PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, Number 4, April 2019, Pages 1797–1804
https://doi.org/10.1090/proc/14373
Article electronically published on December 12, 2018

QUASI-ISOMETRIES AND PROPER HOMOTOPY: THE QUASI-ISOMETRY INVARIANCE OF PROPER 3-REALIZABILITY OF GROUPS

M. CÁRDENAS, F. F. LASHERAS, A. QUINTERO, AND R. ROY

(Communicated by Ken Bromberg)

Abstract. We recall that a finitely presented group G is properly 3-realizable if for some finite 2-dimensional CW-complex X with $\pi_1(X) \cong G$, the universal cover $\widetilde{X}$ has the proper homotopy type of a 3-manifold. This purely topological property is closely related to the asymptotic behavior of the group G. We show that proper 3-realizability is also a geometric property meaning that it is a quasi-isometry invariant for finitely presented groups. In fact, in this paper we prove that (after taking wedge with a single n-sphere) any two infinite quasi-isometric groups of type F_n ($n \geq 2$) have universal covers whose n-skeleta are proper homotopy equivalent. Recall that a group G is of type F_n if it admits a $K(G,1)$-complex with finite n-skeleton.

1. Introduction

Generally, we will be working within the category of locally finite CW-complexes and proper maps. A *proper* map is a map with the property that the inverse image of every compact subset is compact. Thus, two locally finite CW-complexes are said to be proper homotopy equivalent if they are homotopy equivalent and all homotopies involved are proper.

Given an infinite (strongly) locally finite CW-complex Y, a *proper ray* in Y is a proper map $\omega : [0,\infty) \longrightarrow Y$. We say that two proper rays ω, ω' *define the same end* if their restriction to the natural numbers $\omega|_{\mathbb{N}}, \omega'|_{\mathbb{N}}$ are properly homotopic. This equivalence relation gives rise to the notion of *end determined by* ω as the corresponding equivalence class, as well as the space of ends $\mathcal{E}(Y)$ of Y as a compact totally disconnected metrizable space (see [3,14]). The CW-complex Y is *semistable* at the end determined by ω if any other proper ray defining the same end is in fact properly homotopic to ω.

The number of ends of a finitely generated group G represents the number of ends of the (strongly) locally finite CW-complex $\widetilde{X}^1$ for some (equivalently any) CW-complex X with $\pi_1(X) \cong G$ and with finite 1-skeleton. This number equals $0, 1, 2$, or ∞ (finite groups have 0 ends; see [14,21]). Here, $\widetilde{X}$ denotes the universal cover of X. We say that a finitely presented group G is *semistable at each end* (resp., *at infinity*, if G is 1-ended) if the (strongly) locally finite CW-complex $\widetilde{X}^2$ is

Received by the editors April 10, 2017, and, in revised form, July 10, 2018.
2010 *Mathematics Subject Classification.* Primary 57M07; Secondary 57M10, 57M20.
Key words and phrases. Proper homotopy, quasi-isometry, properly 3-realizable, 3-manifold.
This work was partially supported by the project MTM 2015-65397.

so, for some (equivalently any) CW-complex X with $\pi_1(X) \cong G$ and with finite 2-skeleton. Observe that any finite-dimensional locally finite CW-complex is strongly locally finite; see [14, Prop. 10.1.12].

We recall that a finitely presented group G is *properly* 3*-realizable* if for some finite 2-dimensional CW-complex X with $\pi_1(X) \cong G$, the universal cover $\widetilde{X}$ of X has the proper homotopy type of a 3-manifold M (with boundary). Notice that it follows from [6] that $\widetilde{X}$ always has the proper homotopy type of a 4-manifold. This concept was originally motivated by the fact that Lefschetz duality then yields the freeness of the second cohomology $H^2(G;\mathbb{Z}G)$, as it is a direct summand of $H^2_c(\widetilde{X};\mathbb{Z}) \cong H^2_c(M;\mathbb{Z}) \cong H_1(M,\partial M;\mathbb{Z}) \cong \widetilde{H}_0(\partial M;\mathbb{Z})$ (see [15]). This property is a virtual property which does not depend on the choice of X after taking wedge with a single 2-sphere S^2 [1]. It is worth mentioning that direct products and ascending HNN-extensions of infinite finitely presented groups are properly 3-realizable [11, 17]. In fact, any extension of an infinite finitely presented group by another infinite finitely presented group is properly 3-realizable [10]. One-relator groups are also properly 3-realizable [9].

