WEAK $\mathcal{Z}$-STRUCTURES AND ONE-RELATOR GROUPS

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Abstract. Motivated by the notion of boundary for hyperbolic and $CAT(0)$ groups, Bestvina [2] introduced the notion of a (weak) $\mathcal{Z}$-structure and (weak) $\mathcal{Z}$-boundary for a group G of type $\mathcal{F}$ (i.e., having a finite $K(G,1)$ complex), with implications concerning the Novikov conjecture for G. Since then, some classes of groups have been shown to admit a weak $\mathcal{Z}$-structure (see [15] for example), but the question whether or not every group of type $\mathcal{F}$ admits such a structure remains open. In this paper, we show that every torsion free one-relator group admits a weak $\mathcal{Z}$-structure, by showing that they are all properly aspherical at infinity; moreover, in the 1-ended case the corresponding weak $\mathcal{Z}$-boundary has the shape of either a circle or a Hawaiian earring depending on whether the group is a virtually surface group or not. Finally, we extend this result to a wider class of groups still satisfying a Freiheitssatz property.

1. Introduction

We recall that a compact metrizable space W is a compactification of a (path connected) locally compact metrizable space X if it contains a homeomorphic copy $X \subset W$ as a dense open subset. Furthermore, we say that W is a $\mathcal{Z}$-compactification of X if $Z = W - X$ is a $\mathcal{Z}$-set in W, i.e., for every open set $U \subset W$ the inclusion $U - Z \hookrightarrow U$ is a homotopy equivalence; equivalently (if X is an ANR), there is a homotopy $H : W \longrightarrow W$ with $H_0 = id_W$ and $H_t(W) \subset X$ for all $t > 0$. And in this case, we say that Z is a $\mathcal{Z}$-boundary for X. The model example is that in which W is a compact manifold and $Z \subseteq \partial W$ is a closed subset.

From now on all spaces will be ANRs; in fact, we will deal with (connected) locally finite CW-complexes.

In [2] Bestvina introduced the notion of $\mathcal{Z}$-structure and $\mathcal{Z}$-boundary for a group (of type $\mathcal{F}$) as an attempt to generalize the already existing notion of boundary for hyperbolic and $CAT(0)$ groups; namely,

Definition 1.1. A $\mathcal{Z}$-structure on a group G is a pair (W, Z) of spaces satisfying:

(1) W is a contractible ANR,
(2) Z is a $\mathcal{Z}$-set in W,
(3) $X = W - Z$ admits a proper, free and cocompact action by G, and
(4) (nullity condition) For any open cover of W, and any compact subset $K \subseteq X$, all but finitely many translates of K lie in some element of the cover.

Observe that is not necessary that W be finite-dimensional, by [26, 17]. If only conditions (1)-(3) are satisfied, then (W, Z) is called a weak $\mathcal{Z}$-structure on G, and we refer to Z as a (weak) $\mathcal{Z}$-boundary for G.

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An additional condition can be added to the above:

(5) The action of G on X can be extended to W.

If conditions (1)-(5) are satisfied, then (W, Z) is called a $\mathcal{EZ}$-structure on G. It was shown in [11] that the Novikov conjecture holds for any (torsion free) group admitting an $\mathcal{EZ}$-structure. Examples of groups admitting an $\mathcal{EZ}$-structure are (torsion free) δ-hyperbolic and $CAT(0)$ groups, see [3, 2].

Although not stated explicitly, it follows that such a group G as in Definition 1.1 must be of type $\mathcal{F}$ (see [15, Prop. 1.1]). The more conditions on a $\mathcal{Z}$-structure on a group the better; nonetheless, a weak $\mathcal{Z}$-boundary already carries significant information about the group and, when it exists, it is well-defined up to shape, and it is always a first step towards finding a stronger structure on it. The question whether or not every group of type $\mathcal{F}$ admits a (weak) $\mathcal{Z}$-structure still remains open. In this paper, we give a positive answer to this question for a class of groups containing all torsion free one-relator groups. For this, we first use some previous work from [4, 21] to show that they are all properly aspherical at infinity, and then combine this with some recent work in [5] to characterize the corresponding boundary in the 1-ended case. Our main results are Theorems 1.2 and 1.6 below.

Theorem 1.2. *Every finitely generated, torsion free one-relator group G admits a weak $\mathcal{Z}$-structure. Moreover, if G is 1-ended then the corresponding weak $\mathcal{Z}$-boundary has the shape of either a circle or a Hawaiian earring depending on whether G is a virtually surface group or not.*

Bestvina [2] already pointed out that the Baumslag-Solitar group $\langle x, t; t^{-1}xt = x^2\rangle$ admits a $\mathcal{Z}$-boundary homeomorphic to the Cantor-Hawaiian earring, which is shape equivalent to the ordinary Hawaiian earring (see [24] as a general reference for shape theory). It is worth mentioning that for the particular class of Baumslag-Solitar groups a stronger $\mathcal{EZ}$-structure has been recently described in [18].

