(\lambda) = \{(X_i, v_i^\lambda)_{i\in N}\}$, where for $i \in N$, $v_i^\lambda : \times_{i\in N} X_i \rightarrow \mathrm{IR}$, $v_i^\lambda(x) = \lambda^i \cdot v_i = \sum_{j\neq i} \lambda^i_j u_j$. The set of Nash equilibria for this weighted game is denoted by $N(G(\lambda))$.

**Proposition 5.1** *Given the normal-form game $G = \{(X_i, u_i)_{i\in N}\}$ and the weighted games $G(\lambda)$, with $\lambda \in \Gamma^{n-1}$.*

(a) $IA(G) = \bigcap_{\lambda\in\Gamma^{n-1}} N(G(\lambda))$.

(b) $\bigcup_{\lambda\in\Gamma^{n-1}} N(G(\lambda)) \subseteq WA(G)$.

(c) $\bigcup_{\lambda\in\Gamma^{n-1}_+} N(G(\lambda)) \subseteq A(G)$.

(d) *If, for all $i \in N$, $u_i$ are concave in $x_i$ and $X_i$ are convex sets, then*

$$\bigcup_{\lambda\in\Gamma^{n-1}} N(G(\lambda)) = WA(G).$$

(e) *If for all $i \in N$, $u_i$ are strictly concave in $x_i$ and $X_i$ are convex sets, then*

$$\bigcup_{\lambda \in \Gamma_+^{n-1}} N(G(\lambda)) = A(G) = WA(G).$$

As a consequence of case *a)*, the ideal equilibria coincide with the profiles of strategies that are equilibria for any weighted altruistic game and, as a consequence, a Berge equilibrium needs to be a Nash equilibrium for any weighted altruistic game. Nevertheless, as stated in case *c)* for a profile of strategies to be an altruistic equilibrium, it suffices that it be a Nash equilibrium for only one weighted altruistic game. Under the conditions of case *d)*, the existence of weak altruistic equilibria is guaranteed. Moreover, strict concavity conditions of case *e)* guarantee the non-emptiness of the set of altruistic equilibria.

The results in Proposition 5.1 permit the identification of different subsets of the generally wide set of altruistic equilibria by including partial information of the players with respect to the utilities of other players. For instance, while maintaining an altruistic attitude, Player 1 may want to consider that the utility of Player 2 is not less important for her than the utility of Player 3, thus reducing the set of weights for Player 1 to those fulfilling $\lambda_2^1 \geq \lambda_3^1$.

For each player, consider a subset of weights representing the relative importances she may be willing to attach to the payoffs of the rest of players, $\Lambda^i \subseteq \Delta^{n-1}$. Denote by $\Lambda = \times_{i \in N} \Lambda^i$, the set containing the preference information of all the players.

**Definition 5.2** Given the normal-form game $G = \{(X_i, u_i)_{i \in N}\}$ and $\Lambda^i \subseteq \Delta^{n-1}$ the set of preference information of player $i \in N$:

a) An action profile $x^*$ is an altruistic equilibrium for the game $G$ with preference information $\Lambda$ if, for each $i \in N$, $\lambda^i \in \Lambda^i$ exists, such that $v_i^\lambda(x^*) \geq v_i^\lambda(x_i, x_{-i}^*)$ for all $x_i \in X_i$.
b) An action profile $x^*$ is an ideal altruistic equilibrium for the game $G$ with preference information $\Lambda$ if for all $i \in N$ and for all $\lambda^i \in \Lambda^i$, $v_i^\lambda(x^*) \geq v_i^\lambda(x_i, x_{-i}^*)$ for all $x_i \in X_i$.