Also, it has been proved that the class of properly 3-realizable groups is closed under amalgamated products over finite groups [8], and contains the class of all groups whose fundamental pro-group at each end is semistable and pro-(finitely generated free) [17, 18]. Moreover, it is conjectured in [13] that those two classes are the same (and examples of non-proper 3-realizable groups are given), and it has been proved so in [13] under the QSF (*quasi-simply filtered*) property, i.e., roughly speaking, that the corresponding universal cover admits an exhaustion that can be "approximated" by simply connected complexes (see [5]). To this respect, the stronger conjecture that all finitely presented groups are QSF has been posed in [20] (and related papers).

One basic notion in the asymptotic geometry of discrete groups is that of *quasi-isometry.* In general, given two pseudometric spaces (X,d) and (X',d'), a quasi-isometry between them is a (not necessarily continuous) function $f : X \longrightarrow X'$ for which there exist constants $\lambda, \epsilon > 0$ and $C \geq 0$ satisfying:

(i) $\frac{1}{\lambda}d(x,y) - \epsilon \leq d'(f(x),f(y)) \leq \lambda d(x,y) + \epsilon$ for all $x,y \in X$.
(ii) For every $x' \in X'$, there exists $x \in X$ so that $d'(f(x),x') \leq C$.

In the case of a finitely generated group G, consider the word metric on G with respect to some finite generating set $S \subset G$; namely, $d(g,h) = l(h^{-1}g)$, where $l(\theta)$ stands for the length of $\theta \in G$ as a word in $S \cup S^{-1}$. For a (connected) CW-complex Y, consider the following natural pseudometric ρ on Y, where $\rho(x,y)$ is defined as the least integer k so that there is a path in Y joining x to y which meets the interior of $k+1$ different cells.

It is worth mentioning that if G is a finitely generated group of type F_n (together with some, equivalently any, generating set S) and X is a $K(G,1)$-complex with finite n-skeleton, then $\widetilde{X}^n$ and G are quasi-isometric with respect to the (pseudo)metrics considered above (see [14, §18.2]). Again, here $\widetilde{X}$ denotes the universal cover of X. One can think of this as a nexus between the asymptotic geometry and the asymptotic topology of discrete groups, since many topological invariants of $\widetilde{X}$ translate into topological invariants of the group G.

In this context, our main result is the following.

Theorem 1.1. *Let G and H be two infinite finitely generated quasi-isometric groups of type F_n $(n \geq 2)$. Then, for any $K(G,1)$-complex X and any $K(H,1)$-complex Y with finite n-skeleta there exists a proper homotopy equivalence between the universal covers $\widetilde{X^n \vee S^n}$ and $\widetilde{Y^n \vee S^n}$.*

And its corollaries are the following.

Corollary 1.2. *Let G and H be two infinite finitely presented quasi-isometric groups. Then, for any finite 2-dimensional CW-complexes X and Y with $\pi_1(X) \cong G$ and $\pi_1(Y) \cong H$ there exists a proper homotopy equivalence between the universal covers $\widetilde{X \vee S^2}$ and $\widetilde{Y \vee S^2}$.*

This readily follows from the spacial case $n = 2$ of Theorem 1.1, as any finite 2-dimensional CW-complex X, with $\pi_1(X) \cong G$, can be enlarged to a $K(G,1)$-complex W with $W^2 = X$ (see [14, Prop. 7.1.5]).

Corollary 1.3. *Let G and H be two infinite finitely presented quasi-isometric groups. Then, G is properly 3-realizable if and only if H is so. In particular, if $N \leq G$ is a finite normal subgroup, then G is properly 3-realizable if and only if G/N is so.*

Observe that finite groups are all properly 3-realizable (see [1]). Thus, proper 3-realizability becomes a quasi-isometry invariant for finitely presented groups.

2. Proof of the main result

It was proved in [4] that the number of ends and the semistability are quasi-isometry invariants for finitely presented groups. In fact, these results are consequences of the following proposition.