More generally, we may construct the following class $\mathcal{C}$ of finitely presented groups starting off from one-relator group presentations as follows. Let G and H be finitely generated one-relator groups, and assume $\mathcal{P} = \langle X; r\rangle$ and $\mathcal{Q} = \langle Y; s\rangle$ are presentations of G and H with a single relation, respectively. Let $V \subset X$ and $W \subset Y$, together with a bijection $\eta : V \longrightarrow W$, be (possibly empty) subsets not containing all the generators involved in any of the relators of the corresponding presentation. We declare the corresponding amalgamated product $G *_F H$ associated with η (over a free group of rank $card(V) = card(W)$) to be in our class $\mathcal{C}$ together with the obvious presentation for it obtained from $\mathcal{P}$ and $\mathcal{Q}$. It seems natural to consider the class $\mathcal{C}$ of all finitely generated one-relator groups together with those finitely presented groups which can be obtained by successive applications of the construction above, so that $\mathcal{C}$ is closed under amalgamated products (over free subgroups) of the type just described. The group presentations obtained in this way still satisfy a Freiheitssatz property, see [21] for more details. Henceforth, we will refer to those groups as in $\mathcal{C}$ as "generalized" one-relator groups.

Remark 1.3. The first interesting examples of groups in the class $\mathcal{C}$ (other than one-relator groups and their free products) are those groups G given by a presentation of the form $\langle X; r, s\rangle$, where r, s are cyclically reduced words so that $r \in F(Y)$, $Y \subsetneq X$, and $s \in F(X) - F(Y)$ misses at least one generator in Y which occurs in r. Indeed, one can obtain G as an amalgamated product $\langle Y; r\rangle *_F \langle (X - Y) \cup Z; s\rangle$

over a free group of rank $card(Z)$, where $Z \subsetneq Y$ is the subset consisting of those generators in Y which occur in s.

Remark 1.4. From the above remark, one can see that Higman's group H, with presentation $\langle a, b, c, d; a^{-2}b^{-1}ab, b^{-2}c^{-1}bc, c^{-2}d^{-1}cd, d^{-2}a^{-1}da\rangle$, is in $\mathcal{C}$. Indeed, H can be expressed as an amalgamated product (of the type described above) of two copies of $\langle x, y, z; x^{-2}y^{-1}xy, y^{-2}z^{-1}yz\rangle$ over a free subgroup of rank 2 (see [19]).

Remark 1.5. It was also shown in [21] that every finitely presented group G given by a staggered presentation $\mathcal{P}$ is in $\mathcal{C}$. We recall that $\langle X; R\rangle$ is defined to be a *staggered* presentation if there is a subset $X_0 \subset X$ so that both R and X_0 are linearly ordered in such a way that: (i) each relator $r \in R$ contains some $x \in X_0$; (ii) if r, r' are relators with $r < r'$, then r contains some $x \in X_0$ that precedes all elements of X_0 occurring in r', and r' contains some $y \in X_0$ that comes after all those occurring in r (see [22]).

Theorem 1.2 above together with the results in [21] yield the following generalization.

Theorem 1.6. *Every torsion free, 1-ended generalized one-relator group $G \in \mathcal{C}$ admits a weak $\mathcal{Z}$-structure. Moreover, the corresponding weak $\mathcal{Z}$-boundary has the shape of either a circle or a Hawaiian earring depending on whether G is a virtually surface group or not.*

2. Preliminaries

Given a non-compact (strongly) locally finite CW-complex Y, a *proper ray* in Y is a proper map $\omega : [0, \infty) \longrightarrow Y$. Recall that a proper map is a map with the property that the inverse image of every compact subset is compact. We say that two proper rays ω, ω' *define the same end* if their restrictions to the natural numbers $\omega|_{\mathbb{N}}, \omega'|_{\mathbb{N}}$ are properly homotopic. This equivalence relation gives rise to the notion of *end determined by* ω as the corresponding equivalence class, as well as the space of ends $\mathcal{E}(Y)$ of Y as a compact totally disconnected metrizable space (see [1, 13]). The CW-complex Y is *semistable* at the end determined by ω if any other proper ray defining the same end is in fact properly homotopic to ω; equivalently, if the fundamental pro-group $pro - \pi_1(Y, \omega)$ is pro-isomorphic to a tower of groups with surjective bonding homomorphisms (see [13, Prop. 16.1.2]). Recall that the homotopy pro-groups $pro - \pi_n(Y, \omega)$ are represented by the inverse sequences (tower) of groups

$$\pi_n(Y, \omega(0)) \xleftarrow{\phi_1} \pi_n(Y - C_1, \omega(t_1)) \xleftarrow{\phi_2} \pi_n(Y - C_2, \omega(t_2)) \longleftarrow \cdots$$

where $C_1 \subset C_2 \subset \cdots \subset Y$ is a filtration of Y by compact subspaces, $\omega([t_i, \infty)) \subset Y - C_i$ and the bonding homomorphisms ϕ_i are induced by the inclusions and basepoint-change isomorphisms. One can show the independence with respect to the filtration. Also, properly homotopic base rays yield pro-isomorphic homotopy pro-groups $pro - \pi_n$, for all n. If Y is semistable at each end then we will simply say that Y is semistable at infinity, and in this case two proper rays representing the same end yield the same (up to pro-isomorphism) homotopy pro-groups $pro - \pi_n$. We refer to [13, 24] for more details.

