Universidad de Sevilla

Facultad de Matemáticas

Departamento de Estadística e Investigación Operativa

Algunos Problemas en Teoría de Localización

Memoria de Tesis presentada por

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para optar al grado de Doctor en Matemáticas.

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en
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Yolanda Hinojosa Bergillos

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A mi madre

Índice General

Resumen en castellano

Hablando en términos generales, el objetivo de la Teoría de Localización puede resumirse en lo siguiente:

> "Determinar la localización de uno o varios servicios que mejor se ajusten, en algún sentido, a las necesidades de un conjunto de clientes"

No obstante, son muchos los elementos que tendremos que analizar para poder profundizar en el tema.

Nuestro acercamiento a la Teoría de Localización se produce desde el campo de la Investigación Operativa, por lo que nuestro estudio es cuantitativo y analítico. En este sentido, ante cada situación diferente a la que nos enfrentemos, intentaremos adoptar un modelo matemático como ayuda para solucionar un problema de localización. Esto requiere abstracción y frecuentemente la suposición de hipótesis simplificadoras. Por consiguiente, no nos ocupamos de la situación real sino de representaciones de ella y por tanto, la solución obtenida es una solución para el modelo, que puede o puede no ser, una representación exacta del problema de localización del mundo real. De hecho, en ciertas ocasiones, factores cualitativos como armonía o buena fortuna pueden ser de suma importancia en un problema de localización. Este tipo de situaciones son mal modeladas desde un punto de vista cuantitativo. En estos casos, el análisis basado en la consideración matemática es de escaso interés.

Esta memoria se ocupa exactamente del resto de casos donde las decisiones de localización se basan en criterios racionales, objetivos y cuantitativos. Ello explicará nuestra concentración, a lo largo de cada capítulo, en la formulación del problema tratado, en la construcción del modelo matemático adecuado y en el desarrollo de técnicas de resolución apropiadas. Nuestro último objetivo es proveer al decisor de herramientas cuantitativas que le permitan encontrar buenas soluciones ante problemas de localización objetivos y realistas.

Aunque es difícil agrupar los diferentes modelos o problemas relacionados con la Teoría de Localización, ésta podría clasificarse en tres principales áreas: 1) Localización Continua, 2) Localización-asignación, más generalmente denominada Localización Discreta, y 3) Localización en Redes.

La Localización Continua trata problemas en los que el servidor puede ser localizado en cualquier parte de un espacio continuo (normalmente un bidimensional o un tridimensional espacio). La Localización Discreta se ocupa de aquellas situaciones donde las posibles ubicaciones para el servidor son finitas y conocidas de antemano. Por último, la Localización en Redes estudia, como su nombre indica, problemas formulados en grafos o redes. En esta tesis, nos centraremos principalmente en las dos primeras áreas, sobre las que pueden consultarse, excelentes textos: Drezner (1995)[31], Love-Morris y Wesolowsky (1988)[71], Mirchandani y Francis (1990)[78], Francis, McGinnis y White (1992)[40], Hamacher (1995)[50] (en alemán) o Puerto (1996)[88]. Libros más especializados son: Larson y Odoni (1981)[68], Daskin (1995) [28], Hurter y Martinich (1989)[62].

Algunos antecedentes históricos

El origen de la Teoría de Localización puede atribuirse a los geómetras griegos porque fueron quienes primero estudiaron propiedades de puntos singulares con respecto a la configuración de unos puntos dados. Según Wesolowsky (1993)[104], cuando en el 638 se destruyó la Biblioteca de Alejandría, ésta contenía al menos tres soluciones diferentes del, hoy en día más comúnmente conocido, como *problema de Weber* .

Más recientemente, a principios del siglo XVII, Pierre Fermat propuso un problema puramente geométrico: "Dados tres puntos en un plano, hallar un cuarto punto de forma que la suma de su distancia a los tres puntos dados fuera lo más pequeña posible". Este problema fue solucionado geométricamente por Torricelli (1608-1647) aunque algunos autores le atribuyen a Cavallieri (1598-1647) la formulación y la resolución del mismo.

En 1750 Simpson sugirió una generalización del problema al incorporar diferentes pesos asociados a los distintos puntos.

Sin embargo, es a Alfred Weber (1909)[102] a quien se le atribuye el origen moderno de esta teoría. Weber usó este modelo para determinar la localización óptima de una fábrica que suministraba material a un único cliente y se abastecía de materia prima desde dos fuentes distintas. Los tres puntos fijos eran, por tanto, las dos fuentes de abastecimiento por un lado y el mercado (o cliente) por otro. Propuso como criterio para localizar la fábrica la minimización del coste de transporte (ponderado por pesos asociados a cada punto) o equivalentemente, la minimización de la distancia de viaje ponderada. Ésta es hoy, la base del *problema de localización minisum*, frecuentemente llamado *problema de la mediana* o *problema de Weber.*

Debido a la ausencia de computadoras, la mayor parte de los problemas propuestos hasta entonces eran resueltos por procedimientos geométricos. Weiszfeld [103] en 1937 fue el primero que propuso un método analítico, basado en un procedimiento iterativo, para calcular distancias euclídeas a n puntos dados. Su método permaneció desconocido hasta finales de los cincuenta y principios de los sesenta y fue descubierto de nuevo por otros autores. Desde entonces, han aparecido muchas generalizaciones, se han aplicado diferentes criterios de localización y se han dedicado un gran número de artículos al análisis de localización. Arquitectos, ingenieros de computadoras, economistas, ingenieros, investigadores operativos, diseñadores de sistema de transporte, geógrafos técnicos, han descubierto un interés común en su preocupación por la localización de facilidades o servicios.

Mientras los problemas de localización continua datan del siglo XVII (o de los geómetras griegos), el significado de un conjunto finito de posibles localizaciones y la naturaleza discreta de los problemas de decisión en Teoría de Localización no fue completamente reconocido hasta finales de los 50. Según Mirchandani (1990)[77], el ímpetu por el tratamiento formal del problema discreto de localización minisum se debe a Kuehn y Hamburguer(1963)[66], Manne (1964)[73], Hakimi (1964)[46] y Balinski (1965)[5].

Kuehn y Hamburguer (1963)[66] y Manne (1964) [73] fueron los primeros en dar procedimientos heurísticos para la resolución del *problema de localización de plantas simples o sin restricciones de capacidad (SPLP) o (UPLP)* . El (SPLP) es un caso especial de un problema de localización minisum que usualmente aparece en formulaciones discretas. Aunque la primera formulación explícita del SPLP está frecuentemente atribuida a Balinski (1965)[5], para algunos autores, el procedimiento de resolución propuesto por Balinski y Wolfe (1963)[4] fue el primer intento de resolución óptima del SPLP. Por otra parte, Hakimi (1964)[46] introdujo el *problema de la mediana* en una red y con él comenzó el desarrollo de la localización en redes.

Consecuentemente, la teoría de localización discreta, abarca un período de desarrollo de apenas cuatro décadas. No obstante, la literatura en teoría de localización discreta ha crecido rápidamente al igual que la literatura en teoría de localización continua.

Esquema de clasificación

Debido al gran número de problemas de localización que puede ser encontrado en la literatura, resulta útil definir ciertas categorías que puedan usarse para clasificar dichos problemas (ver Hamacher y Nickel (1998)[52] para más detalles) . En tal clasificación habrán de tenerse en cuenta cinco elementos:

1. **Nuevo servidor a localizar**. En un problema general de localización, la variable de decisión es la localización del nuevo servidor. Esta variable viene caracterizada por:

    - *Número y calidad del servicio.* Si hay que ubicar más de un servicio, es necesario especificar las características de cada uno de ellos. Cuando son idénticos, como ocurre, por ejemplo, con los buzones de correo, nos enfrentamos con un modelo multifacilidad. En otro caso, como ocurre con los servicios de salud, podemos encontrarnos con modelos jerárquicos.

En los problemas multifacilidad podemos distinguir dos tipos de modelos: 1) Modelos en los que el número de nuevas ubicaciones ha sido establecido de antemano. Este número puede ser determinado, por ejemplo, por razones económicas o estudios del mercado. Algunos ejemplos de estos modelos son el *problema de la p-mediana* o el *problema del p-centro.* 2)Modelos donde el número de servicios o facilidades que han de ser localizados no está preespecificado de antemano como, por ejemplo, en el *problema de localización de plantas simples (SPLP).*

- *Naturaleza del servicio.* Podemos distinguir tres tipos de servicios: Servicios atractivos, servicios molestos o repulsivos y servicios semirepulsivos.

  Una facilidad es considerada un servicio atractivo cuando representa alguna ventaja para la comunidad donde estará ubicada. Ejemplos de servicios atractivos son escuelas, teatros, bibliotecas, hospitales, estaciones de bomberos, etc.

  No obstante, no todos los servicios son atractivos para la comunidad. Prisiones, instituciones de rehabilitación de drogadictos, plantas nucleares, tuberías de gas o canales que transportan materiales peligrosos, son ejemplos de servicios usualmente rehusados por la población.

  Finalmente, un servicio es considerado semirepulsivo cuando es atractivo para algunos clientes y repulsivo para el resto. Por consiguiente, cuando se modela un problema es muy importante determinar la "atractividad" del servicio a localizar (véase por ejemplo el artículo de Chen et al. (1992)[24]).

- *El "tamaño" del servicio.* El nuevo servicio puede venir representado bien por un punto, por un área de localización o por algún tipo de estructura dada (véase, por ejemplo, Hakimi, Schmeinchel y Labbé (1993)[47], o Carrizosa, Muñoz y Puerto (1998)[21]). Por otra parte, si el servicio que ha de ser localizado es, por ejemplo, una planta de producción o un almacén puede considerarse con restricciones de capacidad o sin ellas.

2. **Espacio de soluciones.** El espacio de soluciones es el marco donde se establece el problema. Contiene como elementos las facilidades existentes o clientes y los nuevos servicios o facilidades que han de ubicarse. La elección de un espacio de solución apropiado es crucial, porque determina aspectos tan importantes como, la exactitud o la posibilidad de resolución del modelo. Los espacios de solución más usuales son:

   - *Espacio Discreto.* Existe un número finito de posibles candidatos y por tanto, los servicios sólo pueden ser localizados en un número finito de ubicaciones potenciales, seleccionadas por algún análisis previo. En muchas ocasiones, el decisor considera que una representación discreta es más realista y es un retrato más exacto del problema (véase, por ejemplo, Mirchandani y Francis (1990) [78]).

   - *Redes.* Muy a menudo, representan redes de comunicación. Los nodos son los elementos importantes representando ciudades o barrios. Los arcos usualmente modelan conexiones entre nodos, como carreteras, calles, conexiones eléctricas, etcétera (véase, por ejemplo, Tansel et al. (1983) [97] o Labbé, Peeters y Thisse (1995)[67])).

   - *Espacio Euclídeo $\mathbb{R}^n$* . Este tipo de espacio es usado cuando el problema presenta aspectos regionales que no pueden ser discretizados (véase, por ejemplo Drezner (1995) [31]). Además, también se suele usar para modelar situaciones de una red cuando el número de arcos y nodos es demasiado grande. El caso $n = 2$ ó 3 tiene un significado físico. Los casos con $n > 3$ se usan para modelar y solucionar problemas de estimación en estadística.

Aunque los espacios solución citados anteriormente son lo más usuales, otros espacios de solución han sido usados en Teoría de Localización, como por ejemplo, la esfera, bastante útil para modelar problemas de localización reales en los que las distancias son demasiado grandes (véase Drezner (1985)[30]).

3. **Facilidades existentes**. En términos de la Teoría de Localización, las facilidades existentes son el conjunto de usuarios que requieren ser servidos. Usualmente, se representan por un conjunto A (de puntos de demanda) y una función de intensidad asociada con cada uno de dichos puntos (a menudo esta intensidad es una medida de probabilidad en A).

   Existen dos formas diferentes de representar las facilidades existentes: 1) por un conjunto finito de puntos del espacio de soluciones, y 2) por regiones dentro del espacio de soluciones.

   En el primer caso, tenemos un conjunto de puntos $A = \{a_1, \cdots, a_M\}$ y un conjunto de pesos representando la importancia (la intensidad) de la demanda generada en cada punto $\{w_1, \cdots, w_M\}$.

   En el modelo regional, se considera una región R (no necesariamente conexa) incluida en el espacio de solución sobre la que hay definida alguna medida de probabilidad, la cual nos da la importancia de cada subconjunto medible de R.

4. **Medida de distancia**. Debido a que la calidad de servicio usualmente depende de la distancia, este elemento es fundamental en problemas de localización. La elección de la función de distancia viene impuesta a menudo por el espacio de solución considerado.

   En un espacio discreto se supone la posibilidad de tomar el camino más corto entre cada posible par cliente-servicio. La distancia más pequeña entre puntos discretos situados en un plano puede ser calculada bien por aproximación, tomando como base las coordenadas de los puntos y la métrica dada, o calculando explícitamente las distancias en una red de carreteras, cuando ésta está predefinida. En el primer caso, las distancias entre los puntos pueden calcularse, dependiendo de la métrica, mediante el uso de una fórmula simple, mientras que en el segundo caso, necesitamos un algoritmo para calcular las distancias más cortas en la red.

En una esfera, el camino más corto entre dos puntos ha de medirse a lo largo del círculo máximo que pasa a través de ellos y será la longitud del arco más pequeño que une a los dos puntos.

En el espacio usual euclídeo $I\!R^n$, nos gustaría encontramos, en la mayoría de los casos, con normas o métricas pero hay muchas otras posibilidades.

Cualquier función $k : I\!R^n \longrightarrow I\!R$ que verifique las siguientes propiedades es llamada una norma:

(a) $k(x) \geq 0 \qquad \forall x \in R^n$

(b) $k(x) = 0 \qquad \text{sii} \quad x = 0$

(c) $k(cx) = |c|k(x) \qquad \forall c \in I\!R \qquad \forall x \in I\!R^n$

(d) $k(x+y) \leq k(x) + k(y) \qquad \forall x, y \in I\!R^n$

Es fácil comprobar que k es una función convexa.

Con estas funciones, las distancias se definen como

$$d(x,y) = k(y-x)$$

y por consiguiente, verifican las siguientes propiedades:

(a) *Positiva* : $\quad d(x,y) \geq 0.$

(b) *Definida* : $\quad d(x,y) = 0 \quad$ sii $\quad x = y.$

(c) *Simetría:* $\quad d(x,y) = d(y,x).$

(d) *Desigualdad Triangular* : $\quad d(x,z) \leq d(x,y) + d(y,z).$

De entre todas las normas, cabe destacar las *normas* l_p, una familia de normas muy usadas en Teoría de Localización (véase, por ejemplo, Love y Morris (1972) [70] o Hansen, Perreur y Thisse (1980) [54]). Dicha familia incluye como casos especiales a las normas euclídea y rectilínea. En la figura 1.1 del

Capítulo 1 de esta memoria se muestran contornos de las bolas unidad de algunas funciones l_p.

El uso de las anteriores propiedades para modelar distancias de viaje han sido criticadas en muchas ocasiones. Por ejemplo, en algunas situaciones la simetría no tiene sentido porque existen itinerarios diferentes para ir y regresar. Witzgall (1964) [105] fue el primero en proponer el uso de calibradores poliédricos perdiéndose así la propiedad de la simetría. A partir de entonces, el uso de calibradores ha sido muy extendido en Teoría de Localización (véase, por ejemplo, Durier y Michelot (1985)[33]). Los calibradores se basan en el llamado funcional de Minkowski asociado a un conjunto convexo y compacto B (en general no simétrico) cuyo interior contiene el origen, definido por:

$$\gamma_B(x) := \inf\{\lambda > 0 \ : \ x \in \lambda B\}.$$

Si B es simétrico, el calibrador es una norma.

Por otra parte, hay veces que no tiene sentido la propiedad "definida" como, por ejemplo, cuando consideramos facilidades regionales no acotadas y $d(R_1, R_2)$ representa la mínima distancia entre las dos regiones. Siempre que $R_1 \cap R_2 \neq \emptyset$, $d(R_1, R_2) = 0$ pero esto no implica que $R_1 = R_2$. En el Capítulo 2 de esta memoria, introducimos el uso de semicalibradores perdiéndose así la propiedad de "definida", así como la de simetría. Los semicalibradores se basan en el funcional de Minkowski asociado a un conjunto convexo y cerrado (no necesariamente compacto) que contiene al origen. Aunque su evaluación matemática no difiere de la referente a conjuntos convexos compactos (véase, por ejemplo, el libro de Hiriart-Urruty y Lemaréchal (1993)[60]) hemos considerado conveniente, desde un punto de vista metodológico, renombrar estas funciones dentro del campo de la Teoría de Localización. Desde el artículo de Witzgall (1964)[105], las funciones calibrador siempre han sido identificadas en Teoría de Localización con calibradores de conjuntos compactos y convexos. Por consiguiente, y para evitar mal entendidos proponemos llamar semicalibradores a los calibradores de conjuntos convexos y cerrados (no necesariamente compactos). Es fácil probar que las "distancias" definidas a partir de semicalibradores no verifican en general la propiedad de "definida".

Si el interior del conjunto contiene al origen y el conjunto es acotado, entonces el semicalibrador es un calibrador . Si, además, el conjunto es simétrico, es una norma. Por consiguiente, los semicalibradores pueden verse como una generalización de los calibradores y las normas, porque contienen a éstos como casos particulares.

5. **Función objetivo.** Debido a nuestro acercamiento cuantitativo a la Teoría de Localización, la mayor parte de los problemas tienen, en términos generales, una formulación como la que sigue:

$$\underset{x=\{x_1,\cdots,x_p\}\in S}{\text{opt}} \qquad F(d(a,(x_1,\cdots,x_p))_{a\in A})$$

donde:

F es la función globalizadora,

opt significa optimizar, ya sea minimizar o maximizar,

S es el espacio de soluciones,

x es el nuevo servicio, pudiendo ser único ($p = 1$) o múltiple ($p > 1$),

A es el conjunto de facilidades existentes,

a representa una facilidad existente general, y

d es la medida de distancia.

La determinación de qué función objetivo ha de ser usada es, a veces, una difícil tarea. Debe tenerse en cuenta que la solución final depende fuertemente de esta elección (véase Carrizosa, Fernández y Puerto (1990)[16]). Por consiguiente, es importante dedicar un poco de esfuerzo a esta parte del proceso de modelización.

Algunos ejemplos de problemas de localización de servicios son:

- *Problema minisum, de la mediana o de Weber* . Dados un conjunto de puntos de demanda A y una medida de probabilidad sobre dicho conjunto μ el problema consiste en minimizar la suma de las distancias al nuevo servicio x.

$$\min_{x \in S} \int_A d(x,a) d\mu(a) \quad ; \qquad A \subset S$$

Cuando A es un conjunto finito, el problema se transforma en

$$\min_{x \in S} \sum_{a \in A} \mu(a) d(x,a)$$

Minimizar la distancia total de viaje es quizá el criterio más usado en localización (véase, por ejemplo, Wesolowsky (1993)[104]).

Una extensión de este problema es el conocido como problema de la *p-mediana* donde el número de nuevos servicios que han de ser localizados es $p > 1$.

El criterio minisum garantiza la eficiencia en la localización y minimiza el coste total de servicio. No obstante, este criterio puede dar lugar a soluciones inaceptables desde el punto de vista del nivel servicio que reciben los clientes ubicados lejos de la facilidad. Por consiguiente, en lugar del criterio minisum, uno puede querer minimizar la máxima distancia recorrida.

- *El problema minimax, del centro o problema de Rawl.* Su formulación es parecida a la anterior, excepto que ahora, el objetivo es la minimización de la máxima distancia ponderada por μ (véase, por ejemplo, Durier (1995)[34]).

$$\min_{x \in S} \sup_{a \in A} \mu(a) d(x,a)$$

Cuando A sea un conjunto compacto, el operador del supremo puede ser reemplazado por el máximo.

Una extensión de este problema es el conocido como problema del *p-centro* donde al igual que ocurría con el anterior, el número de nuevos servicios que han de ser localizados es $p > 1$.

El criterio minimax es un criterio más equitativo que el minisum. Por ejemplo, en caso de la localización de una escuela, un criterio minimax garantizaría que la distancia más grande recorrida por un estudiante fuese tan pequeña como fuera posible. No obstante, el criterio minimax puede provocar servicios muy costosos.

Se han hecho algunos intentos para evitar las deficiencias de los criterios minisum y minimax usados independientemente. Algunos ejemplos de estos intentos son el uso de modelos bicriterio o el problema del centdian.

- *El problema del Cent-dian.* Dado un escalar $\lambda > 0$, la función objetivo es la combinación convexa del problema minisum y del problema minimax. Es decir, el problema consiste en

$$\min_{x \in S} \left(\lambda \int_A d(x,a) d\mu(a) + (1-\lambda) \sup_{a \in A} \mu(a) d(x,a) \right)$$

Este criterio puede entenderse como un compromiso entre la minimización del coste total y la equidad en el servicio. El término *cent-dian* fue introducido por Halpern (1976)[48]. Posteriormente han aparecido generalizaciones de dicho problema, entre las que podemos citar la propuesta por Tamir, Pérez-Brito y Moreno-Pérez (1998)[96]. En ella, los autores generalizan la combinación convexa introduciendo pesos en la función del centro. Asimismo, presentan un algoritmo polinomial para el caso de la $p - facilidad$ en un árbol.

- *El Problema Maximin.* Cuando tenemos que localizar un servicio "ofensivo" o repulsivo parece razonable que uno quiera estar tan alejado de la facilidad como le sea posible. Por tanto, parece apropiado usar criterios "conservadores" como la maximización de la mínima distancia a la facilidad que ha de ser localizada (véase, por ejemplo, Erkut y Verter (1995)[36]). De alguna forma, el problema maximin puede verse como la contraparte repulsiva del problema del centro, donde la función objetivo viene dada por:

$$\max_{x \in S} \inf_{a \in A} \mu(a) d(x,a)$$

Usualmente, los problemas anteriormente mencionados suelen tratarse tanto en espacios continuos como en espacios discretos o redes. El siguiente problema es un problema especial de localización-asignación que generalmente es tratado en espacios discretos.

- *Problema de localización de plantas simples o sin restricciones de capacidad (SPLP) o (UPLP)*. Dado un conjunto finito de posibles localizaciones para las nuevas facilidades (o de posibles reestructuraciones de facilidades ya existentes) el problema consiste en minimizar el coste de producción y el coste de transporte que supone enviar un determinado tipo de producto desde un subconjunto de dichas facilidades a un conjunto de clientes de forma que se satisfagan las demandas de éstos. Las facilidades suelen representar plantas de producción o depósitos de material y pueden tener o no, limitada su capacidad (véase, por ejemplo, Krarup y Pruzan (1983)[65]).

  En contraste con otros problemas prototipo de localización, el (SPLP) permite una mayor flexibilidad en su modelización. Ni el número de plantas que han de ser localizadas ni la estructura de transporte están predeterminadas . Así, aunque la formulación del (SPLP) es básicamente discreta, estática, determinista, de un único tipo de producto, puede modificarse adoptando formulaciones dinámicas, estocásticas, con múltiples tipos de productos, con restricciones de capacidad y con envíos en varias etapas (véase, por ejemplo, Hinojosa, Puerto y Fernández (1999)[59]).

- *El problema estocástico de la mediana en un sistema de colas.* Las llegadas a cada una de las facilidades existentes (las cuales se suponen finitas), se producen según un proceso de Poisson. Dichas llegadas han de ser servidas en la posición del nuevo servidor, siendo el tiempo de servicio, una función de la distancia de viaje.

  Si asumimos que el servidor no tiene limitada su capacidad, el sistema se comporta como una colaa $M/G/1/\infty$ (véase, por ejemplo, Berman et al. (1990)[14]).

El objetivo es encontrar la localización del servidor de forma que se minimice el tiempo de espera, por tanto la función objetivo es minimizar el tiempo medio de respuesta $TR(x)$:

$$\min_{x \in S} TR(x) = Q(x) + T(x)$$

donde, $Q(x)$ es el tiempo medio de permanencia en cola y $T(x)$ es el tiempo medio de viaje.

- *El problema de la mediana con pérdidas para una única facilidad con c canales (c-SFLM).* Al igual que el problema previo, se trata de un problema de colas pero en cual el servidor tiene limitada su capacidad. Suponemos un servidor con c canales o unidades de servicio. Cuando un cliente llega al sistema se pierde si las c unidades de servicio están ocupadas, o bien es atendido inmediatamente, si hay al menos una unidad de servicio libre. Por tanto, el sistema se comporta como una cola $M/G/c/c$ (véase, por ejemplo, Chiu y Larson (1985)[25] o Frenk, Labbé y Zhang (1993)[41]).

  El objetivo es encontrar la localización del servidor con objeto de minimizar la suma ponderada del tiempo medio de respuesta y el coste ocasionado por los clientes rechazados del sistema. Los pesos respectivos son las probabilidades de respuesta inmediata y la probabilidad de rechazo. Por tanto, La función objetivo viene dada por:

$$\min_{x \in S} (1 - \Psi_c(x))\bar{t}(x) + \Psi_c(x)\hat{r}$$

  donde, $\Psi_c(x)$ es la probabilidad de saturación (todas las unidades de servicio ocupadas), $\bar{t}(x)$ es el tiempo medio de viaje para un cliente y $\hat{r} \geq 0$ es el coste que supone rechazar a un cliente.

Los modelos anteriormente citados no son exhaustivos. En la literatura de Teoría de Localización se pueden encontrar muchos más modelos.

Descripción de los diferentes capítulos

Como ya hemos mencionado anteriormente, es posible encontrar una gran variedad de problemas dentro de las diferentes áreas en Teoría de Localización, siendo todos ellos muy interesantes. Sin duda, nos sería imposible cubrir toda esta amplia variedad de problemas. No obstante, no queríamos concentrar esta memoria en el desarrollo de un único área. Por esta razón, hemos querido abordar diversos problemas dentro de los diferentes campos en Teoría de Localización.

Así, en el Capítulo 2 desarrollamos una nueva extensión del problema de localización de un único servicio en el espacio euclídeo $\mathbb{R}^n$ (SFLP) . Este nuevo avance consiste en el uso de semicalibradores como base para la medida de distancias. Como ya se ha dicho hay situaciones en las que la propiedad "definida" de una distancia no tiene sentido. En estos casos, parece razonable el uso de semicalibradores, sin embargo, hasta lo que nosotros sabemos, estas funciones nunca habían sido consideradas antes en Teoría de Localización. Esta nueva extensión al problema, basada en el uso de semicalibradores, representa un marco unificado para el problema SFLP, el cual generaliza aproximaciones previas a problemas de localización en $\mathbb{R}^n$ basadas en calibradores y normas.

En el Capítulo 3 tratamos un problema de localización de plantas simples en un espacio discreto, el cual generaliza el SPLP. Estudiamos un problema bietápico, multiperiodo y multiproducto de localización de plantas con restricciones de capacidad (MMCPL). Como dijimos, el SPLP tiene básicamente una formulación estática, uniproducto y sin restricciones de capacidad, pero ésta podía modificarse con objeto de obtener formulaciones dinámicas, multiproducto, multietápicas y con restricciones de capacidad. Ciertamente, en la literatura se pueden encontrar muchas extensiones de este problema (véase, por ejemplo, Aikens (1985) [2] o Daskin (1995) [31] donde se puede obtener una buena recopilación de este problema y de sus extensiones). Entre dichas extensiones, podemos resaltar dos de ellas. La primera consiste en introducir el aspecto dinámico en el problema y la segunda consiste en la suposición de una cierta estructura en el esquema de transporte (problemas multietápicos). A pesar de ser modelos bastante generales, el marco más natural para estos problemas sería la combinación de estas dos extensiones. Es decir, la consideración común de aspectos multietápicos y multiperiodos. No obstante, hasta lo que nosotros sabemos, esta combinación nunca ha sido estudiada antes, por lo que puede ser considerada como una introducción a un nuevo problema de localización.

En el Capítulo 4 estudiamos un particular problema maximin en un espacio continuo. Tratamos el problema de localizar una ruta lineal peligrosa en el plano con relación a un conjunto de puntos dados. El problema de localizar líneas repulsivas apenas ha sido estudiado en la literatura, podemos citar a Drezner y Wesolowsky (1989)[29] o Hinojosa y Puerto (1999)[58]. Los primeros usan la distancia euclídea como medida de distancia y los segundos una familia particular de normas poliédricas. En esta memoria, consideramos una norma general para medir distancias y caracterizamos la línea solución en el caso general. Además, para prevenir el efecto de desastres motivados por accidentes, hemos asociado un polígono a cada punto. Estos polígonos representan áreas de seguridad alrededor de la posición de los puntos. Se usa como función objetivo el criterio de la maximización de la mínima distancia ponderada a las zonas de protección.

Nuestra extensión generaliza los resultaados previos para este tipo de problemas y también permite usar el criterio del maximim con relación a áreas de protección y no sólo con relación a puntos.

En el Capítulo 5 desarrollamos una extensión del problema de la mediana con pérdidas para una única facilidad con c canales (c-SFLM), la cual llamaremos problema de la mediana con pérdidas y con rechazo para una única facilidad con c canales (c-SFLMR). Como se vió en la sección anterior, en el c-SFLM se estudian propiedades de localización de un modelo de localización con pérdidas con una política de admisión estándar: Si al servidor llega una petición de servicio y hay algún canal desocupado, la petición es atendida inmediatamente. Si todos los canales están ocupados, la petición se pierde.

En nuestra extensión (el c-SFLMR) consideramos una nueva política de admisión con objeto de mejorar la eficiencia global del sistema. Consideramos que los clientes pueden ser discriminados según el grupo al que pertenezcan, de forma que pueden ser rechazados aún si el sistema tiene algunas unidades desocupadas (suponemos que existe un servicio de reserva). A pesar de su importancia, este tipo de políticas preventivas apenas han sido consideradas en la literatura. La razón es la dificultad matemática de estos modelos. Nosotros proponemos un algoritmo que nos permite obtener soluciones óptimas en espacios discretos, en redes y en espacios continuos cuando se usan normas poliédricas como base para medir distancias. Además, damos un procedimiento para obtener buenas soluciones aproximadas en espacios continuos con cualquier norma general.

Asimismo, estudiamos un nuevo modelo con dos servidores (c-TFLMR): Un sistema principal de servicio y un sistema secundario de servicio, de forma que, los clientes rechazados debido a la política de admisión en el sistema de servicio principal son aceptados en el sistema secundario si alguna unidad de servicio está desocupada.

La memoria acaba con un apéndice. Éste, incluye algunos conceptos y resultados de análisis convexo y una descripción de los métodos del subgradiente. Estos resultados son necesitados en el transcurso del Capítulo 2 y el Capítulo 3, respectivamente.

A continuación, damos una descripción un poco más detallada de cada uno de los capítulos desarrollados en esta memoria.

Capítulo 2. Problemas de Localización con semicalibradores

El origen de los resultados presentados en este capítulo proviene del planteamiento del problema de localizar una nueva facilidad con relación a un conjunto de líneas fijas de forma que se minimice la suma de las distancias (véase Robert y Toussaint (1990)[90]). Este problema es el dual de un problema muy conocido consistente en localizar una línea minimizando las distancias con respecto a un conjunto de puntos fijos (véase Morris y Norback (1980)[79] , (1983)[80]; Megiddo y Tamir (1983)[76]; Love, Morris y Wesolowsky (1988)[71]; Schöbel (1998)[94] o Robert y Toussaint (1990) [90]). Después de un primer análisis de este problema uno puede observar que no es muy difícil de solucionar: Se reduce a un problema lineal en $\mathbb{R}^2$, mientras que en $I\!R^n$ es en general un problema convexo de programación cuadrática (véase la sección 2.3 en este capítulo).

No obstante, aunque el problema fue fácil de solucionar nos dimos cuenta que tenía una interesante estructura común que merecía la pena ser investigada. Las distancias de un conjunto no acotado a un punto pueden venir representadas por un calibrador generado por este conjunto. En particular, las distancias desde una recta y las distancias desde una semirecta vienen dadas por el calibrador con relación a un conjunto convexo no compacto. Un ejemplo de esta situación en la vida real puede ser la localización de una facilidad dentro de una región acotada con respecto a un conjunto de cables conductores de electricidad que cruzan la región (suponiendo que los usuarios de la facilidad quieren minimizar los costes de conexión). Nótese que para representar el conjunto de puntos a una distancia cero de los cables (los mismos cables) necesitamos considerar bolas unidad no acotadas. Otro ejemplo es la localización de una facilidad con relación a regiones de demanda minimizando la suma de las distancias desde dichas regiones. En la figura 2.1 de este capítulo se muestra un ejemplo conjuntamente con las curvas de nivel de la distancia con respecto a las regiones.

El marco común en todas estas funciones de "distancia" es que provienen del funcional de Minkowski, pero aplicado a un conjunto convexo no compacto. Es importante resaltar que toda la literatura de localización basada en calibradores de conjuntos acotados puede verse como un caso particular. Hemos decidido llamar a estas funciones semicalibradores. Este término proviene del término calibrador porque estas funciones son positivamente homogéneas y convexas pero en general no son definidas, o sea el conjunto de puntos cuya imagen es cero es diferente del vector nulo.

Nuestro objetivo es explorar las propiedades de estas funciones generadas por el funcional de Minkowski de un conjunto convexo, no compacto y no simétrico y las implicaciones de su uso en el Análisis de Localización. Como mostraremos la mayoría de los resultados ya conocidos pueden ser adaptados a las nuevas medidas de distancias. Los más importantes son aquellos que caracterizan geométricamente el conjunto de soluciones óptimas del problema de localización de una única facilidad y la interpretación geométrica de las medidas dadas por los semicalibradores como la longitud de uno de los caminos más cortos usando direcciones de viaje de su bola unidad.

El capítulo está organizado como sigue. En la sección 2.1 se introduce el concepto de semicalibrador y se ilustra con algunos ejemplos.

Se dice que φ es un "semicalibrador" si existe un conjunto cerrado y convexo C conteniendo al origen tal que,

$$\varphi(x) := \inf\{\lambda > 0 \; : \; x \in \lambda C\} \tag{1}$$

El conjunto C se llamará "bola unidad" asociada a φ y por extensión se le llamará a φ el semicalibrador de C. Para aquellos x no pertenecientes a λC para ningún $\lambda > 0$ se define por convenio que $\varphi(x) = +\infty$.

Posteriormente, se introduce el problema de localización con semicalibradores:

Sea A un conjunto finito de puntos en $I\!R^n$ el cual representa el conjunto de facilidades existentes. Cada facilidad $a \in A$ tiene asociado un semicalibrador φ_a cuya bola unidad C_a es un conjunto convexo y cerrado que contiene al origen. El problema consiste en la localización de una nueva facilidad en $I\!R^n$ de forma que minimice una función objetivo F la cual depende de los semicalibradores con respecto a las facilidades existentes en A.

Matemáticamente, el problema es formulado como,

$$(P_w^\gamma(A)) \qquad \min_{x \in I\!R^n} F(x) = \gamma(w_1 \varphi_{a_1}(x - a_1), \cdots, w_m \varphi_{a_m}(x - a_m))$$

donde $A = \{a_1, \cdots, a_m\}$ es el conjunto de facilidades existentes, $\{w_1, \cdots, w_m\}$ son pesos positivos, φ_{a_i} es el semicalibrador asociado a a_i y γ es un semicalibrador monótono en $I\!R^m$.

Nótese que el problema de la mediana, el problema del centro y el problema del centdian son casos particulares de este problema general de localización.

En la sección 2.2, dedicada a la introducción de algunos preliminares matemáticos, se estudian las relaciones de los semicalibradores con el Análisis Convexo, se prueba que el semicalibrador es una función sublinear, que puede ser evaluado por medio de la función soporte del conjunto polar de su bola unidad y se obtiene su subdiferencial en cualquier punto. Asimismo, en el Teorema 2.2 se da una interpretación geométrica de las distancias medidas por los semicalibradores. Se prueba que la distancia viene dada por la longitud del camino más corto usando direcciones de viaje de su bola unidad. Este resultado generaliza el Teorema 1 de Ward & Wendell (1985)[101].

En la sección 2.3 se da una caracterización geométrica del conjunto de soluciones óptimas del problema general de localización con semicalibradores. Este problema presenta la novedad con respecto a las formulaciones previas en Teoría de Localización de que su valor óptimo puede ser 0 ó ∞. En primer lugar se caracterizan los casos en que esto ocurre, para pasar posteriormente al análisis del caso general. Los resultados obtenidos son extensiones de resultados previamente conocidos en Teoría de Localización cuando se usan calibradores como medidas de distancia (véase Durier (1995)[34]). Por último se incluyen y se caracterizan dos casos de particular interés: El problema de Weber y el problema minimax.

Capítulo 3. Un problema de localización bietápico, multiperiodo y multiproducto con restricciones de capacidad

En muchas situaciones del mundo real donde las grandes compañías manufacturan y distribuyen productos es necesario localizar plantas de producción o almacenes que han de entregar productos a los consumidores finales con objeto de cubrir las demandas de éstos. Éste es el caso, por ejemplo, de piezas de recambio de coches o catálogos de agencias de viaje. Si las localizaciones admisibles para estas facilidades son finitas y conocidas de antemano, entonces nos enfrentamos con un problema discreto de localización de plantas. Estos problemas han sido ampliamente estudiados y hablando en términos generales pueden clasificarse en: 1) Problemas de localización de plantas sin restricciones de capacidad (SPLP); y 2) problemas de localización de plantas con restricciones de capacidad (CPLP). Ambos tipos de problemas pueden ser formulados como problemas de programación entera-mixta (véase Aikens (1985) [2]). Krarup y Pruzan (1983) [65] probaron que incluso el SPLP pertenece a la clase de problemas NP-duros por lo que no es posible obtener soluciones exactas de estos problemas en tiempo polinomial. Véanse, por ejemplo el artículo de Aikens (1985) [2] y los trabajos de Drezner (1995) [31] o Daskin (1995) [28] para una excelente revisión general de estos tipos de problemas y sus extensiones. Como ya dijimos, nosotros proponemos una nueva extensión que combina aspectos dinámicos del problema con aspectos bietápicos (el problema MMCPL), la cual puede ser vista como una introducción a un nuevo problema de localización de plantas.