Proposition 2.1 ([14, Thm. 18.2.11])**.** *Let G and H be two finitely generated groups of type F_n $(n \geq 2)$, let X be a $K(G,1)$-complex with finite n-skeleton, and let Y be a $K(H,1)$-complex Y with finite n-skeleton. If G and H are quasi-isometric, then the universal covers $\widetilde{X^n}$ and $\widetilde{Y^n}$ are proper n-equivalent.*

A proper n-equivalence is used here in the sense of [14]. Namely, we have the following.

Definition 2.2 ([14, §11.1])**.** A proper cellular map $f : X \longrightarrow Y$ between finite-dimensional locally finite CW-complexes is a proper n-equivalence if there is another proper cellular map $g : Y \longrightarrow X$ such that the restrictions $g \circ f|X^{n-1}$ and $f \circ g|Y^{n-1}$ are proper homotopic to the inclusion maps $X^{n-1} \subseteq X$ and $Y^{n-1} \subseteq Y$.

An n-equivalence as in Definition 2.2 is a stronger version of the notion of a proper $(n-1)$-type which extends the classical $(n-1)$-type in ordinary homotopy theory introduced by J. H. C. Whitehead [22]. More precisely, we have the following.

Definition 2.3. Let X and Y be two finite-dimensional locally finite CW-complexes. We say that they have the same proper n-type if there exist proper maps $f : X^{n+1} \longrightarrow Y^{n+1}$ and $g : Y^{n+1} \longrightarrow X^{n+1}$ such that the restrictions $g \circ f|X^n$ and $f \circ g|Y^n$ are proper homotopic to the inclusion maps $X^n \subseteq X^{n+1}$ and $Y^n \subseteq Y^{n+1}$.

Obviously, two n-dimensional CW-complexes are properly n-equivalent if and only if they have the same proper $(n-1)$-type. Recall that, as an immediate consequence of the proper cellular approximation theorem (see [3, 14]), proper 2-equivalences preserve the spaces of ends. In fact, in what follows, all maps can be assumed to be cellular. We have the following lemma.

Lemma 2.4. *Let $\varphi : X \longrightarrow X'$ and $\psi : Y \longrightarrow Y'$ be proper homotopy equivalences between n-dimensional locally finite CW-complexes. Then, X and Y are proper n-equivalent (or, equivalently, they have the same proper $(n-1)$-type) if and only if X' and Y' are so.*

Indeed, if φ and ψ are cellular and $f : X \longrightarrow Y$ is a proper n-equivalence, then so is $\psi \circ f \circ \varphi' : X' \longrightarrow Y'$, where φ' is a (cellular) proper homotopy inverse of φ.

Recall that we work within the category of (strongly) locally finite CW-complexes and proper maps. Although this category is not closed under push-outs, it contains enough ones to allow the basic homotopical constructions. Given a locally finite tree T, by a *properly T-based CW-complex* we mean a pair (X, ω) where $\omega : T \longrightarrow X$ is a proper map. Moreover, (X, ω) is said to be *properly well T-based* if ω is a proper cofibration; i.e., ω has the proper homotopy extension property (for instance, a cellular embedding). By using the mapping cylinder of ω, any properly T-based CW-complex (X, ω) is a proper deformation retract of a properly T-based CW-complex (X', ω) such that $\omega : T \longrightarrow X'$ is properly homotopic to a cellular embedding.

A *relative proper map* $f : (X, \omega) \longrightarrow (Y, \omega')$ is a proper map between properly T-based CW-complexes such that $f \circ \omega = \omega'$. Similarly, a relative proper homotopy between relative proper maps $f, g : (X, \omega) \longrightarrow (Y, \omega')$ is a proper homotopy $H : X \times I \longrightarrow Y$ such that $H_t : (X, \omega) \longrightarrow (Y, \omega')$ is a relative proper map for any t.

In order to deal with all the ends of X simultaneously, we define a *CW-complex under T* to be a properly well T-based CW-complex (X, ω) such that ω is *end-faithful*; i.e., ω induces a homeomorphism $\omega_* : \mathcal{E}(T) \longrightarrow \mathcal{E}(X)$ between the spaces of ends. A *map under T* is a relative proper map $f : (X, \omega) \longrightarrow (Y, \omega')$ between CW-complexes under T. A relative homotopy between maps under T is termed a *homotopy under T*.