Given a CW-complex X, with $\pi_1(X) \cong G$, we will denote by $\widetilde{X}$ the universal cover of X, constructed as prescribed in ([13], §3.2), so that G is acting freely on the CW-complex $\widetilde{X}$ via a cell-permuting left action with $G\backslash\widetilde{X} = X$. The number of ends of an (infinite) finitely generated group G represents the number of ends of the (strongly) locally finite CW-complex $\widetilde{X}^1$, for some (equivalently any) CW-complex X with $\pi_1(X) \cong G$ and with finite 1-skeleton, which is either $1, 2$ or ∞ (finite groups have 0 ends [13, 27]). If G is finitely presented, then G is *semistable at infinity* if the (strongly) locally finite CW-complex $\widetilde{X}^2$ is so, for some (equivalently, any) CW-complex X with $\pi_1(X) \cong G$ and with finite 2-skeleton. Observe that any finite-dimensional locally finite CW-complex is strongly locally finite, see [13].

The following result will be crucial for the proof of the main result in this paper.

Proposition 2.1. *Let $\mathcal{P} = \langle X; r\rangle$ be a finite presentation of a torsion free group with a single (cyclically reduced) relator $r \in F(X)$, and consider the associated 2-dimensional CW-complex $K_{\mathcal{P}}$. Then, the (contractible) universal cover $\widetilde{K}_{\mathcal{P}}$ is properly aspherical at infinity, i.e., for any choice of base ray, the homotopy pro-groups $pro-\pi_n(\widetilde{K}_{\mathcal{P}}) = 0$ are pro-trivial for $n \geq 2$. Furthermore, the fundamental pro-group $pro-\pi_1(\widetilde{K}_{\mathcal{P}})$ is pro-(finitely generated free).*

Observe that the universal cover $\widetilde{K}_{\mathcal{P}}$ above is already known to be contractible (see [8]) and semistable at infinity (see [25]), and hence its homotopy pro-groups do not depend (up to pro-isomorphism) on the choice of the base ray.

Remark 2.2. Recall that the (finite) 2-dimensional CW-complex $K_{\mathcal{P}}$ associated to $\mathcal{P}$ is constructed as follows. The 0-skeleton consists of a single vertex and the 1-skeleton $K^1_{\mathcal{P}}$ consists of a bouquet of circles, one for each element of the basis $x_i \in X$, all of them sharing the single vertex in $K_{\mathcal{P}}$. Finally $K_{\mathcal{P}}$ is obtained from $K^1_{\mathcal{P}}$ by attaching a 2-cell d via a PL map $S^1 \longrightarrow K^1_{\mathcal{P}}$ which spells out the single relator r. Note that every lift in the universal cover $\tilde{d} \subset \widetilde{K}_{\mathcal{P}}$ of the 2-cell $d \subset K_{\mathcal{P}}$ is a disk as r is a cyclically reduced word. Moreover, by the Magnus' Freiheitssatz (see [22, 23]) every subcomplex of the 1-skeleton $K^1_{\mathcal{P}}$ not containing all the 1-cells involved in the relator r lifts in the universal cover $\widetilde{K}_{\mathcal{P}}$ to a disjoint union of trees.

Proposition 2.1 follows immediately from the following lemma, which is an enhancement of [4, Prop. 2.7].

Lemma 2.3. *Let $\mathcal{P} = \langle X; r\rangle$ be a finite, torsion free group presentation with a single (cyclically reduced) relator $r \in F(X)$, and consider the associated 2-dimensional CW-complex $K_{\mathcal{P}}$. Then, the universal cover $\widetilde{K}_{\mathcal{P}}$ is proper homotopy equivalent to another 2-dimensional CW-complex $\widehat{K}_{\mathcal{P}}$ which has a filtration $\widehat{C}_1 \subset \widehat{C}_2 \subset \cdots \subset \widehat{K}_{\mathcal{P}}$ by finite contractible subcomplexes satisfying (for any choice of base ray):*

(a) *The tower $\{1\} \leftarrow \pi_1(\widehat{K}_{\mathcal{P}} - \widehat{C}_1) \leftarrow \pi_1(\widehat{K}_{\mathcal{P}} - \widehat{C}_2) \leftarrow \cdots$ consists of finitely generated free groups of increasing rank, with the bonding maps being the obvious projections, and*
(b) *The tower $\{1\} \leftarrow \pi_n(\widehat{K}_{\mathcal{P}} - \widehat{C}_1) \leftarrow \pi_n(\widehat{K}_{\mathcal{P}} - \widehat{C}_2) \leftarrow \cdots$ is the trivial tower, $n \geq 2$.*

Remark 2.4. In fact, the proper homotopy equivalence in the statement of Lemma 2.3 can be replaced by a "strong" proper homotopy equivalence, i.e., a (possibly infinite) sequence of internal collapses and/or expansions, carried out in a proper fashion. See [4] for more details.