En nuestro estudio modelamos situaciones donde existe un conjunto de clientes con demandas de diferentes productos. Estas demandas deben de ser cubiertas desde un conjunto de almacenes donde los productos se guardan temporalmente. Finalmente, estos productos son enviados a los almacenes desde plantas de producción. Suponemos que tanto almacenes como plantas tienen capacidad limitada. Adicionalmente, estudiamos el problema a través de un horizonte cronológico finito. Nuestra formulación permite tanto la apertura de nuevas facilidades como el cierre de las ya existentes (véase la figura 3.1 en este capítulo).

Bajo esta estructura, el problema consiste en determinar la política óptima para establecer las plantas y los almacenes a través de un horizonte finito de tiempo, con objeto de cubrir la demanda de los clientes y con mínimo coste.

En la sección 3.1 se presenta la formulación matemática del modelo como un problema de programación entera mixta. A pesar de que esta formulación es la usual en los problemas multiperiodo, presentamos una formulación alternativa bastante menos usual pero más conveniente para nuestro propósito de resolución del problema. En el teorema 3.1 probamos que las dos formulaciones son equivalentes. Con esta nueva formulación, el problema sigue siendo un problema de programación entera-mixta con un gran número de variables (por ejemplo, un problema con 50 clientes, 20 almacenes, 20 plantas, 2 tipos diferentes de productos y 4 periodos de tiempo tiene 11360 variables y 764 restricciones). Esto hace que no sea posible obtener soluciones exactas en tiempo polinomial (nuestras experiencias computacionales mostradas al final del capítulo así lo confirman). Por tanto se propone un método alternativo para obtener soluciones aproximadas de este problema.

En la sección 3.2 proponemos una relajación Lagrangiana del problema ($LR(\lambda, \mu)$) que nos permite descomponerlo en dos subproblemas ($LR1(\lambda, \mu)$ y $LR2(\lambda, \mu)$). Tras sucesivas descomposiciones de $LR1(\lambda, \mu)$ y $LR2(\lambda, \mu)$, probamos que estos dos subproblemas pueden ser resueltos de forma óptima mediante la resolución de un número finito de problemas de programación lineal, y por consiguiente también puede ser resuelto óptimamente el problema relajado ($LR(\lambda, \mu)$).

Para cada valor de los multiplicadores λ y μ el valor de la función objetivo de $LR(\lambda, \mu)$ ($v(LR(\lambda, \mu))$) nos da una cota inferior del valor de la función objetivo de nuestro problema original. Por consiguiente, puesto que $v(LR(\lambda, \mu))$ es una función cóncava lineal a trozos, usamos el método del subgradiente para obtener la máxima, o al menos una buena, cota inferior.

En la sección 3.3 desarrollamos un procedimiento heurístico que nos permite obtener una buena solución a nuestro problema original. Este procedimiento toma como punto de partida la mejor solución relajada obtenida por el método del subgradiente.

Por último, en la sección 3.4 realizamos un estudio computacional con datos simulados aleatoriamente para mostrar el buen comportamiento de nuestro modelo (véase la tabla 3.3). No hemos podido comparar los resultados computacionales de nuestro procedimiento con otros procedimientos propuestos en la literatura, porque como ya hemos mencionado, no se ha considerado previamente en la literatura un modelo combinando aspectos dinámicos y bietápicos.

Capítulo 4. Localizando líneas anti-centro en presencia de restricciones de localización

El avance tecnológico experimentado por las sociedades modernas en estos últimos años ha incrementado los peligros, la polución u otros riesgos a los que ciertas comunidades pueden verse expuestas. El transporte de gas y de petróleo por tuberías son ejemplos de tales peligros potenciales. Por esta razón, el diseño de rutas lineales que transportan materiales nocivos o peligrosos se ha convertido en un tema de creciente interés en estos últimos años.

El efecto que tiene un accidente a lo largo de un canal o tubería que transporta material peligroso sobre la población existente puede ser considerado como una función decreciente de la distancia. Así, para prevenir los efectos nocivos, parece razonable proveer a cada población o facilidad de una zona de protección alrededor de ella. Sin embargo, a pesar del área de protección asociada con cada facilidad, un criterio sensato de optimización debería buscar un diseño en el que se maximice la mínima distancia a las áreas de seguridad. La validez del criterio maximin en el diseño de rutas de transporte de material peligroso ha sido criticado algunas veces (en el capítulo de Erkut y Verter en el libro de Drezner (1995)[36] se puede encontrar una buena descripción de este tipo de problemas). Estas críticas dan por hecho que el criterio seleccionado tiene que tener en consideración la noción de riesgo. Estas razones son perfectamente válidas para diseños de envío donde varios conjuntos de rutas pueden ser simultáneamente consideradas con objeto de no exponer repetidamente a la misma población a altos riesgos. Sin embargo, en el diseño de canales o tuberías esta política no puede ser aplicada porque en este caso el diseño debe ser necesariamente fijo. Por consiguiente, en estos casos están justificados los criterios "conservadores" como el max-min.

La localización de líneas en el plano ha sido previamente estudiada por varios autores pero las mayoría de las veces con criterios atractivos. Sin embargo, si el criterio es repulsivo el problema de localización de líneas apenas si ha sido estudiado. Algunas similitudes pueden encontrarse en los artículos de Gopalan et al. (1990) [42], Batta y Chiu (1988) [9] o Sivakumar et al. (1993) [95] entre otros. Estos autores estudian la determinación de rutas para transportar materiales peligrosos en redes de carreteras. Por otra parte, como ya se ha mencionado, la localización de una ruta lineal peligrosa ha sido estudiada por Drezner y Wesolowsky (1989) [29]. En este artículo, estudian la maximización de la mínima distancia euclídea ponderada a un conjunto de puntos. Finalmente, Hinojosa y Puerto (1999) [58] estudian la localización de una línea en el plano maximizando las distancias mínimas ponderadas con relación a un conjunto de polígonos para una familia particular de normas poliédricas. Éstas son las normas cuya bola unidad está inscrita en la bola unidad euclídea. Esta falta de estudios refuerza la importancia del desaarrollo de nuevos tratamientos para este tipo de problemas.

En este capítulo, generalizamos los avances considerados previamente por Drezner y Wesolowsky (1989) [29] e Hinojosa y Puerto (1999) [58]. Consideramos una norma general para medir distancias y caracterizamos la línea solución en el caso general. Además, desarrollamos algoritmos tanto para la distancia Euclídea como para cualquier distancia poliédrica. Las normas poliédricas han sido ampliamente usadas en la literatura de problemas de localización. Por ejemplo, su uso tiene una buena aplicación cuando el impacto dañino causado por la ruta lineal (las tuberías o los canales transportando materiales peligrosos) es difundido por el viento el cual tiene algunas direcciones predominantes que coinciden con las direcciones de viaje de la norma.

En la sección 4.1 se formula el modelo matemático: Sean $P_1, \cdots, P_n$ una familia de al menos dos polígonos disjuntos y acotados en el plano. Denotemos por $v_1^k, \cdots, v_{r_k}^k$ el connjunto de vértices de P_k $\forall\, k = 1, \cdots, n$, cuyas coordenadas son $v_i^k = (x_{v_i^k}, y_{v_i^k})$ y por w_k el peso asociado al polígono P_k o el vértice v_i^k dependiendo del caso. El problema consiste en hallar una línea π dentro del cierre convexo C de un conjunto dado de polígonos P_k $k = 1, \cdots, n$ de forma que se maximice la mínima distancia ponderada desde los polígonos a la línea. Nótese que este problema es no convexo, porque la función objetivo es una función no convexa. Matemáticamente el problema es formulado como,

$$(MAXLIN) \qquad \max_{\pi} \left(\min_{1 \leq k \leq n} w_k \mathrm{d}_B(P_k, \pi) \right)$$

donde $\mathrm{d}_B(P_k, \pi)$ representa la mínima distancia desde P_k a la línea π. Posteriormente se desarrollan varias propiedades que nos permiten caracterizar la solución óptima bajo hipótesis generales basándonos en la relación existente entre cualquier norma y la norma euclídea.

En la sección 4.2 se desarrollan algoritmos que nos permiten resolver el problema cuando son usadas la norma euclídea o cualquier norma poliédrica. Posteriormente se desarrollan dos refinamientos para los algoritmos propuestos que consisten en: 1) un test de acotación y 2) una reducción del número de vértices que han de ser considerados en los diferentes casos. Por último, se realiza un estudio computacional para mostrar la mejora que estas modificaciones producen en la aplicación de los algoritmos (véase las tablas 4.5 y 4.6). Asimismo, se incluyen dos ejemplos en el que se muestra cómo pueden ser aplicados los algoritmos con sus correspondientes modificaciones.

Capítulo 5. Un problema de localización y colas con pérdidas y con rechazo

El origen de este capítulo proviene del modelo previo de localización conocido como problema de la mediana con pérdidas para una única facilidad con c canales (c-SFLM)(véase por ejemplo, Chiu y Larson (1985) [25] o Frenk, Labbé y Zhang (1993) [41]) que ya fue introducido en la sección anterior. Las demandas que requieren ser servidas son de diferentes tipos y llegan según un proceso de Poisson homogéneo independiente con diferentes tasas de entrada. En los artículos de Chiu y Larson (1985) [25] o Frenk, Labbé y Zhang (1993) [41] los autores estudian propiedades de localización del c-SFLM con la política de admisión estándar que ya hemos mencionado anteriormente. Nuestro objetivo es mostrar que aquellas políticas de admisión que permiten rechazar demandas según el grupo a las que éstas pertenecen aún existiendo algún canal libre, mejoran la eficiencia global del sistema mientras que se siguen verificando propiedades similares de localización.

Está claro que para sistemas de emergencia sin un sistema sustituto secundario, nuestro tipo de política puede ser algunas veces inaceptable. Sin embargo, hay situaciones donde este modelo debe ser aplicado naturalmente. Considérese, por ejemplo, el caso de helicópteros usados simultáneamente para urgencias médicas y para transportar órganos humanos que serán usados para realizar algún trasplante.

Estos helicópteros apenas deben ser usados para a transportar heridos leves a hospitales porque deben estar preparados para una emergencia, como el transporte de heridos graves y/o el transporte de órganos humanos donde la rapidez es crucial.

A pesar de su importancia, estas políticas anticipativas apenas han sido consideradas en la literatura debido a la dificultad matemática para manejar estos modelos.

En la sección 5.1 se presenta la formulación matemática del modelo. Se considera la localización de un sistema de servicio de demanda-respuesta en un espacio métrico N en el cual se han establecido c unidades móviles estacionadas en la misma posición. Al sistema llegan n tipos de clientes que requieren ser servidos, siendo el proceso de llegada de los diferentes grupos un proceso de Poisson homogéneo independiente con tasa de llegada λ_i para $i = 1, \cdots, n$ (véase la figura 5.1). Nuestro objetivo es encontrar la localización del sistema de servicio z y las proporciones de clientes aceptados x_i cuando al menos una unidad de servicio está libre de forma que se minimice el coste total esperado por unidad de tiempo. Por tanto, el problema puede ser matemáticamente formulado como:

$$\min_{x \in [0,1]^n, z \in N} F(x, z)$$

siendo

$$F(x, z) = \sum_{i=1}^{n} \lambda_i x_i \Psi_c(x, z) r_i(z) + \sum_{i=1}^{n} (\lambda_i(1 - x_i) + \lambda_i x_i (1 - \Psi_c(x, z))) \hat{r}_i$$

donde $\Psi_c(x, z)$ representa la probabilidad de que exista al menos una unidad de servicio libre. Este valor es conocido en Teoría de Colas como la fórmula de pérdida de Erlang (véase por ejemplo, Medhi (1991)[75]). $r_i(z)$ representa el coste que le supone al sistema la aceptación del cliente i y $\hat{r}_i$ representa el coste de rechazo del cliente i.

En la sección 5.2 se obtienen resultados de localización para el modelo. Se introduce un problema de la mediana relacionado con nuestro problema y se prueba que si existe un conjunto finito dominante D de soluciones óptimas para este problema de la mediana, entonces D también es un conjunto finito dominante para nuestro problema, con lo cual el problema se reduce a la resolución de un número finito de problemas puros de colas que ya fueron resueltos por Carrizosa et al. (1998)[20].

En la sección 5.3 se desarrolla un algoritmo que nos permite solucionar eficazmente el problema en aquellos marcos donde existe un conjunto finito dominante de soluciones óptimas para el problema de la mediana, es decir, en espacios discretos, en redes y en espacios continuos con normas poliédricas. Asimismo, se estudia cómo determinar eficientemente el conjunto finito dominante D en espacios con normas poliédricas.

En la sección 5.4 se da un procedimiento para obtener buenas soluciones aproximadas en espacios continuos con cualquier norma general. Esta aproximación se basa en el hecho de que las normas poliédricas son densas en el conjunto de todas las normas. Basándonos en esto, demostramos que dada una sucesión de normas poliédricas, la sucesión de funciones objetivo obtenidas tomando como medidas de distancia dichas normas, converge uniformemente a la función objetivo de nuestro problema original en un espacio normado cualquiera. Asimismo, damos el valor del máximo error teórico que se puede cometer por esta aproximación (véase la tabla 5.1 en la que se muestran estos errores para algunas normas l_p) e ilustramos el citado procedimiento de aproximación mediante varios ejemplos.

En la sección 5.5 se incluye un sencillo ejemplo que muestra la eficiencia de nuestro modelo en relación al modelo c-SFLM.

En la sección 5.6 se estudia una extensión del modelo, en el cual se consideran dos sistemas de servicio (modelo c-TFLMR): Un sistema principal de servicio y un sistema secundario de servicio, de forma que, los clientes rechazados debido a la política de admisión en el sistema de servicio principal son aceptados en el sistema de servicio secundario si algún canal está libre. El objetivo es encontrar la localización de los dos sistemas de servicio(z_1, z_2) así como, las proporciones de clientes aceptados en el sistema principal de servicio (x_i) cuando al menos un canal está libre, de forma que se minimice el coste total esperado por unidad de tiempo. La formulación matemática del problema viene dada por:

$$\min_{x\in[0,1]^n, z^1\in N, z^2\in N} F(x, z^1, z^2),$$

siendo $F(x, z^1, z^2)$:

$$\Psi_c^1(x, z^1) \sum_{i=1}^{n} \lambda_i x_i (r_i^1(z^1) - \hat{r}_i) + \Psi_c^2(x, z^2) \sum_{i=1}^{n} \lambda_i (1 - x_i)(r_i^2(z^2) - \hat{r}_i) + \sum_{i=1}^{n} \lambda_i \hat{r}_i.$$

donde la notación es similar al modelo anterior, pero teniendo en cuenta la existencia de dos sistemas de servicio.

La función objetivo del modelo c-TFLMR es suma de dos funciones fraccionales no lineales con malas propiedades estructurales. Estas funciones son conocidas por ser extremadamente difíciles de optimizar. Este hecho hace que intentemos explotar algunas propiedades del problema en su faceta de localización (las cuales son similares a las obtenidas para el c-SFLMR).

En la sección 5.7 se obtienen algunos resultados de localización y se desarrolla un procedimiento de optimización heurístico "ad hoc" para la fase de minimización de la política de admisión. Posteriormente, proponemos un procedimiento de ramificación acotación que combina los resultados de localización con métodos de direcciones factibles para solucionar la fase de minimización de la política de admisión. Este procedimiento puede ser aplicado en espacios discretos, en problemas de redes y en problemas continuos con normas poliédricas. En espacios continuos con cualquier norma general este procedimiento puede ser aplicado combinado con el procedimiento previo desarrollado para el c-SFLMR para obtener buenas soluciones aproximadas.

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Departamento de Estadística e Investigación Operativa

Some Problems in Location Theory

Ph. Dissertation

by

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

Roughly speaking, the aim of Location Theory can be summarized in the following sentence :

> "Look for the location of a service that best fits, in some sense, the needs of a set of customers"

However, this description is so vague that more things have to be established to describe the subject of our study.

Our approach to Location Theory proceed from an Operation Research framework. Thus, ours is a quantitative or analytical approach. In this sense, for each different situation we face with, we try to fit a mathematical model as an aid for solving the location problem. Creating useful mathematical models of real location problems always involve abstraction and frequently requires simplifying assumptions. Therefore, we do not deal with the real situation but with ideal representations of it and then the solution obtained is a solution to the model, which may or may be not an accurate representation of the real-world location problem. Indeed, in certain occasions, qualitative factors such as harmony, good fortune or whatever, become very important in a location problem. This kind of situations are badly modeled from a quantitative point of view. In these cases, analysis based on mathematical consideration are of scarce interest. This memory deals exactly with the remaining cases where the location decisions are based on rational, objective and quantitative criteria.

Prior to the development of interest in analytical approaches, location problems were solved by iconic models. That is to say, by models that are a scale representations of the situations to solve. In these models, alternatives have to be judged on the basis of visual effects what make them analyst dependent. As an example of these models we have the Varignon Frame used to solve the traditional *Weber Problem* (see Wesolowsky (1993) [104] for more details).

Nowadays, we are interested in the development of mathematical models. This will explain our concentration on problem formulations, model constructions, and solution techniques for this kind of situations.

There are two different types of mathematical models: 1) descriptive, and 2)normative. In the former, the model is only used to describe and no decision will be made on its basis. Normative models prescribe a course of action that, in some sense, is optimal. Economist and geographers employ the former to explain or describe observable socio-economical phenomena such as inter- and intra-regional resource development, price structures or labor migration. Another examples of these models are queuing models, where effectivity measures are studied to describe only the system. Our aim is different. Most of the models we are interested in are normative, because our ultimate goal is to provide decision-makers with quantitative tools for finding good solutions of objective and realistic locational decision problems.

It is a hard task to classify in wide areas the different topic addressed by the Location Theory. However, location seems to have developed three main areas: 1)Continuous Location, 2)Location-allocation or more generally speaking discrete location, and 3) Network Location.

Continuous Location deals with problems which allow facility locations to be anywhere in a continuous space (commonly a two- or three- dimensional solution space). Discrete Location deals with situations where the possible locations are specified in advance and are a finite number. Network Location problems, as its name indicates, are those that are formulated on graph or networks. In this thesis we are interested mainly in the first and the second one. Excellent textbooks on these subjects are: Drezner (1995)[31], Love, Morris and Wesolowsky (1988)[71], Mirchandani and Francis (1990)[78], Francis, McGinnis and White (1992)[40], Hamacher (1995) (in German)[50] or Puerto (1996)[88] (in Spanish). More specialized books are: Larson and Odoni (1981)[68], Daskin (1995)[28], Hurter and Martinich (1989)[62] or Padberg and Rijal (1996)[83].

1.1 Some historical background

The origin of Location Theory could be credited to the Greek geometers because they were who first studied properties of singular points regarding configurations of given points. However, it is possible to be even more precise. Following Wesolowsky (1993)[104], in 638 A.D. when the Sarracen army destroyed the Alexandrian Library, it contained at least 3 different solution of the nowadays most commonly known as *Weber problem.*

More recently, Pierre Fermat proposed a purely geometric problem in the early 1600s: "Given three points in a plane, find a fourth point such that the sum of its distances to the three given points is as small as possible". This problem was geometrically solved by Torricelli (1608-1647) although some authors credit Cavallieri (1598-1647) with both posing and solving the problem. The solution attributed to Torricelli is as follows:

The three points are joined by lines to make a triangle. Equilateral triangles are constructed on the sides with the vertices pointing outward. The three circles through the vertices of the equilateral triangles intersect at a point, which is the solution of the problem.

This method applies when the triangle formed by the fixed points has no angle greater than 120^o. In 1834, Heinen proved that if an angle is greater than 120^o, the fixed point associated with this angle is the optimal location. In 1750 Simpson suggested generalizing the problem to include different weights. Nevertheless, the modern origin of this theory is usually credited to Alfred Weber who in 1909 [102] wrote the book "Über den standort der Industrien" ("Theory of the location industries"). In this book, the author used this model to determine the optimal location of a factory serving a single client and with two distinct sources for its raw material. Thus, the three fixed points were the two sources of materials with different weights and a weighted market location, respectively. He proposed as criterion for locating the factory the minimization of transport cost or equivalently distance traveled which is in the basis of the *minisum location problem*, frequently referred to as the *median problem* or the *Weber problem.*

Weiszfeld (1937)[103] in his paper "Sur le point lequel la somme des distances de n points donnés est minimum" provided a analytical method based in an iteration procedure for finding Euclidean distances to n fixed points. His method laid unknown until the late fifties and early sixties and was rediscovered by another authors. Since then, various generalizations and several criteria for locating have appeared and lots of papers have been devoted to location analysis. Architects, computer scientists, economists, engineers, operation researchers, transportation system designers, technical geographers have discovered a common interest in a concern for the location of facilities. This continuous growth can be followed in the literature. Plastria (1995) [86] reported a figure showing the number of publications per year since 1960 until 1994 contained in a bibliography restricted to continuous problems and directly related fields.

Whereas the continuous location problems date from the XVII century (or from the Greek geometers), the significance of the finite location set and the discrete nature of locational decision problems were not fully recognized until the late 1950s. According to Mirchandani (1990) [77], the impetus for the formal treatment of discrete minisum locational decision problem must be attributed to Kuehn and Hamburger (1963)[66], Manne (1964)[73], Hakimi (1964)[46] and Balinski (1965)[5].

Kuehn and Hamburger (1963)[66] and Manne (1964)[73] appear to have been the first to give heuristic approaches to the *simple or uncapacitated plant location problem (SPLP) or (UPLP).* The (SPLP) is a special case of a minisum location decision problem usually dealt with in discrete formulations. Although the first explicit formulation of SPLP is frequently attributed to Balinski (1965)[5], the approach proposed by Balinski and Wolfe (1963)[4] appears for some authors to have been the first attempt to solve the SPLP to optimality. On the other hand, Hakimi (1964)[46] introduced the *p-median problem* on a network and he started the development of network location.

Accordingly, discrete location theory, covers a period of development which only spans about four decades. Nevertheless, the literature in discrete location theory is rapidly growing as well as in continuous location theory.

Another important issue, showing that this fields is really an important one, is the fact that the prestigious Mathematical Rewiew/Zentralblatt für Mathematik/Mathematics Abstract decided to include it in their 1991 Mathematics Subject Classification under the number 90B85-Continuous Location and 90B80-Discrete location, within the section 90Bxx, Operations Research and Management Science.

Finally, I would like to mention that the great number of researchers and papers devoted to this field can be also a reason for the journals devoted specifically to this subject: Studies in Locational Analysis and Transportation Science. Needless to say that many other journals are also appropriate to find papers and references to this field.

1.2 A classification scheme

Because of the large number of location problems which can be found in the literature, it is useful to define a number of categories that can be used to classify these problems. In classifying these problems five elements have to be considered:

1. new facility number and characteristics,
2. solution space characteristics,
3. existing facilities,
4. distance measure,
5. objective.

In what follows, we give a description of each one of these categories. Readers interested in a more detailed classification scheme for Location problems are referred to Hamacher and Nickel (1998) [52].

1.2.1 New facility

The location of the new facility is the decision variable of the general location problem. This variable is characterized by

-*Number and quality of the service provided.* If more than one facility have to be located, it will be necessary to specify the characteristics of each one of them. When they are identical, as for instance mail boxes, we face with a multifacility model, otherwise as in the case of health services, we can find hierarchical location problems.

In multifacility problems we can distinguish two kinds of models: 1) Models where it has been decided to establish a predetermined number of new facilities. This number can be determined, for example, by economic reasons or market studies. Examples of these models are the *p-median* or the *p-center problem.* 2) Models were the number of facilities to be located is no pre-specified as, for example, in the *simple plant location problem (SPLP)*

-*Nature of service.* We can distinguish three kinds of services: Attractive services, repulsive or obnoxious services and semi-obnoxious services.

A facility is considered an attractive service when it performs some advantage for the community where it will be located. Examples of attractive services are schools, theaters, libraries, hospitals, fire stations, etc.

Nevertheless, not all the services are attractive for the community. For instance, prisons, drug addict institutions, garbage plant, nuclear plant, solid waste disposal, gas pipeline or channels carrying noxious material are usually refused and the population wants to be as far away as possible from the facility. These facilities perform repulsive, hazardous and obnoxious services.

Finally, a facility is considered a semi-obnoxious service when it is attractive for some clients and repulsive for the remainders. Therefore, in modeling a problem it is very important to determine the attractiveness of the model (see e.g. Chen et al. (1992) [24])

- *The "size" of the facility.* The new facility can be considered to occupy either point locations, area locations or some given structure (see e.g. Hakimi, Schmeinchel and Labbé (1993)[47], or Carrizosa, Muñoz and Puerto (1998)[21]). Besides, if the facility to be located is, for example, a production plant or warehouse it can be either capacitated or uncapacitated.

1.2.2 Solution space

Solution space is the framework where the problem is established. It contains as elements the existing facilities and the new facilities to be located. The choice of an appropriate solution space is crucial, because it determines aspects so important as accuracy or solvability of the model. Some usual solution spaces are:

- *Discrete space.* There exists a finite number of possible candidates and then, the facilities to be located can only be placed at a finite number of potential sites selected via some prior analysis. In most cases decision-makers consider a discrete representation to be a more realistic and a more accurate portrayal of the problem at hand (see e.g. Mirchandani and Francis (1990) [78]).

- *Networks.* Very often, they represent communication networks. The nodes are the important elements representing cities or neighborhoods. The arcs usually model connections between nodes, as roads, streets, wire-connections, etc (see e.g. Tansel et al.(1983) [97] or Labbé, Peeters and Thisse (1995)[67]).

- *Euclidean space $\mathbb{R}^n$.* This kind of space is used when the problem presents regional aspects that do not have to be discretized (see e.g. Drezner (1995) [31]). In addition, it can be also used to model network situations when the number of links and nodes is too huge. The cases $n = 2$ or 3 have a physical meaning.

The cases with $n > 3$ have been used to model and solve estimation problems in statistics.

-*Sphere.* It is very useful to model real location problems with large scale distances (see Drezner (1985) [30]).

-*Embedded network in a continuous space.* This is a solution space where there is a network that models high speed links together with the usual continuous space (see Carrizosa and Rodríguez-Chia (1997) [22]).

-*Functional spaces (Banach spaces).* Recently used to model problems where one looks for the location of trajectories (see Puerto and Rodríguez-Chia (1999) [89]).

1.2.3 Existing facilities

In terms of Location Theory, existing facilities are the set of users who require to be served. It is usually defined by a set A (demand points) and an intensity function associated with each one of these points (very often this intensity is a probability measure on A).

There exist two different ways to represent the existing facilities: 1) by a finite set of points of the solution space, and 2) by regions of the solution space.

In the first case, we face with a set $A = \{a_1, \cdots, a_M\}$ of points and a set of weights representing the importance (intensity) of the demand generated in each point $\{w_1, \cdots, w_M\}$.

In the regional model, we deal with a region R (not necessarily connected) included in the solution space and some probability measure defined which gives importance to each measurable subset of R. However, although realistic the regional facility case leads very often to complex mathematical problems (see e.g. Carrizosa, Conde, Muñoz and Puerto (1995) [19], Carrizosa, Muñoz and Puerto (1998) [21], Hakimi, Schmeichel and Labbé (1993)[47] or Nickel, Puerto and Rodríguez-Chia (1999)[82]). For this reason, sometimes it has been proposed in the literature the aggregation policy. That is to say, concentrating the demand of regions on distinguished points. It should be noted that the assumption made by the aggregation may have important consequences on the final solution.

1.2.4 Distance measures

Due to the fact that quality of service usually depends on distances, this element is a fundamental one in location problems. The choice of the distance function is often forced by the considered solution space.

In a discrete space we presume the availability of the shortest path distances between all relevant points, that is, between each possible facility-client pair. For discrete points on a plane, shortest distances can be computed either by approximation, on the basis of points' coordinates and a given metric, or we can explicitly compute the distances on a road network, when an underlying road network is predefined. In the first case, the distances between points can be calculated by the use of a simple formula based on a metric whereas, in the second, an algorithm is needed to compute the shortest distances on the network.

In a sphere the shortest distance between two points must be measured along the great circle passing through them and is the longitude of the shortest of the two arcs between the points.

In the usual Euclidean space $\mathbb{R}^n$, in most cases, we would like to deal with norms or metrics but other possibilities are also possible.

Any function $k : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfies the following properties is called a norm:

1. $k(x) \geq 0 \qquad \forall x \in R^n$
2. $k(x) = 0 \qquad \text{iff} \quad x = 0$
3. $k(cx) = |c|k(x) \qquad \forall c \in \mathbb{R} \qquad \forall x \in \mathbb{R}^n$
4. $k(x+y) \leq k(x) + k(y) \qquad \forall x, y \in \mathbb{R}^n$

It is straightforward to prove that k is a convex function.

With these functions, distances are defined as

$$d(x,y) = k(y-x)$$

and therefore, they verify the following properties:

1. *Nonnegativity* : $d(x,y) \geq 0.$

2. *Definiteness* : $d(x,y) = 0 \quad \text{iff} \quad x = y.$

3. *Symmetry* : $d(x,y) = d(y,x).$

4. *Triangle inequality* : $d(x,z) \leq d(x,y) + d(y,z).$

The use of these properties when modeling travel distances have been strongly criticized. For instance, in some occasions symmetry has no sense because there exist different itineraries to go and come back. It was Witzgall (1964) [105] who first proposes the use of polyhedral gauges loosing the symmetry property.

On the other hand, the definiteness property sometimes does not make sense as, for example, when we consider non-bounded regional facilities and $d(R_1, R_2)$ stands for the minimum distance from the regional facility R_1 to the regional facility R_2. Whenever $R_1 \cap R_2 \neq \emptyset$, $d(R_1, R_2) = 0$ but it does not imply that $R_1 = R_2$. In Chapter 2 of this memory, we propose the use of semigauges which loose the definiteness property 2.

In the following, we mention a usual family of norms, the l_p *norms*, which include Euclidean and rectilinear norms as special cases and we recall the concept of gauge.

l_p norms.

Perhaps, the most widely used family of measures of distances are those induced by the l_p family of norms (see e.g. Love and Morris (1972) [70] or Hansen, Perreur and Thisse (1980) [54]). The functional form of the l_p norm in the n-dimensional Euclidean space ($\mathbb{R}^n$) is given by:

$$\begin{aligned} l_p(x) &= ||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \qquad x = (x_1, \cdots, x_n) \in \mathbb{R}^n \qquad 1 \leq p < \infty, \\ l_\infty(x) &= ||x||_\infty = \max_{i=1,\cdots,n} |x_i| \qquad x = (x_1, \cdots, x_n) \in \mathbb{R}^n. \end{aligned}$$

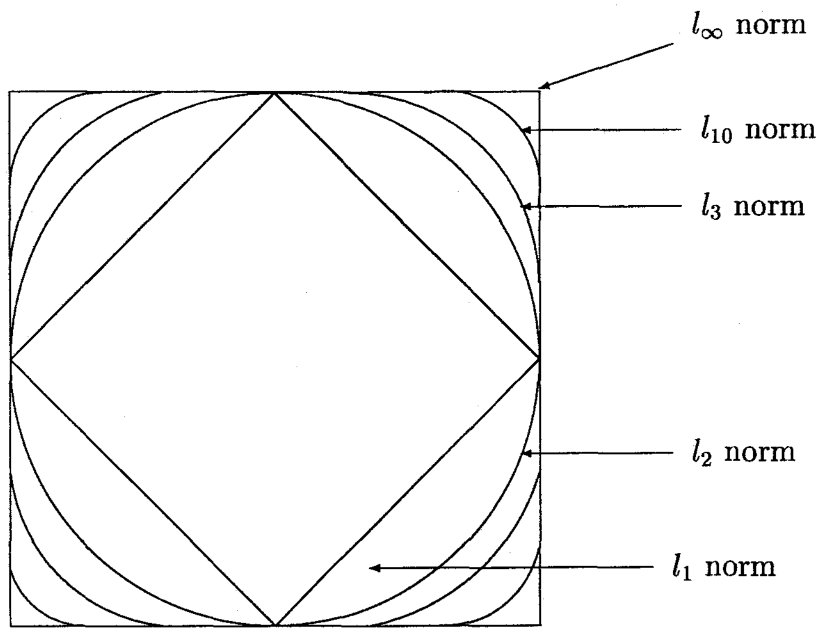


Figure 1.1: l_p norms

In the literature the cases $p = 1, 2, \infty$ have been extensively used and are referred as Manhatan or rectilinear norm ($p = 1$), Euclidean norm ($p = 2$) and Tchebycheff norm ($p = \infty$). When $p < 1$, l_p no longer has the properties of a norm, in fact it looses convexity.

Perhaps the natural distance for most of the people is the Euclidean distance. In some way, this is unfortunate because it is unusual for objects (unless they fly) to travel between any two points in a straight line. Rather, objects must travel through an existing aisle, street or road. In the case where the aisle or street network is rectilinear, the use of the rectilinear distance is clearly more appropriate than the Euclidean distance. The Tchebychev distance is of use in modeling location problems involving the location of items moved in and out of an automated storage and retrieval system (see Francis, McGinnis and White (1992)[40]).

Figure 1.1 shows contours of unit balls of l_p functions. As can be seen, the choice of which p is the most appropriate to fit distances in a particular problem is a question that has to be taken into account (see Love and Morris (1972)[70]).

Gauges.

To avoid critics on the symmetry of norms one can consider gauges (see e.g. Durier and Michelot (1985)[33]). These distance measures allow to reflect cost structures given by non-symmetrical situations. One-way streets, slopes or meteorological phenomena, can be taken into account by these functions.

Gauges are based in the so called Minkowski functional associated with a convex set B. Given a compact, convex set B, the interior of which contains the origin, we define the gauge of B as:

$$\gamma_B(x) := \inf\{\lambda > 0 \ : \ x \in \lambda B\}$$

We say that B is the unit ball of γ.

The gauges have some interesting properties:

1. $\gamma_B(x) \geq 0$ for all $x \in I\!R^n$
2. $\gamma_B(x) = 0$ iff $x = 0$.
3. $\gamma_B(\lambda x) = \lambda \gamma_B(x)$ for all $\lambda \in I\!R^+, \quad x \in I\!R^n$.
4. $\gamma_B(x + y) \leq \gamma_B(x) + \gamma_B(y)$ for all $x, y \in I\!R^n$.

In addition, if B is symmetric then, the gauge γ_B is a norm.

Gauges are in general classified as polyhedral or non-polyhedral. Polyhedral gauges are those generated by polytopes, thus having unit balls with a finite number of extreme points. The remaining are the non-polyhedral or round gauges. These last gauges can also be classified as differentiable or non-differentiable.

Polyhedral gauges are very important because usually they lead to simpler location problems and linear programs.

On the other hand, when the definiteness property does not make sense one can consider gauges with respect to not bounded set. The mathematical evaluation of the gauge of x with respect to a closed (non necessarily compact) convex set does not differ from the one regarding compact convex sets: It is the Minkowski functional of the set (see the book of Hiriart-Urruty & Lemaréchal (1993)[60]). However, it is methodologically convenient to rename those functions within the field of Locational Analysis. Since the seminal paper of Witzgall (1964)[105], gauge functions in Locational Analysis have been identified with gauges of compact, convex sets. Therefore, and in order to avoid misunderstanding we propose to call semigauges to the gauges of closed (non necessarily bounded), convex sets.

- *Semigauges:*

Let C be a closed, convex set containing the origin, the "semigauge" of C is given by

$$\varphi(x) := \inf\{\lambda > 0 \ : \ x \in \lambda C\}.$$

The set C will be called the "unit ball" associated with φ. Remark that for those x not belonging to λC for any $\lambda > 0$ we assume the convention $\varphi(x) = +\infty$. It is straightforward to show from its definition that $\varphi(\cdot)$ verifies:

1. $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^n$.

2. $\varphi(\lambda x) = \lambda \varphi(x)$ for all $\lambda \in \mathbb{R}^+, \quad x \in \mathbb{R}^n$.

3. $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in \mathbb{R}^n$.

It is worth noting that $x = 0$ implies $\varphi(x) = 0$, but the converse in general does not hold. Therefore, definiteness is not verified by semigauges.

If the interior of C contains the origin and C is bounded, φ is a *gauge*. In addition, if C is symmetric, φ is a *norm*. Thus, the semigauges can be seen as a generalization of the gauges and norms, because these are included as special cases.

1.2.5 Objective function

Due to our quantitative approach to location theory, most of the problems that we consider have a formulation as,

$$\begin{array}{cc} \text{opt} & F(d(a, (x_1, \cdots, x_p))_{a \in A}) \\ x = \{x_1, \cdots, x_p\} \in S & \end{array}$$

where:

F is the globalizing function,

opt means optimize, either minimize or maximize,

S is the solution space,

x is the new facility, either single $p = 1$ or multiple $p > 1$,

A is the set of existing facilities,

a stands for a general existing facility, and

d is a measure of distance.

The determination of which objective function has to be used is sometimes a hard task. It should be noted that the final solution strongly depends on that choice (see Carrizosa, Fernández and Puerto (1990)[16]). Therefore, it is important to devote some effort to this part of the modeling process.

