UNIVERSIDAD DE SEVILLA

Facultad de Matemáticas

Dpto. de Ecuaciones Diferenciales y Análisis Númerico

Ecuaciones en derivadas parciales con condiciones de contorno no lineales. Aplicaciones a la dinámica de tumores

February 25, 2009

Autor: **Cristian Morales-Rodrigo**
Director: **Antonio Suárez Fernández**

Tesis presentada por
Cristian Morales Rodrigo
para optar al grado de
Doctor en Matemáticas
por la Universidad de Sevilla

Fdo. Cristian Morales Rodrigo

V° B° del Director del trabajo

Fdo. Antonio Suárez Fernández
Profesor Titular
de la Universidad de Sevilla.

Mathematics is biology's next microscope, only better;
biology is mathematics' next physics, only better.

J.L. Cohen

Agradecimientos.

En primer lugar mi más profundo y sincero agradecimiento a A. Suárez. A él le debo mi introducción en el fascinante mundo de la investigación. Sin duda alguna su dedicación y ayuda han sido inestimables durante todos estos años.

Agradezco a M. Lachowicz el darme la oportunidad de formar parte de su grupo de investigación durante 3 años y el ofrecerme la posibilidad de visitar diversos centros de europa.

También me gustaría agradecer a todos los miembros del Departamento de Ecuaciones Diferenciales y Análisis Numérico su cálida acogida y en especial a M. Delgado, I. Gayte por aceptar ser los avalistas de la tesis, a E. Fernández-Cara por acogerme en su grupo de investigación y a Ara por facilitarme la laboriosa tarea del papeleo.

Agradezco a D. Arcoya, B. Perthame y J.D. Rossi el haber aceptado formar parte de mi tribunal de tesis.

Finalmente agradezco a mi familia el haberme "soportado" durante tanto tiempo.

Contents

CHAPTER 1

Introducción

Recientemente se han propuesto diversos modelos macroscópicos para describir el comportamiento de sistemas vivos en los que algunas de las cantidades a estudiar tienen un comportamiento no lineal en la frontera. En particular nos gustaría recalcar dos ejemplos:
El primero se refiere a una especie de mariposas (ver [70]). Basándose en la información empírica del artículo precedente, en [30] se hace un estudio teórico de una ecuación en derivadas parciales con condiciones de contorno no lineales con el objetivo de explorar los posibles efectos que provocan en una determinada especie los comportamientos particulares de dicha especie en la frontera de su hábitat.
El segundo ejemplo está relacionado con un cáncer no sólido, la leucemia. En este tipo de cáncer, las HSCs (hematopoietic stem cells) juegan un papel relevante en una terapia contra este cáncer. Concretamente, resulta crucial el tiempo que tardan estas células en alcanzar su lugar de origen en el interior del hueso desde que se administran a través de la corriente sanguínea. En [68] se propone un modelo de ecuaciones en derivadas parciales no lineales que involucra a dos poblaciones, las HSCs y un factor quimiotáctico de éstas, es decir, una sustancia química que guía a las HSCs. En este sistema se supone que la producción del factor quimiotáctico tiene un comportamiento no lineal en la frontera.
Finalmente se puede consultar [90] para más ejemplos de EDPs elípticas con condiciones de contorno no lineales.

Esta tesis presenta tres partes:
En la primera haremos un estudio teórico general de las ecuaciones elípticas en derivadas parciales con condiciones de contorno no lineales. Posteriormente, en la segunda, haremos un estudio teórico de ecuaciones elípticas concretas que presentan no linealidades tanto en la ecuación como en la frontera. Finalmente, en la tercera parte, estudiamos modelos concretos de sistemas de ecuaciones en derivadas parciales de tipo parabólico con origen en Biología y dinámica de tumores.

A continuación pasamos a describir brevemente cada uno de los capítulos y posteriormente daremos más detalles de ellos.

a) *Estudio teórico general.* Esta parte engloba a los capítulos 2, 3 y 4.

   En el capítulo 2 se presentan algunos resultados de la teoría de ecuaciones elípticas lineales con condiciones de contorno de tipo mixto así como una versión del teorema de Krein-Rutman, el cual, junto con el principio del máximo, resulta fundamental para determinar la existencia de autovalores principales en problemas de autovalores lineales, es decir, autovalores cuya autofunción asociada se puede escoger positiva. Además el principio del máximo juega un papel relevante en las propiedades del autovalor principal.

   El capítulo 3 se dedica al problema de unicidad de soluciones para ecuaciones elípticas con condiciones de contorno no lineales. En este capítulo presentamos tres teoremas de unicidad complementarios, dos de ellos dan unicidad de solución positiva y otro unicidad de cualquier tipo de solución.

   En el capítulo 4 abordamos tanto al fenómeno de bifurcación como un tema estrechamente relacionado con éste: el de las cotas a priori. Primero daremos condiciones suficientes para la existencia o no existencia de bifurcación desde cero o infinito en problemas elípticos generales con condiciones de contorno no lineales. Posteriormente, estableceremos el comportamiento de una rama de soluciones positivas respecto a una familia de supersoluciones y subsoluciones. Finalizaremos el capítulo con un resultado general de cotas a priori.

b) *Estudio teórico de ecuaciones particulares.* En esta parte se hallan los capítulos 5 y 6. En ella se aplican los resultados de los anteriores capítulos a ecuaciones concretas.

   En el capítulo 5 estudiamos una ecuación elíptica no lineal, la cual presenta una competición entre un término de absorción de la ecuación y un flujo positivo que circula en la frontera. En dicho estudio además de los resultados de capítulos anteriores utilizaremos entre otros el método de la sub-supersoluciones, el principio del barrido de Serrin y el lema del paso de montaña de Ambrosetti y Rabinowitz.

   En el capítulo 6 combinaremos un término cóncavo en la frontera con diferentes tipos de no linealidades en la ecuación: términos convexos, cóncavos y concavo-convexos. Además de los resultados de existencia, también daremos algunos resultados del comportamiento de las soluciones positivas, si es que existen, cuando un parámetro presente en la frontera se hace grande.

c) *Aplicaciones a modelos biológicos.* Esta parte abarca los capítulos 7 y 8. En ella realizamos el estudio de dos sistemas parabólicos en los que, o bien se presentan las variables acopladas en la frontera, o bien la condición de contorno de una de las variables es no lineal. Ambos sistemas comparten además un término de quimiotaxis. Este fenómeno, bastante presente en Biología, se refiere al movimiento de un ente biológico en la dirección del gradiente de una sustancia química.

En el capítulo 7 estudiamos un sistema parabólico-parabólico así como su estacionario asociado que modela una etapa crucial en el crecimiento tumoral; la angiogénesis. En este sistema una de las variables modela un flujo no lineal entrante en la frontera. Probaremos existencia y unicidad de solución global así como varios resultados de convergencia a los estados estacionarios semitriviales. Asimismo, probaremos la existencia de estados de coexistencia, es decir, el caso en el que ambas soluciones son no nulas, y estudiamos además la estabilidad local de las soluciones semitriviales.

En el último capítulo abordamos un sistema parabólico-elíptico propuesto en [83] como un sistema relacionado con la formación de patrones, como puede ser la pigmentación que en la piel presentan diversas especies animales como el tigre o la cebra. Las variables de dicho sistema se hallan acopladas en la frontera. Nosotros daremos, además de la existencia de una única solución global en tiempo, un resultado del comportamiento asintótico de las soluciones cuando el tiempo se hace grande.

A continuación descrimos más exhaustivamente los resultados de cada uno de los capítulos.

1.1. Ecuaciones elípticas lineales

En el capítulo 2 nos dedicamos básicamente, a recopilar información acerca de las ecuaciones elípticas lineales. Esta parte es, en cierta manera, el núcleo principal de la tesis ya que cuánto más profundo sea el conocimiento de las ecuaciones lineales, más información se podrá obtener en el caso no lineal. Los resultados de la primera sección se han extraído principalmente de [36]. Toda la primera sección va encaminada a dar una versión de un teorema de Análisis Funcional, el Teorema de Krein-Rutman, (ver también [79] para una versión no lineal). Dicho teorema es una herramienta fundamental a la hora de probar existencia de autovalores principales, es decir, autovalores cuya autofunción se puede escoger positiva. Se aplicará a los problemas de autovalores de la última sección del capítulo. Para la segunda sección hemos seguido el libro [57] y la tesis [24]. En ella nos ocuparemos de los teoremas de unicidad y regularidad de problemas elípticos lineales de la forma

$$\begin{cases} \mathcal{L}u = f(x) & \text{en } \Omega, \\ \mathcal{B}u = g(x) & \text{sobre } \partial\Omega, \end{cases}$$

donde $\Omega \subset \mathbb{R}^d$ es un dominio acotado de frontera regular con $\partial\Omega = \Gamma_0 \cup \Gamma_1$ y Γ_0, Γ_1 son abiertos y cerrados disjuntos en la topología relativa, podría darse el caso $\Gamma_0 = \emptyset$ ó $\Gamma_1 = \emptyset$. El operador $\mathcal{L}$ es uniformemente elíptico en Ω de la siguiente forma:

$$\mathcal{L} := -\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c,$$

con coeficientes $a_{ij} = a_{ji} \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$, $b_i \in \mathcal{C}^{\alpha}(\overline{\Omega})$ and $c \in \mathcal{C}^{\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$.

El operador $\mathcal{B}$ está definido como sigue

$$\mathcal{B}u := \begin{cases} u & \text{sobre } \Gamma_0, \\ Bu & \text{sobre } \Gamma_1, \end{cases}$$

donde

$$Bu := \frac{\partial u}{\partial n} + b(x)u,$$

$n = (n_1, \dots, n_d)$ denota el vector normal unitario exterior de Γ_1 y $b \in \mathcal{C}^{1+\alpha}(\Gamma_1)$. Esta sección juega un papel relevante a lo largo de la disertación ya que el que una ecuación regularize un solución sirve, si la regularización es fuerte, para definir operadores compactos y aplicar así todas las herramientas del Análisis Funcional que existen para este tipo de operadores.

El contenido de la tercera sección hace referencia a los trabajos [9], [24], [26] y [27]. En esta sección se enuncia una caracterización del principio del máximo para ecuaciones elípticas de segundo orden con condiciones de contorno de tipo mixto $(\mathcal{L}, \mathcal{B}, \Omega)$ en términos de una supersolución positiva estricta y en términos de la positividad de su autovalor principal que notaremos por $\lambda_1(\mathcal{L}, \mathcal{B})$, $\lambda_1(\mathcal{L}, \mathcal{D})$ si $\Gamma_1 = \emptyset$ y $\lambda_1(\mathcal{L}, \mathcal{N})$ para condición frontera de tipo Neumann. Además enumeraremos las múltiples propiedades del autovalor principal mediante perturbaciones en la frontera de Dirichlet, perturbaciones en c y en b. Estas propiedades van a resultar de gran utilidad a la hora de construir sub-supersoluciones para problemas no lineales, la estabilidad de las soluciones y no existencia de éstas.

En la última sección del capítulo consideramos el siguiente problema de autovalores

$$\begin{cases} \mathcal{L}\varphi = \lambda m(x)\varphi & \text{en } \Omega, \\ \varphi = 0 & \text{sobre } \Gamma_0, \\ B\varphi = \lambda r(x)\varphi & \text{sobre } \Gamma_1, \end{cases} \tag{1.1}$$

con $m \in \mathcal{C}^{\alpha}(\overline{\Omega})$ y $r \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$. El caso $\Gamma_1 = \emptyset$, es decir, condiciones de contorno de tipo Dirichlet fue abordado en [61] cuando m cambia de signo y $c \geq 0$, en [94] se generaliza los resultados anteriores para el caso en el que $a_{ij} \in VMO(\Omega) \cap L^{\infty}(\Omega)$ y en [73] se estudia el caso cuando m cambia de signo sin la restricción $c \geq 0$. Si $\Gamma_0 = \emptyset$ y $b \equiv 0$, es decir, condiciones de contorno del tipo Neumann el problema fue abordado en [93] con $c \equiv 0$. Cuando $(m, r) > 0$, es decir, $m \geq 0$, $r \geq 0$ y $(m, r) \neq (0, 0)$, el problema es considerado en [4]; cuando (m, r) cambian de ambos de signo se dan algunas propiedades en [8] y en [102] se estudia dicho problema con $b = 0$, $\Gamma_0 = \emptyset$, $\mathcal{L} = -\Delta$. Más precisamente en [102] se estudia la existencia de autovalores principales del siguiente problema:

$$\begin{cases} -\Delta\varphi = \lambda g(x)\varphi + \mu(\lambda)\varphi & \text{en } \Omega, \\ \dfrac{\partial\varphi}{\partial n} = \lambda h(x)\varphi & \text{sobre } \partial\Omega. \end{cases} \tag{1.2}$$

En dicho artículo se prueba que si $g(x) \not\leq 0$ en Ω y $h(x) \not\leq 0$ sobre $\partial\Omega$ entonces existe un único autovalor principal $\mu_1(\lambda)$ de (1.2) que además satisface

$$\lim_{\lambda \to +\infty} \mu_1(\lambda) = -\infty.$$

Cuando Γ_0, Γ_1 no son necesariamente disjuntos se aborda en [55] el estudio de autovalores principales para el problema de autovalores

$$\begin{cases} -\Delta\varphi - a(x)\varphi = \lambda\varphi & \text{en } \Omega, \\ \varphi = 0 & \text{sobre } \Gamma_0, \\ \dfrac{\partial\varphi}{\partial n} - b(x)u = 0 & \text{sobre } \Gamma_1, \end{cases}$$

con a, b funciones medibles, no necesariamente acotadas.

Nosotros damos una caracterización del autovalor principal de (1.1) cuando $m \equiv 0$, es decir, un problema de autovalores en la frontera, el clásico problema de Steklov. Aunque el resultado sigue prácticamente de [8] y [27].

Teorema 1.1. *a) Si* $(m, r) > 0$ *y*

$$\exists \mu > 0 \ \text{ such that } \ (c + \mu m, b + \mu r) > 0,$$

entonces existe un único autovalor principal para el problema de autovalores (1.1), es simple y su autofunción asociada se puede escoger fuertemente positiva en Ω.

b) Si $m \equiv 0$ *y* $r > 0$ $r(x) > 0$ *en un conjunto de medida* $d-1$ *dimensional no nula; entonces existe el autovalor principal de (1.1) y lo denotaremos por* μ_1 *si y sólo si*

$$\lim_{\lambda \to -\infty} \lambda_1(\mathcal{L}, \mathcal{B} - \lambda r) = \mu_{-\infty} > 0.$$

además su autofunción asociada se puede escoger fuertemente positiva en Ω *y*

$$\lim_{\lambda \to +\infty} \lambda_1(\mathcal{L}, \mathcal{B} - \lambda r) = -\infty.$$

La posibilidad de no existencia de autovalores cuando $r \equiv 1$ abre nuevas posibilidades, por ejemplo, cuando se consideran problemas no lineales con un parámetro en la frontera.

1.2. Unicidad para ecuaciones elípticas con condiciones de contorno no lineales

En el capítulo 3 abordamos el problema de unicidad de soluciones (generalmente positivas) para el problema

$$\begin{cases} \mathcal{L}u = f(x, u) & \text{en } \Omega. \\ u = \varphi(x) & \text{sobre } \Gamma_0, \\ Bu = h(x, u) & \text{sobre } \Gamma_1, \end{cases} \tag{1.3}$$

con $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $\varphi : \Gamma_0 \to \mathbb{R}$ y $h : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ funciones regulares. Nuestro primer resultado es:

Teorema 1.2. *Supongamos que* $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$. *Si las funciones* $u \mapsto f(x, u), h(x, u)$ *son decrecientes entonces existe a lo sumo una solución de (1.3).*

Este resultado es bien conocido cuando $c \geq 0$, $\Gamma_0 = \emptyset$ y $Bu := \beta_0 u + \delta \dfrac{\partial u}{\partial \beta}$ (β un vector exterior no tangente a $\partial\Omega$) con

- o bien $\beta_0 = 1$ y $\delta = 0$ (Caso Dirichlet),
- o bien $\beta_0 = 0$ y $\delta = 1$ (Caso Neumann),
- o bien $\beta_0 > 0$ y $\delta = 1$ (Caso Robin).

Esto puede verse [3] y [92]. En este capítulo generalizamos estos resultados permitiendo condiciones de contorno más generales y además b y c pueden cambiar de signo.

Nuestro segundo resultado es:

Teorema 1.3. *Supongamos que $\varphi \geq 0$ en Γ_0 y que*

$$u \mapsto \frac{f(x,u)}{u}, \frac{g(x,u)}{u}, \tag{1.4}$$

son funciones decrecientes en $(0, +\infty)$ con al menos una de ellas estrictamente decreciente, entonces existe a lo sumo una solución positiva de (1.3).

Este resultado generaliza un teorema clásico, bajo condiciones de Dirichlet homogéneas, (aunque el resultado se puede extender fácilmente para el caso Robin o Neumann) el cual asegura que si en casi todo $x \in \Omega$ la función

$$u \mapsto \frac{f(x,u)}{u} \text{ es decreciente en } (0, +\infty) \tag{1.5}$$

entonces existe a lo sumo una solución positiva de (1.3) ver por ejemplo [20], [21] y [60]. Bajo la condición (1.4), el Teorema 1.3 fue probado en [86, Theorem 4.6.3] cuando $\Gamma_0 = \emptyset$, $\mathcal{L}$ autoadjunto y asumiendo además la existencia de un par de sub-supersoluciones. Ver también [97] para un resultado bajo una condición más restrictiva f/g decreciente.

Finalmente en [40] se da una extensión al resultado clásico en el que se satisface la condición (1.5), y se mostró un resultado que complementa y mejora éste. En este capítulo presentamos el siguiente resultado que generaliza al anterior para condiciones de contorno no lineales.

Teorema 1.4. *Supongamos que $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$, $\varphi \geq 0$ en Γ_0 y que existe $g \in \mathcal{C}^1(0, +\infty) \cap \mathcal{C}([0, +\infty))$, $g(t) > 0$ si $t > 0$ y g' es decreciente, con*

$$u \mapsto \frac{f(x,u)}{g(u)}, \frac{h(x,u)}{g(u)} \text{ son decrecientes en } (0, \infty).$$

Si:

a)

$$\int_0^r \frac{1}{g(t)} < +\infty \text{ para algún } r > 0,$$

entonces existe a lo sumo una solución de (1.3) satisfaciendo

$$u(x) > 0 \quad \text{para todo } x \in \Omega$$

b)

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0,$$

entonces existe a lo sumo una solución fuertemente positiva.

Nos gustaría poner énfasis, como se verá a lo largo de la tesis, que los resultados de unicidad son complementarios.

1.3. Técnicas de bifurcación para ecuaciones elípticas con condiciones de contorno no lineales

En el capítulo 4 nos centramos en el fenómeno de la bifurcación global. Además de un tema relacionado estrechamente con éste, el de las cotas a priori para soluciones positivas. Concretamente, parte de capítulo 4 se dedica a problemas de la forma

$$\left\{ \begin{array}{ll} \mathcal{L}u = \lambda m(x)u + f(x,u) & \text{en } \Omega, \\ u = 0 & \text{sobre } \Gamma_0, \\ Bu = \lambda r(x)u + g(x,u) & \text{sobre } \Gamma_1, \end{array} \right. \tag{1.6}$$

con $(m,r) > 0$, $m \in \mathcal{C}^\alpha(\overline{\Omega})$, $r \in \mathcal{C}^{1+\alpha}(\partial\Omega)$, $f \in \mathcal{C}^\alpha(\overline{\Omega} \times \mathbb{R})$ y $g \in \mathcal{C}(\partial\Omega \times \mathbb{R})$.

a) *No linealidades exclusivamente en la ecuación.* La bifurcación desde cero e infinito de soluciones positivas para el problema (1.6) con condiciones de contorno de tipo Dirichlet fue estudiada en [11, 88] y en [14] para operadores de la forma $-div(a(x,u)\nabla u)$. En el caso de condiciones de contorno similares a (1.6) puede consultarse [5] como referencia estandar.
En el libro [75] se prueba la bifurcación desde cero de soluciones positivas sin necesidad, como en nuestro caso, de que todas las soluciones problema sean positivas. Este hecho se debe a que el autor utiliza los índices de punto fijo en conos mientras que nosotros empleamos, como en [11], la teoría del grado topológico.

b) *No linealidades exclusivamente en la frontera.* La bifurcación desde infinito para operadores autoadjuntos ha sido abordada en [16] (ver también [17]).

c) *No linealidades en la ecuación y en la frontera.* En [100] se aborda el fenómeno de bifurcación desde infinito con no linealidades en la ecuación y en la frontera ambas no linealidades asintóticamente lineales.

Salvo casos concretos, no hemos dado resultados generales de dirección de bifurcación como los que aparecen en [14–16].

En la primera sección del capítulo reescribiremos el problema (1.6) como un problema de punto fijo para un par de operadores compactos, K_1 para la ecuación y K_2 para la frontera. Este tipo de descomposición, en nuestro conocimiento, fue propuesto por primera vez en [4] (ver también [100]). Para la buena definición de este tipo de operadores juega un papel fundamental los resultados del capítulo 2.

Para la segunda sección, bifurcación desde cero, utilizamos, al igual que en [11], la teoría del grado topológico. En nuestro conocimiento no se ha dado un resultado tan general. Previamente al enunciado del teorema definimos $\mathcal{C}$ como

$$\mathcal{C} := \overline{\{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solución positiva de (1.6)}\}}.$$

En la segunda sección probamos que:

Teorema 1.5. (Bifurcación desde cero) *Supongamos que* $(m, r) > 0$,

$$f(x,0) \geq 0 \;\; \forall x \in \Omega, \qquad g(x,0) \geq 0 \;\; \forall x \in \Gamma_1\,, \tag{1.7}$$

existen $c_1, c_2 \in \mathbb{R}$ *tales que*

$$\lim_{s\to 0^+} \frac{f(x,s)}{s} = c_1 \text{ unif. en } \overline{\Omega}, \qquad \lim_{s\to 0^+} \frac{g(x,s)}{s} = c_2 \text{ unif. en } \Gamma_1.$$

Sea $\widetilde{\gamma}_1$, *si existe, el único cero de la aplicación*

$$\mu(\lambda) := \lambda_1(\mathcal{L} - c_1 - \lambda m, \mathcal{B} - c_2 - \lambda r).$$

Entonces $\widetilde{\gamma}_1$, *es un punto de bifurcación desde cero y es el único para soluciones positivas. Además existe un continuo (cerrado y conexo) no acotado* $\mathcal{C}_0 \subset \mathcal{C}$ *emanando desde* $(\widetilde{\gamma}_1, 0)$. *Más aún, si la aplicación* $\mu(\cdot)$ *no se anula entonces no existe bifurcación desde cero para soluciones positivas.*

En la siguiente sección nos centramos en el fenómeno de bifurcación desde infinito. Concretamente probamos que:

Teorema 1.6. (Bifurcación desde infinito) *Supongamos* $(m, r) > 0$, *(1.7) y*

$$\lim_{s\to +\infty} \frac{f(x,s)}{s} = c_1 \text{ unif. en } \overline{\Omega}, \qquad \lim_{s\to +\infty} \frac{g(x,s)}{s} = c_2 \text{ unif. en } \Gamma_1,$$

para algunas constantes $c_1, c_2 \in \mathbb{R}$. *Sea* $\widetilde{\gamma}_1$, *si existe, el único cero de* $\mu(\cdot)$. *Entonces* $\widetilde{\gamma}_1$ *es un punto de bifurcación desde infinito y es el único para soluciones positivas. Además existe un continuo no acotado* $\mathcal{C}_\infty \subset \mathcal{C}$ *bifurcando desde infinito en* $\lambda = \widetilde{\gamma}_1$. *Más aún, si* $\delta_0 > 0$ *es suficientemente pequeño y*

$$\mathcal{J} = [\widetilde{\gamma}_1 - \delta_0, \widetilde{\gamma}_1 + \delta_0] \times \{u \in \mathcal{C}(\overline{\Omega}) : \|u\|_\infty \geq 1\}$$

entonces, o bien

a) $\mathcal{C}_\infty \setminus \mathcal{J}$ *está acotado en* $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$ *y* $\mathcal{C}_\infty \setminus \mathcal{J}$ *encuentra al conjunto* $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$, *o bien*

b) $\mathcal{C}_\infty \setminus \mathcal{J}$ *está no acotado en* $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$.

Finalmente, si $\widetilde{\gamma}_1$ no existe entonces no hay una rama de soluciones positivas bifurcando desde infinito.

La siguiente sección se dedica al fenómeno de bifucación en el caso cóncavo-convexo. Cuando aparecen no linealidades de este tipo en la ecuación el método de bifurcación desde cero se ha empleado en [14], [39] y [38]. En la primera parte nos ocuparemos de ecuaciones en las que el parámetro aparece exclusivamente en la ecuación. Concretamente de ecuaciones de la forma

$$(1.8) \qquad \begin{cases} \mathcal{L}u = \lambda f(x,u) & \text{en } \Omega, \\ u = 0 & \text{sobre } \Gamma_0, \\ Bu = g(x,u) & \text{sobre } \Gamma_1, \end{cases}$$

donde $f \in \mathcal{C}^\alpha(\overline{\Omega} \times \mathbb{R})$, $g \in \mathcal{C}^{1+\alpha}(\Gamma_1 \times (0,+\infty))$. Además supondremos que

$$(1.9) \qquad f(x,0) = 0 \;\; \forall x \in \overline{\Omega}\,, \qquad g(x,0) \geq 0 \;\; \forall x \in \Gamma_1,$$

y

$$(\mathbf{BCC}) \qquad \lim_{s\to 0^+} \frac{f(x,s)}{s} = +\infty \;\; \text{unif. en } \overline{\Omega}, \qquad \lim_{s\to 0^+} \frac{g(x,s)}{s} = 0 \;\; \text{unif. en } \Gamma_1.$$

Bajo las condiciones anteriores probamos lo siguiente:

Teorema 1.7. *Si se verifican las condiciones (1.9) y (BCC) entonces $\lambda = 0$ es un punto de bifurcación desde cero y es el único punto de bifurcación desde cero para soluciones positivas. Además existe un continuo no acotado $\mathcal{C}_0$ de soluciones positivas de (1.8) emanando desde $(0,0)$.*

En la segunda parte de la sección consideraremos ecuaciones en las que el parámetro sólo actúa en la frontera. En concreto de ecuaciones de la forma

$$(1.10) \qquad \begin{cases} \mathcal{L}u = h(x,u) & \text{en } \Omega, \\ u = 0 & \text{sobre } \Gamma_0, \\ Bu = \lambda j(x,u) & \text{sobre } \Gamma_1, \end{cases}$$

donde $h \in \mathcal{C}^\alpha(\overline{\Omega} \times \mathbb{R})$, $j \in \mathcal{C}^{1+\alpha}(\Gamma_1 \times (0,+\infty))$. Adicionalmente supondremos las siguientes condiciones

$$(1.11) \qquad h(x,0) \geq 0 \;\; \forall x \in \overline{\Omega}\,, \qquad j(x,0) = 0 \;\; \forall x \in \Gamma_1,$$

y

$$(\mathbf{BCC2}) \qquad \lim_{s\to 0^+} \frac{h(x,s)}{s} = 0 \;\; \text{unif. in } \overline{\Omega}, \qquad \lim_{s\to 0^+} \frac{j(x,s)}{s} = +\infty \;\; \text{unif. on } \Gamma_1.$$

Bajo las condiciones precedentes probamos lo siguiente:

Teorema 1.8. *Supongamos (1.11) y (BCC2),* *a)* *Si* $\lambda_1(\mathcal{L}, \mathcal{D}) > 0$ *entonces* $\lambda = 0$ *es un punto de bifurcación desde cero y es el único para soluciones positivas de (1.10). Además existe un continuo no acotado* $\mathcal{C}_0$ *de soluciones positivas de (1.10) emanando desde* $(0, 0)$.

b) *Si* $\lambda_1(\mathcal{L}, \mathcal{D}) \leq 0$ *entonces no existe bifurcación de soluciones positivas de (1.10) emanando desde cero.*

En la quinta sección del capítulo ordenaremos las soluciones obtenidas mediante métodos de bifurcación respecto a sus supersoluciones. Para el caso en el que la frontera es de tipo Dirichlet el problema fue tratado en [49], véase [14] para el caso en el que el operador es de la forma $-div(a(x, u)\nabla u)$. Nosotros extendemos el resultado de [49] a ecuaciones de la forma

$$\begin{cases} \mathcal{L}u = f(\lambda, x, u) & \text{en } \Omega, \\ u = 0 & \text{sobre } \Gamma_0, \\ Bu = g(\lambda, x, u) & \text{sobre } \Gamma_1, \end{cases} \tag{1.12}$$

con $f \in \mathcal{C}^{0+1}(\mathbb{R} \times \overline{\Omega} \times \mathbb{R})$ y $g \in \mathcal{C}^{1+\alpha}(\mathbb{R} \times \Gamma_1 \times \mathbb{R})$, $\alpha \in (0, 1)$. En concreto probamos que:

Teorema 1.9. *Sea* $I \subset \mathbb{R}$ *un intervalo y sea* $\Sigma \subset I \times \mathcal{C}^2_{\Gamma_0}(\overline{\Omega})$ *un conjunto conexo de soluciones de (1.12). Consideremos la aplicación continua* $\overline{u} : I \to \mathcal{C}(\overline{\Omega})$ *donde* $\overline{u}(\lambda)$ *es una supersolución estricta de (1.12) para cada* λ*. Si* $u_{\lambda_0} < \overline{u}(\lambda_0)$ *con* $(\lambda_0, u_{\lambda_0}) \in \Sigma$*, entonces* $u_\lambda < \overline{u}(\lambda)$ *para todo* $(\lambda, u_\lambda) \in \Sigma$.

Existe también un resultado análogo para subsoluciones.

La última sección del capítulo 4 la dedicamos al estudio de las estimaciones a priori de soluciones positivas para ecuaciones de la forma

$$\begin{cases} \mathcal{L}u = f(x, u) & \text{en } \Omega, \\ Bu = g(x, u) & \text{sobre } \partial\Omega, \end{cases} \tag{1.13}$$

con $f \in \mathcal{C}(\overline{\Omega} \times [0, +\infty))$ y $g \in \mathcal{C}^{1+\alpha}(\partial\Omega \times [0, +\infty))$. Este tipo de estimaciones son el complemento de secciones anteriores, ya que nos ofrecen una información más precisa de la estructura del continuo de soluciones positivas. La técnica que utilizamos en la prueba tiene su origen en [56]. A grosso modo lo que se hace es aplicar una técnica de blow-up y reducir el problema de cotas a priori a resultados globales para problemas de tipo Liouville. En [56] prueban el resultado para problemas de tipo Dirichlet con la restricción

$$\lim_{t \to +\infty} \frac{f(x, t)}{t^p} = h(x), \tag{1.14}$$

uniformemente en $x \in \overline{\Omega}$ con h continua y estrictamente positiva en $\overline{\Omega}$, $p \in \left(1, \frac{d+2}{d-2}\right)$. Posteriormente en [18] se prueban cotas a priori para condiciones frontera de tipo Robin con $f(x, t) = a(x)g(t)$, $a \in \mathcal{C}^2(\Omega)$ una función que cambia de signo. Si denotamos por

$\Omega_+ = \{x:\ a(x) > 0\}$ y $\Omega_- = \{x:\ a(x) < 0\}$ se verifica $\nabla a(x) \neq 0$ si $x \in \overline{\Omega}_+ \cap \overline{\Omega}_- \subset \Omega$ y $g \in \mathcal{C}^1(\mathbb{R})$ satisface

$$\lim_{t\to+\infty} \frac{g(t)}{t^p} = l > 0,$$

para algún $1 < p < \frac{d+2}{d-1}$, $g'(0) = g(0) = 0$, $g(t) > 0$ si $t > 1$. En [9] mejoran este resultado imponiendo menos regularidad a a, sólo $a \in L^\infty(\Omega)$ y $a(x) = Cdist(x, \partial\Omega_+)^\gamma$ en un entorno de $\partial\Omega_+$ con $\gamma > 0$ y

$$1 < p < \min\left\{\frac{d+1+\gamma}{d-1}, \frac{d+2}{d-2}\right\}.$$

Cuando el problema dispone de no linealidades en la frontera, en [46] las cotas se hacen para sistemas elípticos lineales en las ecuaciones, la no linealidad de la frontera no cambia de signo, en [107] para una ecuación elíptica no lineal con no linealidades exclusivamente en la frontera cambiando de signo y en [50] para una ecuación no lineal con dos no linealidades concretas, una en la ecuación y otra en la frontera que no cambian de signo. Nuestro resultado engloba al de [50] ya que allí se aborda el problema con no linealidades concretas u^q, u^p. De hecho, un teorema tan general con no linealidades en la ecuación y en la frontera no lo hemos encontrado en ninguna referencia. Concretamente probamos que:

Teorema 1.10. *Sea $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ una solución positiva de (1.13). Entonces bajo las hipotesis (1.14),*

$$\lim_{t\to+\infty} \frac{g(x,t)}{t^q} = i(x), \quad \textit{uniformemente en } x \in \partial\Omega,$$

donde $i \in \mathcal{C}^{1+\alpha}(\partial\Omega)$, $\alpha \in (0,1)$ estrictamente positiva, $p \in \left(1, \frac{d+2}{d-2}\right)$, $q \in \left(1, \frac{d}{d-2}\right)$ y $p \neq 2q-1$, se tiene que

$$u(x) \le C(p,q,\Omega), \quad \forall x \in \Omega,$$

donde $C(p,q,\Omega)$ es una constante que depende de p, q, Ω.

1.4. Soluciones positivas de un problema elíptico con una absorción no lineal y un flujo entrante no lineal

En el capítulo 5 abordamos el estudio de las soluciones positivas del problema

$$\left\{\begin{array}{ll} -\Delta u = \lambda u - u^p & \text{en } \Omega, \\ \dfrac{\partial u}{\partial n} = u^r & \text{sobre } \partial\Omega, \end{array}\right. \tag{1.15}$$

con $p, r > 0$, y $\lambda \in \mathbb{R}$ denota el parametro de bifurcación.

En el problema (1.15) hay una competición entre el término de absorción $-u^p$ de la ecuación y el flujo positivo de la frontera u^r. Por tanto, es interesante ver cómo el término lineal, λu, afecta a la existencia de soluciones positivas de (1.15).

En el caso particular $p > 1$, es posible dar una intrepretación ecológica a (1.15), debido a que la ecuación es del tipo logístico y modela la difusión de una especie, cuya densidad viene dada por y y que habita en Ω. La condición de la frontera significa que la especie abandona el hábitat una vez alcanzada su frontera $\partial\Omega$, a una velocidad que depende de una potencia de u, ver [29, 30] para un problema semejante relacionado con la dinámica de poblaciones en el que la no linealidad de la frontera es diferente.

El caso $\lambda = 0$ $p, r > 1$ cambiando $\lambda u - u^p$ por $-au^p$ con $a \in \mathbb{R}$, ha sido tratado en [31, 32, 71, 87] (ver también las referencias de dichos artículos). Para estos valores específicos de λ, p y r, se prueba que si $p < r$ ó $p > 2r - 1$ hay una solución positiva de (1.15) si $a > 0$. Cuando $p = r$ hay solución positiva si $a > |\partial\Omega|/|\Omega|$, y no hay solución positiva de (1.15) si $a < |\partial\Omega|/|\Omega|$. Si $r < p < 2r - 1$ existe $a_0 > 0$ tal que existe solución positiva si $a > a_0$ y no existen soluciones positivas si $a < a_0$. Más aún, si $r < d/(d - 2)$ entonces para casi todo $a \geq a_0$ (1.15) tiene al menos dos soluciones positivas. De hecho, se hace un análisis más detallado en el caso unidimensional y en el caso en el que Ω es una bola (ver también [77] para el caso unidimensional). Este estudio muestra que $p = 2r - 1$ es crítico en muchos aspectos. En particular, las soluciones del problema evolutivo asociado a (1.15) pueden explotar en tiempo finito si y sólo si $p \leq 2r - 1$ (y $a < r$ si $p = 2r - 1$). En [89] se hace un estudio exhaustivo del problema parabólico incluso en el caso $p, r \leq 1$. Más aún, si $p = 2r - 1$, $a = r$ y $d = 1$ entonces existe un equilibrio singular y todas las soluciones positivas del correspondiente problema evolutivo son globales y tienden a dicha solución singular cuando $t \to +\infty$, ver [47].

Finalmente, el caso $r = 1$, $\lambda = 0$, y $p > 1$ o $p < 1$, con un parámetro en la frontera se tratan en [52, 53].

Cuando en vez de un flujo positivo en la frontera, hay un flujo negativo, el problema ha sido estudiado en [28] con $p, r > 1$. Además si una función acotada $g(u)$ aparece en la frontera en vez de u^r, el problema ha sido tratado en [99] y para nolinealidades más generales en [101], donde se hace un análisis local de la bifurcación mediante la reducción de Lyapunov-Schmidt.

En este capítulo estudiamos el problema (1.15) cuando $p, r > 1$, $p \neq 2r - 1$. También consideramos los casos $r = 1$ y $p > 0$; $p = 1$ y $r > 0$; $0 < r < 1 < p$ y $0 < p < 1 < r$. Obsérvese que si $p = r = 1$ entonces el problema es lineal, por tanto existe solución positiva sólo para un valor de λ, el autovalor principal. Nos gustaría recalcar que en la mayoría de los casos consideramos que r es un exponente subcrítico, es decir, $r < d/(d-2)$ si $d \geq 3$. Véase los Teoremas 1.11–1.14 donde enunciamos los principales resultados.

Nuestro principal objetivo es determinar el conjunto de λ's para los que existe solución positiva y también determinar la estabilidad y unicidad de soluciones positivas dependiendo de los valores de p y r. Adicionalmente proporcionamos el comportamiento asintótico de las soluciones cuando $|\lambda|$ se hace grande, en los casos en los que las soluciones existan.

Como sólo estamos interesados en soluciones positivas de (1.15), podemos extender las funciones $\lambda u - u^p$ y u^r para valores negativos de u. De esta manera, toda solución de (1.15) es positiva o nula. Además, cuando $p \geq 1$ el principio del máximo fuerte asegura que toda solución positiva de (1.15) es fuertemente positiva.

Nosotros usamos los autovalores principales para caracterizar la estabilidad de las soluciones respecto del problema parabólico asociado. Diremos que una solución positiva u_0 de (1.15) es *estable* (resp. *inestable*) si el autovalor principal de la linealización de (1.15) en un entorno de u_0 es positivo (resp. negativo), es decir,

$$\lambda_1(-\Delta - \lambda + pu_0^{p-1}, \mathcal{N} - ru_0^{r-1}) > 0 \quad (resp. < 0).$$

También diremos que u_0 es *debilmente estable* si el autovalor es positivo ó nulo, y *neutralmente estable* si es nulo.

En lo que sigue enunciamos los principales resultados del capítulo. El primer resultado se refiere al caso en el que sólo una de las nolinealidades está presente.

Teorema 1.11. *a) Supongamos $r = 1$ y $p \neq 1$. Existe una solución positiva si y sólo si $\lambda > \lambda_1(-\Delta, \mathcal{N} - 1)$. Además,*

(a) Si $p > 1$, la solución es fuertemente positiva, única (denotémosla por u_λ), estable y verifica

$$\lim_{\lambda \searrow \lambda_1(-\Delta, \mathcal{N}-1)} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = +\infty; \tag{1.16}$$

(b) Si $p < 1$, entonces toda familia de soluciones positivas $\{u_\lambda\}$ satisface

$$\lim_{\lambda \searrow \lambda_1(-\Delta, \mathcal{N}-1)} \|u_\lambda\|_\infty = +\infty, \quad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = 0. \tag{1.17}$$

b) Supongamos $p = 1$.

(a) Si $1 < r < d/(d-2)$, existe solución positiva si y sólo si $\lambda < \lambda_1(-\Delta + 1, \mathcal{N})$. Además todas las soluciones positivas son inestables y toda familias de soluciones positivas $\{u_\lambda\}$ verifica

$$\lim_{\lambda \nearrow \lambda_1(-\Delta+1, \mathcal{N})} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = +\infty. \tag{1.18}$$

(b) Si $r < 1$, existe solución positiva si y sólo si $\lambda < \lambda_1(-\Delta + 1, \mathcal{N})$. Además, la solución es única (denotémosla por u_λ), estable y

$$\lim_{\lambda \nearrow \lambda_1(-\Delta+1, \mathcal{N})} \|u_\lambda\|_\infty = +\infty, \quad \lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = 0. \tag{1.19}$$

Teorema 1.12. *Supongamos* $0 < r < 1 < p$. *Existe solución positiva para todo* $\lambda \in \mathbb{R}$. *Además, la solución es única (denotémosla por* u_λ*), estable y*

$$\lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = +\infty. \tag{1.20}$$

Para la prueba del siguiente teorema usaremos métodos variacionales.

Teorema 1.13. *Supongamos* $0 < p < 1 < d/(d-2)$. *Existe solución positiva para todo* $\lambda \in \mathbb{R}$. *Además, para toda familia de soluciones positvas* $\{u_\lambda\}$:

$$\lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = +\infty, \quad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = 0. \tag{1.21}$$

Finalmente, enunciamos el último teorema del capítulo,

Teorema 1.14. *Supongamos* $p, r > 1$.

a) Si $p > 2r - 1$, *existe* $\lambda_0 < 0$ *tal que (1.15) tiene soluciones positivas si y sólo si* $\lambda \geq \lambda_0$. *Además, toda familia de soluciones positivas* $\{u_\lambda\}$ *satisface*

$$\lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = +\infty. \tag{1.22}$$

b) Si $p < 2r - 1$ *y* $r < d/(d-2)$, *existe* $\Lambda_0 \geq 0$ *tal que (1.15) tiene solución positiva siempre que* $\lambda < \Lambda_0$. *Además, si* $\Lambda_0 > 0$, *existen al menos dos soluciones positivas para todo* $\lambda \in (0, \Lambda_0)$ *y al menos una solución positiva para* $\lambda = \Lambda_0$. *Adicionalmente, para toda familia de soluciones positivas* $\{u_\lambda\}$ *tenemos*

$$\lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = +\infty. \tag{1.23}$$

c) Si $p < r$ *o* $p = r$ *y* $|\Omega| > |\partial\Omega|$, *y* $r < d/(d-2)$ *entonces* $\Lambda_0 > 0$. *Adicionalmente, para todo* $\lambda \in (0, \Lambda_0)$ *existe una única solución positiva estable de (1.15).*

La prueba del Teorema 1.14 es más compleja que la de los anteriores. En particular empleamos el principio del barrido de Serrin ([91], pg. 12), la identidad de Picone ([74, Lemma 4.1]) y algunos resultados del problema parabólico asociado a (1.15), [13]. En la figura 1.1 hemos representado los diagramas de bifurcación en todos los casos. Nos gustaría remarcar que, en los casos b), c), f) y h) las soluciones no son necesariamente únicas como se han dibujado.

Es importante recalcar que el comportamiento asintótico de la soluciones cuando $\lambda \nearrow +\infty$ ó $\lambda \searrow -\infty$ desde (1.16) hasta (1.23) son, de hecho, consecuencia de una información más precisa obtenida de las soluciones. En particular, probamos que cuando las soluciones existen para $|\lambda|$ grandes entonces tenemos estimaciones de las forma

$$C_1|\lambda|^\theta \leq \max u \leq C_2|\lambda|^\theta$$

para toda solución positiva de (1.15), con C_1 y C_2 constantes positivas, y el exponente θ dependiendo de p y r. Ver sección 5.3 para la formulación precisa.

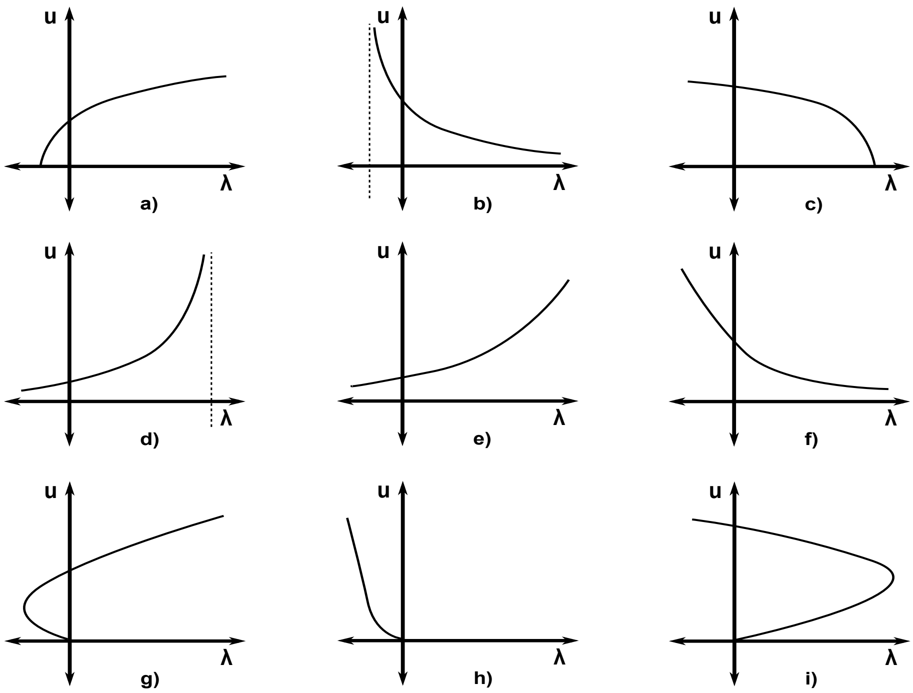

Figure 1.1: Diagramas de bifurcación de (1.15). a) $r = 1 < p$; b) $r = 1 > p$; c) $p = 1 < r < d/(d-2)$; d) $p = 1 > r$; e) $0 < r < 1 < p$; f) $0 < p < 1 < r < d/(d-2)$; g) $p, r > 1$, $p < 2r-1$; h) $p, r > 1$, $p < 2r-1$, $r < d/(d-2)$, $\Lambda_0 = 0$; i) $p, r > 1$, $p < 2r-1$, $r < d/(d-2)$, $\Lambda_0 > 0$.

1.5. Combinando una condición de contorno sublineal con una reacción sublineal o superlineal

A lo largo de sexto capítulo abordamos el problema

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{en } \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda u^q & \text{sobre } \partial\Omega, \end{cases} \tag{1.24}$$

con $p > 0$, $0 < q < 1$, $\lambda \in \mathbb{R}$ el parámetro de bifurcación y $a \in \mathcal{C}^\alpha(\overline{\Omega})$ con $\alpha \in (0,1)$, un peso que prodría cambiar de signo.

La característica principal de (1.24) es la presencia del parámetro en la frontera enfrente de un término no regular, ya que en la en la mayoría de los casos el parámetro actúa en todo el dominio Ω. Nuestro objetivo es ver cómo afecta el parámetro λ en la existencia de soluciones positivas. También daremos, en algunos casos, el comportamiento asintótico de las soluciones positivas de (6.1) cuando $\lambda \to +\infty$.

Cuando $0 < q < 1$, $1 < p \leq p^*$, con p^* el exponente crítico y $a \equiv 1$ el problema (1.24) fue tratado en [50] mediante métodos variacionales y allí se prueba la existencia de un $\Lambda > 0$, tal que para todo $\lambda > \Lambda$ no existen soluciones positivas, mientras que si

$\lambda \in (0, \Lambda)$, entonces existen al menos dos soluciones positivas.

Cuando f y g son funciones que cambian de signo y $0 < q < 1 < p < p^*$, en la ecuación de (1.24) aparece el término $\lambda f(x)u^q$ en vez de $a(x)u^p$ y el término $g(x)u^p$ en la frontera se prueba la existencia de dos soluciones positivas para λ positivo suficientemente pequeño en [105].

En [52] El problema (1.24) se aborda en [52] sin el término $+u$ en la ecuación y con $q = 1$, $a \leq 0$ y $p > 1$, y se prueba la existencia de una única solución positiva bifurcando desde cero en $\lambda = 0$ para $\lambda \in (0, \sigma_1)$, donde $\sigma_1 = +\infty$ si $\partial\Omega \cap \partial\Omega_0 = \emptyset$ y Ω_0 es el interior del conjunto de anulación de a, ó σ_1 es el autovalor principal de

$$\begin{cases} -\Delta\varphi = 0 & \text{en } \Omega_0, \\ \dfrac{\partial\varphi}{\partial n} = \sigma\varphi & \text{sobre } \partial\Omega \cap \partial\Omega_0, \\ \varphi = 0 & \text{sobre } \partial\Omega_0 \cap \Omega, \end{cases}$$

en otro caso. También se estudia el comportamiento asintótico cuando $\lambda \to \sigma_1$, que depende de la posición del conjunto de anulación de a respecto de la frontera.

Cuando en el caso precedente se considera $p \in (0, 1)$, en vez de $p > 1$, entonces el problema es tratado en [53] y se prueba la no existencia de soluciones positivas para $\lambda < 0$, así como una rama de soluciones positivas bifurcando desde infinito en $\lambda = 0$. Además, se dan más detalles del comportamiento de la solución en un entorno de $\lambda = 0$, en concreto

$$\lim_{\lambda \to 0^+} u_\lambda = \lambda^{\frac{-1}{1-p}} \left(\frac{1}{|\partial\Omega|} \int_\Omega a(x) \right)^{\frac{1}{1-p}}.$$

Más aún, se prueba que las soluciones son clásicas, únicas y fuertemente positivas si $\lambda < \lambda_1$ y si $\lambda > \lambda_2 > \lambda_1$ las soluciones presentan núcleos muertos (nótese que en este caso falla el principio del máximo fuerte), es decir, existen conjuntos de medida no nula en los que la solución es nula.

Otros artículos en los que aparecen parámetros en la condición de contorno son por ejemplo [102] aunque para otro tipo de ecuaciones y [54] para sistemas.

A continuación, resumimos los principales resultados obtenidos para (1.24). Lo haremos según los valores del parámetro p. Concretamente, distinguiremos los casos $p = 1$, $p > 1$ y $p < 1$.

Caso $p = 1$. En este caso es crucial μ_1, el único cero, si es que existe, de la aplicación

$$\lambda_1(-\Delta + (1 - a(x)), \mathcal{N} - \lambda).$$

Teorema 1.15. *Sea $p = 1$.*

a) Si $\mu_1 > 0$ entonces existe solución positiva de (1.24) si y sólo si $\lambda > 0$, es única, estable y

$$u_\lambda = \lambda^{\frac{1}{1-q}} z,$$

con z la única solución positiva de

$$\begin{cases} -\Delta z + (1 - a(x))z = 0 & \text{en } \Omega, \\ \dfrac{\partial z}{\partial n} = z^q & \text{sobre } \partial\Omega. \end{cases}$$

b) Si $\mu_1 = 0$ entonces existe solución positiva de (1.24) si y sólo si $\lambda = 0$.

c) Si $\mu_1 < 0$ entonces existe solución positiva de (1.24) si y sólo si $\lambda < 0$. Además todas las soluciones positivas son inestables.

d) Si no existe μ_1, es decir, si $\lambda_1(-\Delta + (1 - a(x)), \mathcal{D}) \leq 0$, entonces el problema (1.24) no tiene soluciones positivas.

En la figura 1.2 dibujamos los posibles diagramas de bifurcación en función del signo de μ_1.

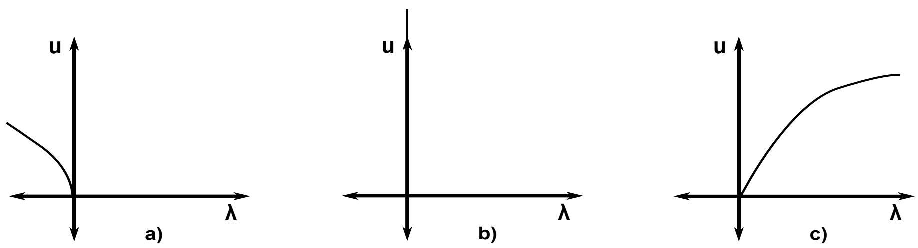

Figure 1.2: *a)* $\mu_1 < 0$; *b)* $\mu_1 = 0$; *c)* $\mu_1 > 0$.

Caso $p > 1$. En este caso resulta conveniente definir los siguientes conjuntos dependientes del peso a,

$$\Omega_+ := \{x \in \Omega \,:\, a(x) > 0\}, \qquad \Omega_- := \{x \in \Omega \,:\, a(x) < 0\}, \qquad \Omega_0 := \Omega \setminus \left(\overline{\Omega}_+ \cap \overline{\Omega}_-\right).$$

Sobre dichos conjuntos supondremos básicamente que las fronteras de Ω_- y Ω_+ son regulares. A partir de aquí, los resultados dependen de si $\Omega_+ = \emptyset$ ó $\Omega_+ \neq \emptyset$.

Caso $\Omega_+ = \emptyset$.

Teorema 1.16. *Si $\Omega_+ = \emptyset$, entonces existe solución positiva de (1.24), u_λ, si y sólo si $\lambda > 0$. Además dicha solución es única, estable y*

$$\lim_{\lambda \searrow 0} \|u_\lambda\|_\infty = 0.$$

También estudiamos el comportamiento asintótico de (1.24) cuando $\lambda \nearrow +\infty$. Nosotros asumimos, por simplicidad que o bien $\partial\Omega_0 = \partial\Omega$ ó $\partial\Omega_0 \cap \partial\Omega = \emptyset$. Básicamente las soluciones tienden a una solución larga. En este teorema resulta crucial el *como a pasa de negativo a cero* por ello se requiere que $a \in \mathcal{C}^1(\overline{\Omega})$ para que ese pasar sea suave.

Teorema 1.17. *Supongamos, adicionalmente, que* $a \in \mathcal{C}^1(\overline{\Omega})$.

- *Si* $\Omega_0 \subset\subset \Omega$ *ó* $\Omega_0 = \emptyset$ *entonces*

$$\lim_{\lambda \nearrow \infty} u_\lambda = z_{\mathcal{D},\Omega},$$

donde $z_{\mathcal{D},\Omega} \in \mathcal{C}^{2+\alpha}(\Omega)$ *es la solución mínima del problema de Dirichlet singular*

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{en } \Omega, \\ u = \infty & \text{sobre } \partial\Omega. \end{cases}$$

- *Si* $\Omega_- \subset\subset \Omega$ *entonces*

$$\lim_{\lambda \nearrow \infty} u_\lambda = z_{\mathcal{D},\Omega_-},$$

donde $z_{\mathcal{D},\Omega_-} \in \mathcal{C}^{2+\alpha}(\Omega)$ *denota la solución mínima del problema de Dirichlet*

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{en } \Omega_-, \\ u = \infty & \text{sobre } \partial\Omega_-. \end{cases}$$

Además,

$$\lim_{\lambda \nearrow +\infty} u_\lambda = \infty,$$

uniformemente en Ω_0.

Caso $\Omega_+ \neq \emptyset$, supondremos además que $|\Omega_+| > 0$. Sea ψ_1 una autofunción principal asociada al problema de autovalores

$$\begin{cases} -\Delta \varphi + \varphi = 0 & \text{en } \Omega, \\ \dfrac{\partial \varphi}{\partial n} = \lambda \varphi & \text{sobre } \partial\Omega. \end{cases}$$

En este caso probamos lo siguiente:

Teorema 1.18. *Supongamos que se cumple una de las siguientes condiciones*

- $\Omega_- = \emptyset$,
- $\Omega_- \subset\subset \Omega$ *ó*
- $\Omega_+ \cap \Omega_0 \subset\subset \Omega$ *y* $\displaystyle\int_\Omega a(x)\psi_1^{p+1} > 0$,

además, supondremos condiciones que aseguren que las soluciones positivas de (1.24) están acotadas si $\lambda < +\infty$ *(ver, por ejemplo Teorema 6.1). Entonces,*

a) existe un continuo de soluciones positivas $\mathcal{C}_0$ *emanando desde cero en* $\lambda = 0$.

b) $\mathcal{P}_\lambda(\mathcal{C}_0) = (-\infty, \Lambda]$ *para algun* $\Lambda > 0$, *donde* $\mathcal{P}_\lambda(\mathcal{C}_0)$ *denota la proyección de* $\mathcal{C}_0$ *sobre el eje* λ.

c) Existen al menos dos soluciones positivas de (1.24) para $\lambda \in (0, \Lambda)$.

d) Existe una única solución positiva de (1.24) que es estable en $(0, \Lambda)$. *Además, dicha solución es la minimal en* $(0, \Lambda)$.