Proof. Indeed, the proof of this lemma is that of [4, Prop.2.7], only that now we extend it, by taking a closer look, so that it covers part (b) here. The proof there goes by induction on the length of the relator $r \in F(X)$ in such a presentation $\mathcal{P} = \langle X; r \rangle$. It consists of a simultaneous double induction argument keeping track of two possible cases, depending on whether there is a generator in X whose exponent sum in r is zero or not, see §3 and §4 in [4] respectively.

In the first case (§3 in [4]), one shows that the induction lies on the fact that $K'_{\mathcal{P}}$, an intermediate cover of the CW-complex $K_{\mathcal{P}}$, is made out, up to homotopy, of blocks $K_{\mathcal{P}'}$, where $\mathcal{P}'$ satisfies the inductive hypothesis. In fact, its universal cover $\widetilde{K_{\mathcal{P}'}}$ is being slightly altered (within their proper homotopy type) to a CW-complexes $\widehat{K}_{\mathcal{P}'}$ so that their copies can be assembled together resulting into a new CW-complex $\widehat{K}_{\mathcal{P}}$ strongly proper homotopy equivalent to the universal cover of $K_{\mathcal{P}}$. This new CW-complex $\widehat{K}_{\mathcal{P}}$ consists of copies of the various CW-complexes $\widehat{K}_{\mathcal{P}'}$ above, glued together along trees (which were already present in the universal cover of $K_{\mathcal{P}}$, that correspond to the intersections of the different copies of $K_{\mathcal{P}'}$ and whose existence is a consequence of the Magnus' Freiheitssatz, see Remark 2.2.

The desired filtration for $\widehat{K}_{\mathcal{P}}$ is then the result of assembling the filtrations we encounter on the various complexes $\widehat{K}_{\mathcal{P}'}$, which already have one by induction, as we grow towards infinity. This can be carefully done in such a way that if two of these CW-complexes $\widehat{K}_{\mathcal{P}'}$ meet along a tree inside $\widehat{K}_{\mathcal{P}}$ then each of the members of the corresponding filtration for each of them intersects that tree in a connected subtree.

Finally, given a compact subset $\widehat{C}_n \subset \widehat{K}_{\mathcal{P}}$ from this resulting filtration, the generalized van-Kampen theorem yields that the fundamental group $\pi_1(\widehat{K}_{\mathcal{P}} - \widehat{C}_n)$ is the free product of a free group together with the various $\pi_1(\widehat{K}_{\mathcal{P}'} - \widehat{C}'_n)$ (finitely generated free by induction), where $\widehat{C}'_n = \widehat{K}_{\mathcal{P}'} \cap \widehat{C}_n \neq \emptyset$.

The novelty here consists of adding part (b) of the statement to the induction hypothesis, and observing that each neighborhood of infinity of the form $U = \widehat{K}_{\mathcal{P}} - \widehat{C}_n$ is an assembly of the various neighborhoods of infinity $U' = \widehat{K}_{\mathcal{P}'} - \widehat{C}'_n$ (with $\widehat{C}'_n = \widehat{K}_{\mathcal{P}'} \cap \widehat{C}_n \neq \emptyset$) together with all those (contractible) copies $\widehat{K}_{\mathcal{P}'} \subset \widehat{K}_{\mathcal{P}}$ which do not intersect $\widehat{C}_n$. Moreover, if two of the neighborhoods of infinity U' (corresponding to two different copies of $\widehat{K}_{\mathcal{P}'}$) intersect inside $\widehat{K}_{\mathcal{P}}$, then they do it along the various components of $T - \widehat{C}'_n$, where $T \subset \widehat{K}_{\mathcal{P}}$ is the corresponding tree along which those copies of $\widehat{K}_{\mathcal{P}'}$ are glued together inside $\widehat{K}_{\mathcal{P}}$. This way, the universal cover $\widetilde{U}$ of $U = \widehat{K}_{\mathcal{P}} - \widehat{C}_n$ is the result of putting together the universal covers $\widetilde{U}'$ of the various neighborhoods of infinity $U' = \widehat{K}_{\mathcal{P}'} - \widehat{C}'_n$ glued along connected subtrees, together with all those copies $\widehat{K}_{\mathcal{P}'} \subset \widehat{K}_{\mathcal{P}}$ which do not intersect $\widehat{C}_n$, each one glued to the rest along a copy of the corresponding tree from the construction indicated above. Thus, the induction hypothesis guarantees that each $\widetilde{U}'$ is a contractible CW-complex and hence part (b) follows for $\widehat{K}_{\mathcal{P}}$.