Several attempts have been proposed in the literature to cope with this difficulty. The most widely accepted consists of establishing a set of rationality principles that our solution must verify and then try to get the objective function by means of an "axiomatic based" process (see e.g. Buhl(1988)[15], Carrizosa, Fernández and Puerto (1990)[16], Carrizosa, Conde, Fernández and Puerto (1994)[17] or Hansen and Roberts (1996)[56]).

Some examples of facility location problems are:

-*Minisum, median or Weber problem.* This problem is a single facility location problem (SFLP). Given a demand set A and a probability measure μ the problem consists of minimizing the expected distance to a new facility x.

$$\min_{x \in S} \int_A d(x,a) d\mu(a) \quad ; \qquad A \subset S$$

When A is a finite set, the problem becomes

$$\min_{x \in S} \sum_{a \in A} \mu(a) d(x,a)$$

Minimizing total distance traveled is perhaps the most popular criterion used in locational decisions (see e.g. Wesolowsky (1993)[104]). This criterion guarantees the efficiency in the location and minimizes the total cost of serving clients.

-*p-Median problem.* In this problem p new facilities are to be located. Existing facilities choose the closest new site for service. The objective is to minimize the expected distance traveled in serving the customers in the existing facilities. Mathematically:

$$\min_{x=\{x_1,\cdots,x_p\}\in S} \int_A \min_{i=1,\cdots,p} d(x_i,a)d\mu(a)$$

When A is a finite set, the problem becomes:

$$\min_{x=\{x_1,\cdots,x_p\}\in S} \sum_{a\in A} \min_{i=1,\cdots,p} d(x_i,a)d\mu(a)$$

This problem usually is dealt with in both, continuous and discrete or network solution spaces. It is a prototype formulation that reflects many realistic location decision problems. It is frequently employed to serve as the starting point for deliberating in cases where, for example, due to other strategic considerations, it has been decided to establish a predetermined number of new facilities to serve new markets or to modify an existing facility structure so as to have exactly p new and/or existing facilities (see e.g. Mirchandani (1990)[77]).

As we have above mentioned the minisum criterion minimizes the total cost of serving clients and guarantees the efficiency in the location. Nevertheless, this criterion may result in solutions which are unacceptable from the point of view of the service level provided to those clients who are located far away from the facility. Therefore, instead of the minisum criterion, one may wish to minimize the maximum distance traveled.

-*Minimax, center or Rawl problem.* Similar to the Weber problem formulation, but now, the minimization of the maximum distance weighted by μ is the objective function (see e.g. Elzinga and Hearn (1972)[35]).

$$\min_{x\in S} \sup_{a\in A} \mu(a)d(x,a)$$

When A is a finite set, the supremum operator can be replaced by the maximum.

The minimax is a more equitable criterion than the minisum. For example, in the case of the location of a school, a minimax criterion would guarantee that the largest distance traveled by a student would be as small as possible. Such an objective is encountered in the design of stadiums and theaters, as well as the location of emergency services such as fire stations, hospitals and civil defense units. Nevertheless, the minimax criterion may lead to very costly service systems.

-*p-Center problem.* In this problem the objective is to open p facilities and to assign each client to exactly one of them such that the maximum weighted distance from any open facility to any of the clients allocated to it, is minimal (see e.g. Handler (1990) [53]). Thus, the problem can be written as:

$$\min_{x=\{x_1,\cdots,x_p\}\in S} \{\sup_{a\in A} \min_{i=1,\cdots,p} d(x_i,a)d\mu(a)\}$$

Some attempts have been made to alleviate the deficiencies of the both, minisum and minimax criteria used alone. Examples of theses attempts are the use of bicriteria models or the cent-dian problem, which minimizes the convex combination of median and center criteria.

-*Cent-dian problem.* Given a scalar $\lambda > 0$, the objective function makes a convex combinations of a sum objective and a maximum objective. That is to say, the problem is,

$$\min_{x\in S} \left(\lambda \int_A d(x,a)d\mu(a) + (1-\lambda) \sup_{a\in A} \mu(a)d(x,a) \right)$$

This criterion can be seen as a compromise between the minimum total cost and the equity in the service. The term *cent-dian* was coined by Halpern (1976)[48] who introduced the λ-centdian as a parametric solution concept based on the bicriteria center/median model in a tree network. He modeled the corresponding trade-off with a convex combination of the unweighted center and weighted median objectives. More recently, Carrizosa et al., in 1994 [17] presented an axiomatic approach justifying the use of the centdian criterion. Tamir, Pérez-Brito and Moreno Pérez in 1998 [96] generalized this convex combination introducing weights in the center function. They presented a polynomial time algorithm for the p-facility case in a tree, where each one of the p facilities is a point.

Halpern in 1978 [49] studied the properties of the λ-centdian in a graph. Pérez-Brito, Moreno-Pérez and Rodríguez in 1997 [84] presented a finite dominating set for the p-facility centdian in a graph, and studied the generalized case.

Usually, the above mentioned problems are considered both in continuous and discrete spaces. A special location-allocation problem which usually is dealt with in discrete spaces is:

-*Simple or uncapacitated plant location problem (SPLP) or (UPLP).* Given a finite set of possible locations for establishing new facilities or redimensioning already existing facilities, the problem deals with the supply of a commodity from a subset of them to a set of clients with a prescribed demand for the commodity. Facilities are assumed to have unlimited capacity such that in principle any facility can satisfy all the demand. For given costs associated with the facilities and with the direct transportation routes from facilities to clients, we seek a minimum cost production/transportation plan (in terms of the number of facilities established, their locations, and the amount shipped from each facility to each client) satisfying all demands (see e.g. Krarup and Pruzan (1983)[65]). The elements of the problem are:

$I = \{1, \cdots, n\}$: The set of customers, indexed by $i \in I$.

$J = \{1, \cdots, m\}$: The set of possible location for facilities, indexed by $j \in J$.

f_j: The fixed cost of establishing the facility j.

d_i: The number of units demanded by customer i.

c_{ij}: The transportation cost of shipping one unit from facility j to client i.

The decision variables are:

$$y_j = \begin{cases} 1 & \text{if facility } j \text{ is open} \\ 0 & \text{otherwise} \end{cases}$$

x_{ij}: Number of units produced at facility j and shipped to client i.

The (SPLP) formulation is the following mixed-integer program:

$$\begin{aligned} \min \quad & \sum_{i \in I}\sum_{j \in J} c_{ij}x_{ij} + \sum_{j \in J} f_j y_j \\ \text{s.t.} \quad & \sum_{j \in J} x_{ij} \geq d_i \qquad \forall i \in I \\ & \sum_{i \in I} x_{ij} \leq \sum_{i \in I} d_i y_j \qquad \forall j \in J \\ & x_{ij} \geq 0 \qquad \forall i \in I,\ j \in J \\ & y_j \in \{0, 1\} \qquad \forall j \in J \end{aligned}$$

In contrast to the other prototype location problems, (SPLP) permits in a sense the broadest framework. Neither the number of plants to be located nor the transportation pattern are predetermined. Furthermore, the basic formulation of (SPLP) lends itself readily to sensitivity analysis. In addition, (SPLP) invites modifications which may permit more "realistic" modeling. While (SPLP) is basically a discrete, static, deterministic, one-product, uncapacitated problem formulation, it can be modified to accommodate dynamic, stochastic, multi-product, capacitated and two-echelon formulations (see e.g. Hinojosa, Puerto and Fernández (1999)[59]).

-*Maximin problem.* When we have to locate an "offensive" or repulsive facility it is clear that one wants to be as far away as possible from the facility. Then, it seems appropriate to use "conservative" criteria such as the maximization of the minimum distance to the facility to be located (see e.g. Erkut and Verter (1995)[36]). In some sense the maximin problem is the repulsive counterpart of the center problem, where the objective function is given by:

$$\max_{x \in S} \inf_{a \in A} \mu(a) d(x, a)$$

-*Stochastic queue median problem.* From each one of the existing facilities, which are assumed to be finite, a Poisson process emits inputs. Assume that these inputs have to be served at the position of the new facility and the service times are functions of the distances traveled.

If we assume that the server has no limit in its capacity the system behaves as a $M/G/1/\infty$ (see e.g. Berman et al. (1990)[14]).

The goal is to find the location of the server in order to minimize the average waiting time of inputs then, the objective function is to minimize the mean response time $TR(x)$:

$$\min_{x\in S} TR(x) = Q(x) + T(x)$$

where, $Q(x)$ is the mean delay in queue and $T(x)$ the expected travel time. $T(x) = \frac{\sum_{a\in A} d(x,a)}{\nu}$ being ν the average speed and the expression for Q is well-known in queuing theory to be:

$$Q(x) = \begin{cases} \dfrac{\lambda \bar{S^2}(x)}{2(1-\lambda\bar{S}(x))} & \text{for } \lambda\bar{S}(x) < 1 \\ +\infty & \text{otherwise} \end{cases}$$

where $\bar{S}(x)$ is the mean service time, $\bar{S^2}(x)$ is the second moment of service time and λ is the arrival or demand rate.

-*c-Server single facility loss median problem (c-SFLM).* As the previous one, this problem is a queuing problem but in which the server has a limit in its capacity. We assume a server with c service units. When a customer arrives it is either lost, if all c service units are busy, or served immediately, if there is at least one free service unit. Then, the system behaves as a $M/G/c/c$ queue (see e.g. Chiu and Larson (1985) [25] or Frenk, Labbé and Zhang (1993)[41]).

The goal is to find the location of the server in order to minimize the weighted sum of the mean response time and the cost of rejecting the customer from the system. The respective weights are the probabilities of immediate response and rejection. The objective function is given by:

$$\min_{x\in S}(1-\Psi_c(x))\bar{t}(x)+\Psi_c(x)\hat{r}$$

where, $\Psi_c(x)$ is the steady state probability of saturation (all service units busy), $\bar{t}(x)$ is the expected one-way travel time to a customer and $\hat{r}\geq 0$ is the cost per rejected customer.

$\Psi_c(x)$ is given by the well-know in Queuing theory, Erlang Loss Formula (e.g., see Larson and Odoni (1981)[68]):

$$\Psi_c(x)=\frac{(\lambda\bar{S}(x))^c/c!}{\sum_{i=0}^{c}(\lambda\bar{S}(x))^i/i!}$$

where $\bar{S}(x)$ is the mean service time and λ is the arrival rate.

It is worth noting that the above mentioned models are not exhaustive. Much more models can be found in the literature of Location Theory.

1.3 Summary

We have already mentioned that within the different areas in Location Theory we can find a large variety of problems, being all of them very interesting. It would be impossible to cover all those problems. Nevertheless, we do not want to concentrate our dissertation on a unique area. For this reason, we have dealt with diverse problems within different fields of Location Theory.

Thus, in Chapter 2 we deal with a new approach to the problem of locating a single facility in the in the Euclidean space $\mathbb{R}^n$ (SFLP). As we have discussed in Section 1.2.4, when the definiteness property of a distance does not make sense it seems reasonable the use of semigauges. Nevertheless, as far as we know, they have never been addressed before. This new approach to the SFLP, based in semigauges, presents a unified framework to the SFLP which generalizes previous approaches to location problems based on gauges and norms.

The concept of semigauge is defined and illustrated by several examples and a general location problem with semigauges is introduced. The minisum, minimax and centdian problems are particular cases of this general location problem. Afterwards, we prove a geometrical interpretation of distances measured by semigauges as the length of one of the shortest path using trip directions of its unit ball. Finally, we deal with the geometrical characterization of the set of optimal solutions of the general location problem with semigauges and we also include two particular interesting cases: the Weber and minimax problems.

In Chapter 3 we deal with a new approach to the single plant location problem (SPLP) in a discrete space. We study a multiperiod two-echelon multicommodity capacitated plant location problem (MMCPL). As we have already written in the above section, the SPLP is basically a discrete, static, one-commodity, uncapacitated problem formulation but it could be modified to accommodate dynamic, multicommodity, capacitated, two-echelon formulations. Indeed, many extensions of this problem can be found in the literature (see e.g. Aikens (1985)[2] or Daskin (1995)[31] for a good overview of this problem and its extensions). Among these extensions, we can emphasize two of them. The first one consists of introducing the dynamic aspect into the problem and the second one consists of the assumption of a certain structure in the transportation pattern (multi-echelon approach). Despite the generality of these models, the most natural framework for these problems is the combination of these two approaches. That is, the joint consideration of multiperiod and multi-echelon aspects. Nevertheless, as far as we know, this approach has never been addressed before and it can be considered as an introduction to a new location problem.

In our approach we model situations where a set of customers exist demanding different products. These demands must be covered from a set of warehouses where products are temporarily stored. Finally, these products are supplied to the warehouses from production plants. We assume that both warehouses and plants have limited capacity. Additionally, we study the problem through a finite time horizon.

Under this structure, the problem consists of determining the policy for sitting the plants and warehouses through a finite time period, in order to cover clients demand and with minimum cost.

We formulate the MMCPL problem as a mixed-integer programming problem. This problem has a large number of variables so that obtaining exact solutions in polynomial time is not possible even for small sizes in number of time periods, plants, warehouses or customers. Therefore, an alternative method is proposed to obtain approximate solutions of this problem.

We develop a lagrangean decomposition approach which permits us decompose the problem into subproblems which are solvable as linear programs. These solutions are used to obtain lower bounds on the optimal objective value. Then, we propose a heuristic method to obtain good feasible solutions using the relaxed solutions.

A computational study is reported showing the behaviour of our method with simulated data.

In Chapter 4 we study a particular maximin problem in a continuous space, which we will call MAXLIN problem. We deal with the problem of locating a linear obnoxious route on the plane with respect to a set of given facilities. The problem of locating obnoxious lines has been hardly studied in the literature, we can refer to Drezner & Wesolowsky (1989)[29] or Hinojosa & Puerto (1999)[58]. The first one uses the Euclidean distance to measure the distance and the second one a particular family of polyhedral norms. In this memory, we consider a general norm to measure distances and we characterize the solution line in the general case. In addition, in order to prevent the effect of disasters motivated by accidents, a polygon has been associated to each facility. These polygons represent security areas around the position of the facilities. The criterion of the maximization of the minimum weighted distance from the protection zones is used as objective function.

Our approach generalizes the previous approaches to this kind of problems and it also allows to use the maximim criterion with respect to protected areas not only with respect to points. A necessary and sufficient condition is stated which ensures the existence of optimal solutions for the MAXLIN problem. General properties for the determination of the optimal line under any general norm are given and algorithms are also developed to solve these problems both, for Euclidean and any polyhedral norms. Two examples are included and computational experience is reported for randomly generated problems. These experiences show the improvements obtained by the successive refinements proposed by the algorithms.

In Chapter 5 we deal with an extension of the c-Server-Single-Facility-Loss-Median service-location model (c-SFLM) which we will call the c-Server-Single-Facility-Loss-Median model with rejection (c-SFLMR). Consider a service system which consists of c mobile units placed at the same position. Requests for service can be of different kinds and they arrive in time as independent homogeneous Poisson processes with different input rates. As we have seen in the above section, in the c-SFLM one studies localization properties of a loss service-location model with a standard admission policy: if a request for service arrives, and some server is idle, it is immediately dispatched to cover the demand. If all the servers are busy the request is lost. The objective is to find the location of the facility that minimizes the expected total cost per unit time with respect to a given reward structure. In our approach (the c-SFLMR) we consider a new admission policy in order to enhance the global efficiency of the system. We consider that the customers can be discriminated according to their group so that they can be rejected even if the system has some idle service units (we suppose that it is handled by a back-up system). Despite their importance these anticipated policies have been hardly considered in the literature. The reason is the mathematical difficulty to handle these models.

Our objective is to find the location of the facility and the proportions of accepted customers when at least one server is idle in order to minimize the expected total cost per unit time. We derive localization results for the c-SFLMR and we prove properties which enable us to solve this problem efficiently. We provide an algorithm to find optimal solutions in discrete spaces, in network and in continuous spaces when polyhedral norms are used as measure of distance. In addition, we give a procedure to obtain good approximated solutions in continuous spaces with any general norm. We show by means of a simple example that drastic reductions in costs can be obtained if anticipated policies are allowed.

Afterwards, we study a new model with two service systems (c-TFLMR): a main service system and a secondary service system, so that, the rejected customers due to the admission policy in the main service system are accepted in the secondary service system if any server is idle. The objective is to find the location of the two service systems as well as the proportions of accepted customers in the main service system when at least a server is idle in order to minimize the expected total cost per unit time. The objective function of the c-TFLMR is sum of two fractional non-linear functions with no good structural properties. These functions are well-known to be extremely hard to optimize. This fact leads us to exploit some properties of the problem in its location face (which are similar to the previous ones obtained for the c-SFLMR) and to develop an "ad hoc" optimization heuristic for the admission policy minimization phase. We propose a branch and bound procedure which combines the localization results with methods of feasible directions to solve the admission policy minimization phase. This procedure can be applied in discrete spaces, in network problems and in continuous problems with polyhedral norms. In continuous spaces with any general norm this procedure can be applied combined with the previous procedure proposed for the c-SFLMR to obtain good approximated solutions.

The dissertation ends with an appendix. It includes several concepts and results of convex analysis and the description of the subgradient methods. These results are needed in the course of Chapter 2 and Chapter 3, respectively.

Chapter 2

Location problems with semigauges

The origin of the results presented in this chapter may be found looking at the problem of locating a new facility with respect to a given set of lines minimizing the sum of the distances (see Robert and Toussaint (1990)[90]). This problem is the dual of a very well-known problem consisting of locating a line minimizing the distances to a given set of points (see Morris & Norback (1980)[79], (1983)[80]; Megiddo & Tamir (1983) [76]; Love, Morris and Wesolowsky (1988)[71]; Schöbel (1998)[94] or Robert and Toussaint (1990)[90]). After a first analysis of this problem one can realize that it is not very hard to solve: it reduces to a linear problem in $\mathbb{R}^2$, while in $\mathbb{R}^n$ it is in general a quadratic convex programming problem (see Section 2.3 for further details).

Nevertheless, although the problem was easy to solve we did realize that there was an interesting common structure worth to be investigated. Distances from a non bounded set to a point can be represented by a gauge generated by this set. In particular, distances from a line and distances from a halfline are given by the gauge with respect to a non-compact convex set. Real-life examples of these situations are the location of a facility within a bounded region with respect to the sewer pipelines which cross the region (assuming that the users want to minimize the connection cost). Notice that in order to represent sets of points at a zero distance from the pipeline (the pipeline itself) we need to consider unbounded unit balls. Otherwise, the points at zero distance shrinks to the origin of the ball. Another example is the location of a facility with respect to regions of demand minimizing the sum of the distances from these regions. Figure 2.1 shows an example together with the distance level curves with respect to the regions.

The common framework on all these "distance" functions is that they proceed from the Minkowski functional but applied to a non-compact, convex set. It is worth noting that all the literature of location analysis based on gauges of bounded sets can be seen as a particular case of this approach. In the following we will call these measures of the distances "semigauges". The term semigauges comes from gauge because these functions are positively homogeneous and convex but in general they are not definite, i.e. the set of points with image into zero is different from the zero vector.

Our aim is to explore the properties of these functions generated by the Minkowski functional of a non-compact, non-symmetric convex set and the implications of their use in Location Analysis. As we will show most of the already known results can be adapted to the new measures of distances. The most important ones are those which geometrically characterize the optimal solution set of the single facility location problem and the geometrical interpretation of the measures given by semigauges as the length of one of the shortest paths using trip directions of its unit ball.

The chapter is organized as follows. In Section 2.1 the concept of semigauge is defined and illustrated by several examples. Afterward, the location problem with semigauges is introduced. Section 2.2 is devoted to some mathematical preliminaries and it includes a proof of the geometrical interpretation of distances measured by semigauges. Section 2.3 deals with the geometrical characterization of the set of optimal solutions of the general location problem with semigauges and it also includes two particularly interesting cases: the Weber and minimax problems. The last section is devoted to the some concluding remarks.

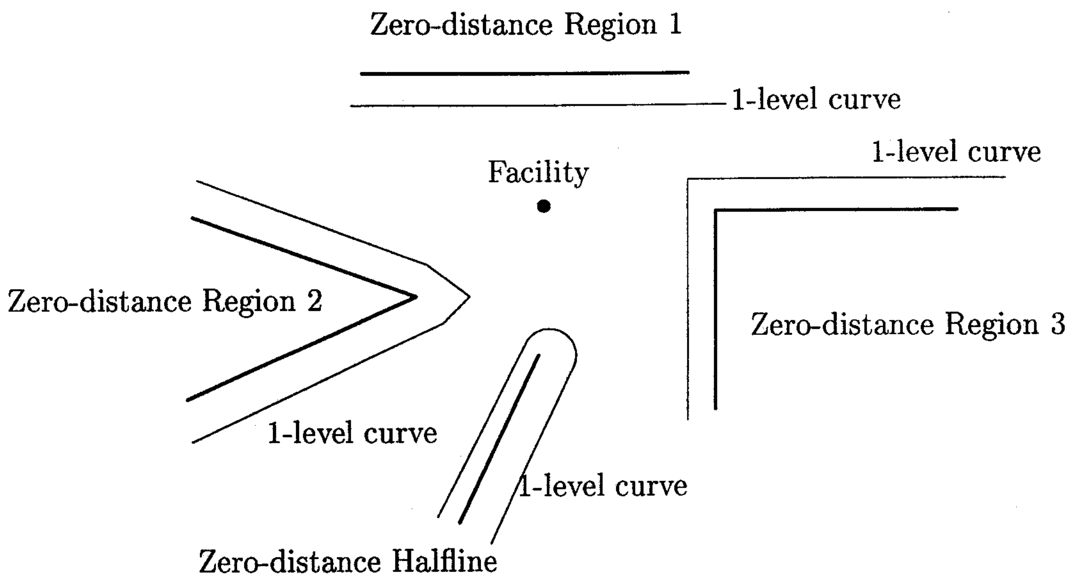


Figure 2.1: Distance level curves with respect to the regions.

2.1 The Model

In this chapter everything takes place in the Euclidean space $\mathbb{R}^n$ where the inner product is denoted by $< \cdot, \cdot >$. In addition, we will consider *semigauges*, so it is necessary to introduce this concept.

We say that φ is a "semigauge" if there exists a closed, convex set C containing the origin such that,

$$\varphi(x) := \inf\{\lambda > 0 \ : \ x \in \lambda C\} \tag{2.1}$$

The set C will be called the "unit ball" associated with φ and by extension we shall call φ the semigauge of C. Remark that for those x not belonging to λC for any $\lambda > 0$ we assume the convention $\varphi(x) = +\infty$. This concept coincides with the definition of gauge usually used in convex analysis (see the book of Hiriart-Urruty & Lemaréchal (1993)[60] (Def. V.1.2.4)). Nevertheless, the term gauge is used in Location Analysis (since the seminal paper of Witzgall (1964)[105]) to denote the functional φ associated to a compact, convex set containing the origin. For this reason, and in order to avoid misunderstanding in the community of locators, we have decided to call the functional φ referred to a closed, convex set (not necessarily bounded) semigauge.

The following lemma, which proof is a straightforward consequence of the definition of semigauge, is included to show properties of the semigauges.

Lemma 2.1 *The semigauge function, $\varphi(\cdot)$, verifies the following properties:*

1. $\varphi(x) \geq 0$ *for all* $x \in \mathbb{R}^n$.

2. $\varphi(\lambda x) = \lambda\varphi(x)$ *for all* $\lambda \in \mathbb{R}^+, \quad x \in \mathbb{R}^n$.

3. $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ *for all* $x, y \in \mathbb{R}^n$.

Remark 2.1 *It is worth noting that $x = 0$ implies $\varphi(x) = 0$, but the converse in general does not hold (see the examples of semigauges in this section). Therefore, definiteness is not verified by semigauges.*

The above remark leads us to consider the concept of *asymptotic cone.*

Definition 2.1 *Let φ be the semigauge of C, the "asymptotic cone" of φ is given by*

$$C_\infty := \{x \in \mathbb{R}^n \ : \ \varphi(x) = 0\} \tag{2.2}$$

For the sake of readability we illustrate the concept of semigauge by several examples.

Examples of semigauges

1. Quadratic semigauges: Take a symmetric positive semi-definite operator Q from $\mathbb{R}^n$ to $\mathbb{R}^n$ and define

$$\varphi(x) := \sqrt{< Qx, x >} \text{ for all } x \in \mathbb{R}^n$$

In $\mathbb{R}^2$ we have,

(a) $\begin{pmatrix} q_{11} & 0 \\ 0 & 0 \end{pmatrix}$ with $q_{11} > 0$. Then $\varphi(x_1, x_2) = \sqrt{q_{11}}|x_1|$

$$C = \{(x_1, x_2) \in R^2 \ : \ \frac{-1}{\sqrt{q_{11}}} \leq x_1 \leq \frac{1}{\sqrt{q_{11}}}\};$$
$$C_\infty = \{(x_1, x_2) \in R^2 \ : \ x_1 = 0\}$$

(b) $\begin{pmatrix} 0 & 0 \\ 0 & q_{22} \end{pmatrix}$ with $q_{22} > 0$. Then $\varphi(x_1, x_2) = \sqrt{q_{22}}|x_2|$

$$C = \{(x_1, x_2) \in R^2 \; : \; \frac{-1}{\sqrt{q_{22}}} \leq x_2 \leq \frac{1}{\sqrt{q_{22}}}\};$$
$$C_\infty = \{(x_1, x_2) \in R^2 \; : \; x_2 = 0\}$$

(c) $\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$ with $q_{11} > 0,\ q_{22} > 0,\ q_{11}q_{22} = q_{12}^2$. Then $\varphi(x_1, x_2) = |\frac{q_{11}}{\sqrt{q_{11}}}x_1 + \frac{q_{12}}{\sqrt{q_{11}}}x_2|$

$$C = \{(x_1, x_2) \in R^2 \; : \; |\frac{q_{11}}{\sqrt{q_{11}}}x_1 + \frac{q_{12}}{\sqrt{q_{11}}}x_2| \leq 1\};$$
$$C_\infty = \{(x_1, x_2) \in R^2 \; : \; x_2 = -\frac{q_{11}}{q_{12}}x_1\}$$

C is a symmetric strip and C_∞ is a line passing through the origin (see Figure 2.2).

2. Let C be an unbounded set defined by intersection of several halfspaces in $\mathbb{R}^n$ and containing the origin, i.e. C can be written as

$$C = \{x \in \mathbb{R}^n \; : < s_i, x > \leq b_i \quad i = 1, \cdots, p\}$$

and $\varphi(\cdot)$ defined by (2.1).
Examples of these sets in $\mathbb{R}^2$ are:

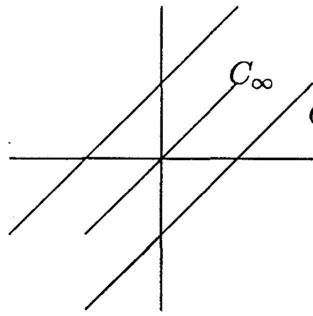
(a) $C = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_1 \leq 1\}; \quad C_\infty = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_1 \leq 0\}$

(b) $C = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_1 \geq -1;\ x_2 \geq -1\};$
$C_\infty = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_1 \geq 0;\ x_2 \geq 0\}$

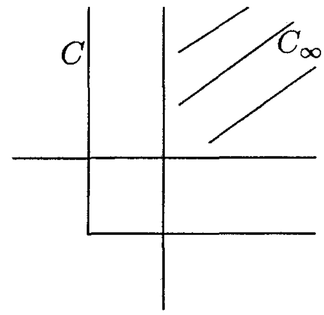
(c) $C = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_1 \geq -1;\ x_2 \geq -1;\ x_2 \geq -x_1 - 1;\ x_2 \leq x_1 + 3\}$
$C_\infty = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_1 \geq 0;\ 0 \leq x_2 \leq x_1\}$

3. Let C be the epigraph of a convex function $f : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, which contains the origin.

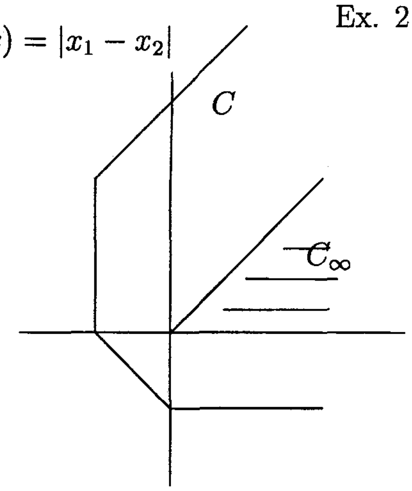
$$C = \{(x, \alpha) \in \mathbb{R}^{n-1} \times \mathbb{R} \; : \; f(x) \leq \alpha\} \text{ with } f \text{ convex and } f(0) \leq 0$$



Ex. 1.(c), $\varphi(x) = |x_1 - x_2|$



Ex. 2.(b)



Ex. 2.(c)

Figure 2.2: Examples of semigauges in $I\!R^2$

and $\varphi(\cdot)$ defined by (2.1).

An example of these sets in $I\!R^2$ is:

(a) $C = \{(x_1, x_2) \in I\!R^2 \,:\, x_2 \geq x_1^2 - 1\}$; $C_\infty = \{(0, x_2) \in I\!R^2 \,:\, x_2 \geq 0\}$

Given a semigauge φ we consider $\varphi(x - y)$ as a measure of the distance from x to y.

Lemma 2.2 *The "distance" defined as $d(y, x) = \varphi(x - y)$ verifies the following properties:*

1. $\varphi(x - y) \geq 0$ *for all* $x, y \in I\!R^n$.

2. $\varphi(x - y) = 0$ *iff* $x - y \in C_\infty$.

3. $\varphi(x-y) \leq \varphi(x-z) + \varphi(z-y)$ *for all* $x, y, z \in I\!R^n$.

The proof is a straightforward consequence of the definition of the "distance".

The following lemma, which proof is a straightforward consequence of their definitions, shows that the gauges and the norms are particular cases of the semigauges.

Lemma 2.3 *If the interior of* C *contains the origin and* C *is bounded,* φ *is a gauge,* $C_\infty = \{0\}$ *and then* $\varphi(x-y) = 0$ *iff* $x = y$. *In addition, if* C *is symmetric,* φ *is a norm,* $\varphi(\lambda x) = |\lambda|\varphi(x)$ *for all* $\lambda \in I\!R$ *and then we have the additional property* $\varphi(x-y) = \varphi(y-x)$.

Problem formulation

Throughout this chapter A is a finite subset of $I\!R^n$ which represents the set of existing facilities. Each facility $a \in A$ has associated a semigauge φ_a whose unit ball C_a is a convex, closed set which contains the origin. The problem deals with the location of one new facility in $I\!R^n$ minimizing an objective function F which depends on the semigauges with respect to the existing facilities in A.

Mathematically the problem is formulated as

$$(P_w^\gamma(A)) \qquad \min_{x \in I\!R^n} F(x) = \gamma(w_1\varphi_{a_1}(x-a_1), \cdots, w_m\varphi_{a_m}(x-a_m))$$

where $A = \{a_1, \cdots, a_m\}$ is the set of existing facilities, $\{w_1, \cdots, w_m\}$ are positive weights, φ_{a_i} is the semigauge associated to a_i and γ is a monotone semigauge on $I\!R^m$, that it to say, $\gamma(u) \leq \gamma(v)$ for every u and v in $I\!R^m$ satisfying $u_i \leq v_i$ for each $i = 1, \cdots, m$ (see Johnson and Nylen (1991)[63] for further details on monotone norms). We denote by C^γ the unit ball of γ.

The following remark shows that interesting, well-know problems in Location Theory are particular cases of this general location problem.

Remark 2.2 *For specific choices of* γ *we obtain a generalization (for semigauges) of some well-known location problems. For example, for* $\gamma(x) = \sum_{i=1}^{m} |x_i|$ *the problem becomes a generalization of the usual Weber problem, for* $\gamma(x) = \max_i |x_i|$ *we get a generalization of the center problem and for* $\gamma(x) = \lambda\sum_{i=1}^{m}|x_i| + (1-\lambda)\max_i |x_i|$ *with* $0 < \lambda < 1$ *we have a generalization of the cent-dian problem.*

2.2 Mathematical preliminaries

In this section, we relate the semigauges with some concepts of convex analysis and we prove a geometrical interpretation of the distances measured by semigauges.

First, we relate the semigauges with some concepts of convex analysis. Further details can be found in the Appendix A at the end of this dissertation and in the books of Hiriart-Urruty & Lemaréchal (1993)[60], or Rockafellar (1970)[91]. In particular, see Appendix A for the definitions of sublinear function, conical hull of a set C, ($cone(C)$), conjugate function of a convex function f, (f^*), polar cone of a nonempty convex cone K, (K^o), polar set of a set C, (C^*), support function of a set, normal cone to a set C at a point x ($N_C(x)$) and for the definition of the subdifferential of a convex function f, (∂f). All these concepts are used in the following results.

Lemma 2.4 *Let φ be the semigauge of C, and C_∞ the asymptotic cone of φ. C_∞ verifies the following properties:*

1. *C_∞ is a closed convex cone.*

2. *If C is symmetric and it is not bounded then C_∞ is a linear subspace of $I\!R^n$.*

Proof.

1. It is easy to check from the definition of C_∞. Further details can be found in the book of Hiriart-Urruty & Lemaréchal (1993)[60].

2. Take $x, y \in C_\infty$ then $\varphi(\lambda x + \mu y) \leq |\lambda|\varphi(x) + |\mu|\varphi(y) = 0$. This implies that $\lambda x + \mu y \in C_\infty \quad \forall \lambda, \mu \in I\!R$. □

Lemma 2.5

- *If φ is a semigauge with unit ball C being a closed, convex set containing the origin, then φ is a closed, sublinear function.*

- *φ is finite if and only if 0 lies in the interior of C.*

- *If zero lies on the boundary of C the set of points where $\varphi(x) = +\infty$ is the complementary set of the conical hull of C, $\mathbb{R}^n \backslash cone(C)$. Therefore, if zero lies on the boundary of C the domain of φ, $dom(\varphi)$, verifies $dom(\varphi) = cone(C)$; otherwise $dom(\varphi) = \mathbb{R}^n$.*

See the book of Hiriart-Urruty & Lemaréchal (1993)[60] for a proof.

Lemma 2.6 *Let C^o_∞ be the polar cone of C_∞ defined in (A.2). The domain of the conjugate function, φ^*, of φ verifies $dom(\varphi^*) \subseteq C^o_\infty$.*

Proof. Remark that if φ is a semigauge of unit ball C and $s \notin C^o_\infty$, that implies $\varphi^*(s) = \infty$. □

It is well-know in Location Theory that the evaluation of the gauge γ_C of a point can be given by means of the support function of the polar set C^*. The same result still holds for general closed convex sets (non necessarily bounded) as can be seen in Theorem 14.5 in Rockafellar's book (1970)[91]. Therefore, this approach gives an alternative way to evaluate semigauges as defined by (2.1).

In the following paragraph we provide examples of polar sets, C^* for the semigauges introduced in Section 2.1.

Examples of polar set

1. For the example 1.(c) of semigauges, we have $\varphi(x_1, x_2) = |\frac{q_{11}}{\sqrt{q_{11}}}x_1 + \frac{q_{12}}{\sqrt{q_{11}}}x_2|$, and the polar set C^* is the segment (see Figure 2.3):

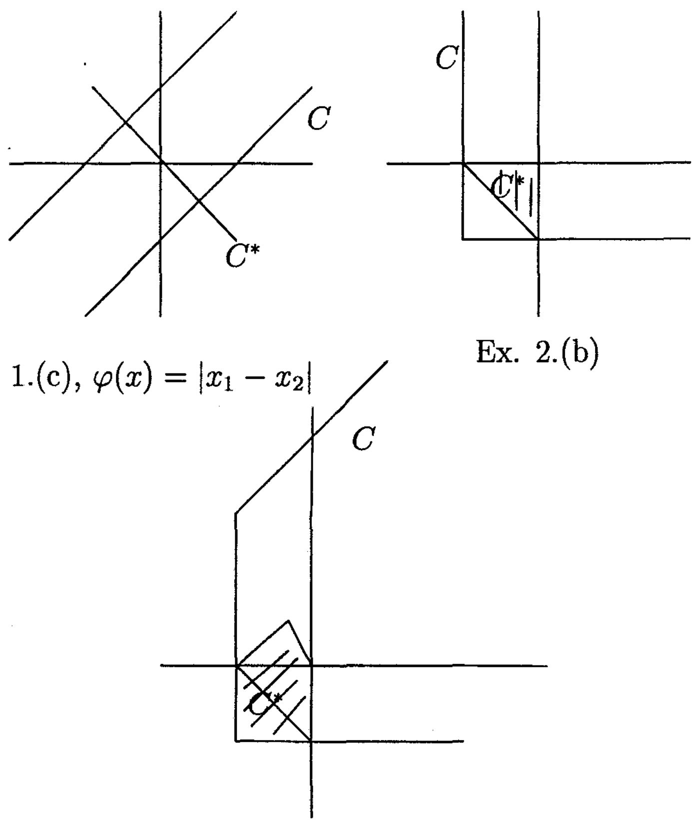
$$C^* = \{(s_1, s_1\frac{q_{12}}{q_{11}}) \ : \ -\frac{\sqrt{q_{11}}}{q_{11}} \leq s_1 \leq \frac{\sqrt{q_{11}}}{q_{11}}\}.$$

2. For the example 2.(b) we obtain,

$$C^* = \{(s_1, s_2) \ : \ s_1 \leq 0; -1 - s_1 \leq s_2 \leq 0\},$$

and for the example 2.(c) we have (see Figure 2.3),

$$C^* = \{(s_1, s_2) \ : \ -1 \leq s_1 \leq 0; s_2 \geq -1; s_2 \leq -s_1; s_2 \leq \frac{s_1}{2} + \frac{1}{2}\}.$$



Ex. 1.(c), $\varphi(x) = |x_1 - x_2|$

Ex. 2.(b)

Ex. 2.(c)

Figure 2.3: Examples of polar sets of "unit balls" in $I\!R^2$

Remark 2.3 *Since φ is the support function of a closed convex C^*, the conjugate φ^* of φ is the indicator function $I_{C^*}(s)$ of C^*.*

$$\varphi^*(s) = I_{C^*}(s) = \begin{cases} 0 & \text{if } s \in C^* \\ +\infty & \text{otherwise} \end{cases} \tag{2.3}$$

Therefore, by (2.3) we have,

$$\partial\varphi^*(s) = \begin{cases} N_{C^*}(s) & \text{if } s \in C^* \\ \emptyset & \text{if } s \notin C^* \end{cases}$$

where $\partial\varphi(s)$ denote the subdifferential of the function φ and $N_C(x)$ denote the normal cone to the set C at the point x. Then, since C^* is closed if $x \in dom(\varphi)$,

$$\begin{aligned} \partial\varphi(x) &= \{s \in I\!R^n \,:\, x \in N_{C^*}(s),\ s \in C^*\} \\ &= \{s \in C^* \,:\, < x, s >= \varphi(x)\} \end{aligned}$$

If $x = 0$, then $\partial\varphi(0) = C^*$ and if $x \in C_\infty$ then $0 \in \partial\varphi(x)$. In the particular case of quadratic semigauges, $\partial\varphi(x) = C^*$ for all $x \in C_\infty$ because C^* is orthogonal to C_∞.