Given a locally finite tree T, relative n-types are defined similarly for properly T-based CW-complexes (X, ω) and (Y, ω') by requiring the homotopies to be relative to the proper maps $\omega : T \longrightarrow X$ and $\omega' : T \longrightarrow Y$. We next show that proper n-equivalences between n-dimensional CW-complexes can be replaced by suitable relative n-equivalences.

Let us start by observing that the argument in the proof given in [19, Chap. 6, §5] (see also [12, I.2.3]) of a theorem of Dold works for proper maps and proper homotopies and it also shows the following result.

Proposition 2.5. *Let $f : X \longrightarrow Y$ be a proper n-equivalence and let $i : A \longrightarrow X$ and $j : A \longrightarrow Y$ be proper cofibrations with* $\dim A \leq n-1$ *and $f \circ i = j$. Then, $f : (X, i) \longrightarrow (Y, j)$ is a relative proper n-equivalence.*

Indeed, as in [19, Chap. 6, §5], the proof of Proposition 2.5 reduces to the following. Given a (relative) proper map $f : X \longrightarrow X$ with $f \circ i = i$ and $f|X^{n-1}$ proper homotopic to the inclusion $X^{n-1} \subseteq X$, it suffices to obtain a (relative) proper map $g : X \longrightarrow X$ with $g \circ i = i$ and such that $g \circ f|X^{n-1}$ is proper homotopic to the inclusion $X^{n-1} \subseteq X$ rel. A. The rest of the proof goes as in [19, Chap. 6, §5], using the fact that the inclusion $X^{n-1} \subseteq X$ is a proper cofibration as well.

Proposition 2.6. *Let $n \geq 2$, and assume that the n-dimensional locally finite CW-complexes X and Y are proper n-equivalent (or, equivalently, they have the same proper $(n-1)$-type). Then, for the inclusion $i : T \longrightarrow X$ of any end-faithful tree $T \subset X$ there exists an n-dimensional locally finite CW-complex $(\overline{Y}, i')$ under T, in the proper homotopy class of Y, such that (X, i) and $(\overline{Y}, i')$ have the same relative $(n-1)$-type.*

Proof. Let $i : T \longrightarrow X$ be the inclusion. Given a proper n-equivalence $f : X \longrightarrow Y$, let $\overline{X} = M_i$ and $\overline{Y} = M_{f \circ i}$ be the corresponding mapping cylinders, and let $\widetilde{f} : \overline{X} \longrightarrow \overline{Y}$ be the proper map induced by f. Notice that $\dim \overline{X} = \dim \overline{Y} = n$. Let $i_1 : T \longrightarrow \overline{X}$ and $i'_1 : T \longrightarrow \overline{Y}$ denote the cofibrations given by the cellular embeddings of T as the top copies in the corresponding mapping cylinders. Notice that the canonical projection $p : (\overline{X}, i_1) \longrightarrow (X, i)$ is a proper map under T which is a proper homotopy equivalence. Hence, by using Dold's theorem [3, Thm. I.4.16] (also directly derived from [19, Chap. 6, §5]) one gets that p is in fact a proper homotopy equivalence under T. Let $q : (X, i) \to (\overline{X}, i_1)$ be a homotopy inverse of p under T. Then, the composite $\overline{f} = \widetilde{f} \circ q : (X, i) \longrightarrow (\overline{Y}, i'_1)$ is a proper map under T, which is a proper n-equivalence by Lemma 2.4. Moreover, Proposition 2.5 yields that $\overline{f}$ is a relative proper n-equivalence (under T) between (X, i) and $(\overline{Y}, i')$ with $i' = i'_1$. Recall that proper n-equivalences induce homeomorphisms between the spaces of ends. □

The next proposition is the proper analogue of a theorem due to J. H. C. Whitehead (see [22, Thm. 14]).