As for the second case (§4 in [4]), in which there is no generator in X whose exponent sum in r is zero, the proof goes somehow the other way around. An auxiliary CW-complex $K_{\mathcal{P}'}$ is built. For such $K_{\mathcal{P}'}$ the induction hypothesis applies since it has a generator whose exponent sum is zero in the presentation $\mathcal{P}'$, which lies under the inductive hypothesis for the previous case (§3 in [4]). As above, its universal cover can be slightly altered (within its proper homotopy type) to a new CW-complex $\widehat{K}_{\mathcal{P}'}$ which is made out of blocks, corresponding to copies of our candidates for the CW-complex $\widehat{K}_{\mathcal{P}}$ in question, glued together along copies of the real line. Given an appropriate filtration $\widehat{C}'_n \subset \widehat{K}_{\mathcal{P}'}$ by compact subsets (provided by the induction hypothesis) satisfying the required properties for $\widehat{K}_{\mathcal{P}'}$, one can get the desired filtration on each copy $\widehat{K}_{\mathcal{P}}$ inside $\widehat{K}_{\mathcal{P}'}$ simply by considering the intersections $\widehat{C}_n = \widehat{K}_{\mathcal{P}} \cap \widehat{C}'_n$. Observe that this procedure may yield different choices for the desired filtration on each of those copies of $\widehat{K}_{\mathcal{P}}$, but they all satisfy the required properties (a)-(b). Indeed, by induction, each neighborhood of infinity in $\widehat{K}_{\mathcal{P}'}$ of the form $U' = \widehat{K}_{\mathcal{P}'} - \widehat{C}'_n$ has finitely generated free fundamental group and trivial higher homotopy groups. From here, the argument is similar to the one given above, concluding that the corresponding neighborhoods of infinity $U = \widehat{K}_{\mathcal{P}} - \widehat{C}_n$ in each copy $\widehat{K}_{\mathcal{P}}$ inside $\widehat{K}_{\mathcal{P}'}$ behave in the same way (as each $\pi_1(U)$ is now a free factor of $\pi_1(U')$). □

A tower of groups $F \equiv (\{1\} \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots)$ consisting of finitely generated free groups of non-decreasing rank and the obvious projections as bonding maps will be said to be "telescopic" (or of telescopic type). One can always associate to any given telescopic tower a 1-ended locally-finite (simply connected) 2-dimensional CW-complex Y_m, $0 \leq m \leq \infty$, whose fundamental pro-group realizes that telescopic tower as follows. Set $Y_0 = \{*\} \times [0, \infty)$ (a copy of $\mathbb{R}_+$). Assume Y_n constructed, $n \in \mathbb{N} \cup \{0\}$. Then, Y_{n+1} consists of the proper wedge of Y_n and a copy $S^1 \times [n, \infty) \cup D^2 \times \{n\}$ of $\mathbb{R}^2$ attached along Y_0. Finally, we set $Y_\infty = \cup_{n \geq 0} Y_n$. Indeed, one can easily check that for some $0 \leq m \leq \infty$ and some filtration $\{J_n\}_{n \geq 1}$ of Y_m, there is a pro-isomorphism $\psi = \{\psi_n\}_{n \geq 1} : pro - \pi_1(Y_m) \longrightarrow F$, where each $\psi_n : \pi_1(Y_m - J_n) \longrightarrow F_n$ is an isomorphism between finitely generated free groups. Observe that the proper homotopy type of Y_m can be represented by a subpolyhedron of $\mathbb{R}^3$, see the figure below.

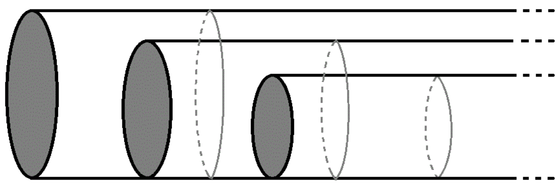

FIGURE 1. Locally finite subpolyhedron of $\mathbb{R}^3$

Corollary 2.5. *With the above notation, in the* 1*-ended case the universal cover* $\widetilde{K}_{\mathcal{P}}$ *is proper homotopy equivalent to either* $Y_1 (= \mathbb{R}^2)$ *or* Y_∞.