En la figura 1.3 hemos representado los posibles diagramas de bifurcación en el caso $p > 1$ vaticinados por los teoremas anteriores.

Figure 1.3: *a)* $\Omega_+ = \emptyset$; *b)* $\Omega_+ \neq \emptyset$.

Caso $0 < p < 1$. En este caso consideramos sólo los casos $a \equiv 1$ o $a \equiv -1$. En esta sección, mediante el método de las sub-supersoluciones, probamos lo siguiente:

Teorema 1.19. *Supongamos* $a \equiv 1$, *entonces existe al menos una solución positiva para todo* $\lambda \in \mathbb{R}$. *Además, si* $p \leq q$ *entonces dicha solución es única. Más aún, si* $p > q$ *entonces para* $\lambda \geq 0$ *la solución es única.*

Teorema 1.20. *Supongamos* $a \equiv -1$, *entonces el problema (1.24) no posee soluciones positivas si* $\lambda \leq 0$. *Si* $\lambda > 0$ *suficientemente grande entonces el problema (1.24) tiene una solución fuertemente positiva. Además, tal solución es única si* $q \leq p$.

1.6. Modelo de Angiogénesis con término de quimiotaxis y flujo no lineal en la frontera

En el capítulo 7 analizamos un sistema de ecuaciones en derivadas parciales que modela un paso crucial en el crecimiento tumoral, la angiogénesis, es decir, el crecimiento de vasos sanguíneos a partir de los vasos preexistentes. La angiogénesis es un proceso que se produce frecuentemente en nuestro organismo y es beneficioso para éste, por ejemplo, en la cura de las heridas. Sin embargo, cuando la angiogénesis está inducida por un tumor indica el principio de lo que se conoce como cascada metastática que es un proceso que una vez concluido es letal para el que lo padece. La angiogénesis, en el caso tumoral, es inducida por el tumor mediante la secreción por éste de sustancias químicas como, por ejemplo, el VEGF (vascular endothelial growth factors), que provocan el crecimiento de la red vascular hacia el tumor. Sugerimos al lector interesado el artículo [80] para conocer más acerca de los múltiples aspectos de la angiogénesis. Nosotros centramos nuestra atención en el comportamiento de dos poblaciones que intervienen en dicho fenómeno: las células endoteliales (ECs), las cuales denotamos por u, que se mueven y se reproducen para generar una nueva red vascular atraídas por una sustancia química generada por el tumor, el TAF, que será denotado por v. Ambas poblaciones interactúan en una región $\Omega \subset \mathbb{R}^d$, $d \geq 1$, que supondremos acotada, conexa y con frontera regular $\partial\Omega$. Específicamente, consideramos el caso

$$\partial\Omega = \Gamma_1 \cup \Gamma_2,$$

con $\Gamma_1 \cap \Gamma_2 = \emptyset$, y Γ_i cerrados y abiertos en la topología relativa de Ω. Suponemos que Γ_2 es la frontera del tumor y Γ_1 la frontera de los vasos sanguíneos, ver figura 1.4, donde hemos representado una situación particular, en este caso el tumor está rodeado por los vasos.

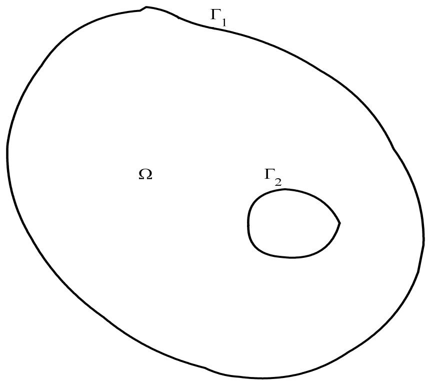

Figure 1.4: Ejemplo particular de dominio Ω.

Supondremos condiciones de contorno Neumann homogéneas en ambas variables en Γ_1, y también para la variable u en Γ_2. Sin embargo, y como una de las principales novedades del modelo, consideraremos que el tumor genera una cantidad de TAF que depende de manera no lineal de la cantidad existente. Específicamente, supondremos que en Γ_2

$$\frac{\partial v}{\partial n} = \mu \frac{v}{1+v},$$

con μ un número real, aunque en la aplicación real μ será una constante positiva. En tal caso μ representa la velocidad a la que se produce el TAF. Suponemos que el tumor genera TAF con un término de producción del tipo Michaelis-Menten, por tanto suponemos un efecto de saturación en la frontera, en contraste con el modelo en [41] donde este término es lineal. Por tanto, estudiamos el siguiente problema parabólico y su estacionario asociado

$$(1.25) \qquad \begin{cases} u_t - \Delta u = -\text{div}(V(u)\nabla v) + \lambda u - u^2 & \text{en } \Omega \times (0,T), \\ v_t - \Delta v = -v - cuv & \text{en } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{sobre } \Gamma_1 \times (0,T), \\ \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial v}{\partial n} = \mu \dfrac{v}{1+v} & \text{sobre } \Gamma_2 \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{en } \Omega, \end{cases}$$

donde $0 < T \leq +\infty$, $\lambda, \mu \in \mathbb{R}$, $c > 0$ y

$$(1.26) \qquad V \in \mathcal{C}^1(\mathbb{R}), \quad V > 0 \text{ en } (0,\infty) \text{ con } V(0) = 0;$$

y u_0, v_0 son dos funciones positivas y no triviales dadas.

Vamos a explicar un poco el modelo. Nosotros asumimos que u posee dos tipos de movimientos: uno indirecto que viene dado por el usual operador de Laplace y uno directo que viene inducido por la presencia del gradiente de v. Aquí V modela la respuesta quimiotáctica de las ECs al quimioatractante TAF, y en este caso esta respuesta depende de la densidad u de una manera no lineal. Este es un modelo de densidad dependiente, en la terminología de [62]. También suponemos que las ECs crecen siguiendo una ley logística. Además, suponemos que el TAF tiene una degradación típicamente lineal, $-v$, y es afectado por u mediante un término de competición $-cuv$, es decir, el TAF es consumido por las células endoteliales y este consumo no tiene un efecto directo salvo a través del término de quimiotaxis. En cierto sentido, el tumor atrae a las ECs, de hecho, se mueven hacia Γ_2 donde se produce el mayor gradiente de TAF.

Modelos similares a (1.25) con condiciones de contorno del tipo Neumann o no flujo han sido estudiados exhaustivamente en los últimos años, ver por ejemplo el artículo recopilatorio [62].

El modelo (1.25) tiene principalmente tres dificultades, debido básicamente a las no linealidades: los términos de reacción, la respuesta quimiotáctica y la condición de contorno. El término logístico ha sido ya usado para modelar el crecimiento y la muerte celular. También, la sensibilidad quimiotáctica no lineal ha sido usada en diversos artículos, ver por ejemplo [63], [85], [104], [64], [23], [34] y las referencias de estos artículos. Nos gustaría mencionar que en [63] la función V es acotada y negativa para valores grandes de u, lo que proporciona cotas de la solución y por tanto previene la acumulación. En todos los artículos anteriores no se considera un crecimiento logístico para u. Para este tipo de crecimiento con $V(u) = u$ nos referimos a los artículos [84], [96] y [103] para los problemas de tipo parabólico y [44], [45] para el caso estacionario.

Sin embargo, un término no lineal en la frontera del tumor no ha sido usado prácticamente en la literatura, de hecho sólo conocemos el artículo [68] en el que se combine una condición de contorno no lineal con un término de quimiotaxis. La presencia de estos términos implica un modelo más complejo y realista. Creemos que el entendimiento del comportamiento de las soluciones puede decidir si un sistema como (1.25) puede ser considerado como un modelo apropiado para fenómenos biológicos complejos. Resumimos los resultados como sigue: Respecto al problema parabólico, se construyen soluciones locales en tiempo mediante los resultados generales de [7] y soluciones globales en tiempo vía cotas uniformes en tiempo en norma $L^\infty(\Omega)$. Más precisamente, mostramos que:

- Existe una única solución positiva local en tiempo de (1.25).
- Si V está acotado, existe una única solución global en tiempo de (1.25).

Con respecto al problema estacionario asociado a (1.25), es claro que existen tres tipos de soluciones: la trivial, las semitriviales $(u, 0)$, $(0, v)$ y las soluciones con ambas componentes positivas, los llamados *estados de coexistencia*. Básicamente, la solución trivial siempre existe, y:

- La solución semitrivial $(u, 0)$ existe si y sólo si $\lambda > 0$. De hecho, en este caso la solución semitrivial es $(\lambda, 0)$.
- Existe un valor $\mu_1 > 0$ tal que la solución semitrivial $(0, v)$ existe si y sólo si $\mu > \mu_1$.

Con respecto a la existencia de estados de coexistencia, necesitamos introducir previamente dos funciones que estarán relacionadas con problemas de autovalores $F : (0, +\infty) \mapsto \mathbb{R}$, $\lambda \mapsto F(\lambda)$ y $\Lambda : (\mu_1, +\infty) \mapsto \mathbb{R}$, $\mu \mapsto \Lambda(\mu)$ tales que:

- Si $\lambda \leq 0$ ó $\mu \leq \mu_1$, entonces no existe ningún estado de coexistencia.
- Suponiendo que $V'(0) > 0$, entonces existe al menos un estado de coexistencia si

$$(\mu - F(\lambda))(\lambda - \Lambda(\mu)) > 0. \tag{1.27}$$

- Suponiendo que $V'(0) = 0$, existe al menos un estado de coexistencia si $\lambda > 0$ y

$$\mu - F(\lambda) > 0. \tag{1.28}$$

En la figura 1.5 hemos representado, según los valores de los parámetros λy μ, las regiones de coexistencia definidas por (1.27) y (1.28).

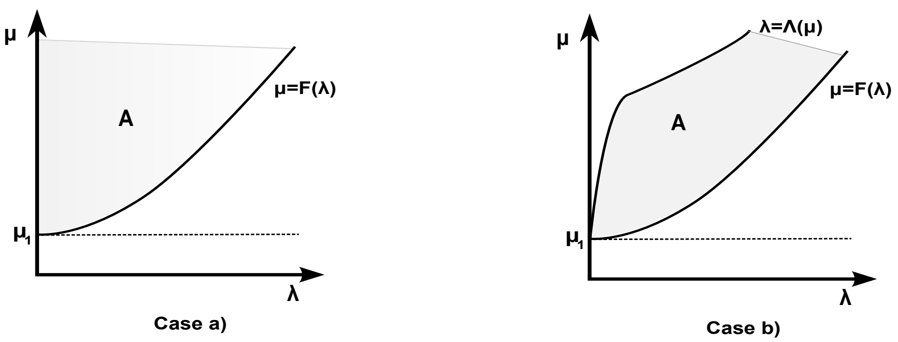

Figure 1.5: Regiones de coexistencia: Caso a) $V'(0) = 0$ y Caso b) $V'(0) > 0$.

Con respecto a la estabilidad local de las soluciones semitriviales, probamos que:

- La solución trivial es estable si $\lambda < 0$ y $\mu < \mu_1$, e inestable si $\lambda > 0$ ó $\mu > \mu_1$.
- $(u, 0)$ es estable si $\mu < F(\lambda)$, e inestable si $\mu > F(\lambda)$.
- $(0, v)$ es estable si $\lambda < \Lambda(\mu)$ (resp. $\lambda < 0$ si $V'(0) = 0$), e inestable si $\lambda > \Lambda(\mu)$ (resp. $\lambda > 0$ si $V'(0) = 0$).

Por tanto, cuando las dos soluciones semitriviales son estables o inestables al mismo tiempo entonces existe al menos un estado de coexistencia. Por consiguiente, las curvas $\mu = F(\lambda)$ y $\lambda = \Lambda(\mu)$ que aperecen en (1.27) y (1.28) son cruciales en el estudio de la existencia de soluciones positivas y las vamos a estudiar en detalle en este capítulo.

Para probar estos resultados usaremos principalmente métodos de bifurcación y de sub-supersoluciones.

En la penúltima sección del capítulo estudiaremos la estabilidad global de $(\lambda, 0)$ y mostraremos que:

- $(0,0)$ es globalmente asintóticamente estable si $\mu < \mu_1$.
- $(\lambda, 0)$ con $\lambda > 0$ es globalmente asintóticamente exponencialmente estable si $\mu < \mu_1$ y

  $$0 < V(s) < Cs^k \qquad |V'(s)| < Cs^{k-1} \tag{1.29}$$

  para todo $s \in (0, \delta_0)$, con $\delta_0 > 0$ suficientemente pequeño, y $k > 1 + d/2$.

La última sección se dedica a dar una interpretación biológica de los resultados obtenidos.

1.7. Sobre un modelo relacionado con la formación de patrones

En el último capítulo de la disertación abordamos el estudio de un modelo quimiotáctico con condiciones de contorno distintas de las típicas no-flujo o Neumann. Concretamente, estudiamos el siguiente sistema

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u(1-u) & \text{en } \Omega \times (0,T), \\ 0 = \Delta v - v + \dfrac{u}{1+u} & \text{en } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} - \chi u \dfrac{\partial v}{\partial n} = r(1-u)\,, \quad \dfrac{\partial v}{\partial n} = r'\left(\dfrac{1}{2} - v\right) & \text{sobre } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{en } \Omega, \end{cases} \tag{1.30}$$

con $\Omega \subset \mathbb{R}^d$ un dominio acotado con frontera regular, μ, r, r' y χ constantes positivas. El modelo (1.30) fue planteado en [83] con la ecuación v también evolutiva. Nosotros consideramos el caso elíptico para v argumentando como en [66] ó [69], es decir, suponemos que la difusión de v es mucho mayor que la difusión de u. Aquí u denota una densidad celular y v un quimioatractante de u, es decir, una sustancia química que induce un movimiento de u en la dirección del gradiente de v. Este tipo de fenómeno se observa, por ejemplo, en la fase de agregación de Dyctiostelium [67]. Aunque en [83] se propone como un modelo relativo a los patrones observados en la pigmentación de la piel de diversos tipos de animales como tigres o cebras. Un modelo parecido con condiciones de contorno no-flujo se ha considerado en [33], donde en vez del término $\frac{u}{1+u}$ se considera un término más general, una función g acotada, la difusión de u es no lineal degenerada en el infinito, concretamente $\nabla \cdot (\alpha(u)\nabla u)$ y el término ∇v está acompañado por $u\beta(u)$ y no por u como en nuestro caso. Allí el autor prueba la existencia de soluciones globales en tiempo bajo restricciones en el cociente α/β. Sin embargo, en nuestro conocimiento sistemas como el de (1.30) no han sido abordados mediante métodos analíticos. Aquí probamos la existencia y unicidad de solución global positiva y estudiamos además el comportamiento asintótico de las soluciones cuando el tiempo se hace largo. Concretamente, probamos lo siguiente:

Teorema 1.21. *Sea $p > d$ y consideramos el dato incial $u_0 \in W^{1,p}(\Omega)$ con $u_0 \geq 0$. Entonces, existe $\tau(\|u_0\|_{W^{1,p}}) > 0$ tal que el problema (1.30) tiene una única solución positiva local en tiempo*

$$(u,v) \in \left(\mathcal{C}([0,\tau]; W^{1,p}(\Omega)) \cap \mathcal{C}^1((0,\tau); \mathcal{C}^{2+\alpha}(\overline{\Omega}))\right)^2,$$

y $u(x,t), v(x,t) \geq 0$ para $(x,t) \in \overline{\Omega} \times [0,\tau]$. Además, la solución depende de manera continua del dato inicial, es decir, si $\mathbf{u}(u_0)$ y $\mathbf{u}(\overline{u}_0)$ denotan las soluciones de (1.30) con datos iniciales u_0, $\overline{u}_0$ respectivamente entonces

$$\|\mathbf{u}(u_0) - \mathbf{u}(\overline{u}_0)\|_{(\mathcal{C}([0,\tau];W^{1,p}))^2} \leq C\|u_0 - \overline{u}_0\|_{W^{1,p}}.$$

Más aún, la solución puede prolongarse indefinidamente hasta alcanzar $T_{max} = +\infty$, es decir, la solución es global en tiempo.

Respecto al comportamiento asintótico tenemos que:

Teorema 1.22. *Si $\min\limits_{x\in\overline{\Omega}} u_0(x) > 0$ y*

$$\gamma_0 := \frac{2\chi}{\left(1 + \frac{\underline{u}_0}{\overline{u}_0}\right)^2} - \mu < 0 \tag{1.31}$$

donde

$$\overline{u}_0 := \max\left\{\max_{x\in\overline{\Omega}} u_0(x), 1\right\}, \qquad \underline{u}_0 := \min\left\{\min_{x\in\overline{\Omega}} u_0(x), 1\right\},$$

entonces, la solución (u,v) de (1.30) satisface

$$\|u(t) - 1\|_\infty + \left\|v(t) - \frac{1}{2}\right\|_{W^{2,p}} \leq -C\epsilon_0^{-1}\ln(\epsilon_0)e^{\gamma_0\epsilon_0 t}, \quad t > 0, \tag{1.32}$$

cualquiera que sea $p > 1$ y $\epsilon_0 := \frac{\underline{u}_0}{\overline{u}_0}$.

Nos gustaría recalcar que el teorema anterior es constructivo, en particular, se compara la solución u con las soluciones de un sistema diferencial ordinario $\underline{u}$, $\overline{u}$ y se prueba que $\underline{u} \leq u(x,t) \leq \overline{u}$ para todo $(x,t) \in \overline{\Omega} \times (0,+\infty)$. Finalmente, y como consecuencia del teorema precedente,

a) si $\mu \geq 2\chi$ (término de reacción fuerte comparado con el de quimiotaxis) entonces, para cualesquiera datos iniciales $u_0 \geq 0$, $u_0 \not\equiv 0$, es decir, datos que tienen sentido biológico, se tiene que:

$$\|u(t) - 1\|_\infty \leq Ce^{-\alpha t}, \qquad \left\|v(t) - \frac{1}{2}\right\|_{W^{2,p}} \leq C_1 e^{-\beta t},$$

para algunas constantes $C, C_1, \alpha, \beta > 0$ que pueden calcularse explícitamente. O lo que es lo mismo, la solución tiende al estado homogéneo estacionario $\left(1, \frac{1}{2}\right)$ de manera exponencial.

b) Si $\mu > \frac{\chi}{2}$ (término de reacción más débil comparado con el de quimiotaxis) entonces, si u_0 está próximo a 1 en norma $L^\infty(\Omega)$ se tiene que:

$$\|u(t)-1\|_\infty \leq C_2 e^{-\gamma t}, \qquad \left\| v(t)-\frac{1}{2}\right\|_{W^{2,p}} \leq C_3 e^{-\delta t},$$

para algunas constantes $C_2, C_3, \gamma, \delta > 0$ que pueden calcularse explícitamente. Este resultado implica en particular la estabilidad local de la solución $\left(1, \frac{1}{2}\right)$.

Probablemente, una de las cuestiones más interesantes de este capítulo es vaticinar qué ocurre ante un término de reacción débil comparado con el de quimiotaxis. La presencia de estados de coexistencia distintos del homogéneo ayudaría a aclarar tal cuestión.

1.8. Problemas abiertos

a) Una cuestión interesante sería estudiar el problema de autovalores (1.1) cuando m y r cambian ambas de signo, especialmente, en el caso en el que no se puede utilizar la caracterización variacional.

b) Otra cuestión es la posibilidad de generalizar los resultados de [55] para el caso en el que no se posee una caracterización variacional del autovalor.

c) En el capítulo 5 desconocemos lo que ocurre cuando $p, r \leq 1$. Aquí, falla el principio del máximo fuerte con lo que existe la posibilidad de existencia de soluciones con núcleos muertos.

d) En el capítulo 6, Teorema 1.18, la condición $\int_\Omega a(x)\psi_1^{p+1} > 0$ parece bastante artificial y se debería poder eliminar. Esta condición se puede interpretar como *a más grande en la parte positiva que en la parte negativa.*

e) En el capítulo 6 si $0 < q, p < 1$ sólo sabemos lo que pasa si hay unicidad de solución positiva.

f) En el capítulo 7 se introduce una función $V(u)$ acotada. Esta función está relacionada con el concepto de *volume filling* que se introdujo para el sistema de Keller-Segel (ver [67]) para evitar la explosión en tiempo finito (ver [66]). Sin embargo, en nuestro caso, ese mecanismo no parece necesario ya que en la ecuación de v la u entra con un signo negativo, al contrario que en el sistema de Keller-Segel.

g) En el capítulo 7 la condición (1.29) parece bastante artificial. Se debería poder probar la convergencia el estado estacionario $(\lambda, 0)$ incluso sin esta restricción.

h) En el capítulo 8 falta estudiar el sistema estacionario asociado a (1.30).

CHAPTER 2

Linear Elliptic Equations

This Chapter is devoted to collect results for linear elliptic equations with mixed boundary conditions. Those results will be used through this work. Special attention will be paid on the properties of the principal eigenvalues i.e. eigenvalues such that the eigenfunction associated can be chosen strongly positive.

2.1. A version of the Krein-Rutman Theorem

Let X a Banach space and $T : D(T) \subset X \to X$ a closed and linear operator. We denote by $\mathcal{L}(X)$ the set of linear and continuous operators acting from X with image in X.

Definition 2.1. *The resolvent of T, denoted by ϱ, is defined by*

$$\varrho(T) := \{\lambda \in \mathbb{C} : \ (\lambda I - T)^{-1} \in \mathcal{L}(X)\}.$$

The complement of ϱ is called the spectrum of T and it is noted as $\sigma(T)$.

Definition 2.2. *We say that $\lambda \in \mathbb{C}$ is an eigenvalue of T if $\lambda I - T$ is not injective. If $x \in X \setminus \{0\}$ satisfies $Tx = \lambda x$ then x is called eigenvector of T.*

Definition 2.3. *The spectral radius of T, denoted by $r(T)$ is defined as*

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Definition 2.4. *Let E a vectorial space and $\leq$ is an order relation, i.e. transitive, reflexive and antisymmetric relation that satisfies the compatibility conditions*

$$\begin{array}{llll} x \leq y & then & x + z \leq y + z & \forall z \in E, \\ x \leq y & then & \lambda x \leq \lambda y & \forall \lambda \geq 0, \end{array}$$

then we say that $(E, \leq)$ is a ordered vector space.

Examples.

a) It is well-known that in the vectorial space $\mathbb{C}$ there is not an order relation $\leq$ that additionally satisfies the previous compatibility conditions.

b) The vectorial space $\mathcal{C}(\overline{\Omega})$ with the order relation $\leq$ given by

$$f \leq g \Longleftrightarrow f(x) - g(x) \leq 0 \;\; \forall x \in \overline{\Omega},$$

is a ordered vector space.

Definition 2.5. *Let E an ordered Banach space, then $x < y$ means $x \leq y$, $x \neq y$. Moreover, the set*

$$[x, y]_E := \{z \in E : \;\; x \leq z \leq y\},$$

is called ordered interval between x and y.

Definition 2.6. *Let $(E, \leq)$ a ordered vector space, the set*

$$E_+ := \{x \in E : \;\; x \geq 0\},$$

is called positive cone of E. Let $x \in E$.

a) We say that x is non-negative if $x \in E_+$.

b) We say that x is positive if $x \in E_+ \setminus \{0\}$.

c) Suppose that $int(E_+) \neq \emptyset$. We say that x is strongly positive if $x \in int(E_+)$.

Definition 2.7. *Let E a Banach space. We say that $(E, \leq)$ is an ordered Banach space if it is a ordered vector space and the positive cone E_+ is closed in norm.*

Definition 2.8. *Let E, F ordered Banach spaces. Let us assume, for simplicity, that $int(F_+) \neq \emptyset$ and let $T : E \to F$ an operator.*

a) We say that T **is positive** *and it will be denoted by $T \geq 0$ if $Tx > 0$ for all $x > 0$.*

b) We say that T **is strongly positive** *and it will be denoted by $T \gg 0$ if $Tx \in int(F_+)$ for all $x > 0$.*

Definition 2.9. *Let $(E, \leq)$ an ordered Banach space. A positive operator is called irreducible if there exists $n_0 \geq 1$ such that T^{n_0} is strongly positive.*

Now, we state a particular version of the Krein-Rutman Theorem that can be found in [36, Theorem 12.3].

Theorem 2.10. (Krein-Rutman Theorem) *Let $(E, \leq)$ an ordered Banach space with $int(E_+) \neq \emptyset$ and $T \in \mathcal{L}(E)$ a positive, compact and irreducible operator then $r(T)$ is an eigenvalue with algebraic multiplicity 1 of T and its dual T^*. The associated eigenspaces are generated by a strongly positive eigenfunction and a strongly positive functional. Moreover, $r(T)$ is the unique eigenvalue of T whose associated eigenfunction can be chosen positive.*

Let us consider the eigenvalue problem

$$\begin{cases} \mathcal{L}u = \lambda u & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω, $\mathcal{L}$, $\mathcal{B}$ are defined as follows

a) $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^2$. Moreover,

$$\partial\Omega := \Gamma_0 \cup \Gamma_1,$$

where Γ_0 and Γ_1 denote two disjoint open and closed sets in the relative topology of $\partial\Omega$.

b) $\mathcal{L}$ is an uniformly elliptic differential operator in Ω of the form

$$\mathcal{L} := -\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c,$$

with coefficients $a_{ij} = a_{ji} \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$, $b_i \in \mathcal{C}^{\alpha}(\overline{\Omega})$ and $c \in \mathcal{C}^{\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$.

c) We define the mixed boundary operator $\mathcal{B}$, by

$$\mathcal{B}u := \begin{cases} u & \text{on } \Gamma_0, \\ Bu & \text{on } \Gamma_1, \end{cases}$$

where $B := \partial_n + b$, $n \in \mathcal{C}^1(\Gamma_1, \mathbb{R}^d)$ stands for the outward normal[1] vector to $\partial\Omega$ and $b \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$.

An easy consequence of the Krein-Rutman Theorem is (see [6]) the following:

Theorem 2.11. *The eigenvalue problem (2.1) has an eigenvalue, $\lambda_1(\mathcal{L}, \mathcal{B})$, that satisfies $Re(\lambda) \geq \lambda_1(\mathcal{L}, \mathcal{B})$ for all $\lambda \in \mathbb{C}$ eigenvalues of (2.1). The eigenvalue $\lambda_1(\mathcal{L}, \mathcal{B})$ is called principal eigenvalue of $(\mathcal{L}, \mathcal{B}, \Omega)$. Moreover, $\lambda_1(\mathcal{L}, \mathcal{B})$ is the only eigenvalue such that the eigenfunction associated can be chosen positive in Ω.*

2.2. Existence, Uniqueness and Regularity

During all the work, at least we say something different, we keep the notation of the previous section. Our first theorem refers to the interior estimates for the equation $\mathcal{L}u = f$.

Theorem 2.12. *Let Ω an open subset of $\mathbb{R}^d$ and $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$, $1 < p < +\infty$ satisfying $\mathcal{L}u = f$ in the pointwise sense, with $\mathcal{L}$ an uniformly elliptic operator whose coefficients satisfy for $\Lambda > 0$*

$$a_{ij} \in \mathcal{C}(\Omega), \quad b_i, c \in L^\infty(\Omega), \quad f \in L^p(\Omega);$$

[1] We assume the normal vector just for simplicity, most of the results of this chapter still true for more general vectors. In particular, outward pointing nowhere tangent vector-field to $\partial\Omega$.

$$|a_{ij}|,\ |b_i|,\ |c| \leq \Lambda$$

for all $i, j = 1, \dots, d$. *Then, for every subdomain* $\Omega' \subset \Omega$ *such that* $dist(\partial\Omega, \Omega') > k > 0$ *we have*

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|u\|_p + \|f\|_p)$$

Now, we state some results related to the existence for linear elliptic problems of the form:

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = g(x) & \text{on } \Gamma_0, \\ Bu = h(x) & \text{on } \Gamma_1. \end{cases} \tag{2.2}$$

The first one refers to the case of $W^{2,p}(\Omega)$ solutions and the second one to the classical solutions.

Theorem 2.13. *Assume* $f \in L^p(\Omega)$, $g \in W^{2-1/p,p}(\Gamma_0)$, $h \in W^{1-1/p,p}(\Gamma_1)$, $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$ *and*

$$a_{ij} \in \mathcal{C}(\overline{\Omega}), \quad b_i, c \in L^\infty(\Omega), \quad f \in L^p(\Omega),$$

for $i, j = 1, \dots, d$, *then the problem (2.2) has a unique solution* $u \in W^{2,p}(\Omega)$. *Moreover,* u *satisfies*

$$\|u\|_{W^{2,p}} \leq C(\|f\|_p + \|g\|_{W^{2-1/p,p}(\Gamma_0)} + \|h\|_{W^{1-1/p,p}(\Gamma_1)}).$$

Theorem 2.14. *Assume* $f \in \mathcal{C}^\alpha(\overline{\Omega})$, $g \in \mathcal{C}^{2+\alpha}(\Gamma_0)$, $h \in \mathcal{C}^{1+\alpha}(\Gamma_1)$ *and* $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$, *then the problem (2.2) has a unique solution* $u \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$. *Moreover,* u *satisfies*

$$\|u\|_{\mathcal{C}^{2+\alpha}(\Omega)} \leq C(\|f\|_{\mathcal{C}^\alpha(\Omega)} + \|g\|_{\mathcal{C}^{2+\alpha}(\Gamma_0)} + \|h\|_{\mathcal{C}^{1+\alpha}(\Gamma_1)}).$$

Theorem 2.15. *Assume* $\Gamma_0 = \emptyset$, $f \in \mathcal{C}(\Omega)$, $h \in W^{1-1/p,p}(\Gamma_1)$, $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$. *Then if* $u \in \mathcal{C}^2(\overline{\Omega})$ *is a solution to (2.2) satisfies*

$$\|u\|_{W^{1,p}} \leq C(\|f\|_p + \|h\|_{L^p(\Gamma_1)}).$$

2.3. Maximum principle and properties of the principal eigenvalue

Let us begin this section with an example that suggests a connection, via the Krein-Rutman Theorem, between the strong maximum principle and the principal eigenvalue.

Let $\mathcal{C}_0^1(\overline{\Omega}) = \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$. Consider the operator

$$T : \mathcal{C}_0^1(\overline{\Omega}) \to \mathcal{C}_0^1(\overline{\Omega})$$

such that $T(f) = u$ where u is the unique solution to the linear equation

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Thanks to the elliptic regularity we know that $T(f) \in \mathcal{C}^{2+\alpha}(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ which is compactly embedded in $\mathcal{C}_0^1(\overline{\Omega})$. Therefore, T is a compact operator. Moreover, we consider the positive cone

$$\left(\mathcal{C}_0^1(\overline{\Omega})\right)_+ = \{f \in \mathcal{C}_0^1(\overline{\Omega}) : \ f(x) \geq 0 \ \forall x \in \overline{\Omega}\},$$

whose interior is given by

$$int\left(\left(\mathcal{C}_0^1(\overline{\Omega})\right)_+\right) = \{f \in \mathcal{C}_0^1(\overline{\Omega}) : \ f(x) > 0 \ \forall x \in \Omega \ and \ \partial_n f(x) < 0 \ \forall x \in \partial\Omega\}.$$

Assume that (2.3) satisfies the strong maximum principle, i.e. for a given $f > 0$, $T(f) \in int\left(\left(\mathcal{C}_0^1(\overline{\Omega})\right)_+\right)$, so T is an irreducible operator. Therefore, thanks to the Krein-Rutman Theorem, there exists an eigen-pair (λ_1, φ_1), $\lambda_1 > 0$ such that $T(\varphi_1) = \lambda_1 \varphi_1$. Moreover, $\varphi_1 \in int\left(\left(\mathcal{C}_0^1(\overline{\Omega})\right)_+\right)$ and is the only eigenvalue with this property. So, we have

$$\begin{cases} \mathcal{L}(\varphi_1) = 1/\lambda_1 \varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

This example shows that, if (2.3) satisfies the strong maximum principle, then the eigenvalue problem associated to (2.3) has a unique positive eigenvalue

$$1/\lambda_1 = \lambda_1(\mathcal{L}, \mathcal{D}) > 0,$$

where $\mathcal{D}$ stands for Dirichlet boundary condition. This relationship is, in fact, as we will see in the next theorem, stronger.

Definition 2.16. *Let $u, v : \Omega \to \mathbb{R}$ the notation $(u, v) \geq 0$ stands for $u \geq 0$ and $v \geq 0$, moreover $(u, v) > 0$ denotes $(u, v) \geq 0$, $(u, v) \neq 0$.*

The next definitions are needed in order to give a characterization of the strong maximum principle for operators $\mathcal{L}$ under mixed boundary conditions $\mathcal{B}$. Such a characterization can be found in [9].

Definition 2.17. *Let $p > d$. We say that $\overline{u} \in W^{2,p}(\Omega)$ is a positive strict supersolution to $(\mathcal{L}, \mathcal{B}, \Omega)$ if $\overline{u} \geq 0$ and $(\mathcal{L}\overline{u}, \mathcal{B}\overline{u}) > 0$.*

Definition 2.18. *Let $p > d$. We say that $u \in W^{2,p}(\Omega)$ is strongly positive in Ω if $u(x) > 0$ for all $x \in \Omega \cup \Gamma_1$ and $\partial_n u(x) < 0$ for all $x \in \Gamma_0$.*

Definition 2.19. *We say that $(\mathcal{L}, \mathcal{B}, \Omega)$ satisfies the strong maximum principle if for some $p > d$, $u \in W^{2,p}(\Omega)$ and $(\mathcal{L}u, \mathcal{B}u) > 0$ then u is strongly positive in Ω.*

Theorem 2.20. (Strong Maximum Principle) *The following assertions are equivalent:*

a) $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$.

b) $(\mathcal{L}, \mathcal{B}, \Omega)$ *has a strict supersolution.*

c) $(\mathcal{L}, \mathcal{B}, \Omega)$ *satisfies the strong maximum principle.*

The rest of the section is devoted to study the properties of the principal eigenvalues of $(\mathcal{L},\mathcal{B},\Omega)$, the proofs can be found in [27]. These properties are the cornerstone of this PhD Thesis.

Proposition 2.21. *If $\Gamma_1\neq\emptyset$ then $\lambda_1(\mathcal{L},\mathcal{B})<\lambda_1(\mathcal{L},\mathcal{D})$, where $\mathcal{D}$ stands for the Dirichlet boundary condition.*

Proposition 2.22. *Let $c_n\in L^\infty(\Omega)$.*

a) If $c_1<c_2$ in a set of positive Lebesgue measure, then

$$\lambda_1(\mathcal{L}+c_1,\mathcal{B})<\lambda_1(\mathcal{L}+c_2,\mathcal{B}).$$

b) If $\lim_{n\to+\infty}c_n=c$ in $L^\infty(\Omega)$ then

$$\lim_{n\to+\infty}\lambda_1(\mathcal{L}+c_n,\mathcal{B})=\lambda_1(\mathcal{L}+c,\mathcal{B}).$$

c) If there exists a set of positive Legesgue measure D such that $\inf_D c_1>0$ then

$$\lim_{\lambda\to+\infty}\lambda_1(\mathcal{L}-\lambda c_1,\mathcal{B})=-\infty.$$

d) If $\inf_\Omega c_1>0$ then

$$\lim_{\lambda\to-\infty}\lambda_1(\mathcal{L}-\lambda c_1,\mathcal{B})=+\infty.$$

Proposition 2.23. *If $b_1,b_2\in\mathcal{C}(\Gamma_1)$ with $b_1<b_2$ then*

$$\lambda_1(\mathcal{L},\mathcal{B}+b_1)<\lambda_1(\mathcal{L},\mathcal{B}+b_2),$$

where

$$(\mathcal{B}+z)u:=\begin{cases}u & \text{on } \Gamma_0,\\ (B+z)u & \text{on } \Gamma_1,\end{cases}$$

for any $z\in\mathcal{C}(\Gamma_1)$.

Definition 2.24. *We denote by $\sigma(L^\infty(\Gamma_1),L^1(\Gamma_1))$ the weak-star topology of $L^\infty(\Gamma_1)$. Given a sequence $b_n\in\mathcal{C}(\Gamma_1)$, $n\geq 1$ we say that*

$$\lim_{n\to+\infty}b_n=b \text{ in } \sigma(L^\infty(\Gamma_1),L^1(\Gamma_1))$$

if

$$\lim_{n\to+\infty}\int_{\Gamma_1}b_n\xi=\int_{\Gamma_1}b\xi,$$

for each $\xi\in L^1(\Gamma_1)$.

Theorem 2.25. *Let $\Gamma_1\neq\emptyset$ and $b_n\in\mathcal{C}(\Gamma_1)$ a sequence such that*

$$\lim_{n\to+\infty}b_n=b \text{ in } \sigma(L^\infty(\Gamma_1),L^1(\Gamma_1)).$$

Let $\lambda_1^n=\lambda_1(\mathcal{L},\mathcal{B}+b_n)$, $\lambda_1=\lambda_1(\mathcal{L},\mathcal{B}+b)$ and φ_n and φ the associated eigenfunctions to λ_1^n and λ_1 respectively. Moreover, we choose φ and φ_n such that $\|\varphi\|_2=\|\varphi_n\|_2=1$. Then, we have

$$\lim_{n\to+\infty}\lambda_1^n=\lambda_1,\qquad \lim_{n\to+\infty}\|\varphi_n-\varphi\|_{W^{1,2}}=0.$$

Theorem 2.26. *Let $\Gamma_1 \neq \emptyset$ and $b_n \in \mathcal{C}(\Gamma_1)$ a sequence such that*

$$\lim_{n\to+\infty} \min_{\Gamma_1} b_n = +\infty.$$

Let $\lambda_1^n = \lambda_1(\mathcal{L}, \mathcal{B}+b_n)$ and φ_n the associated eigenfunction to λ_1^n such that $\|\varphi_n\|_{W^{1,2}} = 1$. Let $\lambda_1^0 = \lambda_1(\mathcal{L}, \mathcal{D})$ and φ_0 the associated eigenfunction to λ_1^0. Then,

$$\lim_{n\to+\infty} \lambda_1^n = \lambda_1^0, \qquad \lim_{n\to+\infty} \|\varphi_n - \varphi_0\|_{W^{1,2}} = 0.$$

Proposition 2.27. *Let Ω_0 a proper subdomain of Ω which satisfies*

$$dist(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0$$

then

$$\lambda_1(\mathcal{L}, \mathcal{B}) < \lambda_1^{\Omega_0}(\mathcal{L}, \mathcal{B}_{\Omega_0}),$$

where the boundary operator $\mathcal{B}_{\Omega_0}$ is defined as

$$\mathcal{B}_{\Omega_0} u = \begin{cases} u & \text{on } \partial\Omega_0 \cap \Omega, \\ \mathcal{B}u & \text{on } \partial\Omega_0 \cap \partial\Omega. \end{cases}$$

2.4. An eigenvalue problem

Through this section we consider the following eigenvalue problem

$$\begin{cases} \mathcal{L}\varphi = \lambda m(x)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma_0, \\ B\varphi = \lambda r(x)\varphi & \text{on } \Gamma_1, \end{cases} \tag{2.4}$$

where $m \in \mathcal{C}^\alpha(\overline{\Omega})$ and $r \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$. The following result provides us with the existence of principal eigenvalue to (2.4). The second paragraph is a characterization of the principal eigenvalue to (2.4) when $m \equiv 0$, i.e., an eigenvalue problem at the boundary, the classical Steklov problem.

Theorem 2.28. *Assume $(m, r) > 0$. Then:*

a) Under condition

$$\exists \mu > 0 \text{ such that } (c + \mu m, b + \mu r) > 0,$$

the eigenvalue problem (2.4) has a unique principal eigenvalue, γ_1, it is simple and its associated eigenfunction can be chosen strongly positive in Ω.

b) If $m \equiv 0$ and $r > 0$ $r(x) > 0$ in a set with positive $d - 1$ dimensional measure, then, there exists the principal eigenvalue to (2.4), denoted by μ_1, if and only if

$$\lim_{\lambda\to-\infty} \lambda_1(\mathcal{L}, \mathcal{B} - \lambda r) = \mu_{-\infty} > 0.$$

Moreover, its associated eigenfunction can be chosen strongly positive in Ω.

Proof. Proof of the first paragraph. Observe that we can assume $(c,b)>0$ otherwise we consider the following eigenvalue problem that is equivalent to (2.4)

$$\begin{cases} \widetilde{\mathcal{L}}u=(\mathcal{L}+\mu m(x))u=(\lambda+\mu)m(x)u=\widetilde{\lambda}m(x)u & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_0,\\ \dfrac{\partial u}{\partial n}+(b(x)+\mu r(x))u=\dfrac{\partial u}{\partial n}+\widetilde{b}(x)u=(\lambda+\mu)r(x)u=\widetilde{\lambda}r(x)u & \text{on } \Gamma_1,\end{cases}$$

where $(\widetilde{c},\widetilde{b})>0$. We define the spaces of functions

$$\begin{array}{rcl} \mathcal{C}^1_{\Gamma_0}(\partial\Omega) & := & \{u\in\mathcal{C}^1(\partial\Omega):\ \ u_{|\Gamma_0}=0\},\\ \mathcal{C}^1_{\Gamma_0}(\overline{\Omega}) & := & \{u\in\mathcal{C}^1(\overline{\Omega}):\ \ u_{|\Gamma_0}=0\}.\end{array}$$

The spaces $\mathcal{C}^1_{\Gamma_0}(\partial\Omega)$ and $\mathcal{C}^1_{\Gamma_0}(\overline{\Omega})$ are closed subspaces of Banach spaces, therefore there are Banach spaces. On one hand, we consider the operator

$$\begin{array}{rcl} K_1:\mathcal{C}^1_{\Gamma_0}(\overline{\Omega}) & \to & \mathcal{C}^1_{\Gamma_0}(\overline{\Omega})\\ f & \mapsto & K_1(f)=u,\end{array}$$

where u is the solution to the problem

$$\begin{cases} \mathcal{L}u=m(x)f & \text{in } \Omega,\\ \mathcal{B}u=0 & \text{on } \partial\Omega.\end{cases}$$

Thanks to Theorem 2.14 K_1 is well-defined and compact. On the other hand, we consider the operator

$$\begin{array}{rcl} K_2:\mathcal{C}^1_{\Gamma_0}(\partial\Omega) & \to & \mathcal{C}^1_{\Gamma_0}(\overline{\Omega})\\ g & \mapsto & K_2(g)=u,\end{array}$$

where u is the solution to the problem

$$\begin{cases} \mathcal{L}u=0 & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_0,\\ Bu=r(x)g & \text{on } \Gamma_1.\end{cases}$$

Arguing as we did for K_1, K_2 is well-defined and compact. Now, we define the trace operator $\gamma(u)=u_{|\Gamma_1}$ and observe that

$$T:=K_1+K_2\cdot\gamma:\mathcal{C}^1_{\Gamma_0}(\overline{\Omega})\to\mathcal{C}^1_{\Gamma_0}(\overline{\Omega}),$$

is a compact operator. Moreover, (λ_1,φ_1) is an eigen-pair to (2.4) if and only if $(1/\lambda_1,\varphi_1)$ is and eigen-pair of T. Let

$$P:=\{u\in\mathcal{C}^1_{\Gamma_0}(\overline{\Omega}):\ \ u(x)\geq 0,\ \ \forall x\in\Omega\}$$

a positive cone. The interior of P is given by

$$int(P)=\{u\in\mathcal{C}^1_{\Gamma_0}(\overline{\Omega}):\ \ u(x)>0,\ \ \forall x\in\Omega\cap\Gamma_1\ \ and\ \ \partial_n u(x)<0\ \ \forall x\in\Gamma_0\}.$$

Since $(c,b)>0$ then $\lambda_1(\mathcal{L},\mathcal{B})>0$. Therefore, thanks to the strong maximum principle, the operator T is strongly positive. Since T strongly positive, compact and irreducible, then the Krein-Rutman Theorem concludes the first paragraph of the Theorem.

We have that μ_1 is the principal eigenvalue to (2.4) if and only if $\mu(\mu_1) = 0$ where

$$\mu(\lambda) := \lambda_1(\mathcal{L}, \mathcal{B} - \lambda r)$$

i.e. for each fixed λ the principal eigenvalue to the problem

$$\begin{cases} \mathcal{L}\varphi = \mu(\lambda)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma_0, \\ (B - \lambda r(x))\varphi = 0 & \text{on } \Gamma_1. \end{cases}$$

Assume that we have

$$\lim_{\lambda \to -\infty} \mu(\lambda) \leq 0$$

then, since $\mu(\lambda)$ is decreasing then the principal eigenvalue does not exist. Assume now that

$$\lim_{\lambda \to -\infty} \mu(\lambda) > 0.$$

Thanks to Proposition 2.23, $\mu(\lambda)$ is analytic, decreasing. So, it suffices to prove that

$$\lim_{\lambda \to +\infty} \mu(\lambda) = -\infty.$$

Suppose the contrary, then

$$\lim_{\lambda \to +\infty} \mu(\lambda) = l,$$

for some $l \in \mathbb{R}$. Since μ is decreasing then $\mu'(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$. Taking into account that μ is not constant then we can assume, for simplicity that there exists $\lambda_0 > 0$ such that $\mu'(\lambda_0) := \alpha < 0$. Next we observe that it is not possible to have $\mu'(y) \leq \mu'(\lambda_0) < 0$ for all $y > \lambda_0$, otherwise, by the mean value Theorem we have

$$\mu(y) - \mu(\lambda_0) = \mu'(\chi)(y - \lambda_0) \leq \alpha(y - \lambda_0),$$

for some $\chi \in (\lambda_0, y)$. Therefore, we obtain

$$l - \mu(\lambda_0) - \alpha\lambda_0 \leq \alpha y.$$

However, the previous inequality does not hold if $y > 0$ large enough. Hence, we infer that there exists $x_0 > \lambda_0$ such that

$$\mu'(x_0) > \mu'(\lambda_0). \tag{2.5}$$

On the other hand since μ is concave and analytic (see [8, section 11]) then $\mu''(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$, in particular the function μ' is non-increasing therefore $\mu'(x_0) \leq \mu'(\lambda_0)$ which is a contradiction with (2.5). ■

Remark 2.29. *Observe that during the proof of Theorem 2.28 we have proved*

$$\lim_{\lambda \to +\infty} \lambda_1(\mathcal{L}, \mathcal{B} - \lambda r) = -\infty.$$

CHAPTER 3

Uniqueness for elliptic equations with nonlinear boundary

In this chapter we present three results of uniqueness of solutions for an elliptic problem with nonlinear boundary conditions. The results of this chapter have been published in [82].

3.1. Preliminaries

Consider

$$\begin{cases} \mathcal{L}u = f(x,u) & \text{in } \Omega, \\ u = \varphi(x) & \text{on } \Gamma_0, \\ Bu = h(x,u) & \text{on } \Gamma_1, \end{cases} \tag{3.1}$$

where $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$, $h : \Gamma_1 \times \mathbb{R} \mapsto \mathbb{R}$ and $\varphi : \Gamma_0 \mapsto \mathbb{R}$ are regular functions.

We present three results of uniqueness of solution of (3.1). Now, we introduce an important change of variables. When $\lambda_1(\mathcal{L},\mathcal{B}) > 0$, there exists $e \gg 0$ (in fact $e(x) > 0$ for all $x \in \overline{\Omega}$) the unique solution of (see Section 3.3)

$$\begin{cases} \mathcal{L}e = 0 & \text{in } \Omega, \\ e = 1 & \text{on } \Gamma_0, \\ Be = 0 & \text{on } \Gamma_1. \end{cases} \tag{3.2}$$

We make the change of variable

$$u := ev,$$

which transforms (3.1) into

$$\begin{cases} \mathcal{L}_1 v = f_1(x,v) & \text{in } \Omega, \\ v = \varphi(x) & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial n} = h_1(x,v) & \text{on } \Gamma_1, \end{cases} \tag{3.3}$$

where

$$\mathcal{L}_1 v := -\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i^1 \frac{\partial v}{\partial x_i}, \tag{3.4}$$

with

$$b_i^1 := \left(b_i - \frac{2}{e} \sum_{j=1}^{d} a_{ij} \frac{\partial e}{\partial x_j} \right),$$

and

$$f_1(x,v) := \frac{f(x,ev)}{e}, \quad h_1(x,v) := \frac{h(x,ev)}{e}. \tag{3.5}$$

Moreover, under the same change of variable, the problem

$$\begin{cases} \mathcal{L}u = \lambda u & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

transforms into

$$\begin{cases} \mathcal{L}_1 v = \lambda v & \text{in } \Omega, \\ \mathcal{N}v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where

$$\mathcal{N}v := \begin{cases} v & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial n} & \text{on } \Gamma_1, \end{cases}$$

and so,

$$\lambda_1(\mathcal{L}_1, \mathcal{N}) = \lambda_1(\mathcal{L}, \mathcal{B}) > 0.$$

3.2. State and proofs of the main results

Our first result is:

Theorem 3.1. *Assume* $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$. *If* $u \mapsto f(x,u), h(x,u)$ *are non-increasing, then there exists at most a solution of* (3.1).

Proof. First observe that if f and h are non-increasing in u, then the functions f_1 and h_1 defined in (3.5) are also non-increasing in v.

Take $v_1 \neq v_2$ two solutions of (3.3) and denote by

$$\Omega_1 := \{x \in \overline{\Omega} : v_1(x) > v_2(x)\}, \quad \text{and} \quad w := v_1 - v_2.$$

Then,

$$\begin{cases} \mathcal{L}_1 w \leq 0 & \text{in } \Omega_1, \\ w = 0 & \text{on } \partial\Omega_1 \cap (\Omega \cup \Gamma_0), \\ \dfrac{\partial w}{\partial n} \leq 0 & \text{on } \partial\Omega_1 \cap \Gamma_1. \end{cases} \tag{3.8}$$

It follows by the maximum principle (see for instance [57, Theorem 3.5]) that the maximum of w has to be attained on $\partial\Omega_1 \cap \Gamma_1$ and that in such point $\partial w/\partial n > 0$ (see [57, Lemma 3.4]), which is a contradiction with $\partial w/\partial n \leq 0$. ∎

Remark 3.2. *Theorem 3.1 is not true if* $\lambda_1(\mathcal{L}, \mathcal{B}) < 0$*. Indeed, consider the logistic equation*

$$\begin{cases} \mathcal{L}u = \lambda u - u^p & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $p > 1$ *and* $\lambda \in \mathbb{R}$*. It is well-known (see for instance [25]) that* (3.9) *possesses the trivial solution* $u \equiv 0$ *for all* $\lambda \in \mathbb{R}$ *and for* $\lambda > \lambda_1(\mathcal{L}, \mathcal{B})$ *possesses another positive solution. Observe that* (3.9) *can be written as*

$$(\mathcal{L} - \lambda)u = -u^p.$$

In this case $f(x, u) = -u^p$ *is decreasing and* $\lambda_1(\mathcal{L} - \lambda, \mathcal{B}) = \lambda_1(\mathcal{L}, \mathcal{B}) - \lambda < 0$ *if* $\lambda > \lambda_1(\mathcal{L}, \mathcal{B})$*.*

Next, we have our second result. Observe that we do not require $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$.

Theorem 3.3. *Assume that* $\varphi \geq 0$ *on* Γ_0 *and*

$$u \mapsto \frac{f(x,u)}{u}, \ \frac{h(x,u)}{u}, \quad \text{are non-increasing in } (0,\infty), \tag{3.10}$$

with at least one of them decreasing. Then there exists at most a positive solution to (3.1)*.*

Proof. First, observe that if u is a positive solution of (3.1) then u is strongly positive. Indeed, since $u \leq \|u\|_\infty$, it follows that

$$\frac{f(x,u)}{u} \geq \frac{f(x, \|u\|_\infty)}{\|u\|_\infty} := -K_1, \qquad \frac{h(x,u)}{u} \geq \frac{h(x, \|u\|_\infty)}{\|u\|_\infty} := -K_2.$$

Take $M > \max\{0, K_1, K_2, -\lambda_1(\mathcal{L}, \mathcal{B})\}$. Then, (3.1) is equivalent to

$$\mathcal{L}u + Mu = f(x,u) + Mu > 0 \quad \text{in } \Omega, \qquad \mathcal{B}u + Mu = h(x,u) + Mu > 0 \quad \text{on } \partial\Omega.$$

Moreover, thanks to the monotonicity properties of the principal eigenvalue we get that

$$\lambda_1(\mathcal{L} + M, \mathcal{B} + M) > \lambda_1(\mathcal{L} + M, \mathcal{B}) = M + \lambda_1(\mathcal{L}, \mathcal{B}) > 0,$$

and so, the strong maximum principle concludes that u is strongly positive.

Take two positive solutions $u_1 \neq u_2$ of (3.1) and define

$$w := u_1 - u_2.$$

Since u_1 is a strongly positive solution of (3.1), then

$$\lambda_1\left(\mathcal{L} - \frac{f(x,u_1)}{u_1}, \mathcal{B} - \frac{h(x,u_1)}{u_1}\right) \geq 0. \tag{3.11}$$

It is not hard to show that

$$\mathcal{L}w - F(x)w = 0 \quad \text{in } \Omega, \qquad \mathcal{B}w - H(x)w = 0 \quad \text{on } \partial\Omega, \tag{3.12}$$

where

$$F(x) := \begin{cases} \dfrac{f(x,u_1) - f(x,u_2)}{u_1 - u_2} & u_1 \neq u_2, \\ D_2 f(x,u_1) & u_1 = u_2, \end{cases} \qquad H(x) := \begin{cases} \dfrac{h(x,u_1) - h(x,u_2)}{u_1 - u_2} & u_1 \neq u_2, \\ D_2 h(x,u_1) & u_1 = u_2. \end{cases}$$

Hence, from (3.12) it follows that 0 is an eigenvalue of the operator $\mathcal{L} - F$ under homogeneous boundary condition $\mathcal{B} - H$, that is

$$0 = \lambda_j(\mathcal{L} - F, \mathcal{B} - H), \quad \text{for some } j \geq 1.$$

On the other hand, thanks to (3.10), it follows that

$$F(x) \leq \frac{f(x,u_1)}{u_1} \quad \text{and} \quad H(x) \leq \frac{h(x,u_1)}{u_1},$$

with at least one of the inequalities strict. Thus,

$$0 = Re(\lambda_j(\mathcal{L} - F, \mathcal{B} - H)) \geq \lambda_1(\mathcal{L} - F, \mathcal{B} - H) > \lambda_1\left(\mathcal{L} - \frac{f(x,u_1)}{u_1}, \mathcal{B} - \frac{h(x,u_1)}{u_1}\right) \geq 0,$$

a contradiction. ∎

Remark 3.4. *If instead of* (3.10)*, we assume that both maps are non-decreasing, we can conclude that if u_1 and u_2 are ordered, then $u_1 = u_2$.*

Finally we state and prove the last theorem of the section

Theorem 3.5. *Assume $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$, $\varphi \geq 0$ on Γ_0 and there exists $g \in \mathcal{C}^1(0,+\infty) \cap \mathcal{C}([0,+\infty))$, $g(t) > 0$ for $t > 0$ and g' non-increasing, such that*

$$u \mapsto \frac{f(x,u)}{g(u)}, \ \frac{h(x,u)}{g(u)} \quad \text{are non-increasing in } (0,\infty). \tag{3.13}$$

If:

a)

$$\int_0^r \frac{1}{g(t)} dt < \infty, \qquad \text{for some } r > 0, \tag{3.14}$$

then there exists at most a solution u to (3.1) *satisfying*

$$u(x) > 0 \text{ for all } x \in \Omega. \tag{3.15}$$

b)

$$\lim_{s \to 0} \frac{s}{g(s)} = 0, \tag{3.16}$$

then there exists at most a strongly positive solution to (3.1).

Proof. Observe again that if f and g satisfy conditions of Theorem 3.5, then there exists a function $g_1 \in \mathcal{C}^1(0,+\infty)\cap\mathcal{C}([0,+\infty))$ such that f_1, h_1 and g_1 satisfy also conditions of Theorem 3.5 (see (3.5)).

a) Assume (3.14) and let v a solution of (3.3) satisfying (3.15). The change of variable

$$w := \int_0^v \frac{1}{g_1(t)}dt \tag{3.17}$$

transforms (3.3) into

$$\begin{cases} \mathcal{L}_1 w = \dfrac{f_1(x,k(w))}{g_1(k(w))} + g_1'(k(w)) \displaystyle\sum_{i,j=1}^d a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} & \text{in } \Omega, \\ w = \widetilde{\varphi} & \text{on } \Gamma_0, \\ \dfrac{\partial w}{\partial n} = \dfrac{h_1(x,k(w))}{g_1(k(w))} & \text{on } \Gamma_1, \end{cases} \tag{3.18}$$

where $\widetilde{\varphi} = \displaystyle\int_0^{\varphi} \frac{1}{g_1(t)}dt$,

$$v = k(w), \tag{3.19}$$

and k satisfies, from (3.17), that $k'(t) = g_1(k(t))$.