We denote as $A_\Lambda(G)$ the set of altruistic equilibria of $G$ with preference information $\Lambda$, and as $IA_\Lambda(G)$ the set of ideal altruistic equilibria of $G$ with preference information $\Lambda$. It is straightforward that the more precise preference information is, the wider the set of ideal equilibria and the smaller the set of equilibria are. That is, if $\Lambda \subseteq \Lambda'$, then

$$IA_{\Lambda'}(G) \subseteq IA_\Lambda(G) \subseteq A_\Lambda(G) \subseteq A_{\Lambda'}(G).$$

In the cases in which the partial preference information of the players is represented by polyhedra, the corresponding altruistic equilibria can be characterized as the equilibria of transformed vector-valued games. This fact is a consequence of the result stated in Theorem 3.3. in Mármol et al. (2017) for general vector-valued games. For each $i \in N$, let $\Lambda^i$ be a polyhedron and $B^i$ the matrix whose rows are the extreme points of $\Lambda^i$. Let $v^i_\Lambda = B^i \cdot u_i$. Denote by $\mathcal{A}_\Lambda = \{(X_i, v^i_\Lambda)_{i \in N}\}$, the transformed vector-valued game. Note that, in this game, the payoff of each player has as many components as extreme points her information polyhedron has.

**Proposition 5.3** *Let $G = \{(X_i, u_i)_{i \in N}\}$ be a normal-form game, such that, for each $i \in N$, $X_i$ is a non-empty subset of a finite-dimensional space, and $u_i$ is concave in $x_i$. Let $\Lambda^i \subseteq \Delta^{n-1}$ be the set of preference information of player i. Consider the vector-valued game $\mathcal{A}_\Lambda = \{(X_i, v^i_\Lambda)_{i \in N}\}$*

*a) An action profile $x^*$ is an altruistic equilibrium for G with preference information $\Lambda$ if and only if it is a weak equilibrium for the game $\mathcal{A}_\Lambda$.*
*b) An action profile $x^*$ is an ideal altruistic equilibrium for G with preference information $\Lambda$ if and only if it is an ideal equilibrium for the game $\mathcal{A}_\Lambda$.*

***Example 4*** A three-person game with preference information. Consider the following three-person game $G = \{(X_i, u_i)_{i=1,2,3}\}$ where $X_i = [-1, 1]$ and the payoff functions are:

$$\begin{aligned} u_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3^2, \\ u_2(x_1, x_2, x_3) &= x_2 - x_3^2, \\ u_3(x_1, x_2, x_3) &= x_1 + x_2^2(x_3 + 1). \end{aligned}$$

This game has no ideal altruistic equilibrium, since if ($a$, $b$, $c$) is an ideal altruistic equilibrium, then, for the third player, the following inequalities must hold for all $x_3 \in [-1, 1]$: $u_1(a, b, c) = a + b + c^2 \geq u_1(a, b, x_3) = a + b + x_3^2$, $u_2(a, b, c) = b - c^2 \geq u_2(a, b, x_3) = b - x_3^2$. If follows that, $c^2 \geq x_3^2$ and $c^2 \leq x_3^2$ for all $x_3 \in [-1, 1]$, which is a contradiction. Hence, there is no ideal altruistic equilibrium. Therefore, no ideal joint equilibrium exists, and no Berge equilibrium exists.

Nevertheless, when preference information is incorporated to the game, ideal altruistic equilibria may exist. Consider the set of preference information representing a situation in which the importance that Player 1 attaches to the utility of Player 2 is no less than that attached to the utility of Player 3, the importance that Player 2 attaches to the utility of Player 1 is no less than that attached to the utility of Player 3, and the importance that Player 3 attaches to the utility of Player 1 is no less than the importance that she attaches to the utility of Player 2:

$$\bar{\Lambda} = \{\lambda \in \Gamma^{n-1} \, : \, \lambda^1_2 \geq \lambda^1_3, \lambda^2_1 \geq \lambda^2_3, \; \lambda^3_1 \geq \lambda^3_2\}.$$

The matrices representing the extreme points of the sets of information for the players are

$$B^1 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}, \quad B^3 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}.$$

Using Theorem 3.3 in Mármol et al. (2017), the equilibria of the game with preference information correspond to the equilibria of the transformed vector-valued game. Note that, in this game, the payoff of each player has as many components as extreme points her information polyhedron has. In the transformed vector-valued game, $\mathcal{A}_{\bar{\Lambda}} = \{(X_i, v^i_{\bar{\Lambda}})_{i=1,2,3}\}$, the two-dimensional utilities for the players are

$$v^1_{\bar{\Lambda}}(x_1, x_2, x_3) = \left(x_2 - x_3^2,\ \frac{1}{2}(x_1 + x_2 - x_3^2 + x_2^2(x_3+1))\right),$$
$$v^2_{\bar{\Lambda}}(x_1, x_2, x_3) = \left(x_1 + x_2 + x_3^2,\ \frac{1}{2}(2x_1 + x_2 + x_3^2 + x_2^2(x_3+1))\right),$$
$$v^3_{\bar{\Lambda}}(x_1, x_2, x_3) = \left(x_1 + x_2 + x_3^2,\ \frac{1}{2}(x_1 + 2x_2)\right).$$

The ideal altruistic equilibria for the game with preference information $\bar{\Lambda}$ correspond to the ideal equilibria of the game $\mathcal{A}_{\bar{\Lambda}}$.