Proof. According to the above, by Lemma 2.3 (a), there is some $0 \leq m \leq \infty$ and a pro-isomorphism $\psi = \{\psi_n\}_n : pro - \pi_1(Y_m) \longrightarrow pro - \pi_1(\widehat{K}_{\mathcal{P}})$, with each $\psi_n : \pi_1(Y_m - J_n) \longrightarrow \pi_1(\widehat{K}_{\mathcal{P}} - \widehat{C}_n)$ being an isomorphism between finitely generated free groups. Moreover, by Lemma 2.3 (b) and [6, Prop. 3.3], there is a proper map $f : Y_m \longrightarrow \widehat{K}_{\mathcal{P}}$ inducing the pro-isomorphism ψ; in fact, f is a weak proper homotopy equivalence, as Y_m is clearly properly aspherical at infinity as well, and hence f induces pro-isomorphisms between all the homotopy pro-groups. Therefore, by the corresponding proper Whitehead theorem (see [10, Thm. 5.5.3] or [1, § 8], for instance) f is in fact a proper homotopy equivalence.

It remains to show that $m = 1$ or ∞. For this, observe that $m > 0$ since otherwise $\widehat{K}_{\mathcal{P}}$ (and hence $\widetilde{K}_{\mathcal{P}}$) would be proper homotopy equivalent to a 3-manifold with a single plane on its boundary (as $Y_0 = \mathbb{R}_+$ thickens to a 3-dimensional half-space), which is not possible by [5, Cor. 5.14]. Furthermore, Y_m (and hence $\widetilde{K}_{\mathcal{P}}$) must be proper homotopy equivalent to a 3-manifold with boundary (by means of a regular neighborhood of the subpolyhedron of $\mathbb{R}^3$ in the figure above) which can only have either two or infinitely many plane boundary components, by [5, Cor. 5.11, 5.14]. The rest of the proof follows from this and the fact that the first option only occurs in the case of a virtually surface group, see [5, Thm. 5.17]. □

Remark 2.6. In terms of [5], every 1-ended, torsion free one-relator group is proper 2-equivalent to either $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{F}_2 \times \mathbb{Z}$ by [5, Thm 5.1], as one relator groups are properly 3-realizable, see [4, 21]; in fact, given a presentation $\mathcal{P}$ as above, the universal cover of $K_{\mathcal{P}}$ itself is proper homotopy equivalent to a 3-manifold (by considering a regular neighborhood of the above subpolyhedron in $\mathbb{R}^3$) with no need to take wedge with a single 2-sphere, thus answering in the affirmative a conjecture posed in [4] (in the torsion free case). Observe that the third option $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ from [5, Thm 5.1] is ruled out by [5, Cor. 5.14], and the first option $\mathbb{Z} \times \mathbb{Z}$ only occurs in the case of a virtually surface group, by [5, Thm. 5.17].

3. Proof of the main results

The purpose of this section is to prove Theorems 1.2 and 1.6. For this, we need the following previous result, which is a combination of other well known results.

Lemma 3.1. *Let X be a locally finite n-dimensional (PL)CW-complex. If the following two conditions hold:*

- (a) *X is inward tame, and*
- (b) *For any choice of base ray, the fundamental pro-group $pro - \pi_1(X)$ is pro-(finitely generated free)*

then the product $X \times I^{2n+5}$ admits a $\mathcal{Z}$-compactification, with $I = [0, 1]$.

We recall that an ANR X is *inward tame* if, for each neighborhood of infinity N there exists a smaller neighbohood of infinity $N' \subset N$ so that, up to homotopy, the inclusion $N' \hookrightarrow N$ factors through a finite complex; equivalently, for every closed neighborhood of infinity N there is a homotopy $H : N \times [0, 1] \longrightarrow N$ with $H_0 = id_N$ and $\overline{H_1(N)}$ compact, see [16]. One can check that inward tameness is a proper homotopy invariant, and we may think of it as pulling the end of X inside X yielding some kind of finite domination at infinity. It is worth noticing that any $\mathcal{Z}$-compactifiable ANR must be inward tame, see [16, Remark 3.8.13]. Also, observe that as a combination of the results in [12] and [14], there is an example of a locally finite 2-dimensional polyhedron X whose product $X \times I^9$ is $\mathcal{Z}$-compactifiable but X itself is not.

Proof. Let I^∞ denote the Hilbert cube. It is well known that the product $Y = X \times I^\infty$ is a Hilbert cube manifold (see [29, 9]) which satisfies again properties (a) and (b) from the statement, as I^∞ is compact and contractible and so X and Y are proper homotopy equivalent. In particular, Y is inward tame. Moreover, the Chapman-Siebenmann obstructions for a Hilbert cube manifold admitting a $\mathcal{Z}$-compactification ([7, Thms 3, 4], see also [16, §3.8.2]) vanish for Y since $pro-\pi_1(Y)$ can be represented by an inverse sequence

$$\pi_1(Y) \longleftarrow \pi_1(N_1) \longleftarrow \pi_1(N_2) \longleftarrow \cdots$$

where $\{N_i\}_i$ is a nested cofinal sequence of neighborhoods of infinity in Y with $\pi_1(N_i)$ a finitely generated free group, $i \geq 1$; in fact, each N_i can be taken as a product $N_i = M_i \times I^\infty$, where M_i is a neighborhood of infinity in X. Thus $Y = X \times I^\infty$ admits a $\mathcal{Z}$-compactification. Finally, the results in [12] show that $X \times I^{2n+5}$ admits a $\mathcal{Z}$-compactification as well. □

We now proceed with the proof of the main results.