Proposition 2.7. *Assume that the n-dimensional locally finite CW-complexes X and Y are proper n-equivalent, $n \geq 2$. Then, for any end-faithful tree $T \subset X$ there exists an end-faithful proper map $\omega : T \longrightarrow Y$ and n-spherical T-objects S^n_α and S^n_β such that the wedges $X \vee S^n_\alpha$ and $Y \vee_\omega S^n_\beta$, obtained via the inclusion $i : T \longrightarrow X$ and ω, are proper homotopy equivalent.*

The proof of Proposition 2.7 requires some terminology from [3]. By an *n-spherical object* S^n_α under T we mean the space obtained from T by attaching $\#\alpha^{-1}(v)$ n-spheres at each vertex $v \in T^0$, where $\alpha : E \longrightarrow T^0$ is a proper map from a countable discrete space E. Similarly, by attaching copies of the n-ball B^n we get the corresponding T-object B^n_α. Any finite-dimensional locally finite CW-complex (X, i) under T is proper homotopy equivalent under T to a T-reduced and T-normalized CW-complex $(\overline{X}, \overline{i})$ in the sense of [3]; i.e., $\overline{X}^1 = S^1_{\delta_1}$ is a 1-spherical T-object and $\overline{X}^n$ is the result of attaching a T-object $B^n_{\delta_n}$ via a proper map $f_n : S^{n-1}_{\delta_n} \longrightarrow \overline{X}^{n-1}$. This follows from [3, Chap. IV, §5].

The classification of spherical objects in [3, Prop. II.4.5] yields a kind of ∞-element; namely, for any n-spherical T-object S^n_α there is relative proper homotopy equivalence $S^n_\alpha \vee S^n_T \simeq S^n_T$. The spherical object S^n_T is defined by attaching exactly one n-sphere at each vertex $v \in T^0$. Here, "$\vee$" denotes the obvious wedge obtained by identifying the corresponding copies of T in S^n_α and S^n_T.

Proof of Proposition 2.7. By the observation above, we can assume that the n-dimensional CW-complexes (X, i) and $(\overline{Y}, i')$ under T given by Proposition 2.6 are

T-reduced and T-normalized. Then, by [23, Satz 6.11] we get n-spherical T-objects S^n_α and S^n_β and a relative proper homotopy equivalence under T between the wedges $X \vee S^n_\alpha$ and $\overline{Y} \vee S^n_\beta$. Thus, by the gluing lemma (see [3, Lemma I.4.9]) applied to a proper homotopy equivalence $f : \overline{Y} \longrightarrow Y$, we get proper homotopy equivalences $X \vee S^n_\alpha \simeq \overline{Y} \vee S^n_\beta \simeq Y \vee_\omega S^n_\beta$, where $\omega = f|T$. □

Remark 2.8. The crucial ingredient for the proof of Proposition 2.7 is [23, Satz 6.11] whose proof is quite involved and makes use of homology with local coefficients in the proper setting. The argument of that proof has been adapted in [2, Thm. II.6.1] for an alternative proof of the classical Whitehead Theorem [22, Thm. 14].

An alternative and purely homotopic proof of Proposition 2.7 for the special case $n = 2$ (and hence of Corollary 1.2, as seen below) follows from [7, Prop. 6.3] whose proof works step by step for (X, i) and $(\overline{Y}, i')$, with the tree T playing now the role of the half-line $\mathbb{R}_{\geq 0}$. The proof relies on the obvious co-H-space structure of the 1-skeleton of a T-normalized and T-reduced CW-complex, which is not the case for the $(n-1)$-skeleton when $n \geq 3$.

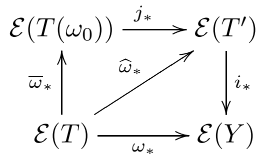
We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let G and H be two infinite finitely generated quasi-isometric groups of type F_n $(n \geq 2)$, and let X be a $K(G, 1)$-complex with finite n-skeleton and Y be a $K(H, 1)$-complex with finite n-skeleton. We may assume that the 0-skeleta of the CW-complexes X and Y reduce to a single vertex, and let T be a maximal tree in the universal cover $\widetilde{X^n}$ of X^n. By combining Propositions 2.1 and 2.7, we find n-spherical T-objects S^n_α and S^n_β and a proper homotopy equivalence $\widetilde{X^n} \vee S^n_\alpha \simeq \widetilde{Y^n} \vee_\omega S^n_\beta$, for some end-faithful proper map $\omega : T \longrightarrow \widetilde{Y^n}$. Let S^n_T be the ∞-element among the n-spherical T-objects. Then, we have

$$\widetilde{X^n} \vee S^n_T \simeq \widetilde{X^n} \vee (S^n_\alpha \vee S^n_T) \simeq \widetilde{Y^n} \vee_\omega (S^n_\beta \vee S^n_T) \simeq \widetilde{Y^n} \vee_\omega S^n_T.$$