To identify the possible ideal altruistic equilibria with this preference information, note that if $(a, b, c)$ is an ideal altruistic equilibrium with information $\bar{\Lambda}$, then, for all $x_1, x_2, x_3 \in [-1, 1]$, the following inequalities must hold: $v^1_{\bar{\Lambda}}(a, b, c) \geqq v^1_{\bar{\Lambda}}(x_1, b, c)$, $v^2_{\bar{\Lambda}}(a, b, c) \geqq v^2_{\bar{\Lambda}}(a, x_2, c)$, $v^3_{\bar{\Lambda}}(a, b, c) \geqq v^3_{\bar{\Lambda}}(a, b, x_3)$. It follows that, $a \geq x_1$, $b \geq x_2$, $b + b^2(c+1) \geq x_2 + x_2^2(c+1)$, and $c^2 \geq x_3^2$ must hold for all $x_1, x_2, x_3 \in [-1, 1]$. Hence

$$IA_{\bar{\Lambda}}(G) = \{(1, 1, 1), (1, 1, -1)\}.$$

Note that (1, 1, 1) is a Nash equilibria of the original game which emerges as an ideal altruistic equilibrium with this specific preference information. That is, even though this action profile does not maximize the payoff of each player, it succeeds in maximizing the vector-valued utilities of the players that represent their preferences with respect to the payoffs of the others with possibly different importances.

Similarly, $(1, 1, -1)$ is not a Nash equilibrium nor an ideal altruistic equilibrium of the original game. However, it constitutes an altruistic equilibrium which is consistent with the preference information provided by the players.

As a conclusion, in this game, there is no equilibrium that allows players to act purely altruistically. However, the inclusion of additional information on the different altruistic attitudes of some players toward others permits the identification of appropriate strategy profiles compatible with their altruistic attitudes. $\square$

## 6 Concluding remarks

The concept of Berge equilibrium is considered in non-cooperative game theory to model the altruistic behavior of the players. It relies on a strong notion of altruism, since it requires very demanding conditions implying collective decisions that seek for the best interest of the other players. As a consequence, its existence is rarely guaranteed for games with more than two players.

In this paper, we introduce some other equilibrium concepts for non-cooperative games that still capture the notion of altruism and that constitute different relaxations of Berge's idea. The approach is based on modeling the non-cooperative decision situation as a vector-valued game in which each player considers the utilities of all the players.

The relationships between Berge equilibria, other existing related concepts, and the equilibrium concepts proposed (joint equilibria, altruistic equilibria, and ideal altruistic equilibria) are explored, and several significant examples are analyzed.

On the other hand, the links between the vector-valued games and the corresponding weighted games open the possibility to identify the new sets of equilibria. More importantly, the approach permit the inclusion of partial and imprecise preference information regarding the relative importance that each player attaches to the utilities of the other players. So that, even in the cases in which neither Berge equilibria, nor ideal altruistic equilibria exist, it is possible to identify sets of equilibria which capture the idea of different altruistic attitudes toward other players, thus providing more accurate and realistic predictions about the equilibria situations that the players will eventually reach.

## Appendix

### *Proof Proposition 3.2*

(a) Let $x^*$ be a Nash equilibrium of the game $G$. Suppose on the contrary that $x^*$ is not a weak joint equilibrium of the game $G$. It follows that $i \in N$ and $x_i \in X_i$ exist, such that $u_j(x_i, x^*_{-i}) > u_j(x^*)$ for all $j \in N$. In particular, for $j = i$, $x_i \in X_i$ exists, such that $u_i(x_i, x^*_{-i}) > u_i(x^*)$. This contradicts the fact that $x^*$ is a Nash equilibrium of $G$.