Proof of Theorem 1.2. Suppose a given torsion free finitely presented group G admits a finite presentation $\mathcal{P} = \langle X; r\rangle$ with a single (cyclically reduced) relator $r \in F(X)$. If G is 2-ended then G must be the group of integers $\mathbb{Z}$ (see [27, Thm. 5.12]) which easily admits a weak $\mathcal{Z}$-structure just by adding two points as its boundary. Assume now G is 1-ended. Then, by Corollary 2.5, the universal cover $\widetilde{K}_{\mathcal{P}}$ is proper homotopy equivalent to either the plane $\mathbb{R}^2$ or the locally finite subpolydedron of $\mathbb{R}^3$ shown in figure 1, which are both easily shown to be inward tame, and hence so is $\widetilde{K}_{\mathcal{P}}$. On the other hand, Proposition 2.1 ensures condition (b) in Lemma 3.1 above. Therefore, the (contractible) CW-complex $\widetilde{K}_{\mathcal{P}} \times I^9$ admits a $\mathcal{Z}$-compactification. Observe that the proper, free and cocompact G action on $\widetilde{K}_{\mathcal{P}}$ yields a proper, free and cocompact G action on $\widetilde{K}_{\mathcal{P}} \times I^9$ in the obvious way, thus providing a weak $\mathcal{Z}$-structure on G whose associated weak $\mathcal{Z}$-boundary has the shape of the $\mathcal{Z}$-boundary of a $\mathcal{Z}$-compactification of either the plane or the subpolyhedron shown in figure 1, see [16, Cor. 3.8.15]. In the case of the plane this $\mathcal{Z}$-boundary has the shape of a circle, and in the second case one can easily show that the corresponding $\mathcal{Z}$-boundary has the shape of a Hawaiian earring, as claimed.

Finally, if G is infinite ended then G decomposes as a free product of groups (as G is torsion free) by the Stallings's structure theorem (see [13,27]). Moreover, being G a one-relator group, it follows from Grushko's theorem that G is a free product of a free group and a one-relator group with at most one end. See [22, Prop. II.5.13] for details. Both factors admit a weak $\mathcal{Z}$-structure and hence so does their free product, by the proof of [28, Thm. 2.9]. □

Just as we did in section §2 with respect to the work in [4], a closer look at the proofs of [21, Thm. 1.13] and [21, Prop. 1.18] yields the following generalization of Proposition 2.1 and Lemma 2.3 (in the 1-ended case).

Proposition 3.2. *Let $\mathcal{P} = \langle X; R\rangle$ be a finite aspherical presentation of a torsion free, 1-ended generalized one-relator group $G \in \mathcal{C}$, with each $r \in R$ being a cyclically reduced word in $F(X)$, and consider the associated 2-dimensional CW-complex $K_{\mathcal{P}}$. Then, the (contractible) universal cover $\widetilde{K}_{\mathcal{P}}$ is properly aspherical at infinity, i.e., for any choice of base ray, the homotopy pro-groups $pro-\pi_n(\widetilde{K}_{\mathcal{P}}) = 0$ are pro-trivial for $n \geq 2$, and the fundamental pro-group $pro-\pi_1(\widetilde{K}_{\mathcal{P}})$ is pro-isomorphic to a telescopic tower*

Thus, the proof of Theorem 1.6 is the same as that of Theorem 1.2 in the 1-ended case.

Remark 3.3. It is worth pointing out that sometimes the strategy followed to prove that some classes of 1-ended groups admit a weak $\mathcal{Z}$-structure includes showing that the fundamental pro-group is pro-(finitely generated free). Under semistability at infinity, this property about the fundamental pro-group amounts to saying that the groups under study are *properly* 3*-realizable*, i.e., they can be realized by a finite 2-dimensional CW-complex whose universal cover is proper homotopy equivalent to a 3-manifold. See [20, Thm. 1.2]) and [5, Thm. 5.22]. The above is the case of this and other papers, see [15] for instance. At the time of writing it is unknown whether there is a relation between proper 3-realizability and the existence of a weak $\mathcal{Z}$-structure.

References

[1] H. J. Baues, A. Quintero. Infinite Homotopy Theory. *K-Monographs in Mathematics*, vol. 6, Kluwer, 2001.

[2] M. Bestvina. Local Homology Properties of Boundaries of Groups. *Michigan Math. J.*, vol. 43, no. 4 (1996), 783-788.