Assume that there exist two solutions $v_1 \neq v_2$ of (3.3) satisfying (3.15). Denote

$$\Omega_1 := \{x \in \overline{\Omega} : v_1(x) > v_2(x)\} \quad \text{and} \quad \Phi := w_1 - w_2,$$

where $v_i = k(w_i)$ $i = 1,2$. Observe that $\Phi > 0$ in Ω_1 thanks to the monotony of k. We have that in Ω_1

$$\begin{aligned} \mathcal{L}_1\Phi = &\left(\frac{f_1(x,k(w_1))}{g_1(k(w_1))} - \frac{f_1(x,k(w_2))}{g_1(k(w_2))}\right) + \\ &+\left(g_1'(k(w_1))\sum_{i,j=1}^d a_{ij}\frac{\partial w_1}{\partial x_i}\frac{\partial w_1}{\partial x_j} - g_1'(k(w_2))\sum_{i,j=1}^d a_{ij}\frac{\partial w_2}{\partial x_i}\frac{\partial w_2}{\partial x_j}\right), \end{aligned} \tag{3.20}$$

$$\Phi = 0 \qquad \text{on } \partial\Omega_1 \cap (\Omega \cup \Gamma_0), \tag{3.21}$$

and

$$\frac{\partial \Phi}{\partial n} = \frac{h_1(x,k(w_1))}{g_1(k(w_1))} - \frac{h_1(x,k(w_2))}{g_1(k(w_2))} \qquad \text{on } \partial\Omega_1 \cap \Gamma_1. \tag{3.22}$$

Observe that,

$$g_1'(k(w_1))\sum_{i,j=1}^d a_{ij}\frac{\partial w_1}{\partial x_i}\frac{\partial w_1}{\partial x_j} - g_1'(k(w_2))\sum_{i,j=1}^d a_{ij}\frac{\partial w_2}{\partial x_i}\frac{\partial w_2}{\partial x_j} =$$

$$g_1'(k(w_1))\sum_{i,j=1}^d a_{ij}\frac{\partial(w_1+w_2)}{\partial x_j}\frac{\partial \Phi}{\partial x_i} + [g_1'(k(w_1)) - g_1'(k(w_2))]\sum_{i,j=1}^d a_{ij}\frac{\partial w_2}{\partial x_i}\frac{\partial w_2}{\partial x_j}.$$

Moreover, thanks to that g_1' is non-increasing and that $\mathcal{L}$ is uniformly elliptic, it follows that

$$[g_1'(k(w_1)) - g_1'(k(w_2))] \sum_{i,j=1}^{d} a_{ij} \frac{\partial w_2}{\partial x_i} \frac{\partial w_2}{\partial x_j} \leq 0.$$

Thus, we get from (3.20) – (3.22) that

$$\begin{cases} \mathcal{L}_2 \Phi \leq 0 & \text{in } \Omega_1, \\ \Phi = 0 & \text{on } \partial\Omega_1 \cap (\Omega \cup \Gamma_0), \\ \dfrac{\partial \Phi}{\partial n} \leq 0 & \text{on } \partial\Omega_1 \cap \Gamma_1, \end{cases} \tag{3.23}$$

where

$$\mathcal{L}_2 \Phi := - \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \left(b_i^1 - g'(k(w_1)) \sum_{j=1}^{d} a_{ij} \frac{\partial (w_1 + w_2)}{\partial x_j} \right) \frac{\partial \Phi}{\partial x_i}.$$

It suffices to apply again the strong maximum principle.

b) Assume now (3.16) and that there exist two strongly positive solutions $v_1 \neq v_2$ of (3.3) with $v_i \in int(P)$, $i = 1, 2$. Let $\Omega_1 := \{x \in \overline{\Omega} : v_1(x) > v_2(x)\}$. We define now for $x \in \Omega_1$

$$\Phi(x) := \int_{v_2(x)}^{v_1(x)} \frac{1}{g_1(t)} dt.$$

First, observe that

$$\Phi = 0 \quad \text{on } \partial\Omega_1 \cap (\Omega \cup \Gamma_0).$$

Indeed, for $x \in \partial\Omega_1 \cap \Omega$ it is clear that $\Phi(x) = 0$. For $x \in \Omega_1$ we have that for some $\xi(x)$ with $v_2(x) \leq \xi(x) \leq v_1(x)$

$$\Phi(x) = \frac{v_1(x) - v_2(x)}{g_1(\xi(x))} \leq \frac{C dist(x)}{g_1(\xi(x))} \to 0,$$

as $dist(x) \to 0$, where $dist(x) = dist(x, \partial\Omega)$ thanks to (3.16). Hence $\Phi = 0$ on $\partial\Omega_1 \cap \Gamma_0$ and $\dfrac{\partial \Phi}{\partial n} \leq 0$. ■

Remark 3.6. *Let us stress the fact that Theorems 3.5 and 3.3 are complementary. Clearly, Theorem 3.3 is not included in Theorem 3.5. In order to show that Theorem 3.5 is not included in Theorem 3.3, we consider the problem*

$$\begin{cases} \mathcal{L}u = a(x) u^q & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = -u^p & \text{on } \Gamma_1, \end{cases}$$

where $0 < q < 1 < p$, $a \in \mathcal{C}^\alpha(\overline{\Omega})$ *is a sign changing function and* $\lambda_1(\mathcal{L}, \mathcal{B}) > 0$. *Observe that the conditions of Theorem 3.3 are not satisfied. However we can apply Theorem 3.5 for* $g(s) = s^q$ *(see also Chapter 6 for another examples).*

3.3. The linear problem

In this section we give a result of existence and uniqueness of a linear problem. We could not find such a result in a general monograph of partial differential equations of elliptic type like [57], so for the sake of completeness we include it here.

Proposition 3.7. *Assume that* $\lambda_1(\mathcal{L},\mathcal{B}) > 0$, $(f,g,h) \in \mathcal{C}^{\alpha}(\overline{\Omega}) \times \mathcal{C}^{1+\alpha}(\Gamma_0) \times \mathcal{C}^{1+\alpha}(\Gamma_1)$, *such that* $f,g,h \geq 0$ *and some of the inequalities strict. Then, there exists a unique strongly positive solution to the linear problem*

$$\begin{cases} \mathcal{L}u = f(x) & \text{in } \Omega, \\ u = g(x) & \text{on } \Gamma_0, \\ Bu = h(x) & \text{on } \Gamma_1. \end{cases} \tag{3.24}$$

Proof. Since Ω is smooth, there exists (see [76, Proposition 3.4]) $\psi \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$ and a constant $\gamma > 0$ such that

$$\frac{\partial \psi}{\partial n} \geq \gamma > 0 \quad \text{on } \Gamma_1. \tag{3.25}$$

We make the following change of variable

$$u := e^{M\psi} v. \tag{3.26}$$

Under this change, (3.24) transforms into

$$\begin{cases} \mathcal{L}_M v = f_M(x) & \text{in } \Omega, \\ v = g_M(x) & \text{on } \Gamma_0, \\ B_M v = h_M(x) & \text{on } \Gamma_1, \end{cases} \tag{3.27}$$

where

$$f_M = f e^{-M\psi}, \quad g_M = g e^{-M\psi}, \quad h_M = h e^{-M\psi},$$

$$\mathcal{L}_M v := -\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i^M \frac{\partial v}{\partial x_i} + c_M(x) v, \qquad B_M v := \frac{\partial v}{\partial n} + b_M(x) v,$$

and

$$b_i^M := \left(b_i - 2M \sum_{j=1}^{d} a_{ij} \frac{\partial \psi}{\partial x_j} \right), \qquad b_M := \left(b(x) + M \frac{\partial \psi}{\partial n} \right),$$

$$c_M := c(x) + M \sum_{i=1}^{d} b_i \frac{\partial \psi}{\partial x_i} - M \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - M^2 \sum_{i,j=1}^{d} a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j}.$$

On the other hand, (3.6) transforms into

$$\begin{cases} \mathcal{L}_M v = \lambda v & \text{in } \Omega, \\ \mathcal{B}_M v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.28}$$

and so,

$$\lambda_1(\mathcal{L},\mathcal{B}) = \lambda_1(\mathcal{L}_M,\mathcal{B}_M).$$

Thanks to (3.25), we can take $M > 0$ large enough such that

$$b_M \geq 0.$$

Now, we focus our attention on solving (3.27). Take a regular function $K(x)$ such that

$$K(x) > \max\{c_M(x), 0\}$$

and consider the unique positive solution (which exists because $b_M, K \geq 0$, see [57, Theorem 6.1]) of

$$\begin{cases} (\mathcal{L}_0 + K(x))w = f_M(x) & \text{in } \Omega, \\ w = g_M(x) & \text{on } \Gamma_0, \\ B_M w = h_M(x) & \text{on } \Gamma_1, \end{cases} \tag{3.29}$$

where

$$\mathcal{L}_0 w := -\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i^M \frac{\partial w}{\partial x_i}.$$

Now, it is evident that a solution v of (3.27) can be written as $v = z + w$ with z solution of

$$\begin{cases} \mathcal{L}_M z = f_1(x) := [K(x) - c_M(x)]w > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma_0, \\ B_M z = 0 & \text{on } \Gamma_1. \end{cases} \tag{3.30}$$

So, it remains to show that (3.30) possesses a unique positive solution, for that we are going to use the classical Riesz Theory. Observe that $\mathcal{L}_M z = f_1(x)$ is equivalent to

$$(\mathcal{L}_M + R)z - Rz = f_1(x) \Longleftrightarrow \frac{1}{R} z - (\mathcal{L}_M + R)^{-1} z = f_2(x) := (\mathcal{L}_M + R)^{-1} (\frac{f_1(x)}{R}) \geq 0,$$

where R is a positive constant sufficiently large so that $\lambda_1(\mathcal{L}_M + R, \mathcal{B}_M) > 0$, and so there exists the inverse of $\mathcal{L}_M + R$ under homogeneous boundary condition $\mathcal{B}_M$. Denoting $r(T)$ the spectral radius of a linear operator T, we get that

$$\frac{1}{R} > r((\mathcal{L}_M + R)^{-1}) = \frac{1}{\lambda_1(\mathcal{L}_M + R, \mathcal{B}_M)} = \frac{1}{\lambda_1(\mathcal{L}_M, \mathcal{B}_M) + R},$$

thanks to $\lambda_1(\mathcal{L}_M, \mathcal{B}_M) > 0$. It now suffices to apply [5, Theorem 3.2] and the result concludes. ■

CHAPTER 4

Bifurcation techniques in elliptic equations with nonlinear boundary

The aim of this chapter is to apply bifurcation techniques to second order uniformly elliptic problems with mixed nonlinear boundary conditions. The general results of this chapter will be used in the next chapters for particular problems. A preliminary version of some of the results of this chapter has been published in [81].

Throughout this chapter we consider the following problem

$$\begin{cases} \mathcal{L}u = f(\lambda, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = g(\lambda, x, u) & \text{on } \Gamma_1, \end{cases} \tag{4.1}$$

where, $f \in \mathcal{C}^{\alpha}(\mathbb{R} \times \overline{\Omega} \times \mathbb{R})$ and $g \in \mathcal{C}^{1+\alpha}(\mathbb{R} \times \Gamma_1 \times \mathbb{R})$, although additional regularity may be required.

In order to use the degree theory we should, at a first step, transform the problem (4.1) into an equivalent fixed point problem. Such a procedure will be done in the following section.

4.1. A Fixed point formulation

For a fixed λ we denote by F_λ and G_λ to the Netmiskii operators associated to f and g respectively (see [10]). Throughout this section we assume

$$(c, b) > 0. \tag{4.2}$$

Let us consider the operator

$$\begin{array}{rcl} K_1 : \mathcal{C}^{\alpha}(\overline{\Omega}) & \to & \mathcal{C}^{2+\alpha}_{\Gamma_0}(\overline{\Omega}) \\ f & \mapsto & K_1(f) = u, \end{array}$$

where u is the solution to the problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = 0 & \text{on } \Gamma_1. \end{cases} \tag{4.3}$$

Thanks to the Schauder theory, the operator K_1 is well defined. Observe that it is possible to extend this operator to the set of functions $\mathcal{C}(\overline{\Omega})$, such as extended operator, in order to simplify the notation, will be again denoted as K_1. Moreover, $K_1 : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ is a compact operator. On the other hand, we consider the operator

$$\begin{array}{rcl} K_2 : \mathcal{C}^{1+\alpha}(\Gamma_1) & \to & \mathcal{C}^{2+\alpha}_{\Gamma_0}(\overline{\Omega}) \\ g & \mapsto & K_2(g) = u, \end{array}$$

where u is the solution to the problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = g & \text{on } \Gamma_1. \end{cases} \tag{4.4}$$

As previously, the operator K_2 is well defined, thanks to the Schauder theory. Moreover, we can extend K_2 to the set of functions $\mathcal{C}(\Gamma_1)$ with image in $W^{1,p}_{\Gamma_0}(\Omega)$, with $p \in (1, \infty)$. Since $W^{1,p}_{\Gamma_0}(\Omega)$ is compactly embedded in $\mathcal{C}_{\Gamma_0}(\Omega)$ for $p > d$ then this new operator, denoted again by $K_2 : \mathcal{C}(\Gamma_1) \to \mathcal{C}_{\Gamma_0}(\overline{\Omega})$, is a compact operator. Next, we define the trace operator:

$$\begin{array}{rcl} \gamma : \mathcal{C}(\overline{\Omega}) & \to & \mathcal{C}(\Gamma_1) \\ u & \mapsto & \gamma(u) = u_{|\Gamma_1}. \end{array}$$

Definition 4.1. *We say that u is a classical solution to (4.1) if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ and $Lu = F_\lambda(u)$ in Ω, $u = 0$ on Γ_0 and $Bu = G_\lambda(u)$ on Γ_1.*

Obviously, if u is a classical solution to (4.1) then

$$u = K_1(F_\lambda(u)) + K_2(G_\lambda(\gamma(u))) \quad in \ \mathcal{C}_{\Gamma_0}(\overline{\Omega}),$$

in fact, the reverse is also true.

Proposition 4.2. *Assume (4.2). If $u = K_1(F_\lambda(u)) + K_2(G_\lambda(\gamma(u)))$ in $\mathcal{C}_{\Gamma_0}(\overline{\Omega})$ then u is a classical solution to (4.1).*

Proof. Thanks to the definition of K_1 and K_2 we have $u \in W^{1,p}_{\Gamma_0}(\Omega)$, therefore, $\widetilde{\gamma}(u) \in W^{1-1/p,p}(\Gamma_1)$, where $\widetilde{\gamma}$ stands for the extension of γ to $W^{1,p}(\Omega)$. The regularity of G_λ assures $G_\lambda(\gamma(u)) \in W^{1-1/p,p}(\Gamma_1)$. Moreover, $F_\lambda(u) \in \mathcal{C}(\overline{\Omega}) \subset L^p(\Omega)$, $1 < p < \infty$ then, the L^p theory for elliptic equations entails $u \in W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is compactly embedded in $\mathcal{C}^{1+\sigma}(\overline{\Omega})$ with $\sigma = 1 - d/p$ then $u \in \mathcal{C}^{1+\sigma}(\overline{\Omega})$. Therefore, picking $\beta = \min\{\alpha, \sigma\}$, we have $F_\lambda(u) \in \mathcal{C}^\beta(\overline{\Omega})$ and $G_\lambda(\gamma(u)) \in C^{1+\beta}(\Gamma_1)$, so, thanks to the Schauder estimates we have $u \in \mathcal{C}^{2+\beta}(\overline{\Omega})$. ∎

4.2. Bifurcation from zero

During this section we consider the nonlinear equation

$$(4.5)\qquad \begin{cases} \mathcal{L}u = \lambda m(x)u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = \lambda r(x)u + g(x,u) & \text{on } \Gamma_1, \end{cases}$$

where λ denotes the bifurcation parameter, $f \in \mathcal{C}^{\alpha}(\overline{\Omega} \times \mathbb{R})$, $g \in \mathcal{C}^{1+\alpha}(\Gamma_1 \times \mathbb{R})$, such that

$$(4.6)\qquad f(x,0) \geq 0 \ \ \forall x \in \Omega, \quad g(x,0) \geq 0 \ \ \forall x \in \Gamma_1,$$

$(m,r) > 0$, $m \in \mathcal{C}^{\alpha}(\overline{\Omega})$, $r \in \mathcal{C}^{1+\alpha}(\Gamma_1)$.

Since we are only interested in non-negative solutions to (4.5), we rewrite (4.5) as a problem with only non-negative solutions. For this purpose, let $u_+ = \max\{u,0\}$. From now on we fix $K \geq 0$ such that $(c+K, b+K) > 0$.

Lemma 4.3. *Let u a solution to*

$$(4.7)\qquad \begin{cases} (\mathcal{L}+K)u = (\lambda m(x)+K)u_+ + f(x,u_+) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ (B+K)u = (\lambda r(x)+K)u_+ + g(x,u_+) & \text{on } \Gamma_1, \end{cases}$$

then $u \geq 0$ in $\overline{\Omega}$.

Proof. Suppose that the set $\Omega' = \{x \in \Omega : \ u(x) < 0\}$ is non-empty. We have

$$\begin{array}{ll} (\mathcal{L}+K)u \geq 0 & \text{in } \Omega', \\ (B+K)u \geq 0 & \text{on } \partial\Omega' \cap \partial\Omega, \\ u = 0 & \text{on } (\partial\Omega' \cap \Omega) \cup \Gamma_0. \end{array}$$

Then by [6] it holds $u \equiv 0$ in Ω', a contradiction. ∎

Through this section we assume that

$$(\mathbf{B0})\qquad \lim_{s\to 0^+} \frac{f(x,s)}{s} = 0 \ \text{ unif. in } \overline{\Omega}, \qquad \lim_{s\to 0^+} \frac{g(x,s)}{s} = 0 \ \text{ unif. on } \Gamma_1.$$

Let us denote by M, R, F and G the Netmiskii operators associated to $m(x)u$, $r(x)u$, f and g respectively. Consider the maps $\Phi_\lambda, \Phi_\lambda^t : \mathcal{C}_{\Gamma_0}(\overline{\Omega}) \to \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ defined by

$$\begin{aligned} \Phi_\lambda(u) = \ & u - K_1(\lambda M(u_+) + Ku_+ + F(u_+)) \\ & - K_2(\lambda R(\gamma(u_+)) + K\gamma(u_+) + G(\gamma(u_+))), \end{aligned}$$

$$\begin{aligned} \Phi_\lambda^t(u) = \ & u - tK_1(\lambda M(u_+) + Ku_+ + F(u_+)) \\ & - tK_2(\lambda R(\gamma(u_+)) + K\gamma(u_+) + G(\gamma(u_+))), \end{aligned}$$

for $t \geq 0$. Thanks to Proposition 4.2 and Lemma 4.3, u is a classical non-negative solution to (4.7) if and only if $\Phi_\lambda(u) = 0$ in $\mathcal{C}_{\Gamma_0}(\overline{\Omega})$.

We define the application

$$\delta^K(\lambda) := \lambda_1(\mathcal{L} + K - \lambda M, \mathcal{B} + K - \lambda R). \tag{4.8}$$

Since $\delta^K(\lambda)$ is decreasing there exists at most one γ_1^K such that

$$\delta^K(\gamma_1^K) = 0.$$

Obviously, by the choice of K that we have done $\gamma_1^K > 0$ exists. Let us denote by φ_1^K a strongly positive eigenfunction associated to γ_1^K. In the next two sections we assume the existence of

$$\gamma_1^0 = \gamma_1, \qquad \varphi_1^0 = \varphi_1.$$

Lemma 4.4. *Let $\Lambda \subset (-\infty, \gamma_1)$ a compact interval. Then, there exists $\delta > 0$ such that $\Phi_\lambda^t(u) \neq 0$ $\forall u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $0 < \|u\|_{\mathcal{C}(\overline{\Omega})} = \|u\| < \delta$, $\forall \lambda \in \Lambda$ and $\forall t \in [0,1]$.*

Proof. Suppose that there exist $\lambda_n, t_n \in \mathbb{R}$ and $u_n \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ such that $\lambda_n \to \overline{\lambda} \in \Lambda$, $t_n \to \overline{t} \in [0,1]$, $\|u_n\| \to 0$ and $\Phi_{\lambda_n}^{t_n}(u_n) = 0$. By Lemma 4.3 $u_n \geq 0$ and dividing by $\|u_n\|$ we obtain

$$v_n = t_n K_1 \left(\frac{\lambda_n M(u_n) + K u_n + F(u_n)}{\|u_n\|} \right) + t_n K_2 \left(\frac{\lambda_n R(\gamma(u_n)) + K\gamma(u_n) + G(\gamma(u_n))}{\|u_n\|} \right), \tag{4.9}$$

where $v_n = \frac{u_n}{\|u_n\|}$. Thanks to (B0) we have that

$$\left\| \frac{\lambda_n M(u_n) + K u_n + F(u_n)}{\|u_n\|} \right\| \leq C, \qquad \left\| \frac{\lambda_n R(\gamma(u_n)) + K\gamma(u_n) + G(\gamma(u_n))}{\|u_n\|} \right\|_{\Gamma_1} \leq C,$$

where $\|\cdot\|_{\Gamma_1}$ stands for the norm associated to the space $\mathcal{C}(\Gamma_1)$. Since K_1 and K_2 are compact operators in $\mathcal{C}_{\Gamma_0}(\overline{\Omega})$, then the sequence v_n is a relatively compact in $\mathcal{C}_{\Gamma_0}(\overline{\Omega})$. Therefore, we can suppose that $v_n \to \overline{v}$ in $\mathcal{C}(\overline{\Omega})$. By (B0), we have

$$\frac{F(u_n)}{\|u_n\|} \to 0 \text{ in } \mathcal{C}(\overline{\Omega}), \quad \frac{G(\gamma(u_n))}{\|u_n\|} \to 0 \ \text{ on } \mathcal{C}(\Gamma_1).$$

Passing to the limit in (4.9) and taking into account that K_1 and K_2 are closed operators we conclude that

$$\overline{v} = \overline{t}[K_1(\overline{\lambda} M(\overline{v}) + K\overline{v}) + K_2(\overline{\lambda} R(\gamma(\overline{v})) + K\gamma(\overline{v}))],$$

so $\overline{v}$ is the solution to the eigenvalue problem

$$\begin{cases} (\mathcal{L} + K)\overline{v} = \overline{t}(\overline{\lambda} m(x) + K)\overline{v} & \text{in } \Omega, \\ \overline{v} = 0 & \text{on } \Gamma_0, \\ (B + K)\overline{v} = \overline{t}(\overline{\lambda} r(x) + K)\overline{v} & \text{on } \Gamma_1. \end{cases}$$

Since $u_n \geq 0$, $\|v_n\| = 1$ and by the maximum principle, $\overline{v}$ is a strongly positive function in Ω. Therefore,

$$0 = \lambda_1(\mathcal{L} - \overline{\lambda}\,\overline{t} m(x) + K(1 - \overline{t}), \mathcal{B} + K(1 - \overline{t}) - \overline{\lambda}\,\overline{t} r(x)).$$

Let us observe that $\bar{t} \neq 1$, otherwise $\overline{\lambda} = \gamma_1$ which is not possible by the choice of Λ. On one hand, thanks to Proposition 2.23 we have

$$0 > \lambda_1(\mathcal{L} - \overline{\lambda}\,\bar{t}m(x), \mathcal{B} - \overline{\lambda}\,\bar{t}r(x)) + K(1 - \bar{t}). \tag{4.10}$$

On the other hand, since $\overline{\lambda t}$ is finite and the function $\delta^0(\lambda)$ (see (4.8)) is analytic then there exists $M > 0$ such that

$$\lambda_1(\mathcal{L} - \overline{\lambda}\,\bar{t}m(x), \mathcal{B} - \overline{\lambda}\,\bar{t}r(x)) > -M$$

So, picking $K \geq \frac{M}{1-\bar{t}}$ we arrive at a contradiction with (4.10). ∎

Remark 4.5. *Assume that (λ_n, u_n) are solutions to (4.5) with $(\lambda_n, u_n) \to (\gamma_1, 0)$ in $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$. Then, arguing as in Lemma 4.4 we have*

$$\lim_{n\to+\infty} \frac{u_n}{\|u_n\|} = \varphi_1 \text{ unif. in } \overline{\Omega}.$$

Definition 4.6. *Let $F : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega})$, $R > 0$ and $B_R = \{u \in \mathcal{C}(\overline{\Omega}) : \ \|u\|_{\mathcal{C}(\overline{\Omega})} < R\}$. Then, $\deg(F, B_R, 0)$ stands for the degree of F on B_R with respect to 0, and $i(F, u_0, 0)$ denotes the index of an isolated solution u_0 to the equation $F(u) = 0$.*

Corollary 4.7. *If $\lambda < \gamma_1$, then $i(\Phi_\lambda, 0, 0) = 1$.*

Proof. Let us take $\Lambda = \{\lambda\}$. Thanks to Lemma 4.4, we know that $\exists\delta > 0$ such that $\forall u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\| \in (0, \delta)$ we have $\Phi^t_\lambda(u) \neq 0$, $\forall t \in [0, 1]$. Therefore, by the homotopy invariance of the degree, we infer

$$i(\Phi_\lambda, 0, 0) = deg(\Phi^1_\lambda = \Phi_\lambda, B_\delta, 0) = deg(\Phi^0_\lambda = I, B_\delta, 0) = 1.$$

∎

Lemma 4.8. *Let $\lambda > \gamma_1$. Then, there exists $\delta > 0$ such that $\forall u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\| \in (0, \delta)$, $\Phi_\lambda(u) \neq \tau\varphi^K_1$, $\forall\tau \geq 0$.*

Proof. Let us assume that there exist sequences $\tau_n \geq 0$, $u_n \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ such that $\|u_n\| \to 0$ and $\Phi_\lambda(u_n) = \tau_n\varphi^K_1$. Thanks to Proposition 4.2 and similar arguments that we have employed in Lemma 4.3, we have that u_n is a strongly positive classical solution to the problem

$$\begin{cases} (\mathcal{L} + K)u_n = \lambda m(x)u_n + f(x, u_n) + Ku_n + \gamma^K_1\tau_n m(x)\varphi^K_1 & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_0, \\ (B + K)u_n = \lambda r(x)u_n + Ku_n + g(x, u_n) + \gamma^K_1\tau_n r(x)\varphi^K_1 & \text{on } \Gamma_1. \end{cases}$$

On one hand, by (B0), we have that for all $\epsilon > 0$ there exists $n_0(\epsilon)$ such that for all $n \geq n_0(\epsilon)$ we get

$$\begin{cases} (\mathcal{L} + K)u_n > \lambda m(x)u_n + Ku_n - \epsilon u_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_0, \\ (B + K)u_n > \lambda r(x)u_n + Ku_n - \epsilon u_n & \text{on } \Gamma_1. \end{cases}$$

Hence, u_n is a strict supersolution to $(\mathcal{L} - \lambda m(x) + \epsilon, \mathcal{B} - \lambda r(x) + \epsilon, \Omega)$, so

$$\delta_\epsilon(\lambda) = \lambda_1(\mathcal{L} - \lambda m(x) + \epsilon, \mathcal{B} - \lambda r(x) + \epsilon) > 0. \tag{4.11}$$

On the other hand, since $\lambda > \gamma_1$ then $\delta_0(\lambda) < 0$. Moreover, by Propositions 2.22 and 2.23 we infer that there exists $\epsilon_0 > 0$ such that $\delta_{\epsilon_0}(\lambda) < 0$. So, picking $\epsilon = \epsilon_0 > 0$ there is a contradiction with (4.11). ∎

Corollary 4.9. *If $\lambda > \gamma_1$ then $i(\Phi_\lambda, 0, 0) = 0$.*

Proof. Let $\epsilon \in (0, \delta)$ where δ is given in Lemma 4.8. Since Φ_λ is bounded in $\overline{B}_\epsilon$, then by Lemma 4.8, there exists $a > 0$ such that $\Phi_\lambda(u) \neq ta\varphi_1^K$, for all $u \in \overline{B}_\epsilon$, for all $t \in [0, 1]$. Hence,

$$i(\Phi_\lambda, 0, 0) = deg(\Phi_\lambda, B_\epsilon, 0) = deg(\Phi_\lambda - a\varphi_1^K, B_\epsilon, 0) = 0.$$

∎

Let

$$\mathcal{C} := \overline{\{(\lambda, u) \in \mathbb{R} \times \mathcal{C}_{\Gamma_0}(\overline{\Omega}) : \ \Phi_\lambda(u) = 0, \ u \neq 0\}}.$$

Then,

Theorem 4.10. *Assume $(m, r) > 0$, (4.6) and (B0). If γ_1 exists then it is a bifurcation point from the trivial solution, and it is the only one for positive solutions. Moreover, there exists an unbounded continuum (closed and connected set) $\mathcal{C}_0 \subset \mathcal{C}$ of positive solutions of (4.5) emanating from $(\gamma_1, 0)$. Furthermore, if γ_1 does not exist then the bifurcation from zero does not occur.*

Proof. Owing to Corollaries 4.7 and 4.9 there is a change in the index of the solution zero to Φ_λ for λ in a neighborhood of γ_1, this shows that a new solution around the zero solution appears. The existence of an unbounded continuum is consequence of the global bifurcation Theorem of Rabinowitz (see [11, Proposition 3.5]). Now, we prove the uniqueness of the bifurcation point γ_1 as well as the last part of the Theorem. Assume that there exists a pair (λ_j, u_j), $j \in \mathbb{N}$ of positive solutions to (4.5) satisfying

$$(\lambda_j, u_j) \to (\overline{\lambda}, 0) \text{ in } \mathbb{R} \times \mathcal{C}(\overline{\Omega}).$$

We consider $v_j = \frac{u_j}{\|u_j\|}$ and we argue as in Lemma 4.4, so we conclude that there exists a subsequence (denoted with the same index) such that $v_j \to \overline{v}$ uniformly in $\overline{\Omega}$ where $\overline{v} > 0$ is the solution to

$$\begin{cases} \mathcal{L}\overline{v} = \overline{\lambda} m(x)\overline{v} & \text{in } \Omega, \\ \overline{v} = 0 & \text{on } \Gamma_0, \\ B\overline{v} = \overline{\lambda} r(x)\overline{v} & \text{on } \Gamma_1. \end{cases}$$

Therefore, thanks to the uniqueness of the principal eigenvalue γ_1, $\overline{\lambda} = \gamma_1$. ∎

Corollary 4.11. *Assume $(m,r)>0$, (4.6),*

$$\lim_{s\to 0^+}\frac{f(x,s)}{s}=c_1 \text{ unif. in } \overline{\Omega},\quad \lim_{s\to 0^+}\frac{g(x,s)}{s}=c_2 \text{ unif. on } \Gamma_1,$$

for some c_1, $c_2\in\mathbb{R}$. Let $\widetilde{\gamma}_1$, if there exists, the unique zero of

$$\mu(\lambda)=\lambda_1(\mathcal{L}-c_1-\lambda M,\mathcal{B}-c_2-\lambda R).$$

Then $\widetilde{\gamma}_1$ is a bifurcation point from the trivial solution, and it is the only one for positive solutions. Moreover, there exists an unbounded continuum $\mathcal{C}_0\subset\mathcal{C}$ of positive solutions of (4.5) emanating from $(\widetilde{\gamma}_1,0)$. Furthermore, if the application $\mu(\cdot)$ does not have any zero then the bifurcation from zero does not occur.

Proof. Consider the problem

$$\begin{cases}\mathcal{L}_1u=\lambda m(x)u+f_1(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_0,\\ B_2u=\lambda r(x)u+g_2(x,u) & \text{on } \Gamma_1,\end{cases}$$

where $\mathcal{L}_1=\mathcal{L}-c_1$, $B_2=B-c_2$, $f_1(x,u)=f(x,u)-c_1u$ and $g_2(x,u)=g(x,u)-c_2u$. Since f_1, g_2 satisfy (B0) then the conditions of Theorem 4.10 are satisfied. ■

4.3. Bifurcation from infinity

During this section we consider the nonlinear equation (4.5). We assume (4.6). Moreover, we suppose that

$$\textbf{(BI)}\qquad \lim_{s\to+\infty}\frac{f(x,s)}{s}=0 \text{ unif. in } \overline{\Omega},\qquad \lim_{s\to+\infty}\frac{g(x,s)}{s}=0 \text{ unif. on } \Gamma_1.$$

Lemma 4.12. *The condition (BI) implies that*

$$\lim_{s\to+\infty}\frac{\max\limits_{t\in[0,s]}f(x,t)}{s}=0\quad \text{unif. in } \overline{\Omega},$$

and

$$\lim_{s\to+\infty}\frac{\max\limits_{t\in[0,s]}g(x,t)}{s}=0\quad \text{unif. on } \Gamma_1.$$

Proof. We prove just one of the assertions, the other one follows in the same fashion. We have that for all $\epsilon>0$ there exists $s_0(\epsilon)>0$ such that

$$\left|\frac{f(x,s)}{s}\right|<\epsilon,\ \ \forall s\geq s_0,\ \forall x\in\overline{\Omega}. \tag{4.12}$$

We have

$$\left|\frac{\max\limits_{t\in[0,s]}f(x,t)}{s}\right|\leq\frac{\max\limits_{t\in[0,s_0]}|f(x,t)|+\max\limits_{t\in[s_0,s]}|f(x,t)|}{s}.$$

On one hand,

$$\left|\frac{\max_{t\in[0,s_0]} f(x,t)}{s}\right| \leq \frac{\max_{t\in[0,s_0]} |f(x,t)|}{s} < \epsilon, \tag{4.13}$$

for all $s \geq s_1 \geq s_0$ and for all $x \in \overline{\Omega}$. On the other hand, taking into account (4.12) we obtain

$$\left|\frac{\max_{t\in[s_0,s]} f(x,t)}{s}\right| \leq \frac{\max_{t\in[s_0,s]} |f(x,t)|}{s} < \epsilon, \tag{4.14}$$

for all $s \geq s_1$ and for all $x \in \overline{\Omega}$. Finally, thanks to (4.13) and (4.14) we conclude.

■

Lemma 4.13. *Let $\Lambda \subset (-\infty, \gamma_1)$ a compact subset then there exists $R > 0$ such that $\Phi^t_\lambda(u) \neq 0$ for all $u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\| > R$, $\forall \lambda \in \Lambda$ and $\forall t \in [0,1]$.*

Proof. Assume the contrary, then there exist sequences $\lambda_n \to \overline{\lambda}$, $t_n \to \overline{t}$ and $u_n \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $lim_{n\to+\infty}\|u_n\| = +\infty$ such that $\Phi^{t_n}_{\lambda_n}(u_n) = 0$. Owing to Lemma 4.3 we have $u_n \geq 0$. Therefore, dividing by $\|u_n\|$ we get
(4.15)

$$v_n = t_n K_1 \left(\frac{\lambda_n M(u_n) + K u_n + F(u_n)}{\|u_n\|}\right) + t_n K_2 \left(\frac{\lambda_n R(\gamma(u_n)) + K\gamma(u_n) + G(\gamma(u_n))}{\|u_n\|}\right).$$

Thanks to (BI) and the continuity of f and g we have that

$$\left\|\frac{\lambda_n M(u_n) + K u_n + F(u_n)}{\|u_n\|}\right\| \leq C, \qquad \left\|\frac{\lambda_n R(\gamma(u_n)) + K\gamma(u_n) + G(\gamma(u_n))}{\|u_n\|}\right\|_{\Gamma_1} \leq C.$$

Since K_1 and K_2 are compact operators in $\mathcal{C}_{\Gamma_0}(\overline{\Omega})$, then the sequence v_n is a relatively compact in $\mathcal{C}_{\Gamma_0}(\overline{\Omega})$. Therefore, we can suppose that $v_n \to \overline{v}$ in $\mathcal{C}(\overline{\Omega})$. By Lemma 4.12, we have

$$\frac{F(u_n)}{\|u_n\|} \to 0 \text{ in } \mathcal{C}(\overline{\Omega}), \quad \frac{G(\gamma(u_n))}{\|u_n\|} \to 0 \text{ on } \mathcal{C}(\Gamma_1).$$

Therefore, taking into account that K_1, K_2 are closed operators, we can pass to the limit in (4.15) and we obtain

$$\overline{v} = \overline{t}[K_1(\overline{\lambda} M(\overline{v}) + K\overline{v}) + K_2(\overline{\lambda} R(\gamma(\overline{v})) + K\gamma(\overline{v}))].$$

Since $\overline{v}$ is a strongly positive function in Ω. Now, we can argue exactly as in the end of Lemma 4.4 to reach a contradiction with the choice of the set Λ. ■

Let $z \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ then we define the following operators

$$\Psi_\lambda(z) = \begin{cases} \|z\|^2 \Phi_\lambda\left(\frac{z}{\|z\|^2}\right) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

$$\Psi^t_\lambda(z) = \begin{cases} \|z\|^2 \Phi^t_\lambda\left(\frac{z}{\|z\|^2}\right) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Corollary 4.14. *If $\lambda < \gamma_1$ then $i(\Psi_\lambda, 0, 0) = 1$.*

Proof. We take $\Lambda = \{\lambda\}$. Thanks to Lemma 4.13, we know that $\exists\delta > 0$ such that $\forall u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\| \in (0, \delta)$, $\delta < R^{-1}$ we have $\Psi_\lambda^t(u) \neq 0$, $\forall t \in [0, 1]$. Therefore, by the homotopy invariance of the degree, we infer

$$i(\Psi_\lambda, 0, 0) = deg(\Psi_\lambda^1 = \Psi_\lambda, B_\delta, 0) = deg(\Psi_\lambda^0 = I, B_\delta, 0) = 1.$$

■

Lemma 4.15. *If $\lambda > \gamma_1$ then there exists $R > 0$ such that $\forall u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\| > R$ we have $\Phi_\lambda(u) \neq \tau\varphi_1^K$ for all $\tau \geq 0$.*

Proof. Let us assume that there exist sequences $\tau_n \geq 0$, $u_n \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ such that $\|u_n\| \to +\infty$ and $\Phi_\lambda(u_n) = \tau_n\varphi_1^K$. Arguing as in Proposition 4.2 we have that u_n is a strongly positive classical solution to the problem

$$\begin{cases} (\mathcal{L} + K)u_n = \lambda m(x)u_n + f(x, u_n) + Ku_n + \gamma_1^K\tau_n m(x)\varphi_1^K & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_0, \\ (B + K)u_n = \lambda r(x)u_n + g(x, u_n) + Ku_n + \gamma_1^K\tau_n r(x)\varphi_1^K & \text{on } \Gamma_1. \end{cases}$$

Next, we divide by $\|u_n\|$ and we have

$$(4.16) \qquad \begin{cases} (\mathcal{L} + K)v_n = \lambda m(x)v_n + \frac{f(x,u_n)}{\|u_n\|} + Kv_n + \frac{\tau_n}{\|u_n\|}\gamma_1^K m(x)\varphi_1^K & \text{in } \Omega, \\ v_n = 0 & \text{on } \Gamma_0, \\ (B + K)v_n = \lambda r(x)v_n + \frac{g(x,u_n)}{\|u_n\|} + Kv_n + \frac{\tau_n}{\|u_n\|}\gamma_1^K r(x)\varphi_1^K & \text{on } \Gamma_1. \end{cases}$$

In what follows we prove that the sequence $\frac{\tau_n}{\|u_n\|}$ is bounded. Let $(\lambda_1^*, \varphi_1^*)$ the principal eigenvalue and a positive eigenfunction associated to the eigenvalue problem

$$\begin{cases} (\mathcal{L} + K)^*\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma_0, \\ (B + K)^*\varphi = 0 & \text{on } \Gamma_1, \end{cases}$$

where $(\mathcal{L}+K)^*$ and $(\mathcal{B}+K)^*$ denote the formally adjoint operators of $\mathcal{L}+K$ and $\mathcal{B}+K$ respectively. Now, we apply the Green's formula (see [6, pg. 235]) to obtain

$$(4.17) \quad ((\mathcal{L} + K)v_n, \varphi_1^*) - (v_n, (\mathcal{L} + K)^*\varphi_1^*) = -\int_{\Gamma_1} \rho(\varphi_1^*(B + K)v_n - v_n(B + K)^*\varphi_1^*)$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$,

$$\rho := (n_a|n)(n|n)$$

with $(\cdot|\cdot)$ the usual scalar product in $\mathbb{R}^d$ and n_a is the outer conormal vector respect to $\mathcal{L}$. We observe that, since $\mathcal{L}$ is an uniformly elliptic operator then

$$\rho = (n|n)\sum_{i=1}^{d}(n_a)_i n_i = (n|n)\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}n_j n_i \geq \alpha(n|n)^2 > 0$$

for some $\alpha > 0$. Let us define the functions

$$h(x, u_n, v_n) := \lambda m(x) v_n + \frac{f(x, u_n)}{\|u_n\|} + K v_n,$$

$$j(x, u_n, v_n) := \lambda r(x) v_n + \frac{g(x, u_n)}{\|u_n\|} + K v_n.$$

Observe that from (4.17) we have

$$\begin{aligned} \lambda_1^* \int_\Omega v_n \varphi_1^* \;&= \int_\Omega h(x, u_n, v_n)\varphi_1^* + \int_{\Gamma_1} \rho j(x, u_n, v_n)\varphi_1^* + \\ &+ \gamma_1^K \frac{\tau_n}{\|u_n\|} \left(\int_\Omega m(x)\varphi_1^K \varphi_1^* + \int_{\Gamma_1} \rho r(x) \varphi_1^K \varphi_1^* \right). \end{aligned}$$

Therefore, from the above equality we deduce that the sequence

$$\frac{\tau_n}{\|u_n\|} \text{ is bounded.}$$

Now, extracting a subsequence we can pass to the limit in (4.16) and we infer that $v_n \to \overline{v}$ in $\mathcal{C}(\overline{\Omega})$ where $\overline{v}$ is a solution to

$$\text{(4.18)} \qquad \begin{cases} \mathcal{L}\overline{v} = \lambda m(x)\overline{v} + \gamma_1^K \overline{\tau} m(x) \varphi_1^K & \text{in } \Omega, \\ \overline{v} = 0 & \text{on } \Gamma_0, \\ B\overline{v} = \lambda r(x) \overline{v} + \gamma_1^K \overline{\tau} r(x) \varphi_1^K & \text{on } \Gamma_1, \end{cases}$$

where $\overline{\tau} \geq 0$. If $\overline{\tau} = 0$ then $\lambda = \gamma_1$ therefore $\overline{\tau} > 0$ and $\overline{v}$ is a positive supersolution to $(\mathcal{L} - \lambda m, \mathcal{B} - \lambda r, \Omega)$. Hence, by the strong maximum principle

$$\lambda_1(\mathcal{L} - \lambda M, \mathcal{B} - \lambda R) > 0,$$

which is a contradiction with the choice of λ. ∎

Corollary 4.16. *If $\lambda > \gamma_1$ then $i(\Psi_\lambda, 0, 0) = 0$.*

Proof. Thanks to Lemma 4.15, there exists $R > 0$ such that $\Phi_\lambda(u) \neq t\|u\|^2 \varphi_1^K$ for all $u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\| > R$, $\forall t \in [0, 1]$. Therefore, $\Psi_\lambda(z) \neq t\varphi_1^K$ for all $z \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $z \in B_{R^{-1}}$. Hence, thanks to the homotopy invariance of the degree we have

$$i(\Psi_\lambda, 0, 0) = deg(\Psi_\lambda, B_\epsilon, 0) = deg(\Psi_\lambda - \varphi_1^K, B_\epsilon, 0) = 0,$$

where $\epsilon < R^{-1}$. ∎

Theorem 4.17. *Assume $(m, r) > 0$, (4.6) and (BI). If γ_1 exists then it is a bifurcation point from infinity, and it is the only one for positive solutions. Moreover, there exists an unbounded continuum $\mathcal{C}_\infty \subset \mathbb{R} \times \mathcal{C}(\overline{\Omega})$ of positive solutions to (4.5). Furthermore, if $\delta_0 > 0$ is small enough and*

$$\mathcal{J} = [\gamma_1 - \delta_0, \gamma_1 + \delta_0] \times \{u \in \mathcal{C}(\overline{\Omega}) : \|u\|_\infty \geq 1\},$$

then either,

a) $\mathcal{C}_\infty \setminus \mathcal{J}$ *is bounded in* $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$ *and* $\mathcal{C}_\infty \setminus \mathcal{J}$ *meets the set* $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$*, or*

b) $\mathcal{C}_\infty \setminus \mathcal{J}$ *is unbounded in* $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$.

Finally, if γ_1 *does not exist then the bifurcation from infinity does not occur.*

Proof. We should argue as in Theorem 4.10 (see also [88]). ∎

4.4. Bifurcation in the concave-convex case

Through this section we consider the problems

$$\begin{cases} \mathcal{L}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = g(x,u) & \text{on } \Gamma_1, \end{cases} \tag{4.19}$$

$$\begin{cases} \mathcal{L}u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = \lambda j(x,u) & \text{on } \Gamma_1, \end{cases} \tag{4.20}$$

where $f, h \in \mathcal{C}^\alpha(\overline{\Omega} \times \mathbb{R})$, $g, j \in \mathcal{C}^{1+\alpha}(\Gamma_1 \times (0, +\infty))$. We also assume, during this section the following conditions

$$f(x,0) = 0 \ \ \forall x \in \overline{\Omega}\,, \qquad g(x,0) \geq 0 \ \ \forall x \in \Gamma_1, \tag{4.21}$$

$$h(x,0) \geq 0 \ \ \forall x \in \overline{\Omega}\,, \qquad j(x,0) = 0 \ \ \forall x \in \Gamma_1, \tag{4.22}$$

(**BCC**) $$\lim_{s \to 0^+} \frac{f(x,s)}{s} = +\infty \ \text{ unif. in } \overline{\Omega}, \qquad \lim_{s \to 0^+} \frac{g(x,s)}{s} = 0 \ \text{ unif. on } \Gamma_1.$$

(**BCC2**) $$\lim_{s \to 0^+} \frac{h(x,s)}{s} = 0 \ \text{ unif. in } \overline{\Omega}, \qquad \lim_{s \to 0^+} \frac{j(x,s)}{s} = +\infty \ \text{ unif. on } \Gamma_1.$$

Remark 4.18. *We would like to point out that by the condition (4.22) (resp. (4.21)) the regularity of* j *(resp.* g*) at zero is irrelevant. Since* u *a positive solution to (4.19) (resp. (4.20)) never attains the value zero on* Γ_1*. Indeed, if* u *attains the value zero in* $x_0 \in \Gamma_1$ *then the minimum of* u *is attained on* Γ_1 *and since* $u(x_0) = 0$ *then thanks to (4.22) (resp. (4.21)) we have*

$$\frac{\partial u}{\partial n}(x_0) \geq 0,$$

a contradiction with the Hopf Lemma.

We have, by the previous remark, that u is a classical non-negative solution to (4.19) if and only if $\Psi_\lambda(u) = 0$ in $\mathcal{C}_{\Gamma_0}(\overline{\Omega})$, where $\Psi_\lambda : \mathcal{C}_{\Gamma_0}(\overline{\Omega}) \to \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ is defined as

$$\Psi_\lambda(u) = u - K_1(\lambda F(u_+) + Ku_+) - K_2(G(\gamma(u_+)) + K\gamma(u_+)).$$

Also, we consider

$$\Psi_\lambda^t(u) = u - tK_1(\lambda F(u_+) + Ku_+) - tK_2(G(\gamma(u_+)) + K\gamma(u_+)).$$

Analogously, u is a classical non-negative solution to (4.20) if and only if $\theta_\lambda(u) = 0$ in $\mathcal{C}_{\Gamma_0}(\overline{\Omega})$, where $\theta_\lambda : \mathcal{C}_{\Gamma_0}(\overline{\Omega}) \to \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ is defined as

$$\theta_\lambda(u) = u - K_1(H(u_+) + Ku_+) - K_2(\lambda J(\gamma(u_+)) + K\gamma(u_+)),$$

where H and J stands for the Netmiskii operators associated to h and j respectively. Also, we define $\theta^t_\lambda(u) = u - tK_1(H(u_+) + Ku_+) - tK_2(\lambda J(\gamma(u_+)) + K\gamma(u_+))$.

Lemma 4.19. *If $\lambda < 0$ then there exists $\delta > 0$ such that for all $u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\|_{\mathcal{C}(\overline{\Omega})} = \|u\| \in (0, \delta)$ we have $\Psi^t_\lambda(u) \neq 0$, for all $t \in [0, 1]$.*

Proof. Suppose the contrary, then there exist sequences $t_n \in [0, 1]$, $u_n \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ such that $t_n \to \bar{t}$, $\|u_n\| \to 0$ with $\Psi^{t_n}_\lambda(u_n) = 0$. Dividing by $\|u_n\|$, we obtain

$$v_n = \lambda t_n K_1 \left(\frac{F(u_n)}{\|u_n\|} + Kv_n \right) + t_n K_2 \left(\frac{G(\gamma(u_n))}{\|u_n\|} + K\gamma(v_n) \right),$$

where $v_n = \frac{u_n}{\|u_n\|}$. Therefore, v_n is a solution to the problem

$$\begin{cases} (\mathcal{L} + K)v_n = t_n\lambda \frac{f(x,u_n)}{\|u_n\|} + t_n K v_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \Gamma_0, \\ (B + K)v_n = t_n \frac{g(x,u_n)}{\|u_n\|} + t_n K v_n & \text{on } \Gamma_1. \end{cases}$$

Taking into account that $\lambda < 0$ and (BCC) we have that for all $\epsilon > 0$ there exists $n_0(\epsilon)$ such that for all $n \geq n_0(\epsilon)$,

$$\lambda \frac{f(x, u_n)}{\|u_n\|} + Kv_n \leq 0 \text{ in } \Omega$$

and

$$\frac{g(x, u_n)}{\|u_n\|} + Kv_n < (K + \epsilon)v_n \text{ on } \Gamma_1.$$

Therefore $v_n \geq 0$ satisfies that

$$\begin{cases} (\mathcal{L} + K)v_n \leq 0 & \text{in } \Omega, \\ v_n = 0 & \text{on } \Gamma_0, \\ (B + K)v_n < t_n(K + \epsilon)v_n & \text{on } \Gamma_1. \end{cases}$$

Now, picking $K > 0$ large enough we get $\lambda_1(\mathcal{L} + K, \mathcal{B} + K(1 - t_n) - t_n\epsilon) > 0$. Hence $v_n \equiv 0$ for all $n \geq n_0$ which contradicts the fact that $\|v_n\| = 1$. ■

Lemma 4.20. *If $\lambda > 0$ then there exists $\delta > 0$ such that for all $u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\| \in (0, \delta)$ and for all $\tau \geq 0$ we have $\Psi_\lambda(u) \neq \tau\xi^K_1$ where ξ^K_1 is a positive eigenfunction associated to $\lambda_1(\mathcal{L} + K, \mathcal{B} + K) > 0$.*

Proof. Let us assume that there exist sequences $u_n \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u_n\| \to 0$ and $\tau_n \geq 0$, $\Psi_\lambda(u_n) = \tau_n\xi^K_1$. It is clear, by the strong maximum principle, that u_n it is a strongly positive classical solution to the problem

$$\begin{cases} (\mathcal{L} + K)u_n = \lambda f(x, u_n) + Ku_n + \lambda_1(\mathcal{L} + K, \mathcal{B} + K)\tau_n\xi^K_1 & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_0, \\ (B + K)u_n = g(x, u_n) + Ku_n & \text{on } \Gamma_1. \end{cases}$$

Take $\epsilon > 0$, and $M > \lambda_1(\mathcal{L}, \mathcal{B}+\epsilon)$. Since $u_n \to 0$ in $\mathcal{C}(\overline{\Omega})$, we have, taking into account (BCC), that there exists n_0 such that for all $n \geq n_0$

(4.23)
$$\begin{cases} (\mathcal{L}+K)u_n = \lambda f(x,u_n) + Ku_n + \lambda_1(\mathcal{L}+K,\mathcal{B}+K)\tau_n\xi_1^K > (M+K)u_n & \text{in } \Omega,\\ u_n = 0 & \text{on } \Gamma_0,\\ (B+K)u_n = g(x,u_n) + Ku_n > (K-\epsilon)u_n & \text{on } \Gamma_1. \end{cases}$$

Therefore, u_n is a positive strict supersolution to $(\mathcal{L}-M, \mathcal{B}+\epsilon, \Omega)$, then $\lambda_1(\mathcal{L}-M, \mathcal{B}+\epsilon) > 0$ which implies $M < \lambda_1(\mathcal{L}, \mathcal{B}+\epsilon)$, a contradiction. ∎

Theorem 4.21. *Under conditions (4.21) and (BCC), $\lambda = 0$ is a bifurcation point from the trivial solution and it is the only one for positive solutions of (4.19). Moreover, there exists an unbounded continuum $\mathcal{C}_0$ of positive solutions emanating from $(0,0)$.*

Proof. It is possible, thanks to Lemmas 4.19 and 4.20, arguing as Theorem 4.10 to prove that there exists an unbounded continuum $\mathcal{C}_0$ of positive solutions of (4.19). We only need to prove the uniqueness of the bifurcation point. By Lemma 4.19 we can prove that the bifurcation from the trivial solution does not occur for points of the form $(\lambda_0, 0)$, $\lambda_0 < 0$. Let us assume that there exist a sequence $(\lambda_n, u_{\lambda_n}) \in \mathbb{R} \times \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ verifying $(\lambda_n, u_{\lambda_n}) \to (\lambda_0, 0)$ in $\mathbb{R} \times \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\lambda_0 > 0$. Then, for sufficiently large M we have

$$\mathcal{L}u_{\lambda_n} = \lambda_n f(x, u_{\lambda_n}) > \lambda_n M u_{\lambda_n} \text{ in } \Omega, \quad Bu_{\lambda_n} = g(x, u_{\lambda_n}) > -\epsilon u_{\lambda_n} \text{ on } \Gamma_1.$$

At this point we only need to argue as in Lemma 4.20 to obtain a contradiction. ∎

Now, we study the bifurcation from zero of (4.20). In this case we need to assume that

$$\lambda_1(\mathcal{L}, \mathcal{D}) > 0. \tag{4.24}$$

Thanks to (4.24) it is possible, by Theorem 2.26, to fix $K \geq 0$ satisfying additionally that

$$\lambda_1(\mathcal{L}, \mathcal{B}+K) > 0. \tag{4.25}$$

Lemma 4.22. *If $\lambda < 0$ then there exists $\delta > 0$ such that for all $u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\|_{\mathcal{C}(\overline{\Omega})} = \|u\| \in (0,\delta)$ we have $\theta_\lambda^t(u) \neq 0$, for all $t \in [0,1]$.*

Proof. Suppose the contrary, then there exist sequences $t_n \in [0,1]$, $u_n \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ such that $t_n \to \bar{t}$, $\|u_n\| \to 0$ with $\theta_\lambda^{t_n}(u_n) = 0$. Picking $v_n = \frac{u_n}{\|u_n\|}$ we have that v_n is a positive solution to the problem

$$\begin{cases} (\mathcal{L}+K)v_n = t_n \frac{h(x,u_n)}{\|u_n\|} + t_n K v_n & \text{in } \Omega,\\ v_n = 0 & \text{on } \Gamma_0,\\ (B+K)v_n = t_n \lambda \frac{j(x,u_n)}{\|u_n\|} + t_n K v_n & \text{on } \Gamma_1. \end{cases}$$

Taking into account that $\lambda < 0$ and (BCC2) we have that for all $\epsilon > 0$ there exists $n_0(\epsilon)$ such that for all $n \geq n_0(\epsilon)$,

$$\lambda \frac{j(x,u_n)}{\|u_n\|} + K v_n \leq 0 \text{ in } \Omega$$

and

$$\frac{h(x,u_n)}{\|u_n\|} + K v_n < (K+\epsilon) v_n \text{ on } \Gamma_1.$$

Therefore $v_n \geq 0$ satisfies that

$$\begin{cases} (\mathcal{L}+K)v_n \leq t_n(K+\epsilon) & \text{in } \Omega, \\ v_n = 0 & \text{on } \Gamma_0, \\ (B+K)v_n \leq 0 & \text{on } \Gamma_1. \end{cases}$$

Now, since v_n is a positive strict subsolution to $(\mathcal{L}+(1-t_n)K - t_n\epsilon, \mathcal{B}+K, \Omega)$ then $0 > \lambda_1(\mathcal{L}+(1-t_n)K - t_n\epsilon, \mathcal{B}+K) \geq \lambda_1(\mathcal{L}-\epsilon, \mathcal{B}+K)$, a contradiction with (4.25).