(b) Let $x^*$ be a Berge equilibrium of the game $G$. Suppose on the contrary that $x^*$ is not a weak joint equilibrium of the game $G$. That is, $j \in N$ and $x_j \in X_j$ exist, such that $u_i(x_j, x^*_{-j}) > u_i(x^*)$ for all $i \in N$. In particular, for $i \neq j$, it follows that $u_i(x_j, x^*_i, x^*_{-ij}) > u_i(x^*)$, where $x^*_{-ij}$ stands for the strategy combination of all players except players i and j. Hence, for $\hat{x}_{-i} = (x_j, x^*_{-ij}) \in X_{-i}$, $u_i(x^*_i, \hat{x}_{-i}) > u_i(x^*)$. This contradicts the fact that $x^*$ is a Berge equilibrium of $G$.

(c) If $x^*$ is an ideal joint equilibrium of the game $G$, then for all $i \in N$ $u_j(x^*) \geq u_j(x_i, x^*_{-i})$ for all $j \in N$ and for all $x_i \in X_i$. In particular, for $j = i$, $u_i(x^*) \geq u_i(x_i, x^*_{-i})$ for all $x_i \in X_i$. That is, $x^*$ is a Nash equilibrium of $G$.

(d) If $x^*$ is a Nash equilibrium and a Berge equilibrium of the game $G$, then given $i \in N, u_i(x^*) \geq u_i(x_i, x^*_{-i})$ for all $x_i \in X_i$, and for all $j \in N, u_j(x^*) \geq u_j(x^*_j, x_{-j})$ for all $x_{-j} \in X_{-j}$. Consider $j \neq i$, it follows that $u_j(x^*) \geq u_j(x^*_j, x_{-j}) = u_j(x^*_j, x_i, x_{-ij})$ for all $x_i \in X_i$ and for all $x_{-ij} \in X_{-ij}$. In particular, for $x_{-ij} = x^*_{-ij}$, $u_j(x^*) \geq u_j(x^*_j, x_i, x^*_{-ij}) = u_j(x_i, x^*_{-i})$ for all $x_i \in X_i$. Therefore, $x^*$ is an ideal joint equilibrium of the game $G$.

□

***Proof Proposition 4.2***

Results (a) and (b) follow from the definitions. To prove (c), let $x^*$ be a Berge equilibrium of $G$. For each $i \in N$, consider $j \neq i$. Since $x^*$ is a Berge equilibrium, it follows that $u_j(x^*) \geq u_j(x_j^*, x_{-j})$ for all $x_{-j} \in X_{-j}$. Hence, $u_j(x^*) \geq u_j(x_j^*, x_{-j}) = u_j(x_j^*, x_i, x_{-ij})$ for all $x_i \in X_i$ and for all $x_{-ij} \in X_{-ij}$. In particular, for $x_{-ij} = x_{-ij}^*$, $u_j(x^*) \geq u_j(x_j^*, x_i, x_{-ij}^*) = u_j(x_i, x_{-i}^*)$ for all $x_i \in X_i$. Therefore, $x^*$ is an ideal altruistic equilibrium.

□

***Example 2*** The three-person game is represented by

$$s_1^3 : \begin{array}{c} \\ s_1^1 \\ s_2^1 \end{array} \begin{array}{c} \begin{array}{cc} s_2^1 & s_2^2 \end{array} \\ \begin{pmatrix} (4,4,4) & (1,4,1) \\ (4,1,1) & (2,2,1) \end{pmatrix} \end{array} \quad s_2^3 : \begin{array}{c} \\ s_1^1 \\ s_2^1 \end{array} \begin{array}{c} \begin{array}{cc} s_2^1 & s_2^2 \end{array} \\ \begin{pmatrix} (1,1,4) & (1,2,2) \\ (2,1,2) & (3,3,3) \end{pmatrix} \end{array}.$$

The payoff functions for the players in the mixed extension of the game are

$$\begin{aligned} u_1(p,q,r) &= 3 - 2p - q - r + pq + pr + 3qr, \\ u_2(p,q,r) &= 3 - p - 2q - r + pq + 3pr + qr, \\ u_3(p,q,r) &= 3 - p - q - 2r + 3pq + pr + qr. \end{aligned}$$