In what follows, we show that in order to evaluate the semigauge of a closed, convex set C one can use a geometrical interpretation of the length of certain shortest paths.

We denote by *ext* A the set of extreme points of the set A, by A^{ext} the set of extreme directions and by $conv(A)$ the convex hull of A (see Appendix A for the definition of extreme points, extreme directions and the definition of convex hull of a set).

Definition 2.2 *A convex set C has lineality zero if it contains no lines.*

Theorem 2.1 *Let C be a closed convex set with non-zero linearity. Then,*

$$C = conv(extC_0) + cone((C_0)_\infty^{ext}) + L$$

with $C = C_0 + L$ where L is the linearity subspace of C and $C_0 = C \cap L^\perp$ is a convex set which has linearity zero, with $L^\perp$ being the orthogonal subspace to L.

Proof. By Theorem 18.5 of Rockafellar's book (1970)[91] any closed convex set C_0 with linearity zero can be expressed as $A = conv(extA) + cone(A_\infty^{ext})$. On the other hand, if C is a non-empty convex set which contains a space L with non-zero linearity , one can obviously write C as the direct sum $C = C + (C \cap L^\perp)$, where $L^\perp$ is the orthogonal subspace to L, being zero the linearity of the set $C \cap L^\perp$ in this expression (see Rockafellar's book (1970)[91]). Combining both results the thesis of the theorem follows. □

For the purpose of giving a interpretation of the evaluation of any semigauge as the shortest path between points using only displacement in the directions given by the extreme points of C_0 we can use a simplified representation of a convex set.

$$C = conv(extC_0) + cone(C_\infty) \tag{2.4}$$

C_0 being a closed convex subset of C such that $C = C_0 + L$, where L is the linearity subspace of C and $C_0 = C \cap L^\perp$ is a convex set which has linearity zero, with $L^\perp$ being the orthogonal subspace to L. Notice that in that representation, we have replaced the extreme directions of C_∞ by any direction of this cone. With this representation the following result can be proved.

Theorem 2.2 *Let C be a closed, convex set containing the origin, the semigauge of C is given by*

$$\varphi(x) = \inf\{\sum_{i\in I}\lambda_i \ : \ x = \sum_{i\in I}\lambda_i x_i + \sum_{j\in J}\mu_j d_j; \quad \lambda_i, \mu_j \geq 0, \ \forall\, i \in I, j \in J\}$$

with $C = C_0 + L$, C_0 and L as before, $x_i \in ext\, C_0\ \forall\, i \in I$ and $d_j \in C_\infty\ \forall\, j \in J$ and $|I| + |J| \leq n+1$.

Proof.

$$\varphi(x) = \inf\{\alpha > 0 \ : \ x \in \alpha C\}$$

Therefore, using the representation of C given in (2.4) we obtain:

$$\varphi(x) = \inf\{\alpha > 0 \ : \ \frac{x}{\alpha} = \sum_{i\in I}\lambda_i x_i + \sum_{j\in J}\mu_j d_j; \quad \sum_{i\in I}\lambda_i = 1; \ \lambda_i, \mu_j \geq 0, \ \forall\, i \in I, j \in J\}$$

with $x_i \in ext\, C_0\ \forall\, i \in I$, $d_j \in C_\infty\ \forall\, j \in J$ and by Caratheodory's Theorem $|I| + |J| \leq n+1$. Thus, taking $\lambda'_i = \alpha\lambda_i$ and $\mu'_j = \alpha\mu_j$ we have:

$$\begin{aligned} \varphi(x) &= \inf\{\alpha > 0\ :\ x = \sum_{i\in I}\lambda'_i x_i + \sum_{j\in J}\mu'_j d_j;\ \ \sum_{i\in I}\lambda'_i = \alpha;\ \lambda'_i, \mu'_j \geq 0,\ \forall\, i \in I, j \in J\} \\ &= \inf\{\sum_{i\in I}\lambda'_i\ :\ x = \sum_{i\in I}\lambda'_i x_i + \sum_{j\in J}\mu'_j d_j;\ \ \lambda'_i, \mu'_j \geq 0,\ \forall\, i \in I, j \in J\} \end{aligned}$$

□

This result is an extension of the Theorem 1 of Ward & Wendell (1985)[101] stating that any block norm of a point x is the length of one of the shortest path from the origin to x in the trip directions of the unit ball. In our theorem, we show that the same interpretation can be applied to a particular expansion of the point x in terms of extreme points of a specific convex subset of the unit set and directions of C_∞.

2.3 Geometrical description of optimal sets

In this chapter we will study essentially the set of points x at which the function F attains its minimum. As it is usual, the set of minimizers of $P^\gamma_w(A)$ will be denoted by $M^\gamma_w(A)$.

Our aim is to characterize geometrically the set of optimal solutions $M^\gamma_w(A)$ of the problem $(P^\gamma_w(A))$ introduced in Section 2.1. In doing that, we recall the concept of elementary convex set (see Durier & Michelot (1985)[33]) . For all $a \in A$ we denote by C_a the unit ball of φ_a, by C^*_a the polar set of C_a and by C^a_∞ the asymptotic cone of φ_a. In the same way we denote by C^γ the unit ball of γ and by C^γ_∞ the asymptotic cone of γ.

Definition 2.3 *Let $\pi = (p_a)_{a\in A}$ be a family of elements of $\mathbb{R}^n$ such that $p_a \in C^*_a$ for each $a \in A$ and let*

$$C_\pi = \bigcap_{a\in A}(a + N_a(p_a))$$

being $N_a(p_a)$ the normal cone to C^*_a at p_a. A nonempty convex set C is said to be an elementary convex set if there exists a family π such that $C_\pi = C$.

As an illustration of this definition we will see some examples for semigauges. In doing that we will explain how to construct the normal cone to a convex set C^* at p.

$$x \in N_{C^*}(p) \quad \text{iff} \quad < x, p >= \varphi(x) \quad \text{with } p \in C^*$$

If $p = 0$ that implies, $N_{C^*}(0) = C_\infty$.
If p is in the boundary of C^*, $N_{C^*}(p)$ is the convex cone with vertex at the origin which contains the exposed face of C defined by $\{x \in C \ : < p, x >= 1\}$.

Examples of elementary convex sets

1. For the example 1.(c) of semigauge we have

   If p_1 is the extreme point of C^* given by $p_1 = (-\frac{q_{11}}{\sqrt{q_{11}}}, -\frac{q_{12}}{\sqrt{q_{11}}})$ then,
   $x \in N_{C^*}(p_1)$ iff $x_1 < -\frac{q_{12}}{q_{11}} x_2$
   In this case, $N_{C^*}(p_1)$ is one of the halfspaces defined by C_∞.

   If p_2 is the extreme point of C^* given by $p_2 = (\frac{q_{11}}{\sqrt{q_{11}}}, \frac{q_{12}}{\sqrt{q_{11}}})$ then,
   $x \in N_{C^*}(p_2)$ iff $x_1 > -\frac{q_{12}}{q_{11}} x_2$
   In this case, $N_{C^*}(p_2)$ is the other halfspace defined by C_∞.

   If p_3 is not an extreme point of C^*,
   $x \in N_{C^*}(p_3)$ iff $x_1 = -\frac{q_{12}}{q_{11}} x_2$
   In this case, $N_{C^*}(p_3)$ is the asymptotic cone C_∞.

   If $\varphi_a(x) = \varphi(x)$ for all $a \in A$ the elementary convex sets are parallel strips or lines passing through the existing facilities (see Figure 2.4).
   If $\varphi_a(x) \neq \varphi_b(x)$ with $a \neq b$ the elementary convex sets are polygons (eventually unbounded), their sides and their vertices (see Figure 2.4).

2. Obtaining the elementary convex sets for the remainder examples only consists of computing the normal cone to C^* at any point of its boundary (see Figure 2.4). In these cases the elementary convex sets are polygons, their sides and some of their vertices. It is worth noting that the existing facilities may not

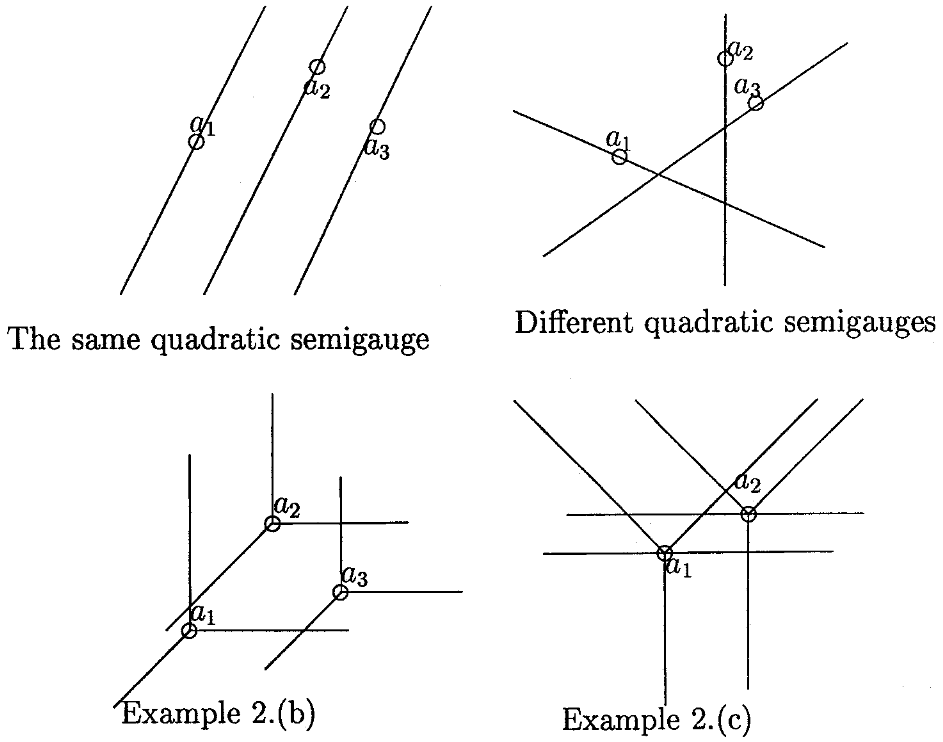


Figure 2.4: Examples of elementary convex sets

be elementary convex sets. In fact, these points are elementary convex sets only if they are intersections of normal cones corresponding to different C_a^*.

The general problem

The general problem $(P_w^\gamma(A))$ introduced in Section 2.1, with γ being a monotone semigauge has the novelty with respect to previous formulations in Location Theory that its optimal value may be 0 or ∞. First of all, we characterize these cases and then we proceed to the general case.

Let $A_1, A_2, \cdots, A_m$ be sets in $\mathbb{R}^n$. We denote by $\bigotimes_{i=1}^{m} A_i$ the cartesian product of the A_i's. This is to say

$$\bigotimes_{i=1}^{m} A_i = A_1 \times A_2 \times \cdots \times A_m \subset \mathbb{R}^{n \times m}.$$

and we denote by $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Lemma 2.7

1. *If* $\bigcup_{x\in \mathbb{R}^n}\left(\bigotimes_{i=1}^{m} w_{a_i}\varphi_{a_i}(x-a_i)\right)\bigcap C_\infty^\gamma \neq \emptyset$, *then* $M_w(A) = \{x\in \mathbb{R}^n : \bigotimes_{i=1}^m w_{a_i}\varphi_{a_i}(x-a_i)\in C_\infty^\gamma\}$ *and the optimal objective value for* $F(\cdot)$ *is* $F(x^*)=0$ *with* $x^*\in M_w(A)$.

2. *If* $\bigcup_{x\in \mathbb{R}^n}\left(\bigotimes_{i=1}^{m} w_{a_i}\varphi_{a_i}(x-a_i)\right) \subset \mathbb{R}^m\backslash cone(C^\gamma)$, *then* $M_w(A) = \mathbb{R}^n$ *and the optimal objective value for* $F(\cdot)$ *is* ∞.

In the following, we will assume that $\bigcup_{x\in \mathbb{R}^n}\left(\bigotimes_{i=1}^{m} w_{a_i}\varphi_{a_i}(x-a_i)\right)\bigcap C_\infty^\gamma = \emptyset$ and $\bigcup_{x\in \mathbb{R}^n}\left(\bigotimes_{i=1}^{m} w_{a_i}\varphi_{a_i}(x-a_i)\right) \not\subset \mathbb{R}^m\backslash cone(C^\gamma)$.

Let Q and $\bar{Q}$ be sets in $\mathbb{R}^m$ defined by $Q=\{u\in\mathbb{R}^m : u_i>0 \quad \forall i=1,\cdots,m\}$ and $\bar{Q}=\{u\in\mathbb{R}^m : u_i\geq 0 \quad \forall i=1,\cdots,m\}$. We assume γ to be a monotone semigauge on Q, i.e., $\gamma(u)\leq\gamma(v)$ for every u and v in $\mathbb{R}^m$ satisfying $0\leq u_i\leq v_i$ for each $i=1,\cdots,m$ (see Johnson and Nylen (1991)[63] for further details on monotone norms).

Proposition 2.1 *Suppose* γ *is a monotone semigauge on* Q. *Then the function*

$$\gamma^+(u)=\gamma(\max(u_1,0),\cdots,\max(u_m,0))$$

is a sublinear, monotone function in $\mathbb{R}^m$. *Moreover,* $\partial\gamma^+(u)\subset\bar{Q} \quad \forall u\in\bar{Q}$.

Remark 2.4 *If* γ *is a norm, it is known that* $\partial\gamma^+(u)=\partial\gamma(u)\subset\bar{Q}$ *if* $u\in Q$ *and that* $\partial\gamma^+(u)=\partial\gamma(u)\cap\bar{Q}$ *if* $u\in\bar{Q}\backslash Q$ *(see Durier (1995)[34]).*

Since γ^+ *is a sublinear function, then* γ^+ *is the support function of the polar set of its "unit ball". Let* $B=\{u\in\mathbb{R}^m : \gamma^+(u)\leq 1\}$ *be the unit ball of* γ^+, *and let* B^* *be the polar set of* B. *The subdifferential set of the function* γ^+ *at a point* $u\in\bar{Q}, \quad u\neq 0$ *is:*

$$\partial\gamma^+(u)=\{\lambda=(\lambda_1,\cdots,\lambda_m) : \lambda_i\geq 0 \quad \forall i,\ \lambda\in B^*,\ <\lambda,u>=\gamma^+(u)\}.$$

In order to give a geometrical description of the set $M^{\gamma}_{w}(A)$, we will use the condition $x \in M^{\gamma}_{w}(A)$ iff $0 \in \partial F(x)$. These results are extensions of previously known results (see Durier (1995)[34]). Our generalization extends the use of norms to semigauges. The proofs run similarly in both cases.

We need to start with a characterization of the subdifferential set $\partial F(x)$, which makes use of the Theorem VI.4.3.1 in Hiriart-Urruty & Lemaréchal book (1993)[60] (see Appendix A). Notice that this result requires monotonicity of the outer component of the composition between the functions. This is the reason to introduce γ^+ because γ is not monotone in the whole $I\!R^m$.

Lemma 2.8 *An element* $x^* \in I\!R^n$ *belongs to* $\partial F(x)$ *if and only if there exist* $p_{a_1} \in C^*_{a_1}, \cdots, p_{a_m} \in C^*_{a_m}$, $\lambda = (\lambda_1, \cdots, \lambda_m) \in B^*$ *with* $\lambda_i \geq 0 \quad \forall i = 1, \cdots, m$ *such that,*

$x \in a_i + N_{a_i}(p_{a_i}), \quad \forall\, i = 1, \cdots, m, \quad \sum_{i=1}^{m} \lambda_i w_{a_i} \varphi_{a_i}(x - a_i) = F(x)$ *and*

$x^* = \sum_{i=1}^{m} \lambda_i w_{a_i} p_{a_i}.$

Proof. Let $\Phi(x) = (w_{a_1}\varphi_{a_1}(x - a_1), \cdots, w_{a_m}\varphi_{a_m}(x - a_m))$, then $F(x) = (\gamma^+ \circ \Phi)(x)$, F being a convex function.

$$\partial F(x) = \left\{ \sum_{i=1}^{m} \lambda_i q_{a_i} \;:\; (\lambda_1, \cdots, \lambda_m) \in \partial \gamma^+(\Phi(x)),\; q_{a_i} \in \partial(w_{a_i}\varphi_{a_i}(x - a_i))\ \forall i \right\}$$

$\partial \gamma^+(\Phi(x)) = \{\lambda = (\lambda_1, \cdots, \lambda_m) \;:\; \lambda \in B^*,\ \lambda_i \geq 0,\ <\lambda, \Phi(x)> = \gamma^+(\Phi(x)) = F(x)\}$;

$q_{a_i} = w_{a_i} p_{a_i}$ with $p_{a_i} \in \partial \varphi_{a_i}(x - a_i)$; $\partial \varphi_{a_i}(x - a_i) = \{p_{a_i} \in C^*_{a_i} \;:\; x \in a_i + N_{a_i}(p_{a_i})\}$.

□

Let us now introduce some notation. For $J \subseteq [1, m]$, $J \neq \emptyset$, $p = (p_{a_i})_{i \in J}$ with $p_{a_i} \in C^*_{a_i}$ we let,

$$C_J(p) = \{x \in I\!R^n \;:\; \forall i \in J, < p_{a_i}, x - a_i > = \varphi_{a_i}(x - a_i)\}$$

or equivalently,

$$C_J(p) = \bigcap_{i \in J} (a_i + N_{a_i}(p_{a_i})) \quad \text{elementary convex set},$$

and for $\lambda = (\lambda_1, \cdots, \lambda_m) \in B^*$ we let,

$$D_J(\lambda) = \{x \in \mathbb{R}^n \ : \ F(x) = -\sum_{i\in J} \lambda_i w_{a_i} < p_{a_i}, a_i >\}.$$

Lemma 2.9 *A point $x \in \mathbb{R}^n$ belongs to $M_w^\gamma(A)$ iff exist $J \subseteq [1,m]$, $J \neq \emptyset$ $(p_{a_i})_{i\in J}$ with $p_{a_i} \in C_{a_i}^*$ and $\lambda = (\lambda_1, \cdots, \lambda_m) \in B^*$ with $\lambda_i > 0 \quad \forall i \in J, \quad \lambda_i = 0 \quad \forall i \notin J$ satisfying $\sum_{i\in J} \lambda_i w_{a_i} p_{a_i} = 0$ such that,*

$$x \in C_J(p) \cap D_J(\lambda)$$

Proof. Let $x \in \mathbb{R}^n$, $x \in M_w^\gamma(A)$ iff $0 \in \partial F(x)$, therefore applying the lemma above, there exist

$p_{a_1} \in C_{a_1}^*, \cdots, p_{a_m} \in C_{a_m}^*, \quad \lambda = (\lambda_1, \cdots, \lambda_m) \in B^*$ with $\lambda_i \geq 0 \quad \forall i = 1, \cdots, m$ such that,

$$x \in a_i + N_{a_i}(p_{a_i}), \quad \forall\, i = 1, \cdots, m, \tag{2.5}$$

$$\sum_{i=1}^m \lambda_i w_{a_i} \varphi_{a_i}(x - a_i) = F(x) \tag{2.6}$$

$$0 = \sum_{i=1}^m \lambda_i w_{a_i} p_{a_i}. \tag{2.7}$$

(2.5) implies that $< p_{a_i}, x - a_i >= \varphi_{a_i}(x - a_i)\}$, then by (2.6) we have that, $F(x) = \sum_{i=1}^m \lambda_i w_{a_i} < p_{a_i}, x > - \sum_{i=1}^m \lambda_i w_{a_i} < p_{a_i}, a_i >$ and by (2.7) we obtain

$F(x) = -\sum_{i\in J} \lambda_i w_{a_i} < p_{a_i}, a_i >$. Let $J = \{\lambda_i \ : \ \lambda_i > 0\}$, then $x \in C_J(p) \cap D_J(\lambda)$.

Remark that $J \neq \emptyset$ because if $\lambda_i = 0\ \forall i$, then $F(x) = 0$ and we have assumed that $\bigcup_{x\in\mathbb{R}^n} \left(\bigotimes_{i=1}^m w_{a_i} \varphi_{a_i}(x - a_i) \right) \bigcap C_\infty^\gamma = \emptyset$ and then $F(x) \neq 0$ for all x. □

The next theorem deals with the simultaneous characterization of all the points of the optimal set, $M_w^\gamma(A)$. The proof runs parallel to Theorem 4.3 in Durier (1995)[34].

Theorem 2.3

1. *If $M_w^\gamma(A)$ is nonempty, then there exist $J \subseteq [1,m]$, $J \neq \emptyset$, $(p_{a_i})_{i\in J}$ with $p_{a_i} \in C_{a_i}^*$ and $\lambda = (\lambda_1, \cdots, \lambda_m) \in B^*$ with $\lambda_i > 0 \quad \forall i \in J, \quad \lambda_i = 0 \quad \forall i \notin J$ satisfying $\sum_{i\in J} \lambda_i w_{a_i} p_{a_i} = 0$ such that,*

$$M_w^\gamma(A) = C_J(p) \cap D_J(\lambda)$$

2. *If there exist $J \subseteq [1,m]$, $J \neq \emptyset$, $(p_{a_i})_{i\in J}$ with $p_{a_i} \in C^*_{a_i}$ and $\lambda = (\lambda_1, \cdots, \lambda_m) \in B^*$ with $\lambda_i > 0 \quad \forall i \in J, \quad \lambda_i = 0 \quad \forall i \notin J$ satisfying $\sum_{i\in J} \lambda_i w_{a_i} p_{a_i} = 0$ such that,*

$$C_J(p) \cap D_J(\lambda) \neq \emptyset,$$

then

$$M^\gamma_w(A) = C_J(p) \cap D_J(\lambda).$$

Proof. Firstly, according to lemma above if (J, p, λ) is a triplet satisfying the conditions then, $M^\gamma_w(A) \supseteq C_J(p) \cap D_J(\lambda)$.

On the other hand, if $x \in M^\gamma_w(A)$ applying the lemma above there exists a triplet (J, p, λ) associated to x. Let $\bar{x} \in M^\gamma_w(A)$ with $x \neq \bar{x}$, then we have,

$F^* = F(x) = -\sum_{i\in J} \lambda_i w_{a_i} < p_{a_i}, a_i >=^{(1)} \sum_{i\in J} \lambda_i w_{a_i} < p_{a_i}, \bar{x} - a_i > \leq^{(2)}$

$\sum_{i\in J} \lambda_i w_{a_i} \varphi_{a_i}(\bar{x} - a_i) \leq \max_{\mu=(\mu_1,\cdots,\mu_m)\in B^*} \sum_{i=1}^{m} \mu_i w_{a_i} \varphi_{a_i}(\bar{x} - a_i) =$

$\max_{\mu\in B^*} < \mu, (w_{a_1}\varphi_{a_1}(\bar{x} - a_1), \cdots, w_{a_m}\varphi_{a_m}(\bar{x} - a_m)) >=^{(3)}$

$\gamma(w_{a_1}\varphi_{a_1}(\bar{x} - a_1), \cdots, w_{a_m}\varphi_{a_m}(\bar{x} - a_m)) = F(\bar{x})$

(1) is obtained because $\sum_{i\in J} \lambda_i w_{a_i} p_{a_i} = 0$

(2)is due to $p_{a_i} \in C^*_{a_i}$ and to φ_{a_i} is the support function of $C^*_{a_i}$ and

(3) is verified because γ is the support function of B^*.

If $\bar{x} \in M^\gamma_w(A)$ then inequalities become equalities and then,

$\forall i \in J \quad < p_{a_i}, \bar{x} - a_i >= \varphi_{a_i}(\bar{x} - a_i)$, i.e., $\bar{x} \in \cap_{i\in J}(a_i + N_{a_i}(p_{a_i}))$ and

$F(\bar{x}) = -\sum_{i\in J} \lambda_i w_{a_i} < p_{a_i}, a_i >$, i.e., $\bar{x} \in C_J(p) \cap D_J(\lambda)$ and this implies

$M^\gamma_w(A) \subseteq C_J(p) \cap D_J(\lambda)$. □

Two remarkable applications (consequences) of Theorem 2.3 leads to two characterization of optimal solutions sets of well-known problems in Location Theory.

The Weber problem with semigauges

The mathematical formulation is,

$$\min_{x \in I\!R^n} F(x) = \sum_{i=1}^{m} w_{a_i} \varphi_{a_i}(x - a_i)$$

being φ_{a_i} the semigauge of a closed, convex set C_{a_i} the interior of which contains the origin.

This problem was already solved by Durier & Michelot (1985)[33] for the case that the functions φ_{a_i} are gauges. Nevertheless the results obtained by Durier & Michelot are still valid for the general case of semigauge functions except for the fact that the optimal solution set $M_w(A)$ need not be a bounded set. In addition, in the general formulation it may happen that the optimal value is zero.

Lemma 2.10

1. *If* $\bigcap_{i=1}^{m}(a_i + C_{\infty}^{a_i}) \neq \emptyset$ *then* $M_w(A) = \bigcap_{i=1}^{m}(a_i + C_{\infty}^{a_i})$ *and the optimal objective value for* $F(\cdot)$ *is* $F(x^*) = 0$ *with* $x^* \in M_w(A)$.

2. *If* $\bigcap_{i=1}^{m}(a_i + cone(C_{a_i})) = \emptyset$ *then* $M_w(A) = I\!R^n$ *and the optimal objective value for* $F(\cdot)$ *is* $+\infty$.

Proof.

1. If $x^* \in \bigcap_{i=1}^{m}(a_i + C_{\infty}^{a_i}) \Longrightarrow x^* - a_i \in C_{\infty}^{a_i} \quad \forall i = 1, \cdots, m$. Then, $\varphi_{a_i}(x^* - a_i) = 0$ $\forall i = 1, \cdots, m$ and therefore $F(x^*) = 0$.

2. If $\bigcap_{i=1}^{m}(a_i + cone(C_{a_i})) = \emptyset$, $\forall x \in I\!R^n \ \exists a_{i_0}$ such that $x - a_{i_0} \notin cone(C_{a_{i_0}})$, then $\varphi_{a_{i_0}}(x - a_{i_0}) = +\infty$ and therefore $F(x) = +\infty \quad \forall x \in I\!R^n$.

□

Theorem 2.4 *If* $\bigcap_{i=1}^{m}(a_i + C_{\infty}^{a_i}) = \emptyset$ *and* $\bigcap_{i=1}^{m}(a_i + cone(C_{a_i})) \neq \emptyset$ *then* $M_w(A)$ *is a closed elementary convex set* C_π *given by a family* $\pi = (p_{a_i})_{a_i \in A}$ *such that* $\sum_{i=1}^{m} w_{a_i} p_{a_i} = 0$.

Conversely, let C_π *be an elementary convex set associated with a family* $\pi = (p_{a_i})_{a_i \in A}$ *such that* $\sum_{i=1}^{m} w_{a_i} p_{a_i} = 0$ *then,* $C_\pi = M_w(A)$.

See Durier & Michelot (1985)[33] for a proof. Notice that $M_w(A)$ need not be compact because it may exist $x \in C_{\infty}^{a_i}$ with $||x|| \longrightarrow \infty$.

The minimax problem

As a consequence of the main characterization theorem, we obtain a geometrical description of the set of optimal solutions for the particular case of the minimax problem. Mathematically the problem is formulated as

$$(P_w^{l_\infty}(A)) \qquad \min_{x \in I\!R^n} F(x) = \max_{i=1,\cdots,m} w_{a_i} \varphi_{a_i}(x - a_i)$$

In this particular case the function $\gamma = l_\infty$ is a norm and then we apply Theorem 2.3 when γ is a norm.

We introduce the following notation. For $J \subseteq [1, m]$, $J \neq \emptyset$, and $\tau > 0$ we let

$$E_J(\tau) = \{x \in I\!R^n \; : \; \forall i \in J, \; w_{a_i}\varphi_{a_i}(x - a_i) = \tau \text{ and } \forall i \notin J, \; w_{a_i}\varphi_{a_i}(x - a_i) \leq \tau\}$$

Corollary 2.1

1. *(a) If* $\bigcap_{i=1}^{m}(a_i + C_{\infty}^{a_i}) \neq \emptyset$ *then* $M_w^{l_\infty}(A) = \bigcap_{i=1}^{m}(a_i + C_{\infty}^{a_i})$ *and the optimal objective value for* $F(\cdot)$ *is* $F(x^*) = 0$ *with* $x^* \in M_w^{l_\infty}(A)$.

   (b) If $\bigcap_{i=1}^{m}(a_i + cone(C_{a_i})) = \emptyset$ *then* $M_w(A) = I\!R^n$ *and the optimal objective value for* $F(\cdot)$ *is* $+\infty$.

2. *If* $\bigcap_{i=1}^{m}(a_i + C_{\infty}^{a_i}) = \emptyset$ *and* $\bigcap_{i=1}^{m}(a_i + cone(C_{a_i})) \neq \emptyset$ *then,*

   (a) If $M_w^{l_\infty}(A)$ *is nonempty, then there exist* $\tau > 0$, $J \subseteq [1, m]$, $J \neq \emptyset$, $(p_{a_i})_{i \in J}$ *with* $p_{a_i} \in C_{a_i}^*$ *and* $\lambda = (\lambda_1, \cdots, \lambda_m)$ *with* $\lambda_i > 0 \; \forall i \in J$, $\lambda_i = 0 \; \forall i \notin J$ *satisfying* $\sum_{i \in J} \lambda_i = 1$, $\sum_{i=1}^{m} \lambda_i w_{a_i} p_{a_i} = 0$ *such that,*

   $$M_w^{l_\infty}(A) = C_J(p) \cap E_J(\tau).$$

   Moreover τ *is the optimal value of* $F(x)$ *and*

   $$\tau = -\sum_{i \in J}{}_{i}^{\lambda} \; w_{a_i} < p_{a_i}, a_i >$$

(b) *If there exist* $\tau > 0$, $J \subseteq [1,m]$, $J \neq \emptyset$, $(p_{a_i})_{i\in J}$ *with* $p_{a_i} \in C^*_{a_i}$ *and* $\lambda = (\lambda_1, \cdots, \lambda_m) \in B^*$ *with* $\lambda_i > 0\ \forall i \in J$, $\lambda_i = 0\ \forall i \notin J$ *satisfying* $\sum_{i\in J} \lambda_i = 1$, $\sum_{i=1}^{m} \lambda_i w_{a_i} p_{a_i} = 0$ *such that,*

$$C_J(p) \cap E_J(\tau) \neq \emptyset$$

then

$$M_w^{l_\infty}(A) = C_J(p) \cap E_J(\tau)$$

and $\tau = -\sum_{i\in J} \frac{\lambda}{i} w_{a_i} < p_{a_i}, a_i >$ *is the optimal value of* F.

Finally, we finish this section with some particular cases.

Some particular cases.

1. Consider the Weber problem in $I\!R^2$ with quadratic semigauges.

$$(P_w(A)) \qquad \min_{x\in I\!R^2} F(x) = \sum_{i=1}^{m} w_{a_i} \varphi_{a_i}(x - a_i)$$

being $\varphi_{a_i}(x) = \sqrt{< Qx, x >}$ where Q is a symmetric positive semi-definite matrix in $I\!R^{2\times 2}$.

First of all, we study the case where $\varphi_{a_i}(x) = \varphi(x) \quad \forall i = 1, \cdots, m$. Then, we analyze the case where $\varphi_{a_i}(x) \neq \varphi_{a_j}(x)$ for some $i \neq j$.

(a) Assume $\varphi_{a_i}(x) = \varphi(x) \quad \forall i = 1, \cdots, m$. We denote by C the unit ball of φ.

Let t be the orthogonal line to C_∞ and θ the angle between t and the axis OX.

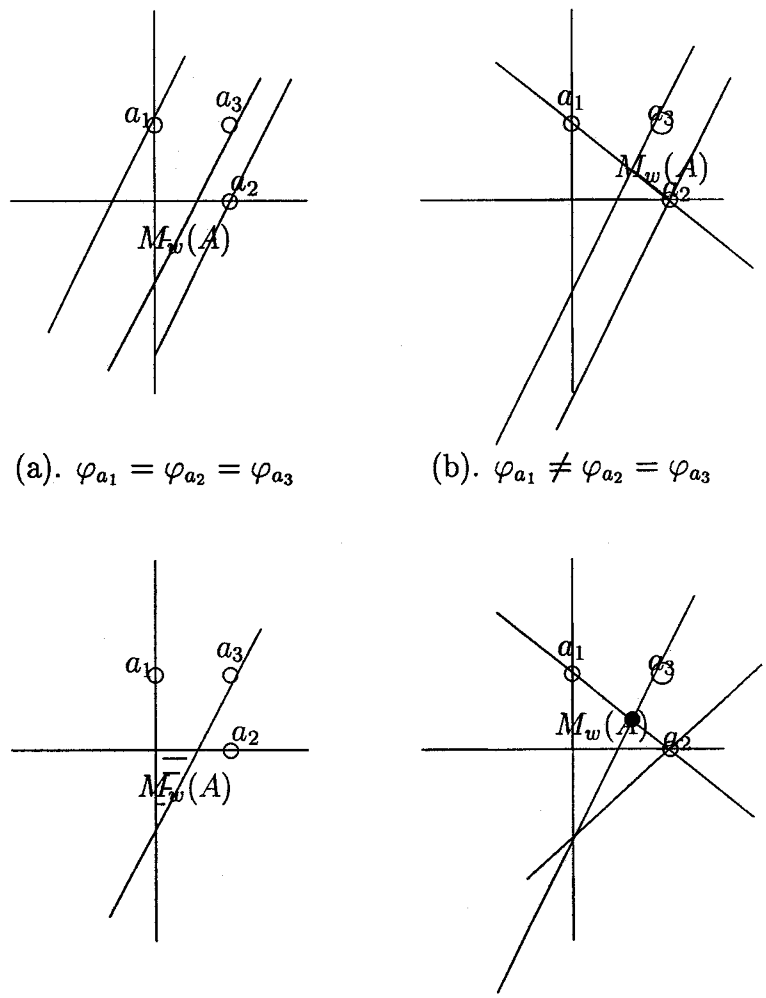
For the semigauge of example 1.(a) we have, $t:\ x_2 = 0$.

For the semigauge of example 1.(b) we have, $t:\ x_1 = 0$.

For the semigauge of example 1.(c) we have, $t:\ x_1 = \frac{q_{11}}{q_{12}} x_2$.

Let $x = (x_1, x_2) \in I\!R^2$. The orthogonal projection of x onto the line t is given by:

$$x(\theta) = x_1 \cos\theta + x_2 \sin\theta$$



(a). $\varphi_{a_1} = \varphi_{a_2} = \varphi_{a_3}$ (b). $\varphi_{a_1} \neq \varphi_{a_2} = \varphi_{a_3}$

(c). $\varphi_{a_1} \neq \varphi_{a_2} \neq \varphi_{a_3}$ (d). $\varphi_{a_1} \neq \varphi_{a_2} \neq \varphi_{a_3}$

Figure 2.5: Examples of the different cases

The problem $P_w(A)$ is equivalent to $P_w(A(\theta))$ given by:

$$(P_w(A(\theta))) \qquad \min_{x(\theta)\in I\!R} F(x(\theta)) = \sum_{i=1}^{m} w_{a_i}|x(\theta) - a_i(\theta)|$$

Therefore, the original problem in $I\!R^2$ becomes a Weber problem in $I\!R$.