■

Lemma 4.23. *If $\lambda > 0$ then there exists $\delta > 0$ such that for all $u \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u\| \in (0,\delta)$ and for all $\tau \geq 0$ we have $\theta_\lambda(u) \neq \tau\phi_1^K$ where ϕ_1^K is a positive eigenfunction associated to $\mu_1^K > 0$, the unique zero of*

$$\sigma(\mu) = \lambda_1(\mathcal{L}+K, \mathcal{B}+K-\mu).$$

Proof. Let us assume that there exist sequences $u_n \in \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\|u_n\| \to 0$ and $\tau_n \geq 0$, $\theta_\lambda(u_n) = \tau_n\phi_1^K$. It is clear, by the strong maximum principle, that u_n it is a strongly positive classical solution to the problem

$$\begin{cases} (\mathcal{L}+K)u_n = h(x,u_n) + Ku_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_0, \\ (B+K)u_n = \lambda j(x,u_n) + Ku_n + \mu_1^K \tau_n \phi_1^K & \text{on } \Gamma_1. \end{cases}$$

Take $\epsilon > 0$, and M such that $\lambda_1(\mathcal{L}, \mathcal{B}-M) + \epsilon < 0$. Since $u_n \to 0$ in $\mathcal{C}(\overline{\Omega})$, we have, taking into account (BCC), that there exists n_0 such that for all $n \geq n_0$

$$\begin{cases} (\mathcal{L}+K)u_n = \lambda h(x,u_n) + Ku_n > (K-\epsilon)u_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_0, \\ (B+K)u_n = \lambda j(x,u_n) + Ku_n + \mu_1^K\phi_1^K > (M+K)u_n & \text{on } \Gamma_1. \end{cases} \tag{4.26}$$

Therefore, u_n is a positive strict supersolution to $(\mathcal{L}+\epsilon, \mathcal{B}-M, \Omega)$, then $\lambda_1(\mathcal{L}+\epsilon, \mathcal{B}-M) > 0$, a contradiction by the choice of M. ■

Theorem 4.24. *Assume (4.22) and (BCC2).*

a) If $\lambda_1(\mathcal{L},\mathcal{D}) > 0$ then $\lambda = 0$ is a bifurcation point from the trivial solution and it is the only one for positive solutions. Moreover, there exists an unbounded continuum $\mathcal{C}_0$ of positive solutions of (4.20) emanating from $(0,0)$.

b) If $\lambda_1(\mathcal{L}, \mathcal{D}) \leq 0$ then the bifurcation from zero does not occur.

Proof. It is possible, thanks to Lemmas 4.22 and 4.23, arguing as Theorem 4.10 to prove that there exists an unbounded continuum $\mathcal{C}_0$ of positive solutions of (4.20). Now we prove the uniqueness of the bifurcation point. By Lemma 4.22 we can prove that the bifurcation from the trivial solution does not occur for points of the form $(\lambda_0, 0)$, $\lambda_0 < 0$. Let us assume that there exist a sequence $(\lambda_n, u_{\lambda_n}) \in \mathbb{R} \times \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ verifying $(\lambda_n, u_{\lambda_n}) \to (\lambda_0, 0)$ in $\mathbb{R} \times \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ with $\lambda_0 > 0$. Then, for sufficiently large M and n we have

$$\mathcal{L}u_{\lambda_n} = h(x, u_{\lambda_n}) > -\epsilon u_{\lambda_n} \text{ in } \Omega, \quad Bu_{\lambda_n} = \lambda_n j(x, u_{\lambda_n}) > \lambda_n M u_{\lambda_n} \text{ on } \Gamma_1.$$

At this point we only need to argue as in Lemma 4.23 to obtain a contradiction. Finally we show that if

$$\lambda_1(\mathcal{L}, \mathcal{D}) \leq 0$$

then the bifurcation from zero does not occur. Suppose that there exists (λ_n, u_n) of positive solutions to (4.20) such that $(\lambda_n, u_{\lambda_n}) \to (\lambda_0, 0)$ in $\mathbb{R} \times \mathcal{C}_{\Gamma_0}(\overline{\Omega})$ then for all $\epsilon > 0$ there exists n_0 such that for all $n \geq n_0$ we have

$$0 = \lambda_1 \left(\mathcal{L} - \frac{f(x, u_n)}{u_n}, \mathcal{B} - \lambda_n \frac{g(x, u_n)}{u_n} \right) \leq \lambda_1(\mathcal{L} - \epsilon, \mathcal{D}) < 0,$$

which concludes the Theorem. ∎

4.5. Behavior of a continuum of solutions respect to a supersolution

Through this section we study the behavior of a continuum of solutions provided by bifurcation methods respect to a supersolution. Let us consider the problem

$$\text{(4.27)} \qquad \begin{cases} \mathcal{L}u = f(\lambda, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = g(\lambda, x, u) & \text{on } \Gamma_1, \end{cases}$$

where $f \in \mathcal{C}^{0+1}(\mathbb{R} \times \overline{\Omega} \times \mathbb{R})$ and $g \in \mathcal{C}^{1+\alpha}(\mathbb{R} \times \Gamma_1 \times \mathbb{R})$.

Lemma 4.25. *Let $\overline{u} \in \mathcal{C}^2(\overline{\Omega})$ a strict supersolution to (4.27) and let u a solution to (4.27). Then $\overline{u} - u \notin \partial P$, where P is the positive cone (see Section 2.4).*

Proof. Assume the contrary, then $v = \overline{u} - u \in \partial P$, therefore $\overline{u}(x) \geq u(x)$ for all $x \in \overline{\Omega}$. Let l_1 and l_2 the Lipschitz constants associated to f and g in $\{\lambda\} \times \overline{\Omega} \times [\min_{\overline{\Omega}} u, \max_{\overline{\Omega}} \overline{u}]$ and $\{\lambda\} \times \Gamma_1 \times [\min_{\Gamma_1} u, \max_{\Gamma_1} \overline{u}]$ respectively. Take $r > \max\{l_1, l_2\}$ large enough such that $\lambda_1(\mathcal{L} + r, \mathcal{B} + r) > 0$. We have

$$\begin{array}{llll} (\mathcal{L} + r)v & = f(\lambda, x, \overline{u}) - f(\lambda, x, u) + rv & > 0 & \text{in } \Omega, \\ v & = 0 & \geq 0 & \text{on } \Gamma_0, \\ (B + r)v & = g(\lambda, x, \overline{u}) - g(\lambda, x, u) + rv & > 0 & \text{on } \Gamma_1, \end{array}$$

Therefore, by the strong maximum principle we infer that $v \in int(P)$, a contradiction. ∎

Theorem 4.26. *Let $I \subset \mathbb{R}$ an interval, and let $\Sigma \subset I \times \mathcal{C}^2_{\Gamma_0}(\overline{\Omega})$ a connected set of solutions of (4.27). Consider the continuous application $\overline{u} : I \to \mathcal{C}(\overline{\Omega})$, where $\overline{u}(\lambda)$ stands for a strict supersolution to (4.27) for each λ. If $u_{\lambda_0} < \overline{u}(\lambda_0)$ for some $\lambda_0 \in I$ with $(\lambda_0, u_{\lambda_0}) \in \Sigma$, then $u_\lambda < \overline{u}(\lambda)$ for all $(\lambda, u_\lambda) \in \Sigma$.*

Proof. Consider the continuous application

$$\begin{array}{rcl} T : I \times \mathcal{C}^1_{\Gamma_0}(\overline{\Omega}) & \to & \mathcal{C}^1(\overline{\Omega}) \\ (\lambda, u_\lambda) & \mapsto & T(\lambda, u_\lambda) = \overline{u}(\lambda) - u_\lambda \end{array}$$

Since T is continuous and Σ is a connected set then $T(\Sigma)$ is a connected set. Therefore, thanks to Lemma 4.25, $T(\Sigma) \cap \partial P = \emptyset$, then either $T(\Sigma) \subset int(P)$ or $T(\Sigma)$ is outside of P. Taking into account that $T(\lambda_0, u_{\lambda_0}) \in P$ then $T(\Sigma) \subset int(P)$, i.e. $u_\lambda < \overline{u}(\lambda)$ for all $(\lambda, u_\lambda) \in \Sigma$. ∎

The same kind of results are also true for subsolutions. To be more precise, arguing as in Theorem 4.26, we can prove the following.

Theorem 4.27. *Let $I \subset \mathbb{R}$ an interval, and let $\Sigma \subset I \times \mathcal{C}^2_{\Gamma_0}(\overline{\Omega})$ a connected set of solutions of (4.27). Consider the continuous application $\underline{u} : I \to \mathcal{C}_{\Gamma_0}(\overline{\Omega})$, where $\underline{u}(\lambda)$ stands for a strict subsolution to (4.27) for each λ. If $\underline{u}(\lambda_0) < u_{\lambda_0}$ for some $\lambda_0 \in I$ with $(\lambda_0, u_{\lambda_0}) \in \Sigma$ then, $\underline{u}(\lambda) < u_\lambda$ for all $(\lambda, u_\lambda) \in \Sigma$.*

4.6. A priori Bounds

This section is devoted to the a priori bounds for equations of the form

$$\begin{cases} \mathcal{L}u = f(x,u) & \text{in } \Omega, \\ Bu = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{4.28}$$

where $f \in \mathcal{C}(\overline{\Omega} \times [0, +\infty))$, $g \in \mathcal{C}^{1+\alpha}(\overline{\Omega} \times [0, +\infty))$. Moreover, we suppose that there exist $p \in \left(1, \frac{d+2}{d-2}\right), q \in \left(1, \frac{d}{d-2}\right)$ such that

$$\lim_{t \to +\infty} \frac{f(x,t)}{t^p} = h(x) \ \text{ unif. in } \overline{\Omega}, \tag{4.29}$$

where $h \in \mathcal{C}(\overline{\Omega})$ strictly positive in $\overline{\Omega}$ and

$$\lim_{t \to +\infty} \frac{g(x,t)}{t^q} = i(x) \ \text{ unif. on } \Gamma_1, \tag{4.30}$$

where $i \in \mathcal{C}^{1+\alpha}(\partial\Omega)$ strictly positive on $\partial\Omega$.

Theorem 4.28. *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ a positive solution of (4.28), then under the conditions (4.29), (4.30) with $p \in \left(1, \frac{d+2}{d-2}\right), q \in \left(1, \frac{d}{d-2}\right)$ and $p \neq 2q - 1$ we have*

$$u(x) \leq C(p, q, \Omega), \quad \forall x \in \Omega,$$

where $C(p, q, \Omega)$ is a constant depending on p, q and Ω.

Proof. Without losing generality we suppose $\mathcal{L} = -\Delta$, the general case follows the same spirit as in our proof, see [56]. Assume that there exists a sequence of functions u_j solutions to (4.28) such that

$$M_j = \max_{\overline{\Omega}} u_j \to +\infty \text{ when } j \to +\infty.$$

Since $M_j = u_j(x_j)$ with $x_j \in \overline{\Omega}$ we may assume, after extracting a subsequence, that $x_j \to x_0 \in \overline{\Omega}$. Now, we distinguish between two cases $x_0 \in \Omega$ or $x_0 \in \partial\Omega$.
Case 1. $x_0 \in \Omega$: Let us make the change of variables

$$y = \frac{x - x_j}{\lambda_j}.$$

The scalars λ_j are chosen $\lambda_j > 0$ for all j and $\lambda_j \to 0$ when $j \to +\infty$, later on we will give them explicitly. Let us define the blow-up function v_j as

$$v_j(y) = \frac{u_j(x)}{M_j} = \frac{u_j(\lambda_j y + x_j)}{M_j}. \tag{4.31}$$

Since, $d(x_0, \partial\Omega) = k > 0$ then for all $j \geq j_0$ we have $B_c(x_j) \subset \Omega$, $c < k$ where $B_c(x_j)$ stands for the open ball centered at x_j and radius c. Therefore, the function v_j is well defined in $B_{c/\lambda_j}(0)$. Moreover, we have

$$0 \leq v_j(y) \leq v_j(0) = 1 \quad \forall y \in B_{c/\lambda_j}(0).$$

So, for a fixed $R > 0$ we have that $\forall j \geq j_R$ the function v_j is well defined in $B_R(0)$. From now on we suppose that $j \geq j_R$. After some algebra, v_j satisfies the following equation in $B_R(0)$

$$-\Delta v_j = \lambda_j^2 M_j^{-1} f(\lambda_j y + x_j, M_j v_j) \text{ in } B_R(0). \tag{4.32}$$

Thanks to (4.29) we have

$$\lim_{j\to+\infty} |\lambda_j^2 M_j^{-1} f(\lambda_j y + x_j, M_j v_j) - \lambda_j^2 M_j^{-1} h(\lambda_j y + x_j)(M_j v_j)^p| = 0, \tag{4.33}$$

for all $y \in B_R(0)$. Therefore, we choose λ_j such that

$$\lambda_j^2 M_j^{p-1} = 1,$$

so $\lambda_j = M_j^{\frac{1-p}{2}}$. Taking into account the L^p estimates we obtain

$$\|v_j\|_{W^{2,p}(B_R(0))} \leq C,$$

Hence, $v_j \rightharpoonup v$ in $W^{2,p}(B_R(0))$, this implies

$$-\Delta v_j \rightharpoonup -\Delta v \text{ in } L^p(B_R(0)). \tag{4.34}$$

Since, $W^{2,p}(B_R(0))$ is compactly embedded in $\mathcal{C}^1(\overline{B_R(0)})$ then $v_j \to v$ in $\mathcal{C}^1(\overline{B_R(0)})$, so

$$M_j^{-p} f(\lambda_j y + x_j, M_j v_j) \to h(x_0) v^p \text{ in } L^p(B_R(0)). \tag{4.35}$$

From (4.34) and (4.35) we obtain

$$-\Delta v = h(x_0)v^p \ \text{ in } B_R(0),$$

for each $R > 0$. Thanks to a standard argument is possible to prove that $v \in W^{2,p}_{loc}(\mathbb{R}^d)$ and $-\Delta v = h(x_0)v^p$ in $\mathbb{R}^d$. Taking into account that $v^p \in \mathcal{C}^1(\mathbb{R}^d)$ then, thanks to the elliptic regularity, $v \in \mathcal{C}^2(\mathbb{R}^d)$. Making a new change of variables $w = v/h(x_0)$ we have $-\Delta w = w^p$ in $\mathbb{R}^d$, w non-negative and $w(0) = 1/h(x_0)$, this is a contradiction with [56, Theorem 1.2].

Case 2. $x_0 \in \partial\Omega$: After a new change of variables we may assume that $\partial\Omega$ is on the hyperplane $x^d = 0$ (see [9]). We have that v_j is well defined for $j \geq j_R$ in

$$H_{R,j} := B_R(0) \cap \left\{ y^d > \frac{-x_j^d}{\lambda_j} \right\}.$$

We know that if $y \in H_{R,j}$ then $0 \leq v_j(y) \leq v_j(0) = 1$. Now, we distinguish between three different cases.

- Case 2.1. Assume that the sequence x_j^d/λ_j is not bounded from upper. Therefore, after extracting a subsequence, we have $x_j^d/\lambda_j \to +\infty$, so, $H_{R,j} \to B_R(0)$ and we may argue as in Case 1.

- Case 2.2. Assume that the sequence x_j^d/λ_j is not bounded from below. Therefore, after extracting a subsequence, we have $x_j^d/\lambda_j \to 0$, thus, the set $H_{R,j}$ is approximating to $B_R(0) \cap \{x \in \mathbb{R}^d : y^d > 0\} = B_R \cap H$. Now we distinguish between two cases:

  a) Assume $p > 2q - 1$. Then v_j verifies the following equation:
  (4.36)
  $$\begin{cases} -\Delta v_j = \lambda_j^2 M_j^{-1} f(\lambda_j y + x_j, M_j v_j) & \text{in } H \cap B_R, \\ \dfrac{\partial v_j}{\partial n} + \lambda_j b(\lambda_j y + x_j) v_j = \lambda_j M_j^{-1} g(\lambda_j y + x_j, M_j v_j) & \text{on } \partial H \cap B_R. \end{cases}$$

  We choose $\lambda_j = M_j^{(1-p)/2}$. Taking into account (4.33) and $\|v_j\|_\infty = 1$ we obtain

  $$\|\lambda_j^2 M_j^{-1} f(\lambda_j y + x_j, M_j v_j)\|_{p, B_R \cap H} \leq \|h\|_{\infty, B_R \cap H}. \tag{4.37}$$

  Since, $\frac{1-p}{2} + q - 1 < 0$ ($p > 2q - 1$), we can argue as previously for sufficiently large j and we infer

  $$\|\lambda_j^2 M_j^{-1} g(\lambda_j y + x_j, M_j v_j)\|_{p, \partial H \cap B_R} \leq \epsilon. \tag{4.38}$$

  Thanks to Theorem 2.15 and taking into account (4.37) and (4.38), we get

  $$\|v_j\|_{W^{1,p}(B_R \cap H)} \leq C.$$

  Next, by a bootstrap argument, we have

  $$\|v_j\|_{W^{2,p}(B_R \cap H)} \leq C.$$

Now, we can argue as in Case 1 and we prove that there exists $v \in \mathcal{C}^2(\overline{H})$ with $v(0) = 1$ and v solution to

$$\begin{cases} -\Delta v = v^p & \text{in } H, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial H. \end{cases} \tag{4.39}$$

Applying the reflection principle we have that $v \in \mathcal{C}^2(\overline{H})$, $v(0) = 1$ and v solution to $-\Delta v = v^p$ in $\mathbb{R}^d$, a contradiction with [56, Theorem 1.2].

b) Assume $p < 2q - 1$. Then, as we did previously v_j satisfies (4.36). We pick $\lambda_j = M_j^{1-q}$ then, since $2(1-q) - 1 + p < 0$ we can argue as in the case $p > 2q - 1$ and we have that there exists $v \in \mathcal{C}^2(\overline{H})$ with $v(0) = 1$ solution to

$$\begin{cases} -\Delta v = 0 & \text{in } H, \\ \dfrac{\partial v}{\partial n} = v^q & \text{on } \partial H. \end{cases} \tag{4.40}$$

This is a contradiction with [65, Theorem 1.2].

- Case 2.3. Assume that the sequence x_j^d/λ_j is bounded from upper and below. Then, after extracting a subsequence, we have $x_j^d/\lambda_j \to s$, $s > 0$. Therefore, $H_{R,j}$ approximate to $B_R(0) \cap \{x \in \mathbb{R}^d : y^d > -s\} = B_R \cap H_{-s}$. If $p > 2q - 1$ then we can proceed as in Case 2.2 and prove that there exists $v \in \mathcal{C}^2(\overline{H}_{-s})$ with $v(0) = 1$ solution to

$$\begin{cases} -\Delta v = v^p & \text{in } H_{-s}, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial H_{-s}. \end{cases} \tag{4.41}$$

Using the reflection principle through $\{y^d = -s\}$ we obtain that v is solution to $-\Delta v = v^p$ in $\mathbb{R}^d$, a contradiction. If $p < 2q - 1$ then we can argue as Case 2.2 and infer that there exists $v \in \mathcal{C}^2(\overline{H}_{-s})$ with $v(0) = 1$, solution to

$$\begin{cases} -\Delta v = 0 & \text{in } H_{-s}, \\ \dfrac{\partial v}{\partial n} = v^q & \text{on } \partial H_{-s}. \end{cases} \tag{4.42}$$

After a linear change of variables we get that there exists a solution $w \in \mathcal{C}^2(\overline{H})$ with $w(0) = c > 0$, solution to

$$\begin{cases} -\Delta w = 0 & \text{in } H, \\ \dfrac{\partial w}{\partial n} = w^q & \text{on } \partial H, \end{cases} \tag{4.43}$$

a contradiction.

Remark 4.29. *Observe that if $p < 2q-1$ and we were able to prove that the maximum of u solution to (4.28) is attained on $x \in \partial\Omega$ then we may assume less on h, for example, h can be a sign changing function or even a negative function. Indeed, following the proof of Theorem 4.28 we arrive at Case 2.2 b) and the scaling $\lambda_j = M_j^{1-q}$ arrives, independently of h, in (4.40).*

Remark 4.30. *Theorem 4.28 also holds for mixed boundary problems. The proof of this fact follows the same lines of Theorem 4.28 the only difference is that we have to consider the case when* $x_0 \in \Gamma_0$ *and we arrive, after applying the scaling argument, to standard Lioville problems.*

CHAPTER 5

Nonnegative solutions to an elliptic problem with nonlinear absorption and a nonlinear incoming flux on the boundary

In this chapter we perform an extensive study of the existence, uniqueness (or multiplicity) and stability of nonnegative solutions to the semilinear elliptic equation $-\Delta u = \lambda u - u^p$ in Ω, with the nonlinear boundary condition $\partial u/\partial n = u^r$ on $\partial\Omega$. Here Ω is a smooth bounded domain $\mathbb{R}^d$ with outward unit normal n, λ is a real parameter and $p, r > 0$. Additionally we present the precise behaviour of solutions for large $|\lambda|$ in the cases where they exist. In the proofs we use techniques as bifurcation and sub-supersolutions that were introduced in the previous chapters. Moreover, we apply variational methods in the particular case $0 < p < 1 < r < d/(d-2)$. The results of this chapter have been published in [51].

5.1. Preliminaries

Consider a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with regular boundary $\partial\Omega$. We are interested in the positive solutions to the problem

$$(5.1) \qquad \begin{cases} -\Delta u = \lambda u - u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = u^r & \text{on } \partial\Omega, \end{cases}$$

where $p, r > 0$, $\lambda \in \mathbb{R}$ denotes the bifurcation parameter and n is the outward unit normal vector field to $\partial\Omega$.

Since it will be used in the sequel, we recall the definition of a positive weak solution to (5.1).

Definition 5.1. *A positive weak solution to (5.1) is a positive function $u \in H^1(\Omega)$ such* *that*

$$\int_\Omega \nabla u \cdot \nabla \varphi - \int_{\partial\Omega} u^r \varphi = \int_\Omega (\lambda u - u^p)\varphi \quad \forall \varphi \in H^1(\Omega).$$

The next lemma shows that, if $0 < p < 1 < r < d/(d-2)$, positive solutions to (5.1) are in fact classical.

Lemma 5.2. *Let $u \in H^1(\Omega)$ a positive weak solution to (5.1), where $0 < p < 1 < r < d/(d-2)$. Then, $u \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$, where $\alpha \in (0,1)$.*

Proof. We only sketch the main points. First, since $0 < r < d/(d-2)$, a Moser iteration as in [42] gives us that $u \in L^\infty(\Omega)$. Thus, it follows that u is a weak solution to the following type of problem

$$\begin{cases} -\Delta u + u = f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = g(u) & \text{on } \partial\Omega, \end{cases}$$

where $f(u) \in L^\infty(\Omega)$ and $g(u) \in L^\infty(\partial\Omega)$. Next, we apply Theorem 2.15 and we argue exactly like in the section 4.1 (the construction of the extended operators for K_1 and K_2). So we get $u \in W^{1,p}(\Omega)$ for all $p > 1$, therefore $\gamma(u) \in W^{1,1-1/p}(\partial\Omega)$ and $g(u) \in W^{1,1-1/p}(\Omega)$. Thanks to the Theorem 2.13 we infer $u \in W^{2,p}(\Omega)$ for all $p > 1$. So, having in mind the Sobolev embedding $u \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$, we infer $f(u) \in \mathcal{C}^{\alpha}(\overline{\Omega})$, $g(u) \in \mathcal{C}^{1+\alpha}(\partial\Omega)$ and the result follows from Theorem 2.14. ■

The following lemma provides a pointwise lower estimate for all solutions to (5.1) when $p > 1$ and $\lambda > 0$.

Lemma 5.3. *Assume that $p > 1$ and $r > 0$. Then, if u is a positive solution to (5.1) with $\lambda > 0$, we have*

$$u > \lambda^{1/(p-1)}. \tag{5.2}$$

Proof. It is clear that if u is a positive solution to (5.1), then it is a strict supersolution to the problem

$$\begin{cases} -\Delta v = \lambda v - v^p & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Moreover, $\underline{u} = \epsilon > 0$ is a strict subsolution to (5.3) for small ϵ. Additionally, thanks to the strong maximum principle, we know that $\epsilon < u$. Thus, the sub and supersolutions method entails $\epsilon < v < u$. Since for $\lambda > 0$, $\lambda^{1/(p-1)}$ is the unique solution to (5.3), the result follows. ■

We close this section with two results based on bifurcation for (5.1). In the first one we deal with bifurcation from infinity, the proof is a direct consequence of Theorem 4.17.

Proposition 5.4. *Assume $r < 1 = p$ (resp. $p < 1 = r$). There exists and unbounded continuum $\mathcal{C}_\infty \subset \mathbb{R} \times \mathcal{C}(\overline{\Omega})$ of positive solutions to (5.1) bifurcating from infinity at*

$\lambda = \lambda_1(-\Delta+1, \mathcal{N})$ (resp. at $\lambda = \lambda_1(-\Delta, \mathcal{N}-1)$). Moreover, this is the unique bifurcation point from infinity. Furthermore, if $\delta_0 > 0$ is small enough and $\mathcal{J} = [\lambda_1 - \delta_0, \lambda_1 + \delta_0] \times \{u \in \mathcal{C}(\overline{\Omega}) : \|u\|_\infty \geq 1\}$ with $\lambda_1 = \lambda_1(-\Delta + 1, \mathcal{N})$ (resp. $\lambda_1 = \lambda_1(-\Delta, \mathcal{N} - 1)$), then either

- *$\mathcal{C}_\infty \setminus \mathcal{J}$ is bounded in $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$ and $\mathcal{C}_\infty \setminus \mathcal{J}$ meets the set $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$, or*
- *$\mathcal{C}_\infty \setminus \mathcal{J}$ is unbounded in $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$.*

The next result is related to bifurcation from the trivial solution. Previously we give the precise meaning of bifurcation direction *subcritical* and *supercritical.*

Definition 5.5. *We say that at the bifurcation point $(\lambda_1, 0)$ the bifurcation direction is* subcritical *(resp.* supercritical*) if for every sequence $\{\lambda_j, u_j\}$ of positive solutions to (5.1) with $\lambda_j \to \lambda_1$ and $\|u_j\|_\infty \to 0$ as $j \to +\infty$, we have $\lambda_j < \lambda_1$ (resp. $\lambda_j > \lambda_1$) for large j.*

Proposition 5.6. *Assume $p \geq 1$ and $r > 1$. There exists an unbounded continuum $\mathcal{C}_0 \subset \mathbb{R} \times \mathcal{C}(\overline{\Omega})$ of positive solutions to (5.1) emanating from the trivial solution at $\lambda = \lambda_1(-\Delta + 1, \mathcal{N})$ when $p = 1$ or at $\lambda = 0$ when $p > 1$. Moreover, this is the unique bifurcation point from the trivial solution, and with respect to the bifurcation direction:*

a) if $p = 1 < r$, then the bifurcation direction is subcritical;

b) if $1 < p < r$ (resp. $p > r$) then the bifurcation direction is supercritical (resp. subcritical);

c) if $p = r$ then the bifurcation direction is supercritical (resp. subcritical) for $|\Omega| > |\partial\Omega|$ (resp. $|\Omega| < |\partial\Omega|$).

Proof. The existence of an unbounded continuum $\mathcal{C}_0$ is a consequence of Theorem 4.10. We now show the bifurcation direction in the second and third paragraphs (the remaining case can be proved similarly). Take a sequence of solutions (λ_j, u_j) such that $\lambda_j \to 0$ and $\|u_j\|_\infty \to 0$ as $j \to +\infty$. We integrate the Eq. (5.1) and we obtain

$$-\int_{\partial\Omega} u_j^r + \int_\Omega u_j^p = \lambda_j \int_\Omega u_j.$$

Now, we divide by $\|u_j\|_\infty^p$:

$$-\|u_j\|_\infty^{r-p} \int_{\partial\Omega} \left(\frac{u_j}{\|u_j\|_\infty}\right)^r + \int_\Omega \left(\frac{u_j}{\|u_j\|_\infty}\right)^p = \lambda_j \|u_j\|_\infty^{1-p} \int_\Omega \frac{u_j}{\|u_j\|_\infty}, \tag{5.4}$$

and take into account that $u_j/\|u_j\|_\infty \to 1$ in $\mathcal{C}(\overline{\Omega})$ (see Remark 4.5). Thus we deduce that if $1 < p < r$, then $\lambda_j > 0$ for large j, while if $p > r$, $\lambda_j < 0$ for large j, which concludes the second paragraph. When $p = r$, the left-hand side of (5.4) converges to $-|\partial\Omega| + |\Omega|$. Thus $sgn(\lambda_j) = sgn(|\Omega| - |\partial\Omega|)$ for large j and $|\Omega| \neq |\partial\Omega|$, which proves the last paragraph. ∎

5.2. State and proof of the main results

In the first result of the section we include, for the sake of clarity, all the stability results.

Lemma 5.7. *Let u_0 a positive solution to (5.1).*

a) If $p \geq 1$ and $r \leq 1$ and $(p,r) \neq (1,1)$, then u_0 is stable.

b) If $p = 1$ and $r > 1$, then u_0 is unstable.

c) If $1 < p \leq r$ and $\lambda \leq 0$, then u_0 is unstable.

Proof. We have to compute the sign of $\lambda_1(-\Delta - \lambda + pu_0^{p-1}, \mathcal{N} - ru_0^{r-1})$. For that, it is well known (see for instance [48, Lemma 2.2]) that this eigenvalue is positive (resp. negative) if there exists a strict supersolution (resp. subsolution), that is, a positive function v such that

$$\begin{cases} (-\Delta - \lambda + pu_0^{p-1})v \geq 0 \ \ (resp. \leq 0) & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} - ru_0^{r-1}v \geq 0 \ \ (resp. \leq 0) & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

and at least of the inequalities is strict. Observe that taking $v = u_0$ we have

$$\begin{cases} -\Delta u_0 - \lambda u_0 + pu_0^p = (p-1)u_0^p & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial n} - ru_0^r = (1-r)u_0^r & \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

whence we deduce the first and second paragraphs. For the last paragraph, take $v = u_0^q$, with $1 < p \leq q \leq r$. We have that

$$\frac{\partial v}{\partial n} - ru_0^{r-1}v = (q-r)u_0^{q+r-1} \leq 0 \text{ on } \partial\Omega,$$

and in Ω,

$$(-\Delta - \lambda + pu_0^{p-1})v = q(1-q)u_0^{q-2}|\nabla u_0|^2 + \lambda u_0^q(q-1) + u_0^{p+q-1}(p-q) < 0.$$

This concludes the proof. ■

Now, we deal with the main theorems of this chapter.

Theorem 5.8. *a) Assume $r = 1$ and $p \neq 1$. There exists a positive solution if, and only if, $\lambda > \lambda_1(-\Delta, \mathcal{N} - 1)$. Moreover,*

(a) if $p > 1$, the solution is strongly positive, unique (denoted by u_λ), stable and verifies

$$\lim_{\lambda \searrow \lambda_1(-\Delta,\mathcal{N}-1)} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = +\infty; \tag{5.7}$$

(b) if $p < 1$, then every family of positive solutions $\{u_\lambda\}$ satisfies

$$\lim_{\lambda \searrow \lambda_1(-\Delta,\mathcal{N}-1)} \|u_\lambda\|_\infty = +\infty, \quad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = 0. \tag{5.8}$$

b) Assume $p = 1$.

(a) If $1 < r < d/(d-2)$, there exists a positive solution if, and only if, $\lambda < \lambda_1(-\Delta + 1, \mathcal{N})$. Moreover, all positive solutions are unstable and for every family of positive solutions $\{u_\lambda\}$ it holds

$$\lim_{\lambda \nearrow \lambda_1(-\Delta+1,\mathcal{N})} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = +\infty. \tag{5.9}$$

(b) If $r < 1$, there exists a positive solution if an only if $\lambda < \lambda_1(-\Delta + 1, \mathcal{N})$. Moreover, the solution is strongly positive, unique (denoted by u_λ), stable and

$$\lim_{\lambda \nearrow \lambda_1(-\Delta+1,\mathcal{N})} \|u_\lambda\|_\infty = +\infty, \quad \lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = 0. \tag{5.10}$$

Proof. We divide the proof in four cases, according to whether $r = 1$ and $p > 1$ or $p < 1$, and $p = 1$, $r > 1$ or $r < 1$.

Case $r = 1 < p$. First, we show that $\lambda > \lambda_1(-\Delta, \mathcal{N} - 1)$ is a necessary condition for the existence of positive solutions. Denote φ_1 a positive eigenfunction associated to $\lambda_1(-\Delta, \mathcal{N} - 1)$. Then, on multiplying (5.1) by φ_1 and integrating by parts, we get

$$(\lambda_1(-\Delta, \mathcal{N} - 1) - \lambda) \int_\Omega u\varphi_1 = -\int_\Omega u^p \varphi_1.$$

Thus $\lambda > \lambda_1(-\Delta, \mathcal{N} - 1)$. To show the existence of solutions when $\lambda > \lambda_1(-\Delta, \mathcal{N} - 1)$, we use the method of sub and supersolutions. On one hand, the function $\overline{u} = M\varphi_1$ is a supersolution to (5.1) for

$$M = \frac{(\lambda - \lambda_1(-\Delta, \mathcal{N} - 1))^{1/(p-1)}}{\delta_0}, \tag{5.11}$$

where $0 < \delta_0 = \min_{x \in \overline{\Omega}} \varphi_1(x)$. On the other hand, $\underline{u} = \epsilon\varphi_1$ is a subsolution to (5.1) if

$$\epsilon^{p-1}\varphi_1^{p-1} \leq \lambda - \lambda_1(-\Delta, \mathcal{N} - 1) \text{ in } \Omega.$$

Therefore, it is enough to take $\epsilon > 0$, to assure the existence of a positive solution. The uniqueness follows directly by Theorem 3.3.
Now, thanks to the uniqueness and the way the supersolution was built, see (5.11), we conclude that

$$\lim_{\lambda \searrow \lambda_1(-\Delta, \mathcal{N}-1)} \|u_\lambda\|_\infty = 0,$$

and by Lemma 5.3,

$$\lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = +\infty.$$

The stability follows by the first paragraph of Lemma 5.7.

Case $p < 1 = r$. Arguing as in the previous case we can prove that the condition $\lambda > \lambda_1(-\Delta, \mathcal{N} - 1)$ is necessary to have positive solutions. Next, we apply Proposition 5.4, so an unbounded continuum $\mathcal{C}_\infty$ of positive solutions to (5.1) bifurcates at $\lambda = \lambda_1(-\Delta, \mathcal{N} - 1)$. It suffices to show that this continuum does not meet the set $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$. Assume that there exists a sequence (λ_j, u_j) of solutions to (5.1) such that $\lambda_j \to \lambda_0 \geq \lambda_1(-\Delta, \mathcal{N} - 1)$ and $\|u_j\|_\infty \to 0$ as $j \to \infty$. Take $M \geq \lambda_j - \lambda_1(-\Delta, \mathcal{N} - 1)$. For j large enough, we have that $u_j^p > Mu_j$ and so

$$\begin{cases} -\Delta u_j < (\lambda_j - M)u_j & \text{in } \Omega, \\ \dfrac{\partial u_j}{\partial n} = u_j & \text{on } \partial\Omega, \end{cases}$$

which implies $\lambda_1(-\Delta - \lambda_j + M, \mathcal{N} - 1)$, that is, $\lambda_1(-\Delta, \mathcal{N} - 1) - \lambda_j + M < 0$, a contradiction. Finally by Theorem 5.20 we have that

$$\lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = 0,$$

for every family $\{u_\lambda\}$ of nonnegative solutions.

Case $p = 1 < r < d/(d-2)$. Consider φ_1 a positive eigenfunction associated to $\lambda_1(-\Delta + 1, \mathcal{N})$. On multiplying (5.1) by φ_1 and integrating we get

$$-\int_{\partial\Omega} u^r \varphi_1 = (\lambda - \lambda_1(-\Delta + 1, \mathcal{N})) \int_\Omega u\varphi_1,$$

and hence $\lambda < \lambda_1(-\Delta + 1, \mathcal{N})$. Now, we can apply Proposition 5.6 and deduce the existence of an unbounded continuum $\mathcal{C}_0$ bifurcating subcritically at $\lambda = \lambda_1(-\Delta + 1, \mathcal{N})$. Thanks to the a priori bounds, Theorem 4.28, we deduce the existence of at least a positive solution for every $\lambda < \lambda_1(-\Delta + 1, \mathcal{N})$. By the second paragraph of Lemma 5.7, every solution is unstable. Finally, by Theorem 5.20 we have that

$$\lim_{\lambda \searrow +\infty} \|u_\lambda\|_\infty = +\infty,$$

for every family $\{u_\lambda\}$ of positive solutions.

Case $r < 1 = p$. Arguing as in the previous case we can show that $\lambda < \lambda_1(-\Delta + 1, \mathcal{N})$ is necessary to have positive solutions. By Proposition 5.4, there exists an unbounded continuum $\mathcal{C}_\infty$ of positive solutions bifurcating from infinity at $\lambda = \lambda_1(-\Delta + 1, \mathcal{N})$. Assume that there exists a sequence (λ_j, u_j) of solutions to (5.1) such that $\lambda_j \to \lambda_0 \leq \lambda_1(-\Delta + 1, \mathcal{N})$ and $\|u_j\|_\infty \to 0$ as $j \to \infty$. Take $M > 0$ large enough so that $-\lambda_j + \lambda_1(-\Delta + 1, \mathcal{N} - M) \leq 0$ which is possible thanks to Remark 2.29. For this value of M and large j we have that $u_j^r > Mu_j$, and so

$$\begin{cases} (-\Delta - \lambda_j + 1)u_j = 0 & \text{in } \Omega, \\ \dfrac{\partial u_j}{\partial n} - Mu_j > 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $\lambda_1(-\Delta + 1 - \lambda_j, \mathcal{N} - M) > 0$, a contradiction. This completes the proof of the existence of solutions. Since, every positive solution is in fact strongly positive then, we can apply Theorem 3.3 and get uniqueness of positive solution. The stability follows by the first paragraph of Lemma 5.7. Moreover, by Theorem 5.22 we have that

$$\lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = 0,$$

for the unique positive solution. ■

Theorem 5.9. *Assume* $0 < r < 1 < p$. *There exists a positive solution for all* $\lambda \in \mathbb{R}$. *Moreover the solution is unique (denoted by* u_λ*), stable and*

$$\lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = +\infty. \tag{5.12}$$

Proof. We use the method of sub and supersolutions to prove existence existence of positive solution. For a fixed $\lambda \in \mathbb{R}$, we choose $K_0 \in \mathbb{R}$ such that $\lambda_1(-\Delta, \mathcal{N} - K_0) < \lambda$ (this is possible according to Remark 2.29). Define $\underline{u} = \epsilon\varphi_1$ with $\epsilon > 0$ and φ_1 a positive eigenfunction associated to $\lambda_1(-\Delta, \mathcal{N} - K_0)$. Then, $\underline{u}$ is subsolution to (5.1) if

$$\begin{array}{lll} \epsilon^{p-1}\varphi_1^{p-1} & \leq \lambda - \lambda_1(-\Delta, \mathcal{N} - K_0) & \text{in } \Omega, \\ K_0(\epsilon\varphi_1)^{1-r} & \leq 1 & \text{on } \partial\Omega. \end{array}$$

Thus it suffices to take ϵ small enough. As supersolution we take $\overline{u} = M\varphi > 0$ with $M > 0$ sufficiently large. Then, $\overline{u}$ is a supersolution to (5.1) if

$$\begin{array}{lll} M^{p-1}\varphi_1^{p-1} & \geq \lambda - \lambda_1(-\Delta, \mathcal{N} - K_0) & \text{in } \Omega, \\ K_0(M\varphi_1)^{1-r} & \geq 1 & \text{on } \partial\Omega, \end{array}$$

which is true for M large.
The uniqueness is consequence of Theorem 3.3 and the stability follows by Lemma 5.7. Moreover, thanks to Theorem 5.22 and Lemma 5.3, we have

$$\lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = +\infty.$$

■

Now, we consider the case $0 < p < 1 < r < d/(d-2)$, by contrast with the previous cases, we will use variational arguments to prove the existence of positive solutions.

Theorem 5.10. *Assume* $0 < p < 1 < r < d/(d-2)$. *There exists a positive solution for all* $\lambda \in \mathbb{R}$. *Moreover, for every family of positive solutions* $\{u_\lambda\}$:

$$\lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = +\infty, \quad \lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = 0. \tag{5.13}$$

Proof. We consider in $H^1(\Omega)$ the functional whose critical points coincide with weak solutions to (5.1) :

$$F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u_+|^2 + \frac{1}{p+1}\int_\Omega |u|^{p+1} - \frac{1}{r+1}\int_{\partial\Omega} |u_+|^{r+1},$$

where $u_+ = \max\{u, 0\}$. Since, r is subcritical, $r < d/(d-2)$, it is well known that F is well defined in $H^1(\Omega)$ and $\mathcal{C}^1$ in $H^1(\Omega)$. By means of the Mountain Pass Theorem (see [12]), we are showing that there exists at least a nontrivial critical point $u \in H^1(\Omega)$ of F, which will be a nontrivial weak solution to (5.1). According to Lemma 5.2, $u \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$ will be a classical solution to (5.1). Thus, we have only to prove that the geometric conditions to apply the Mountain Pass Theorem hold:

Lemma 5.11. *Assume* $0 < p < 1 < r < d/(d-2)$. *Then:*

a) There exists a constant c such that for ρ small enough, $F(u) \geq c\rho^3$ if $\|u\|_{H^1(\Omega)} = \rho$.

b) There exists v_0 with large H^1-norm such that $F(v_0) < 0$.

c) F verifies the Palais-Smale condition.

Proof. Assume that there exists a sequence u_n such that

$$\|u_n\|_{H^1(\Omega)} = \rho_n \to 0 \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{F(u_n)}{\rho_n^3} \leq 0. \tag{5.14}$$

Let $v_n = u_n/\rho_n$. Since $\|v_n\|_{H^1(\Omega)} = 1$ we can extract as subsequence (denoted again with the same index) such that

$$\begin{array}{ll} v_n \rightharpoonup v_0 & \text{weakly in } H^1(\Omega), \\ v_n \to v_0 & \text{strongly in } L^2(\Omega),\ L^{p+1}(\Omega),\ L^{r+1}(\partial\Omega). \end{array}$$

From (5.14) we obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{\rho_n} \Bigg(& \frac{1}{2}\int_\Omega |\nabla v_n|^2 - \frac{\lambda}{2}\int_\Omega |(v_n)_+|^2 + \\ & + \frac{\rho_n^{p-1}}{p+1}\int_\Omega |v_n|^{p+1} - \frac{\rho_n^{r-1}}{r+1}\int_{\partial\Omega} |(v_n)_+|^{r+1} \Bigg) \leq 0. \end{aligned} \tag{5.15}$$

Since $p < 1$, the weak limit v_0 satisfies

$$\int_\Omega |v_0|^{p+1} = 0,$$

so $v_0 \equiv 0$ in Ω. Going back to (5.15), we get

$$\int_\Omega |\nabla v_n|^2 \to 0,$$

a contradiction with $\|v_n\|_{H^1(\Omega)} = 1$.

For the second paragraph we take a function v such that $v_+ \not\equiv 0$ on $\partial\Omega$ and observe that

$$\lim_{t\to+\infty} F(tv) = -\infty.$$

Hence it suffices with setting $v_0 = tv$ for a large t.

For the last paragraph, let u_n a Palais-Smale sequence, that is a sequence such that

$$|F(u_n)| \le C \qquad and \qquad F'(u_n) \to 0.$$

We have to prove that it contains a strongly convergent subsequence. To this end let us first check that it is bounded. Assume that this is not the case, that is, there exists a subsequence such that $\|u_n\|_{H^1(\Omega)} \to \infty$. Let

$$v_n = \frac{u_n}{\|u_n\|_{H^1(\Omega)}}.$$

Sine v_n is bounded in $H^1(\Omega)$ there exists a subsequence (as before it is denoted with the same index) such that

$$\begin{aligned} v_n &\rightharpoonup v_0 \quad \text{weakly in } H^1(\Omega),\\ v_n &\to v_0 \quad \text{strongly in } L^2(\Omega),\ L^{p+1}(\Omega),\ L^{r+1}(\partial\Omega). \end{aligned}$$

On the other hand, since $F(u_n)$ is bounded and $F'(u_n) \to 0$ we get

$$\begin{aligned} \frac{F(u_n)}{\|u_n\|_{H^1(\Omega)}} - \frac{1}{2}\langle F'(u_n), v_n\rangle &= \left(\frac{1}{p+1} - \frac{1}{2}\right)\|u_n\|^p_{H^1(\Omega)}\int_\Omega |v_n|^{p+1}\\ &+ \left(\frac{1}{2} - \frac{1}{r+1}\right)\|u_n\|^r_{H^1(\Omega)}\int_{\partial\Omega} |(v_n)_+|^{r+1} \to 0. \end{aligned}$$

Hence

$$v_0 \equiv 0 \quad in \ \ \Omega. \tag{5.16}$$

In addition,

$$\left\langle F'(u_n), \frac{u_n}{\|u_n\|^2_{H^1(\Omega)}}\right\rangle \to 0.$$

That is,

$$\int_\Omega |\nabla v_n|^2 - \lambda\int_\Omega |(v_n)_+|^2 + \|u_n\|^{p-1}_{H^1(\Omega)}\int_\Omega |v_n|^{p+1} - \|u_n\|^{r-1}_{H^1(\Omega)}\int_{\partial\Omega} |(v_n)_+|^{q+1} \to 0,$$

and taking into account (5.16), we get that

$$\int_\Omega |\nabla v_n|^2 \to 0,$$

again a contradiction. Thus $\{u_n\}$ is bounded in $H^1(\Omega)$. We may pass to a subsequence which verifies

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega),\\ u_n &\to u_0 \quad \text{strongly in } L^2(\Omega),\ L^{p+1}(\Omega),\ L^{r+1}(\partial\Omega). \end{aligned}$$

Since $\langle F'(u_n), u_n\rangle \to 0$, $\langle F'(u_n), u_0\rangle \to 0$, we have

$$\int_\Omega |\nabla u_n|^2 - \lambda\int_\Omega |(u_n)_+|^2 + \int_\Omega |u_n|^{p+1} - \int_{\partial\Omega} |(u_n)_+|^{r+1} \to 0$$

and

$$\int_\Omega \nabla u_n \cdot \nabla u_0 - \lambda \int_\Omega (u_n)_+ u_0 + \int_\Omega |u_n|^{p-1} u_n u_0 - \int_{\partial\Omega} |(u_n)_+|^{r-1} u_n u_0 \to 0.$$

And thanks to the weak convergence of u_n we arrive at

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 = \int_\Omega |\nabla u_0|^2,$$

which proves that u_n converges strongly to u_0. This completes the proof. ∎

In what follows we consider the case $p, r > 1$. This case, as we will see, is more involved than the previous ones.

Theorem 5.12. *Assume $p, r > 1$.*

a) If $p > 2r - 1$, there exists $\lambda_0 < 0$ such that (5.1) has a positive solution if, and only if, $\lambda \geq \lambda_0$. Moreover, for every family of positive solutions $\{u_\lambda\}$ it holds

$$\lim_{\lambda \nearrow +\infty} \|u_\lambda\|_\infty = +\infty. \tag{5.17}$$

b) If $p < 2r - 1$ and $r < d/(d-2)$, there exists $\Lambda_0 \geq 0$ such that (5.1) has positive solution if $\lambda < \Lambda_0$. Moreover, if $\Lambda_0 > 0$, there exist at least two positive solutions for $\lambda \in (0, \Lambda_0)$ and at least a positive solution for $\lambda = \Lambda_0$. In addition, for every family of solutions $\{u_\lambda\}$ we have

$$\lim_{\lambda \searrow -\infty} \|u_\lambda\|_\infty = +\infty. \tag{5.18}$$

c) If $p < r$ or $p = r$ and $|\Omega| > |\partial\Omega|$, and $r < d/(d-2)$ then $\Lambda_0 > 0$. In addition, for every $\lambda \in (0, \Lambda_0)$ there exists a unique positive stable solution to (5.1).

Proof. First, we state the sweeping method of Serrin ([91], pg. 12) which allows to obtain some a priori bounds for the solutions to (5.1).

Lemma 5.13. *Let u be a solution to (5.1). Assume that there exists a family $\overline{u}_M \in C^1(\overline{\Omega})$ of strict supersolutions to (5.1) for $M \in [M_0, M_1]$ such that $\overline{u}_M$ is a continuous and increasing function in M and $u \leq \overline{u}_{M_1}$ in $\overline{\Omega}$. Then, it holds*

$$u \leq \overline{u}_{M_0} \quad \text{in } \overline{\Omega}.$$

Now, we state and prove a non-existence result.

Lemma 5.14. *Assume $p, r > 1$. Then:*

a) If $p = r$ and $|\Omega| \leq |\partial\Omega|$, problem (5.1) does not have positive solutions for $\lambda \geq 0$.

b) Assume that $p \leq 2r - 1$. Then, there exists $\Lambda_1 > 0$ such that problem (5.1) does not have positive solutions for $\lambda \geq \Lambda_1$.

c) *Assume that* $p > 2r - 1$. *Then, there exists* $\Lambda_2 < 0$ *such that problem (5.1) does not have positive solutions for* $\lambda \leq \Lambda_2$.

Proof. Let u be a positive solution to (5.1) with $p = r$. Then, multiplying (5.1) by $1/u^r$ and integrating by parts, we get

$$-r\int_\Omega u^{-r-1}|\nabla u|^2 - |\partial\Omega| + |\Omega| = \lambda\int_\Omega u^{1-r}.$$

The first paragraph follows.

Assume that there exists a sequence $\lambda_n \nearrow \infty$ with corresponding solutions u_n of (5.1). Consider the parabolic problem

$$\begin{cases} w_t - \Delta w = -w^p & \text{in } \Omega\times(0,T),\\ \dfrac{\partial w}{\partial n} = w^r & \text{on } \partial\Omega\times(0,T),\\ w(x,0) = w_0 & \text{in } \Omega. \end{cases} \tag{5.19}$$

We know by [13, Theorem 2.3], that if $p \leq 2r-1$ then all positive solutions to (5.19) blow-up in a finite time $T > 0$ provided $\inf_{\overline{\Omega}} w_0$ is large enough. If we prove that u_n is a supersolution of (5.19) for large n, then $u_n(x) > w(x,t)$ for all $t \in (0,T)$ which is clearly a contradiction. Observe that u_n is supersolution of (5.19) if $u_n > w_0$. By Lemma 5.3, we have $u_n > \lambda_n^{1/(p-1)}$. Thus for large enough n, we may set $w_0 = \lambda_n^{\frac{1}{p-1}}$, which concludes the proof of the second paragraph.

Assume now that $p > 2r-1$. We want to show that for λ negative enough, there are no positive solutions to (5.1). First, we claim that there exists a positive function $U \in C(\overline{\Omega})$ such that

$$u_\lambda \leq U \tag{5.20}$$

for all family of positive solutions to (5.1) with $\lambda \leq 0$. Assume (5.20). Consider the problem

$$\begin{cases} v_t - \Delta v = \lambda v - v^p & \text{in } \Omega\times(0,T),\\ \dfrac{\partial v}{\partial n} = v^r & \text{on } \partial\Omega\times(0,T),\\ v(x,0) = v_0(x) > 0 & \text{in } \Omega, \end{cases} \tag{5.21}$$

and denote $v(t;v_0)$ its positive solution. It is clear that

$$u_\lambda = v(t;u_\lambda) \leq v(t;U),$$

and so it suffices to prove $\|v(t,U)\|_\infty \to 0$ as $t \to \infty$ for negative enough λ. To this aim, it suffices to construct a global supersolution of (5.21) which goes to zero at infinity. Since $p > 2r-1$, for every initial datum $w_0 \in L^\infty(\Omega)$, (5.19) has a positive solution w, which is globally bounded (cf. [13]). Consider $\overline{v} := e^{-\mu t}w$ for some fixed $\mu > 0$. It is not hard to show that $\overline{v}$ is a supersolution of (5.21) provided that

$$\begin{aligned} w^{p-1}(e^{-(p-1)\mu t} - 1) - \mu &\geq \lambda && \text{in } \Omega\times(0,T),\\ e^{\mu(r-1)t} &\geq 1 && \text{on } \partial\Omega\times(0,T). \end{aligned}$$

Since w is bounded, there exists $\lambda_0 < 0$ such that for $\lambda \le \lambda_0$ the two inequalities hold. Thus, it remains to prove (5.20). For that, we are going to use Lemma 5.13 and so we need to construct now a family of strict supersolutions to (5.1). For the particular case $\lambda = 0$ a different supersolution was used in [71] and [106]. Take

$$\overline{u}_M := M(\phi + M^{-\sigma})^{-\beta}$$

where $\beta = 2/(p-1)$, $\phi > 0$ is such that

$$\begin{cases} -\Delta\phi = \lambda_1\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $M, \sigma > 0$ are to be chosen. It is clear that $\overline{u}_M$ is a continuous and increasing function in M. After some calculations, we have that $\overline{u}_M$ is strict supersolution to (5.1) provided that

$$M^{p-1} - \beta(1+\beta)|\nabla\phi|^2 - \beta(\phi + M^{-\sigma})\lambda_1\phi - \lambda(\phi + M^{-\sigma})^2 \ge 0 \quad \text{in } \Omega,$$

and

$$-\beta\frac{\partial\phi}{\partial n} > M^{r-1-\sigma[\beta(1-r)+1]} \quad \text{on } \partial\Omega.$$

Taking into account that $-\partial\phi/\partial n \ge c_1 > 0$ on $\partial\Omega$, it is not hard to show that there exists $M_0 > 0$ (independent of λ) such that for $M \ge M_0$ and $\lambda \le 0$ both inequalities are satisfied, provided that (recall $p > 2r - 1$)

$$r - 1 - \sigma[\beta(1-r) + 1] < 0$$

that is

$$\sigma > \frac{(p-1)(r-1)}{p - 2r + 1}.$$

On the other hand, given a positive solution u_λ to (5.1) with $\lambda \le 0$, there exists $M(\lambda) > M_0$ sufficiently large such that $u_\lambda < \overline{u}_{M(\lambda)}$. So, we can apply Lemma 5.13 and conclude that $u_\lambda \le \overline{u}_{M_0} := U$. This proves (5.20) and completes the proof of Lemma 5.14.

■

We are now ready to come to the proof of Theorem 5.12.

Case $p > 2r - 1$. From Proposition 5.6 it follows that there exists an unbounded continuum $\mathcal{C}_0$ of positive solutions bifurcating at $\lambda = 0$ subcritically (observe that $p > r$ in this case).

Take $I = [\Lambda_2, K]$, with $K > \Lambda_2$ arbitrary where Λ_2 is given by Lemma 5.14. We have a continuous map $\overline{u} : I \to C^1(\overline{\Omega})$, $\lambda \mapsto \overline{u}(\lambda)$ where $\overline{u}(\lambda)$ is the strict supersolution of (5.1) which has been constructed above. Moreover, we have a connected set $\mathcal{C}_0$ such that for λ_0 small enough $u_{\lambda_0} < \overline{u}(\lambda_0)$ for $(\lambda_0, u_{\lambda_0}) \in \mathcal{C}_0$. Then by Theorem 4.25 we obtain that $u_\lambda < \overline{u}(\lambda)$ for all $(\lambda, u_\lambda) \in \mathcal{C}_0$ and $\lambda \in I$. This implies that the projection on the real axis of the continuum $\mathcal{C}_0$ is $[\lambda_2, +\infty)$ for some $\lambda_2 < 0$.

To complete the proof, set

$$\lambda_0 := \inf\{\lambda \in \mathbb{R} : (5.1) \text{ has a positive solution}\}.$$

Thanks to Lemma 5.14 we know that $-\infty < \lambda_0 < 0$. Now, we want to show that there exists a solution for all $\lambda \geq \lambda_0$. Indeed, for $\lambda > \lambda_0$, we can take $\lambda_1 \in (\lambda_0, \lambda)$ such that the corresponding solution u_{λ_1} (which exists thanks to the definition of λ_0) is subsolution of (5.1) for this λ. Again, as supersolution we can take $\overline{u}(\lambda)$. Thus there exists a solution for every $\lambda > \lambda_0$.