*a) Nash equilibria.* To determine the Nash equilibria, we describe the best response of each player to the actions of the other two (as shown in Fig. 3)

$$R_p^1(q,r) = \begin{cases} 0 & if \ (q,r) \neq (1,1) \\ [0,1] & if \ (q,r) = (1,1) \end{cases}, \quad R_q^2(p,r) = \begin{cases} 0 & if \ (p,r) \neq (1,1) \\ [0,1] & if \ (p,r) = (1,1) \end{cases},$$
$$R_r^3(p,q) = \begin{cases} 0 & if \ (p,q) \neq (1,1) \\ [0,1] & if \ (p,q) = (1,1) \end{cases}.$$

This game has two Nash equilibria: $(p,q,r) = (1,1,1)$ and $(p,q,r) = (0,0,0)$, that is, the first pure strategies of each player $\left(s_1^1, s_1^2, s_1^3\right)$ and the second pure strategies of each player $\left(s_2^1, s_2^2, s_2^3\right)$.

*b) Berge equilibria.* The identification of Berge equilibria entails the search for the solutions to the following problems:

$$\begin{aligned} &Max_{0 \leq q,r \leq 1} \ u_1(p,q,r) = 3 - 2p + (-1+p)q + (-1+p)r + 3qr, \\ &Max_{0 \leq p,r \leq 1} \ u_2(p,q,r) = 3 - 2q + (-1+q)p + (-1+q)r + 3pr, \\ &Max_{0 \leq p,q \leq 1} \ u_3(p,q,r) = 3 - 2r + (-1+r)p + (-1+r)q + 3pq. \end{aligned}$$

By applying the first-order optimality conditions, the best responses of two players to the actions of the other player are obtained: $R_{q,r}^1(p) = (1,1)$ for all $p \in [0,1]$, $R_{p,r}^2(q) = (1,1)$ for all $q \in [0,1]$, and $R_{p,q}^3(r) = (1,1)$ for all $r \in [0,1]$.

Figure 4 represents these best responses. Only one Berge equilibrium exists which coincides with one of the Nash equilibria $(p, q, r) = (1, 1, 1)$. This profile of strategies is thus an ideal joint equilibria, and therefore, it is also an ideal altruistic equilibria.

*c) Altruistic equilibria.* We will now show that other ideal altruistic equilibria exist. Consider (0, 0, 0). $u_j(0, 0, 0) = 3$ for $j = 1, 2, 3$. Since $p, q, r \geq 0$, we have

$$u_2(0,0,0) \geq u_2(p,0,0) = 3 - p, \text{ and } u_3(0,0,0) \geq u_3(p,0,0) = 3 - p,$$
$$u_1(0,0,0) \geq u_1(0,q,0) = 3 - q, \text{ and } u_3(0,0,0) \geq u_3(0,q,0) = 3 - q,$$
$$u_1(0,0,0) \geq u_1(0,0,r) = 3 - r, \text{ and } u_2(0,0,0) \geq u_2(0,0,r) = 3 - r.$$

It follows that $(0,0,0) \in IA(G)$. Since (0, 0, 0) is also a Nash equilibrium, then $(0,0,0) \in IJ(G)$. Note that (0, 0, 0) is not a Berge equilibrium nor is it Pareto optimal, since $u_j(0,0,0) = 3 < u_j(1,1,1) = 4$ for $j = 1, 2, 3$, that is, (0, 0, 0) is dominated.

Consider now the profile of strategies $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. The following equalities hold:

$$u_j\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = u_j\left(p, \frac{1}{4}, \frac{1}{4}\right) = \frac{37}{16} \text{ for } j = 2, 3;$$
$$u_j\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = u_j\left(\frac{1}{4}, q, \frac{1}{4}\right) = \frac{37}{16} \text{ for } j = 1, 3;$$
$$u_j\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = u_j\left(\frac{1}{4}, \frac{1}{4}, r\right) = \frac{37}{16} \text{ for } j = 1, 2.$$

It follows that $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \in IA(G)$. Note that $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is an ideal altruistic equilibrium, that is not a Nash equilibrium nor a Berge equilibrium.

As a consequence of Theorem 2.3 in Mármol et al. (2017), the set of weak altruistic equilibria can be identified as those profiles of strategies that mutually belong to the best responses of each player to the functions of the rest of the players. The best response of Player $i$ consists of the values between the best responses to the payoff functions of Players $j$ and $k$ with $j, k \neq i$.