The optimality condition for $b(\theta) \in A(\theta)$ for the problem $P_w(A(\theta))$ is:

$$b(\theta) \in M_w(A(\theta)) \quad \text{iff} \quad \begin{cases} \sum_{a_i(\theta)<b(\theta)} w_{a_i} \leq \sum_{b(\theta)\leq a_i(\theta)} w_{a_i} \\ \text{or} \\ \sum_{a_i(\theta)\leq b(\theta)} w_{a_i} \geq \sum_{b(\theta)<a_i(\theta)} w_{a_i} \end{cases}$$

The optimal solutions of $P_w(A(\theta))$ are defined by the set of medians of $P_w(A(\theta))$. It is easy to see that if $\sum_{a_i(\theta)\leq b(\theta)} w_{a_i} - \sum_{b(\theta)<a_i(\theta)} w_{a_i} > 0$ the unique optimal solution of $P_w(A(\theta))$ is $b(\theta)$. Otherwise the set of optimal solutions is the interval $[b(\theta), suc(b(\theta))]$ where $suc(b(\theta)) = \min\{a_i(\theta) : a_i(\theta) > b(\theta)\}$. In the first case, the solution of $P_w(A)$ is the orthogonal line to t which contains b, i.e., $y - b_2 = -\frac{1}{\tan\theta}(x - b_1)$ (see Figure 2.5.(a)). In the latter, the set of optimal solutions of $P_w(A)$ is the strip delimited by the lines
$y - b_2 = -\frac{1}{\tan\theta}(x - b_1)$ and $y - suc(b)_2 = -\frac{1}{\tan\theta}(x - suc(b)_1)$.

(b) Assume $\varphi_{a_i}(x) \neq \varphi_{a_j}(x)$ for some $i \neq j$ In this case, we have:

$$\varphi_{a_i}(x) = \max\{< s, x > : \ s \in ext(C^*_{a_i})\}$$

where $ext(C^*_{a_i})$ stands for the set of extreme points of $C^*_{a_i}$. Therefore, for the different possibilities of quadratic semigauges in $I\!R^2$, we obtain:

For the semigauge of example 1.(b)

$$\varphi(x) = \max\{-\frac{1}{\sqrt{q_{22}}}x_2, \frac{1}{\sqrt{q_{22}}}x_2\}$$

For the semigauge of example 1.(a) ($q_{12} = 0$) and 1.(c)

$$\varphi(x) = \max\{-(\frac{q_{11}}{\sqrt{q_{11}}}x_1 + \frac{q_{12}}{\sqrt{q_{11}}}x_2), (\frac{q_{11}}{\sqrt{q_{11}}}x_1 + \frac{q_{12}}{\sqrt{q_{11}}}x_2)\}$$

Then, we can rewrite the original problem as a linear problem.

$$\sum_{i=1}^{m} w_{a_i}\varphi_{a_i}(x-a_i) = \sum_{i=1}^{m} w_{a_i} \max\{u_{a_i}(x-a_i) \ : \ u_{a_i} \in ext(C^*_{a_i})\}$$

Let z be an auxiliary variable such that

$$\max\{u_{a_i}(x-a_i) \ : \ u_{a_i} \in ext(C^*_{a_i})\} \leq z_{a_i}$$

then the problem becomes:

$$\begin{array}{ll} \min & \sum_{i=1}^{m} w_{a_i} z_{a_i} \\ \text{s.t.} & u_{a_i}(x-a_i) \leq z_{a_i} \quad \forall i=1,\cdots,m; \ \forall u_{a_i} \in ext(C^*_{a_i}) \\ & x \in I\!R^2, \quad z_{a_i} \geq 0 \quad \forall i=1,\cdots,m. \end{array}$$

In the next case we geometrically describe the set of optimal solution when there are three existing facilities and $\varphi_{a_i}(x) \neq \varphi_{a_j}(x)$ for some $i \neq j$.

2. Our second case is devoted to geometrically describe the set of optimal solutions of the classical Fermat problem (although with quadratic semigauges rather than Euclidean distances). Given $\{a_1, a_2, a_3\}$ three points on the plane find the points minimizing the sum of the "distances" to the three given points. We know (see Theorem 2.4) that,

$$M_w(A) = \bigcap_{a_i \in A} (a_i + N_{a_i}(p_{a_i})) \quad \text{with} \quad \sum_{i=1}^{m} w_{a_i} p_{a_i} = 0.$$

Therefore, the optimal solution set is an elementary convex set. In addition, the function $\varphi_{a_i}(x-a_i)$ increases in the direction orthogonal to the line $a_i + C^{a_i}_{\infty}$ $\forall i = 1,2,3$. This implies that, if $\varphi_{a_i}(x) \neq \varphi_{a_j}(x)$ for some $i \neq j$, $M_w(A)$ has to be a bounded elementary convex set. Otherwise $\varphi_{a_{i_0}}(x - a_{i_0})$ goes to infinity for some $i_0 = 1,2,3$.

- If $\varphi_{a_i} \neq \varphi_{a_j}$ $\forall i \neq j$, $i,j = 1,2,3$, only two cases can occur:

  The lines $a_i + C^{a_i}_{\infty}$ $i = 1,2,3$ intersect at a unique point. This point is the minimizer (the Fermat point) and it is at zero "distance" from the a_i's.

The lines, $a_i + C_\infty^{a_i}$ $i = 1, 2, 3$ intersect delimiting a triangle. In this case, the bounded elementary convex sets are: 1) that triangle; 2) their sides and 3) their vertices.

The optimal point is that vertex giving the smaller objective function (see Figure 2.5.(d)). If two vertices have the same minimum value the whole side is optimal. If three vertices have the same value the optimal points are all the points of the triangle (see Figure 2.5.(c)).

- If $\varphi_{a_1} = \varphi_{a_2} \neq \varphi_{a_3}$, the unique bounded elementary convex sets are the segment on the line $a_3+C_\infty^3$ bounded by the lines $a_1+C_\infty^1$ and $a_2+C_\infty^2$ and its extreme points. If the objective value is the same at both points the optimal solution set is the whole segment (see Figure 2.5.(b)). Otherwise the "Fermat" point is the vertex with minimum value.

2.4 Concluding remarks

In this chapter we deal with a new approach to the single facility location problem based on semigauges. This approach generalizes previous approaches to location problems based on gauges and it can be seen geometrically as if the existing facilities are chosen between a given set of dimensional structures. We characterize the whole set of optimal solutions and provide extensions to new models of location problems. Although the development of algorithms to solve these problems exceeds the scope of the chapter some hints can be suggested to solve them in specific cases. For instance, for problems whose unit balls are polytopes the objective functions admit linear representations within the elementary convex sets. Thus, solving the problems reduces to solve a finite number of linear programs. This fact together with the convexity of the considered objective functions permits to solve these problems following a descent scheme in polynomial time (see for instance the examples in the above section).

Using the denseness of the closed polytopes in the family of closed convex sets we can also solve, up to a given accuracy, any problem whose unit balls are closed convex sets.

Chapter 3

A multiperiod two-echelon multicommodity capacitated plant location problem

In many real world situations where large companies manufacture and distribute products it is necessary to locate production plants or warehouses to deliver goods or products to final customers in order to meet their demands. This is the case of health-care products, spare parts of cars or catalogues in travel agencies. If the admissible locations of these facilities are finite and known in advance, we cope with a discrete plant location problem. These problems have been widely studied and roughly speaking are classified into: 1) Uncapacitated or Simple plant location problems (UPLP) or (SPLP); and 2) Capacitated plant location problems (CPLP). Both kinds of problems can be formulated as mixed integer programming problems (see Aikens (1985) [2]). Nevertheless, obtaining their exact solutions in polynomial time is not possible because Krarup and Pruzan (1983) [65] prove that even the UPLP belongs to the class of NP-hard problems. See e.g. the paper by Aikens (1985) [2] and the boos of Drezner (1995) [31] or Daskin (1995) [28] for a good overview of these kinds of problems and their extensions.

Among these extensions we shall focus on two of them. The first one consists of introducing the dynamic aspect into the problem. In this case, not only the transportation plan but also the time-staged establishment of the facilities are decision variables (see e.g. Warszawski's (1973) [100], Van Roy & Erlenkotter (1982) [99] or more recently Chardaire et al. (1996) [23]). The second one consists of the assumption of a certain structure in the transportation pattern. This is to say, the transportation follows a two step path. These models have been hardly studied in the classical literature of location (see e.g. Kaufman et al. (1977) [64], Tcha and Lee (1984) [98],Pirkul and Jayaraman (1996) [85] or Barros and Labbé (1994) [6] and [7]), although recently a detailed analysis can be found in Daskin (1995) [28]or Marín (1996) [74]. The main difference in these models is that the products are delivered from the production plants to the warehouses and then from the warehouses to the final customers (or retailers). Therefore, the decision problem consists of locating the plants and the warehouses and determining the amount of the different products that will be delivered from each open plant to each open warehouse and from each open warehouse to each final customer. Obviously, capacities may or may not be considered. That structure has been sometimes called two-echelon approach (see Marín (1996) [74], Crainic and Delorme (1993) [26] , Barros and Labbé (1994) [6]).

Despite the generality of these models, the most natural framework for these problems is the combination of these two approaches. That is, the joint consideration of multiperiod and multi-echelon aspects (see Figure 3.1). Nevertheless, as far as we know, this approach has never been addressed before and it can be considered as an introduction to a new location problem.

In this chapter, we deal with a multiperiod two-echelon multicommodity capacitated location problem. We assume that the capacities of plants and warehouses, as well as demands and transportation costs change over T time periods. We do not consider holding decisions. Our goal is to minimize the total cost for meeting demands for the different products specified over time at various customer locations. Although no-real application has motivated this model, it perfectly applies to those situations where intermediate distribution and seasonal demand exist. The formulation permits both the opening of new facilities and the closing of existing

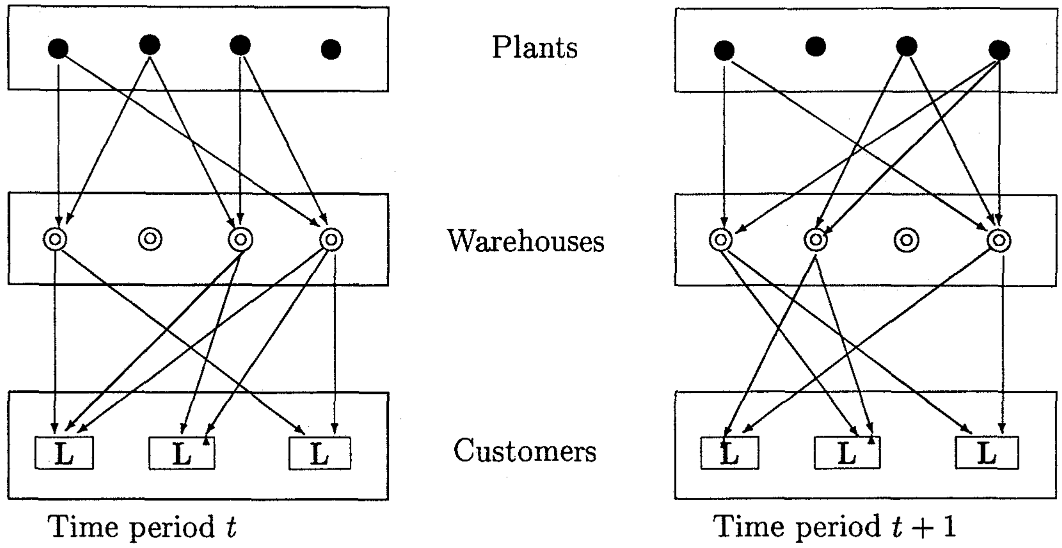


Figure 3.1: The model

ones. This is a very large mixed integer programming problem. For instance, for a problem with 50 customers, 20 warehouses, 20 plants, 2 different products and 4 time periods we shall have 11360 variables and 764 constraints. Our computational experience shows that solving exactly this problem by branch and bound needs prohibitive CPU time (see Table 3.3). Therefore, we propose an alternative approach. We present a Lagrangean relaxation scheme incorporating a dual ascent method together with a heuristic construction phase method which shows in computational test to provide good feasible solutions for this problem (see Barceló et al.(1991)[8], Erlenkotter (1978)[37], Guignard and Rosenwein (1989)[44], Guignard and Opaswongkarn (1990)[45] or Holmberg and Jornsten (1996)[61] for similar analysis in different problems). As commented above, no comparative testing with other procedures in the literature can be reported because as far as we know, this is the first time that this problem has been addressed.

The chapter is organized as follows. In Section 3.1 the mathematical formulation of the model is presented, together with a suitable reformulation more convenient for our optimization purposes. In Section 3.2 a Lagrangean relaxation is proposed for this problem and its solution is presented. Section 3.3 develops the heuristic phase of our solution method. The section 3.4 is devoted to the computational results and the last section to the conclusions.

3.1 The model

The multiperiod two-echelon multicommodity capacitated plant location problem we deal with has the objective of minimizing the total cost for meeting demands of the different products specified over time at various customer locations. The version of the problem that we consider in the chapter assumes the following hypotheses. There are not holding decisions. The sets of customers and products, together with the feasible locations for the facilities (plants and warehouses) are considered fixed and known beforehand. Therefore, they do not change over the time horizon. It is usual to consider seasons or months as a typical period length for this kind of problem. Then, the time horizon is chosen in accordance with the period lengths and the planning horizon. In addition, we will denote by:

$I = \{1, \cdots, n\}$ set of customers, indexed by $i \in I$.

$L = \{1, \cdots, q\}$ set of product types, indexed by $l \in L$.

$J = \{1, \cdots, m\}$ set of possible location for warehouses, indexed by $j \in J$.

$K = \{1, \cdots, p\}$ set of possible location for plants, indexed by $k \in K$.

At the beginning of the first time period there exists a subset K_c (respectively J_c) of the whole set of feasible locations for the plants (respectively warehouses) where operating facilities are established. These facilities can be closed at the end of any time period $t \in \{1, ..., T\}$, but once closed they cannot be reopened. We denote by K_o (respectively J_o) the set of feasible locations where there does not exist open plants (respectively warehouses). These facilities can be opened at the beginning of any time period and it is also assumed that if they were open they would not be closed. This hypothesis is quite reasonable. In real-life applications the opening/closing of final retailers usually leads to a loss of market because customers require a certain regularity to patronize a particular facility. In addition, this phenomenon increases the operating cost.

Additionally, we assume that a minimum number of plants and warehouses must be open at the first and last time period which ensures a minimum coverage of the demand at the beginning and after the time horizon. Let us denote by ND^1, ND^T (respectively NC^1, NC^T) the minimum number of warehouses (respectively plants) open at the beginning of the first time period and at the end of the last time period.

We consider the situation where both plants and warehouses have limited capacity which depends on the time period. We denote by:

$W_j^t :=$ capacity of warehouse j at time period t.

$C_k^t :=$ capacity of plant k at time period t.

$d_{il}^t :=$ demand of product l at customer i during time period t.

Finally, we assume a cost structure that includes both transportation costs of goods and maintenance costs. For the elements of this problem we will use the following notation:

$c_{ijl}^t :=$ transportation cost per unit of product l from warehouse j to customer i at time period t.

$b_{jkl}^t :=$ transportation cost per unit of product l from plant k to warehouse j at time period t.

$f_j^t :=$ operating cost of a warehouse open at position j during time period t.

$g_k^t :=$ operating cost of a plant open at position k during time period t.

Notice that we do not explicitly have an installation or setup cost for the facilities. This is because the facilities which belong to the set J_c (respectively K_c) are already open at the beginning of the first period. Therefore, they do not have installation of setup cost. On the second hand, we consider that the facilities which belong to J_o (respectively K_o) have a fixed setup cost which is charged at period T. This is possible because, once these facilities are opened they will never be closed until the end of the planning horizon. The decision variables of the problem are:

$x_{ijl}^t :=$ fraction (regarding to d_{il}^t) of product l delivered to customer i from warehouse j at time period t.

$y_{jkl}^t :=$ fraction (regarding to W_j^t) of product l sent to warehouse j from plant k at time period t.

$$u_j^t = \begin{cases} 1 & \text{if warehouse } j \text{ is open at the beginning of time period } t \\ 0 & \text{otherwise} \end{cases}$$

$$v_k^t = \begin{cases} 1 & \text{if plant } k \text{ is open at the beginning of time period } t \\ 0 & \text{otherwise} \end{cases}$$

In addition, we denote by $v(\cdot)$ the optimal objective value of Problem $(\cdot)$. Using these conventions, the mathematical formulation of this problem is:

$$(MMCPL) \qquad \min g(x,y,u,v) := \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{q} c_{ijl}^t x_{ijl}^t d_{il}^t + \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} b_{jkl}^t y_{jkl}^t W_j^t + \sum_{t=1}^{T}\sum_{j=1}^{m} f_j^t u_j^t + \sum_{t=1}^{T}\sum_{k=1}^{p} g_k^t v_k^t$$

s.t.:

$$\sum_{j=1}^{m} x_{ijl}^t \geq 1 \qquad \forall i, \forall l, \forall t \tag{3.1}$$

$$\sum_{i=1}^{n}\sum_{l=1}^{q} d_{il}^t x_{ijl}^t \leq W_j^t u_j^t \qquad \forall j, \forall t \tag{3.2}$$

$$\sum_{k=1}^{p} W_j^t y_{jkl}^t \geq \sum_{i=1}^{n} d_{il}^t x_{ijl}^t \qquad \forall j, \forall\ , l, \forall t \tag{3.3}$$

$$\sum_{j=1}^{m}\sum_{l=1}^{q} W_j^t y_{jkl}^t \leq C_k^t v_k^t \qquad \forall k, \forall t \tag{3.4}$$

$$\sum_{j=1}^{m} u_j^1 \geq ND^1 \qquad \sum_{j=1}^{m} u_j^T \geq ND^T \tag{3.5}$$

$$\sum_{k=1}^{p} v_k^1 \geq NC^1 \qquad \sum_{k=1}^{p} v_k^T \geq NC^T \tag{3.6}$$

$$u_j^1 = 1 \quad \forall j \in J_c; \quad u_j^t \geq u_j^{t+1} \quad \forall j \in J_c \quad \forall t; \quad u_j^t \leq u_j^{t+1} \quad \forall j \in J_o \quad \forall t \tag{3.7}$$

$$v_k^1 = 1 \quad \forall k \in K_c; \quad v_k^t \geq v_k^{t+1} \quad \forall k \in K_c \quad \forall t; \quad v_k^t \leq v_k^{t+1} \quad \forall k \in K_o \quad \forall t \tag{3.8}$$

$$x_{ijl}^t,\, y_{jkl}^t \geq 0 \qquad \forall i, \forall j, \forall k, \forall l, \forall t \tag{3.9}$$

$$u_j^t,\, v_k^t \in \{0,1\} \qquad \forall j, \forall k, \forall t \tag{3.10}$$

The constraints (3.1) and (3.3) refer to the demand. The constraints (3.1) guarantee meeting the demand of each customer for each one of the products in each time period t. Notice that warehouse j needs an amount of product l equal to the sum of the amounts of this product that it delivers to the final customers. The constraints (3.2) and (3.4) refer to capacity. (3.2) assures that the total number of units delivered from warehouse j is not greater than its capacity in each time period t. (3.4) is similar to (3.2) but focused towards plants rather that warehouses. The constraints (3.5) and (3.6) state the minimum number of warehouses and plants that must be open at the first and last time period. The constraints (3.7) and (3.8) describe the sets $J = J_0 \cup J_c$ and $K = K_o \cup K_c$.

This is a formulation often used for multiperiod models (see e.g. Roodman & Schwarz (1977) [92] or Chardaire et al. (1996) [23]). Thus, this formulation simplifies the understanding process of our model to the readers accustomed to read papers in this field. Nevertheless, from the point of view of our resolution approach, it is more convenient to deal with an alternative formulation which will be proven equivalent (see Theorem 3.1). To this end, the following variables z_j^t and ζ_k^t are introduced.

$$\forall\ j \in J_o, \forall\ t \quad z_j^t = \begin{cases} 1 & \text{if a warehouse is established at } j \text{ at the beginning} \\ & \text{of time period } t \\ 0 & \text{otherwise} \end{cases}$$

$$\forall j \in J_c,\ \forall t < T-1 \quad z_j^t = \begin{cases} 1 & \text{if existing warehouse at } j \text{ is removed at the end of time period } t \\ 0 & \text{otherwise} \end{cases}$$

$$\forall\ j \in J_c \quad z_j^T = \begin{cases} 1 & \text{if warehouse } j \text{ is open during all the planning horizon} \\ 0 & \text{otherwise} \end{cases}$$

ζ_k^t is analogously defined for the set of plants.

And the costs are defined as follows:

$F_j^t = \sum_{r=t}^{T} f_j^r$, total cost of warehouse j being established in time period t $\forall t$ and $\forall j \in J_o$.

$F_j^t = \sum_{r=1}^{t} f_j^r$, total cost of warehouse j removed at the end of time period t $\forall t$ and $\forall j \in J_c$.

$G_k^t = \sum_{r=t}^{T} g_k^r$, total cost of plant k being established in time period t $\forall t$, $\forall k \in K_o$.

$G_k^t = \sum_{r=1}^{t} g_k^r$, total cost of plant k removed at the end of time period t $\forall t$, $\forall k \in K_c$.

Let $T_{jt} = \begin{cases} \{1,\cdots,t\} & \text{if } j \in J_o \\ \{t,\cdots,T\} & \text{if } j \in J_c \end{cases}$ and $T_{kt} = \begin{cases} \{1,\cdots,t\} & \text{if } k \in K_o \\ \{t,\cdots,T\} & \text{if } k \in K_c \end{cases}$.

The $MMCPL$ problem can be reformulated in terms of the variables z_j^t, ζ_k^t and the costs F_j^t, G_k^t, namely $MMCPL'$ problem:

$$(MMCPL') \quad \min f(x,y,z,\zeta) := \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{q} c_{ijl}^t x_{ijl}^t d_{il}^t + \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} b_{jkl}^t y_{jkl}^t W_j^t + \sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j^t + \sum_{t=1}^{T}\sum_{k=1}^{p} G_k^t \zeta_k^t$$

s.t.:

$$\sum_{j=1}^{m} x_{ijl}^{t} \geq 1 \qquad \forall i, \forall l, \forall t \tag{3.1}$$

$$\sum_{i=1}^{n}\sum_{l=1}^{q} d_{il}^{t} x_{ijl}^{t} \leq W_j^t \sum_{\tau \in T_{jt}} z_j^{\tau} \qquad \forall j, \forall t \tag{3.2'}$$

$$\sum_{k=1}^{p} W_j^t y_{jkl}^t \geq \sum_{i=1}^{n} d_{il}^t x_{ijl}^t \qquad \forall j, \forall l, \forall t \tag{3.3}$$

$$\sum_{j=1}^{m}\sum_{l=1}^{q} W_j^t y_{jkl}^t \leq C_k^t \sum_{\tau \in T_{kt}} \zeta_k^{\tau} \qquad \forall k, \forall t \tag{3.4'}$$

$$\sum_{j \in Jo} z_j^1 + \sum_{j \in Jc}\sum_{t=1}^{T} z_j^t \geq ND^1; \qquad \sum_{j \in Jo}\sum_{t=1}^{T} z_j^t + \sum_{j \in Jc} z_j^T \geq ND^T \tag{3.5'}$$

$$\sum_{k \in Ko} \zeta_k^1 + \sum_{k \in Kc}\sum_{t=1}^{T} \zeta_k^t \geq NC^1 \qquad \sum_{k \in Ko}\sum_{t=1}^{T} \zeta_k^t + \sum_{k \in Kc} \zeta_k^T \geq NC^T \tag{3.6'}$$

$$\sum_{t=1}^{T} z_j^t = 1 \quad \forall j \in J_c; \qquad \sum_{t=1}^{T} z_j^t \leq 1 \quad \forall j \in J_o \tag{3.7'}$$

$$\sum_{t=1}^{T} \zeta_k^t = 1 \quad \forall k \in K_c; \qquad \sum_{t=1}^{T} \zeta_k^t \leq 1 \quad \forall k \in K_o \tag{3.8'}$$

$$x_{ijl}^t, y_{jkl}^t \geq 0 \qquad \forall i, \forall j, \forall k, \forall l, \forall t \tag{3.9}$$

$$z_j^t, \zeta_k^t \in \{0,1\} \qquad \forall j, \forall k, \forall t \tag{3.10'}$$

Theorem 3.1 *If (x, y, u, v) is a feasible solution of the $MMCPL$ problem then there exists a feasible solution (x', y', z, ζ) of the $MMCPL'$ problem such that $g(x, y, u, v) = f(x', y', z, \zeta)$. Conversely, if (x', y', z, ζ) is a feasible solution of the $MMCPL'$ problem then there exists a feasible solution (x, y, u, v) of the $MMCPL$ problem such that $f(x', y', z, \zeta) = g(x, y, u, v)$.*

Proof. Let (x, y, u, v) be a feasible solution of the $MMCPL$ problem. We define (x', y', z, ζ) as:

$x' = x$; $\quad y' = y$;

$$\forall j \in J_o \quad \begin{array}{l} z_j^1 = u_j^1 \\ z_j^t = u_j^t - u_j^{t-1} \quad \forall t > 1 \end{array} \qquad \text{and} \qquad \forall j \in J_c \quad \begin{array}{l} z_j^T = u_j^T \\ z_j^t = u_j^t - u_j^{t+1} \quad \forall t < T \end{array};$$

$$\forall k \in K_o \quad \begin{array}{l} \zeta_k^1 = v_k^1 \\ \zeta_k^t = v_k^t - v_k^{t-1} \;\; \forall t > 1 \end{array} \quad \text{and} \quad \forall k \in K_c \quad \begin{array}{l} \zeta_k^T = v_k^T \\ \zeta_k^t = v_k^t - v_k^{t+1} \;\; \forall t < T \end{array}.$$

By constraints (3.7) (respectively (3.8)) and (3.10) we obtain $z_j^t \in \{0,1\} \;\; \forall j, \forall t$ (respectively $\zeta_k^t \in \{0,1\} \;\; \forall k, \forall t$). In addition, by the definition of the variables z and ζ, we have:

$$u_j^t = \sum_{\tau \in T_{jt}} z_j^\tau \qquad \text{and} \qquad v_k^t = \sum_{\tau \in T_{kt}} \zeta_k^\tau.$$

Therefore, since (x, y, u, v) is a feasible solution of $MMCPL$, it is straightforward to substitute u and v in the constraints of $MMCPL$ to check that (x', y', z, ζ) verify the constraints of $MMCPL'$. Hence, (x', y', z, ζ) is a feasible solution of $MMCPL'$.

On the other hand, by the definition of F_j^t and G_k^t we have:

$$\forall j \in J_o \quad f_j^t = \begin{cases} F_j^t - F_j^{t+1} & \forall t < T \\ F_j^T & \text{if } t = T \end{cases} \quad \text{and } \forall j \in J_c \quad f_j^t = \begin{cases} F_j^t - F_j^{t-1} & \forall t > 1 \\ F_j^1 & \text{if } t = 1 \end{cases}$$

$$\forall\, k \in K_o \quad g_k^t = \begin{cases} G_k^t - G_k^{t+1} & \forall t < T \\ G_k^T & \text{if } t = T \end{cases} \quad \text{and } \forall\, k \in K_c \quad g_k^t = \begin{cases} G_k^t - G_k^{t-1} & \forall t > 1 \\ G_k^1 & \text{if } t = 1 \end{cases}$$

Then, we obtain:

$$\begin{aligned} \sum_{t=1}^{T}\sum_{j=1}^{m} f_j^t u_j^t &= \sum_{j \in J_o}\left[\sum_{t=1}^{T-1}\left(F_j^t - F_j^{t+1}\right)\sum_{\tau=1}^{t} z_j^\tau + F_j^T \sum_{\tau=1}^{T} z_j^\tau\right] \\ &\quad + \sum_{j \in J_c}\left[F_j^1 \sum_{\tau=1}^{T} z_j^\tau + \sum_{t=2}^{T}\left(F_j^t - F_j^{t-1}\right)\sum_{\tau=t}^{T} z_j^\tau\right] \\ &= \sum_{j \in J_o}\left[\sum_{t=1}^{T} F_j^t z_j^t\right] + \sum_{j \in J_c}\left[\sum_{t=1}^{T} F_j^t z_j^t\right] = \sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j^t; \end{aligned} \tag{3.11}$$

and in the same way,

$$\sum_{t=1}^{T}\sum_{k=1}^{p} g_k^t v_k^t = \sum_{t=1}^{T}\sum_{k=1}^{p} G_k^t \zeta_k^t. \tag{3.12}$$

Therefore, $g(x, y, u, v) = f(x', y', z, \zeta)$.

Conversely, let (x', y', z, ζ) be a feasible solution of the $MMCPL'$ problem. We define (x, y, u, v) as:

$$x = x'; \quad y = y'; \quad u_j^t = \sum_{\tau \in T_{jt}} z_j^\tau \;\forall j, t; \quad v_k^t = \sum_{\tau \in T_{kt}} \zeta_k^\tau \;\forall j, t.$$

By constraints (3.7') (respectively (3.8')) and (3.10') we obtain $u_j^t \in \{0,1\}$ $\forall j, \forall t$ (respectively $v_k^t \in \{0,1\} \quad \forall k, \forall t$). In addition, by the definition of the variables u and constraints (3.7') we obtain $u_j^1 = \sum_{t=1}^{T} z_j^t = 1 \quad \forall j \in J_c$ and by (3.10') we obtain $u_j^t \geq u_j^{t+1} \quad \forall j \in J_c \quad \forall t$ and $u_j^t \leq u_j^{t+1} \quad \forall j \in J_o \quad \forall t$. Then, the variables u fulfill the constraints (3.7). In the same way, by (3.8') and (3.10') one can prove that the variables v verify the constraints (3.8). Substituting z and ζ by u and v in the remainder constraints of $MMCPL'$ it is straightforward that u and v fulfill the constraints of $MMCPL$. Therefore, (x, y, u, v) is a feasible solution of $MMCPL$ and by (3.11) and (3.12) we have, $f(x', y', z, \zeta) = g(x, y, u, v)$. □

Theorem 3.1 proves that both formulations are equivalent in the sense that they provide the same set of optimal solutions. From now on, we will always deal with the $MMCPL'$ problem.

The $MMCPL'$ problem is a mixed-integer programming problem which includes as a particular instance the UPLP. Since our problem includes as a particular case the UPLP and this problem is NP-hard (see Krarup and Pruzan (1983) [65]) one cannot expect to solve exactly large sizes of the $MMCPL'$ problem in polynomial time. For this reason, we will adopt a heuristic method to solve $MMCPL'$ for those instances. It is based on: 1) using a Lagrangean relaxation; and 2)using an "ad hoc" procedure obtaining a feasible solution from the solutions of the relaxed problems.

3.2 Decomposition of the problem: Lagrangean relaxation.

In this section, we consider a relaxation of the $MMCPL'$ problem obtained relaxing the constraints which ensure that demands are met. To this end, we associate non-negative multipliers $\mu_{il}^t \geq 0$ to the constraints (3.1) and $\lambda_{jl}^t \geq 0$ to the constraints

(3.3). Therefore the relaxed problem $LR(\lambda,\mu)$ is:

$$LR(\lambda,\mu) \qquad \min f_{\lambda,\mu}(x,y,z,\zeta) := \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{q} c^t_{ijl} x^t_{ijl} d^t_{il} + \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} b^t_{jkl} y^t_{jkl} W^t_j$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{m} F^t_j z^t_j + \sum_{t=1}^{T}\sum_{k=1}^{p} G^t_k \zeta^t_k + \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{l=1}^{q} \mu^t_{il}\left(1-\sum_{j=1}^{m} x^t_{ijl}\right)$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{l=1}^{q} \lambda^t_{jl}\left(\sum_{i=1}^{n} d^t_{il} x^t_{ijl} - \sum_{k=1}^{p} W^t_j y^t_{jkl}\right)$$

s.t.: (3.2'), (3.4'), (3.5'), (3.6'), (3.7), (3.8'), (3.9'), (3.10')

A little thought about Problem $LR(\lambda,\mu)$ leads us to separate it into two subproblems, $LR1(\lambda,\mu)$ and $LR2(\lambda,\mu)$. These two problems are the following:

$$LR1(\lambda,\mu) \qquad \min \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{q}\left(c^t_{ijl} d^t_{il} + \lambda^t_{jl} d^t_{il} - \mu^t_{il}\right) x^t_{ijl} + +\sum_{t=1}^{T}\sum_{j=1}^{m} F^t_j z^t_j$$

s.t.: (3.2'), (3.5'), (3.7'), $x^t_{ijl} \geq 0, \quad z^t_j \in \{0,1\}$;

and

$$LR2(\lambda,\mu) \qquad \min \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\left(b^t_{jkl} W^t_j - \lambda^t_{jl} W^t_j\right) y^t_{jkl} + \sum_{t=1}^{T}\sum_{k=1}^{p} G^t_k \zeta^t_k$$

s.t.: (3.4'), (3.6'), (3.8'), $y^t_{jkl} \geq 0, \quad \zeta^t_k \in \{0,1\}$.

These problems can be solved independently and their solutions can be used to solve $LR(\lambda,\mu)$. Once the problems $LR1(\lambda,\mu)$ and $LR2(\lambda,\mu)$ have been solved, the value of $LR(\lambda,\mu)$ is given by the following proposition whose proof is obvious.

Proposition 3.1

$$v(LR(\lambda,\mu)) = v(LR1(\lambda,\mu)) + v(LR2(\lambda,\mu)) + \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{l=1}^{q} \mu^t_{il}$$

Analysis of $LR1(\lambda,\mu)$

First of all, in order to solve $LR1(\lambda,\mu)$ we will leave constraints (3.5') aside. Then, $LR1(\lambda,\mu)$ can be separated into m subproblems and once the solution of each subproblem is obtained, we will obligate constraints (3.5') to be fulfilled in such a way that the optimal value for $LR1(\lambda,\mu)$ is obtained. This last step is justified in Proposition 3.3.

Provided that (3.5') is removed $LR1(\lambda,\mu)$ can be separated into the following m subproblems, one for each $j = 1,\dots,m$:

$$LR1_j(\lambda,\mu) \qquad \min \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{l=1}^{q}\left(c_{ijl}^t d_{il}^t + \lambda_{jl}^t d_{il}^t - \mu_{il}^t\right) x_{ijl}^t + \sum_{t=1}^{T} F_j^t z_j^t$$

s.t.:

$$\sum_{i=1}^{n}\sum_{l=1}^{q} d_{il}^t x_{ijl}^t \le W_j^t \sum_{r\in T_{jt}} z_j^r \qquad \forall t$$

$$\sum_{t=1}^{T} z_j^t \le 1 \text{ if } j\in J_o \quad \textit{or} \quad \sum_{t=1}^{T} z_j^t = 1 \text{ if } j \in J_c$$

$$x_{ijl}^t \ge 0 \qquad \forall i,\ \forall l,\ \forall t$$

$$z_j^t \in \{0,1\} \qquad \forall t$$

These problems are associated to each warehouse $j \in J$ and we can solve them independently.

In order to solve $LR1_j(\lambda,\mu)$, we distinguish two cases depending on either $j \in J_o$ or $j \in J_c$ because of the relationships that hold among the z_j^t variables on each case.

1. Let us assume $j \in J_o$. For each $t_0 = 1,\cdots,T$ let $LR1_{jt_0}(\lambda,\mu)$ be the following problem:

$$LR1_{jt_0}(\lambda,\mu) \qquad \sum_{t=t_0}^{T}\sum_{i=1}^{n}\sum_{l=1}^{q}\left(c_{ijl}^t d_{il}^t + \lambda_{jl}^t d_{il}^t - \mu_{il}^t\right) x_{ijl}^t + F_j^{to}$$

s.t.:

$$\sum_{i=1}^{n}\sum_{l=1}^{q} d_{il}^t x_{ijl}^t \le W_j^t \qquad \forall t \ge t_0$$

$$x_{ijl}^t \ge 0 \forall i,\ \forall l,\ \forall t \ge t_0.$$

2. Let us assume $j \in J_c$. For each $t_0 = 1, \cdots, T$ let $LR1_{jt_0}(\lambda, \mu)$ be the following problem:

$$LR1_{jt_0}(\lambda,\mu) \qquad \min \sum_{t=1}^{t_0}\sum_{i=1}^{n}\sum_{l=1}^{q} \left(c_{ijl}^t d_{il}^t + \lambda_{jl}^t d_{il}^t - \mu_{il}^t\right) x_{ijl}^t + F_j^{t_0}$$

s.t.:

$$\sum_{i=1}^{n}\sum_{l=1}^{q} d_{il}^t x_{ijl}^t \le W_j^t \qquad \forall t \le t_0$$

$$x_{ijl}^t \ge 0 \forall i,\ \forall l,\ \forall t \le t_0.$$

Proposition 3.2

1. *If* $j \in J_o$, $\quad v(LR1_j(\lambda,\mu)) = \min\left\{ \min_{1\le t_0 \le T} v(LR1_{jt_0}(\lambda,\mu)),\ 0\right\}$.
2. *If* $j \in J_c$, $\quad v(LR1_j(\lambda,\mu)) = \min_{1\le t_0 \le T} v(LR1_{jt_0}(\lambda,\mu))$.

Proof.

1. If $j \in J_o$, it may occur that: a) for some $t_0 = 1, \cdots, T$, $z_j^{t_0} = 1$ and $z_j^t = 0$ for all $t \neq t_0$ (this assumption corresponds to the hypothesis that warehouse j is open at time period t_0); or b)$z_j^t = 0$ for all t (this assumption corresponds to the hypothesis that warehouse j is never open).

   In case a) we have, $z_j^{t_0} = 1$, $\quad z_j^t = 0$ for all $t \neq t_0$. Then, $\sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j = F_j^{t_0}$ and $\sum_{\tau \in T_{jt}} z_j^\tau = \begin{cases} 1 & \text{if } t \ge t_0 \\ 0 & \text{otherwise} \end{cases}$.