[3] M. Bestvina, G. Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.*, vol. 4, no. 3 (1991), 469-481.

[4] M. Cárdenas, F. F. Lasheras, A. Quintero, D. Repovš. One-relator groups and proper 3-realizability *Rev. Mat. Iberoamericana*, vol. 25, no. 2 (2009), 739-756.

[5] M. Cárdenas, F. F. Lasheras, A. Quintero, R. Roy. A topological equivalence relation for finitely presented groups. *J. Pure Appl. Algebra*, vol. 224, no. 7 (2020), 25 pp.

[6] M. Cárdenas, F. Muro, A. Quintero. The proper L-S category of Whitehead manifolds. *Top. Appl.*, vol. 153 (2005), 557-579.

[7] T. A. Chapman, L. C. Siebenmann. Finding a boundary for a Hilbert cube manifold. *Acta Math.*, vol. 137, no. 3-4 (1976), 171-208.

[8] E. Dyer, A. T. Vasquez. Some small aspherical spaces. *J. Austral. Math. Soc.*, 16 (1973), 332-352.

[9] R. D. Edwards. Characterizing infinite-dimensional manifolds topologically (after Henryk Torunczyk). *Séminaire Bourbaki (1978/79)*, Exp. No. 540, *Lecture Notes in Math.* 770 (1980), Springer, 278-302.

[10] D. A. Edwards, H. M. Hastings. Čech and Steenrod homotopy theories with applications to geometric topology. *Lecture Notes in Mathematics*, vol. 542. Springer-Verlag, 1976.

[11] F. T. Farrell, J-F. Lafont. *EZ*-structures and topological applications. *Comment. Math. Helv.*, vol. 80 (2005), 103-121.

[12] S. C. Ferry. Stable compactifications of polyhedra. *Michigan Math. J.*, vol. 47, no. 2 (2000), 287-294.

[13] R. Geoghegan. Topological Methods in Group Theory. *Graduate Texts in Mathematics* 243, Springer, 2008.

[14] C. R. Guilbault. A non-$\mathcal{Z}$-compactifiable polyhedron whose product with the Hilbert cube is $\mathcal{Z}$Z-compactifiable. *Fund. Math.*, vol. 168, no. 2 (2001), 165-197.

[15] C. R. Guilbault. Weak $\mathcal{Z}$-structures for some classes of groups. *Algebr. Geom. Topol.*, vol. 14, no. 2 (2014), 1123-1152.

[16] C. R. Guilbault. Ends, shapes, and boundaries in manifold topology and geometric group theory. *Topology and geometric group theory*, *Springer Proc. Math. Stat.*, vol. 184 (2016), Springer, 45-125.

[17] C. R. Guilbault, M. A. Moran. Proper homotopy types and $\mathcal{Z}$-boundaries of spaces admitting geometric group actions. *Expo. Math.*, vol. 37, no. 3 (2019), 292-313.

[18] C. R. Guilbault, M. A. Moran, C. J. Tirel. Boundaries of Baumslag-Solitar groups. *Algebr. Geom. Topol.*, vol. 19 (2019), 2077-2097.

[19] G. Higman. A finitely generated infinite simple group. *J. London Math. Soc.*, vol. 16 (1973), 332-352.

[20] F. F. Lasheras. Ascending HNN-extensions and properly 3-realisable groups. *Bull. Austral. Math. Soc.*, vol. 72 (2005), 187-196.

[21] F. F. Lasheras, R. Roy. Relating the Freiheitssatz to the asymptotic behavior of a group. *Rev. Mat. Iberoamericana*, vol. 29, no. 1 (2013), 75-89.
[22] R. C. Lyndon, P. E. Schupp. Combinatorial Group Theory. *Springer-Verlag, Berlin*, 1977.
[23] W. Magnus. Über diskontinuierliche Gruppen mit einer definierenden Relation. (Der FHS) *J. reine angew. Math.*, vol. 163 (1930), 141-165.
[24] S. Mardešic, J. Segal. Shape Theory. *North-Holland*, 1982.
[25] M. Mihalik, S. Tschantz. One relator groups are semistable at infinity. *Topology*, vol. 31, no. 4 (1992), 801-804.
[26] M. A. Moran. Finite-dimensionality of $\mathcal{Z}$-boundaries. *Groups Geom. Dyn.*, vol. 10, no. 2 (2016), 819-824.
[27] P. Scott, C. T. C. Wall. Topological methods in group theory. *Homological Group Theory, London Math. Soc. Lecture Notes*, Cambridge Univ. Press, Cambridge (1979), 137-204.
[28] C. J. Tirel. $\mathcal{Z}$-structures on product groups. *Algebr. Geom. Topol.*, vol. 11 (2011), 2587-2625.
[29] J. E. West. Factoring the Hilbert cube. *Bull. Amer. Math. Soc.*, vol. 76 (1970), 116-120.

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