For this game, the best responses are

$$R_p^2(q,r) = \begin{cases} 0 & \text{if } -1+q+3r<0 \\ [0,1] & \text{if } -1+q+3r=0, \\ 1 & \text{if } -1+q+3r>0 \end{cases} R_p^3(q,r) = \begin{cases} 0 & \text{if } -1+3q+r<0 \\ [0,1] & \text{if } -1+3q+r=0 \\ 1 & \text{if } -1+3q+r>0 \end{cases}.$$

Analogously for Player 2

$$R_q^1(p,r) = \begin{cases} 0 & \text{if } -1+p+3r<0 \\ [0,1] & \text{if } -1+p+3r=0, \\ 1 & \text{if } -1+p+3r>0 \end{cases} R_q^3(p,r) = \begin{cases} 0 & \text{if } -1+3p+r<0 \\ [0,1] & \text{if } -1+3p+r=0 \\ 1 & \text{if } -1+3p+r>0 \end{cases}.$$

And for Player 3

$$R_r^1(p,q)=\begin{cases}0 & if \quad -1+p+3q<0\\ [0,1] & if \quad -1+p+3q=0,\\ 1 & if \quad -1+p+3q>0\end{cases} R_r^2(p,q)=\begin{cases}0 & if \quad -1+3p+q<0\\ [0,1] & if \quad -1+3p+q=0,\\ 1 & if \quad -1+3p+q>0\end{cases}.$$

We will show that in this game, no more ideal altruistic equilibria exist. Moreover, the set of altruistic equilibria coincide with the set of ideal altruistic equilibria.

To identify the complete set of altruistic equilibria, we first consider the region where $q, r < \frac{1}{4}$. In this case, the best response of Player 1 is $p = 0$. Now, with $p = 0$ and $q < \frac{1}{4}$, the best response of Player 3 is necessarily $r = 0$, and it follows that since $p = 0$ and $r = 0$, then the best response of Player 2 is $q = 0$. Hence, $\{(0, 0, 0)\}$ is the only (weak) altruistic equilibrium in this region. Analogously, when $q, r > \frac{1}{4}$, the best response of Player 1 is $p = 1$. It follows, since $p = 1$ and $q > \frac{1}{4}$, that $r = 1$, and since $p = 1$ and $r = 1$, then $q = 1$. Hence, $\{(1, 1, 1)\}$ is the only (weak) altruistic equilibrium in this region. If $q < \frac{1}{4}$ and $r > \frac{1}{4}$, then the best response of Player 1 is $p \in [0, 1]$. Two cases can be considered now. First, if $p < \frac{1}{4}$, since $q < \frac{1}{4}$, then the best response of Player 3 is $r = 0$, that is a contradiction. Second, if $p \geq \frac{1}{4}$, since $r > \frac{1}{4}$, the best response of Player 2 is $q = 1$, that is a contradiction. Hence, no (weak) altruistic equilibria exist in this case. Analogously, when $q > \frac{1}{4}$ and $r < \frac{1}{4}$, no altruistic equilibria exist. If $q = r = \frac{1}{4}$, then the best response of Player 1 is $p \in [0, 1]$. If $p = \frac{1}{4}$, then $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is an ideal altruistic equilibria. When $p < \frac{1}{4}$, then since $r = \frac{1}{4}$, the best response of Player 2 must be $q = 0$, which is a contradiction. When $p > \frac{1}{4}$, then since $r = \frac{1}{4}$, the best response of Player 2 must be $q = 1$, which is a contradiction. If $q = \frac{1}{4}$ and $r > \frac{1}{4}$, then the best response of Player 1 is $p = 1$. Now, for $q = \frac{1}{4}$ and $p = 1$, the best response of Player 3 is $r = 1$. However, for $q = 1$ and $r = 1$, the best response of Player 1 is necessarily $p = 1$, which is a contradiction. Using a similar reasoning for the other case, it can be shown that no more (weak) altruistic equilibria exist.

Hence, only three weak altruistic equilibria exist which coincide with the three ideal altruistic equilibria identified above

$$IA(G) = A(G) = WA(G) = \left\{(1,1,1), (0,0,0), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right\}.$$

Figure 5 represents the corresponding best responses. The ideal altruistic equilibria are indicated with stars.