   In addition, $x_{ijl}^t \ge 0 \quad \forall i,\ \forall l,\ \forall t \ge t_0$ and $x_{ijl}^t = 0 \quad \forall i,\ \forall l,\ \forall t < t_0$. Applying these transformations to Problem $LR1_j(\lambda,\mu)$ it becomes Problem $LR1_{jt_0}(\lambda,\mu)$.

   In case b) $z_j^t = 0$ for all t. Then, the objective function of Problem $LR1_j(\lambda,\mu)$ is zero.

   Thus, $v(LR1_j(\lambda,\mu))$ is the minimum between all these possibilities:

$$v(LR1_j(\lambda,\mu)) = \min\left\{ \min_{1\le t_0 \le T} v(LR1_{jt_0}(\lambda,\mu)),\ 0\right\}$$

2. If $j \in J_c$, we have for some $t_0 = 1, \cdots, T$, $z_j^{t_0} = 1$ and $z_j^t = 0$ for all $t \neq t_0$ (this assumption corresponds to the hypothesis that warehouse j is closed at the end of t_0).

   Assume $z_j^{t_0} = 1$, $z_j^t = 0$ for all $t \neq t_0$. Then, $\sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j = F_j^{t_0}$ and

   $$\sum_{r \in T_{jt}} z_j^r = \begin{cases} 1 & \text{if } t \leq t_0 \\ 0 & \text{otherwise} \end{cases} .$$

   In addition, $x_{ijl}^t \geq 0 \quad \forall i, \ \forall l, \ \forall t \leq t_0$ and $x_{ijl}^t = 0 \quad \forall i, \ \forall l, \ \forall t > t_0$. Applying these transformations to Problem $LR1_j(\lambda, \mu)$ it becomes Problem $LR1_{jt_0}(\lambda, \mu)$.

   Thus, $v(LR1_j(\lambda, \mu))$ is the minimum between these possibilities:

   $$v(LR1_j(\lambda, \mu)) = \min_{1 \leq t_0 \leq T} v(LR1_{jt_0}(\lambda, \mu))$$

□

Notice that to solve $LR1_j(\lambda, \mu)$ we have only needed to solve T independent, continuous linear programming problems, $LR1_{jt_0}(\lambda, \mu)$. In addition, solving $LR1_j(\lambda, \mu)$ for each $j = 1, \cdots, m$ we obtain the time period in which the warehouse j has to be opened (if this is the case) or closed and therefore, the value of the integer variables z_j^t for all j, t. Let J^* be the set of indexes of those warehouses which belong to J_o and have never been opened after this process.

In order to solve $LR1(\lambda, \mu)$, we obligate the constraints (3.5') to be fulfilled. We denote by

$$\Delta 1(j) := \begin{cases} v(LR1_{j1}(\lambda, \mu)) - v(LR1_j(\lambda, \mu)) & \text{if } j \in J_o \text{ and } z_j^1 = 0 \\ +\infty & \text{otherwise} \end{cases}$$

First of all, we check the number of warehouses open at $t = 1$. If this number is less than ND^1 we calculate $\Delta 1(j)$ for each $j \in J_o$ closed at $t = 1$ and we open, one at a time, those warehouses $j \in J_o$ with the smallest increment $\Delta 1(j)$ until the constraint is fulfilled. Let J^1 denote the set of warehouses open in this way at $t = 1$.

Once, we have done that for $t = 1$ we proceed in the same way with the time period T. We denote by

$$\Delta T(j) := \begin{cases} \min_{t_0} v(LR1_{jt_0}(\lambda, \mu)) & \text{if } j \in J^* \setminus J^1 \\ v(LR1_{jT}(\lambda, \mu)) - v(LR1_j(\lambda, \mu)) & \text{if } j \in J_c \text{ and } z_j^T = 0 \\ +\infty & \text{otherwise} \end{cases}$$

We check the number of warehouses open at T. If this number is less than ND^T, we compute $\Delta T(j)$ for each $j \in J^* \setminus J^1$ and for each $j \in J_c$ closed at T. We select, one at a time, those warehouses with the smallest increment $\Delta T(j)$ until the constraint is fulfilled. Let J^T denote the set of warehouses chosen in this way.

If there exists a warehouse $j \in J^* \cap J^T$ such that, $\min_{t_0} v(LR1_{jt_0}(\lambda, \mu)) = v(LR1_{j1}(\lambda, \mu))$ then, we set $J^1 = J^1 \backslash \{j_0\}$ where j_0 is such that $\Delta 1(j_0) = \max_{j \in J^1 \backslash J^*} \Delta 1(j)$.

The following result proves that this procedure gives us an optimal solution of $LR1(\lambda, \mu)$.

Proposition 3.3

$$v(LR1(\lambda, \mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda, \mu)) + \sum_{j \in J^1} \Delta 1(j) + \sum_{j \in J^T} \Delta T(j).$$

Proof. Once Problem $LR1_j(\lambda, \mu)$ has been solved for each $j = 1, \cdots, m$, four different cases may occur:

1. The constraints (3.5') are fulfilled. In this case $J^1 = \emptyset$, $J^T = \emptyset$. Therefore, it is obvious by the decomposition of the problem into the m independent subproblems $LR1_j(\lambda, \mu)$, that:

$$v(LR1(\lambda, \mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda, \mu)).$$

2. The number of warehouses open at $t = 1$ is less than ND^1 and the number of warehouses open at T is greater than or equal to ND^T. In this case, $J^T = \emptyset$ and the optimal solution of $LR1_j(\lambda, \mu)$ is obtained by opening at $t = 1$ the

number of required warehouses with the smallest increment in the objective function. This increment is given by $\Delta 1(j)$. Therefore,

$$v(LR1(\lambda,\mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda,\mu)) + \sum_{j\in J^1} \Delta 1(j).$$

3. The number of warehouses open at $t = 1$ is greater than or equal to ND^1 and the number of warehouses open at T is less than ND^T. In this case, $J^1 = \emptyset$ and we have to fulfill the requirement on ND^T. This can be done with those warehouses belonging to J^* and J_c closed before T.

   If $j \in J^*$, which means that this warehouse is never open, we have, $v(LR1_j(\lambda,\mu))$ $= 0$ and $v(LR1_{jt_0}(\lambda,\mu)) \geq 0$ for all $t_0 = 1,\cdots,T$. Then, if the warehouse j would have been opened, the smallest increment for the objective function would have been given by $\Delta T(j)$.

   On the other hand, let's assume that $j \in J_c$ was closed before the time period T. If it would not have been closed, the minimum increment for the objective function would have been $\Delta T(j)$. Then,

$$v(LR1(\lambda,\mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda,\mu)) + \sum_{j\in J^T} \Delta T(j).$$

4. The number of warehouses open at $t = 1$ (respectively at T) is less than ND^1 (respectively ND^T). We have to fulfill the constraints on ND^1 (respectively ND^T) in such a way that the increment of the objective function is minimum.

   We start by opening the warehouses j with minimum $\Delta 1(j)$ at $t = 1$ until the constraint on ND^1 is fulfilled. For those $j \in J^* \cap J^1$, this implies that they are open at T as well. Once the constraint on ND^1 is fulfilled, we proceed in a similar way, but instead, using $\Delta T(j)$ to fulfill the constraint on ND^T. This process produces the smallest increment in the objective function in such a way that the constraints on ND^1, ND^T are fulfilled (except in the case that $j \in J^* \cap J^T$ exists).

In the case of $j \in J^* \cap J^T$, to fulfill the constraint on ND^T we choose a warehouse $j \in J^*$ and the minimum increment is given by opening it at $t = 1$. In this situation we can close one warehouse which belongs to $J^1 \setminus J^*$ reducing the objective function and the constraint on ND^1 is still fulfilled. The maximal reduction is given by that $j \in J^1 \setminus J^*$ with the maximum $\Delta 1(j)$ value. Using this policy we obtain at the end an optimal solution of $LR1(\lambda, \mu)$ and its value is:

$$v(LR1(\lambda,\mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda,\mu)) + \sum_{j \in J^1} \Delta 1(j) + \sum_{j \in J^T} \Delta T(j). \qquad \square$$

Analysis of $LR2(\lambda, \mu)$

In order to solve $LR2(\lambda, \mu)$, we use the same strategy as for $LR1(\lambda, \mu)$. Once (3.6') is removed, we separate $LR2(\lambda, \mu)$, into p subproblems $LR2_k(\lambda, \mu)$, one for each $k = 1, \cdots, p$. Thus, we solve the problem for each plant. These p subproblems will be separated into T subproblems, $LR2_{kt_0}(\lambda, \mu)$ for each $t_0 \in \{1, \cdots, T\}$. Then, we solve each one of them as we did for $LR1(\lambda, \mu)$.

Proposition 3.4

1. $v(LR2_k(\lambda,\mu)) = \min\left\{\min_{1 \le t_0 \le T} v(LR2_{kt_0}(\lambda,\mu)),\ 0\right\} \quad \forall\, k \in K_o.$

2. $v(LR2_k(\lambda,\mu)) = \min_{1 \le t_0 \le T} v(LR2_{kt_0}(\lambda,\mu)) \quad \forall\, k \in K_c.$

3. $v(LR2(\lambda,\mu)) = \sum_{k=1}^{p} v(LR2_k(\lambda,\mu)) + \sum_{k \in K^1} \Delta' 1(k) + \sum_{k \in K^T} \Delta' T(k)$, *where* K^1, K^T, $\Delta' 1(k)$ *and* $\Delta' T(k)$ *are defined in the same way as* J^1, J^T, $\Delta 1(j)$ *and* $\Delta T(j)$ *respectively.*

The proof is similar to the proof of propositions 3.2 and 3.3, once one substitutes warehouses by plants. Therefore, the proof is left out.

Obtaining lower bounds

The solution of $LR(\lambda, \mu)$ for each set of multipliers verifies the following well-known relation. Further details can be found in the Appendix A at the end of this memory and in the paper of Fisher (1981) [39].

$$v(MMCPL') \geq v(DL) := \max_{\lambda,\mu} v(LR(\lambda, \mu)).$$

Since $v(LR(\lambda, \mu))$ is a piecewise linear, concave function, we can use a subgradient approximation scheme to get the maximum or at least a good lower bound. Nevertheless, it may happen that this solution is not feasible (i.e., it does not verify the relaxed constraints) for the $MMCPL'$ problem. Therefore, we approximate (DL) by several choices of multipliers and, using the better solution, we construct a feasible solution by means of a heuristic approach.

For each pair of fixed multipliers λ, μ the function $v(LR(\lambda, \mu))$ is a piecewise linear, concave function because it can be written as a pointwise infimum of affine-linear functions. Therefore, we can obtain the subdifferential set of $v(LR(\cdot, \cdot))$ at any point.

Let $X^*(\lambda, \mu)$ be the whole set of extreme optimal solutions of $LR(\lambda, \mu)$, and denote by $e(\lambda, \mu)$ any of its elements. That is to say, $e(\lambda, \mu) = (x, y, z, \zeta) \in X^*(\lambda, \mu)$. Thus, we can write,

$$v(LR(\lambda, \mu)) = f_{\lambda,\mu}(e(\lambda, \mu)) \qquad \text{for any } e(\lambda, \mu) \in X^*(\lambda, \mu)$$

where $f_{\lambda,\mu}$ was already defined as the objective function of Problem $LR(\lambda, \mu)$.

Then, a subgradient of the function $f_{\lambda,\mu}$ at λ, μ is given by

$$\partial f_{\lambda,\mu}(e(\lambda, \mu)) = \begin{bmatrix} \sum_i d_{il}^t x_{ijl}^t - \sum_k W_j^t y_{jkl}^t \\ 1 - \sum_j x_{ijl}^t \end{bmatrix} \qquad \text{for any} \quad t,\ i,\ j,\ l.$$

We use the subgradient method (see Appendix A for the description of the subgradient method) to get a lower bound for $MMCPL'$. The selection of the initial set of multipliers is crucial because the quality of the first solution depends very much on this choice. It should be noted that for an appropriate choice of multipliers λ_{jl}^t

and μ_{il}^t, the solution of $LR(\lambda, \mu)$ must be close to a feasible solution. Otherwise, some of the constraints (3.1) or (3.3) would be violated and the corresponding term in the objective function would obtain worse values. For this reason, we propose the following set of initial multipliers:

1. Subproblem $LR2(\lambda, \mu)$.

$$\lambda_{jl}^t = \max_k b_{jkl}^t \quad \text{for all } j,\ l,\ t.$$

   Once we know λ_{jl}^t, we describe the multipliers for $LR1(\lambda, \mu)$.

2. Subproblem $LR1(\lambda, \mu)$.

$$\mu_{il}^t = \max_j (c_{ijl}^t + \lambda_{jl}^t) d_{il}^t \quad \text{for all } i,\ l,\ t.$$

In addition, as Barros and Labbé suggest in [6], these results can be improved if the region of variation of the multipliers is reduced. In what follows, we apply this technique to our problem.

Let (LP) denote the continuous relaxation of the $MMCPL'$ problem and (DLP) the continuous dual of (LP). Then, the mathematical formulation of Problem (DLP) is:

$$(DLP) \qquad \max \quad \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{l=1}^{q} \mu_{il}^t + \sum_{j \in J_c} \gamma_j - \sum_{j \in J_o} \gamma_j + ND^1\phi^1 + ND^T\phi^T$$
$$+ \sum_{k \in K_c} \delta_k - \sum_{k \in K_o} \delta_k + NC^1\psi^1 + NC^T\psi^T$$

s.t.:

$$\mu_{il}^t - d_{il}^t\alpha_j^t - d_{il}^t\lambda_{jl}^t \leq c_{ijl}^t d_{il}^t \qquad \forall i,\, j,\, l,\, t \tag{3.13}$$

$$W_j^t\lambda_{jl}^t - W_j^t\beta_k^t \leq W_j^t b_{jkl}^t \qquad \forall j,\, k,\, l,\, t \tag{3.14}$$

$$\left(\sum_{r=1}^{T} W_j^r\right)\alpha_j^1 - \gamma_j + \phi^T + \phi^1 \leq F_j^1 \qquad \forall j \in J_o \tag{3.15}$$

$$\left(\sum_{r=t}^{T} W_j^r\right)\alpha_j^t - \gamma_j + \phi^T \leq F_j^t \qquad \forall j \in J_o, \quad t = 2, \cdots, T \tag{3.16}$$

$$\left(\sum_{r=1}^{T} W_j^r\right)\alpha_j^T + \gamma_j + \phi^1 + \phi^T \leq F_j^T \qquad \forall j \in J_c \tag{3.17}$$

$$\left(\sum_{r=1}^{t} W_j^r\right)\alpha_j^t - \gamma_j + \phi^1 \leq F_j^t \qquad \forall j \in J_c, \quad t = 1, \cdots, T-1 \tag{3.18}$$

$$\left(\sum_{r=1}^{T} C_k^r\right)\beta_k^1 - \delta_k + \psi^T + \psi^1 \leq G_k^1 \qquad \forall k \in K_o \tag{3.19}$$

$$\left(\sum_{r=t}^{T} C_k^r\right)\beta_k^t - \delta_k + \psi^T \leq G_k^t \qquad \forall k \in K_o, \quad t = 2, \cdots, T \tag{3.20}$$

$$\left(\sum_{r=1}^{T} C_k^r\right)\beta_k^T + \delta_k + \psi^1 + \psi^T \leq G_k^T \qquad \forall k \in K_c \tag{3.21}$$

$$\left(\sum_{r=1}^{t} C_k^r\right)\beta_k^t - \delta_k + \psi^1 \leq G_k^t \qquad \forall k \in K_c, \quad t = 1, \cdots, T-1 \tag{3.22}$$

$$\gamma_j \text{ unrestricted } \forall\, j \in J_c; \qquad \gamma_j \geq 0\, \forall\, j \in J_o \tag{3.23}$$

$$\delta_k \text{ unrestricted } \forall\, k \in K_c; \qquad \delta_k \geq 0\, \forall\, k \in K_o \tag{3.24}$$

$$\mu_{il}^t,\, \alpha_j^t,\, \lambda_{jl}^t,\, \beta_k^t \geq 0 \qquad \forall i, \forall j, \forall k, \forall l, \forall t \tag{3.25}$$

$$\phi^1,\, \phi^T,\, \psi^1,\, \psi^T \geq 0 \tag{3.26}$$

The feasible region of (DLP) is used to obtain bounds on the range of variation of multipliers (λ, μ).

Lemma 3.1 *A reduced feasible region for the multipliers* (λ, μ) *is:*

$$0 \leq \lambda_{jl}^t \leq \min_k \left\{ b_{jkl}^t + \frac{G_k^t}{\sum_{r \in \overline{T_{kt}}} C_k^r} \right\} \qquad \forall j, \forall l, \forall t.$$

$$\min_j\{(c^t_{ijl}+\lambda^t_{jl})d^t_{il}\} \le \mu^t_{il} \le \min_j\left\{(c^t_{ijl}+\frac{F^t_j}{\sum\limits_{r\in\overline{T_{jt}}} W^r_j}+\lambda^t_{jl})d^t_{il}\right\} \quad \forall i, \forall l, \forall t.$$

Proof. The constraints of (DLP) allow us to establish a reduced feasible region for the dual variables.

$$\mu^t_{il} \le (c^t_{ijl}+\alpha^t_j+\lambda^t_{jl})d^t_{il} \qquad \forall i, \forall j, \forall l, \forall t \tag{3.27}$$

$$\lambda^t_{jl} \le b^t_{jkl}+\beta^t_k \qquad \forall j, \forall k, \forall l, \forall t \tag{3.28}$$

In addition, it holds,

$$\text{if } j\in J_o \implies \alpha^t_j \le \frac{F^t_j}{\sum\limits_{r=t}^{T} W^r_j}+\frac{\gamma_j}{\sum\limits_{r=t}^{T} W^r_j} \quad \text{with } \gamma_j \ge 0$$

$$\text{if } j\in J_c \implies \begin{cases} \alpha^t_j \le \dfrac{F^t_j}{\sum\limits_{r=1}^{t} W^r_j}-\dfrac{\gamma_j}{\sum\limits_{r=1}^{t} W^r_j} & \text{if } \gamma_j \le 0 \\ \alpha^t_j \le \dfrac{F^t_j}{\sum\limits_{r=1}^{t} W^r_j} & \text{if } \gamma_j \ge 0 \end{cases}$$

$$\text{if } k\in K_o \implies \beta^t_k \le \frac{G^t_k}{\sum\limits_{r=t}^{T} C^r_k}+\frac{\delta_k}{\sum\limits_{r=t}^{T} C^r_k} \quad \text{with } \delta_k \ge 0$$

$$\text{if } k\in K_c \implies \begin{cases} \beta^t_k \le \dfrac{G^t_k}{\sum\limits_{r=1}^{t} C^r_k}-\dfrac{\delta_k}{\sum\limits_{r=1}^{t} C^r_k} & \text{if } \delta_k \le 0 \\ \beta^t_k \le \dfrac{G^t_k}{\sum\limits_{r=1}^{t} C^r_k} & \text{if } \delta_k \ge 0 \end{cases}$$

Therefore, if we assume $j\in J_o$ and $k\in K_c$ with $\delta_k \le 0$, we have:

$$\mu^t_{il} \le \left(c^t_{ijl}+\frac{F^t_j}{\sum\limits_{r=t}^{T} W^r_j}+\frac{\gamma_j}{\sum\limits_{r=t}^{T} W^r_j}+b^t_{jkl}+\frac{G^t_k}{\sum\limits_{r=1}^{t} C^r_k}-\frac{\delta_k}{\sum\limits_{r=1}^{t} C^r_k}\right)d^t_{il}.$$

Then, if μ_{il}^t achieves its maximum value, the following two terms appear in the objective function of (DLP),

$$\left(\frac{d_{il}^t}{\sum_{r=t}^{T} W_j^r} - 1\right)\gamma_j \text{ and } \left(1 - \frac{d_{il}^t}{\sum_{r=1}^{t} C_k^r}\right)\delta_k \qquad \text{with } \gamma_j \geq 0,\ \delta_k \leq 0.$$

Now, since we have that $\frac{d_{il}^t}{\sum_{r=t}^{T} W_j^r} < 1$, and $\frac{d_{il}^t}{\sum_{r=1}^{t} C_k^r} < 1$, we can conclude that $\gamma_j = 0$ and $\delta_k = 0$.

Analogously, if we assume that either $j \in J_o$ and $k \in K_o$ or $j \in J_c$ and $k \in K_o$ or $k \in K_c$ we obtain the same conclusion.

From (3.27) and (3.28) we obtain:

$$\begin{aligned} \mu_{il}^t &\leq \min_j \left\{ (c_{ijl}^t + \frac{F_j^t}{\sum_{r \in \overline{T_{jt}}} W_j^r} + \lambda_{jl}^t) d_{il}^t \right\} \quad \forall i, \forall l, \forall t \\ \lambda_{jl}^t &\leq \min_k \left\{ b_{jkl}^t + \frac{G_k^t}{\sum_{r \in \overline{T_{kt}}} C_k^r} \right\} \quad \forall j, \forall l, \forall t \end{aligned}$$

On the other hand, since (DLP) is a maximization problem we can use (3.13) to have,

$$\begin{aligned} \mu_{il}^t &= (c_{ijl}^t + \alpha_j^t + \lambda_{jl}^t) d_{il}^t \geq (c_{ijl}^t + \lambda_{jl}^t) d_{il}^t \\ &\geq \min_j \left\{ (c_{ijl}^t + \lambda_{jl}^t) d_{il}^t \right\} \qquad \forall i, \forall l, \forall t \end{aligned}$$

In conclusion, a reduced feasible region of multipliers can be given as follows,

$$0 \leq \lambda_{jl}^t \leq \min_k \left\{ b_{jkl}^t + \frac{G_k^t}{\sum_{r \in \overline{T_{kt}}} C_k^r} \right\} \qquad \forall j, \forall l, \forall t.$$

Then, once the value of λ_{jl}^t is known we get for μ_{il}^t the following region,

$$\min_j\{(c_{ijl}^t + \lambda_{jl}^t)d_{il}^t\} \leq \mu_{il}^t \leq \min_j \left\{ (c_{ijl}^t + \frac{F_j^t}{\sum_{r \in \overline{T_{jt}}} W_j^r} + \lambda_{jl}^t)d_{il}^t \right\} \quad \forall i, \forall l, \forall t.$$

□

3.3 Heuristic to construct a feasible solution

In the previous section, we develop an ascent procedure to generate a good solution for the relaxed problem (DL). This solution is very often infeasible for our original problem $MMCPL'$. Therefore, we must develop an alternative procedure that starting from that solution constructs a good feasible solution for $MMCPL'$.

We propose the following scheme that consists of two different steps. The first step looks for capacities in each time period t. Both for plants and warehouses. Once these capacities have been established for meeting the demand, the second step looks for the best transportation plan between plants and warehouses, and between warehouses and customers. A detailed description of this procedure is given in the following paragraph.

STEP 1.

For each time period t compute the total capacity of all the open warehouses as well as the total demand in t. Let us denote by C_t the difference between the demand and the capacity in this time period.

Arrange in non increasing sequence with respect to C_t all those time periods where the capacity of the plants is not enough to cover the demand.

For every time period t_0 arranged according to the above process assign to all the warehouses j which are closed at t_0 the index

$$I(j, t_0) = [v(LR1_{jt_0}(\lambda, \mu)) - \overline{v}(LR1_j(\lambda, \mu))] \times \left[\max\left\{ \frac{C_{t_0}}{W_j^{t_0}}, 1 \right\} \right],$$

where

$$\overline{v}(LR1_j(\lambda,\mu)) := \begin{cases} v(LR1_{j1}(\lambda,\mu)) & \text{if } j \in J^1 \\ \Delta T(j) + v(LR1_j(\lambda,\mu)) & \text{if } j \in J^T \\ v(LR1_j(\lambda,\mu)) & \text{otherwise} \end{cases}$$

Remark. *The reason for the above ordering is that the greater C_t, the larger the number of warehouses that have to be opened and this affects the remaining time periods. Then, $I(j,t_0)$ gives us a cost index of the cost which reflects the fact that warehouse j is opened at the time period t_0 rather than in the time period where it is currently open. To see this interpretation, just consider that $v(LR1_{jt_0}(\lambda,\mu)) - \overline{v}(LR1_j(\lambda,\mu))$ is the increment in the objective function if the warehouse j is opened at t_0 and $\max\left\{\frac{C_{t_0}}{W_j^{t_0}}, 1\right\}$ is the number of times that one should open the warehouse j to satisfy the uncovered demand.*

The process consists of opening those warehouses in non decreasing order of the index $I(\cdot,t_0)$ until the demand in that time period is fulfilled.

Once the process is finished, if there is excess capacity, one verifies whether there exists open warehouses whose capacity is less than or equal to the excess. If this happens, one should close those warehouses with greatest index among those verifying that their capacity is less than the excess of capacity of the whole process. This swapping process continues until all the open warehouses at t_0 have a capacity greater than or equal to C_{t_0}.

The same procedure has to be applied to the opening of plants. Obviously the capacity of the open plants in each time period has to be enough to satisfy the demand of the warehouses. The demand of the warehouses coincides with the demand of all the customers in a considered time period. The only difference in this step with respect to the previous one is that the index $I(k,t_0)$ is now computed based on $v(LR2_k(\lambda,\mu))$.

STEP 2.

Once the warehouses and plants open in each time period t_0 is known, we can replace the values of these binary variables in the formulation of $MMCPL'$. Therefore, $MMCPL'$ is a continuous linear program that can be easily solved.

These two steps give us a feasible solution for $MMCPL'$. In this process, we are intensively using the solution of our relaxed problems.

3.4 Computational Study

The computational results presented in this section were designed to obtain the performance of our algorithm with respect to several test problems. The considered model combining dynamic aspects with multi-echelon location problems have not been considered previously in the literature. For this reason, no comparisons with other computational tests can be reported.

The computational study has been performed in a subcomplex (virtual machine) with 6 processors and 1/2 Gb of RAM of a machine HP Exemplar SPP-1000 Series. The code has been written in C++ and uses subroutines of IMSL to solve linear programs. In addition, CPLEX 6.0 has been used to obtain exact solutions of medium sized problems by branch & bound using the parameters defined by default by the solver. The data have been generated randomly. The transportation cost c^t_{ijl} and b^t_{jkl} have been computed being proportional to the Euclidean distance among the location of final customers and warehouses, and plants and warehouses, respectively. The locations of all the facilities were uniformly distributed in the square $[1,15] \times [1,15]$. In addition, we assume that all these costs experience an increment between 10% and 25% in each time period (inflation rate,...).

The maintenance costs of warehouses and plants have been generated according to a uniform distribution $U(600, 1000)$. The demand follows a uniform distribution $U(0, 20)$. Finally, the capacity of warehouses and plants has been drawn uniformly in $U(50, 80)$.

The minimum number of plants and warehouses open at the first and the last time period depends on the difference between the total demand requested in each

	Customers	Warehouses			Plants			Products
	I	J_o	J_c	J	K_o	K_c	K	L
P1	10	4	1	5	4	1	5	2
P2	10	5	2	7	5	2	7	3
P3	20	7	3	10	7	3	10	2
P4	20	8	4	12	8	4	12	3
P5	30	10	5	15	10	5	15	2
P6	50	14	6	20	14	6	20	2
P7	75	25	15	40	25	15	40	2

Table 3.1: Description of Test Problems

time period and the average of the capacity of warehouses (respectively plants) in that time period.

Table 3.1 describes the test problems that have been solved. Planning horizons from 1 to 4 periods have been considered. For each planning horizon we have solved 7 different classes of problems. Each one of them differs by the data structure assigned to the parameters in their formulations. These problems are named **P1** to **P7**. In this table, column **Customer** denotes the number of customers (I). The column **Warehouses** includes the number of warehouses distinguishing the number of warehouses that can be open in future time periods (J_o), the number of currently open warehouses (J_c) and $J = J_o + J_c$. The column **Plants** includes the same information with respect to the number of plants. Finally, the column **Products** indicates the number of different commodities used in the problem.

Table 3.2 shows the size of each test problem for the considered planning horizons (**T=1, T=2, T=3, T=4**). The row "NV" describes the number of decision variables and the row "NC" the number of constraints for each time period in the $MMCPL'$ problem.

Finally, Table 3.3 shows the results for the considered planning horizons (**T=1, T=2, T=3, T=4**) . For each planning horizon and problem class, at least 10

		P1	P2	P3	P4	P5	P6	P7
T=1	NV	160	371	620	1176	1380	2840	9280
	NC	52	81	102	146	152	222	394
T=2	NV	320	742	1240	2352	2760	5680	18560
	NC	94	148	184	268	274	404	704
T=3	NV	480	1113	1860	3528	4140	8520	27840
	NC	134	213	264	388	394	584	1014
T=4	NV	640	1484	2480	4704	5520	11360	37120
	NC	174	278	344	508	514	764	1324

Table 3.2: Size of Test Problems

instances have been solved and the average results are reported. In this table, "H-Gap" denotes the percentage gap between the feasible solution obtained applying the heuristic and the greatest lower bound obtained in each instance between the continuous and the Lagrangean relaxation of MMCPL'. "Worst-H" denotes the worst result used to compute the average H-Gap. "N" is the number of iterations needed by the heuristic algorithm and "CPU-H" is the average time in seconds used for these iterations. "E-Gap" denotes the percentage gap with respect to the exact solution of the problem obtained using CPLEX. This gap has been obtained with respect to 10 exact solutions in each case. "Worst-E" denotes the worst result used to compute the average E-Gap, and CPU-E is the average time in seconds used by CPLEX to solve the problems. Notice that the values of "E-Gap" are not complete. The reason for the missing values is that to obtain the exact solutions CPLEX solver needs prohibitive computational times. For instance, for Problem P4 with T=4 CPLEX took around 5000 seg. of CPU while for P5 with T=4 CPLEX was even not able to obtain the exact solution in many cases. Summarizing the results shown in Table 3.3, the heuristic method that we propose to solve the multiperiod two-echelon multicommodity capacitated plant location problem provides solutions whose gaps (H-Gap) range between 0.24 to 5%. It is worth noting that these gaps are computed with respect to lower bounds of the optimal values. The variability

		P1	P2	P3	P4	P5	P6	P7
T=1	H-Gap	0.4266	0.6012	0.8028	0.3358	0.9352	0.7442	0.9630
	Worst-H	1.8420	1.9861	2.0145	1.7342	2.1467	2.0688	2.4362
	CPU-H	17	17	22	35	35	68	2889
	N	820	442	437	494	428	462	906
	E-Gap	0.3903	0.1793	0.1986	0.3199	0.3138	* (1)	*
	Worst-E	1.9517	0.8965	0.8618	1.1391	0.9232	*	*
	CPU-E	0.12	0.36	0.97	3.3	2.53	*	*
T=2	H-Gap	0.6132	0.4121	0.2468	1.0629	1.0563	1.0155	1.1598
	Worst-H	1.8234	1.5136	1.4762	2.2042	2.1968	2.4326	2.5143
	CPU-H	43	36	72	96	173	282	7650
	N	845	369	549	474	828	736	738
	E-Gap	0.1063	0.1897	0.2037	0.8160	0.9606	*	*
	Worst-E	1.0450	1.1040	1.4665	1.2432	3.1316	*	*
	CPU-E	0.28	1.29	36.2	25.41	63.71	*	*
T=3	H-Gap	2.2613	2.2821	3.7425	2.6172	3.4055	2.6157	3.623
	Worst-H	3.6531	3.7133	4.8941	3.9502	4.5167	4.0101	4.6899
	CPU-H	101	131	115	369	216	811	9124
	N	786	820	389	863	387	748	472
	E-Gap	0.6045	2.2135	2.7941	1.0798	1.7346	*	*
	Worst-E	1.8435	3.3818	3.4343	1.4833	2.6560	*	*
	CPU-E	2.35	17.54	432.46	411.66	3988	*	*
T=4	H-Gap	3.6978	4.3105	4.8207	2.3071	4.4896	2.2023	4.5081
	Worst-H	4.9105	5.9048	6.2153	3.8104	5.9109	3.6717	6.0984
	CPU-H	119	139	257	339	343	1097	14089
	N	739	425	570	371	425	489	390
	E-Gap	0.8474	1.3176	1.6553	1.0092	1.402	*	*
	Worst-E	2.7363	3.6528	2.7082	2.3836	2.5517	*	*
	CPU-E	6.94	80.67	1507	5138	6555	*	*

(1) * means that no exact optimal solutions are available.

Table 3.3: Computational Results.

in the H-Gap depends on the quality of the lower bound found in each case. In those cases where exact optimal solutions have been obtained the gaps (E-Gap) are much smaller and more stable. In these cases E-Gap ranges between 0.17 to 2.7. Finally, the CPU-E time is smaller than CPU-H for small-sized problems. However, when the size of the problem increases CPU-H is more stable and much smaller than CPU-E. These results show that our heuristic behaves acceptably well to solve this kind of problems.

3.5 Concluding remarks

The multiperiod two-echelon multicommodity capacitated plant location problem combines many features previously considered in the field of Locational Analysis which, as far as we know, have never been studied all together. Despite its difficulty, this is a natural model to formulate all those large-scale distribution models with seasonal demand.

In this chapter, we propose a heuristic method to solve this problem. Our method is based on a Lagrangean relaxation which provides solutions (possibly infeasible for the original problem) but verifying the integrality constraints. In a second step, starting with these solutions we build feasible solutions of our original problem. We report computational results which show the gaps between the solutions that we propose and lower bounds of the optimal solution and exact solutions. The values of the gaps (H-Gap, E-Gap) and the computational times shown in Table 3.3 indicate that our heuristic is acceptable to solve the multiperiod two-echelon multicommodity capacitated plant location problem.

Chapter 4

Locating anti-center lines in presence of locational constraints

The technological advance experimented by modern societies in recent years has augmented the potential dangers, pollution or other hazards that certain communities can suffer. Transportation of gas and petroleum by pipelines are examples of such potential dangers. For this reason, the design of linear routes of noxious or hazardous materials has become a topic of increasing interest in recent years, and it has an enormous impact on the designs of gas pipeline or channels carrying noxious material and also in certain problems in robotics (see Gouzenes (1984)[43]).

The effect that an accident along an obnoxious channel or pipeline has on an existing population can be considered a decreasing function of the distance. Thus, to prevent the nocive effects, it seems reasonable to provide each population or facility with a security zone around it. See Fernández et al. (1997)[38] for a similar use of security areas in a different location problem. However, despite the protection associated with each facility, a sensible optimization criterion should look for a design maximizing the minimum distance regarding the security zones. The validity of the maximin criterion for the designs of transportation plans for dangerous material has been sometimes criticized (see the chapter of Erkut & Verter in the book of Drezner (1995) for a comprehensive review [36]). These critics assume that the selected criterion has to consider the notion of risk. These reasons are perfectly valid for shipment designs where several set of routes can be simultaneously considered in order not to repeatedly expose the same population to high risk. Nevertheless, in the designs of channels or pipelines this policy cannot be applied because in this case the designs must be necessarily fixed and the risk-share criterion cannot be implemented on the same basis. Therefore, “conservative” criteria, such as the max-min, are justified.

The localization of lines on the plane has been previously studied by several authors but in most cases with attractive criteria (a survey can be found in the paper of Robert and Toussaint (1990)[90]) . The first approach was due to Mack-innon & Barber (1972)[72], later on Morris & Norback (1980)[79] looked for the line minimizing the weighted sum of the distances to a set of given points on the plane. Some generalizations of this problem were addressed by the same authors in another paper several years later (1983)[80]. More recently, Lee & Wu (1986)[69] and Agarwal & Sharir (1994)[1] developed efficient algorithms for several problems of locating lines on the plane. Recently, Schöbel (1998)[94] studies the location of least distance lines on the plane with minisum and minimax criteria.

However, if a repulsive criterion is taken into account the problem of locating lines has been hardly studied. Similarities can be found in the papers of Gopalan et al. (1990)[42], Batta & Chiu (1988)[9] or Sivakumar et al. (1993)[95] among others. These authors study the determination of routes for transporting hazardous materials on road networks. On the other hand, the location of a linear route has been studied by Drezner & Wesolowsky (1989)[29]. In this paper, they study the maximization of the minimum weighted Euclidean distance to a set of points. Finally, Hinojosa & Puerto (1999)[58] study the location of a line on the plane maximizing the minimum weighted distances with respect to a set of polygons for a particular family of polyhedral norms. These are the norms whose unit ball is inscribed in the Euclidean unit ball. This lack of studies reinforces the importance of new approaches to this kind of problems.

In this chapter, we generalize the previously considered approaches of Drezner & Wesolowsky (1989)[29] and Hinojosa & Puerto (1999)[58]. We consider a general norm to measure distances and characterize the solution line in the general case. In addition, we develop algorithms both for the Euclidean and any polyhedral distance. Polyhedral norms have been widely used in the literature of location problems, see for instance Ward and Wendell (1985)[101], Durier (1990)[32], Hamacher & Nickel (1995)[50], Nickel (1995)[81], Carrizosa & Puerto (1995)[18] among others. The use of the Euclidean norm in these problems is clear. Moreover, the use of polyhedral norms can be also justified from a practical point of view. For instance, their use applies very well assuming that the harmful impact made by the linear route (pipelines or channels carrying noxious materials) to be located is spread out by the wind which has some predominant directions which coincide with the travel directions of the norm.

The chapter is organized as follows. Section 4.1 states the models and its formulation as a mathematical program. In addition, several properties are developed to characterize the optimal solution under general hypotheses. Section 4.2 is devoted to the development of the algorithms and refinements based on bounding tests. It also includes two examples and a computational report which compares the behaviour of the different algorithms. The last section is devoted to the conclusions of this chapter.