Finally, we show that there exists a solution for $\lambda = \lambda_0$. Take (λ_j, u_j) a sequence of solutions such that $0 > \lambda_j > \lambda_0$ and $\lambda_j \to \lambda_0$. Since we have an a priori bound for all solutions, namely $u_j < U$ (see (5.20)), it is standard to pass to the limit to obtain that $u_j \to u_0$ with u_0 a solution to (5.1) for $\lambda = \lambda_0$. Since $\lambda_0 < 0$, it cannot be a bifurcation point from the trivial solution, and hence $u_0 \not\equiv 0$. This completes the proof.

Case $p < 2r - 1$**.** Thanks to Proposition 5.6, there exists an unbounded continuum $\mathcal{C}_0$ of positive solutions to (5.1) which emanates from zero at $\lambda = 0$, and by Theorem 4.28 the solutions are bounded for bounded λ. Thus, since there are no positive solutions for large λ, we conclude the existence of $\lambda_0 \geq 0$ such that there exists at least a positive solution to (5.1) for $\lambda < \lambda_0$. Moreover, (5.18) follows by Theorem 5.22.

Now define

$$\Lambda_0 := \sup\{\lambda \in \mathbb{R} : \ (5.1) \text{ has a positive solution}\}.$$

We already know that $0 \leq \Lambda_0 < +\infty$, and clearly there are no solutions for $\lambda > \Lambda_0$. It remains to show that when $\Lambda_0 > 0$ there exist at least two positive solutions for all $\lambda \in (0, \Lambda_0)$ and one positive solution for $\lambda = \Lambda_0$.

We first show that a minimal positive solution exists if $\lambda \in (0, \Lambda_0)$. Fix such a λ. We have that there exists $\overline{\lambda} \in (\lambda, \Lambda_0)$ such a positive solution $u_{\overline{\lambda}}$ of (5.1) exists. It is clear that $u_{\overline{\lambda}}$ is a supersolution to (5.1) for all $\lambda \leq \overline{\lambda}$. On the other hand, $\underline{u} = \varepsilon$ is a subsolution for small $\varepsilon > 0$. Thus there exists at least a positive solution for every $\lambda \in (0, \Lambda_0)$.

Moreover, we have that any positive solution u_λ verifies $u_\lambda > \lambda^{1/(p-1)}$, thanks to Lemma 5.3. Hence, the existence of a minimal solution to (5.1) follows. It will be denoted by u_λ.

We now show the existence of a second solution when $\lambda \in (0, \Lambda_0)$. We are proving for this aim that our problem is in the general setting of [5] (we refer there for the definitions to be used in the sequel). Let P be the cone of positive functions of $\mathcal{C}(\overline{\Omega})$. With the ordering induced by P, $\mathcal{C}(\overline{\Omega})$ is an ordered Banach space with a normal cone which has nonempty interior, see Example 1.10 in [5]. Consider the interval $I = [-1, \Lambda_0 + 1]$ and let

$$\beta > \sup_{\lambda \in I} \|u(\lambda)\|_\infty,$$

being $u(\lambda)$ any solution to (5.1). This is possible since we have a priori bounds for the solutions when λ runs in finite intervals (cf. Theorem 4.28). Take $K > 0$ a constant to be chosen later, we have that (5.1) can be rewritten as

$$\begin{cases} (-\Delta + K)u = (\lambda + K)u - u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = u^r & \text{on } \partial\Omega. \end{cases}$$

We want to show that solving our problem is equivalent to find fixed points of a nonlinear operator. For that, let $\mathcal{K}_1 : \mathcal{C}^\alpha(\overline{\Omega}) \mapsto \mathcal{C}^{2+\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, be the operator such that $f \mapsto u = \mathcal{K}_1 f$ where u is the unique solution to

$$\begin{cases} (-\Delta + K)u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

This operator can be extended to a linear, compact and strongly positive map, denoted again by $\mathcal{K}_1$, $\mathcal{K}_1 : \mathcal{C}(\overline{\Omega}) \mapsto \mathcal{C}^1(\overline{\Omega})$, see [5, Theorem 4.2]. Consider now the operator $\mathcal{K}_2 : \mathcal{C}^{1+\alpha}(\partial\Omega) \mapsto \mathcal{C}^{2+\alpha}(\overline{\Omega})$, $g \mapsto u = \mathcal{K}_2 g$, where u is the unique solution to

$$\begin{cases} (-\Delta + K)u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases}$$

Now, by [4], $\mathcal{K}_2$ can be extended to a linear compact map from $\mathcal{C}(\partial\Omega)$ to $\mathcal{C}(\overline{\Omega})$. It is not difficult to prove that u is solution to (5.1) if, and only if,

$$u = F(u, \lambda) = \mathcal{K}_1((\lambda + K)u - u^p) + \mathcal{K}_2(\gamma(u^r)),$$

where $\gamma : \mathcal{C}(\overline{\Omega}) \mapsto \mathcal{C}(\partial\Omega)$ is the trace operator.

Moreover, $F : \mathcal{C}(\overline{\Omega}) \times \mathbb{R} \to \mathcal{C}(\overline{\Omega})$ is a differentiable operator, which is compact on bounded sets, and it is strongly increasing for fixed λ if K is large enough and u is restricted to bounded sets. In addition, the partial derivatives,

$$\partial_u F(u_0, \lambda_0)\xi = \mathcal{K}_1((\lambda + K) - p u_0^{p-1})\xi + \mathcal{K}_2(r\gamma(u_0^{r-1}))\gamma(\xi)$$

and

$$\partial_\lambda F(u_0, \lambda_0)\mu = \mu \mathcal{K}_1 u_0,$$

are strongly positive if K is selected large enough. Indeed, observe that since $\lambda \in I$, then $\sup_{\lambda \in I} \|u(\lambda)\|_\infty < \beta$ for any positive solution $u(\lambda)$ of (5.1), and so K can be taken large to make the partial derivatives strongly positive. Hence, F satisfies hypothesis (H) of [5] p. 680, and so we can apply [5, Theorem 20.9] (see the arguments after Proposition 20.8 and [9, Theorem 7.4]) and conclude the existence of at least two positive solutions for $\lambda \in (0, \Lambda_0)$ and at least a positive solution for $\lambda = \Lambda_0$.

We quote for its use in the next section that, denoting by $\rho = r(u_0, \lambda_0)$ the spectral radius of $\partial_u F(u_0, \lambda_0)$, then ρ satisfies

$$\lambda_1(-\Delta + \frac{1}{\rho}(p u_0^{p-1} - \lambda_0), \mathcal{N} + \frac{r}{\rho} u_0^{r-1}) = K(\frac{1}{\rho} - 1). \tag{5.22}$$

Case $p < r$ or $p = r$ and $|\Omega| > |\partial\Omega|$. First of all, notice that $p < 2r - 1$ in this case. Thus there exists a solution for every $\lambda < \Lambda_0$, for a certain $\Lambda_0 \geq 0$. Since a supercritical bifurcation takes place at $\lambda = 0$ (Proposition 5.1) we have $\Lambda_0 > 0$. Thus only the uniqueness of the stable solution for $\lambda \in (0, \Lambda_0)$ remains to be proved. We adapt the argument used in [58].

The following result provides us with a complete picture of the structure of the set of positive solutions near a stable or neutrally stable solution.

Lemma 5.15. *Let* (λ_0, u_0) *be a positive solution to (5.1) with* $\lambda = \lambda_0$.

a) If

$$\lambda_1(-\Delta - \lambda_0 + pu_0^{p-1}, \mathcal{N} - ru_0^{r-1}) > 0, \tag{5.23}$$

then, there exists $\varepsilon > 0$ *and a differentiable mapping* $u : I = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \mapsto P$ *such that* $u(\lambda_0) = u_0$ *and* $(\lambda, u(\lambda))$ *is a positive solution to (5.1) for each* $\lambda \in I$. *Moreover, the mapping* $\lambda \mapsto u(\lambda)$ *is increasing and there exists a neighborhood* $\mathcal{V}$ *of* (λ_0, u_0) *in* $\mathbb{R}\times P$ *such that if* $(\lambda, u) \in \mathcal{V}$ *is a solution to (5.1), then* $(\lambda, u) = (\lambda, u(\lambda))$ *for some* $\lambda \in I$.

b) If

$$\lambda_1(-\Delta - \lambda_0 + pu_0^{p-1}, \mathcal{N} - ru_0^{r-1}) = 0, \tag{5.24}$$

let Φ_0 *be the principal eigenfunction associated with* $\lambda_1(-\Delta-\lambda_0+pu_0^{p-1}, \mathcal{N}-ru_0^{r-1})$. *Then, there exists* $\varepsilon > 0$ *and a twice continuously differentiable mapping* $(\lambda, u) : J = (-\varepsilon, \varepsilon) \mapsto \mathbb{R} \times P$ *such that* $(\lambda(0), u(0)) = (\lambda_0, u_0)$ *and for each* $s \in J$, $(\lambda(s), u(s))$ *is a positive solution to (5.1). Moreover,* $\lambda'(0) = 0$, $u(s) = u_0 + s(\Phi_0 + v(s))$ *where* $v \in \mathcal{C}^1((-\varepsilon, \varepsilon), C(\overline{\Omega}))$ *satisfies* $v(0) = 0$, *and finally*

$$\lambda''(0) = \frac{\displaystyle\int_\Omega p(p-1)u_0^{p-2}\Phi_0^3 - \int_{\partial\Omega} r(r-1)u_0^{r-2}\Phi_0^3}{\displaystyle\int_\Omega u_0\Phi_0}, \tag{5.25}$$

for $s \simeq 0$. *In addition, there exists a neighborhood* $\mathcal{W}$ *of* (λ_0, u_0) *in* $\mathbb{R} \times P$ *such that if* $(\lambda, u) \in \mathcal{W}$ *is a solution to (5.1), then* $(\lambda, u) = (\lambda(s), u(s))$ *for some* $s \in J$. *Also,*

$$\operatorname{sgn} \lambda'(s) = \operatorname{sgn} \lambda_1(-\Delta - \lambda(s) + pu(s)^{p-1}, \mathcal{N} - ru(s)^{r-1}). \tag{5.26}$$

Proof. By (5.22), if (5.23) holds, 1 is not an eigenvalue of $\partial_u F(u_0, \lambda_0)$, and so $Id - \partial_u F(u_0, \lambda_0)$ is a topological isomorphism. Hence we can apply [5, Proposition 20.6] and conclude the first paragraph.

Again by (5.22), if (5.24) holds, 1 is an eigenvalue with positive eigenfunction of $\partial_u F(u_0, \lambda_0)$, so we can apply [5, Propositions 20.7 and 20.8].

Finally, to prove (5.26), observe that from [5, Proposition 20.8] it follows that

$$\operatorname{sgn} \lambda'(s) = \operatorname{sgn}(1 - r(u(s), \lambda(s))).$$

Taking into account (5.22) it is not hard to show that

$$\operatorname{sgn}(1 - r(u(s), \lambda(s))) = \operatorname{sgn}\lambda_1(-\Delta - \lambda(s) + pu(s)^{p-1}, \mathcal{N} - ru(s)^{r-1}).$$

This completes the proof. ∎

We now analyze the behavior of the branch of solutions near a point (λ_0, u_0) such that (5.24) holds. The fact that $\lambda'(0) = 0$ shows that this is actually a turning point of the branch of positive solutions (cf. Corollary 5.18 below). We are elucidating in what follows the direction of the turning. The essential ingredient is a Picone's type identity (see [19, Section 4] and [74, Lemma 4.1], for instance).

Lemma 5.16. (Picone's Identity) *Let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ be such that $v/u \in \mathcal{C}(\overline{\Omega})$, $\Upsilon : [0,\infty) \mapsto \mathbb{R}$ an arbitrary $\mathcal{C}^1$ function and $\mathcal{L}$ and uniformly elliptic and selfadjoint operator of second order. Then*

$$\int_\Omega \Upsilon(\frac{v}{u})(v\mathcal{L}u - u\mathcal{L}v) = -\int_\Omega \Upsilon'(\frac{v}{u})u^2\left|\nabla\left(\frac{v}{u}\right)\right|^2 - \int_{\partial\Omega}\Upsilon(\frac{v}{u})[v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}]. \tag{5.27}$$

Then we have the following important result.

Proposition 5.17. *Assume $p \le r$. Let (λ_0, u_0) be a positive solution to (5.1) with $\lambda = \lambda_0$, such that* (5.24) *holds. Then $\lambda''(0) < 0$, where $\lambda''(0)$ is defined in* (5.25).

Proof. To determine the sign of $\lambda''(0)$, we use the Picone's identity (5.27) with $\Upsilon(t) = t^2$, $v = \Phi_0, u = u_0$ and $\mathcal{L} = -\Delta$, to obtain

$$(p-1)\int_\Omega u_0^{p-2}\Phi_0^3 < (r-1)\int_{\partial\Omega} u_0^{r-2}\Phi_0^3. \tag{5.28}$$

From (5.28) and as $p \le r$ we can infer that $\lambda''(0) < 0$. This concludes the proof.

■

As an easy consequence of Lemma 5.15 (in particular relations (5.25) and (5.26)) and Proposition 5.17, we obtain:

Corollary 5.18. *Let (λ_0, u_0) be a positive solution to (5.1) with $\lambda = \lambda_0 > 0$, such that $\lambda_1(-\Delta - \lambda_0 + pu_0^{p-1}, \mathcal{N} - ru_0^{r-1}) = 0$. Then, there exists $\varepsilon > 0$ such that for each $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, (5.1) has two positive solutions, one of them stable and the other one unstable. Moreover, there exists a neighborhood $\mathcal{N}$ of (λ_0, u_0) in $\mathbb{R} \times P$ such that (5.1) does not have a positive solution in $\mathcal{N}$ for $\lambda > \lambda_0$.*

We are finally ready to prove the uniqueness of the stable solution.

Theorem 5.19. *Assume that $p \le r$. Then, the minimal solution is the unique positive stable solution to* (5.1) *for all $\lambda \in (0, \Lambda_0)$.*

Proof. We first show that the minimal solution u_λ is stable for all $\lambda \in (0, \Lambda_0)$. It is well known (see [5, Proposition 20.4]) that the minimal solution is weakly stable, i.e.,

$$\lambda_1(-\Delta - \lambda + pu_\lambda^{p-1}, \mathcal{N} - ru_\lambda^{r-1}) \ge 0 \qquad \text{for all } \lambda \in (0, \Lambda_0). \tag{5.29}$$

On the other hand, in a neighborhood $\mathcal{N}$ of $(\lambda, u) = (0,0)$, there exists a unique positive solution for fixed λ. Since the minimal solution exists for all $\lambda \in (0, \Lambda_0)$, the unique solution coincides with the minimal, so by Corollary 5.18 there exists $\underline{\lambda}$ such that for all $0 < \lambda \leq \underline{\lambda}$ we have that

$$\lambda_1(-\Delta - \lambda + pu_\lambda^{p-1}, \mathcal{N} - ru_\lambda^{r-1}) > 0.$$

Now, we can produce this branch to the right to reach a value $\lambda_0 \leq \Lambda_0$ such that $\lambda_1(-\Delta - \lambda + pu_\lambda^{p-1}, \mathcal{N} - ru_\lambda^{r-1}) > 0$ for all $\lambda < \lambda_0$ and

$$\lambda_1(-\Delta - \lambda_0 + pu_{\lambda_0}^{p-1}, \mathcal{N} - ru_{\lambda_0}^{r-1}) = 0. \tag{5.30}$$

If $\lambda_0 = \Lambda_0$ we have proved that the minimal solution is stable for all $\lambda < \Lambda_0$. So assume that $\lambda_0 < \Lambda_0$. Thanks to (5.29) and Corollary 5.18, there exists a value $\lambda_1 \in (\lambda_0, \Lambda_0)$ such that

$$\lambda_1(-\Delta - \lambda_1 + pu_{\lambda_1}^{p-1}, \mathcal{N} - ru_{\lambda_1}^{r-1}) > 0,$$

and by Lemma 5.15, first paragraph, we can continue the branch to the left of λ_1. Denote

$$\Gamma = \{(\lambda, u(\lambda)) : \ \lambda \leq \lambda_1\}.$$

Now two possibilities may arise:

a) There exists $\lambda_2 < \lambda_1$ such that $\lambda_1(-\Delta - \lambda_2 + pu(\lambda_2)^{p-1}, \mathcal{N} - ru(\lambda_2)^{r-1}) = 0$.

b) The branch Γ can be continued for all $\lambda \leq \lambda_1$ with $\lambda_1(-\Delta - \lambda + pu(\lambda)^{p-1}, \mathcal{N} - ru(\lambda)^{r-1}) > 0$.

If the first possibility holds, then Corollary 5.18 is contradicted. In the second possibility, Γ does not reach negative values of λ by Lemma 5.7. So, again two situations are possible:

a) The branch Γ meets the real axis $\{(\lambda, 0)\}$.

b) The branch Γ reaches the minimal solution at some point $(\lambda_3, u_{\lambda_3})$.

If Γ meets the axis $\{(\lambda, 0)\}$, since we know that the unique bifurcation point from the trivial solution is $\lambda = 0$, then Γ reaches at $(0, 0)$. But, as remarked before, in a neighborhood $\mathcal{N}$ of $(\lambda, u) = (0, 0)$ there exists a unique solution, in fact the minimal solution. So, the second possibility occurs. If λ_3 is such that u_{λ_3} satisfies (5.30), Corollary 5.18 leads to a contradiction. However, if λ_3 is such that u_{λ_3} satisfies $\lambda_1(-\Delta-\lambda_3+pu_{\lambda_3}^{p-1}, \mathcal{N}-ru_{\lambda_3}^{r-1}) > 0$, we know that in a neighborhood $\mathcal{M}$ of $(\lambda_3, u_{\lambda_3})$ there exists a unique solution, a contradiction. This contradiction shows that the minimal solution u_λ is stable for all $\lambda \in (0, \Lambda)$ and neutrally stable for $\lambda = \Lambda$.

Now, assume that for some $\lambda_0 \in (0, \Lambda_0)$ there exists a second stable solution $v_0 > u_{\lambda_0}$. We argue as in the first part of the proof. By Lemma 5.15, first paragraph, there exists a branch, say Γ', of stable solutions of the form $(\lambda(s), v(s))$, $s \in I$, with $\lambda(0) = \lambda_0$, $v(0) = v_0$. Moreover, we can continue this branch to the left until there exists a value λ^* in which it is noncontinuable. Since, by Lemma 5.7, third paragraph, all solutions are unstable for $\lambda \leq 0$, it follows that $\lambda^* \geq 0$.

If $\lambda^* > 0$, we would have thanks to Lemma 5.15, first paragraph, that $\lambda_1(-\Delta - \lambda^* + pv_{\lambda^*}^{p-1}, \mathcal{N} - rv_{\lambda^*}^{r-1}) = 0$, and we arrive at a contradiction with Corollary 5.18. Hence $\lambda^* = 0$. Moreover, the branch Γ' has to degenerate at $(0,0)$, otherwise we could continue it thanks to Lemma 5.15, first paragraph. However, this contradicts the uniqueness of solutions for $\lambda \sim 0$, and the uniqueness of the stable solution is proved. ∎

5.3. Behavior of solutions for large $|\lambda|$

This section is devoted to present the behavior of all positive solutions to (5.1) when $\lambda \nearrow \infty$ or $\lambda \searrow -\infty$. All the proofs are based on the well-known blow-up argument of Gidas and Spruck, [56]. An essential role in them is played by a nonexistence result for problems with nonlinear boundary conditions in a half-space obtained in [65]. The proofs of the Theorems in this section can be found in [51, section 5].

We begin by considering the behavior of the positive solutions for $\lambda \to +\infty$ in the case $p < 1 \le r$, assuming that r is subcritical.

Theorem 5.20. *Assume that $0 < p < 1 \le r < d/(d-2)$. For every $\lambda_0 > 0$, there exist positive constants C_1, C_2 such that, for every nonnegative solution u to (5.1) with $\lambda \ge \lambda_0$, we have*

$$C_1 \lambda^{-\frac{1}{1-p}} \le \max_{\Omega} u \le C_2 \lambda^{-\frac{1}{1-p}}. \tag{5.31}$$

The next theorem refers to the case $p > 1$ in the cases where positive solutions exist for large λ, that is $r \le 1$ or $r > 1$ and $p > 2r - 1$ (we recall that no solutions exist for large λ if $p = 1$).

Theorem 5.21. *Assume that $p > 1$ and $r \le 1$ or $1 < r < d/(d-2)$ and $p > 2r - 1$. For every $\lambda_0 > 0$, there exists a positive constant C such that, for every nonnegative solution u to (5.1) with $\lambda \ge \lambda_0$, we have*

$$\lambda^{-\frac{1}{1-p}} \le \max_{\Omega} u \le C \lambda^{-\frac{1}{1-p}}. \tag{5.32}$$

We now turn to consider the cases where positive solutions exist for large negative λ, namely $r < 1$, $p \ge 1$ and $1 < r < d/(d-2)$, $p < 2r - 1$. We collect them both in a single statement.

Theorem 5.22. *Assume that $0 < r < 1 \le p$ or $1 < r < d/(d-2)$ and $p < 2r - 1$. For every $\lambda_0 < 0$, there exist positive constants C_1, C_2 such that, for every nonnegative solution u to (5.1) with $\lambda \le \lambda_0$, we have*

$$C_1 (-\lambda)^{\frac{1}{2(r-1)}} \le \max_{\Omega} u \le C_2 (-\lambda)^{\frac{1}{2(r-1)}}. \tag{5.33}$$

CHAPTER 6

Combining a sublinear boundary condition with a sublinear or superlinear reaction

During this chapter we study the existence, uniqueness (or multiplicity) and stability of positive solutions to the semilinear elliptic equation $-\Delta u + u = a(x)u^p$ in Ω, with the nonlinear boundary condition $\partial u/\partial n = \lambda u^q$ on $\partial\Omega$. Here Ω is a smooth bounded domain of $\mathbb{R}^d$ with outward unit normal vector n, λ is a real parameter, $q \in (0,1)$, $p > 0$ and a may be a sign changing function.

6.1. Preliminaries

Consider a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with regular boundary $\partial\Omega$. We are interested in the positive solutions to the problem

$$\left\{ \begin{array}{ll} -\Delta u + u = a(x)u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda u^q & \text{on } \partial\Omega, \end{array} \right. \tag{6.1}$$

where $p > 0$, $0 < q < 1$, $\lambda \in \mathbb{R}$ denotes the bifurcation parameter, n is the outward unit normal vector field to $\partial\Omega$ and $a \in \mathcal{C}^\alpha(\overline{\Omega})$ with $\alpha \in (0,1)$.
We define the sets

$$\Omega_+ := \{x \in \Omega \,:\, a(x) > 0\}\,, \qquad \Omega_- := \{x \in \Omega \,:\, a(x) < 0\}\,, \qquad \Omega_0 := \Omega \setminus \left(\overline{\Omega}_+ \cap \overline{\Omega}_-\right).$$

During this chapter we will assume the following hypotheses about the previous sets.

- Ω_+ and Ω_- are open sets and with regular boundary, for instance $\mathcal{C}^2$.
- If $K \subset \Omega_+$ is a compact set then there exists $\delta > 0$ such that $a(x) \geq \delta$. Analogously, if $K \subset \Omega_-$ is a compact set then there exists $\delta < 0$ such that $a(x) \leq \delta$.

The following Theorem establishes a priori bounds for (6.1) for p subcritical and it will useful when we apply bifurcation techniques.

Theorem 6.1. *Suppose that one of the following assertions is true.*

- $\Omega_- \cup \Omega_0 = \emptyset$, $1 < p < \dfrac{d+2}{d-2}$.
- *There exists a function* $\alpha : \overline{\Omega}_+ \to \mathbb{R}^+$ *which is continuous and bounded away from zero and a constant* $\gamma \geq 0$ *such that*

$$a(x) = \alpha(x)\,(dist(x, \partial\Omega_+))^\gamma \quad in\ \Omega_+,$$

$$1 < p < \min\left\{\frac{d+1+\gamma}{d-1}, \frac{d+2}{d-2}\right\}.$$

Then for each compact interval $\Lambda \subset \mathbb{R}$ *there exists a positive constant such that*

$$\|u_\lambda\|_\infty \leq C,$$

for every solution (λ, u_λ) *with* $\lambda \in \Lambda$.

Proof. The case $\Omega_- \cup \Omega_0 = \emptyset$, $1 < p < \dfrac{d+2}{d-2}$ is very close to what we have done in Theorem 4.28 but some remarks are needed. Observe that the boundary condition is not regular at zero. However solutions to (6.1) are separated from zero at least on the boundary. To see this claim, let u a positive solution to (6.1). If there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$ then the minimum of u is attained on the boundary, therefore by the Hopf Lemma

$$\frac{\partial u}{\partial n}(x_0) < 0 = \lambda u(x_0)^q,$$

a contradiction. Also, we would like to point out that although the condition (4.30) is not satisfied it is not important since we never arrive at a Lioville problem on the boundary because $p > 2q - 1$. With this observations in mind it is possible to repeat the proof of Theorem 4.28 and conclude the result.

The proof in the second case is more involved. We separate the proof in two steps.

Step 1. *A priori bounds in* $\overline{\Omega}_+$. For this step we can repeat with minor changes the proof of [9, Lemma 4.2, Theorem 4.3].

Step 2. *A priori bounds in* Ω. Let

$$R := \sup_{\lambda \in \Lambda} \sup_{x \in \overline{\Omega}_+} u(x) < +\infty.$$

We consider $Q = \Omega \setminus \overline{\Omega}_+$ and the problem

$$(6.2) \qquad \begin{cases} -\Delta z + z = a(x) z^p & \text{in } Q, \\ z = R & \text{on } \partial Q \cap \Omega, \\ \dfrac{\partial z}{\partial n} = \lambda z^q & \text{on } \partial Q \cap \partial\Omega. \end{cases}$$

We claim that there exists a unique bounded solution to (6.2). Then, since the solutions to (6.2) are supersolutions to (6.1) and by the uniqueness of (6.2) we have

$$\|u_\lambda\|_\infty \leq \|z_\lambda\|_\infty.$$

It remains to prove the existence and uniqueness of (6.2) to this aim we use the method of sub-supersolutions together with Theorem 3.5. Observe that $\underline{z} = 0$ is a strict subsolution to (6.1). On the other hand, let e the unique bounded solution to the problem

$$\begin{cases} -\Delta e + e = 1 & \text{in } Q, \\ e = 1 & \text{on } \partial Q \cap \Omega, \\ \dfrac{\partial e}{\partial n} = 1 & \text{on } \partial Q \cap \partial\Omega, \end{cases}$$

then $\overline{z} = Me$ is a strict supersolution where

$$M = \max\left\{(|\lambda|\|e\|_\infty)^{\frac{1}{1-q}}, R\right\}.$$

The uniqueness for (6.2) follows taking $g(s) = s^q$ in Theorem 3.5. ■

Lemma 6.2. *Let $p = 1$,*

a) If $\lambda > 0$ then the positive solutions to (6.1) are stable.

b) If $\lambda < 0$ then the positive solutions to (6.1) are unstable.

Let $p > 1$,

a) If $\Omega_+ = \emptyset$ then for every $\lambda > 0$ the positive solutions to (6.1) are stable.

b) If $\Omega_+ \neq \emptyset$ then for every $\lambda \leq 0$ the positive solutions to (6.1) are unstable.

Proof. Let $p = 1$ and u_λ a positive solution to (6.1) for $\lambda > 0$. We have to compute the sign of $\lambda_1(-\Delta + 1 - pa(x)u_\lambda^{p-1}, \mathcal{N} - \lambda q u_\lambda^{q-1})$. If $\lambda > 0$

$$\begin{aligned} 0 &= \lambda_1(-\Delta + 1 - a(x), \mathcal{N} - \lambda u_\lambda^{q-1}) \\ &< \lambda_1(-\Delta + 1 - a(x), \mathcal{N} - \lambda q u_\lambda^{q-1}), \end{aligned}$$

concluding the first paragraph. The next one follows in the same fashion. Let $p > 1$ For the first paragraph we observe that

$$\begin{aligned} 0 &= \lambda_1(-\Delta + 1 - a(x)u_\lambda^{p-1}, \mathcal{N} - \lambda u_\lambda^{q-1}) \\ &< \lambda_1(-\Delta + 1 - a(x)pu_\lambda^{p-1}, \mathcal{N} - \lambda q u_\lambda^{q-1}), \end{aligned}$$

concluding the paragraph. For the last one it is enough to prove that the problem

$$(-\Delta + 1 - a(x)pu_\lambda^{p-1}, \mathcal{N} - \lambda q u_\lambda^{q-1}, \Omega)$$

admits a positive strict subsolution. To this end we take $\underline{u} = u_\lambda^r$ with $r > 1$ to be determined, where u_λ is a solution to (6.1), then

$$(-\Delta + 1 - a(x)pu_\lambda^{p-1})u_\lambda^r = ra(x)u_\lambda^{p+r-1} - pa(x)pu_\lambda^{p+r-1} - r(r-1)u_\lambda^{r-2}|\nabla u_\lambda|^2$$

Therefore, if we pick $r = p$ then $(-\Delta + 1 - a(x)pu_\lambda^{p-1})\underline{u} \leq 0$ in Ω. On the other hand, we have

$$\left(\frac{\partial}{\partial n} - \lambda q u_\lambda^{q-1}\right) u_\lambda^p = p\lambda u_\lambda^{p+q-1} - q\lambda u_\lambda^{p+q-1} < 0 \text{ on } \partial\Omega.$$

This concludes the result. ■

6.2. Case $0 < q < 1 = p$

We begin with a non-existence result.

Lemma 6.3. *Let μ_1 the unique zero, if exists, of the application*

$$\sigma(\lambda) = \lambda_1(-\Delta + b(x), \mathcal{N} - \lambda),$$

where $b(x) = 1 - a(x)$. Then:

a) If $\mu_1 > 0$ then the problem (6.1) has not a positive solution for $\lambda \leq 0$.

b) If $\mu_1 < 0$ then the problem (6.1) has not a positive solution for $\lambda \geq 0$.

c) If $\mu_1 = 0$ then the problem (6.1) has not a positive solution for $\lambda \neq 0$.

d) If $\lambda_1(-\Delta + b, \mathcal{D}) \leq 0$, i.e. no existence of μ_1, then the problem (6.1) does not have a positive solution.

Proof. Let φ_1 a positive eigenfunction associated to μ_1. Taking φ_1 as a test function in the problem (6.1) we get

$$-\int_\Omega b(x)u\varphi_1 = -\int_\Omega \Delta\varphi_1 u = \int_\Omega \nabla u \cdot \nabla\varphi_1 - \int_{\partial\Omega} \frac{\partial\varphi_1}{\partial n} u,$$

$$-\int_\Omega b(x)u\varphi_1 = -\int_\Omega \Delta u\varphi_1 = \int_\Omega \nabla u \cdot \nabla\varphi_1 - \int_{\partial\Omega} \frac{\partial u}{\partial n} \varphi_1.$$

Therefore, subtracting the above equalities we have

$$\lambda \int_{\partial\Omega} u^q\varphi_1 = \mu_1 \int_{\partial\Omega} u\varphi_1,$$

and from the previous equality we deduce the first, second and third paragraphs. For the last one, suppose that there exists a positive solution. We take ψ_1 as a test function in (6.1) with ψ_1 a positive eigenfunction associated to $\lambda_1(-\Delta + b(x), \mathcal{D}) \leq 0$ and we obtain

$$\begin{aligned} 0 &= \int_\Omega \nabla u \cdot \nabla\psi_1 - \int_{\partial\Omega} \frac{\partial u}{\partial n}\psi_1 + \int_\Omega b(x)u\psi_1 \\ &= \lambda_1(-\Delta + b(x), \mathcal{D}) \int_\Omega u\psi_1 + \int_{\partial\Omega} \frac{\partial\psi_1}{\partial n} u \end{aligned}$$

By the Hopf lemma the above equality leads to a contradiction. ■

Remark 6.4. *In the previous Theorem a more computable condition than the sign of μ_1 is the sign of $\lambda_1(-\Delta + b, \mathcal{N})$, i.e.*

$$sgn(\mu_1) = sgn(\lambda_1(-\Delta + b, \mathcal{N})).$$

Theorem 6.5. *Assume that $\mu_1 > 0$ then the problem (6.1) has a positive solution, u_λ, if and only if $\lambda > 0$. Moreover this solution is unique, stable and we have*

$$u_\lambda = \lambda^{\frac{1}{1-q}} z,$$

where z is the unique positive solution to

$$\begin{cases} -\Delta z + (1 - a(x))z = 0 & in\ \Omega, \\ \dfrac{\partial z}{\partial n} = z^q & on\ \partial\Omega. \end{cases}$$

Proof. The uniqueness follows from Theorem 3.3 and by Theorem 4.24 there exists an unbounded continuum of positive solutions to (6.1) emanating from zero at $\lambda = 0$. Let φ_1 a strongly positive eigenfunction associated to μ_1 then $\overline{u} = M\varphi_1$ is, for sufficiently large M, a strict supersolution to (6.1). Therefore, by Theorem 4.26, the continuum of solutions is bounded on finite λ-intervals which leads to the existence of solution for every $\lambda > 0$. For the computation of u_λ we observe

$$-\Delta \left(\lambda^{\frac{1}{1-q}} z\right) + (1 - a(x))\lambda^{\frac{1}{1-q}} z = 0,$$

and

$$\frac{\partial \left(\lambda^{\frac{1}{1-q}} z\right)}{\partial n} = \lambda^{\frac{1}{1-q}} z^q = \lambda \left(\lambda^{\frac{1}{1-q}} z\right)^q,$$

this concludes the Theorem. ∎

Theorem 6.6. *Suppose that $\mu_1 = 0$ then the problem (6.1) has a positive solution if and only if $\lambda = 0$.*

Proof. The Theorem is a consequence of Theorem 4.24 together with Lemma 6.3. ∎

Theorem 6.7. *Suppose that $\mu_1 < 0$ then the problem (6.1) has a positive solution if and only if $\lambda < 0$. Moreover, every positive solution to (6.1) is unstable and*

$$\lim_{\lambda \to 0^-} \|u_\lambda\|_\infty = 0.$$

Proof. By Theorem 4.24 there exists an unbounded continuum $\mathcal{C}_0$ of positive solutions to (6.1) emanating from zero at $\lambda = 0$. Thanks to Theorem 6.3 the branch emanates subcritically and does not cross the λ-axis at $\lambda = 0$. Therefore we just need to prove that the problem does not have a bifurcation point from infinity. Assume the contrary, then there exists a sequence (λ_n, u_n) such that $\lambda_n \to \lambda_0 < 0$ and $\lim_{n\to+\infty} \|u_n\|_\infty = +\infty$. Let us define the function

$$v_n = \frac{u_n}{\|u_n\|_\infty},$$

Arguing as in Lemma 4.13 we obtain that $v_n \to v$ in $\mathcal{C}(\overline{\Omega})$, where v is a solution to

$$\begin{cases} -\Delta v + (1-a(x))v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

but, taking φ_1 as a test function in (6.3), a positive eigenfunction associated to $\lambda_1(-\Delta + (1-a(x)), \mathcal{N})$,we deduce, after integrating twice by parts, that $v \equiv 0$, a contradiction.

■

6.3. Case $0 < q < 1 < p$

Through this section we deal with the case $0 < q < 1 < p$. We separate the statements depending on the non-trivial weight function a, non-positive, non-negative and sign changing.

6.3.1. Case $\Omega_+ = \emptyset$

Theorem 6.8. *Assume $\Omega_+ = \emptyset$. There exists a positive solution to (6.1) if and only if $\lambda > 0$. Moreover, the solution is strongly positive, unique (denoted by u_λ), stable and*

$$\lim_{\lambda \searrow 0^+} \|u_\lambda\|_\infty = 0.$$

Proof. First we show $\lambda > 0$ is necessary for the existence of non-trivial and non-negative solutions. Assume that $\lambda \le 0$ then $-\Delta u + u \le 0$ in Ω and $\partial u/\partial n \le 0$ on $\partial\Omega$ therefore, by the maximum principle, $u \le 0$ in $\overline{\Omega}$. To show the existence of positive solutions when $\lambda > 0$ we use the method of sub-supersolutions. First, thanks to Remark 2.29 we can take $\overline{\lambda} > 0$ such that

$$\mu(\overline{\lambda}) := \lambda_1(-\Delta + 1, \mathcal{N} - \overline{\lambda}) < 0.$$

Let φ_1 a positive eigenfunction associated to $\mu(\overline{\lambda})$ such that $\|\varphi_1\|_\infty = 1$. The function $\underline{u} := \epsilon\varphi_1$ is a subsolution to (6.1) if

$$\mu(\overline{\lambda}) \le a(x)\epsilon^{p-1}\varphi_1^{p-1},$$

$$\epsilon \le \left(\frac{\lambda}{\overline{\overline{\lambda}}}\right)^{\frac{1}{1-q}}.$$

Picking $\epsilon > 0$ sufficiently small both conditions are satisfied simultaneously. Let μ_1 such that $\lambda_1(-\Delta + 1, \mathcal{N} - \mu_1) = 0$ and φ_1 a positive eigenfunction associated to μ_1 with $\|\varphi_1\|_\infty = 1$. The function $\overline{u} := M\varphi_1$ is a supersolution to (6.1) provided that

$$0 \ge a(x)M^p\varphi_1^p,$$

$$M \ge \left(\frac{\lambda c_L^{q-1}}{\mu_1}\right)^{\frac{1}{1-q}},$$

with $0 < c_L = \min_{x\in\overline{\Omega}} \varphi_1(x)$. Therefore taking

$$M = \max\left\{\left(\frac{\lambda c_L^{q-1}}{\mu_1}\right)^{\frac{1}{1-q}}, \epsilon c_L^{-1}\right\},$$

we have that $\overline{u}$ is a supersolution and also $\underline{u} \leq \overline{u}$. Taking into account that, by the strong maximum principle, positive solutions are strongly positive then the uniqueness follows from Theorem 3.3. Finally, by the choice of M (observe that $\lim_{\lambda \to 0^+} M(\lambda) = 0$), we conclude that

$$\lim_{\lambda \searrow 0} \|u_\lambda\|_\infty = 0.$$

■

Remark 6.9. *Let u_λ the solution to (6.1) with $a = a_-$. As a consequence of the previous Theorem we know that for $\lambda > 0$ sufficiently small*

$$\left(\frac{\lambda}{\overline{\lambda}}\right)^{\frac{1}{1-q}} \varphi_1 \leq u_\lambda,$$

where φ_1 is an eigenfunction associated to $\mu(\overline{\lambda})$ such that $\|\varphi_1\|_\infty$.

In the next Theorem we will study the asymptotic behavior of u_λ when λ goes to infinity. The results depend on the position of Ω_0. We assume, for simplicity, that either $\Omega_0 \subset\subset \Omega$ or $\Omega_- \subset\subset \Omega$.

Theorem 6.10. *Let us suppose in addition that $a \in \mathcal{C}^1(\overline{\Omega})$.*

- *If $\Omega_0 \subset\subset \Omega$ or $\Omega_0 = \emptyset$ then*

$$\lim_{\lambda \nearrow \infty} u_\lambda = z_{\mathcal{D},\Omega},$$

where $z_{\mathcal{D},\Omega} \in \mathcal{C}^{2+\alpha}(\Omega)$ is the minimum solution to the singular Dirichlet problem

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega. \end{cases}$$

- *If $\Omega_- \subset\subset \Omega$ then*

$$\lim_{\lambda \nearrow \infty} u_\lambda = z_{\mathcal{D},\Omega_-},$$

where $z_{\mathcal{D},\Omega_-} \in \mathcal{C}^{2+\alpha}(\Omega)$ denotes the minimum solution to the Dirichlet problem

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{in } \Omega_-, \\ u = \infty & \text{on } \partial\Omega_-. \end{cases}$$

Moreover,

$$\lim_{\lambda \nearrow +\infty} u_\lambda = \infty,$$

uniformly in Ω_0.

Proof. The proof is rather similar to Theorem 3 and Theorem 4 of [52] and therefore we omit it. ■

In what follows we consider the case $\Omega_+ \neq \emptyset$.

6.3.2. Case $\Omega_+ \neq \emptyset$

Theorem 6.11. *Assume $\Omega_+ \neq \emptyset$. There exists an unbounded continuum $\mathcal{C}_0 \subset \mathbb{R} \times \mathcal{C}(\overline{\Omega})$ of positive solution to (6.1) emanating from the trivial solution at $\lambda = 0$. Moreover, this is the only bifurcation point from the trivial solution and the bifurcation direction is supercritical.*

Proof. Except the bifurcation direction the rest follows from Theorem 4.21. Let us prove that the bifurcation direction is supercritical. Assume that there exists a sequence $\lambda_n \leq 0$ such that

$$(\lambda_n, u_{\lambda_n}) \to (0,0) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}).$$

Therefore, for $n \geq n_0(\epsilon)$ we have

$$-\Delta u_{\lambda_n} + u_{\lambda_n} = a(x) u_{\lambda_n}^p \leq \|a\|_\infty \epsilon u_{\lambda_n} \text{ in } \Omega,$$

$$\frac{\partial u_{\lambda_n}}{\partial n} \leq 0 \text{ on } \partial\Omega.$$

Picking $\epsilon = \epsilon_0$ such that $\epsilon_0 \|a\|_\infty < 1$ we obtain, from the maximum principle that

$$\lambda_1(-\Delta + 1 - \epsilon_0 \|a\|_\infty, \mathcal{N}) < 0,$$

a contradiction. ∎

Lemma 6.12. *Assume $|\Omega_+| > 0$. Moreover, we assume that one of the following conditions holds:*

- $\Omega_- = \emptyset$,
- $\Omega_- \subset\subset \Omega$ *or*
- $\Omega_+ \cup \Omega_0 \subset\subset \Omega$ *and* $\int_\Omega a(x)\psi_1^{p+1} > 0$, *where ψ_1 stands for a principal eigenfunction associated to the Steklov problem*

$$\begin{cases} -\Delta\varphi + \varphi = 0 & \text{in } \Omega, \\ \dfrac{\partial \varphi}{\partial n} = \lambda\varphi & \text{on } \partial\Omega. \end{cases}$$

Then there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$ the problem (6.1) does not have positive solutions.

Proof. Case $\Omega_- = \emptyset$. Let us assume the contrary. Let v_1 the unique solution to the problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = u^q & \text{on } \partial\Omega. \end{cases} \tag{6.4}$$

For $\lambda > 1$ the function $u_\lambda \lambda^{-1}$, where u_λ stands for a positive solution to (6.1), is a supersolution to the problem (6.4). Therefore $\lambda v_1 < u_\lambda$. On the other hand, by Propositions 2.23 and 2.22, we know that

$$0 = \lambda_1(-\Delta + 1 - a(x)u_\lambda^{p-1}, \mathcal{N} - \lambda u_\lambda^{q-1}) < \lambda_1(-\Delta + 1 - \lambda^{p-1}a(x)v_1^{p-1}, \mathcal{N}). \tag{6.5}$$

Since a is positive in a set with positive measure then

$$\lim_{\lambda \to +\infty} \lambda_1(-\Delta + 1 - \lambda^{p-1}a(x)v_1^{p-1}, \mathcal{N}) = -\infty,$$

which is a contradiction with (6.5).

Case $\Omega_- \subset\subset \Omega$. Assume that for all $\lambda > 0$ there exists a positive solution to (6.1). Let v_1 the unique positive solution to the problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega_+ \cup \Omega_0, \\ u = 0 & \text{on } \partial(\Omega_+ \cup \Omega_0) \cap \Omega, \\ \dfrac{\partial u}{\partial n} = u^q & \text{on } \partial\Omega. \end{cases} \tag{6.6}$$

For $\lambda > 1$ the function $u_\lambda \lambda^{-1}$, where u_λ denotes a positive solution to (6.1), is a supersolution to (6.6) in $\Omega_+ \cup \Omega_0$. Therefore $\lambda v_1 < u_\lambda$ in $\Omega_+ \cup \Omega_0$ for all $\lambda > 1$. We observe that

$$\begin{aligned} 0 = \lambda_1(-\Delta + 1 - a(x)u_\lambda^{p-1}, \mathcal{N} - \lambda u_\lambda^{q-1}) &< \lambda_1^{\Omega_+ \cup \Omega_0}(-\Delta + 1 - a(x)u_\lambda^{p-1}, \mathcal{D}, \mathcal{N} - \lambda u_\lambda^{q-1}) \\ &< \lambda_1^{\Omega_+ \cup \Omega_0}(-\Delta + 1 - \lambda^{p-1}a(x)v_1^{p-1}, \mathcal{D}, \mathcal{N}). \end{aligned} \tag{6.7}$$

On the other hand, since $|\Omega_+| > 0$ then

$$\lim_{\lambda \to +\infty} \lambda_1^{\Omega_+ \cup \Omega_0}(-\Delta + 1 - \lambda^{p-1}a(x)v_1^{p-1}, \mathcal{D}, \mathcal{N}) = -\infty,$$

which is a contradiction with (6.7).

Case $\Omega_+ \cup \Omega_0 \subset\subset \Omega$. Let us suppose that there exists a solution u_λ to (6.1) that exists for all $\lambda > 0$. We use, as in the previous chapter, the Picone identity (see for instance (5.27)).

$$\begin{aligned} &\int_\Omega \left(\frac{\psi_1}{u_\lambda}\right)^p [\psi_1(-\Delta u_\lambda + u_\lambda) - u_\lambda(-\Delta \psi_1 + \psi_1)] \\ &\quad = -\int_\Omega p \left(\frac{\psi_1}{u_\lambda}\right)^{p-1} u_\lambda^2 \left|\nabla \left(\frac{\psi_1}{u_\lambda}\right)\right|^2 - \int_{\partial\Omega} \left(\frac{\psi_1}{u_\lambda}\right)^p \left[\psi_1 \frac{\partial u_\lambda}{\partial n} - u_\lambda \frac{\partial \psi_1}{\partial n}\right]. \end{aligned}$$

Therefore,

$$\int_\Omega \psi_1^{p+1} a(x) = -\int_\Omega p \left(\frac{\psi_1}{u_\lambda}\right)^{p-1} u_\lambda^2 \left|\nabla \left(\frac{\psi_1}{u_\lambda}\right)\right|^2 - \lambda \int_{\partial\Omega} \psi_1^{p+1} u_\lambda^{q-p} + \int_{\partial\Omega} \psi_1^{p+1} \mu_1 u_\lambda^{1-p},$$

with μ_1 the eigenvalue associated to ψ_1. We know that $u_\lambda \geq u_{[\lambda,a_-]}$ where $u_{[\lambda,a_-]}$ is the unique positive solution to

$$\begin{cases} -\Delta u + u = a_-(x)u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda u^q & \text{on } \partial\Omega. \end{cases} \tag{6.8}$$

Since, by Theorem 6.10, we have

$$\lim_{\lambda \nearrow \infty} u_{[\lambda,a_-]} = z_{\mathcal{D},\Omega}$$

then, for all $\epsilon > 0$ there exists $\lambda_0 > 0$ such that $u_\lambda^{1-p}(x) < \epsilon$ for all $x \in \partial\Omega$ and $\lambda \geq \lambda_0$. Hence, for $\lambda \geq \lambda_0$ we obtain

$$\int_\Omega a(x)\psi_1^{p+1} < \epsilon,$$

a contradiction. ∎

Let

$$\Lambda := \sup\{\lambda \in \mathbb{R} \,:\, (6.1) \text{ has a positive solution}\}.$$

From now on, we assume that we are under the hypotheses of Lemma 6.12. Therefore, $\Lambda < +\infty$.

Lemma 6.13. *Assume that we are under the hypotheses of Lemma 6.12 then for all $\lambda \in (0, \Lambda)$ problem $(6.1)_\lambda$ has a solution.*

Proof. Let $\mu \in (\lambda, \Lambda)$ and u_μ a solution associated to $(6.1)_\mu$. Clearly $\overline{u} = u_\mu$ is a supersolution to $(6.1)_\lambda$. As a subsolution we can take $\underline{u} = u_{[\lambda,a_-]}$. By the uniqueness of solution to (6.8) we have $u_{[\lambda,a_-]} < u_\mu$. Hence, we conclude that there exists a solution u_λ to $(6.1)_\lambda$. ∎

Lemma 6.14. *Under the hypotheses of Lemma 6.12 the problem $(6.1)_\lambda$ has a minimal solution $u_{\lambda,*}$ for all $\lambda \in (0, \Lambda)$.*

Proof. From the previous lemma we know that $(\underline{u}, \overline{u}) = (u_{[\lambda,a_-]}, u_\mu)$ is a pair of sub-supersolutions then there exists a minimal solution $u_{\lambda,*}$ in $[u_{[\lambda,a_-]}, u_\mu]$. Moreover if u_λ is a solution to $(6.1)_\lambda$ then $u_\lambda \geq u_{[\lambda,a_-]}$ therefore $u_{\lambda,*}$ is the minimal solution to $(6.1)_\lambda$.

∎

Lemma 6.15. *Assume the hypotheses of Lemma 6.12 then for all $\lambda \in (0, \Lambda)$ the minimal solution $u_{\lambda,*}$ is weakly stable i.e.*

$$\sigma_1 := \lambda_1(-\Delta + 1 - pa(x)u_{\lambda,*}^{p-1}, \mathcal{N} - \lambda q u_{\lambda,*}^{q-1}) \geq 0.$$

Proof. Suppose that $\sigma_1 < 0$. Let ξ_1 a positive eigenfunction associated to σ_1. We claim that $u_{\lambda,*} - \alpha\xi_1$ is a strict supersolution to $(6.1)_\lambda$ for some $\alpha > 0$ sufficiently small. Indeed,

$$\begin{aligned} &-\Delta(u_{\lambda,*} - \alpha\xi_1) + (u_{\lambda,*} - \alpha\xi_1) - a(x)(u_{\lambda,*} - \alpha\xi_1)^p \\ &\quad = a(x)(u_{\lambda,*}^p - \alpha p u_{\lambda,*}^{p-1}\xi_1 - (u_{\lambda,*} - \alpha\xi_1)^p) - \alpha\sigma_1\xi_1 = a(x)(O(\alpha\xi_1)) - \alpha\sigma_1\xi_1 > 0. \end{aligned}$$

On the other hand, since $s\mapsto s^q$ is concave then

$$(u_{\lambda,*}-\alpha\xi_1)^q\le u_{\lambda,*}^q-\alpha q u_{\lambda,*}^{q-1}\xi_1$$

and hence

$$\begin{aligned}&\frac{\partial(u_{\lambda,*}-\alpha\xi_1)}{\partial n}-\lambda(u_{\lambda,*}-\alpha\xi_1)^q\\&=\lambda u_{\lambda,*}^q-\alpha\xi_1\lambda q u_{\lambda,*}^{q-1}-\lambda(u_{\lambda,*}-\alpha\xi_1)^q\ge 0\end{aligned}$$

We have that $u_{\lambda,*}-\alpha\xi_1$ is a strict supersolution to $(6.1)_\lambda$. Moreover it is possible to pick $\alpha>0$ such that $u_{\lambda,*}-\alpha\xi_1>u_{[\lambda,a_-]}$. Therefore, $(\overline{u},\underline{u})=(u_{\lambda,*}-\alpha\xi_1,u_{[\lambda,a_-]})$ are a pair of sub-supersolutions to $(6.1)_\lambda$. Hence, there exists a solution u_λ to $(6.1)_\lambda$ such that $u_\lambda<u_{\lambda,*}$, a contradiction. ■

In order to eliminate the possibility that the branch comes back to zero i.e. the possibility of a loop (note that u^q is not a differentiable in zero, so the local bifurcation Theorem of Crandall-Rabinowitz can not be applied here) the following lemma is required.

Lemma 6.16. *Assume that we are under the hypothesis of Lemma 6.12 then there exists a constant $\delta>0$ such that for all $\lambda\in(0,\Lambda)$ there exists at most a positive solution to (6.1) which also satisfies $\|u\|_\infty<\delta$.*

Proof. Suppose that there exists a second solution w then

$$w:=u_{\lambda,*}+v,$$

for some $v>0$ and $\|w\|_\infty<\delta$. We claim that for every $\lambda\in[0,\Lambda]$ there exists $\beta>0$ such that

(6.9) $$\lambda_1(-\Delta+1-pa_-u_{[\lambda,a_-]}^{p-1},\mathcal{N}-q\lambda u_{[\lambda,a_-]}^{q-1})>\beta.$$

Indeed, we define the function

$$\theta_\lambda:=\lambda^{\frac{1}{q-1}}u_{[\lambda,a_-]},$$

Having in mind that θ_λ is a solution to the problem

$$\begin{cases}-\Delta\theta_\lambda+\theta_\lambda=\lambda^{\frac{p-1}{1-q}}a_-(x)\theta_\lambda^p & \text{in } \Omega,\\ \dfrac{\partial\theta_\lambda}{\partial n}=\theta_\lambda^q & \text{on } \partial\Omega,\end{cases}$$

we get that

(6.10) $$\lim_{\lambda\to 0^+}\theta_\lambda=u \text{ in } \mathcal{C}^2(\overline{\Omega},$$

where u is the unique positive solution to

$$\begin{cases}-\Delta u+u=0 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n}=u^q & \text{on } \partial\Omega.\end{cases}$$

Now, we observe that there exists $\xi > 0$

$$0 = \lambda_1(-\Delta + 1, \mathcal{N} - u^{q-1}) < 2\xi < \lambda_1(-\Delta + 1, \mathcal{N} - qu^{q-1}),$$

By (6.10) and the continuity of the eigenvalue respect to perturbations we have that there exists $\lambda_m > 0$ such that

$$\lambda_1(-\Delta + 1 - pa_- u^{p-1}_{[\lambda,a_-]}, \mathcal{N} - q\lambda u^{q-1}_{[\lambda,a_-]}) > \xi > 0,$$

for all $\lambda \in [0, \lambda_m]$. On the other hand, since $u_{[\lambda,a_-]}$ is stable and the application

$$\sigma(\lambda) = \lambda_1(-\Delta + 1 - pa_- u^{p-1}_{[\lambda,a_-]}, \mathcal{N} - q\lambda u^{q-1}_{[\lambda,a_-]})$$

is analytic on $(0, +\infty)$ then

$$\sup_{\lambda\in[\lambda_m,\Lambda]} \sigma(\lambda) > \gamma > 0,$$

this proves the claim. We take $\delta > 0$ sufficiently small, in particular such that

$$a_+ p\delta^{p-1} < \beta.$$

Then, by (6.9) we have

$$\lambda_1(-\Delta + 1 - pa_- u^{p-1}_{[\lambda,a_-]} - a_+ p\delta^{p-1}, \mathcal{N} - \lambda q u^{q-1}_{[\lambda,a_-]}) > 0. \tag{6.11}$$

We claim that v is a positive strict subsolution to

$$(-\Delta + 1 - pa_- u^{p-1}_{[\lambda,a_-]} - a_+ p\delta^{p-1}, \mathcal{N} - \lambda q u^{q-1}_{[\lambda,a_-]}, \Omega)$$

which is a contraction with (6.11). Indeed, by the mean value Theorem, there exists $z \in (u_{\lambda,*}, u_{\lambda,*} + v)$ such that

$$a(x)((u_{\lambda,*} + v)^p - u^p_{\lambda,*}) = a(x)pz^{p-1}v,$$

and therefore, taking into account that $u_{[\lambda,a_-]} < u_{\lambda,*}$, we have

$$\begin{aligned}
&(-\Delta + 1 - pa_- u^{p-1}_{[\lambda,a_-]} - a_+ p\delta^{p-1})v \\
&\quad = a(x)((u_{\lambda,*} + v)^p - u^p_{\lambda,*}) + (-a_+ p\delta^{p-1} - pa_- u^{p-1}_{[\lambda,a_-]})v \\
&\quad = \left(a_+(pz^{p-1} - p\delta^{p-1}) + a_-(pz^{p-1} - pu^{p-1}_{[\lambda,a_-]})\right) v < 0 \text{ in } \Omega.
\end{aligned}$$

By the concavity of $s \mapsto s^q$

$$(u_{\lambda,*} + v)^q \leq u^q_{\lambda,*} + qu^{q-1}_{\lambda,*} v,$$

and hence

$$\begin{aligned}
\left(\frac{\partial}{\partial n} - \lambda q u^{q-1}_{[\lambda,a_-]}\right) v &= \lambda(u_{\lambda,*} + v)^q - \lambda u^q_{\lambda,*} - \lambda q u^{q-1}_{[\lambda,a_-]} v \\
&\leq \lambda q u^{q-1}_{\lambda,*} v - \lambda q u^{q-1}_{[\lambda,a_-]} v \leq 0 \text{ on } \partial\Omega.
\end{aligned}$$

■

In what follows we state, for the reader's convenience, the adaptation of Lemma 5.15 to our case

Lemma 6.17. *Let* (λ_0, u_0) *be a positive solution to (6.1) with* $\lambda = \lambda_0$.

a) If

$$\lambda_1(-\Delta + 1 - pa(x)u_0^{p-1}, \mathcal{N} - q\lambda_0 u_0^{q-1}) > 0, \tag{6.12}$$

then, there exists $\varepsilon > 0$ *and a differentiable mapping* $u : I = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \mapsto P$ *such that* $u(\lambda_0) = u_0$ *and* $(\lambda, u(\lambda))$ *is a positive solution to (5.1) for each* $\lambda \in I$. *Moreover, the mapping* $\lambda \mapsto u(\lambda)$ *is increasing and there exists a neighborhood* $\mathcal{V}$ *of* (λ_0, u_0) *in* $\mathbb{R} \times P$ *such that if* $(\lambda, u) \in \mathcal{V}$ *is a solution to (6.1), then* $(\lambda, u) = (\lambda, u(\lambda))$ *for some* $\lambda \in I$.

b) If

$$\lambda_1(-\Delta + 1 - pa(x)u_0^{p-1}, \mathcal{N} - q\lambda_0 u_0^{q-1}) = 0, \tag{6.13}$$

let Φ_0 *be the principal eigenfunction associated with* $\lambda_1(-\Delta - pa(x)u_0^{p-1}, \mathcal{N} - q\lambda_0 u_0^{q-1})$. *Then, there exists* $\varepsilon > 0$ *and a twice continuously differentiable mapping* $(\lambda, u) : J = (-\varepsilon, \varepsilon) \mapsto \mathbb{R} \times P$ *such that* $(\lambda(0), u(0)) = (\lambda_0, u_0)$ *and for each* $s \in J$, $(\lambda(s), u(s))$ *is a positive solution to (6.1). Moreover,* $\lambda'(0) = 0$, $u(s) = u_0 + s(\Phi_0 + v(s))$ *where* $v \in \mathcal{C}^1((-\varepsilon, \varepsilon), C(\overline{\Omega}))$ *satisfies* $v(0) = 0$, *and finally*

$$\lambda''(0) = \frac{\displaystyle\int_\Omega p(1-p)a(x)u_0^{p-2}\Phi_0^3 + \int_{\partial\Omega} \lambda_0 q(1-q)u_0^{q-2}\Phi_0^3}{\displaystyle\int_{\partial\Omega} u_0^q \Phi_0}, \tag{6.14}$$

for $s \simeq 0$. *In addition, there exists a neighborhood* $\mathcal{W}$ *of* (λ_0, u_0) *in* $\mathbb{R} \times P$ *such that if* $(\lambda, u) \in \mathcal{W}$ *is a solution to (6.1), then* $(\lambda, u) = (\lambda(s), u(s))$ *for some* $s \in J$. *Also,*

$$\operatorname{sgn} \lambda'(s) = \operatorname{sgn} \lambda_1(-\Delta - pa(x)u(s)^{p-1}, \mathcal{N} - q\lambda(s)u(s)^{q-1}). \tag{6.15}$$

As a consequence of the Picone identity we have

Proposition 6.18. *Let* (λ_0, u_0) *be a positive solution to (6.1) with* $\lambda = \lambda_0$, *such that (6.13) holds. Then* $\lambda''(0) < 0$, *where* $\lambda''(0)$ *is defined in (6.14).*

Proof. The Picone's identity (5.27) with $\Upsilon(t) = t^2$, $v = \Phi_0$, $u = u_0$ and $\mathcal{L} = -\Delta + I$, entails

$$\int_\Omega (1-p)a(x)u_0^{p-2}\Phi_0^3 \leq -\int_{\partial\Omega} \lambda_0(1-q)u_0^{q-2}\Phi_0^3 \tag{6.16}$$

From (6.16) and as $p > 1 > q$ we deduce that $\lambda''(0) < 0$. This concludes the proof.