The set of weak joint equilibria of the three-person game can be obtained using Theorem 2.3 in Mármol et al. (2017). It consists of

$$\begin{aligned} WJ(G) = & \{(0,0,0)\} \cup \{(p,q,r) : -1+q+3r \geq 0, -1+3q+r \geq 0, -1+p+3r \geq 0, \\ & -1+3p+r \geq 0, -1+p+3q \geq 0, -1+3p+q \geq 0, p,q,r \in [0,1]\}. \end{aligned}$$

The set of weak altruistic equilibria is a proper subset of the set of weak joint equilibria. For instance, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (1, 1, \frac{3}{4}) \in WJ(G) \setminus WA(G)$. $\square$

***Example 3*** The three-person game is represented by

$$s_1^3 : \begin{array}{c|cc} & s_2^1 & s_2^2 \\ \hline s_1^1 & (4,4,1) & (1,4,0) \\ s_2^1 & (4,1,0) & (2,2,0) \end{array} \quad s_2^3 : \begin{array}{c|cc} & s_2^1 & s_2^2 \\ \hline s_1^1 & (1,1,0) & (1,2,0) \\ s_2^1 & (2,1,0) & (3,3,1) \end{array}.$$

The payoff functions are

$$\begin{aligned} u_1(p,q,r) &= 3 - 2p - q - r + pq + pr + 3qr, \\ u_2(p,q,r) &= 3 - p - 2q - r + pq + 3pr + qr, \\ u_3(p,q,r) &= 1 - p - q - r + pq + pr + qr. \end{aligned}$$

The best responses of Player 1 and Player 2 to the actions of the other two coincide with those in Example 2. The individual best response of Player 3 to the actions of Players 1 and 2 is the following:

$$R_r^3(p,q) = \begin{cases} 1 & \textit{if } p+q > 1 \\ [0,1] & \textit{if } p+q = 1 \\ 0 & \textit{if } p+q < 1 \end{cases}.$$

This game has two Nash equilibria: $(p,q,r) = (1,1,1)$ and $(p,q,r) = (0,0,0)$. To determine the Berge equilibria, note that the best responses of two players to the actions of the other player coincide with those of the last example for Players 1 and 2. The best response of Players 1 and 2 to the actions of Player 3 is

$$R_{p,q}^3(r) = \begin{cases} (0,0) & \textit{if } r \in [0, \frac{1}{2}] \\ (1,1) & \textit{if } r \in [\frac{1}{2}, 1] \end{cases}.$$

Only one Berge equilibrium exists which coincides with one of the Nash equilibria $(p,q,r) = (1,1,1)$.

Since $0 \leq p,q,r \leq 1$, we have

$$\begin{aligned} &u_2(1,1,1) = 4 \geq u_2(p,1,1) = 4 - p, \text{ and } u_3(1,1,1) = 1 \geq u_3(p,1,1) = p, \\ &u_1(1,1,1) = 4 \geq u_1(1,q,1) = 4 - q, \text{ and } u_3(1,1,1) = 1 \geq u_3(1,q,1) = q, \\ &u_1(1,1,1) = 4 \geq u_1(1,1,r) = 4 - r, \text{ and } u_2(1,1,1) = 4 \geq u_2(1,1,r) = 4 - r. \end{aligned}$$

This profile of strategies is, thus, an ideal altruistic equilibrium, and, since it is a Nash equilibrium, it is also an ideal joint equilibrium.

In this case, the ideal altruistic equilibria coincide with the two Nash equilibria, since they correspond to any best response of each player to each of the utility functions of the other players. There are other altruistic equilibria that are not ideal altruistic equilibria. For instance, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is a weak altruistic equilibria, since, for each player, at least one of the utility functions of the other players do not change when a deviation of her strategy occurs, since $u_2(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = u_2(\frac{1}{4}, \frac{1}{4}, r) = \frac{37}{16}$, $u_1(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = u_1(\frac{1}{4}, q, \frac{1}{4}) = \frac{37}{16}$, $u_j(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = u_j(\frac{1}{4}, \frac{1}{4}, r) = \frac{37}{16}$ for $j = 1, 2$. □

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