We may assume that ω is a cellular map, so that $\omega(T^0) \subset (T')^0$, where $T' \subset \widetilde{Y}$ is a maximal tree in the universal cover of Y^n. Then, it is clear that $\widetilde{Y^n} \vee_\omega S^n_T$ coincides with $\widetilde{Y^n} \vee S^n_{\omega_0}$, where $S^n_{\omega_0}$ is the n-spherical T'-object determined by $\omega_0 = \omega|T^0$, i.e., the one obtained by attaching $\#\omega_0^{-1}(v)$ n-spheres at each vertex $v \in \omega_0(T^0) \subset (T')^0$. Let $T(\omega_0) \subset T'$ denote the subtree generated by the vertices in the image of ω_0. We will show below that the space of ends of $T(\omega_0)$ coincides with the space of ends of T', and hence $S^n_{\omega_0}$ is proper homotopy equivalent under T' to the ∞-element $S^n_{T'}$ in the class of n-spherical T'-objects, by the classification of spherical objects in [3, Prop. II.4.5]. Thus, by the gluing lemma ([3, Lemma I.4.9]), $\widetilde{X^n} \vee S^n_T \simeq \widetilde{Y^n} \vee_\omega S^n_T = \widetilde{Y^n} \vee S^n_{\omega_0} \simeq \widetilde{Y^n} \vee S^n_{T'}$. Finally, we observe that $\widetilde{X^n} \vee S^n_T = \widetilde{X^n \vee S^n}$, and similarly $\widetilde{Y^n} \vee S^n_{T'} = \widetilde{Y^n \vee S^n}$.

It remains to check that the spaces of ends $\mathcal{E}(T(\omega_0))$ of $T(\omega_0)$ and $\mathcal{E}(T')$ of T' are the same. For this, let $\widehat{\omega} : T \longrightarrow T'$ be the proper cellular extension of ω_0 defined by setting, for any edge $\sigma = (v, v')$, $\widehat{\omega}(\sigma)$ to be the single arc in T' from $\omega(v)$ to $\omega(v')$. Observe that the image of $\widehat{\omega}$ is the subtree $T(\omega_0)$ above. Moreover, we have a commutative diagram



where i and j are the corresponding inclusions and $\overline{\omega}$ is the restriction of $\widehat{\omega}$ onto its image. In particular, $\overline{\omega}_*$ is surjective. The induced maps i_* and ω_* are homeomorphisms, as i and ω are end-faithful, and hence so is $\widehat{\omega}_*$. It follows then that $\overline{\omega}_*$ is injective, and hence $\overline{\omega}_*$ and j_* are also homeomorphisms. □

Proof of Corollary 1.3. Let X and Y be finite 2-dimensional CW-complexes with $\pi_1(X) \cong G$ and $\pi_1(Y) \cong H$. By Corollary 1.2, there is a proper homotopy equivalence $\widetilde{X \vee S^2} \simeq \widetilde{Y \vee S^2}$. Assume that G is a properly 3-realizable group. Then, $\widetilde{X \vee S^2} \simeq \widetilde{Y \vee S^2}$ has the proper homotopy type of a 3-manifold, by [1, Prop. 1.3]. Since $\pi_1(Y \vee S^2) \cong H$, it follows that H is a properly 3-realizable group as well.

The second part follows immediately from a well-known corollary of Švarc-Milnor's Lemma; see [14, Thm. 18.2.15] or [16]. □

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Departamento de Geometría y Topología, Fac. Matemáticas, Universidad de Sevilla, C/. Tarfia s/n 41012-Sevilla, Spain
Email address: `mcard@us.es`

Departamento de Geometría y Topología, Fac. Matemáticas, Universidad de Sevilla, C/. Tarfia s/n 41012-Sevilla, Spain
Email address: `lasheras@us.es`

Departamento de Geometría y Topología, Fac. Matemáticas, Universidad de Sevilla, C/. Tarfia s/n 41012-Sevilla, Spain
Email address: `quintero@us.es`

College of Arts and Sciences, New York Institute of Technology, Old Westbury, New York 11568-8000
Email address: `rroy@nyit.edu`