■

Theorem 6.19. *Suppose that we are under the hypothesis of Lemma 6.12. Additionally we assume that one of the following conditions is satisfied:*

- $\Omega_- \cup \Omega_0 = \emptyset$, $1 < p < \dfrac{d+2}{d-2}$ *or*
- *There exists a function* $\alpha : \Omega_+ \to \mathbb{R}^+$ *which is continuous and bounded away from zero and a constant* $\gamma \geq 0$ *such that*

$$a(x) = \alpha(x)\left(dist(x, \partial\Omega_+)\right)^{\gamma}, \quad in\ \Omega_+,$$

and

$$1 < p < \min\left\{\frac{d+1+\gamma}{d-1}, \frac{d+2}{d-2}\right\}.$$

Then,

- $\mathcal{P}_\lambda(\mathcal{C}_0) = (-\infty, \Lambda]$, *where* $\mathcal{P}_\lambda(\mathcal{C}_0)$ *stands for the projection over the* λ*-axis of* $\mathcal{C}_0$.
- *There exist at least two positive solutions to (6.1) in* $(0, \Lambda)$.
- *There exists a unique positive solution to (6.1) which is stable in* $(0, \Lambda)$.

Proof. The proof is exactly as in Theorem 5.12. ■

6.4. Case $0 < p, q < 1$

Theorem 6.20. *Assume* $a \equiv 1$. *Then, there exists at least a positive solution to (6.1) for all* $\lambda \in \mathbb{R}$. *Moreover, if* $p \leq q$ *then the solution is unique. Furthermore, if* $p > q$ *then the solution is unique for* $\lambda \geq 0$.

Proof. First we prove the existence of positive solutions to (6.1) by the method of sub-supersolutions. Let φ_1 a positive eigenfunction associated to $\lambda_1(-\Delta + 1, \mathcal{D})$ such that $\|\varphi_1\|_\infty = 1$. The function $\underline{u} := \epsilon\varphi_1$ is a subsolution to (6.1) if

$$\epsilon \leq \lambda_1(-\Delta + 1, \mathcal{D})^{-\frac{1}{1-p}}, \qquad \epsilon\frac{\partial\varphi_1}{\partial n} \leq 0.$$

Observe that the second inequality holds by the Hoph Lemma. On the other hand, the function $\overline{u} := Mz$, where z is the unique positive solution to the problem

$$\begin{cases} -\Delta z + z = 1 & \text{in } \Omega, \\ \dfrac{\partial z}{\partial n} = 1 & \text{on } \partial\Omega, \end{cases}$$

is a supersolution to (6.1) provided that

$$M \geq \|z\|_\infty^{\frac{1}{1-p}}, \qquad M \geq (|\lambda|\|z\|_\infty^q)^{\frac{1}{1-q}}.$$

It is possible to pick M satisfying $\underline{u} \leq \overline{u}$ and also the previous inequalities. Now, we deal with the uniqueness. Suppose that $p \leq q$ then it is enough to apply the Theorem 3.5 with $g(s) = s^q$. Let us assume now that $\lambda \geq 0$ and $p > q$, in this case we can apply Theorem 3.5 with $g(s) = s^p$. ■

Theorem 6.21. *Assume $a \equiv -1$. The problem (6.1) does not have a positive solution if $\lambda \leq 0$. For λ large enough the problem (6.1) have a positive solution. Moreover, such a solution is unique if $q \leq p$.*

Proof. If $\lambda \leq 0$ then, by the maximum principle, every solution to (6.1) satisfies $u \leq 0$. For the existence of positive solutions we use the method of sub-supersolutions. We know that there exists $\overline{\lambda} > 0$ such that

$$\mu(\overline{\lambda}) := \lambda_1(-\Delta + 1, \mathcal{N} - \overline{\lambda}) < 0.$$

Let φ_1 a positive eigenfunction associated to $\mu(\overline{\lambda})$ with $\|\varphi_1\|_\infty = 1$. The function $\underline{u} := \epsilon\varphi_1$ is a subsolution to (6.1) provided that

$$\mu(\overline{\lambda})(\epsilon c_L)^{1-p} \leq -1\,, \qquad \epsilon \leq \left(\frac{\lambda}{\overline{\lambda}}\right)^{\frac{1}{1-q}},$$

where $0 < c_L = \min_{x\in\overline{\Omega}} \varphi_1$. Therefore, if λ large enough, it is possible to pick ϵ such that the previous inequalities are satisfied simultaneously. On the other, the function $\overline{u} = M\varphi_1$, where φ_1 stands for a positive eigenfunction associated to $\mu(\mu_1) = 0$ such that $\|\varphi_1\|_\infty = 1$ is a supersolution to (6.1) if

$$0 \geq -(M\varphi_1)^{p-1}\,, \qquad M \geq c_L^{-1}\left(\frac{\lambda}{\mu_1}\right)^{\frac{1}{1-q}}.$$

Moreover, picking M large enough it is also satisfied that $\underline{u} \leq \overline{u}$. For the uniqueness if $q \leq p$ then it is enough to apply Theorem 3.5 with $g(s) = s^p$. ■

CHAPTER 7

An angiogenesis model with nonlinear chemotactic response and flux at the tumor boundary

In this chapter we consider a parabolic problem as well as its stationary counterpart of a model arising in angiogenesis. The problem includes a chemotaxis type term and a nonlinear boundary condition at the tumor boundary. We show that the parabolic problem admits a unique global in time solution. Moreover, under some restrictions, we show the convergence to the steady-states even with explicit rate of convergence. Furthermore, by bifurcation methods, we show the existence of coexistence states and also we study the local stability of the semi-trivial states. The results of this chapter, except the part of convergence to steady-states, have been submitted for publication in [37].

7.1. Introduction

In this chapter we analyze a system modelling a crucial step in the tumor growth process: the angiogenesis. We only focus our attention on the behavior of two populations involved in such process: the endothelial cells (ECs), denoted by u, which move and reproduce to generate a new vascular net attracted by a chemical substance generated by the tumor (TAF), denoted by v. They interact in a region $\Omega \subset \mathbb{R}^d$, $d \geq 1$, that is assumed to be bounded and connected and with a regular boundary $\partial\Omega$. Specifically, we consider the case

$$\partial\Omega = \Gamma_1 \cup \Gamma_2,$$

with $\Gamma_1 \cap \Gamma_2 = \emptyset$, being Γ_i closed and open in the relative topology of $\partial\Omega$. We assume that Γ_2 is the tumor boundary and Γ_1 is the blood vessel boundary. Hence, we study the following parabolic problem and its stationary counterpart

$$\text{(7.1)} \qquad \begin{cases} u_t - \Delta u = -\mathrm{div}(V(u)\nabla v) + \lambda u - u^2 & \text{in } \Omega \times (0,T), \\ v_t - \Delta v = -v - cuv & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_1 \times (0,T), \\ \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial v}{\partial n} = \mu \dfrac{v}{1+v} & \text{on } \Gamma_2 \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where $0 < T \leq +\infty$, $\lambda, \mu \in \mathbb{R}$, $c > 0$ and

$$\text{(7.2)} \qquad V \in \mathcal{C}^1(\mathbb{R}), \quad V > 0 \text{ in } (0,\infty) \text{ with } V(0) = 0;$$

and u_0 and v_0 are non-negative and non-trivial given functions.

The structure of this chapter is as follows. In Section 2, we introduce some results of functional analysis. In Section 3, we prove the existence of local solution in time and then the global existence in time. The next section is devoted to the stationary problem and also to the local stability of the semi-trivial solutions. In the fifth section we study the global stability of the solution $(\lambda, 0)$. Finally, in the last section, we briefly discuss some biological implications of our results.

7.2. Some results of Functional Analysis

We remind some facts about Sobolev spaces, the interpolation theory and the functional setting which is used in this chapter for reader's convenience. This frame is a particular case of [7].

1) For Sobolev spaces with non-integer index, the so-called the Slobodeckii spaces, it holds

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded $\mathcal{C}^2$-domain.*

a) ([7], pg. 25) If s_0, $s_1 \in \mathbb{R}^+ \setminus \mathbb{N}$, $s_1 \leq s_0$, then

$$W^{s_0,p}(\Omega) \hookrightarrow W^{s_1,p}(\Omega) \ \ \forall p \in [1,\infty].$$

b) ([1]. Th. 7.58) Let $s_0 > 0$, $1 < p < q < \infty$ and $s_1 = s_0 - \frac{d}{p} + \frac{d}{q}$. If $s_1 \geq 0$, then

$$W^{s_0,p}(\Omega) \hookrightarrow W^{s_1,q}(\Omega).$$

2) With respect to the interpolation theory, we remind that if E_0 and E_1 are two normed spaces continuously embedded into a topological space $\mathcal{E}$, we can defined the real interpolation for $0 < \theta < 1$ and $1 \leq p \leq \infty$, which we denote $(E_0, E_1)_{\theta,p}$ (see [95], Def. 22.1). It is true that

a) $(E_0, E_1)_{\theta,p} \subset E_0 + E_1$.

b) $E_0 \cap E_1 \hookrightarrow (E_0, E_1)_{\theta,p}$.

c) ([95] Lemma 22.2) If $0 < \theta < 1$ and $1 \le p \le q \le \infty$, then

$$(E_0, E_1)_{\theta,p} \hookrightarrow (E_0, E_1)_{\theta,q}.$$

Finally, due to $(E_0, E_1)_{\theta,1} \subset (E_0, E_1)_{\theta,p} \subset E_0 + E_1$, it follows from [95], Lemma 25.2 ii) that if $(E_0, E_1)_{\theta,p}$ is a Banach space, then

$$\exists C > 0, \text{ such that } \|a\|_{(E_0,E_1)_{\theta,p}} \le C\|a\|_{E_0}^{1-\theta} \|a\|_{E_1}^{\theta} \quad \forall a \in E_0 \cap E_1. \tag{7.3}$$

3) Consider now a parabolic problem in the form

$$\begin{cases} z_t + \mathcal{A}z = F(z) & \text{in } \Omega \times (0,T), \\ \mathcal{B}z = G(z) & \text{on } \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) & \text{in } \Omega, \end{cases} \tag{7.4}$$

where

$$\mathcal{A}z := -\sum_{j,k=1}^{d} \partial_j(a_{jk}\partial_k z + a_j z) + b_j\partial_j z + a_0 z$$

is a general second order lineal operator in divergence form with $\mathcal{L}(\mathbb{R}^r)$-valued coefficients which we suppose continuous on $\bar{\Omega}$, and

$$\mathcal{B}z := \sum_{j,k=1}^{d} n_j\gamma(a_{jk}\partial_k z + a_j z) + c\gamma(z)$$

is a boundary operator where γ denotes the trace operator and c is continuous on $\partial\Omega$. Here $\Omega \subset \mathbb{R}^d$ is a bounded regular domain and r is the number of equations of (7.4). We know that $(\mathcal{A}, \mathcal{B})$ is normally elliptic (see the definition of this concept in [7], pg. 18) since there exists $a \in \mathcal{C}(\overline{\Omega}, \mathcal{L}(\mathbb{R}^r))$ such that

$$a_{jk} = a\,\delta_{jk} \text{ (the Kronecker symbol)}, \quad 1 \le j,k \le d,$$

the spectrum of a verifies

$$Re[\sigma(a(z))] > 0 \quad \forall z \in \bar{\Omega},$$

and the boundary condition is a Neumann boundary condition of each component of $\partial\Omega$ (see [7] pg. 21).

The interpolation theory will be used in this chapter for $r = 1$, that is, only for one equation. We write the results in this case. If we denote

$$W_{\mathcal{B}}^{s,\gamma} := \begin{cases} \{z \in W^{s,\gamma}(\Omega) : \ \mathcal{B}z = 0\} & \text{if } 1 + \frac{1}{\gamma} < s \le 2 \\ W^{s,\gamma}(\Omega) & \text{if } -1 + 1/\gamma < s < 1 + 1/\gamma \\ (W^{-s,\gamma'}(\Omega))' & \text{if } -2 + 1/\gamma < s \le -1 + 1/\gamma \end{cases}, \tag{7.5}$$

then it holds

Theorem 7.2. *([7], Th. 5.2; Th. 7.2) Suppose that $(\mathcal{A}, \mathcal{B})$ is a normally elliptic Neumann problem on Ω with $\mathcal{C}^1$-coefficients, $1 < p < \infty$ and $0 < \theta < 1$. Then* *a)*

$$(L^p, W^{2,p}_{\mathcal{B}})_{\theta,p} = W^{2\theta,p}_{\mathcal{B}}, \quad 2\theta \in (0,2) \setminus \{1, 1+1/p\}.$$

b) If $-2+1/p < s_0 < s_1 < 1+1/p$, *and* $s_\theta := (1-\theta)s_0 + \theta s_1 \notin \mathbb{N}$, *then,*

$$(W^{s_0,p}_{\mathcal{B}}, W^{s_1,p}_{\mathcal{B}})_{\theta,p} = W^{s_\theta,p}_{\mathcal{B}}.$$

When $(\mathcal{A},\mathcal{B})$ is a normally elliptic problem on Ω with $\mathcal{C}^1$-coefficients, it is possible to construct an interpolation-extrapolation scale of spaces; we put $E_0 = L^p$, $E_1 = W^{2,p}_{\mathcal{B}}$, and E_{-1} a completion of E_0 (see [7], pg. 29), we define

$$E_\theta = (E_0, E_1)_{\theta,p} = W^{2\theta}_{p,\mathcal{B}} \quad \text{for} \quad 2\theta \in (0,2) \setminus \{1, 1+1/p\}, \tag{7.6}$$

and we can extend the definition inductively for $E_{k+\theta}$. Then, there exists a family of operators, $A_\theta \in \mathcal{L}(E_{1+\theta}, E_\theta)$, being A_θ the negative infinitesimal generator of an analytic semigroup on E_θ. The semigroup e^{-tA_θ} is defined $e^{-tA_\theta} : E_\theta \to E_\theta$ (see [7], pg. 28-30).

7.3. The parabolic problem

7.3.1. Local existence

We are interested in the positive solutions of the system of PDEs (7.1), where c is a positive constant and $\lambda, \mu \in \mathbb{R}$.
First, we will get the existence and uniqueness of a local positive solution. This will be proved by using the abstract theory of quasilinear parabolic problems done in [7]. After that, in the next section, we will study the global existence.

Theorem 7.3. *Let* $p > d$ *and suppose (7.2), that the initial data* $(u_0, v_0) \in (W^{1,p}(\Omega))^2$ *and* $u_0 \geq 0$, $v_0 \geq 0$ *a.e. in* Ω. *Then, problem (7.1) has a unique non-negative local in time classical solution*

$$(u,v) \in (\mathcal{C}([0,T_{max}); W^{1,p}(\Omega)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T_{max})))^2,$$

where T_{max} *denotes the maximal existence time. Moreover, if there exists a function* $w : (0,+\infty) \to (0,+\infty)$ *such that, for each* $T > 0$,

$$\|(u(t), v(t))\|_\infty \leq w(T), \qquad 0 < t < \min\{T, T_{max}\}, \tag{7.7}$$

then $T_{max} = +\infty$.

Proof. We will prove that problem (7.1) is included in the frame of [7]. Let $\delta_0 > 0$ be and denote $D_0 := (-\delta_0, +\infty) \times (-\delta_0, +\infty)$ which is an open set containing the range of the solutions u and v. For $r = 2$ (number of equations) we define r^2 functions $a_{jk} \in \mathcal{C}^2(D_0; \mathcal{L}(\mathbb{R}^2))$ for $1 \leq j,k \leq r$ in the following way. For each $(\eta_1, \eta_2) \in D_0$,

$$a_{jk}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if } j \neq k; \qquad a_{jk}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -V(\eta_2) & 1 \end{pmatrix} \quad \text{if } j = k;$$

then, we put

$$\mathcal{A}\begin{pmatrix} v \\ u \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix} := -\sum_{j,k=1}^{d} \partial_j \left(a_{jk}\begin{pmatrix} v \\ u \end{pmatrix}\begin{pmatrix} \partial_k v \\ \partial_k u \end{pmatrix}\right) = -\begin{pmatrix} \operatorname{div}(\nabla v) \\ \operatorname{div}(\nabla u - V(u)\nabla v) \end{pmatrix}.$$

For the boundary conditions, we denote (c_{ij}) $1 \le i,j \le 2$ the following function matrix:

$$c_{11}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{cases} 0 & \text{on } \Gamma_1 \\ -\dfrac{\mu}{1+\eta_1} & \text{on } \Gamma_2 \end{cases} \qquad c_{12}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{cases} 0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \end{cases}$$

$$c_{21}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{cases} 0 & \text{on } \Gamma_1 \\ \dfrac{\mu V(\eta_2)}{1+\eta_1} & \text{on } \Gamma_2 \end{cases} \qquad c_{22}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{cases} 0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \end{cases}.$$

Then, we write

$$\mathcal{B}\begin{pmatrix} v \\ u \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix} := \left[\sum_{j=1}^{d} n_j \gamma \left(a_{jk}\begin{pmatrix} v \\ u \end{pmatrix}\begin{pmatrix} \partial_k v \\ \partial_k u \end{pmatrix}\right) + \gamma\left(\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix}\right)\right] =$$

$$\begin{pmatrix} \dfrac{\partial v}{\partial n} - \dfrac{\mu v}{1+v} \\ -V(u)\dfrac{\partial v}{\partial n} + \dfrac{\partial u}{\partial n} + \dfrac{\mu V(u) v}{1+v} \end{pmatrix}.$$

This couple $(\mathcal{A},\mathcal{B})$ is a linear boundary value problem normally elliptic.

For the reaction term, we define the function $f \in \mathcal{C}^2(D_0;\mathbb{R}^2)$ by

$$f\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_1 - c\eta_1\eta_2 \\ \lambda\eta_2 - \eta_2^2 \end{pmatrix}.$$

Then, (7.1) can be written as the following quasilinear parabolic boundary value problem

$$\begin{cases} \partial_t\begin{pmatrix} v \\ u \end{pmatrix} + \mathcal{A}\begin{pmatrix} v \\ u \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix} = f\begin{pmatrix} v \\ u \end{pmatrix} & \text{in } \Omega \times (0,\infty), \\ \mathcal{B}\begin{pmatrix} v \\ u \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } \partial\Omega \times (0,\infty), \\ \begin{pmatrix} v(x,0) \\ u(x,0) \end{pmatrix} = \begin{pmatrix} v_0(x) \\ u_0(x) \end{pmatrix} & \text{on } \Omega. \end{cases} \tag{7.8}$$

Then, Theorems 14.4 and 14.6 of [7] are applicable. The first one says that there exists a unique maximal weak $W^{1,p}$-solution which is a $W^{1,q}$-solution for each $q \in (1,\infty)$ ([7], Coroll. 14.5). The second one asserts that the solution is a classical solution and the equation is verified point-wise because the boundary operator is equal to 0.

The non-negativity of the solution follows from [7], Theorem 15.1. In fact, the hypothesis (15.3) is verified for $r = 2$ because $V(0) = 0$ and, so, the non-negativity of u holds. But, if $u \geq 0$, then the maximum principle applied to the problem

$$\left\{\begin{array}{ll} v_t - \Delta v = -v - cuv & \text{in } \Omega \times (0,T), \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_1 \times (0,T), \\ \dfrac{\partial v}{\partial n} = \mu \dfrac{v}{1+v} & \text{on } \Gamma_2 \times (0,T), \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{array}\right. \tag{7.9}$$

implies that $v \geq 0$.

To reach the result about the global solution, we can invoke [7], Theorem 15.5 which is applicable because $(\mathcal{A}, \mathcal{B})$ is a lower triangular system. For these systems and if a_{jk} is a diagonal matrix, f is independent of the gradient and if there exists a function $w : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|(u(t), v(t))\|_\infty \leq w(T), \;\; 0 \leq t \leq T < \infty, \;\; t < T_{max},$$

then $T_{max} = \infty$, supposed that the solution is bounded away from $\partial\Omega$ for each $T > 0$.

■

7.3.2. Global existence

In this subsection we are going to prove that the local positive solution (u, v) of the system (7.1) exists for every $t > 0$, i.e., it is a global solution. For that, in the following lemmas our purpose is to prove the criterium (7.7). Since we are prolonging the local solutions obtained in the previous section, during this section we assume that

$$(u_0, v_0) \in (W^{1,p}(\Omega))^2,$$

for some $p > d$. In order to get uniform bounds in time, that will also exclude the possibility of blow-up at infinity, we need to assume

$$V \in L^\infty(\mathbb{R}). \tag{7.10}$$

The proof of the uniform bounds in time is carried out by a recursive procedure (see for instance [64], where a similar argument is used.)

From now, C will denote a constant that may vary from line to line and does not depend on time.

The following lemma states that v is uniformly bounded in time via the well-known method of sub and supersolutions (see for example [86]).

Lemma 7.4. *There exists a constant $C > 0$ such that the v-solutions of (7.1) satisfy*

$$\|v(t)\|_{C(\overline{\Omega})} \leq C, \;\; \forall t \in [0, T_{max}).$$

Proof. Observe that the v-solutions of the problem (7.1) are subsolutions of

$$\begin{cases} w_t - \Delta w = -w & \text{in} \quad \Omega \times (0, T_{max}), \\ \dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \Gamma_1 \times (0, T_{max}), \\ \dfrac{\partial w}{\partial n} = |\mu| & \text{on} \quad \Gamma_2 \times (0, T_{max}), \\ v(x,0) = v_0(x) & \text{in} \quad \Omega. \end{cases} \tag{7.11}$$

Now, let φ be the solution of the stationary problem

$$\begin{cases} -\Delta \varphi + \varphi = 0 \text{ in } \Omega \\ \dfrac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma_1, \ \dfrac{\partial \varphi}{\partial n} = |\mu| \text{ on } \Gamma_2. \end{cases}$$

It is well known that $\varphi > 0$ in $\overline{\Omega}$. Taking $K > 0$ big enough, $K\varphi$ is a supersolution of (7.11). Therefore, $v \leq w \leq K\varphi$ and the conclusion easily follows. ■

Lemma 7.5. *The solution of the u-equation of (7.1) satisfies*

$$\|u(t)\|_2 \leq C, \quad \forall t \in [0, T_{max}),$$

where C a positive constant.

Proof. Multiplying the v-equation by v and integrating in Ω, we obtain

$$\frac{d}{2dt}\int_\Omega v^2 + \int_\Omega v^2 + \int_\Omega |\nabla v|^2 = \int_{\Gamma_2} \frac{\mu v^2}{1+v} - \int_\Omega cuv^2.$$

Adding the term $-\dfrac{C\varepsilon}{2}\displaystyle\int_\Omega v^2$ on both sides of the equality, taking into account Lemma 7.4 and multiplying the before equality by $e^{(2-C\varepsilon)t}$, we have

$$\frac{d}{dt}\left(e^{(2-C\varepsilon)t}\int_\Omega v^2\right) + 2e^{(2-C\varepsilon)t}\int_\Omega |\nabla v|^2 \leq 2|\mu|C|\Gamma_2|e^{(2-C\varepsilon)t},$$

where $|\Gamma_2|$ denotes the $d-1$ dimensional Lebesgue measure of Γ_2. Therefore, integrating in $(0,t)$ and multiplying by $e^{-(2-C\varepsilon)t}$ we get

$$\begin{aligned} &\int_\Omega v^2(t) + e^{-(2-C\varepsilon)t} 2\int_0^t (e^{(2-C\varepsilon)s}\int_\Omega |\nabla v|^2) ds \leq \\ &\left(2|\mu|C|\partial\Omega|\int_0^t e^{(2-C\varepsilon)s} ds\right) e^{-(2-C\varepsilon)t} + \int_\Omega v_0^2 e^{-(2-C\varepsilon)t}. \end{aligned} \tag{7.12}$$

In particular, from (7.12) we obtain

$$e^{-(2-C\varepsilon)t}\int_0^t (e^{(2-C\varepsilon)s}\int_\Omega |\nabla v|^2) ds \leq C(\|v_0\|_2). \tag{7.13}$$

Next, we multiply the u-equation by u and we integrate in the space variable

$$\frac{d}{dt}\int_\Omega u^2 = -2\int_\Omega |\nabla u|^2 + 2\int_\Omega V(u)\nabla u \cdot \nabla v - 2\int_{\Gamma_2} V(u)\frac{\mu uv}{1+v} + \int_\Omega 2\lambda u^2 - 2\int_\Omega u^3.$$

Adding $2\int_\Omega u^2$ on both sides of the equality we have

$$\frac{d}{dt}\int_\Omega u^2+2\int_\Omega u^2=-2\int_\Omega|\nabla u|^2+2\int_\Omega V(u)\nabla u\cdot\nabla v-2\int_{\Gamma_2}V(u)\frac{\mu uv}{1+v}+2\int_\Omega((\lambda+1)u^2-u^3).$$

Owing to $|V(s)|\le C$ for almost every $s\in\mathbb{R}$ and the inequality $(\lambda+1)s^2-s^3\le C=C(\lambda)$ for all $s\ge 0$, we deduce

$$\frac{d}{dt}\int_\Omega u^2+2\int_\Omega u^2\le-2\int_\Omega|\nabla u|^2+C\int_\Omega 2|\nabla u|\,|\nabla v|+C\int_{\Gamma_2}|\mu|\frac{uv}{1+v}+2C|\Omega|. \tag{7.14}$$

The boundary term is bounded by the following expressions

$$C|\mu|\int_{\Gamma_2}\frac{uv}{1+v}\le C|\mu|\int_{\Gamma_2}u\le\varepsilon\int_{\Gamma_2}u^2+C(\varepsilon),$$

using that for every $\varepsilon>0$ there exists $C(\varepsilon)>0$ such that $s\le\varepsilon s^2+C(\varepsilon)$ for every $s\in\mathbb{R}$. Moreover,

$$\varepsilon\int_{\Gamma_2}u^2\le C\left(\varepsilon\int_\Omega u^2+\varepsilon\int_\Omega|\nabla u|^2\right).$$

Taking account the previous estimates in (7.14) we get

$$\frac{d}{dt}\int_\Omega u^2+(2-C\varepsilon)\int_\Omega u^2\le(C\varepsilon-2)\int_\Omega|\nabla u|^2+C\int_\Omega 2|\nabla u|\,|\nabla v|+C(\varepsilon).$$

Then,

$$\frac{d}{dt}\int_\Omega u^2+(2-C\varepsilon)\int_\Omega u^2\le((C+1)\epsilon-2)\int_\Omega|\nabla u|^2+C(\epsilon)\int_\Omega|\nabla v|^2+C(\varepsilon).$$

Multiplying this inequality by $e^{(2-C\varepsilon)t}$ we have

$$\frac{d}{dt}\left(e^{(2-C\varepsilon)t}\int_\Omega u^2\right)\le C(\varepsilon)e^{(2-C\varepsilon)t}\int_\Omega|\nabla v|^2+Ce^{(2-C\varepsilon)t},$$

and integrating in $(0,t)$ we obtain

$$e^{(2-C\varepsilon)t}\int_\Omega u^2-\int_\Omega u_0^2\le C(\varepsilon)\int_0^t(e^{(2-C\varepsilon)s}\int_\Omega|\nabla v|^2)ds+C\int_0^t e^{(2-C\varepsilon)s}ds.$$

At this point, thanks to (7.13),

$$\int_\Omega u^2\le\|u_0\|_2^2e^{-(2-C\varepsilon)t}+C(\|v_0\|_2)+C$$

and the Lemma is easily concluded. ∎

Lemma 7.6. *Let $\gamma\in(1,\infty)$, and $t\in[t_0,T_{max})$ for $t_0>0$ small enough. If $\|u(t)\|_\gamma\le C$, then*

$$\|v(t)\|_{W^{1,p(\gamma,d)}}\le C,$$

where

$$p(\gamma,d)=\gamma\,\frac{d}{d-1+\varepsilon\gamma}\quad\forall\varepsilon>0.$$

Proof. System (7.8) has a local classical solution $(u(t), v(t))$ defined in $(0, T_{max})$. So, we pose the nonhomogeneous linear problem

$$
\text{(7.15)} \qquad \begin{cases} v_t - \Delta v + v = f(t) := -cu(t)v(t) & \text{in } \Omega \times (0, T_{max}) \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_1 \times (0, T_{max}), \\ \dfrac{\partial v}{\partial n} = g(t) := \mu \dfrac{v(t)}{1+v(t)} & \text{on } \Gamma_2 \times (0, T_{max}), \\ v(x,0) = v_0(x), & \text{in } \Omega, \end{cases}
$$

which we simply denote

$$
\text{(7.16)} \qquad \begin{cases} v_t - \mathcal{A}_0 v = f(t) & \text{in } \Omega \times (0, T_{max}) \\ \mathcal{B}_0 v = g(t) & \text{on } \Gamma \times (0, T_{max}), \\ v(x,0) = v_0(x), & \text{in } \Omega, \end{cases}
$$

$(\mathcal{A}_0, \mathcal{B}_0)$ generates an analytical semigroup whose generator is $A_0 \in \mathcal{L}(L^\gamma(\Omega))$ with $\operatorname{dom} A_0 = ker\, \mathcal{B}_0$. The operator $\mathcal{B}_0 \rfloor_{Ker\, \mathcal{A}_0} : Ker\, \mathcal{A}_0 \to W^{1-1/\gamma,\gamma}(\partial\Omega)$ is an homeomorphism and we denote $\mathcal{B}_0^c$ its inverse operator. The generalized variation-of-constants formula gives, for $2\alpha \in (1/\gamma, 1 + 1/\gamma)$,

$$
\text{(7.17)} \qquad v(t) = e^{-tA_{\alpha-1}} v_0 + \int_0^t e^{-(t-\tau)A_{\alpha-1}} (f(\tau) + A_{\alpha-1}\mathcal{B}_0^c g(\tau))\, d\tau,
$$

and (7.17) is well defined for $(f,g) \in \mathcal{C}((0,T_{max}), W^{2\alpha-2}_{\gamma,\mathcal{B}_0} \times \partial W^{2\alpha}_\gamma)$ ([7], pg. 63). Note that it follows from (7.8) of [7] and (7.5) that for $-2 \le s < 0$,

$$
\text{(7.18)} \qquad L^\gamma(\Omega) \hookrightarrow W^{s,\gamma}_{\mathcal{B}_0}.
$$

We choose $\beta := 1 + \frac{1}{\gamma} - \varepsilon < 2\alpha < 1 + \frac{1}{\gamma}$. Owing to Theorem 7.2, there exists some $0 < \theta < 1$ such that

$$
(W^{2\alpha-2,\gamma}_{\mathcal{B}_0}, W^{2\alpha,\gamma}_{\mathcal{B}_0})_{\theta,\gamma} = W^{\beta,\gamma}_{\mathcal{B}_0} = W^{\beta,\gamma}.
$$

So, by (7.3), it holds that

$$
\forall w \in W^{2\alpha-2,\gamma}_{\mathcal{B}_0} \cap W^{2\alpha,\gamma}_{\mathcal{B}_0} = W^{2\alpha,\gamma}_{\mathcal{B}_0}, \qquad \|w\|_{W^{\beta,\gamma}} \le C\, \|w\|^\theta_{W^{2\alpha,\gamma}_{\mathcal{B}_0}}\, \|w\|^{1-\theta}_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}}.
$$

If we remind (7.6), $W^{2\alpha,\gamma}_{\mathcal{B}_0} = E_\alpha$, $W^{2\alpha-2,\gamma}_{\mathcal{B}_0} = E_{\alpha-1}$, $A_{\alpha-1} : E_\alpha \to E_{\alpha-1}$ and the semigroup $e^{-tA_{\alpha-1}} : E_{\alpha-1} \to E_{\alpha-1}$. So, for $f \in W^{2\alpha-2,\gamma}_{\mathcal{B}_0} = E_{\alpha-1}$,

$$
\|e^{-(t-\tau)A_{\alpha-1}} f\|_{W^{\beta,\gamma}} \le C \|e^{-(t-\tau)A_{\alpha-1}} f\|^\theta_{W^{2\alpha,\gamma}_{\mathcal{B}_0}}\, \|e^{-(t-\tau)A_{\alpha-1}} f\|^{1-\theta,\gamma}_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}}.
$$

Since $A_{\alpha-1}$ is the generator of an analytic semigroup then it holds that ([7], Remark 8.6.c))

$$
\|w\|_{W^{2\alpha,\gamma}_{\mathcal{B}_0}} \le C \|A_{\alpha-1} w\|_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}} \qquad \forall w \in E_\alpha,
$$

due to $[0, +\infty) \subset \rho(-A_0)$ (ρ is the the resolvent set); this claim holds following Theorem 8.5 and (3.1) of [7]. So

$$
\|e^{-(t-\tau)A_{\alpha-1}} f\|_{W^{\beta,\gamma}} \le C \|A_{\alpha-1} e^{-(t-\tau)A_{\alpha-1}} f\|^\theta_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}}\, \|e^{-(t-\tau)A_{\alpha-1}} f\|^{1-\theta}_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}}.
$$

Finally, thanks to [59], Theor 1.3.4,

$$\|A_{\alpha-1}e^{-(t-\tau)A_{\alpha-1}}f\|^{\theta}_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}} \le C\,(t-\tau)^{-\theta}e^{-\delta(t-\tau)\theta}\,\|f\|^{\theta}_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}},$$

$$\|e^{-(t-\tau)A_{\alpha-1}}f\|^{1-\theta}_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}} \le C\,e^{-\delta(t-\tau)(1-\theta)}\,\|f\|^{1-\theta}_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}},$$

and it results

$$\|e^{-(t-\tau)A_{\alpha-1}}f\|_{W^{\beta,\gamma}} \le C\,e^{-\delta(t-\tau)}(t-\tau)^{-\theta}\,\|f\|_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}}, \tag{7.19}$$

for each $\delta \in (0,1)$. Taking norm $\|\cdot\|_{W^{\beta,\gamma}}$ on both sides of (7.17) and using (7.19), we get

$$\|v(t)\|_{W^{\beta,\gamma}} \le C\,t^{-\theta}\,\|v_0\|_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}} + \int_0^t \quad e^{-\delta(t-\tau)}(t-\tau)^{-\theta}(\|f(\tau)\|_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}}+ \\ +\|A_{\alpha-1}\mathcal{B}_0^c g(\tau)\|_{W^{2\alpha-2,\gamma}_{\mathcal{B}_0}}).$$

Taking into account Lemma 2.2, (7.18) and the Sobolev embedding $W^{\beta,\gamma}(\Omega) \hookrightarrow W^{1,p(d,\gamma)}(\Omega)$, we deduce the Lemma. ■

Lemma 7.7. *Let $\alpha \in [1,+\infty)$. There exists $C>0$ such that $u(t) \in L^{\alpha}(\Omega)$ and*

$$\|u(t)\|_{\alpha} \le C \quad \forall t \in [t_0, T_{max}).$$

Proof. Fix $\alpha > 2$. Multiplying the u-equation by $\alpha u^{\alpha-1}$ and integrating in Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^\alpha = \; & \frac{-4(\alpha-1)}{\alpha}\int_\Omega |\nabla(u^{\alpha/2})|^2 + \alpha(\alpha-1)\int_\Omega V(u)u^{\alpha-2}\nabla v\cdot\nabla u - \\ & -\alpha\int_{\Gamma_2} V(u)\mu\frac{v}{1+v}u^{\alpha-1} + \lambda\alpha\int_\Omega u^\alpha - \alpha\int_\Omega u^{\alpha+1}.\end{aligned} \tag{7.20}$$

We add $\displaystyle\int_\Omega u^\alpha$ on both sides of the equality. Besides, we estimate the boundary term as follows

$$-\alpha\int_{\Gamma_2} V(u)\mu\frac{v}{1+v}u^{\alpha-1} \le C\int_{\Gamma_2} u^{\alpha-1}.$$

At this point, we use the following inequality. Given $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that $s^{\alpha-1} \le \varepsilon s^\alpha + C(\varepsilon)\ \ \forall s \in \mathbb{R}$. Then,

$$\begin{aligned}\int_{\Gamma_2} u^{\alpha-1} & \le \varepsilon\int_{\Gamma_2} u^\alpha + C(\varepsilon)|\Gamma_2| \\ & \le C\left(\varepsilon\int_\Omega (u^{\alpha/2})^2 + \varepsilon\int_\Omega |\nabla(u^{\alpha/2})|^2\right) + C.\end{aligned}$$

Using the Sobolev trace embedding and taking account the above estimate, in (7.20), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^\alpha + \int_\Omega u^\alpha \le \; & \frac{-4(\alpha-1)}{\alpha}\int_\Omega |\nabla(u^{\alpha/2})|^2 + \alpha(\alpha-1)\int_\Omega V(u)u^{\alpha-2}\nabla v\cdot\nabla u + \\ & + \int_\Omega((\alpha\lambda+1)u^\alpha - \alpha u^{\alpha+1}) + C\varepsilon\int_\Omega u^\alpha + C\varepsilon\int_\Omega |\nabla(u^{\alpha/2})|^2 + C.\end{aligned}$$

By the estimate $(\alpha\lambda+1)s^\alpha - \alpha s^{\alpha+1} \le C = C(\lambda,\alpha)$ for every $s \ge 0$ we get,

$$\frac{d}{dt}\int_\Omega u^\alpha + (1-C\varepsilon)\int_\Omega u^\alpha \le$$

$$\le \left(\frac{-4(\alpha-1)}{\alpha} + C\varepsilon\right)\int_\Omega |\nabla(u^{\alpha/2})|^2 + \alpha(\alpha-1)\int_\Omega V(u)u^{\alpha-2}\nabla v\cdot\nabla u + C.$$

An easy computation gives

$$\alpha(\alpha-1)\int_\Omega V(u)u^{\alpha-2}\nabla v\cdot\nabla u = 2(\alpha-1)\int_\Omega u^{\frac{\alpha}{2}-1}V(u)\nabla v\cdot\nabla(u^{\alpha/2})$$

and replacing it in the previous inequality, we obtain

$$\frac{d}{dt}\int_\Omega u^\alpha + (1-\varepsilon)\int_\Omega u^\alpha \le$$

$$\le \left(\frac{-4(\alpha-1)}{\alpha} + \varepsilon\right)\int_\Omega |\nabla(u^{\alpha/2})|^2 + 2(\alpha-1)\int_\Omega u^{\frac{\alpha}{2}-1}V(u)\nabla v\cdot\nabla(u^{\alpha/2}) + C. \tag{7.21}$$

Now, we deal with the second term in the right hand side,

$$\begin{aligned} \|u^{\frac{\alpha}{2}-1}V(u)\nabla v\cdot\nabla(u^{\alpha/2})\|_1 &\le C\|u^{\frac{\alpha}{2}-1}\|_{\theta(p)}\|\nabla v\|_p\|\nabla(u^{\alpha/2})\|_2 \\ &\le C(\varepsilon)\|u^{\frac{\alpha}{2}-1}\|^2_{\theta(p)}\|\nabla v\|_p^2 + \varepsilon\|\nabla(u^{\alpha/2})\|_2^2, \end{aligned} \tag{7.22}$$

where

$$\theta(p) = \frac{2p}{p-2},\ p>2.$$

Replacing (7.22) into (7.21), and considering ε small enough, we obtain

$$\frac{d}{dt}\|u^{\alpha/2}\|_2^2 + (1-C\varepsilon)\|u^{\alpha/2}\|_2^2 \le C(\varepsilon)\|u^{\frac{\alpha}{2}-1}\|^2_{\theta(p)}\|\nabla v\|_p^2 + C. \tag{7.23}$$

Now, we begin a recursive algorithm. Taking

$$\gamma_0 := 2,$$

by Lemma 7.5 $u(t)\in L^2(\Omega)$, $\|u(t)\|_2 \le C$. By Lemma 7.6, $v(t)\in W^{1,p(\gamma_0,d)}(\Omega)$, so, $\|\nabla v\|_p^2$ is finite. Choosing $\alpha \le 2+\frac{4}{\theta(p)}$ we assure that $u^{\frac{\alpha}{2}-1}\in L^\theta(\Omega)$. So, for $2<\alpha\le 2+\frac{2\gamma_0}{\theta(p(\gamma_0,d))}$ we have

$$\frac{d}{dt}\|u^{\alpha/2}(t)\|_2^2 + (1-\varepsilon)\|u^{\alpha/2}(t)\|_2^2 \le C.$$

Therefore,

$$\|u^{\alpha/2}(t)\|_2^2 \le \|u(t_0)^{\alpha/2}\|_2^2 + C.$$

We have just proved that $\|u(t)\|_\alpha \le C$ for $2<\alpha\le 2+\frac{2\gamma_0}{\theta(p(\gamma_0,d))}$.
Now, we define

$$\gamma_1 := 2 + \frac{2\gamma_0}{\theta(p(\gamma_0,d))}.$$

Owing to the previous reasoning, we have that $u(t)\in L^{\gamma_1}(\Omega)$. So, by Lemma 7.6 $v(t)\in W^{1,p(\gamma_1,d)}(\Omega)$. If we choose $\alpha\le 2+\frac{2\gamma_1}{\theta(p(\gamma_1,d))}$ we will assure that $u^{\alpha/2-1}(t)\in L^\theta(\Omega)$, and by (7.23) we obtain that for $2 < \alpha \leq 2 + \frac{2\gamma_1}{\theta(p(\gamma_1,d))}$, $u(t) \in L^\alpha(\Omega)$ and $\|u(t)\|_\alpha \leq C$. By a recursive algorithm we get that $u(t) \in L^\alpha(\Omega)$ and $\|u(t)\|_\alpha \leq C$, for every α with $2 < \alpha \leq 2 + \frac{2\gamma_n}{\theta(p(\gamma_n,d))}$, and

$$\gamma_n = 2 + \frac{2\gamma_{n-1}}{\theta(p(\gamma_{n-1},d))}.$$

Using that

$$p(\gamma_{n-1},d) = \gamma_{n-1}\frac{d}{d-1+\varepsilon\gamma_{n-1}},$$

we have that

$$\gamma_n = \gamma_{n-1}\left(1 - \frac{2\varepsilon}{d}\right) + \frac{2}{d}.$$

The limit of γ_n is $\frac{1}{\varepsilon}$, so for $\varepsilon > 0$ as small as we want, we have that $u(t) \in L^\alpha(\Omega)$ and $\|u(t)\|_\alpha \leq C$ for all $\alpha \in [1,+\infty)$. ■

Remark 7.8. *Consequently, by Lemmas 7.6 and 7.7, we have obtained that* $u(t) \in L^p(\Omega)$, $\nabla v \in L^p(\Omega)^N$, *for every* $p \in [1,+\infty)$.

In the following result we obtain a better bound of u, a L^∞-bound. Let $p > 1$ and define

$$B := -\Delta + I,$$

with domain

$$D(B) := \left\{u \in W^{2,p}(\Omega) \, : \partial u/\partial n = 0 \text{ on } \partial\Omega\right\}.$$

For each $\beta \geq 0$, define the sectorial operator B^β (see [59]) and

$$X_\beta := D(B^\beta) \quad \text{with the norm } \|u\|_{X_\beta} := \|B^\beta u\|_p.$$

Lemma 7.9. *Let* $2\beta < 1$, *then for* $t \in [t_0, T_{max})$ *we have*

$$\|u(t)\|_{X_\beta} \leq C$$

Proof. We have that

$$u(t) = e^{-tB}u_0 + \int_0^t e^{-(t-\tau)B}(-\nabla\cdot(V(u)\nabla v) + (\lambda+1)u - u^2)d\tau,$$

and so

$$\|u(t)\|_{X_\beta} \leq \|e^{-tB}u_0\|_{X_\beta} + \int_0^t \|e^{-(t-\tau)B}(-\nabla\cdot(V(u)\nabla v) + (\lambda+1)u - u^2)d\tau\|_{X_\beta}.$$

First, by [59, Theorem 1.4.3]

$$\|e^{-tB}u_0\|_{X_\beta} \leq Ct^{-\beta}e^{-\delta t}\|u_0\|_p,$$

and

$$\|e^{-(t-\tau)B}((\lambda+1)u - u^2)\|_{X_\beta} \leq (t-\tau)^{-\beta}e^{-\delta(t-\tau)}((\lambda+1)\|u\|_p + \|u^2\|_p)$$

where $\delta \in (0,1)$.

Moreover, by [64, Lemma 2.1] we obtain

$$\begin{aligned}\|e^{-(t-\tau)B}(-\nabla\cdot(V(u)\nabla v)\|_{X_\beta} \leq\ & C\|e^{-(t-\tau)\Delta}(-\nabla\cdot(V(u)\nabla v)\|_{X_\beta}\\ \leq\ & C(\varepsilon)(t-\tau)^{-1/2-\beta-\epsilon}e^{-\delta(t-\tau)}\|V(u)\nabla v\|_p\end{aligned}$$

where $\epsilon > 0$ such that $-1/2-\beta-\epsilon > -1$. Then,

$$\begin{aligned}\|u(t)\|_{X_\beta} \leq\ & Ct^{-\beta}e^{-\delta t}\|u_0\|_p + C\int_0^t [(t-\tau)^{-1/2-\beta-\epsilon}e^{-\delta(t-\tau)}\|V(u)\nabla v\|_p +\\ & +(t-\tau)^{-\beta}e^{-\delta(t-\tau)}((\lambda+1)\|u\|_p + \|u^2\|_p)]d\tau.\end{aligned} \tag{7.24}$$

Now, observe that

$$\|V(u)\nabla v\|_p \leq C\|\nabla v\|_p.$$

Finally, thanks to Lemmas 7.6 and 7.7 we easily conclude the result from (7.24).

■

Theorem 7.10. *We have that*

$$\|u(t)\|_\infty \leq C \quad \textit{for all } t\in[0,T_{max}),$$

and consequently we have proved the global existence.

Proof. Let $p > N$, $2\beta \in \left(\frac{d}{p},1\right)$. Since $2\beta > d/p$ we have by [59, Theorem 1.6.1] that

$$X_\beta \hookrightarrow \mathcal{C}(\overline{\Omega}).$$

Thanks to Lemma 7.9 we have that $\|u(t)\|_\infty < C$ for $t > t_0 > 0$. Moreover, the local existence Theorem yields $\|u(t)\|_\infty < C$ for $t < t_0$. Therefore, $\|u(t)\|_\infty < C$ for all $t \geq 0$. Then, this result and Lemma 7.4 prove the global existence criterium (see (7.7)).

■

7.4. Steady-states

Consider now the stationary problem

$$\begin{cases} -\Delta u = -\mathrm{div}(V(u)\nabla v) + \lambda u - u^2 & \text{in } \Omega,\\ -\Delta v = -v - cuv & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_1,\\ \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial v}{\partial n} = \mu\dfrac{v}{1+v} & \text{on } \Gamma_2.\end{cases} \tag{7.25}$$

First, we need to introduce some notations. For $\alpha \in (0,1)$ we denote

$$X_1 := \{w \in \mathcal{C}^{2+\alpha}(\overline{\Omega}) : \partial w/\partial n = 0 \text{ on } \partial\Omega\}, \quad X_2 := \{w \in \mathcal{C}^{2+\alpha}(\overline{\Omega}) : \partial w/\partial n = 0 \text{ on } \Gamma_1\}$$

and finally

$$X := X_1 \times X_2.$$

Moreover, given a function $c \in \mathcal{C}(\overline{\Omega})$ we denote by

$$c_M := \max_{\overline{\Omega}} c(x), \qquad c_L := \min_{\overline{\Omega}} c(x).$$

We are interested in solutions $(u, v) \in X$ of (7.25) with both components non-negative and non-trivial. Observe that thanks to the strong maximum principle, any component, u or v, of a non-negative and non-trivial solution is in fact positive in the whole domain $\overline{\Omega}$.

Consider functions $m \in \mathcal{C}^\alpha(\overline{\Omega})$, $g \in \mathcal{C}^{1+\alpha}(\Gamma_2)$ and the eigenvalue problem

$$\begin{cases} -\Delta\phi + m\phi = \lambda\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial\phi}{\partial n} + g\phi = 0 & \text{on } \Gamma_2. \end{cases} \tag{7.26}$$

We are only interested in the principal eigenvalue of (7.26), i.e., the eigenvalues which have an associated positive eigenfunction. In the following result we remind its main properties, see also Chapter 2, [4] and [27].

Lemma 7.11. *Problem* (7.26) *admits a unique principal eigenvalue, which will be denoted by* $\lambda_1(-\Delta + m; \mathcal{N}, \mathcal{N} + g)$. *Moreover, this eigenvalue is simple, and any positive eigenfunction,* ϕ, *verifies* $\phi \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$. *In addition,* $\lambda_1(-\Delta + m; \mathcal{N}, \mathcal{N} + g)$ *is separately increasing in* m *and* g; *if* $m = K\varphi$ *with* $\varphi > 0$ *in* Ω *then*

$$\lim_{K \to +\infty} \lambda_1(-\Delta + K\varphi; \mathcal{N}, \mathcal{N} + g) = +\infty. \tag{7.27}$$

Moreover, we are going to consider the following Steklov eigenvalue problem with eigenvalue on the boundary

$$\begin{cases} -\Delta\phi + m\phi = 0 & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial\phi}{\partial n} + g\phi = \mu\phi & \text{on } \Gamma_2. \end{cases} \tag{7.28}$$

It is well-known that there exists a principal eigenvalue of (7.28), we denote it by

$$\mu_1(-\Delta + m; \mathcal{N}, \mathcal{N} + g).$$

It is clear that μ is a principal eigenvalue of (7.28) if, and only if,

$$0 = \lambda_1(-\Delta + m; \mathcal{N}, \mathcal{N} + g - \mu),$$

and that thanks to Lemma 7.11

$$0 > \lambda_1(-\Delta + m; \mathcal{N}, \mathcal{N} + g - \mu) \Longleftrightarrow \mu > \mu_1(-\Delta + m; \mathcal{N}, \mathcal{N} + g), \tag{7.29}$$

and analogously,

$$0 < \lambda_1(-\Delta + m; \mathcal{N}, \mathcal{N} + g - \mu) \Longleftrightarrow \mu < \mu_1(-\Delta + m; \mathcal{N}, \mathcal{N} + g).$$

Finally, we need to introduce the next notation: for a solution U_0 of a nonlinear equation, we say that it is *stable* if the first eigenvalue of the linearization around U_0 is positive, and *unstable* if it is negative.

7.4.1. Semi-trivial solutions

Apart from the trivial solution $(u, v) = (0, 0)$ of (7.25), there exist the semi-trivial solutions. It is clear that if $v \equiv 0$, then u verifies

$$-\Delta u = \lambda u - u^2 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

that is $u \equiv \lambda$. Hence, $(\lambda, 0)$ is a semi-trivial solution of (7.25).

On the other hand, when $u \equiv 0$ then v satisfies the equation

$$\text{(7.30)} \qquad \begin{cases} -\Delta v + v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial v}{\partial n} = \mu \dfrac{v}{1+v} & \text{on } \Gamma_2. \end{cases}$$

This equation was analyzed in [98] with $\Gamma_1 = \emptyset$; we include a proof for reader's convenience and some useful estimates.

Proposition 7.12. *There exists a positive solution of (7.30) if, and only if,*

$$\mu > \mu_1 := \mu_1(-\Delta + 1; \mathcal{N}, \mathcal{N}).$$

Moreover, if the solution exists, it is the unique positive solution, and we denote it by θ_μ. Furthermore, θ_μ is stable for $\mu > \mu_1$, i.e. ,

$$\text{(7.31)} \qquad \lambda_1(-\Delta + 1; \mathcal{N}, \mathcal{N} - \mu(1/(1+\theta_\mu))^2) > 0.$$

Finally,

$$\text{(7.32)} \qquad \frac{1}{\|\varphi_1\|_\infty}\left(\frac{\mu}{\mu_1} - 1\right)\varphi_1 \le \theta_\mu \le \frac{1}{(\varphi_1)_L}\left(\frac{\mu}{\mu_1} - 1\right)\varphi_1, \quad \textit{in } \Omega,$$

where φ_1 is a positive eigenfunction associated to μ_1.

Proof. Observe that if v is a positive solution of (7.30) we get

$$0 = \lambda_1\left(-\Delta + 1; \mathcal{N}, \mathcal{N} - \mu\frac{1}{1+v}\right),$$

and so $\mu > 0$. Moreover,

$$0 = \lambda_1\left(-\Delta + 1; \mathcal{N}, \mathcal{N} - \mu\frac{1}{1+v}\right) > \lambda_1(-\Delta + 1; \mathcal{N}, \mathcal{N} - \mu)$$

and then by (7.29), $\mu > \mu_1$.