4.1 The model and properties

Let $P_1, \cdots, P_n$ be a family of at least two disjoint bounded polygons on the plane. Denote by $v_1^k, \cdots, v_{r_k}^k$ the vertices of P_k $\forall\, k = 1, \cdots, n$, whose coordinates are $v_i^k = (x_{v_i^k}, y_{v_i^k})$ and by w_k the weight associated with the polygon P_k or the vertex v_i^k depending of the case. Consider for any $X, Y \in I\!R^2$ the distance $d_B(X,Y) = \gamma_B(X-Y)$ where $\gamma_B(X)$ is a norm defined by

$$\gamma_B(X) := \inf\{|\lambda| \ : \ X \in \lambda B\}$$

being B a symmetrical with respect to the origin, convex compact set in the plane the interior of which contains the origin. Finally, given any set S denote by $extS$ the set of extreme points of S and by $int(S)$ the interior of S.

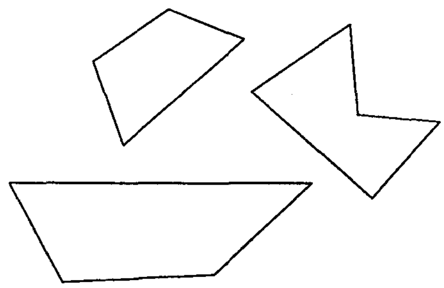


Figure 4.1: A problem without optimal solution

With this notation the problem we deal with consists of finding a line π inside the convex hull C of the given family of polygons P_k $k = 1, \cdots, n$ maximizing the minimum weighted distance from the polygons to the line. Notice that the problem considered in this chapter is non convex because the objective function is not convex. Mathematically the problem is formulated as

$$(MAXLIN) \qquad \max_{\pi} \left(\min_{1 \leq k \leq n} w_k \mathrm{d}_B(P_k, \pi) \right)$$

where $\mathrm{d}_B(P_k, \pi)$ stands for the minimum distance from P_k to the line π.

This is a well-defined problem provided that the set of feasible lines is not empty, where a line is feasible if it does not intersect any polygon. A necessary and sufficient condition which ensures feasibility of this problem is given in the following lemma. In Figure 4.1 we show an example of a unfeasible $MAXLIN$ problem.

Lemma 4.1 *The $MAXLIN$ problem has an optimal solution if and only if there exists a partition of the family of polygons into two families having empty convex hull intersection (see Figure 4.2).*

Proof. Let C_1 and C_2 be the convex hulls of the two families considered. Then $C_1 \cap C_2 = \phi$ and therefore $\mathrm{d}(C_1, C_2) > 0$. Hence, a line π can be drawn between C_1 and C_2.

Conversely, when an optimal solution exists, the line which contains the solutions separates the family into two, each one in a different half-space, thus having empty intersection. □

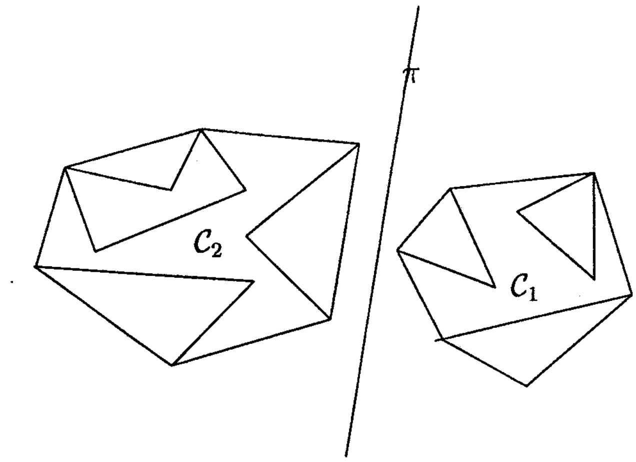


Figure 4.2: A necessary and sufficient condition

From now on, we will assume that the $MAXLIN$ problem has non-empty feasible region which can be obtained under the hypothesis of Lemma 4.1. First of all, it should be noted that there exists always a vertex that achieves the minimum distance from a line π to a polygon P_k if $P_k \cap \pi = \emptyset$. In addition, we can assume without loss of generality that P_k is a convex polygon because if P_k is not convex the vertices not belonging to its convexification can be discarded. Thus, in what follows when we refer to vertices, we will be only considering those belonging to the convexification of P_k . For the sake of readability, we will denote in some occasions and when no confusion is possible, the vertices by v_i and its associated weight by w_i with $i = 1, \cdots, N := \sum_{k=1}^{n} r_k$.

Let us denote by f^* and π^*, respectively the optimum objective value and the optimal line of the $MAXLIN$ problem, i. e.

$$f^* = f(\pi^*) = \max_{\pi} f(\pi) \tag{4.1}$$

where

$$f(\pi) = \min_{1 \leq k \leq n} w_k \mathrm{d}_B(P_k, \pi) \tag{4.2}$$

Before describing the different cases which determine an optimal solution of the $MAXLIN$ problem, we will state several technical results which will be needed later.

We denote by $\mathsf{d}_2(X,Y)$ the Euclidean distance from X to Y and by $\mathsf{d}_b(X,Y)$ the distance measure by γ_B along the direction given by $b \in extB$. This is to say

$$\mathsf{d}_b(X,Y) := \gamma_b(X-Y)$$

where

$$\gamma_b(X) := \begin{cases} |\lambda| & \text{if } X = \lambda b \quad \lambda \in I\!R \\ \infty & \text{otherwise} \end{cases}$$

Let r be a line, it is straightforward that (see Schöbel (1998)[94] for more details),

$$\mathsf{d}_B(X,r) = \min_{b \in extB} \mathsf{d}_b(X,r). \tag{4.3}$$

Lemma 4.2 *Let r be a straight line with slope m and let d_B be the distance defined by the norm γ_B. Then, there exists a constant $K(m,\gamma_B) > 0$ such that*

$$\mathsf{d}_B(X,r) = K(m,\gamma_B)\mathsf{d}_2(X,r) \qquad \forall\, X \in I\!R^2$$

A proof of this lemma is given in the paper of Schöbel (1998)[94].

This lemma shows that the constant $K(m,\gamma_B)$ depends only on the slope of r and on the norm under consideration. Therefore, once the line r and the constant value $K(m,\gamma_B)$ are known we always can obtain the distance induced by any norm using the Euclidean norm.

Lemma 4.3 *Let s_b be the slope of the straight line with direction $b \in extB$. Then,*

- *If $m \neq \infty$,*

$$K(m,\gamma_B) = \begin{cases} \min\limits_{b \in extB} \left\{ \dfrac{\sqrt{1+s_b^2}\sqrt{1+m^2}}{|m-s_b|\mathsf{d}_2(0,b)} \right\} := K_{b_0} & \text{if } s_b \neq \infty \quad \forall\, b \in extB \\ \min \left\{ K_{b_0}, \dfrac{\sqrt{1+m^2}}{\mathsf{d}_2(0,b_1)} \right\} & \text{if } s_{b_1} = \infty \text{ for some } b_1 \in extB \end{cases}$$

- *If $m = \infty$,* $\quad K(m,\gamma_B) = \min\limits_{b \in extB} \left\{ \dfrac{\sqrt{1+s_b^2}}{\mathsf{d}_2(0,b)} \right\}$

Proof. Recall that from (4.3) $\mathsf{d}_B(X,r) = \min_{b \in extB} \mathsf{d}_b(X,r)$. We can assume without loss of generality that $s_b \neq m$. Otherwise, this is to say, if $s_b = m$ then $\mathsf{d}_b(X,r) = \infty$. Therefore, the minimum distance is not achieved in that direction. Let $X = (x_1, x_2) \in I\!R^2$.

- If $m \neq \infty$, let r be the straight line with equation $\mathsf{y} = m\mathsf{x} + n$. Then we have,

$$\mathsf{d}_2(X, r) = \frac{|x_2 - mx_1 - n|}{\sqrt{1+m^2}}$$

Let r_b be the line with equation $\mathsf{y} - x_2 = s_b(\mathsf{x} - x_1)$. This is to say, the line passing through X with slope $s_b \neq \infty$. Let Z be the intersection between r_b and r. Therefore, the coordinates of Z are:

$$Z = (\frac{x_2 - s_b x_1 - n}{m - s_b}, \frac{mx_2 - ms_b x_1 - ns_b}{m - s_b})$$

The point Z verifies that $\mathsf{d}_b(X, r) = \dfrac{\mathsf{d}_2(X, Z)}{\mathsf{d}_2(0, b)}$. After some algebra we obtain:

$$\mathsf{d}_2(X, Z) = \frac{\sqrt{1+s_b^2}}{|m - s_b|}|x_2 - mx_1 - n| = \frac{\sqrt{1+s_b^2}}{|m - s_b|}\sqrt{1+m^2}\,\mathsf{d}_2(X, r)$$

Then, one concludes that:

$$\mathsf{d}_B(X, r) = \min_b \left\{ \frac{\sqrt{1+s_b^2}\sqrt{1+m^2}}{|m - s_b|\mathsf{d}_2(0, b)} \right\} \mathsf{d}_2(X, r) \quad \text{if } s_b \neq \infty \quad \forall b \in extB$$

If there exists b_1 with $s_{b_1} = \infty$ then r_{b_1} is given by $\mathsf{x} = x_1$ and $Z = (x_1, mx_1 + n)$. Therefore,

$$\mathsf{d}_{b_1}(X, r) = \frac{\mathsf{d}_2(X, Z)}{\mathsf{d}_2(0, b_1)} = \frac{|x_2 - mx_1 - n|}{\mathsf{d}_2(0, b_1)} = \frac{\sqrt{1+m^2}}{\mathsf{d}_2(0, b_1)}\mathsf{d}_2(X, r).$$

- If $m = \infty$, let us denote by r the line $\mathsf{x} = a$. Then,

$$\mathsf{d}_2(X, r) = |x_1 - a|, \quad \text{and} \quad Z = (a, s_b(a - x_1) + x_2)$$

Thus,

$$\mathsf{d}_b(X, r) = \frac{\mathsf{d}_2(X, Z)}{\mathsf{d}_2(0, b)} = \frac{\sqrt{1+s_b^2}|x_1 - a|}{\mathsf{d}_2(0, b)} = \frac{\sqrt{1+s_b^2}}{\mathsf{d}_2(0, b)}\mathsf{d}_2(X, r)$$

and one concludes that:

$$\mathsf{d}_B(X, r) = \min_b \left\{ \frac{\sqrt{1+s_b^2}}{\mathsf{d}_2(0, b)} \right\} \mathsf{d}_2(X, r)$$

□

Lemma 4.4 *Let* r *be a line and* X, Y *be two points in* $\mathbb{R}^2$ *then*

$$w_X \mathsf{d}_B(X,r) \ \mathcal{R} \ w_Y \mathsf{d}_B(Y,r) \text{ iff } w_X \mathsf{d}_2(X,r) \ \mathcal{R} \ w_Y \mathsf{d}_2(Y,r)$$

where $\mathcal{R} \in \{=,>,<\}$.

The proof is a direct consequence of Lemma 4.2. □

Definition 4.1 *The weighted midpoint of* $X = (x_1, x_2)$, $Y = (y_1, y_2)$ *is the point* m_{XY} *whose coordinates are:*

$$\left(\frac{w_X x_1 + w_Y y_1}{w_X + w_Y}, \frac{w_X x_2 + w_Y y_2}{w_X + w_Y} \right) \tag{4.4}$$

Lemma 4.5 *The weighted midpoint belongs to the segment* $[X,Y]$ *and verifies that*

$$w_X \mathsf{d}_2(X, m_{XY}) = w_Y \mathsf{d}_2(Y, m_{XY}).$$

Proof. It is straightforward from the definition of the midpoint.

Lemma 4.6 *Let* r *be a line which separates two points* X *and* Y, *then:*

$$w_X \mathsf{d}_B(X,r) = w_Y \mathsf{d}_B(Y,r) \text{ if and only if } r \text{ passes through } m_{XY}.$$

Proof. Suppose that $w_X \mathsf{d}_B(X,r) = w_Y \mathsf{d}_B(Y,r)$. Let X_0 be the intersection point between r and the segment $[X,Y]$. Applying Lemma 4.4 we have that

$$w_X \mathsf{d}_2(X,r) = w_Y \mathsf{d}_2(Y,r)$$

Besides, by similarity between triangles the following equation holds

$$\frac{\mathsf{d}_2(X,X_0)}{\mathsf{d}_2(X,r)} = \frac{\mathsf{d}_2(Y,X_0)}{\mathsf{d}_2(Y,r)}$$

Multiplying and dividing the left- and right-hand sides by w_X and by w_Y respectively, we have:

$$w_X \mathsf{d}_2(X,X_0) = w_Y \mathsf{d}_2(Y,X_0)$$

which by definition gives us that $X_0 = m_{XY}$.

Conversely if r passes through m_{XY}, by similarity between triangles the following equation holds

$$\frac{d_2(X, m_{XY})}{d_2(X, r)} = \frac{d_2(Y, m_{XY})}{d_2(Y, r)}$$

Multiplying and dividing the left- and right-hand sides by w_X and by w_Y respectively, we have:

$$w_X d_2(X, r) = w_Y d_2(Y, r)$$

Then, by Lemma 4.4 $w_X d_B(X, r) = w_Y d_B(Y, r)$. □

The following results establish how to get an optimal line π^* of the $MAXLIN$ problem for any distance d_B induced by a norm γ_B.

Lemma 4.7 *There exist at least two vertices v_i, v_j belonging to different polygons so that they are separated by π^* and they are at the same weighted distance to π^*.*

Proof. The proof easily follows taking into account that in other case it would be possible to improve the objective value doing a translation or rotation of π^* towards the farthest vertex which contradicts the optimality of π^*. □

Theorem 4.1 *Let v_i, v_j be two vertices belonging to different polygons. Any optimal line is either 1) determined by the tangency points of the two balls of γ_B centered at v_i and v_j with radii $d_B(v_i, m_{v_i v_j})$ and $d_B(v_j, m_{v_i v_j})$ respectively; or 2) it is at the same weighted distance to three of the vertices.*

Proof. First of all, remark that if P belongs to the segment $[v_i, v_j]$ then the two balls of γ_B centered at v_i and v_j with radii $d_B(v_i, P)$ and $d_B(v_j, P)$ respectively are tangent at least in P. This is true, because B is symmetrical with respect to its center.

Let π^* be an optimal line. By Lemma 4.7 there exists at least two vertices, namely v_i, v_j, belonging to different polygons separated by π^* so that $w_i d_B(v_i, \pi^*) = w_j d_B(v_j, \pi^*)$, thus, by Lemma 4.6 π^* has to pass through $m_{v_i v_j}$. Let $\Pi_{m_{v_i v_j}}$ be the set of lines which pass through $m_{v_i v_j}$. For any $\pi \in \Pi_{m_{v_i v_j}}$ we have:

$$d_B(v_l, \pi) \leq d_B(v_l, m_{v_i v_j}) \qquad \text{for } l = i, j.$$

Then, the maximum objective function value can be obtained for any line π belonging to $\Pi_{m_{v_i v_j}}$ which verifies $d_B(v_l, \pi) = d_B(v_l, m_{v_i v_j})$ for $l = i, j$, that is to say, for any line π tangent to the balls of γ_B centered at v_l with radii $d_B(v_l, m_{v_i v_j})$ for $l = i, j$.

Therefore, if $w_k d_B(v_k, \pi) \geq w_l d_B(v_l, \pi)$, $l = i, j$ for any vertex v_k the optimal line is determined by the tangency points of the two balls above mentioned. This concludes the proof for the first case.

On the other hand, if there exists a vertex v_k (suppose without loss of the generality that v_k and v_i belong to the different halfspaces with respect to π) so that $w_k d_B(v_k, \pi) < w_i d_B(v_i, \pi)$ π is not an optimal line because it would be possible to improve the objective function value rotating π on $m_{v_i v_j}$ towards v_i until $w_k d_B(v_k, \pi) = w_i d_B(v_i, \pi)$. After that, π is also at the same weighted distance to v_k and v_i and then, π is at the same weighted distance to the three vertices v_i, v_j and v_k. This concludes the proof because this is precisely the second case considered.
□

Remark 4.1 *The line defined in the second case of Theorem 4.1 is at the same weighted distance to the three vertices v_i, v_j and v_k, then, by Lemma 4.6 it has to pass through $m_{v_i v_j}$ and through $m_{v_i v_k}$. Hence, this line is unique and it does not depend on the considered norm.*

Theorem 4.1 shows how one can construct an optimal line of the $MAXLIN$ problem. We distinguish two cases: 1) those lines determined by two vertices v_i, v_j belonging to different polygons, that is to say, the lines determined by the tangency points of the two balls of γ_B centered at v_i and v_j with radii $d_B(v_i, m_{v_i v_j})$ and $d_B(v_j, m_{v_i v_j})$ respectively and 2) those lines determined by three vertices. In the later case the line does not depend on the considered norm, as it was shown in Remark 4.1.

The following results, characterize an optimal line depending on the relative position of the vertices. These results will be used in the following section to develop algorithms to solve the problem both for Euclidean and polyhedral distances.

For the Euclidean distance, we have the following property,

Corollary 4.1 . *(Case of Euclidean distance) If there exist only two vertices* v_i, v_j *belonging to different polygons separated by* π^* *such that* $w_i \mathrm{d}_2(v_i, \pi^*) = w_j \mathrm{d}_2(v_j, \pi^*)$ *then* π^* *must be orthogonal to the segment* $[v_i, v_j]$ *passing for its weighted midpoint.*

Proof. The proof is a direct consequence of the first case of Theorem 4.1. □

In the same way, if we consider polyhedral distances we have the corresponding result characterizing an optimal line π^*. Notice that in this case the line may not be unique.

Let B be a symmetrical with respect to the origin, convex polyhedron the interior of which contains the origin. Let $\{b_g : g = 0, \cdots, 2r\}$ with $b_{2r} = b_0$ and $b_{g+r} = -b_g$ be the set of extreme points of B numbered counterclockwise. For each $g = 0, \cdots, 2r-1$, C_g denotes the convex cone pointed at the origin generated by b_g and b_{g+1} and $Q_g := C_g \cup (C_{g+r})$ for each $g = 0, \cdots, r-1$. Notice that $Q_{g+r} = Q_g$ for each $g = 0, \cdots, r-1$. Let $\{f_g : g = 0, \cdots, 2r-1\}$ be the set of directions of the faces of B. We assume that $f_g := b_g - b_{g+1}$ corresponds to the face determined by the extreme points b_g and b_{g+1} . For each $g = 0, \cdots, r-1$, let us denote by C'_g the convex cone with apex at the origin defined by f_g and f_{g+1} and let $Q'_g = C'_g \cup (-C'_g)$. Notice that $Q'_{g+r} = Q'_g$ for each $g = 0, \cdots, r-1$ (see Figure 4.3 for an example of these elements in the case of hexagonal norm).

Corollary 4.2 . *(Case of Polyhedral distances) If there exist only two vertices* v_i, v_j *belonging to different polygons separated by* π^* *such that* $w_i \mathrm{d}_B(v_i, \pi^*) = w_j \mathrm{d}_B(v_j, \pi^*) = f^*$ *then only one of the following assertions holds.*

1. *If* $v_j \in v_i + int(Q_g)$ *for some* $g = 0, \cdots, r-1$, *then* π^* *must be parallel to* f_g *passing through* $m_{v_i v_j}$.

2. *If* $v_j = v_i + \lambda b_g$ *for some* $\lambda \in I\!R$ *and some* $g = 1, \cdots, r$ *and there is no polygon* P_k *such that* $w_k \mathrm{d}_B(P_k, m_{v_i v_j} + Q'_{g-1}) < f^*$ *then* π^* *can be any line passing through* $m_{v_i v_j}$ *and included in* $m_{v_i v_j} + Q'_{g-1}$.

3. *If* $v_j = v_i + \lambda b_g$ *for some* $\lambda \in I\!R$ *and some* $g = 1, \cdots, r$ *and there exists a polygon* P_k *such that* $w_k \mathrm{d}_B(P_k, m_{v_i v_j} + Q'_{g-1}) < f^*$ *then* π^* *can be any line*

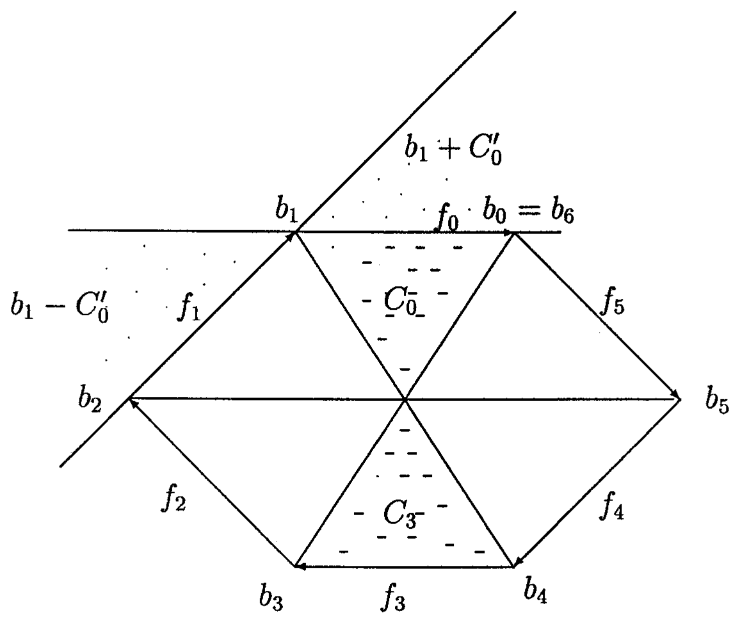


Figure 4.3: Example of Hexagonal norm

included in $m_{v_iv_j} + Q'_{g-1}$ *verifying that its weighted distance to* P_k *is greater than or equal to* f^*.

Proof.

1 and 2. The proof is a direct consequence of the first case of Theorem 4.1 (see Figure 4.4.(a) and 4.4.(b)).

3. Applying 2, if there is no polygon at a distance less than f^* from $m_{v_iv_j} + Q'_{g-1}$ any line r included in $m_{v_iv_j} + Q'_{g-1}$ is a solution line. However, since by hypothesis there is a polygon P_k such that $w_k d_B(P_k, m_{v_iv_j} + Q'_{g-1}) < f^*$ then the lines r included in $m_{v_iv_j} + Q'_{g-1}$ at a distance less than f^* from P_k cannot be optimal solutions because in this case the minimum distance is obtained from P_k (see Figure 4.4.(c)).

□

Corollaries 4.1 and 4.2 characterize the situation where there are only two vertices at the same distance of the optimal line π^*. Besides, as a consequence of Remark 4.1, if there are more than two vertices the line determined by them is unique. This remark together with Corollaries 4.1 and 4.2 show that all the cases for constructing

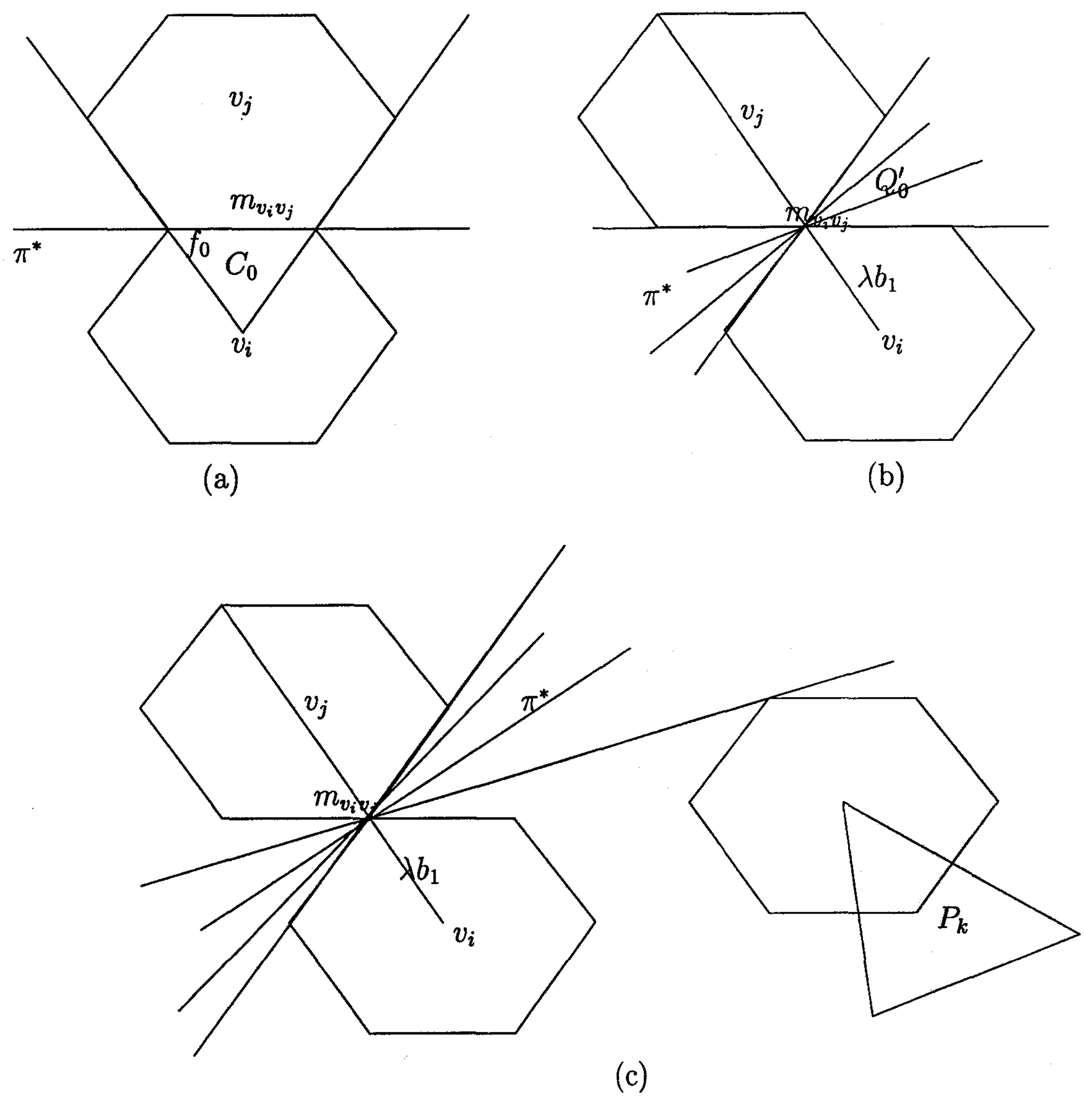


Figure 4.4: Determination of the optimal line with the hexagonal norm.

an optimal line have been explicitly considered.

4.2 The algorithms

In this section we develop algorithms for solving the $MAXLIN$ problem when the Euclidean or any polyhedral norm is used to measure distances. In order to solve the $MAXLIN$ problem for the Euclidean distance we follow a similar approach to the one given by Drezner & Wesolowsky (1989)[29] but taking into account that in our case the line that we are looking for cannot intersect any polygon. In addition, we develop a different algorithm for polyhedral norms which makes use of the relationships stated by Lemmas 4.2, 4.4 and Remark 4.1. The final part of this section is devoted to develop improvements of the basic procedures of the algorithm making use of several bounding rules. Two examples and a report with computational results are also included.

For the sake of readability we will describe some statements of the algorithms whose are actually procedures (for instance, weighted Euclidean distance from the vertex v_i to the line π or test the intersection between π and P_k). We only mention them as simple instructions in the algorithms.

In this description the following notation is used:

Notation

- t, the orthogonal line to the particular solution line π which goes by the origin.
- θ, the angle between the line t and the OX axis.
- $\pi(\theta)$, the orthogonal projection of π onto t.
- $v_i(\theta)$, the orthogonal projection of v_i onto t, i.e.,

$$v_i(\theta) = x_{v_i} \cos\theta + y_{v_i} \operatorname{sen}\theta$$

- $m_{v_iv_j}(\theta)$, the orthogonal projection of the weighted midpoint of v_i, v_j onto t, i.e.,

$$m_{v_iv_j}(\theta) = \frac{w_iv_i(\theta) + w_jv_j(\theta)}{w_i + w_j}$$

It is obvious that we need to know the angle θ to calculate $\pi(\theta)$, $v_i(\theta)$ and $m_{v_iv_j}(\theta)$. The following result determines the angle θ for the different cases.

Lemma 4.8

1. *Under the hypotheses of Corollary 4.1, let $v_i = (x_{v_i}, y_{v_i})$, $v_j = (x_{v_j}, y_{v_j})$ be the two vertices which determine π. Assuming that $x_{v_i} \leq x_{v_j}$ the angle θ is given by:*

$$\theta = \begin{cases} \theta_{v_iv_j} = \arccos\left(\dfrac{x_{v_j} - x_{v_i}}{\mathrm{d}_2(v_i, v_j)}\right) & \textit{if } y_{v_i} \leq y_{v_j} \\ \theta = -\theta_{v_iv_j} & \textit{otherwise} \end{cases}$$

2. *Under the hypotheses of Corollary 4.2, let v_i, v_j be the two vertices which determine π. The angle θ is given either by $\frac{\pi}{2} + \alpha_g$ where $\alpha_g \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the angle that f_g forms with the OX axis or by an angle included in the orthogonal cone to Q'_{g-1} (which corresponds according to cases 2. and 3. of Corollary 4.2).*

3. *Let v_i, v_j, v_k be the three vertices which determine π. If π separates the three vertices and v_i and v_j belong to the same halfspace with respect to π then, the angle θ is given by*

$$\theta = \arctan\left(-\frac{w_iw_j(x_{v_i} - x_{v_j}) + w_iw_k(x_{v_i} - x_{v_k}) + w_kw_j(x_{v_k} - x_{v_j})}{w_iw_j(y_{v_i} - y_{v_j}) + w_iw_k(y_{v_i} - y_{v_k}) + w_kw_j(y_{v_k} - y_{v_j})}\right).$$

Proof. The proofs for 1. and 2. are a direct consequences of Corollary 4.1 and Corollary 4.2 respectively. Then, we will prove only the third case:
Under the hypotheses of the third case the following equations hold:

$$w_i[\pi(\theta) - x_{v_i}\cos\theta - y_{v_i}\operatorname{sen}\theta] = -w_k[\pi(\theta) - x_{v_k}\cos\theta - y_{v_k}\operatorname{sen}\theta] \quad (4.5)$$

$$w_j[\pi(\theta) - x_{v_j}\cos\theta - y_{v_j}\operatorname{sen}\theta] = -w_k[\pi(\theta) - x_{v_k}\cos\theta - y_{v_k}\operatorname{sen}\theta] \quad (4.6)$$

Isolating $\pi(\theta)$ in (4.5), we have

$$\pi(\theta) = \frac{(w_i x_{v_i} + w_k x_{v_k})\cos\theta + (w_i y_{v_i} + w_k y_{v_k})\operatorname{sen}\theta}{w_i + w_k} \tag{4.7}$$

and a few algebra yields

$$\tan\theta = -\frac{w_i w_j (x_{v_i} - x_{v_j}) + w_i w_k (x_{v_i} - x_{v_k}) + w_k w_j (x_{v_k} - x_{v_j})}{w_i w_j (y_{v_i} - y_{v_j}) + w_i w_k (y_{v_i} - y_{v_k}) + w_k w_j (y_{v_k} - y_{v_j})} \tag{4.8}$$

□

1. Weighted Euclidean distance from the vertex v_s to the line π: Once θ is obtained for the diverse cases, $\pi(\theta)$ is given by $\pi(\theta) = m_{v_i,v_j}(\theta)$ where v_i and v_j belong to different halfspaces with respect to π .
   The weighted Euclidean distance from any vertex v_s to π is $w_s \mathsf{d}_2(v_s, \pi) = |w_s(v_s(\theta) - \pi(\theta))|$.

2. Test on the intersection between the line π and the polygon P_k.

   2.1 Obtain $w_k(v_i^k(\theta) - \pi(\theta))$ for all $v_i^k \in P_k$.

   2.2 If $sig(w_k(v_i^k(\theta) - \pi(\theta))) \neq sig(w_k(v_j^k(\theta) - \pi(\theta)))$ for some $i \neq j$ then the line π intersects the polygon P_k.
   Otherwise π and P_k do not intersect.

In what follows we develop exact algorithms to solve the $MAXLIN$ problem. We distinguish two cases corresponding to the different distances considered, namely the Euclidean or the polyhedral norms.

The proposed algorithms consist of two steps. Step 1 determines the candidate line when only two vertices are considered at the same time, while Step 2 selects the best candidate line considering all the triplets of vertices. Finally, the maximum objective value between the ones obtained in Steps 1 and 2 is selected.

We denote by E_r^1 and E_r^2 the two halfspaces defined by the line r and by $l_{x_1 x_2}$ the line containing the segment $[x_1, x_2]$.

Algorithm 4.1 .(Case of Euclidean distance)

STEP 1. (Lines determined only by two vertices).

For each pair of vertices v_i^p, v_j^s with $p \neq s$ do:

- Consider the candidate line π determined by the pair v_i^p, v_j^s and compute the weighted Euclidean distance, $D := w_p \mathrm{d}_2(v_i^p, \pi)$, from v_i^p to π.
- If the weighted Euclidean distance from π to each polygon is greater than or equal to D, let $f(v_i^p, v_j^s) := D$.
  Otherwise let $f(v_i^p, v_j^s) := 0$.

STEP 2. (Lines determined by three vertices).

For each pair of vertices, v_i^p, v_j^s do:

- Obtain the vertex $v_k^r \in E^1_{l_{v_i^p v_j^s}}$ (respectively $v_k^r \in E^2_{l_{v_i^p v_j^s}}$), $r \neq p, s$ which is at the minimum weighted Euclidean distance from $l_{v_i^p v_j^s}$.
- Obtain the line π_1 (respectively π_2) which is at the same weighted Euclidean distance to v_i^p, v_j^s and $v_k^r \in E^1_{l_{v_i^p v_j^s}}$ (respectively $v_k^r \in E^2_{l_{v_i^p v_j^s}}$).
- Let $\pi \in \arg\max_{l=1,2} w_p \mathrm{d}_2(v_i^p, \pi_l)$.
  If π does not intersect any polygon and $w_k \mathrm{d}_2(P_k, \pi) \geq w_p \mathrm{d}_2(v_i^p, \pi)$ $k = p, s$, let $f'(v_i^p, v_j^s) := w_p \mathrm{d}_2(v_i^p, \pi)$. Otherwise let $f'(v_i^p, v_j^s) := 0$.
  If $f'(v_i^p, v_j^s) = 0$ let $\pi' \in \arg\min_{l=1,2} w_p \mathrm{d}_2(v_i^p, \pi_l)$.
  If π' does not intersect any polygon and $w_k \mathrm{d}_2(P_k, \pi') \geq w_p \mathrm{d}_2(v_i^p, \pi')$ $k = p, s$, let $f'(v_i^p, v_j^s) := w_p \mathrm{d}_2(v_i^p, \pi')$.

STEP 3. (Optimal line)

Let $f^* = \max \left\{ \max_{(v_i^p, v_j^s)} \left(f(v_i^p, v_j^s) \right), \max_{(v_i^p, v_j^s)} \left(f'(v_i^p, v_j^s) \right) \right\}$.

If $f^* = 0$ then the problem is not feasible. Otherwise the solution line π^* is determined by the vertices which give the maximum value of f^*.

The algorithm to solve the $MAXLIN$ problem using polyhedral distances is in essence similar to the previous one. The main difference consists of determining the solution when it depends on two vertices v_i^p, v_j^s such that $v_i^p = v_j^s + \lambda b_g$ for some $\lambda \in \mathbb{R}$ and some $b_g \in extB$. That is to say, v_i^p, v_j^s are collinear on a fundamental direction of the norm γ_B. It is due to the fact that in these cases it may exist a cone whose lines are all at the same distance from these two vertices. Since all this process is done in *STEP 1* of the following algorithm, Algorithm 4.1 is valid as soon as the corresponding modifications on *STEP 1* are taken into account.

For this reason, the algorithm for the case of polyhedral distance only needs to specify the new *STEP 1*. In addition, by Lemmas 4.2 and 4.3 given any line π there exists a constant $K(\gamma_B, \pi)$ such that $\mathrm{d}_B(v_i^p, \pi) = K(\gamma_B, \pi)\mathrm{d}_2(v_i^p, \pi) \quad \forall \; v_i^p \in \mathbb{R}^2$. This will be used in the algorithm for computing $f(v_i^p, v_j^s)$ and $f'(v_i^p, v_j^s)$. In *STEP 1* we distinguish two cases depending of the candidate line π. If the two considered vertices are not collinear on fundamental directions then, π is unique, whereas if they are collinear on fundamental directions then, π is not unique and it belongs to a cone.

To simplify the presentation we introduce some additional notation that we will use in the following algorithm.

Let $f(v_i, v_j) = \max_{\pi} \min\{\mathrm{d}_B(\pi, v_i), \mathrm{d}_B(\pi, v_j)\}$. Let $I(v_i, v_j, P_k)$ be the set of the slopes of those lines belonging to $m_{v_i v_j} + Q'_{g-1}$ (where g is given by the equation $v_j = v_i + \lambda b_g$) whose distances to the polygon P_k are greater than or equal to $f(v_i, v_j)$ and let $I(v_i, v_j)$ be the whole set of admissible slopes of the lines belonging to $m_{v_i v_j} + Q'_{g-1}$. This is to say,

$$I(v_i, v_j) = \bigcap_{k=1}^{n} I(v_i, v_j, P_k)$$

Algorithm 4.2 .(Case of Polyhedral distances)

STEP 1. (Lines determined only by two vertices).