To prove the existence of solution, we apply the sub-supersolution method. Take φ_1 a positive eigenfunction associated to μ_1. Then $(\underline{v}, \overline{v}) = (\varepsilon\varphi_1, M\varphi_1)$ is sub-supersolution of (7.30) if

$$\varepsilon = \frac{\frac{\mu}{\mu_1} - 1}{\|\varphi_1\|_\infty} \quad \text{and} \quad K \ge \frac{\frac{\mu}{\mu_1} - 1}{(\varphi_1)_L}.$$

The uniqueness follows by an standard argument. Indeed, observe that the map $s \mapsto \mu s/s(1+s) = \mu/(1+s)$ is decreasing.

To prove the stability, linearizing (7.30) around θ_μ, we need to prove that

$$\lambda_1(-\Delta + 1; \mathcal{N}, \mathcal{N} - \mu(1/(1+\theta_\mu))^2) > 0.$$

For that, observe that $\overline{v} = \theta_\mu$ is a strict-supersolution of the following problem

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial v}{\partial n} - \mu \left(\dfrac{1}{1+\theta_\mu} \right)^2 v = 0 & \text{on } \Gamma_2. \end{cases}$$

■

The next result provides us with a priori bounds of the solutions of (7.25) and bounds in the space X.

Lemma 7.13. *Let (u, v) a coexistence state of (7.25). Then,*

$$u \le \lambda \quad \textit{and} \quad v \le \theta_\mu. \tag{7.33}$$

Moreover, consider that $(\lambda, \mu) \in \mathcal{K} \subset \mathbb{R}^2$ compact. Then, there exists a constant C (independent of λ and μ) such that for any solution (u, v) of (7.25) we have

$$\|(u, v)\|_X \le C.$$

Proof. That $v \le \theta_\mu$ is clear. On the other hand, observe that the first equation of (7.25) can be written as

$$-\Delta u = -V'(u)\nabla u \cdot \nabla v - V(u)\Delta v + \lambda u - u^2 = -V'(u)\nabla u \cdot \nabla v - V(u)(v + cuv) + \lambda u - u^2.$$

If we denote by $\overline{x} \in \overline{\Omega}$ such that $u(\overline{x}) = \max_{\overline{\Omega}} u$, we have that (see Lemma 2.1 in [78])

$$u(\overline{x}) \le \lambda - \frac{V(u(\overline{x}))}{u(\overline{x})} v(\overline{x})[1 + cu(\overline{x})], \tag{7.34}$$

whence we can conclude that $u \le \lambda$. This completes the proof of (7.33).

Suppose $(\lambda, \mu) \in \mathcal{K} \subset \mathbb{R}^2$ compact and let (u, v) be a solution of (7.25). Then, we have that u and v are bounded in $L^\infty(\Omega)$ for some constant C not depending on λ or μ. Now, going back to the v-equation and using the $W^{1,p}(\Omega)$-estimates, (see Theorem 2.15) we get

$$\|v\|_{W^{1,p}} \le C(\| - v - cuv\|_p + \|\mu v/(1+v)\|_{p,\partial\Omega}) \le C.$$

On the other hand, since $W^{1,p}(\Omega) \hookrightarrow W^{1-1/p,p}(\partial\Omega)$ we have that

$$\|\mu v/(1+v)\|_{W^{1-1/p,p}(\partial\Omega)} \le C\|\mu v/(1+v)\|_{W^{1,p}} \le C,$$

where we have used that the map $G(s) = s/(1+s)$ is differentiable, $G(0) = 0$, $|G'| \le C$ and the chain rule in Sobolev space. Hence, for p large

$$\|v\|_{C^1(\overline{\Omega})} \le C\|v\|_{W^{2,p}} \le C(\| - v - cuv\|_p + \|\mu v/(1+v)\|_{W^{1-1/p,p}(\partial\Omega)}) \le C.$$

But, the u-equation in (7.25) can be written as follows

$$-\Delta u + V'(u)\nabla u \cdot \nabla v = \lambda u - u^2 - V(u)(v + cuv)$$

and thus, u is bounded in $W^{2,p}(\Omega)$ for all $p > 1$, and so in $\mathcal{C}^1(\overline{\Omega})$. Now, again using the v-equation and the Schauder Theory in Hölder spaces (see [57]), v is bounded in X_2, and finally u in X_1 with constants independent of λ and μ. ■

As an easy consequence of the above result, we have

Corollary 7.14. *If $\lambda \le 0$ or $\mu \le \mu_1$ then (7.25) does not possess positive solution.*

The following result will be crucial in the existence result:

Proposition 7.15. *a) Assume that $V'(0) > 0$ and fix $\lambda > 0$. Then, there exists $\mu_0(\lambda)$ such that (7.25) does not possess coexistence states for $\mu \ge \mu_0(\lambda)$.*

b) Fix $\mu > \mu_1$. Then, there exists $\lambda_0(\mu)$ such that (7.25) does not possess coexistence states for $\lambda \ge \lambda_0(\mu)$.

Proof. 1. Fix $\lambda > 0$ and assume that there exist a sequence $\mu_n \to \infty$ and coexistence states (u_n, v_n) of (7.25). Denote by $x_n \in \overline{\Omega}$ such that $u_n(x_n) = \|u_n\|_\infty$. Then, by (7.34) we have

$$\|u_n\|_\infty + \frac{V(\|u_n\|_\infty)}{\|u_n\|_\infty} v_n(x_n)(1 + c\|u_n\|_\infty) \le \lambda. \tag{7.35}$$

Moreover, we know that $u_n \le \lambda$, and so

$$-\Delta v_n + (1 + c\lambda)v_n \ge 0,$$

and so, by a similar argument to the proof of Proposition 7.12, we get that

$$v_n(x) \ge \left(\frac{\mu_n}{\mu_1} - 1 - c\lambda\right)\varphi_1(x),$$

and so by (7.35), we have

$$\|u_n\|_\infty + \frac{V(\|u_n\|_\infty)}{\|u_n\|_\infty}(1 + c\|u_n\|_\infty)\left(\frac{\mu_n}{\mu_1} - 1 - c\lambda\right)\varphi_1(x_n) \le \lambda.$$

Since $\varphi_1 \ge \delta > 0$ in $\overline{\Omega}$ and $1 + c\|u_n\|_\infty \ge 1$, we obtain that $V(\|u_n\|_\infty)/\|u_n\|_\infty \to 0$, which is impossible due to $V'(0) > 0$.

2. Denote by $u_m = \min_{x\in\Omega} u(x)$. With a similar argument to the used one to prove (7.34) we get (see again Lemma 2.1 in [78])

$$\lambda \le u_m + \frac{V(u_m)}{u_m} v(x_m)(1 + cu_m).$$

Since $v \leq \theta_\mu$, if $\lambda \to \infty$ then $u_m \to \infty$. On the other hand, since v is solution of the second equation of (7.25) we have

$$0 = \lambda_1 \left(-\Delta + 1 + cu; \mathcal{N}, \mathcal{N} - \frac{\mu}{1+v} \right) \geq \lambda_1(-\Delta + 1 + cu_m; \mathcal{N}, \mathcal{N} - \mu).$$

But observe that $\lambda_1(-\Delta + 1 + cu_m; \mathcal{N}, \mathcal{N} - \mu) \to \infty$ as $\lambda \to \infty$, a contradiction. ■

Finally, another eigenvalue problem is analyzed

$$\begin{cases} -\Delta\phi + \operatorname{div}(V'(0)\nabla\theta_\mu\phi) = \lambda\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \Gamma_1 \cup \Gamma_2. \end{cases} \tag{7.36}$$

Fix $\mu > \mu_1$, we denote the principal eigenvalue as $\Lambda(\mu)$ and extend $\Lambda(\mu) = 0$ for $\mu \leq \mu_1$. Observe that $\Lambda(\mu) = 0$ when $V'(0) = 0$. In the following result we show some properties of $\Lambda(\mu)$.

Proposition 7.16. *Assume that $V'(0) > 0$. Then*

$$\lim_{\mu \to \infty} \Lambda(\mu) = \infty.$$

Proof. Under a change of variable $\Phi = e^{\frac{V'(0)}{2}\phi}$ (see [41] for references about this change of variable), we get

$$\Lambda(\mu) = \lambda_1 \left(-\Delta + \frac{V'(0)^2}{4}|\nabla\theta_\mu|^2 + \frac{V'(0)}{2}\theta_\mu; \mathcal{N}, \mathcal{N} + \frac{V'(0)}{2}\mu\frac{\theta_\mu}{1+\theta_\mu} \right).$$

And so, using (7.32) and (7.27) we obtain that

$$\Lambda(\mu) > \lambda_1 \left(-\Delta + \frac{V'(0)}{2} \frac{\frac{\mu}{\mu_1} - 1}{\|\varphi_1\|_\infty}\varphi_1; \mathcal{N}, \mathcal{N} \right) \to \infty$$

as $\mu \to \infty$. ■

Now, finally we denote by

$$F(\lambda) := \mu_1(-\Delta + 1 + c\lambda; \mathcal{N}, \mathcal{N}).$$

It is clear that $F(0) = \mu_1$, and that $\lambda \mapsto F(\lambda)$ is increasing and $\lim_{\lambda \to +\infty} F(\lambda) = +\infty$.

The main result is:

Theorem 7.17. *a) Assume that $V'(0) > 0$. Then there exists at least a coexistence state if*

$$(\lambda - \Lambda(\mu))(\mu - F(\lambda)) > 0. \tag{7.37}$$

b) Assume that $V'(0) = 0$. Then there exists at least a coexistence state if $\lambda > 0$ and

$$\mu > F(\lambda). \tag{7.38}$$

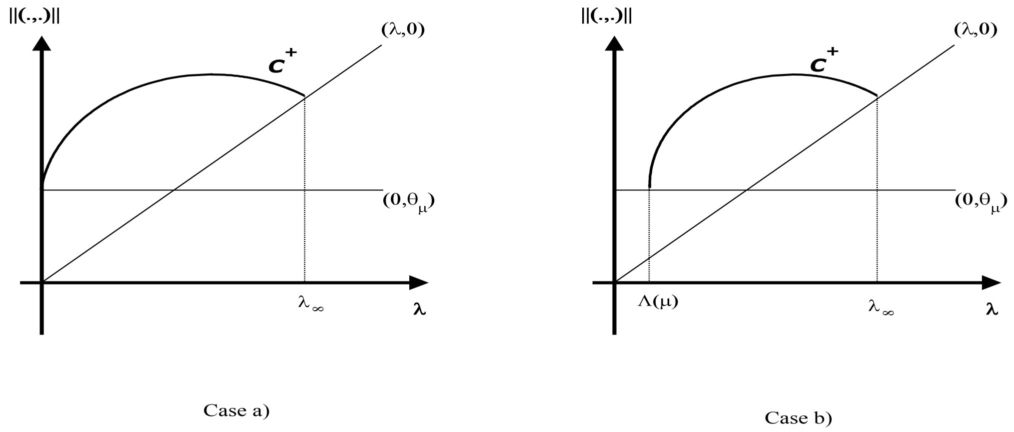

Figure 7.1: Bifurcation diagrams: Case a) $V'(0) = 0$ and Case b) $V'(0) > 0$.

Proof. We fix $\mu > \mu_1$ and consider λ as bifurcation parameter, see Figure 7.1 where we have drawn the two bifurcation diagrams.

First, we apply the Crandall-Rabinowitz theorem [35] in order to find the bifurcation point from the semi-trivial solution $(0, \theta_\mu)$. Consider the map $\mathcal{F} : \mathbb{R} \times X_1 \times X_2 \mapsto \mathcal{C}^\alpha(\overline{\Omega}) \times \mathcal{C}^\alpha(\overline{\Omega}) \times \mathcal{C}^\alpha(\Gamma_2)$ defined by

$$\mathcal{F}(\lambda, u, v) := \left(-\Delta u + \mathrm{div}(V(u)\nabla v) - \lambda u + u^2, -\Delta v + v + cuv, \frac{\partial v}{\partial n} - \mu \frac{v}{1+v} \right).$$

It is clear that $\mathcal{F}$ is regular, that $\mathcal{F}(\lambda, 0, \theta_\mu) = 0$ and

$$D_{(u,v)}\mathcal{F}(\lambda_0, u_0, v_0) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\Delta\xi + \mathrm{div}(V'(u_0)\xi\nabla v_0 + V(u_0)\nabla\eta) - \lambda\xi + 2u_0\xi \\ -\Delta\eta + \eta + cu_0\eta + cv_0\xi \\ \dfrac{\partial \eta}{\partial n} - \mu(\dfrac{1}{1+v_0})^2\eta \end{pmatrix}.$$

Hence, for $(u_0, v_0) = (0, \theta_\mu)$ and $\lambda_0 = \Lambda(\mu)$ and we get that

$$Ker[D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu)] = span\{(\Phi_1, \Phi_2)\}$$

where Φ_1 is an eigenfunction associated to $\Lambda(\mu)$ and Φ_2 is the unique solution of

$$\begin{cases} (-\Delta + 1)\Phi_2 = -c\theta_\mu\Phi_1 & \text{in } \Omega, \\ \dfrac{\partial \Phi_2}{\partial n} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial \Phi_2}{\partial n} - \mu(1/(1+\theta_\mu))^2\Phi_2 = 0 & \text{on } \Gamma_2. \end{cases}$$

Observe that Φ_2 is well-defined by (7.31). Hence, $dim(Ker[D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu)]) = 1$.

On the other hand, observe that

$$D_{\lambda(u,v)}\mathcal{F}(\lambda_0, u_0, v_0)\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}-\xi\\ 0\\ 0\end{pmatrix}.$$

We can show that $D_{\lambda(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu)(\Phi_1, \Phi_2)^t \notin R(D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu))$. Indeed, suppose that there exists $(\xi, \eta) \in X$ such that $D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu)(\xi, \eta)^t = (-\Phi_1, 0, 0)$, and so

$$-\Delta\xi + V'(0)\mathrm{div}(\xi\nabla\theta_\mu) - \lambda_0\xi = -\Phi_1 \quad \text{in } \Omega, \quad \partial\xi/\partial n = 0 \quad \text{on } \partial\Omega.$$

Under the change of variable $\xi = e^{V'(0)\theta_\mu}\varsigma$, the above equation is transformed into

$$\text{(7.39)} \qquad \begin{cases} -\mathrm{div}(e^{V'(0)\theta_\mu}\nabla\varsigma) - \lambda_0 e^{V'(0)\theta_\mu}\varsigma = -\Phi_1 & \text{in } \Omega,\\ \dfrac{\partial\varsigma}{\partial n} = 0 & \text{on } \Gamma_1,\\ \dfrac{\partial\varsigma}{\partial n} + V'(0)\mu\dfrac{\theta_\mu}{1+\theta_\mu}\varsigma = 0 & \text{on } \Gamma_2.\end{cases}$$

In a similar way, since Φ_1 is an eigenfunction associated to $\Lambda(\mu)$ we can make the change of variable $\Phi_1 = e^{\alpha\theta_\mu}\psi_1$, and (7.36) transforms into

$$\text{(7.40)} \qquad \begin{cases} -\mathrm{div}(e^{\alpha\theta_\mu}\nabla\psi_1) = \lambda_0 e^{\alpha\theta_\mu}\psi_1 & \text{in } \Omega,\\ \dfrac{\partial\psi_1}{\partial n} = 0 & \text{on } \Gamma_1,\\ \dfrac{\partial\psi_1}{\partial n} + V'(0)\mu\dfrac{\theta_\mu}{1+\theta_\mu}\psi_1 = 0 & \text{on } \Gamma_2.\end{cases}$$

Now, multiplying (7.39) by ψ_1 and (7.40) by ς, and subtracting we get

$$0 = \int_\Omega \Phi_1\psi_1,$$

an absurdum.

It can be showed that $R(D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu))$ has co-dimension 1.

Hence, the point $(\lambda, u, v) = (\Lambda(\mu), 0, \theta_\mu)$ is a bifurcation point from the semi-trivial solution $(0, \theta_\mu)$.

Now, we can apply Theorem 4.1 of [72] and conclude the existence of a continuum $\mathcal{C}^+ \subset \mathbb{R} \times X_1 \times X_2$ of positive solutions of (7.25) emanating from the point $(\lambda, u, v) = (\Lambda(\mu), 0, \theta_\mu)$ such that:

i) $\mathcal{C}^+$ is unbounded in $\mathbb{R} \times X_1 \times X_2$; or

ii) there exists $\lambda_\infty \in \mathbb{R}$ such that $(\lambda_\infty, \lambda_\infty, 0) \in cl(\mathcal{C}^+)$; or

iii) there exists $\overline{\lambda} \in \mathbb{R}$ such that $(\overline{\lambda}, 0, 0) \in cl(\mathcal{C}^+)$.

Alternative iii) is not possible. Indeed, if a sequence of positive solutions $(\lambda_n, u_n, v_n) \in cl(\mathcal{C}^+)$ such that $\lambda_n \to \overline{\lambda}$ and $(u_n, v_n) \to (0, 0)$ uniformly, then denoting by

$$V_n = \frac{v_n}{\|v_n\|_\infty},$$

and using the elliptic regularity, we have that $V_n \to V \geq 0$ and non-trivial in $\mathcal{C}^2(\overline{\Omega})$ with

$$-\Delta V + V = 0 \quad \text{in } \Omega, \quad \frac{\partial V}{\partial n} = 0 \quad \text{on } \Gamma_1, \quad \frac{\partial V}{\partial n} = \mu V \quad \text{on } \Gamma_2,$$

and so $\mu = \mu_1$, a contradiction.

Fixed $\mu > \mu_1$, we know by Corollary 7.14 and Proposition 7.15 that (7.25) does not possess positive solution if $\lambda \leq 0$ or λ is large. Moreover, by Proposition 7.13 it follows that $\mathcal{C}^+$ is bounded in X uniformly on compact subintervals of λ. Hence, alternative i) does not occur. Therefore, alternative ii) holds. When this alternative occurs, there exists a sequence (λ_n, u_n, v_n) of solutions of (7.25) such that $(\lambda_n, u_n, v_n) \to (\lambda_\infty, \lambda_\infty, 0)$. Denoting by

$$V_n = \frac{v_n}{\|v_n\|_\infty},$$

we obtain that $V_n \to V$ in $\mathcal{C}^2(\overline{\Omega})$ with

$$(-\Delta + 1 + c\lambda_\infty)V = 0 \quad \text{in } \Omega, \quad \partial V/\partial n = 0 \quad \text{on } \Gamma_1, \quad \partial V/\partial n = \mu V \quad \text{on } \Gamma_2,$$

that is, $\mu = F(\lambda_\infty)$. So, we can conclude the existence of a coexistence state for

$$\lambda \in (\min\{\Lambda(\mu), \lambda_\infty\}, \max\{\Lambda(\mu), \lambda_\infty\}).$$

Observe that if $V'(0) = 0$ then $\Lambda(\mu) = 0$. This completes the proof of the theorem. ■

In Figure 7.2 we have drawn the coexistence regions, denoted by A, defined by (7.37) and (7.38) in Theorem 7.17 in both cases: Case a) $V'(0) = 0$ and Case b) $V'(0) > 0$.

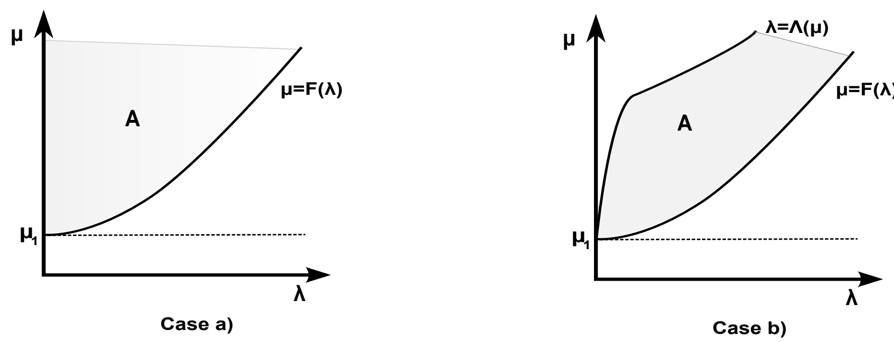

Figure 7.2: Coexistence regions: Case a) $V'(0) = 0$ and Case b) $V'(0) > 0$.

7.4.2. Local stability

We study the local stability of the trivial and semi-trivial solutions.

Proposition 7.18. *a) The trivial solution of (7.25) is stable if $\lambda < 0$ and $\mu < \mu_1$ and unstable if $\lambda > 0$ or $\mu > \mu_1$.*

b) Assume that $\lambda > 0$. The semi-trivial solution $(\lambda, 0)$ is stable if $\mu < F(\lambda)$ and unstable if $\mu > F(\lambda)$.

c) Assume that $\mu > \mu_1$. The semi-trivial solution $(0, \theta_\mu)$ is stable if $\lambda < \Lambda(\mu)$ and unstable if $\lambda > \Lambda(\mu)$.

Proof. We only prove the third paragraph in the result, the other ones are similarly followed. Observe that the stability of $(0, \theta_\mu)$ is given by the real parts of the eigenvalues for which the following problem admits a solution $(\xi, \eta) \in X \setminus \{(0,0)\}$

$$\tag{7.41} \begin{cases} -\Delta\xi + \mathrm{div}(V'(0)\nabla\theta_\mu\xi) - \lambda\xi = \sigma\xi & \text{in } \Omega, \\ (-\Delta + 1 + c\theta_\mu)\eta = \sigma\eta & \text{in } \Omega, \\ \dfrac{\partial\eta}{\partial n} = \mu\left(\dfrac{1}{1+\theta_\mu}\right)^2 \eta & \text{on } \Gamma_2. \end{cases}$$

Assume that $\xi \equiv 0$, then for some $j \geq 1$ and we have

$$\begin{aligned} \sigma &= \lambda_j\left(-\Delta + 1 + c\theta_\mu; \mathcal{N}, \mathcal{N} - \mu\left(\frac{1}{1+\theta_\mu}\right)^2\right) \\ &> \lambda_1\left(-\Delta + 1; \mathcal{N}, \mathcal{N} - \mu\left(\frac{1}{1+\theta_\mu}\right)^2\right) > 0. \end{aligned}$$

Now suppose that $\xi \not\equiv 0$, then from the first equation of (7.41) we get that $\lambda + \sigma$ is a real eigenvalue associated to (7.36). Since $\lambda < \Lambda(\mu)$ it follows that $\sigma > 0$.

Assume that $\lambda > \Lambda(\mu)$. Then,

$$\sigma_1 := \lambda_1(-\Delta + \mathrm{div}(V'(0)\nabla\theta_\mu) - \lambda; \mathcal{N}, \mathcal{N}) < 0.$$

Denote by ξ a positive eigenfunction associated to σ_1, that is

$$-\Delta\xi + \mathrm{div}(V'(0)\nabla\theta_\mu\xi) - \lambda\xi = \sigma_1\xi \quad \text{in } \Omega, \qquad \partial\xi/\partial n = 0 \quad \text{on } \partial\Omega.$$

Since $\sigma_1 < 0$, then

$$\lambda_1\left(-\Delta + 1 + c\theta_\mu - \sigma_1; \mathcal{N}, \mathcal{N} - \mu\left(\frac{1}{1+\theta_\mu}\right)^2\right) > 0,$$

and so there exists η solution of

$$(-\Delta + 1 + c\theta_\mu)\eta = \sigma_1\eta \quad \text{in } \Omega, \quad \frac{\partial\eta}{\partial n} = \mu\left(\frac{1}{1+\theta_\mu}\right)^2 \eta \quad \text{on } \Gamma_2.$$

Then, $\sigma_1 < 0$ is an eigenvalue of (7.41) with the eigenfunction associated (ξ, η), so $(0, \theta_\mu)$ is unstable. ∎

7.5. Convergence to the semi-trivial solution $(\lambda, 0)$

Now, we deal with the convergence to the semitrivial steady-state $(\lambda, 0)$. Through this section we assume the condition (7.10).

Lemma 7.19. *Let $\gamma \in (1, +\infty)$, $\beta \in (1, 1+1/\gamma)$, $\mu \in [0, \mu_1)$ and $0 < \delta < \rho < \alpha(\mu)$ where $\alpha(\mu) \in (0, 1]$ is the application*

$$\alpha(\mu) := \lambda_1(-\Delta + 1; \mathcal{N}, \mathcal{N} - \mu).$$

Then, there exists $C > 0$ such that, for $t > 0$, the v-solution to (7.1) satisfies

$$\begin{aligned} \|v(t)\|_\gamma &\leq Ce^{-\rho t}\|v_0\|_\gamma \\ \|v(t)\|_{W^{\beta,\gamma}} &\leq C(1 + t^{-\theta})e^{-\delta t}\|v_0\|_\gamma, \end{aligned}$$

where $\theta = \theta(\beta) \in (0, 1)$.

Proof. The solutions to the problem

$$\left\{ \begin{array}{ll} w_t - \Delta w + w = 0 & \text{in } \Omega \times (0, T_{max}), \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \Gamma_1 \times (0, T_{max}), \\ \dfrac{\partial w}{\partial n} = \mu w & \text{on } \Gamma_2 \times (0, T_{max}), \\ w(x, 0) = v_0(x) & \text{in } \Omega, \end{array} \right. \tag{7.42}$$

are supersolutions to the v-equation of (7.1), therefore $v \leq w$. Now, we define the sectorial operator

$$A_p := -\Delta + I.$$

Therefore,

$$w(x, t) = e^{-tA_p}v_0$$

Picking $D(A_p) = W^{2,p}_{\mathcal{B}(\mu)}$, where

$$\mathcal{B}(\mu)u = \left\{ \frac{\partial u}{\partial n} \text{ on } \Gamma_1,\ \frac{\partial u}{\partial n} - \mu u \text{ on } \Gamma_2 \right\}$$

and thanks to [59, Theorem 1.3.4] we have the first assertion. For the second assertion we take into account (7.18) and we argue as in Lemma 7.6 to infer

$$\|v(t)\|_{W^{\beta,\gamma}} \leq e^{-\delta t}t^{-\theta}\|v_0\|_\gamma + Ce^{-\delta t}\int_0^t e^{\delta\tau}(t-\tau)^{-\theta}(\|f(\tau)\|_\gamma + \|g(\tau)\|_\gamma)d\tau. \tag{7.43}$$

where f and g are as in Lemma 7.6. Observe that we have

$$\|f(\tau)\|_\gamma \leq C\|v\|_\gamma \leq Ce^{-\rho\tau}\|v_0\|_\gamma, \tag{7.44}$$

$$\|g(\tau)\|_\gamma \leq \|v\|_\gamma \leq Ce^{-\rho\tau}\|v_0\|_\gamma \tag{7.45}$$

Putting the estimates (7.44) and (7.45) in (7.43) we obtain

$$\|v(t)\|_{W^{\beta,\gamma}} \leq Ce^{-\delta t}t^{-\theta}\|v_0\|_\gamma + Ce^{-\delta t}\|v_0\|_\gamma \int_0^t e^{(\delta-\rho)\tau}(t-\tau)^{-\theta}d\tau.$$

Now, by the choice of δ and ρ we have $\displaystyle\int_0^\infty e^{(\delta-\rho)\tau}(t-\tau)^{-\theta}d\tau = C < +\infty$ and the Lemma easily follows. ■

Our purpose is to show the convergence to the steady states for u. To this end we distinguish separately the cases $\lambda = 0$, $\lambda > 0$.

7.5.1. Case $\lambda = 0$.

Lemma 7.20. *Let $\tau, k > 0$ and $y \in C(\tau,+\infty) \cap L^1(\tau,+\infty)$, $y' \in L^1(\tau,+\infty)$. If*

$$\lim_{t\to+\infty} \int_t^{t+k} (|y(s)| + |y'(s)|)ds = 0$$

then $\displaystyle\lim_{t\to+\infty} |y(t)| = 0$.

Proof. Let us assume that $\displaystyle\lim_{t\to+\infty} |y(t)| \neq 0$, then there exists a sequence $\{t_n\}_{n\in\mathbf{N}}$, $t_n \to +\infty$, such that

$$|y(t_n)| > C > 0, \quad \forall n \geq n_0.$$

We pick $\theta \in (0,k]$, then for all $n \geq n_0$ we have

$$\big||y(t_n+\theta)| - |y(t_n)|\big| \leq |y(t_n+\theta) - y(t_n)| \leq \int_{t_n}^{t_n+\theta} |y'(s)|ds \leq \int_{t_n}^{t_n+k} |y'(s)|ds.$$

Therefore $|y(s)| > C/2$ for all $s \in [t_n, t_n+k]$, $n \geq n_0$. The last assertion contradicts the fact that

$$\lim_{n\to+\infty} \int_{t_n}^{t_n+k} |y(s)|ds = 0.$$

■

Theorem 7.21. *Assume that $0 \leq \mu < \mu_1$ and $\lambda = 0$, then*

$$\lim_{t\to+\infty} \|u(t)\|_{W^{m,p}} = 0,$$

for any $m < 1$ and $p \geq 2$.

Proof. After integrating in the space variable the u-equation of (7.1) we get

$$\begin{aligned}\int_\Omega u_t &= \int_{\partial\Omega}\left(\frac{\partial u}{\partial n} - V(u)\frac{\partial v}{\partial n}\right) - \int_\Omega u^2 \\ &= -\mu\int_{\Gamma_2} \frac{V(u)v}{1+v} - \int_\Omega u^2.\end{aligned}$$

So, integrating the last expression in the time variable between (τ, t) we obtain

$$\mu \int_\tau^t \int_{\Gamma_2} \frac{V(u)v}{1+v} + \int_\tau^t \int_\Omega u^2 = \int_\Omega u(\tau) - \int_\Omega u(t). \tag{7.46}$$

In particular from (7.46) we have

$$\int_\tau^t \int_\Omega u^2 \leq \|u(\tau)\|_1 \quad \forall t > \tau. \tag{7.47}$$

On the other hand, multiplying the u-equation of (7.1) by u and integrating in the space variable we obtain

$$\begin{aligned} \frac{d}{2dt} \int_\Omega u^2 &= \int_\Omega \left(-|\nabla u|^2 + V(u)\nabla v \cdot \nabla u - u^3\right) - \mu \int_{\Gamma_2} \frac{V(u)uv}{1+v} \\ &\leq (\epsilon - 1) \int_\Omega |\nabla u|^2 + C(\epsilon) \int_\Omega |\nabla v|^2 - \mu \int_{\Gamma_2} \frac{V(u)uv}{1+v} - \int_\Omega u^3. \end{aligned} \tag{7.48}$$

Therefore, we infer

$$\frac{d}{2dt} \int_\Omega u^2 + (1-\epsilon) \int_\Omega |\nabla u|^2 \leq C(\epsilon) \|v\|^2_{W^{1,2}},$$

and after integrating in time, thanks to Lemma 7.19 we obtain

$$\int_\Omega u(t)^2 - \int_\Omega u(\tau)^2 + (1-\epsilon) \int_\tau^t \int_\Omega |\nabla u|^2 \leq C(\epsilon) \int_\tau^t (1 + s^{-\theta})^2 e^{-2\delta s} \|v_0\|_p^2$$

In particular we deduce

$$\int_\tau^t \int_\Omega |\nabla u|^2 \leq C \quad \forall t > \tau.$$

Now, using the fact that $\|u(t)\|_{C(\overline{\Omega})} \leq C$ for all $t > 0$, we have

$$\left| \frac{d}{2dt} \int_\Omega u^2 \right| \leq C \int_\Omega |\nabla u|^2 + C(\epsilon) \|v\|^2_{W^{1,2}} + C\mu \int_{\Gamma_2} \frac{V(u)v}{1+v} + C \int_\Omega u^2$$

Thanks to (7.46), we get

$$\int_\tau^t \left| \frac{d}{2dt} \int_\Omega u^2 \right| \leq C \quad \forall t > \tau. \tag{7.49}$$

Finally, estimates (7.47) and (7.49) together with Lemma 7.20 entail

$$\lim_{t \to +\infty} \|u(t)\|_2 = 0. \tag{7.50}$$

Also thanks to $\|u(t)\|_{C(\overline{\Omega})} \leq C$ for all $t > 0$ we obtain

$$\lim_{t \to +\infty} \|u(t)\|_p = 0, \quad \forall p > 2. \tag{7.51}$$

We observe that Lemma 7.9 together with the embedding $X_\beta \hookrightarrow W^{k,p}$ (see for instance [59, Theorem 1.6.1]) assures that

$$\|u(t)\|_{W^{k,p}} \leq C \quad \forall k < 1,\ p \geq 2. \tag{7.52}$$

Next, the Gagliardo-Nirenberg inequality entails

$$\|u(t)\|_{W^{m,p}} \leq C\|u(t)\|^{\theta}_{W^{k,p}}\|u(t)\|_p^{1-\theta},$$

for $m < k\theta$, $\theta \in (0,1)$. Therefore, we have

$$\lim_{t\to+\infty}\|u(t)\|_{W^{m,p}} \leq C\lim_{t\to+\infty}\|u(t)\|_p^{1-\theta} = 0. \tag{7.53}$$

■

Remark 7.22. *Let us point out that if we pick m such that $m - d/p > 0$ then $W^{m,p}(\Omega)$ is embedded $\mathcal{C}(\overline{\Omega})$.*

7.5.2. Case $\lambda > 0$.

In order to do that we impose the following condition. Assume that there exists t_0 such that

$$u(t) > \delta_0 > 0,\ \forall t > t_0 > 0. \tag{H}$$

Next, we show the long time behavior for u under the hypothesis (H) and after we will give sufficient conditions on $V(u)$ that imply (H).

Theorem 7.23. *Let $0 \leq \mu < \mu_1$ and assume the the hypothesis (H) is satisfied, then there exists $\theta > 0$ such that*

$$\|u(t) - \lambda\|_{W^{m,p}} \leq Ce^{-\theta t}, \tag{7.54}$$

for all $t \geq t_0$ and any $m < 1$, $p \geq 2$.

Proof. On multiplying the u-equation by $u - \lambda$ we have

$$\begin{aligned} \frac{d}{2dt}\int_\Omega (u-\lambda)^2 = &\ -\int_\Omega |\nabla u|^2 + \int_\Omega V(u)\nabla v\cdot\nabla u - \\ &-\mu\int_{\Gamma_2}\frac{v}{1+v}V(u)(u-\lambda) - \int_\Omega u(u-\lambda)^2 \\ \leq &\ -\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\|V\|_\infty^2}{2}\int_\Omega |\nabla v|^2 + \\ &+\mu\|V(u)(u-\lambda)\|_{2,\Gamma_2}\left(\int_{\Gamma_2}\frac{v^2}{(1+v)^2}\right)^{1/2} - \int_\Omega u(u-\lambda)^2. \end{aligned} \tag{7.55}$$

Having in mind that $(1+v)^2 \geq 1$, the hypothesis (H) and the Sobolev trace embedding

$$W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$$

we get

$$\frac{d}{dt}\int_\Omega (u-\lambda)^2 + 2\delta_0\int_\Omega (u-\lambda)^2 \leq C\|v\|^2_{W^{1,2}} + \mu C\|v\|_{W^{1,2}} \tag{7.56}$$

Easily, from Lemma 7.19 we can deduce

$$\|u(t) - \lambda\|_2^2 \leq Ce^{-\theta_1 t}$$

for $0 < \theta_1 < \min\{2\delta_0, \beta\}$. At this point we can argue exactly as in the end of Theorem 7.21 to conclude. ∎

In the rest of the chapter we give conditions on V that imply (H). Such conditions on V involve only the behavior of V around zero. Roughly speaking we require a superlinear grow of V around zero. From now on we assume that there exist $C, \delta_0 > 0$, $k > 1 + d/2$, $j > d/2$ such that

$$(H1) \qquad 0 < V(s) < Cs^k\,, \qquad |V'(s)| \le Cs^j$$

for all $s \in (0, \delta_0)$.

Remark 7.24. *The condition* $(H1)$ *is satisfied, for example, for functions*

$$V(u) = \frac{u^\alpha}{1 + u^\alpha}$$

with $\alpha > 1 + d/2$.

Lemma 7.25. *Assume that* $0 \le \mu < \mu_1$ *and that* $(H1)$ *is satisfied then* (H) *is verified.*

Proof. Let $\delta > 0$ a constant to be fixed. Given a function f, we define the negative part of f as follows $f_- := \min\{f, 0\}$. Our purpose is to show that $\|(u-\delta)_-(t)\|_\infty \le \delta/2$ for every $t > t_0$ which implies this claim. In order to obtain the previous estimate we multiply the u-equation by $(u-\delta)_-$ and we integrate in the space variable to obtain

$$\begin{aligned}
\frac{d}{2dt}\int_\Omega (u-\delta)_-^2 &= -\int_\Omega (\nabla u - V(u)\nabla v)\cdot\nabla(u-\delta)_- + \\
&\quad + \int_{\partial\Omega}\left(\frac{\partial u}{\partial n} - V(u)\frac{\partial v}{\partial n}\right)(u-\delta)_- + \int_\Omega u(\lambda - u)(u-\delta)_- \\
&= -\int_\Omega |\nabla(u-\delta)_-|^2 + \int_\Omega V(u)\nabla v\cdot\nabla(u-\delta)_- - \\
&\quad - \int_{\Gamma_2} V(u)\mu\frac{v}{1+v}(u-\delta)_- + \int_\Omega u(\lambda-u)(u-\delta)_- \\
&= -\int_\Omega |\nabla(u-\delta)_-|^2 + \int_{\Omega_\delta} V(u)\nabla v\cdot\nabla(u-\delta)_- - \\
&\quad - \mu\int_{\Gamma_\delta}\frac{v}{1+v}V(u)(u-\delta)_- + \int_\Omega u(\lambda-u)(u-\delta)_-,
\end{aligned}$$

where

$$\Omega_\delta := \{x \in \Omega : \; u(x) < \delta\}\,, \quad \Gamma_\delta := \{x \in \Gamma_2 : \; u(x) < \delta\}.$$

Next, we get

$$\begin{aligned}
\frac{d}{2dt}\int_\Omega (u-\delta)_-^2 &\le (\epsilon - 1)\int_\Omega |\nabla(u-\delta)_-|^2 + C(\epsilon)\int_{\Omega_\delta} V^2(u)|\nabla v|^2 - \\
&\quad - \mu\int_{\Gamma_\delta}\frac{v}{1+v}V(u)(u-\delta)_- + \int_\Omega u(\lambda-u)(u-\delta)_- \\
&\le (\epsilon - 1)\int_\Omega |\nabla(u-\delta)_-|^2 + C(\epsilon)\sup_{s\in(0,\delta)} V^2(s)\int_\Omega |\nabla v|^2 - \\
&\quad - \mu\int_{\Gamma_\delta}\frac{v}{1+v}V(u)(u-\delta)_- + \int_\Omega u(\lambda-u)(u-\delta)_-.
\end{aligned}$$

Let

$$f(\delta) := \sup_{s\in(0,\delta)} V^2(s).$$

We obtain

$$\frac{d}{2dt}\int_\Omega (u-\delta)_-^2 \leq \ (\epsilon-1)\int_\Omega |\nabla(u-\delta)_-|^2 + C(\epsilon)f(\delta)\int_\Omega |\nabla v|^2 + \mu\widetilde{\epsilon}\int_{\Gamma_2}\frac{v^2}{(1+v)^2}+$$
$$+\mu C(\widetilde{\epsilon})\int_{\Gamma_\delta} V(u)^2(u-\delta)_-^2 + \int_\Omega u(\lambda-u)(u-\delta)_-.$$

Thanks to the Sobolev trace embedding $W^{1,2}(\Omega)\hookrightarrow L^2(\partial\Omega)$ and having in mind that $(v+1)^2\geq 1$, we have

$$\int_{\Gamma_2} V(u)^2(u-\delta)_-^2 \leq C\left(\int_\Omega V^2(u)(u-\delta)_-^2+\right.$$
$$\left.+\int_\Omega (2(u-\delta)_-^2 V'(u)^2 + 2V^2(u))|\nabla(u-\delta)_-|^2\right),$$

$$\mu\widetilde{\epsilon}\int_{\Gamma_2}\frac{v^2}{(1+v)^2} \leq C\mu\widetilde{\epsilon}\|v\|^2_{W^{1,2}}.$$

Therefore, we obtain

$$\frac{d}{2dt}\int_\Omega (u-\delta)_-^2 \leq \ (\epsilon-1)\int_\Omega |\nabla(u-\delta)_-|^2 + C(\epsilon)f(\delta)\int_\Omega |\nabla v|^2 + C\widetilde{\epsilon}\|v\|^2_{W^{1,2}}+$$
$$+C(\widetilde{\epsilon})\int_\Omega V^2(u)(u-\delta)_-^2 + C(\widetilde{\epsilon})\int_\Omega (2(u-\delta)_-^2 V'(u)^2 + 2V^2(u))|\nabla(u-\delta)_-|^2+$$
$$+\int_\Omega u(\lambda-u)(u-\delta)_-.$$

Let

$$g(\delta) := \sup_{s\in(0,\delta)} (2(s-\delta)_-^2 V'(s)^2 + 2V^2(s)).$$

Hence, taking into account that $-\delta < (u-\delta)_-$, we get

$$\frac{d}{2dt}\int_\Omega (u-\delta)_-^2 \leq \ (\epsilon + C(\widetilde{\epsilon})g(\delta)-1)\int_\Omega |\nabla(u-\delta)_-|^2 + (C(\epsilon)f(\delta)+\mu\widetilde{\epsilon})\|v\|^2_{W^{1,2}}+$$
$$+\int_\Omega u(u-\delta)_-\left(\lambda - u - C(\widetilde{\epsilon})\frac{V^2(u)}{u}\delta\right)$$

Observe that if

$$C(\widetilde{\epsilon})g(\delta) < 1-\epsilon \tag{7.57}$$

and

$$C(\widetilde{\epsilon})\frac{V^2(s)}{s}\delta \leq \lambda-\delta \ \ \forall s\in(0,\delta) \tag{7.58}$$

then, thanks to the strong maximum principle for the u-equation we have $u(\tau) > \delta_0 > \delta$ for arbitrary $\tau > 0$ small enough. Thereofore, applying Lemma 7.19 we obtain after integrating in the time variable that

$$\|(u-\delta)_-(t)\|_2^2 \leq (2C(\epsilon)f(\delta)+2\mu\widetilde{\epsilon})C(\beta),$$

for $t > \tau > 0$. We try to prove that

$$\|(u-\delta)_-(t)\|_2^2 \le \delta^{2\alpha},$$

for some $\alpha > 1$. To this end, we impose the following conditions

$$2C(\epsilon)f(\delta)C(\beta) \le \frac{\delta^{2\alpha}}{2} \tag{7.59}$$

and

$$2\mu\widetilde{\epsilon}C(\beta) \le \frac{\delta^{2\alpha}}{2}. \tag{7.60}$$

We have to check that there exist $\epsilon, \widetilde{\epsilon}, \delta > 0$ such that conditions (7.57)-(7.60) are satisfied simultaneously. Let us observe that the inequality (7.60) is satisfied for

$$\widetilde{\epsilon} \le \frac{\delta^{2\alpha}}{2\mu C(\beta)},$$

so,

$$C(\widetilde{\epsilon}) = C(\beta, \mu)\delta^{-2\alpha}.$$

Then, we pick $\epsilon = 1/2$, thus $C(\epsilon) = 1/2$. Thanks to $(H1)$, we have

$$f(\delta)C(\beta) = \sup_{s\in(0,\delta)} V^2(s)C(\beta) \le C\delta^{2k}C(\beta).$$

Hence, for $\alpha < k$ and δ sufficiently small the inequality (7.59) is satisfied . Now, owing to $(H1)$, we observe that

$$C(\widetilde{\epsilon})\frac{V^2(s)}{s}\delta \le C(\widetilde{\epsilon}) \sup_{s\in(0,\delta)} \frac{V^2(s)}{s}\delta \le C(\widetilde{\epsilon})\delta^{2k},$$

So, condition (7.58) can be assured for $\alpha < k$ and δ small enough. Now, it is straightforward to see that condition (7.57) it is also satisfied for $1 < \alpha < \min\{k, 1+j\}$. Next we use interpolation between L^p spaces to obtain

$$\begin{aligned} \|(u-\delta)_-\|_{2/\theta_1} &\le \|(u-\delta)_-\|_2^{\theta_1}\|(u-\delta)_-\|_\infty^{1-\theta_1} \\ &\le \delta^{\alpha\theta_1}\delta^{1-\theta_1} = \delta^{1+(\alpha-1)\theta_1} \end{aligned}$$

Next, we apply [7, Theorem 7.2] for $s_0 = 0$ and $s_1 = 1$. So, we infer

$$\|(u-\delta)_-\|_{W^{\theta,2/\theta_1}} \le C\|(u-\delta)_-\|_{W^{1,2/\theta_1}}^{\theta}\|(u-\delta)_-\|_{2/\theta_1}^{1-\theta},$$

In order to have $W^{\theta,2/\theta_1}(\Omega) \hookrightarrow L^\infty(\Omega)$ we pick θ_1 such that

$$\theta - \frac{d\theta_1}{2} > 0 \tag{7.61}$$

Thus, $\theta > \frac{d\theta_1}{2}$. It should be satisfied also that

$$(1-\theta)(1+(\alpha-1)\theta_1) > 1. \tag{7.62}$$

So,

$$1 < \left(1 - \frac{d\theta_1}{2}\right)(1 + (\alpha - 1)\theta_1).$$

After some algebra, it is possible to find $\theta, \theta_1 \in (0,1)$ satisfying (7.61) and (7.62) if and only if $\alpha > 1 + \frac{d}{2}$. Therefore using the uniform bound in time in $W^{1,p}(\Omega)$ (that is a consequence of [7, Theorem 15.5]) and the Sobolev embedding we obtain

$$\|(u - \delta)_-(t)\|_\infty \leq \frac{\delta}{2},$$

for $t \geq \tau > 0$. The last estimate concludes easily the Lemma. ■

7.6. Interpretation

In this chapter we have analyzed a problem modelling the angiogenesis. For that, we have included a nonlinear chemotactic sensitivity, a logistic term to model the growth rate of the ECs and a nonlinear term at the boundary of the tumor. We have shown the validity of the model proving the existence and the uniqueness of the global positive solution of the model.

Let us interpret some of our results. Fix the growth rate of ECs, that is, fix $\lambda > 0$; then $F(\lambda)$ is also fixed. Hence, we can define $\mu_1(\lambda)$ and $\mu_2(\lambda)$ as

$$\mu_1(\lambda) := \min\{\mu : \lambda = \Lambda(\mu)\}, \qquad \mu_2(\lambda) := \max\{\mu : \lambda = \Lambda(\mu)\}.$$

It is clear that $\mu_1(\lambda) = \mu_2(\lambda) = +\infty$ if $V'(0) = 0$. On the other hand, remember that in the case $V'(0) > 0$ for $\mu > \mu_0(\lambda)$ there does not exist any coexistence state, see Proposition 7.15. Our results of local stability, Proposition 7.18, show that when μ is large enough, $\Lambda(\mu) > \lambda$, then the semi-trivial solution $(0, \theta_\mu)$ is linearly asymptotically stable, and so, this solution indicates the long time behavior of the parabolic problem.

Taking $V'(0) > 0$ but small, we get that $F(\lambda) < \mu_1(\lambda)$.

With this notation, we know that:

a) In both cases ($V'(0) = 0$ and $V'(0) > 0$) for μ small, that is $\mu < F(\lambda)$, the TAF disappears, i.e., if the TAF generated by the tumor is consumed by the ECs.

b) If $V'(0) = 0$ and $\mu > F(\lambda)$, then both populations, TAF and ECs, coexist and so the angiogenesis occurs.

c) If $V'(0) > 0$ and $\mu \in (F(\lambda), \mu_1(\lambda))$, then both populations again coexist and so the angiogenesis occurs. However, when μ is much larger, $\mu > \max\{\mu_2(\lambda), \mu_0(\lambda)\}$, then the ECs tends to the extinction, and so the angiogenesis does not occur.

Hence, for μ large there is a drastic change of behavior between the cases $V'(0) = 0$ and $V'(0) > 0$. Indeed, when $V'(0) > 0$ and μ is large, the ECs are sensible to the TAF, they move towards the tumor and there, as μ is large, the TAF is also large, the two populations compete and they come into contact and the ECs die. However, when $V'(0) = 0$, the ECs are nearly insensible to the TAF call, and so they do not move quickly towards the tumor and so they do not come into contact with TAF. In summary, in this model a case of little sensitivity response and a tumor generating a lot of TAF drives the system to the extinction of ECs in the domain, due to the competition between the TAF and the ECs.

CHAPTER 8

On a model related to pattern formation

This chapter deals with a nonlinear system of parabolic-elliptic type with a logistic source term related to pattern formation. We prove the existence of a unique positive global in time classical solution. Moreover it is proved, under the assumption of sufficiently strong logistic dumping, that there is only one nonzero equilibrium, and all the solutions to the non-stationary tend to this steady-state for large times.

8.1. Preliminaries

During this work we consider the following system of equations

$$
(8.1) \qquad \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u(1-u) & \text{in } \Omega \times (0,T), \\ 0 = \Delta v - v + \dfrac{u}{1+u} & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} - \chi u \dfrac{\partial v}{\partial n} = r(1-u)\,, \quad \dfrac{\partial v}{\partial n} = r' \left(\dfrac{1}{2} - v \right) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}
$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with regular boundary, μ, r, r' and χ denote positive constants. Since we are interested only in non-negative solutions we assume that $u_0(x) \geq 0$ in Ω. First we observe that

$$
\chi \nabla \cdot (u \nabla v) = \chi (\nabla u \cdot \nabla v + u \Delta v) = \chi \left(\nabla u \cdot \nabla v + u \left(v - \frac{u}{1+u} \right) \right).
$$

So, we may rewrite the system (8.1) as follows

$$
(8.2) \qquad \begin{cases} u_t = \Delta u - \chi \nabla u \cdot \nabla v + \chi u \left(\dfrac{u}{1+u} - v \right) + \mu u(1-u) & \text{in } \Omega \times (0,T), \\ 0 = \Delta v - v + \dfrac{u}{1+u} & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = \chi u r' \left(\dfrac{1}{2} - v \right) + r(1-u)\,, \quad \dfrac{\partial v}{\partial n} = r' \left(\dfrac{1}{2} - v \right) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}
$$

Next, in order to avoid the singularity of $\dfrac{u}{u+1}$ we consider the penalized function

$$h(u) := \frac{u_+}{u_+ + 1}$$

and we consider the system

$$\text{(8.3)}\qquad \begin{cases} u_t = \Delta u - \chi \nabla u \cdot \nabla v + \chi u \left(h(u) - v + \dfrac{\mu}{\chi}(1-u) \right) & \text{in } \Omega \times (0,T), \\ 0 = \Delta v - v + h(u) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = \chi u r' \left(\dfrac{1}{2} - v \right) + r(1-u)\,, \quad \dfrac{\partial v}{\partial n} = r' \left(\dfrac{1}{2} - v \right) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Observe that we may consider the equation for u as a linear equation

$$U_t - \Delta U = -\chi \nabla U a(x,t) + \chi U b(x,t) \text{ in } \Omega,$$

$$\frac{\partial U}{\partial n} = \chi U r' c(x,t) + r(1-U) \text{ on } \partial\Omega.$$

where

$$\begin{aligned} a(x,t) &:= \nabla v, \\ b(x,t) &:= h(u) - v + \frac{\mu}{\chi}(1-u), \\ c(x,t) &:= r' \left(\frac{1}{2} - v \right). \end{aligned}$$

Thus, if $b(x,t) \in L^\infty(\Omega \times (0, T_{max}))$ then, by the maximum principle, $U(x,t) = u(x,t) \geq 0$ in $\Omega \times [0,T)$ and a solution to (8.3) is a solution (8.2) and viceversa. Now, having in mind that for positive solutions the systems (8.2) and (8.3) are equivalent if $u, v \in L^\infty(\Omega \times (0, T_{max}))$ we will show the local existence theorem for (8.3). Previously we prove the following result

Lemma 8.1. *Let $1 < \beta < 2\alpha < 1 + \frac{1}{p}$ and $(W_{\mathcal{B}}^{2\alpha-2,p}, A_{\alpha-1})$ the interpolation-extrapolation scales generated by $A_0 := -\Delta + I$ and the real interpolation functor then*

$$\|e^{-tA_{\alpha-1}} u\|_{W^{\beta,p}} \leq C e^{-\nu t} t^{-\theta} \|u\|_{W_{\mathcal{B}}^{2\alpha-2,p}},$$

with $\theta(\beta) \in (0,1)$ and $\nu \in (0,1)$.

Proof. By Theorem 7.2 we have $(W_{\mathcal{B}}^{2\alpha,p}, W_{\mathcal{B}}^{2\alpha-2,p})_{\theta,p} = W_{\mathcal{B}}^{\beta,p}$, therefore, using (7.3) there exists $\theta \in (0,1)$ such that

$$\|e^{-tA_{\alpha-1}} u\|_{W_{\mathcal{B}}^{\beta,p}} \leq \|e^{-tA_{\alpha-1}} u\|_{W_{\mathcal{B}}^{2\alpha,p}}^{\theta} \|e^{-tA_{\alpha-1}} u\|_{W_{\mathcal{B}}^{2\alpha-2,p}}^{1-\theta}.$$

Next, we apply [7, Theorem 8.5] together with [7, (3.1)] and we get

$$\|e^{-tA_{\alpha-1}} u\|_{W_{\mathcal{B}}^{\beta,p}} \leq \|(I + A_{\alpha-1}) e^{-tA_{\alpha-1}} u\|_{W_{\mathcal{B}}^{2\alpha-2,p}}^{\theta} \|e^{-tA_{\alpha-1}} u\|_{W_{\mathcal{B}}^{2\alpha-2,p}}^{1-\theta}.$$

Taking into account that $I+A_{\alpha-1}$ and $A_{\alpha-1}$ are sectorial operators with $Re\sigma(I+A_{\alpha-1}) = 1 + Re\sigma(A_{\alpha-1}) = 2$ then we invoke [59, Theorem 1.3.4]. Thus, there exists $\nu \in (0,1)$, $\alpha \in (0,2)$ such that

$$\begin{aligned}\|e^{-tA_{\alpha-1}}u\|_{W^{\beta,p}_{\mathcal{B}}} &\leq \|(I+A_{\alpha-1})e^{-t(I+A_{\alpha-1})}e^{tI}u\|^{\theta}_{W^{2\alpha-2,p}_{\mathcal{B}}}\|e^{-tA_{\alpha-1}}u\|^{1-\theta}_{W^{2\alpha-2,p}_{\mathcal{B}}}\\ &= e^{\theta t}\|(I+A_{\alpha-1})e^{-t(I+A_{\alpha-1})}u\|^{\theta}_{W^{2\alpha-2,p}_{\mathcal{B}}}\|e^{-tA_{\alpha-1}}u\|^{1-\theta}_{W^{2\alpha-2,p}_{\mathcal{B}}}\\ &\leq e^{(\theta(1-\alpha)-\nu(1-\theta))t}t^{-\theta}\|u\|_{W^{2\alpha-2,p}_{\mathcal{B}}}.\end{aligned}$$

Finally, we pick $1-\alpha = -\nu$ and use the fact that $W^{\beta,p}_{\mathcal{B}} = W^{\beta,p}$ to conclude the proof. ■

Lemma 8.2. *Let $1 < \beta < 2\alpha < 1 + \frac{1}{p}$ then there exists $\theta < 1$ such that $X_\theta \hookrightarrow W^{\beta,p}$, where $X_\theta := D((I + A_{\alpha-1})^\theta)$*

Proof. Arguing in the same manner as we did in the previous lemma we have that there exists $\theta < 1$ such that

$$\|u\|_{W^{\beta,p}} \leq C\|(I+A_{\alpha-1})u\|^{\theta}_{W^{2\alpha-2,p}}\|u\|^{1-\theta}_{W^{2\alpha-2,p}}.$$

Finally the lemma can be concluded with the use of [59, pg. 28, Exer. 11]. ■

8.2. Global existence in time

Theorem 8.3. *Let $p > d$ and consider the initial data $u_0 \in W^{1,p}(\Omega)$ with $u_0 \geq 0$. Then there exists $\tau(\|u_0\|_{W^{1,p}})$ such that the system (8.3) has a unique positive local in time solution*

$$(u,v) \in \left(\mathcal{C}([0,\tau]; W^{1,p}(\Omega)) \cap \mathcal{C}^1((0,\tau); \mathcal{C}^{2+\alpha}(\overline{\Omega}))\right)^2,$$

and $u(x,t), v(x,t) \geq 0$ for $(x,t) \in \overline{\Omega} \times [0,\tau]$. Moreover, the solution depends continuously on the initial data, i.e. if $\mathbf{u}(u_0)$ and $\mathbf{u}(\overline{u}_0)$ denote the solutions to (8.3) with inial data u_0 and $\overline{u}_0$ respectively then

$$\|\mathbf{u}(u_0) - \mathbf{u}(\overline{u}_0)\|_{(\mathcal{C}([0,\tau];W^{1,p}))^2} \leq C\|u_0 - \overline{u}_0\|_{W^{1,p}}.$$

Proof. The proof of the Theorem it is based on a standard fixed point argument. Let

$$X_T := \mathcal{C}([0,T]; W^{1,p}(\Omega)).$$

For each $f \in X_T$ we consider the operator

$$\begin{aligned} S : X_T &\rightarrow \mathcal{C}([0,T]; W^{2,p}(\Omega))\\ f &\mapsto S(f) = v, \end{aligned}$$

where v is the unique solution to

$$\begin{cases} -\Delta v + v = h(f) & \text{in } \Omega \times (0,T),\\ \dfrac{\partial v}{\partial n} = r'\left(\dfrac{1}{2} - v\right) & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Moreover for each $t \in [0,T]$, thanks to Theorem 2.13, the following estimate is satisfied

$$\|v(t)\|_{W^{2,p}} \le C(t)\left(\|h(f)\|_p + \|1/2r'\|_{W^{1-1/p,p}(\partial\Omega)}\right). \tag{8.4}$$

Next, we consider the operator

$$\begin{array}{rcl} H : X_T & \to & X_T \\ f & \mapsto & H(f) = u, \end{array}$$

where u is the unique solution to the linear parabolic problem

$$\begin{cases} u_t - \Delta u = -\chi\nabla f\cdot\nabla v + \chi f(h(f)-v) + \mu f(1-f) & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial n} = \chi f r'\left(\dfrac{1}{2}-v\right) + r(1-f) & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{8.5}$$

Let $2\alpha \in \left(1, 1+\frac{1}{p}\right)$. Thanks to the generalized variations of constants formula we can rewrite (8.5) in the following manner

$$u(t) = e^{-tA_{\alpha-1}}u_0 + \int_0^t e^{-(t-\tau)A_{\alpha-1}}(\theta(f,v) + A_{\alpha-1}\mathcal{B}^c\gamma g(f,v))d\tau, \tag{8.6}$$

where $\gamma : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$ denotes the trace operator and

$$\begin{array}{rcl} \theta(f,v) & := & -\chi\nabla f\cdot\nabla v + \chi f(h(f)-v) + \mu f(1-f), \\ g(f,v) & := & \chi f r'\left(\frac{1}{2}-v\right) + r(1-f). \end{array}$$

Let us point out that, since $A_{\alpha-1}\mathcal{B}^c \in \mathcal{L}(W^{2\alpha-1-1/p,p}(\partial\Omega), W^{2\alpha-2,p}_{\mathcal{B}})$ together with the embedding $W^{1-1/p,p}(\partial\Omega) \hookrightarrow W^{2\alpha-1-1/p,p}(\partial\Omega)$ then we can assert that $A_{\alpha-1}\mathcal{B}^c\gamma$ is well defined for $g(f,v) \in W^{1,p}(\Omega)$. Next, we define the closed set

$$B_R^T := \{f \in \mathcal{C}([0,T]; W^{1,p}(\Omega)) \; : \; \|f\|_{X_T} \le R\}.$$

Now, we have to verify that the conditions of the Banach fixed point Theorem are satisfied for H.

Step 1. For a given $f \in B_R^T$ there exist $R, T > 0$ such that $H(f) \subset B_R^T$.
From (8.6), thanks to the embedding $W^{\beta,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ we get

$$\begin{array}{rl} \|u(t)\|_{W^{1,p}} \le C\|u_0\|_{W^{1,p}} + C\displaystyle\int_0^t & e^{-\nu(t-\tau)}(t-\tau)^{-\theta}(\|\theta(f,v)\|_{W^{2\alpha-2,p}_{\mathcal{B}}} + \\ & +\|A_{\alpha-1}\mathcal{B}^c\gamma g(f,v)\|_{W^{2\alpha-2,p}_{\mathcal{B}}})d\tau. \end{array}$$

Taking into account the embedding $W^{1-1/p,p}(\partial\Omega) \hookrightarrow W^{2\alpha-1-1/p,p}(\partial\Omega)$ we have

$$\begin{array}{rl} \|A_{\alpha-1}\mathcal{B}^c\gamma g(f,v)\|_{W^{2\alpha-2,p}(\partial\Omega)} & \le C\|\gamma g(f,v)\|_{W^{2\alpha-1-1/p}(\partial\Omega)} \\ & \le C\|\gamma g(f,v)\|_{W^{1-1/p,p}(\partial\Omega)} \\ & \le C\|g(f,v)\|_{W^{1,p}} \end{array}$$

also, having in mind [7, (7.5),(7.8)], we have $L^p := W^{0,p}_{\mathcal{B}} \hookrightarrow W^{2\alpha-1,p}_{\mathcal{B}}$. Thus, we infer

$$\|u(t)\|_{W^{1,p}} \le C\|u_0\|_{W^{1,p}} + C\int_0^t e^{-\nu(t-\tau)}(t-\tau)^{-\theta}(\|\theta(f,v)\|_p + \|g(f,v)\|_{W^{1,p}})d\tau. \tag{8.7}$$

On one hand we have

$$\|\theta(f,v)\|_{L^p} \le \chi(\|\nabla f\cdot\nabla v\|_p + \|fh(f)\|_p + \|fv\|_p) + \mu(\|f\|_p + \|f^2\|_p). \tag{8.8}$$

Now, we estimate each term of (8.8) separately

$$\begin{aligned}\|\nabla f\cdot\nabla v\|_p &\le \|\nabla v\|_\infty\|\nabla f\|_p\\ &\le C\|v\|_{W^{1,\infty}}\|f\|_{W^{1,p}}\\ &\le C\|v\|_{W^{2,p}}\|f\|_{W^{1,p}}\\ &\le C(\|h(f)\|_p + C(r'))\|f\|_{W^{1,p}}\\ &\le C(\|h(f)\|_\infty + C(r'))\|f\|_{W^{1,p}}\\ &\le C(\|f\|_{W^{1,p}} + C(r'))\|f\|_{W^{1,p}}.\end{aligned}$$

The remaining terms of (8.8) can be estimated in a similar way to obtain

$$\|\theta(f,v)\|_p \le C(\chi,\mu,r',\|f\|_{W^{1,p}}), \tag{8.9}$$

with $C(\chi,\mu,r',\|f\|_{W^{1,p}})$ an increasing function on its arguments. On the other hand we have

$$\|g(f,v)\|_{W^{1,p}} \le \frac{\chi r'}{2}\|f\|_{W^{1,p}} + \chi r'\|fv\|_{W^{1,p}} + \|r\|_{W^{1,p}} + r\|f\|_{W^{1,p}}. \tag{8.10}$$

The term $\|fv\|_{W^{1,p}}$ is estimated as follows

$$\begin{aligned}\|fv\|_{W^{1,p}} &\le C\|f\|_{W^{1,p}}\|v\|_{W^{1,\infty}}\\ &\le C\|f\|_{W^{1,p}}(\|h(f)\|_p + C(r'))\\ &\le C\|f\|_{W^{1,p}}(\|f\|_{W^{1,p}} + C(r')).\end{aligned}$$

So, we obtain

$$\|g(f,v)\|_{W^{1,p}} \le C(\chi,r,r',\|f\|_{W^{1,p}}), \tag{8.11}$$

where $C(\chi,r,r',\|f\|_{W^{1,p}})$ is an increasing function on its arguments. Now, we plug (8.9) and (8.11) in (8.7) to obtain

$$\begin{aligned}\|u(t)\|_{W^{1,p}} &\le C\|u_0\|_{W^{1,p}} + C(R)\int_0^t Ce^{-\nu(t-\tau)}(t-\tau)^{-\theta}d\tau\\ &\le C\|u_0\|_{W^{1,p}} + C(R)T^{1-\theta}.\end{aligned}$$

Thus, choosing $R > C\|u_0\|_{W^{1,p}}$ and $\tau_0 = T(R)$ sufficiently small then $\|u\|_{X_{\tau_0}} \le R$. Moreover, $\|u\|_{X_T} \le R$ for all $T \le \tau_0$. Now, we fix $R > C\|u_0\|_{W^{1,p}}$ and $T \le \tau_0$ is free to our disposal.

Step 2. H is contractive.

Let $f_1, f_2 \in B_R^T$ then, $u_1 = H(f_1) \in B_R^T$ and $u_2 = H(f_2) \in B_R^T$ and

$$u_1(t) - u_2(t) = \int_0^t e^{-(t-\tau)A_{\alpha-1}}((\theta(f_1,v_1) - \theta(f_2,v_2)) + A_{\alpha-1}\mathcal{B}^c\gamma(g(f_1,v_1) - g(f_2,v_2)))d\tau.$$

So, we obtain

$$\begin{aligned}\|u_1(t) - u_2(t)\|_{W^{1,p}} \le \int_0^t \; & e^{-\nu(t-\tau)}(t-\tau)^{-\theta}(\|\theta(f_1,v_1) - \theta(f_2,v_2)\|_p +\\ & + \|g(f_1,v_1) - g(f_2,v_2)\|_{W^{1,p}})d\tau.\end{aligned} \tag{8.12}$$

On one hand we have

(8.13)
$$\begin{aligned}\|\theta(f_1,v_1)-\theta(f_2,v_2)\|_p \leq\ & \chi\|\nabla f_2\cdot\nabla v_2-\nabla f_1\cdot\nabla v_1\|_p+\chi\|f_1h(f_1)-f_2h(f_2)\|_p+\\ &+\mu\|f_1-f_2\|_p+\mu\|f_2^2-f_1^2\|_p.\end{aligned}$$

Now, we estimate each term of (8.13) separately

$$\begin{aligned}\|\nabla f_2\cdot\nabla v_2-\nabla f_1\cdot\nabla v_1\|_p &\leq \|\nabla(f_2-f_1)\cdot\nabla v_2\|_p+\|\nabla f_1\cdot\nabla(v_2-v_1)\|_p\\ &\leq C(\|v_2\|_{W^{2,p}}\|f_1-f_2\|_{W^{1,p}}+R\|v_1-v_2\|_{W^{2,p}})\\ &\leq C(C(R)\|f_1-f_2\|_{W^{1,p}}+R\|h(f_1)-h(f_2)\|_p).\end{aligned}$$

Taking into account that $\|h(f_1)-h(f_2)\|_p \leq \|(f_1)_+-(f_2)_+\|_\infty \leq \|f_1-f_2\|_\infty$ we get

$$\|\nabla f_2\cdot\nabla v_2-\nabla f_1\cdot\nabla v_1\|_p \leq C(R)\|f_1-f_2\|_{W^{1,p}}.$$

For the rest of the terms of (8.13) we can argue in a similar way to obtain

$$\|\theta(f_1,v_1)-\theta(f_2,v_2)\|_p \leq C(R)\|f_1-f_2\|_{W^{1,p}}. \tag{8.14}$$

On the other hand we have

$$\|g(f_1,v_1)-g(f_2,v_2)\|_{W^{1,p}} \leq \left(\frac{\chi r}{2}+r\right)\|f_1-f_2\|_{W^{1,p}}+\chi r'\|f_2v_2-f_1v_1\|_{W^{1,p}}.$$

We deduce

$$\begin{aligned}\|f_2v_2-f_1v_1\|_{W^{1,p}} &\leq \|v_2(f_2-f_1)\|_{W^{1,p}}+\|f_1(v_2-v_1)\|_{W^{1,p}}\\ &\leq \|v_2\|_{W^{1,\infty}}\|f_1-f_2\|_{W^{1,p}}+\|f_1\|_{W^{1,p}}\|v_2-v_1\|_{W^{1,\infty}}\\ &\leq C(R)\|f_1-f_2\|_{W^{1,p}}.\end{aligned}$$

Thus, we get

$$\|g(f_1,v_1)-g(f_2,v_2)\|_{W^{1,p}} \leq C(R)\|f_1-f_2\|_{W^{1,p}}. \tag{8.15}$$

Finally, we put the estimates (8.14) and (8.15) in (8.12) to obtain

$$\|u_1-u_2\|_{X_T} \leq C(R)T^{1-\theta}\|f_1-f_2\|_{X_T}.$$

Hence, taking T sufficiently small we prove that H is contractive.

Now we deal with the regularity of the solution. Let us fix any $t\in(0,\tau)$ then the equation for u has the abstract representation

$$\frac{du}{dt}+(I+A_{\alpha-1})u=f(x,t),\quad u(0)=u_0.$$

Thus, thanks to [59, Theorem 3.5.2] $\frac{du}{dt}(t)\in X_\theta$ with $\theta<1$. In particular, we have $\frac{du}{dt}(t)\in W^{\beta,p}$ for some $\beta>1$, $p>d$. So, we obtain $u\in\mathcal{C}^1((0,\tau);W^{1,p}(\Omega))$ and since the v-equation preserves the regularity in time then $v\in\mathcal{C}^1((0,\tau);W^{1,p}(\Omega))$. We observe that

$$\begin{cases}-\Delta v(t)+v(t)=h(u)(t) & \text{in }\Omega,\\ \dfrac{\partial v}{\partial n}(t)+r'v(t)=\dfrac{r'}{2} & \text{on }\partial\Omega.\end{cases}$$

Taking into account that $h(u)(t) \in \mathcal{C}^{\alpha}(\overline{\Omega})$ then the elliptic regularity assures $v(t) \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$. So, we have proved that $v \in \mathcal{C}^1((0,\tau);\mathcal{C}^{2+\alpha}(\overline{\Omega}))$. Now we rewrite the u-equation as follows

$$\begin{cases} -\Delta u(t) + \nabla u(t)\cdot\nabla v(t) = f(t) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(t) + \left(r - \chi r'\left(\dfrac{1}{2} - v\right)\right)u(t) = r & \text{on } \partial\Omega, \end{cases}$$

where

$$f(t) := (uh(u) - uv + \mu u(1-u) - u_t)(t).$$

Since $f(t) \in \mathcal{C}^{\alpha}(\overline{\Omega})$, $r - \chi r'\left(\frac{1}{2} - v\right)(t) \in \mathcal{C}^{1+\alpha}(\partial\Omega)$ and $\nabla v(t) \in \mathcal{C}^{\alpha}(\overline{\Omega})$ then elliptic regularity entails $u(t) \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$.

Next we observe that the positivity of (u,v) is consequence of the maximum principle for parabolic equations.

At the end we show the continuity respect to the initial data, for this purpose we argue in the following manner. Let $R > C(\|u_0\|_{W^{1,p}} + \|\overline{u}_0\|_{W^{1,p}})$. We have

$$\begin{aligned} u(u_0)(t) &= e^{-tA_{\alpha-1}}u_0 + \int_0^t e^{-(t-\tau)A_{\alpha-1}}(\theta(u(u_0),v(u_0)) + A_{\alpha-1}\mathcal{B}^c\gamma g(u(u_0),v(u_0)))d\tau \\ &= e^{-tA_{\alpha-1}}u_0 + H(u(u_0)) - e^{-tA_{\alpha-1}}u_0. \end{aligned}$$

Hence, we infer

$$\|(u(u_0) - u(\overline{u}_0))(t)\|_{W^{1,p}} \le 2\|e^{-tA_{\alpha-1}}(u_0 - \overline{u}_0)\|_{W^{1,p}} + \|H(u(u_0) - H(u(\overline{u}_0))\|_{W^{1,p}}.$$

Taking supremum on time, thanks to the contractivity of H, we obtain

$$\|u(u_0) - u(\overline{u}_0)\|_{X_T} \le C\|u_0 - \overline{u}_0\|_{W^{1,p}} + k\|u(u_0) - u(\overline{u}_0)\|_{X_T},$$

with $k < 1$. Also we have

$$\begin{aligned} \|(v(u_0) - v(\overline{u}_0))(t)\|_{W^{2,p}} &\le C(t)\|h(u(u_0))(t) - h(\overline{u}_0(u_0))(t)\|_{L^p} \\ &\le C(t)\|u(u_0) - u(\overline{u}_0)\|_{W^{1,p}}. \end{aligned} \tag{8.16}$$

From (8.16) the proof of the continuity can be easily concluded. ∎

Now we deal with the issue of global in time solutions. To this end we have just to show that $\|u(t)\|_{W^{1,p}} \le C(t)$ for all $t < T_{max}$ where T_{max} stands for the maximal interval of existence. We observe that

$$\begin{cases} -\Delta v + v = \dfrac{u}{1+u} & \text{in } \Omega\times(0,T_{max}), \\ \dfrac{\partial v}{\partial n} + r'v = \dfrac{r'}{2} & \text{on } \partial\Omega\times(0,T_{max}). \end{cases}$$

Since $\left\|\frac{u(t)}{1+u(t)}\right\|_{\infty} \le 1$ then $\|v(t)\|_{W^{2,p}} \le C$ for all $t \in [0,T_{max})$. Next we put the formula of generalized variations of constants to obtain

$$u(t) = e^{-tA_{\alpha-1}}u_0 + \int_0^t e^{-(t-\tau)A_{\alpha-1}}(\theta(u,v) + A_{\alpha-1}\mathcal{B}^c g(u,v))d\tau.$$

So,

$$\begin{aligned}\|u(t)\|_{W^{1,p}} &\leq C\|u_0\|_{W^{1,p}} + C\int_0^t e^{-\nu(t-\tau)}(t-\tau)^{-\theta}(\|\theta(u,v)\|_{L^p} + \|g(u,v)\|_{W^{1,p}})\\ &\leq C\|u_0\|_{W^{1,p}} + C\int_0^t e^{-\nu(t-\tau)}(t-\tau)^{-\theta}\|u(\tau)\|_{W^{1,p}}d\tau.\end{aligned}$$

Finally the proof of global existence concludes with use of Gronwall's Lemma.

8.3. Asymptotic behavior

Theorem 8.4. *If* $\min_{x\in\overline{\Omega}} u_0(x) > 0$ *and*

$$\gamma_0 := \frac{2\chi}{\left(1+\frac{\underline{u}_0}{\overline{u}_0}\right)^2} - \mu < 0 \tag{8.17}$$

where

$$\overline{u}_0 := \max\left\{\max_{x\in\overline{\Omega}} u_0(x), 1\right\}, \qquad \underline{u}_0 := \min\left\{\min_{x\in\overline{\Omega}} u_0(x), 1\right\},$$

then the solution (u,v) *to (8.1) fulfills*

$$\|u(t)-1\|_\infty + \left\|v(t)-\frac{1}{2}\right\|_{W^{2,p}} \leq -C\epsilon_0^{-1}\ln(\epsilon_0)e^{\gamma_0\epsilon_0 t}, \;\; t>0, \tag{8.18}$$

for any $p>1$ *and* $\epsilon_0 := \frac{\underline{u}_0}{\overline{u}_0}$.

Proof. Let

$$\begin{aligned}F_1(u,v) &:= \chi u\left(\frac{u}{1+u} - \frac{v}{1+v}\right) + \mu u(1-u)\\ F_2(u,v) &:= \chi v\left(\frac{v}{1+v} - \frac{u}{1+u}\right) + \mu v(1-v)\end{aligned}$$

and $(\overline{u},\underline{u}) = (\overline{u}(t),\underline{u}(t))$ the solution to the following system of differential equations

$$\begin{cases}\overline{u}_t = F_1(\overline{u},\underline{u}),\\ \underline{u}_t = F_2(\overline{u},\underline{u}),\end{cases} \tag{8.19}$$

with initial data $(\overline{u}(0),\underline{u}(0)) = (\overline{u}_0,\underline{u}_0)$. Let us decompose the proof of the Theorem into several steps.

Step 1. We first claim that

$$0<\underline{u}\leq\overline{u} \qquad \text{in } (0,T_{max}). \tag{8.20}$$

where $T_{max}>0$ denotes the maximal existence time. Having in mind that F_1 and F_2 are regular functions then, at least locally the system (8.19) has a unique solution in $(0,T_{max})$. Since $F_1(0,\underline{u}) = 0$ and $\overline{u}_0>0$ then $\overline{u}(t)>0$ for all $t\in(0,T_{max})$. Arguing in the same manner we can prove that $\underline{u}(t)>0$ for all $t\in(0,T_{max})$. Finally, it remains to prove $\underline{u}\leq\overline{u}$. Suppose that $\overline{u}_0 = \underline{u}_0$ then trivially we have $\overline{u}(t) = \underline{u}(t) = 1$ for all $t\geq 0$.

Suppose that $\overline{u}_0 > \underline{u}_0$. If $\underline{u} \not\leq \overline{u}$ then there would exist $t_0 > 0$ such that $\overline{u}(t_0) = \underline{u}(t_0)$. But then by the uniqueness $\overline{u}(t) = \underline{u}(t) = v(t)$ for all $t \geq 0$ where v solves $v_t = \mu v(1-v)$, $v(t_0) = \overline{u}(t_0)$, this means $\overline{u}_0 = \underline{u}_0$, a contradiction.

Step 2. We have

$$\underline{u} \leq 1 \leq \overline{u} \qquad \text{in } (0, T_{max}). \tag{8.21}$$

Since $\frac{x}{x+1}$ is an increasing function, by (8.19) and Step 1, $\overline{u}$ satisfies $\overline{u}_t \geq \mu\overline{u}(1-\overline{u})$. Since $\overline{u}(0) \geq 1$, by comparison we get $\overline{u} \geq 1$. In the same fashion we prove $\underline{u} \leq 1$.

Step 3. We next show that under assumption (8.17) we have

$$\overline{u}(t) - \underline{u}(t) \leq -\epsilon_0^{-1} \ln(\epsilon_0) e^{\gamma_0 \epsilon_0 t}. \tag{8.22}$$

On multiplying the first equation in (8.19) by $\frac{1}{\overline{u}}$ and the second one by $\frac{1}{\underline{u}}$, we obtain

$$\frac{\overline{u}_t}{\overline{u}} = \chi\left(\frac{\overline{u}}{1+\overline{u}} - \frac{\underline{u}}{1+\underline{u}}\right) + \mu(1-\overline{u}) \quad \text{in } (0, T_{max}),$$

and

$$\frac{\underline{u}_t}{\underline{u}} = \chi\left(\frac{\underline{u}}{1+\underline{u}} - \frac{\overline{u}}{1+\overline{u}}\right) + \mu(1-\underline{u}) \quad \text{in } (0, T_{max}),$$

for $t > 0$, respectively. Subtracting the previous equations we get

$$\frac{d}{dt}\left(\ln\frac{\overline{u}}{\underline{u}}\right) = 2\chi\left(\frac{\overline{u}}{1+\overline{u}} - \frac{\underline{u}}{1+\underline{u}}\right) - \mu(\overline{u} - \underline{u}). \tag{8.23}$$

Next, we observe that the mean value theorem entails

$$2\chi\left(\frac{\overline{u}}{1+\overline{u}} - \frac{\underline{u}}{1+\underline{u}}\right) = \frac{2\chi}{(1+\xi(t))^2}(\overline{u} - \underline{u}), \tag{8.24}$$

where $\xi(t) \in (\underline{u}(t), \overline{u}(t))$. Let

$$\gamma(t) := \left(\frac{2\chi}{(1+\xi(t))^2} - \mu\right)$$

Therefore

$$\frac{d}{dt}\left(\ln\frac{\overline{u}}{\underline{u}}\right) = \gamma(t)(\overline{u} - \underline{u}). \tag{8.25}$$

Next, we claim that $\underline{u}(t) \geq \epsilon_0$. Suppose the contrary then the set

$$A := \{t \in [0, T_{max}) \,:\, \underline{u}(t) < \epsilon_0\}$$

in not empty and bounded from below because $0 \notin A$. Therefore, there exists $t_0 := \inf A$. It is not difficult to infer that $t_0 > 0$. Thanks to (8.17) there exists $k > 0$ such that

$$\frac{2\chi}{(1+\epsilon_0 - k)^2} - \mu \leq 0.$$

By the continuity of $\underline{u}$ and the definition of t_0 we have that there exists $\delta_0(k) > 0$ such that $\underline{u}(t_0+\delta) > \epsilon - k$ for all $\delta \leq \delta_0$. More generally, we have $\underline{u}(t) > \epsilon_0 - k$ for all $t \in [0, t_0+\delta_0]$. Since $\xi(t) \geq \underline{u}(t) \geq \epsilon_0 - k$ for all $t \in [0, t_0+\delta_0]$ then $\gamma(t) \leq 0$ for all $t \in [0, t_0+\delta_0]$. Hence, from (8.25) we get

$$\frac{\overline{u}(t)}{\underline{u}(t)} \leq \frac{\overline{u}(0)}{\underline{u}(0)} = \epsilon_0^{-1}, \quad \forall t \in [0, t_0+\delta_0].$$

Therefore, taking into account that $\overline{u}(t) \geq 1$, the previous inequality asserts

$$\epsilon_0 \leq \underline{u}(t) \ \ \forall t \in [0, t_0+\delta_0]. \tag{8.26}$$

From (8.26) we infer $\inf A \geq t_0 + \delta_0$, a contradiction. As a consequence of the previous proof we have that $\gamma(t) \leq \gamma_0 < 0$. Now, we observe that the mean value theorem entails

$$\overline{u} - \underline{u} = e^{\ln \overline{u}} - e^{\ln \underline{u}} = e^{\ln \hat{u}(t)}(\ln \overline{u} - \ln \underline{u}) = \hat{u}(t) \ln\left(\frac{\overline{u}}{\underline{u}}\right), \tag{8.27}$$

where $\hat{u}(t) \in (\underline{u}(t), \overline{u}(t))$. Substituting the term (8.27) in (8.25) we deduce

$$\frac{d}{dt}\left(\ln\left(\frac{\overline{u}}{\underline{u}}\right)\right) = \gamma(t)\hat{u}(t)\ln\left(\frac{\overline{u}}{\underline{u}}\right) \leq \gamma_0\epsilon_0 \ln\left(\frac{\overline{u}}{\underline{u}}\right).$$

Upon integration this yields

$$\ln\left(\frac{\overline{u}(t)}{\underline{u}(t)}\right) \leq -\ln(\epsilon_0)e^{\gamma_0\epsilon_0 t}.$$

The above inequality prove that $T_{\max} = +\infty$. Moreover substituting the estimate (8.27) in the previous inequality we get

$$\overline{u}(t) - \underline{u}(t) \leq -\epsilon_0^{-1}\ln(\epsilon_0)e^{\gamma_0\epsilon_0 t}.$$

Step 4. We now establish a connection between (8.19) and (8.17) by showing that

$$\underline{u}(t) \leq u(x,t) \leq \overline{u}(t) \qquad \forall\,(x,t) \in \overline{\Omega} \times (0,\infty). \tag{8.28}$$

We consider $\overline{U} = u - \overline{u}$ and $\underline{U} = u - \underline{u}$ which satisfy:

$$\overline{U}_t - \Delta\overline{U} = -\chi\nabla\left(\overline{U}\cdot\nabla v\right) - \chi\overline{u}\left(\frac{\overline{u}}{1+\overline{u}} - \frac{\underline{u}}{\underline{u}+1}\right) + \chi\overline{u}\left(\frac{u}{1+u} - v\right) + \mu u(1-u) - \mu\overline{u}(1-\overline{u})$$

since

$$-\chi\overline{u}\left(\frac{\overline{u}}{1+\overline{u}} - \frac{\underline{u}}{\underline{u}+1}\right) + \chi\overline{u}\left(\frac{u}{1+u} - v\right) = \chi\overline{u}\left(\frac{u}{1+u} - \frac{\overline{u}}{1+\overline{u}} + \frac{\underline{u}}{\underline{u}+1} - v\right),$$

$$= \chi\overline{u}\left(\frac{1}{(1+\xi_1)^2}\overline{U} + \frac{\underline{u}}{\underline{u}+1} - v\right),$$

and

$$\mu u(1-u) - \mu\overline{u}(1-\overline{u}) = \mu(1-\overline{u}-u)\overline{U}$$

where $\xi_1(x,t) \in (\min\{u,\overline{u}\}, \max\{u,\overline{u}\})$. Then $\overline{U}$ satisfies

$$\overline{U}_t - \Delta\overline{U} = -\chi\nabla\cdot\left(\overline{U}\nabla v\right) + \chi\overline{u}\left(\frac{1}{(1+\xi_1)^2}\overline{U} + \frac{\underline{u}}{\underline{u}+1} - v\right) + \mu(1-\overline{u}-u)\overline{U}$$

i.e.

$$\overline{U}_t - \Delta\overline{U} = -\chi\nabla\cdot\left(\overline{U}\nabla v\right) + \overline{U}\left(\frac{\chi\overline{u}}{(1+\xi_1)^2} + \mu(1-\overline{u}-u)\right) + \chi\overline{u}\left(\frac{\underline{u}}{\underline{u}+1} - v\right). \tag{8.29}$$

Notice that

$$\begin{aligned}
&\int_\Omega \left(-\Delta\overline{U} + \chi\nabla\cdot(\overline{U}\nabla v)\right)\overline{U}_+ = \\
&= \int_\Omega |\nabla\overline{U}_+|^2 - \chi\int_\Omega \overline{U}\nabla v\cdot\nabla\overline{U}_+ - \int_{\partial\Omega}\left(\frac{\partial\overline{U}}{\partial n} - \chi\overline{U}\frac{\partial v}{\partial n}\right)\overline{U}_+ \\
&= \int_\Omega |\nabla\overline{U}_+|^2 - \frac{\chi}{2}\int_\Omega \nabla v\cdot\nabla\overline{U}_+^2 - \int_{\partial\Omega}\left(r(1-u) + \chi\overline{u}\frac{\partial v}{\partial n}\right)\overline{U}_+ \\
&= \int_\Omega |\nabla\overline{U}_+|^2 - \frac{\chi}{2}\int_\Omega\left(\frac{u}{1+u} - v\right)\overline{U}_+^2 - \frac{\chi}{2}\int_{\partial\Omega}\frac{\partial v}{\partial n}\overline{U}_+^2 - \int_{\partial\Omega}\left(r(1-u) + \chi\overline{u}\frac{\partial v}{\partial n}\right)\overline{U}_+ \\
&= \int_\Omega |\nabla\overline{U}_+|^2 + \frac{\chi}{2}\int_\Omega\left(v - \frac{u}{1+u}\right)\overline{U}_+^2 - \int_{\partial\Omega}\left(r(1-u) + \chi r'\left(\frac{\overline{U}_+}{2} + \overline{u}\right)\left(\frac{1}{2} - v\right)\right)\overline{U}_+.
\end{aligned}$$

In what follows we try to prove that the boundary term in the above equality is non-positive. To this end, it is enough if $u \geq 1$ and $v \geq 1/2$ whenever $\overline{U}_+ \neq 0$. Since $\overline{U}_+ \neq 0$ whenever $u \geq \overline{u}$ then by step 2, $u \geq \overline{u} \geq 1$. On the other hand, let $x_{min}(t)$ the point of the boundary where v attains the minimum. If the minimum of v if attained on the boundary then, by the Hopf Lemma, we have

$$\frac{\partial v}{\partial n}(x_{min}(t),t) = r'\left(\frac{1}{2} - v(x_{min}(t),t)\right) \leq 0.$$

Hence, $v(x_{min}(t),t) \geq 1/2$ and the boundary term in non-positive. If $x_{min}(t) \in \Omega$ then

$$0 \leq -\Delta v(x_{min}(t),t) = \frac{u(x_{min}(t),t)}{1+u(x_{min}(t),t)} - v(x_{min}(t),t).$$

Therefore,

$$\frac{u(x_{min}(t),t)}{1+u(x_{min}(t),t)} \leq v(x_{min}(t),t). \tag{8.30}$$

Taking into account that $\overline{U}_+ \neq 0$ then $u(x_{min}) \geq \overline{u} \geq 1$. Hence, from (8.30) we deduce

$$\frac{1}{2} \leq v(x_{min}(t),t)$$

and, as a consequence, the boundary term is non-positive. We take $\overline{U}_+$ as test function in (8.29) to obtain

$$\frac{d}{2dt}\int_\Omega \overline{U}_+^2 + \int_\Omega |\nabla\overline{U}_+|^2 \leq \int_\Omega \overline{U}_+^2 g(u,v,\overline{u}) + \chi\int_\Omega \overline{U}_+\overline{u}\left(\frac{\overline{u}}{1+\overline{u}} - v\right), \tag{8.31}$$

with

$$g(u,v,\overline{u}) := \frac{\chi}{2}\left(\frac{u}{1+u}-v\right)+\frac{\chi\overline{u}}{(1+\xi_1)^2}+\mu(1-\overline{u}-u).$$

We try to estimate the last term in the right hand side of (8.31). We observe

$$\begin{aligned}\chi\int_\Omega \overline{U}_+\overline{u}\left(\frac{\underline{u}}{1+\underline{u}}-v\right) &\leq \chi\int_\Omega \overline{U}_+\overline{u}\left(\frac{\underline{u}}{1+\underline{u}}-v\right)_+\\ &\leq \chi\epsilon_0^{-1}\left(\int_\Omega \overline{U}_+^2+\int_\Omega\left(v-\frac{\underline{u}}{1+\underline{u}}\right)_-^2\right),\end{aligned} \tag{8.32}$$

where in the last inequality we used that $f_+ = -(-f)_-$. We note that

$$-\Delta v + v - \frac{\underline{u}}{1+\underline{u}} = \underline{U}\frac{1}{(1+\xi_2)^2}$$

for some $\xi_2(x,t)\in(\min\{\underline{u},u\},\max\{\underline{u},u\})$. After multiplying the previous expression by $(v-\frac{\underline{u}}{1+\underline{u}})_-$ we get

$$\begin{aligned}&\int_\Omega\left|\nabla\left(v-\frac{\underline{u}}{1+\underline{u}}_-\right)\right|^2-\int_{\partial\Omega}r'\left(\frac{1}{2}-v\right)\left(v-\frac{\underline{u}}{1+\underline{u}}\right)_-+\int_\Omega\left(v-\frac{\underline{u}}{1+\underline{u}}\right)_-^2=\\ &=\int_\Omega \underline{U}\frac{1}{(1+\xi_2)^2}\left(v-\frac{\underline{u}}{1+\underline{u}}\right)_-\\ &\leq\int_\Omega \underline{U}_-\frac{1}{(1+\xi_2)^2}\left(v-\frac{\underline{u}}{1+\underline{u}}\right)_-\\ &\leq\frac{1}{2}\int_\Omega \underline{U}_-^2+\frac{1}{2}\int_\Omega\left(v-\frac{\underline{u}}{1+\underline{u}}\right)_-^2\end{aligned}$$

Taking into account that the boundary term is non-positive then, from the above inequality we deduce

$$\frac{1}{2}\int_\Omega\left(v-\frac{\underline{u}}{1+\underline{u}}\right)_-^2\leq\frac{1}{2}\int_\Omega \underline{U}_-^2. \tag{8.33}$$

Plugging the estimate (8.33) into (8.32) we obtain

$$\chi\int_\Omega \overline{U}_+\left(\frac{\underline{u}}{1+\underline{u}}-v\right)\leq\chi\epsilon_0^{-1}\left(\int_\Omega\overline{U}_+^2+\int_\Omega\underline{U}_-^2\right).$$

The previous inequality provides with the following bound in (8.31)

$$\frac{d}{2dt}\int_\Omega\overline{U}_+^2\leq\int_\Omega\overline{U}_+^2 g(u,v,\overline{u})+\chi\epsilon_0^{-1}\left(\int_\Omega\overline{U}_+^2+\int_\Omega\underline{U}_-^2\right).$$

Since $g(u,v,\overline{u})\leq C$ then

$$\frac{d}{2dt}\int_\Omega\overline{U}_+^2\leq C\left(\int_\Omega\overline{U}_+^2+\int_\Omega\underline{U}_-^2\right).$$

In the same fashion we have

$$\frac{d}{2dt}\int_\Omega\underline{U}_-^2\leq C\left(\int_\Omega\overline{U}_+^2+\int_\Omega\underline{U}_-^2\right).$$

Adding the above inequalities and taking into account that $(\overline{U}_0)_+ = (\underline{U}_0)_- = 0$, we may invoke Gronwall's Lemma to achieve

$$\overline{U}_+ = \underline{U}_- = 0 \tag{8.34}$$

which proves the step.

Step 5. Since $\underline{u} \le u \le \overline{u}$ and $\underline{u} \le 1 \le \overline{u}$ then

$$\|u(t) - 1\|_\infty \le \overline{u} - \underline{u} \le -\epsilon_0^{-1} \ln(\epsilon_0) e^{\gamma_0 \epsilon_0 t} \quad \forall t > 0. \tag{8.35}$$

Next we observe that $\theta = \frac{1}{2}$ is the unique solution to the problem

$$\begin{cases} -\Delta\theta + \theta = \dfrac{1}{2} & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial\theta}{\partial n} = r'\left(\dfrac{1}{2} - \theta\right) & \text{on } \partial\Omega \times (0, +\infty). \end{cases}$$

Therefore $z := v - \theta$ satisfies

$$\begin{cases} -\Delta z + z = \dfrac{u}{1+u} - \dfrac{1}{2} & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial z}{\partial n} + r'z = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{8.36}$$

and elliptic regularity asserts

$$\|z(t)\|_{W^{2,p}} \le C \left\| \frac{u(t) - 1}{2(1 + u(t))} \right\|_\infty \le C\|u(t) - 1\|_\infty,$$

concluding the result. ■

Corollary 8.5. *If $\mu \ge 2\chi$ then the solution (u, v) to (8.1) is globally exponentially asymptotically stable and converges to the homogeneous steady-state $(1, \frac{1}{2})$. Moreover, if $\mu \ge 2\chi$, then the previous homogeneous steady-state is the only positive solution to the the steady-state problem associated to (8.1).*

Proof. Assume $\min u_0 = 0$ then, by the strong maximum principle, $\min u(\tau) > 0$ for arbitrary $\tau > 0$ small as desired. Next, we observe that $\gamma_0 < 2\chi - \mu \le 0$ and thanks to Theorem 8.4 we conclude the first part. The second part is a direct consequence of the global stability. ■

Another consequence of Theorem 8.4 is the next Corollary

Corollary 8.6. *If $\mu > \frac{\chi}{2}$ then the solution $(1, \frac{1}{2})$ to (8.1) is locally exponentially asymptotically stable.*

Bibliography

[1] R. Adams, *Sobolev spaces.* Academic Press, New York, 1975.

[2] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial difeerential equations satisfying general boundary conditions I.* Comm. Pure Appl. Math. **12** (1959), 623–727.

[3] H. Amann, *A uniqueness theorem for nonlinear elliptic boundary value problems.* Arch. Rational Mech. Anal. **44** (1971/72), 178–181.

[4] H. Amann, *Nonlinear elliptic equations with nonlinear boundary conditions.* In: "New Developments in differential equations" (W. Eckhaus, ed.), Math Studies **21**, pp. 43–63, North-Holland, Amsterdam 1976.

[5] H. Amann, *Fixed points equations and nonlinear eigenvalue problems in ordered Banach spaces.* SIAM Rev. **18** (1976), 620–709.

[6] H. Amann, *Dual semigroups and second order linear elliptic boundary value problems.* Israel J. Math. **45** (1983), 225–254.

[7] H. Amann, *Nonhomogeneous Linear and Quasilinear Elliptic and Parabolic Boundary Value Problems.* Eds. Schmeisser and Triebel, Function Spaces, Differential Operators and Nonlinear Analysis, pp. 9–126, Teubner Texte zur Mathematik **133** 1993.

[8] H. Amann, *Maximum principles and principal eigenvalues.* In: Ten Mathematical Essays on Approximation in Analysis and Topology, J. Ferrera, J. López-Gómez, F. R. Ruíz del Portal (editors), pp. 1–60, Elsevier 2005.

[9] H. Amann and J. López-Gómez, *A priori bounds and multiple solutions for superlinear indefinite elliptic problems.* J. Differential Equations **146** (1998), 336–374.

[10] A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis.* Cambridge University Press. Cambridge 1993.

[11] A. Ambrosetti and P. Hess, *Positive solutions of asymptotically linear elliptic problems.* J. Math. Anal. Appl. **73** (1980), 411–422.

[12] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications.* J. Funct. Anal. **14** (1973), 349–381.

[13] F. Andreu, J.M. Mazón, J. Toledo and J.D. Rossi, *Porous media equation with absorption and a nonlinear boundary condition.* Nonlinear Anal. **49** (2002), 541–563.

[14] D. Arcoya, J. Carmona and B. Pellaci, *Bifurcation for some quasi-linear operators.* Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), 733–766.

[15] D. Arcoya and J.L. Gámez, *Bifurcation theory and related problems: anti-maximum principle and resonance.* Comm. Partial Diferential Equations **26** (2001), 1879–1911.

[16] J. M. Arrieta, R. Pardo and A. Rodríguez-Bernal, *Bifurcation and stability of equilibria with asymptotically linear boundary conditions at infinity.* Proc. Roy. Soc. Edinburh Sect. A **137**, 225–252.

[17] J. M. Arrieta, R. Pardo and A. Rodríguez-Bernal, *Equilibria and global dynamics of a problem with bifurcation from infinity.* J. Differential Equations, in press.

[18] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems.* Topol. Methods Nonlinear Anal. **4** (1994), 59–78.

[19] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems.* NoDEA Nonlinear Differential Equations Appl. **2** (1995), 553–572.

[20] H. Brezis and S. Kamin, *Sublinear elliptic equations in* $\mathbb{R}^N$. Manuscripta Math. **74** (1992), 87–106.

[21] H. Brezis and L. Oswald, *Remarks on sublinear elliptic equations.* Nonlinear Anal. **10** (1986), 55–64.

[22] J. E. Cohen, *Mathematics is biology's next microscope, only better; biology is mathematics' next physics, only better.* PLos Biology **2** (12) (2004), e439.

[23] V. Calvez and J.A. Carrillo, *Volume effects in the Keller-Segel model: energy estimates preventing blow-up.* J. Math. Pures Appl. **86** (2006), 155–175.

[24] S. Cano-Casanova, *Dependencia continua respecto a variaciones del domonio y condiciones de frontera de las soluciones positivas de una clase general de problemas de contorno sublineales elípticos*, PhD. Dissertation, March 2001.

[25] S. Cano-Casanova, *Existence and structure of the set of positive solutions of a general class of sublinear elliptic non-classical mixed boundary value problems.* Nonlinear Anal. **49** (2002), 361–430.

[26] S. Cano-Casanova and J. López-Gómez, *Continuous dependence of principal eigenvalues with respect to perturbations of the domain around its Dirichlet boundary.* Nonlinear Anal. **47** (2001), 1797–1808.

[27] S. Cano-Casanova and J. López-Gómez, *Properties of the principal eigenvalues of a general class of non-classical mixed boundary value problems.* J. Differential Equations **178** (2002), 123–211.

[28] S. Cano-Casanova, *On the positive solutions of the logistic weighted elliptic BVP with sublinear mixed boundary conditions.* In: Cano-Casanova S., López-Gómez J., Mora-Corral (eds.) Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology, pp. 1–15, World Scientific, Singapore 2005.

[29] R.S. Cantrell and C. Cosner, *On the effects of nonlinear boundary conditions in diffusive logistic equations on bounded domains.* J. Differential Equations **231** (2006), 768–804.

[30] R.S. Cantrell and C. Cosner, *Density dependent behavior at habitat boundaries and the allee effect.* Bull. Math. Biol. **69** (2007), 2339–2360.

[31] M. Chipot, M. Fila and P. Quittner, *Stationary solutions, blow up and convergence to steacionary solutions for a semilinear parabolic equation with nonlinear boundary conditions.* Acta Math. Univ. Commenianae **60** (1991), 35–103.

[32] M. Chipot and P. Quittner, *Equilibria, connecting orbits and a priori bounds for semilinear parabolic equations with nonlinear boundary conditions.* J. Dynam. Differential Equations **16** (2004), 91–138.

[33] T. Cieslak, *Quasilinear nonuniformly parabolic system modelling chemotaxis.* J. Math. Anal. Appl. **326** (2007), 1410–1426.

[34] T. Cieslak and C. Morales-Rodrigo, *Quasilinear non-uniformly parabolic-elliptic system modelling chemotaxis with volume filling effect. Existence and uniqueness of global-in-time solutions.* Topol. Methods Nonlinear Anal. **29** (2007), 361–381.

[35] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues.* J. Funct. Anal. **8** (1971), 321–340.

[36] D. Daners and P. Koch-Medina, *Abstract evolution equations, periodic problems and applications.* Pitman Research Notes in Mathematics Series **279**, Longman Scientific & Technical, Harlow 1992.

[37] M. Delgado, I. Gayte, C. Morales-Rodrigo and A. Suárez, *An angiogenesis model with nonlinear chemotactic response and flux at the tumor boundary.* Submitted.

[38] M. Delgado and A. Suárez, *Nonnegative solutions for the degenerate logistic indefinite sublinear problem.* Nonlinear Anal. **52** (2003), 127–141.

[39] M. Delgado and A. Suárez, *Nonnegative solutions for the degenerate logistic indefinite sublinear problem: the slow diffusion case.* Houston J. Math. Anal. **29** (2003), 801–823.

[40] M. Delgado and A. Suárez, *On the uniqueness of positive solution of an elliptic equation.* Appl. Math. Lett. **18** (2005), 1089–1093.

[41] M. Delgado and A. Suárez, *Study of an elliptic system arising from angiogenesis with chemotaxis and flux at the boundary.* J. Differential Equations **244** (2008), 3119–3150.

[42] M. Del Pino and M. Flores, *Asymptotic behavior of bests constants and extremals for trace embeddigns in expanding domains.* Comm. Partial Differential Equations **26** (2001), 2189–2210.

[43] R. Dillon, P.K. Maini and H.G. Othmer, *Pattern formation in generalised Turing systems: I. Steady-state patterns in systems with mixed boundary conditions.* J. Math. Biol. **32** (1994), 345–393.

[44] L. Dung, *Coexistence with chemotaxis.* SIAM J. Math. Anal. **32** (2000), 504–521.

[45] L. Dung and H.L. Smith, *Steady states of models of microbial growth and competition with chemotaxis.* J. Math. Anal. Appl. **229** (1999), 295–318.

[46] J. Fernández Bonder and J.D. Rossi, *Existence for an elliptic system with nonlinear boundary conditions via fixed point methods.* Adv. Differential Equations **6** (2001), 1–20.

[47] M. Fila, J.J.L. Velázquez and M. Winkler, *Blow-up on the boundary for a semilinear parabolic problem.* In: Progress in Nonlinear Differential Equations and Their Applications **64**, pp. 137–150. Birkhäuser, Basel 2005.

[48] J.M. Fraile, P. Koch Medina, J. López-Gómez and S. Merino, *Elliptic eigenvalue problems and unbounded continua of positive solutions ofa semilinear elliptic equation.* J. Differential Equations **127** (1996), 295–319.

[49] J. L. Gámez, *Sub- and super-solutions in bifurcation problems.* Nonlinear Anal. **28** (1997), 625–632.

[50] J. García-Azorero, I. Peral and J.D. Rossi, *A convex-concave problem with a nonlinear boundary condition.* J. Differential Equations **198** (2004), 91–128.

[51] J. García-Melián, C. Morales-Rodrigo, J.D. Rossi and A. Suárez, *Nonnegative solutions to an elliptic problem with nonlinear absorption and a nonlinear incoming flux on the boundary.* Ann. Mat. Pura Appl. **187** (2008), 459–486.

[52] J. García-Melián, J.D. Rossi and J. Sabina de Lis, *A bifurcation problem governed by the boundary condition I.* NoDEA Nonlinear Differential Equation Appl. **14** (2007), 499–525.

[53] J. García-Melián, J.D. Rossi and J. Sabina de Lis, *A bifurcation problem governed by the boundary condition II.* Proc. London Math. Soc. **94** (2007), 1–25.

[54] J. García-Melián, J.D. Rossi and J. Sabina de Lis, *An elliptic system with bifurcation parameters on the boundary conditions.* Preprint.

[55] J. García-Melián, J.D. Rossi and J. Sabina de Lis, *Existence and uniqueness of positive solutions to elliptic problems with sublinear mixed boundary conditions.* To appear in Communications in Contemporary Mathematics.

[56] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations.* Comm. Partial Differential Equations **6** (1981), 883–901.

[57] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order.* Springer, Heidelberg 1983.

[58] R. Gomez-Reñasco and J. López-Gómez,*The effect of varying coefficients on the dynamics of a class of superlinear indefinite reaction-diffusion equations.* J. Differential Equations **167** (2000), 36–72.

[59] D. Henry, *Geometric theory of semilinear parabolic equations.* Lecure Notes Math. **840**, Springer 1981.

[60] P. Hess, *On uniqueness of positive solutions of nonlinear elliptic boundary value problems.* Math. Z. **154** (1977), 17–18.

[61] P. Hess and T. Kato, *On some linear and nonlinear eigenvalue problems with and indefinite weight function.* Comm. Partial Differential Equations **5** (1980), 999–1030.

[62] T. Hillen and K. J. Painter, *A user's guide to PDE models for chemotaxis.* J. Math. Biol. **58** (2009), 183–217.

[63] T. Hillen and K. J. Painter, *Global existence for a parabolic chemotaxis model with preventing of overcrowding.* Adv. Appl. Math. **26** (2001), 280–301.

[64] D. Horstmann and M. Winkler, *Boundedness vs Blow-up in a chemotaxis sytem.* J. Differential Equations **215** (2005), 52–107.

[65] B. Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition.* Differential Integral Equations **7** (1994), 301–313.

[66] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Amer. Math. Soc. **329** (1992), 819–824.

[67] E.F. Keller and L.A. Segel, *Initiation of slime mold aggregation viewed as an instability.* J. Theor. Biol. **26** (1970), 399–415.

[68] A. Kettemann and M. Neuss-Radu, *Derivation and analysis of a system modeling the chemotactic movement of hematopoietic stem cells.* J. Math. Biol. **56** (2008), 579–610.

[69] R. Kowalczyk, *Preventing blow-up in a chemotaxis model.* J. Math. Anal. Appl. **305** (2005), 566-588.

[70] M. Kuussaari, I. Saccheri, M. Camara and I. Hanski, *Allee effect and population dynamics in the Glanville fritillary butterfly.* Oikos **82** (1998), 384–392.

[71] A.W. Leung and Q. Zhang, *Reaction diffusion equations with non-linear boundary conditions, blow up and steady states.* Math. Methods Appl. Sci. **21** (1998), 1593–1617.

[72] J. López-Gómez, *Nonlinear eigenvalues and global bifurcation: Application to the search of positive solutions for general Lotka-Volterra reaction-diffusion systems with two species.* Differential Integral Equations **7** (1994), 1427–1452.

[73] J. López-Gómez, *The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems.* J. Differential Equations **127** (1996), 263–294.

[74] J. López-Gómez, *On the existence of positive solutions for some indefinite superlinear elliptic problems.* Comm. Partial Differential Equations **22** (1997), 1787–1804.

[75] J. López-Gómez, *Spectral Therory and Nonlinear Funcional Analysis.* Research Notes in Mathematics 426, CRC Press, Boca Raton 2001.

[76] J. López-Gómez, *Classifying smooth supersolutions for a general class of elliptic boundary value problems.* Adv. Differential Equations. **8** (2003), 1025–1042.

[77] J. López-Gómez, V. Márquez and N. Wolanski *Dynamic behaviour of positive solutions to reaction-diffusion problems with nonlinear absorption through the boundary.* Rev. Un. Mat. Argentina **38** (1993), 196–209.

[78] Y. Lou and W. M. Ni, *Diffusion vs cross-diffusion: an elliptic approach.* J. Differential Equations **154** (1999), 157–190.

[79] R. Mahadevan, *A note on a non-linear Krein-Rutman theorem.* Nonlinear Anal. **67** (2007), 3084–3090.

[80] N. V. Mantzaris, S. Webb and H. G. Othmer, *Mathematical modeling of tumor-induced angiogenesis.* J. Math. Biol. **49** (2004), 111–187.

[81] C. Morales-Rodrigo and A. Suárez, *Some elliptic problems with nonlinear boundary conditions.* In: Cano-Casanova S., López-Gómez J., Mora-Corral (eds.) Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology, pp. 175–199, World Scientific, Singapore 2005.

[82] C. Morales-Rodrigo and A. Suárez, *Uniqueness of solution for elliptic problems with nonlinear boundary conditions.* Comm. Appl. Nonlinear Anal. **13** (3) (2006), 69–78.

[83] M.R. Myerscough, P.K. Maini and K.J. Painter, *Pattern formation in a generalised chemotactic model.* Bull. Math. Biol. **60** (1998), 1–26.

[84] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, *Exponential attractor for a chemotaxis-growth system of equations.* Nonlinear Anal. **51** (2002), 119–144.

[85] K. J. Painter and T. Hillen, *Volume-filling and quorum-sensing in models for chemosensitive movement.* Can. Appl. Math. Q. **10** (2002), 501–543.

[86] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations.* Plenum Press, New York 1992.

[87] P. Quittner, *On global existence and stationary solutions for two classes of semilinear parabolic problems.* Comment. Math. Univ. Carolinae **34** (1993), 105–124.

[88] P.H. Rabinowitz, *On bifurcation from infinity.* J. Differential Equations **14** (1973), 462–475.

[89] A. Rodríguez-Bernal and A. Tajdine, *Nonlinear balance for reaction-diffusion equations under nonlinear boundary conditions: dissipativity and blow-up.* J. Differential Equations **169** (2001), 332–372.

[90] J.D. Rossi, *Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem.* In: Chipot, M., Quittner, P. (eds.) Handbook of Differential Equations: Stationary partial differential Equations **2**, pp. 311–406. Elsevier, Amsterdam 2005.

[91] J. Serrin, *Nonlinear Elliptic Equations of Second Order.* In: AMS Symposium in Partial Differential Equations, Berkeley, August 1971.

[92] J. Serrin, *A remark on the preceding paper of Amann.* Arch. Rational Mech. Anal. **44** (1972), 182–186.

[93] S. Senn and P. Hess, *On positive solutions of a linear elliptic eigenvalue problem with Neumann boundary conditions.* Math. Ann. **258** (1982), 459–470.

[94] K. Taira, *Logistic Dirichlet problems with discontinuous coefficients.* J. Math. Pures Appl. **82** (2003), 1137–1190.

[95] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces.* Lecture Notes of the Unione Matematica Italiana **3**, Springer, Berlin, 2007.

[96] J.I. Tello and M. Winkler, *A chemotaxis system with logistic source.* Comm. Partial Differential equations **32** (2007), 849–877.

[97] K. Umezu, *Global positive solution branches of positone problems with nonlinear boundary conditions.* Differential Integral Equations **13** (2000), 669–686.

[98] K. Umezu, *Nonlinear elliptic boundary value problems suggested by fermentation.* NoDEA Nonlinear Differential Equations Appl. **7** (2000), 143–155.

[99] K. Umezu, *Behaviour and stability of positive solutions of nonlinear elliptic boundary value problems arising in populutation dynamics.* Nonlinear Anal. **49** (2002), 817–840.

[100] K. Umezu, *Bifurcation from infinity for asymptotically linear elliptic eigenvalue problems.* J Math. Anal. Appl. **267** (2002), 651–664.

[101] K. Umezu, *Local bifurcation analysis and stability of steady-state solutions for diffusive equations with nonlinear boundary conditions.* Commun. Appl. Anal. **8** (2004), 533–547.

[102] K. Umezu, *On eigenvalue problems with Robin type boundary conditions having indefinite coefficients.* Appl. Anal. **85** (2006), 1313–1325.

[103] M. Winkler, *Chemotaxis with logistic source: very weak global solutions and their boundedness properties.* J. Math. Anal. Appl. **348** (2008), 708–729.

[104] D. Wrzosek, *Global attractor for a chemotaxis model with prevention of overcrowding.* Nonlinear Anal. **59** (2004), 1293–1310.

[105] T. F. Wu, *A semilinear problem involving nonlinear boundary condition and sign-changing potential.* Electron. J. Differential Equations (2006), No 131.

[106] Y. Wu and M. Wang, *Existence and nonexistence of global solution of nonlinear parabolic equation with nonlinear boundary condition.* Chin. Ann. Math. **16B** (1995), 371–378.

[107] M. Zhu, *On elliptic problems with indefinite superlinear boundary conditions.* J. Differential Equations **193** (2003), 180–195.