1. (Vertices not collinear on fundamental directions).

   For each pair of vertices v_i^p, v_j^s with $p \neq s$ verifying that $v_j^s \in v_i^p + int(Q_g)$ for some $g = 0, \cdots, r-1$ do STEP 1 in Algorithm 4.1 assigning $f(v_i^p, v_j^s) := K(\gamma_B, \pi)D$ if the weighted Euclidean distance from π to each polygon is greater than or equal to D.

Otherwise, $f(v_i^p, v_j^s) := 0$.

2. (Vertices collinear on fundamental directions).

   For each pair of vertices v_i^p, v_j^s with $p \neq s$ such that $v_j^s = v_i^p + \lambda b_g$ for some $\lambda \in \mathbb{R}$ and some $g = 1, \cdots, r$ do:

   - Consider the candidate cone Q'_{g-1} and compute $D := w_p \mathrm{d}_2(v_i^p, m_{v_i v_j} + Q'_{g-1})$.
   - Determine the interval of admissible slopes, $I(v_i^p, v_j^s, P_k)$, for each polygon P_k distinguishing the following cases:

     Case 1. P_k intersects the two lines bounding the cone $m_{v_i v_j} + Q'_{g-1}$, (so that $I(v_i^p, v_j^s) = \phi$).

     Case 2. P_k only intersects one of the lines bounding the cone $m_{v_i v_j} + Q'_{g-1}$.

     Case 3. P_k is included in $m_{v_i v_j} + Q'_{g-1}$.

     Case 4. P_k does not intersect the cone $m_{v_i v_j} + Q'_{g-1}$ but $w_k \mathrm{d}_2(P_k, m_{v_i v_j} + Q'_{g-1}) < D$.
   - Let $I(v_i^p, v_j^s) = \bigcap_{k=1}^{n} I(v_i^p, v_j^s, P_k)$.

     If $I(v_i^p, v_j^s) \neq \phi$ then $f(v_i^p, v_j^s) := K(\gamma_B, \pi)D$, being π any line in the cone Q'_{g-1}.

     Otherwise $f(v_i^p, v_j^s) := 0$.

STEP 2. (Lines determined by three vertices).

Do STEP 2 of Algorithm 4.1 assigning $f'(v_i^p, v_j^s) := K(\gamma_B, \pi) w_p \mathrm{d}_2(v_i^p, \pi)$ or $f'(v_i^p, v_j^s) := K(\gamma_B, \pi') w_p \mathrm{d}_2(v_i^p, \pi')$.

STEP 3. (Optimal line)

Do STEP 3 of Algorithm 4.1.

The overall complexity of these algorithms is dominated by the different number of lines that have to be considered. Since, we consider lines determined by triplets or pairs of vertices, this complexity is $\mathcal{O}(N^3)$, where N is the total number of vertices. Then, we have to test if these lines intersect some polygon or if they are closer to another polygon which is linear on the total number N of vertices. Since, Algorithm 4.1 only performs these operations its complexity is $\mathcal{O}(N^4)$. On the other hand, Algorithm 4.2 has also to consider for each line determined by a pair of vertices the cone which contains those vertices. This operation is $\mathcal{O}(r)$, where r is the number of fundamental directions of the norm. In addition, we also must consider for each one of these lines the intersections with polygons or if they are closer to another polygon which is linear in N. Therefore, Algorithm 4.2 has a complexity of $\mathcal{O}(\max\{N^4, N^3 r\})$.

However, from a practical point of view the complexity of the algorithms is smaller because it can be improved using several reduction results. In the final part of this section we develop two improvements of the proposed algorithms for solving $MAXLIN$ which consist of 1) a bounding test and 2) a reduction on the number of vertices that have to be considered in the different cases. The behaviour of these modifications have been analyzed in a computational study, included at the end of this section. This report shows the efficiency with respect to the original algorithms.

Bounding test

The first improvement that we propose is based on the objective value of the best solution found until the last iteration. This value can be used as a lower bound for pruning some comparisons. The following modifications may be introduced in the algorithms. Let f denote the incumbent objective value of the $MAXLIN$ problem.

In the *STEP 1* of Algorithm 4.1 and 4.2:

Let $f := 0$.

If the pair v_i, v_j determines a feasible solution, i.e. $f(v_i, v_j) > 0$ then $f(v_i, v_j)$ is a lower bound for f. Thus, given another pair of vertices v'_i, v'_j, let π' be the line determined by the pair v'_i, v'_j, if $w'_i \mathrm{d}_B(v'_i, \pi') < f(v_i, v_j)$ then reject the pair (v'_i, v'_j).

Otherwise update $f := \max\{f, f(v'_i, v'_j)\}$.

In the *STEP 2* of Algorithm 4.1 and 4.2:

Start from f updated in *STEP 1.*

Let $E^1_{l_{v_iv_j}}$, $E^2_{l_{v_iv_j}}$ be the two halfspaces defined by the line that contains the segment $[v_i, v_j]$. If there exists a line r_1 (respectively r_2) determined by the vertices v_i, v_j and $v_{k_1} \in E^1_{l_{v_iv_j}}$ (respectively $v_{k_2} \in E^2_{l_{v_iv_j}}$) verifying $w_i \mathrm{d}_B(v_i, r_1) < f$ (respectively $w_i \mathrm{d}_B(v_i, r_2) < f$) then do not consider any vertex in $E^1_{l_{v_iv_j}}$ (respectively in $E^2_{l_{v_iv_j}}$).

If it holds for $E^1_{l_{v_iv_j}}$ and $E^2_{l_{v_iv_j}}$ then reject the pair v_i, v_j.

Otherwise update $f := \max\{f, f'(v_i, v_j)\}$.

With these modifications the third step of the algorithms can be removed because when the algorithms finish the Step 2, f gives the optimal objective value and the vertices which achieve this value determine all the optimal solution lines.

The second improvement of the algorithms is obtained by introducing a reduction on the number of vertices that have to be considered. This modification is based on the following lemma where we use again the relationship given in Lemma 4.4. In the following, we will denote by $E(r, X)$ the halfspace defined by the line r containing the point X.

Lemma 4.9

1. *In the application of STEP 1 of the algorithms, it holds:*
   - *(a) (Case of Euclidean distance).*
     Let $l_{m_{v_iv_j}}$ be the halfline with origin at $m_{v_iv_j}$ containing the vertex v_j then, no vertex v_k verifying $v_k \in l_{m_{v_iv_j}}$ and $w_k \mathrm{d}_2(v_k, m_{v_iv_j}) \geq w_j \mathrm{d}_2(v_j, m_{v_iv_j})$ has to be considered together with v_i.
   - *(b) (Case of Polyhedral distances).*
     - *i. If $v_j \in v_i + int(Q_g)$ and π is the parallel line to f_g passing through $m_{v_iv_j}$ then no vertex v_k verifying $v_k \in (v_i + int(Q_g)) \cap E(\pi, v_j)$ and $w_k \mathrm{d}_2(v_k, \pi) \geq w_j \mathrm{d}_2(v_j, \pi)$ has to be considered together with v_i.*
     - *ii. If $v_j = v_i + \lambda b_g$ for some $\lambda \in I\!R$ and some $g = 1, \cdots, r$ and π_{g-1}, π_g are two lines passing through $m_{v_iv_j}$ and parallel to f_{g-1} and f_g respectively then no vertex v_k verifying $v_k \in (v_i + Q_{g-1}) \cap E(\pi_{g-1}, v_j)$*

and $w_k \mathrm{d}_2(v_k, \pi_{g-1}) \geq w_j \mathrm{d}_2(v_j, \pi_{g-1})$ *or*
$v_k \in (v_i + Q_g) \cap E(\pi_g, v_j)$ *and* $w_k \mathrm{d}_2(v_k, \pi_g) \geq w_j \mathrm{d}_2(v_j, \pi_g)$ *has to be considered together with* v_i.

2. *Let* $E^1_{l_{v_i^p v_j^s}}$ *and* $E^2_{l_{v_i^p v_j^s}}$ *be the two halfspaces with respect to the line* $l_{v_i^p v_j^s}$ *containing the segment* $[v_i^p, v_j^s]$. *In the application of STEP 2 of the algorithms:*

   (a) *If there exists a vertex* v_k *belonging to the same polygon than either* v_i^p *or* v_j^s *and belonging to* $int(E^1_{l_{v_i^p v_j^s}})$ *(respectively* $int(E^2_{l_{v_i^p v_j^s}})$ *) then, it is not necessary to evaluate the pair* v_i^p, v_j^s *with* v_k^r, *for any* v_k^r *belonging to* $int(E^1_{l_{v_i^p v_j^s}})$ *(respectively* $int(E^2_{l_{v_i^p v_j^s}})$ *).*

   (b) *If there exists a polygon* P_r *such that* $P_r \cap l_{v_i^p v_j^s} \neq \emptyset$ *then it is not necessary to evaluate the pair* v_i^p, v_j^s *with* v_k^r *for any* $v_k^r \in P_r$. *In addition, in the particular case that equal weights are used, if there exists a polygon* P_r *such that* $int(P_r) \cap l_{v_i^p v_j^s} \neq \emptyset$ *then reject the pair* v_i^p, v_j^s.

Proof.

1.(a) By Corollary 4.1 the line π' determined by v_i and v_k has to be orthogonal to $l_{m_{v_i v_j}}$ (thus, parallel to π) passing through $m_{v_i v_k}$.

If $v_k \in l_{m_{v_i v_j}}$ and $w_k \mathrm{d}_2(v_k, \pi) \geq w_j \mathrm{d}_2(v_j, \pi)$ then, $m_{v_i v_k} \in l_{m_{v_i v_j}}$. Therefore, $w_j \mathrm{d}_2(v_j, \pi') \leq w_i \mathrm{d}_2(v_i, \pi') = w_k \mathrm{d}_2(v_k, \pi')$ which contradicts that π' is a solution line because v_j is (weighted) closer to π' than v_k.

1.(b).i Using a similar argument to the one in 1.(a) based now in Corollary 4.2 (case 1) we do not have to consider any vertex $v_k \in (v_i + int(Q_g)) \cap E(\pi_g, v_j)$ with $w_k \mathrm{d}_2(v_k, \pi) \geq w_j \mathrm{d}_2(v_j, \pi)$.

1.(b).ii Using the same arguments that in 1.(b).i, we do not have to consider either the vertices
$v_k \in (v_i + int(Q_g)) \cap E(\pi_g, v_j)$ with $w_k \mathrm{d}_2(v_k, \pi_g) \geq w_j \mathrm{d}_2(v_j, \pi_g)$ or
$v_k \in (v_i + int(Q_{g-1})) \cap E(\pi_{g_1}, v_j)$ with $w_k \mathrm{d}_2(v_k, \pi_{g-1}) \geq w_j \mathrm{d}_2(v_j, \pi_{g-1})$.
Finally the vertices $v_k = v_i + \lambda b_g$ for some $\lambda \in I\!R$ can be eliminated by the same reasons but using now Corollary 4.2 (case 2).

2.(a) Consider without loss of generality that $v_i^p \in P_p$ and that there exists $v_k^p \in P_p \cap int(E^1_{l_{v_i^p v_j^s}})$. Every solution line determined by v_i^p, v_j^s and a third vertex on this configuration has to pass between the segment $[v_i^p, v_j^s]$ and the third vertex. Therefore, the line must intersect P_p or it has to be (weighted) closer to v_k^p than to v_i^p. In both cases that line does not have to be considered.

2.(b) If $P_r \cap l_{v_i^p v_j^s} \neq \emptyset$, every solution line determined by the pair v_i^p, v_j^s and a third vertex $v_k^r \in P_r$ has to pass between $l_{v_i^p v_j^s}$ and the third vertex. Therefore the line intersect P_r.
In addition, in the particular case that equal weights are used, the line π determined by the pair v_i^p, v_j^s and a third vertex has to be parallel to the segment $[v_i^p, v_j^s]$. Therefore if $int(P_r) \cap l_{v_i^p v_j^s} \neq \emptyset$, π has to be (weighted) closer to P_r than to v_i^p. □

In the particular case that Euclidean distance and equal weights are used, it is possible to introduce a major reduction on the number of vertices that have to be considered in the *STEP 1* of Algorithm 4.1. This modification is based on Theorem 4.2.

For each pair of vertices v_i, v_j, let us assume an orthogonal reference system centered at the midpoint of v_i and v_j, being the axis OY the line containing the segment $[v_i, v_j]$. For the scope of Theorem 4.2, consider that the coordinates of all the points are referred to this new reference system.

Denote by $sig(x)$ the sign of $x \in \mathbb{R}$, this is to say,

$$sig(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Theorem 4.2 *If in STEP 1 of Algorithm 4.1 the pair of vertices v_i, v_j, have been evaluated then it is not necessary to evaluate the pair v_i, v_k for any vertex v_k, verifying simultaneously the following conditions.*

1. $sig(y_{v_i} - y_{v_k}) = sig(y_{v_i} - y_{v_j})$

2. *The vertex* v_k *does not belong to the circle of radius* $r := |y_{v_i}|$ *centered at the origin.*

Proof. We will prove that if the vertex v_k verifies the conditions *1* and *2* then the line π determined by v_i, v_k cannot be an optimal solution of the $MAXLIN$ problem because v_j is closer to π than both v_k and v_i .

The coordinates (in the new reference system) of v_i and v_j are $v_i = (0, y_{v_i})$, $v_j = (0, -y_{v_i})$ where, without loss of generality, we can to consider $y_{v_i} > 0$.

Under the hypotheses of Corollary 4.1 π has to be orthogonal to the line containing the segment $[v_i, v_k]$ passing for the midpoint of v_i, v_k. Thus, the equation of π is:

$$\mathsf{y} - \frac{y_{v_i} + y_{v_k}}{2} = \frac{x_{v_k}}{y_{v_i} - y_{v_k}}\left(\mathsf{x} - \frac{x_{v_k}}{2}\right)$$

By Condition *1*, we have $y_{v_i} - y_{v_k} > 0$. Then,

$$\mathsf{d}_2(v_i, \pi) = \frac{y_{v_i}^2 + y_{v_k}^2 + x_{v_k}^2 - 2y_{v_i}y_{v_k}}{2(y_{v_i} - y_{v_k})C}$$

where $C = \sqrt{1 + \left(\dfrac{x_{v_k}}{y_{v_i} - y_{v_k}}\right)^2}$. In addition,

$$\mathsf{d}_2(v_j, \pi) = \left|\frac{-3y_{v_i}^2 + y_{v_k}^2 + x_{v_k}^2 + 2y_{v_i}y_{v_k}}{2(y_{v_i} - y_{v_k})C}\right|$$

and using that $y_{v_i} - y_{v_k} > 0$ it gives,

$$\mathsf{d}_2(v_j, \pi) < \mathsf{d}_2(v_i, \pi) \Longleftrightarrow |-3y_{v_i}^2 + y_{v_k}^2 + x_{v_k}^2 + 2y_{v_i}y_{v_k}| < y_{v_i}^2 + y_{v_k}^2 + x_{v_k}^2 - 2y_{v_i}y_{v_k}$$

Now, depending on the sign of $-3y_{v_i}^2 + y_{v_k}^2 + x_{v_k}^2 + 2y_{v_i}y_{v_k}$, it yields

- If $-3y_{v_i}^2 + y_{v_k}^2 + x_{v_k}^2 + 2y_{v_i}y_{v_k} > 0$, then:

  $$\mathsf{d}_2(v_j, \pi) < \mathsf{d}_2(v_i, \pi) \Longleftrightarrow 4y_{v_i}(y_{v_i} - y_{v_k}) > 0$$

  Therefore, in this case the vertex v_j is closer to π than v_i if and only if $(y_{v_i} - y_{v_k}) > 0$ which coincides with $sig(y_{v_i} - y_{v_j})$ as Condition *1* asserts.

- If $-3y_{v_i}^2 + y_{v_k}^2 + x_{v_k}^2 + 2y_{v_i}y_{v_k} < 0$, then:

  $$\mathsf{d}_2(v_j, \pi) < \mathsf{d}_2(v_i, \pi) \Longleftrightarrow y_{v_k}^2 + x_{v_k}^2 > y_{v_i}^2$$

Hence, the vertex v_j is closer to π than v_i if and only if Condition *1* is verified and v_k is not in the circle centered at the origin with radius $r = |y_{v_i}|$. □

4.2.1 Examples

Example 4.1 *Let* P_1 *be the polygon with vertices* $v_1^1 = (2,8)$, $v_2^1 = (4,10)$, $v_3^1 = (5,8)$ *and* $v_4^1 = (2,6)$; P_2 *the polygon with vertices* $v_1^2 = (6,5)$, $v_2^2 = (8,6)$, $v_3^2 = (10,6)$ *and* $v_4^2 = (8,4)$ *and* P_3 *the triangle with vertices* $v_1^3 = (0,2)$, $v_2^3 = (2,2)$ *and* $v_3^3 = (1,0)$. *Consider that the weights are all equal to 1.*

We solve this example both for the Euclidean and rectilinear distances (see Figures 4.5 and 4.6 respectively).

1. Case of Euclidean distance:

   STEP 1 (Algorithm 4.1).

   In Table 4.1 we show the solutions that are obtained by the first step of Algorithm 4.1. The column "Pair" indicates the two vertices which are considered to construct the line. The column "Solution" describes the solution line associated with the considered pair where $m_{v_i^p v_j^s}$ stands for the midpoint of the considered pair and θ is the angle between the OX axis and the orthogonal line to the solution line. The column "D" gives the Euclidean distance from v_i^p to the solution line. Finally, the column "Eliminated pairs" includes the set of vertices v_k^r that do not have to be considered together with v_i^p or v_j^s (consequence of Theorem 4.2). By the application of this improvement we only need to consider 6 pairs out of 40.

   STEP 2 (Algorithm 4.1).

   In Table 4.2 we show the solutions given by the second step of Algorithm 4.1. The number of triplets to be considered has been reduced by the application of Lemma 4.9 (part 2).

   The column "Triplets" indicates the three vertices used to construct the solution. The column "Part. Sol." describes the line which is at the same

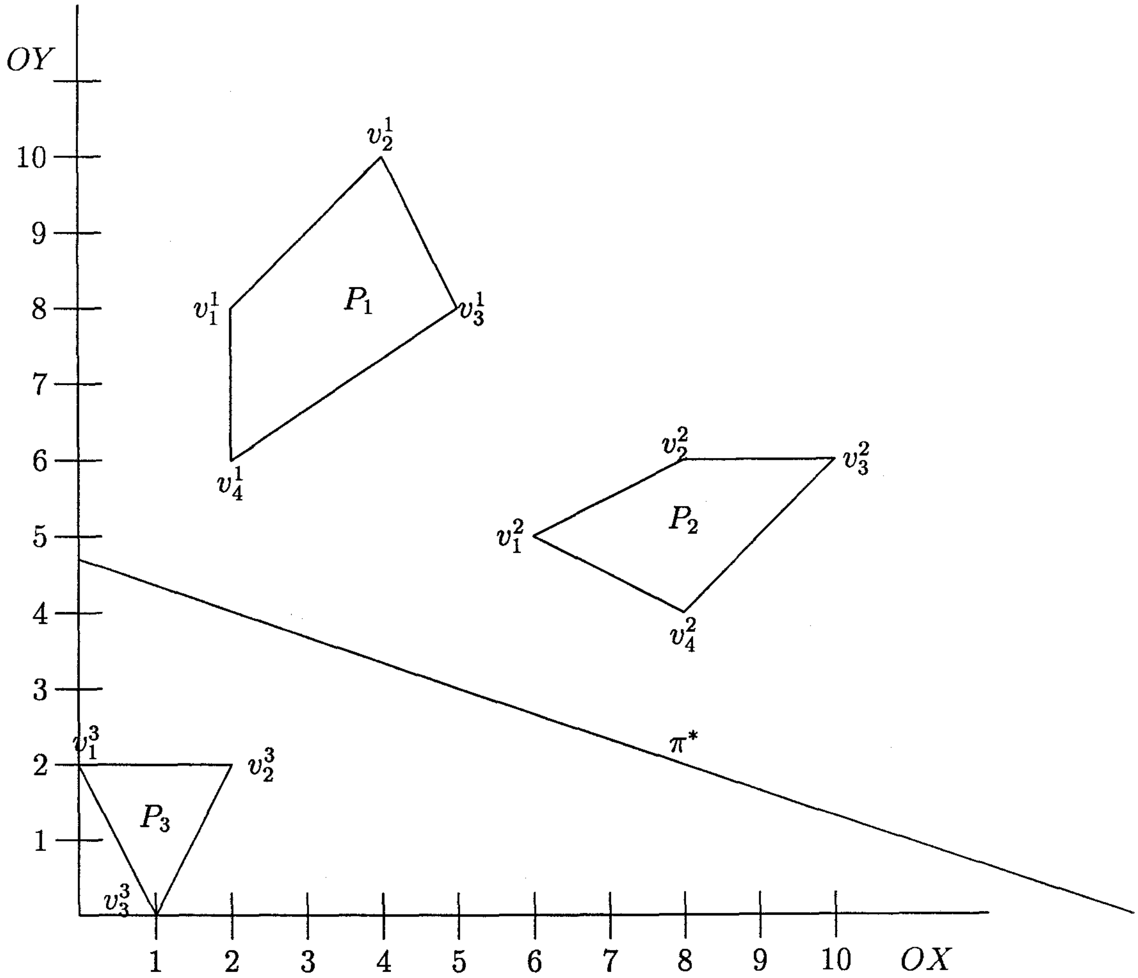


Figure 4.5: Example 4.1 with the Euclidean distance

Pair		Solution		D	$f(v_i^p, v_j^s)$	f	Eliminated pairs	
v_i^p	v_j^s	$m_{v_i^p v_j^s}$	θ					v_k^r
(2,8)	(6,5)	$(4, \frac{13}{2})$	-0.6435	2.5	0	0	(2,8)	Rest of vertices
							(6,5)	(4,10), (0,2), (2,2), (1,0)
(4,10)	(8,6)	(6,8)	-0.7854	2.8284	0	0	(4,10)	Rest of vertices
							(8,6)	(2,6), (0,2), (2,2), (1,0)
(5,8)	(6,5)	$(\frac{11}{2}, \frac{13}{2})$	-1.249	1.5811	0	0	(5,8)	Rest of vertices
							(6,5)	(2,6)
(2,6)	(10,6)	(6,6)	0	4	0	0	(10,6)	Rest of vertices
(2,6)	(8,4)	(5,5)	-0.3217	3.1623	0	0	(2,6)	(2,2), (1,0)
							(8,4)	Rest of vertices
(2,6)	(0,2)	(1,4)	1.1071	2.2361	0	0		

Table 4.1: Step 1 of Algorithm 4.1 of Example 4.1

weighted distances from the three vertices in the column triplets. The angle θ and $m_{v_i^p v_k^r}$ have been previously defined. In the third column D denotes the Euclidean distance from the three vertices to the solution line. The fourth column describes the vertices eliminated by the bounding test. The column "Solution" gives the solution line, if any, determined by the first two vertices in triplets.

Notice that once we consider the pair [(5,8),(2,2)] we do not have to consider the pairs [(5,8),(1,0)] and [(2,2),(1,0)]. This is due to the fact that (5,8),(2,2),(1,0) are collinear. For the same reasons the pairs [(6,5),(0,2)] and [(8,6),(0,2)] do not have to be considered once the pair [(6,5),(8,6)] is analyzed.

<u>OPTIMAL SOLUTION</u> (case of Euclidean distance): The optimal line is determined by the triplet $v_4^1 = (2,6)$, $v_4^2 = (8,4)$ and $v_2^3 = (2,2)$.

- Optimal objective value: 1.8973.

- Optimal line $\pi^* \; : \; \mathsf{y} - 4 = -\frac{1}{3}(\mathsf{x} - 2)$.

2. Case of Rectilinear distance:

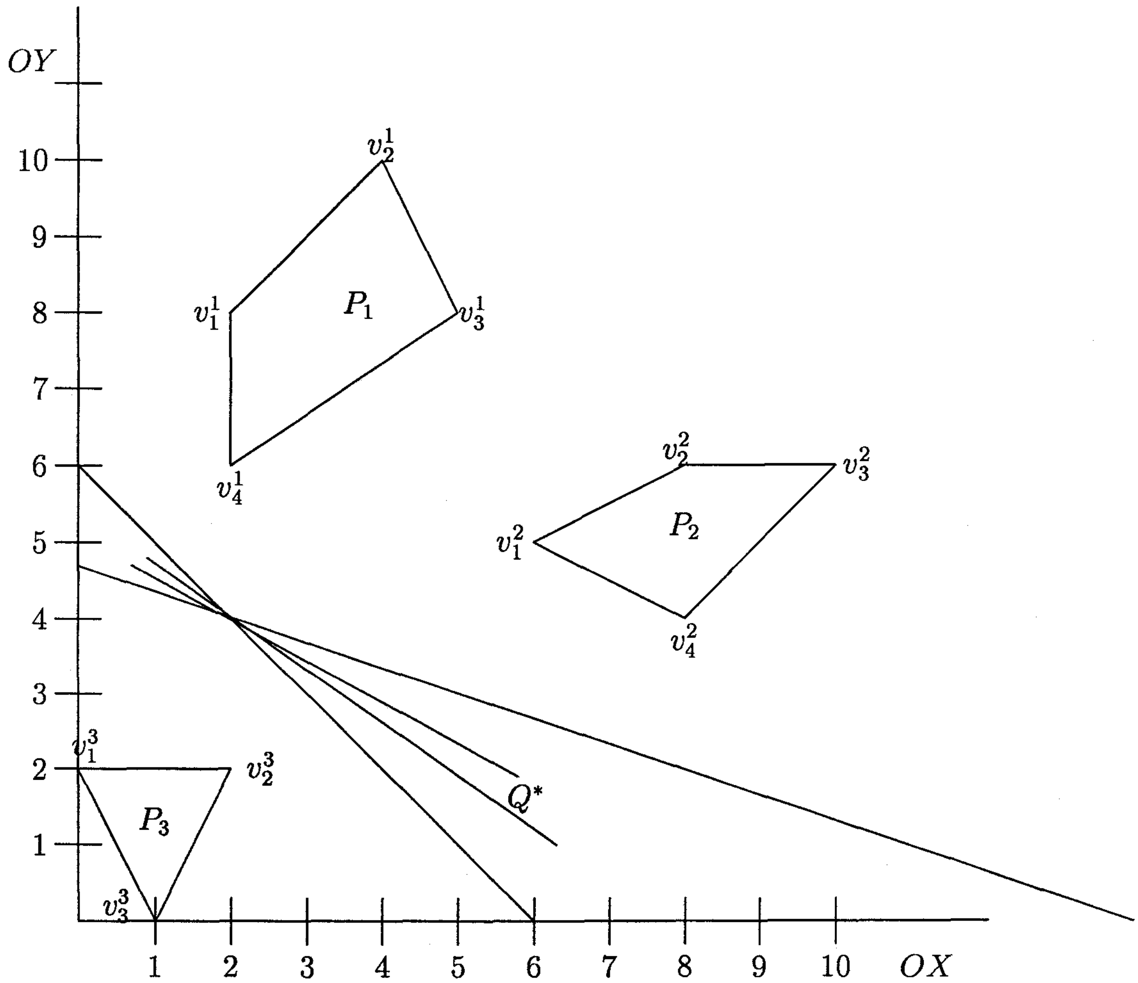


Figure 4.6: Example 4.1 with the Rectilinear distance

Triplets		Part. Sol.		D	Bounding Test	Solution		$f'(v_i^p, v_j^s)$	f
v_i^p, v_j^s	v_k^r	θ	$m_{v_i^p, v_k^r}$			θ	$m_{v_i^p, v_k^r}$		
(5,8), (2,6)	(6,5)	−0.9828	$(\frac{11}{2}, \frac{13}{2})$	1.5254		−0.9828	$(\frac{5}{2}, 5)$	1.1094	1.1094
	(8,6)		$(\frac{13}{2}, 7)$	1.6641					
	(10,6)		$(\frac{15}{2}, 7)$	2.2188					
	(8,4)		$(\frac{13}{2}, 6)$	2.4961					
	(0,2)		$(\frac{5}{2}, 5)$	1.1094					
	(2,2)		$(\frac{7}{2}, 5)$	1.6641					
	(1,0)		(3,4)	2.2188					
(5,8),(2,2)	(6,5)	−0.4636	$(\frac{11}{2}, \frac{13}{2})$	1.1181		−0.4636	$(\frac{11}{2}, \frac{13}{2})$	1.1181	1.1181
	(8,6)		$(\frac{13}{2}, 7)$	1.7888					
	(10,6)		$(\frac{15}{2}, 7)$	2.6833					
	(8,4)		$(\frac{13}{2}, 6)$	2.2361					
(2,6), (8,4)	(0,2)	1.249	(1,4)	2.2136		1.249	(2,4)	1.8973	1.8973
	(2,2)		(2,4)	1.8973					
	(1,0)		$(\frac{3}{2}, 3)$	3.0042					
(6,5), (8,6)	(2,8)	−1.1071	$(4, \frac{13}{2})$	2.2361					1.8973
	(4,10)		$(5, \frac{15}{2})$	2.6833					
	(5,8)		$(\frac{11}{2}, \frac{13}{2})$	1.5652	Rest of vert.				
(0,2), (2,2)	(2,8)	1.5708	(1,5)	3					1.8973
	(4,10)		(2,6)	4					
	(5,8)		$(\frac{5}{2}, 5)$	3					
	(2,6)		(1,4)	2					
	(6,5)		$(3, \frac{7}{2})$	1.5	Rest of vert.				

Table 4.2: Step 2 of Algorithm 4.1 of Example 4.1

STEP 1 (Algorithm 4.2).

In Table 4.3 we show the solutions that are obtained by the first step of Algorithm 4.2. The column "Pair" indicates the two vertices which are considered to construct the line. The column "Solution" describes the solution line or the solution cone (depending on the case) associated with the considered pair. $m_{v_i^p v_j^s}$ has been previously defined and "slopes" are the slopes of the particular solutions determined by the vertices in the column Pair. The column "D" represents the Euclidean distance to the solution. The column "Admiss. slopes" expresses the slopes of the lines which are actually candidates to solution. Finally, the column "Eliminated pairs" includes the set of vertices v_k^r that do not have to be considered together with v_i^p or v_j^s (consequence of Lemma 4.9 (part 1.(b))). This elimination makes necessary to consider 11 pairs out of 40.

Pair		Solution		D	Admiss. slopes	$f(v_i^p, v_j^s)$	f	Eliminated pairs	
v_i^p	v_j^s	$m_{v_i^p v_j^s}$	slopes						v_k^r
(2,8)	(6,5)	$(4, \frac{13}{2})$	1	2.4748	$\emptyset$	0	0	(2,8)	(8,6), (10,6), (8,4)
								(6,5)	(4,10)
(2,8)	(0,2)	(1,5)	−1	2.8284	$\emptyset$	0	0	(2,8)	(1,0)
								(0,2)	(4,10), (5,8), (8,6) (10,6), (8,4)
(2,8)	(2,2)	(2,5)	[−1,1]	2.1213	$\emptyset$	0	0	(2,2)	(4,10), (5,8), (8,6) (10,6), (8,4)
(4,10)	(8,6)	(6,8)	1	2.8284	$\emptyset$	0	0	(4,10)	(10,6), (8,4)
(4,10)	(1,0)	$(\frac{5}{2}, 5)$	−1	4.5962	$\emptyset$	0	0	(1,0)	(8,6), (10,6)
(5,8)	(6,5)	$(\frac{11}{2}, \frac{13}{2})$	1	1.4142	$\emptyset$	0	0	(5,8)	(8,6), (10,6), (8,4)
								(6,5)	(2,6)
(5,8)	(1,0)	(3,4)	−1	4.2426	$\emptyset$	0	0		
(2,6)	(8,6)	(5,6)	$\mathbb{R} \setminus (-1,1)$	2.1213	$\emptyset$	0	0	(2,6)	(10,6), (8,4)
(2,6)	(0,2)	(1,4)	−1	2.1213	$\emptyset$	0	0	(2,6)	(1,0)
								(0,2)	(6,5)
(2,6)	(2,2)	(2,4)	[−1,1]	1.4142	$[-1, -\frac{1}{3}]$	2	2	(2,2)	(6,5)
(6,5)	(1,0)	$(\frac{7}{2}, \frac{5}{2})$	−1	3.5355	$\emptyset$	0	2	(1,0)	(8,4)

Table 4.3: Step 1 of Algorithm 4.2 of Example 4.1

STEP 2 (Algorithm 4.2).

In Table 4.4 we show the solutions given by the second step of Algorithm 4.2. Table 4.4 is similar to Table 4.2 apart from column "d_1" that gives the rectilinear distance ($d_1 = K(m, \gamma_B)D$).

OPTIMAL SOLUTION (case of Rectilinear distance): The solution is determined by the pair $v_4^1 = (2,6)$, $v_2^3 = (2,2)$.

- Optimal objective value: 2.

- The cone of optimal solution Q^* is bounding by the lines $y - 4 = -\frac{1}{3}(x - 2)$ and

  $y - 4 = -(x - 2)$.

Triplets		Part. Sol.		d_1	Bounding Test	Solution		$f'(v_i^p, v_j^s)$	f
v_i^p, v_j^s	v_k^r	θ	$m_{v_i^p, v_k^r}$			θ	$m_{v_i^p, v_k^r}$		
$(5,8),(2,6)$	$(6,5)$	-0.9828	$(\frac{11}{2}, \frac{13}{2})$	1.8333	Rest of vert.				2
$(5,8),(2,2)$	$(6,5)$	-0.4636	$(\frac{11}{2}, \frac{13}{2})$	1.25	Rest of vert.				2
$(2,6),(8,4)$	$(0,2)$	1.249	$(1,4)$	2.3333		1.249	$(2,4)$	2	2
	$(2,2)$		$(2,4)$	2					
	$(1,0)$		$(\frac{3}{2}, 3)$	3.1666					
$(6,5),(8,6)$	$(2,8)$	-1.1071	$(4, \frac{13}{2})$	2.5					2
	$(4,10)$		$(5, \frac{15}{2})$	3					
	$(5,8)$		$(\frac{11}{2}, \frac{13}{2})$	1.75	Rest of vert.				
$(0,2),(2,2)$	$(2,8)$	1.5708	$(1,5)$	3					2
	$(4,10)$		$(2,6)$	4					
	$(5,8)$		$(\frac{5}{2}, 5)$	3					
	$(2,6)$		$(1,4)$	2					
	$(6,5)$		$(3, \frac{7}{2})$	1.5	Rest of vert.				

Table 4.4: Step 2 of Algorithm 4.2 of Example 4.1

4.2.2 Computational results

In order to compare the behaviour of the algorithms, once the above modifications have been implemented, we have performed a computational study for the problem with Euclidean distance, equal weights and points rather than polygonal regions. This computational study is reported in Tables 4.5 and 4.6. We did 20 simulations for each sample size (where n stands for the number of points). We show the average of the number of evaluations needed to find the optimal solution.

Table 4.5 shows only the evaluations performed in STEP 1, while Table 4.6 reports the behaviour of the whole algorithm. We distinguish four rows describing the number of evaluations of Algorithm 4.1 without modifications (original), with the modifications based on Theorem 4.2 (circle), with the bounding test (bound) and with the two improvements (circle-bound). Since the modifications induced by Theorem 4.2 only applies to *STEP 1*, its influence is only significant in Table 4.5. Finally, Table 4.6 also shows that the major computational effort in these algorithms focuses on *STEP 2*, where the number of evaluations is $\mathcal{O}(n^3)$. As can be seen the percentage of reduction on the total number of evaluations increases

	$n = 10$	$n = 25$	$n = 50$	$n = 75$	$n = 100$
$ORIGINAL$	108	865	3595	8291	15079
$CIRCLE$	56	410	1575	3637	6482
$BOUND$	97	739	3299	7631	13944
$CIRCLE - BOUND$	51	355	1442	3347	6020

Table 4.5: Computational experience of Step 1 of Algorithm 4.1

	$n = 10$	$n = 25$	$n = 50$	$n = 75$	$n = 100$
$ORIGINAL$	468	7765	62388	210855	500150
$CIRCLE$	416	7310	60368	206201	491553
$BOUND$	312	3057	16546	43208	86214
$CIRCLE - BOUND$	266	2673	14689	38924	78290

Table 4.6: Computational experience of Step 1 and 2 of Algorithm 4.1

with the number of considered points. Notice that with 100 points the number of evaluations reduces by 85 % .

4.3 Concluding remarks

The problem dealt with in this chapter belongs to a very important class of problems dealing with the location of pipelines or channels of hazardous or noxious materials. We incorporate the existence of protected zones around each existing facility to prevent population from the effects caused by these materials.

We develop in this chapter different algorithms which solve these problems, both for Euclidean and polyhedral distances and compare their computational efficiency. These analysis show that reduction of over 80% of the computational effort with respect to the enumerative approach is achieved with our algorithms. Thus, it is possible to solve these problems even for moderately large number of polygons.

Chapter 5

A loss queuing-location problem with rejection

The origin of this chapter comes from a previous model of facility location with spatially dispersed customers called c-Server-Single-Facility-Loss-Median (c-SFLM) (see e.g., Chiu and Larson (1985)[25] or Frenk, Labbé and Zhang (1993) [41]). This model is a loss queuing-location model where a service system with c mobile units placed at the same position is considered. Request for service can be of different kinds and they arrive in time as independent homogeneous Poisson processes with different input rates. In the papers of Chiu and Larson (1985)[25] or Frenk, Labbé and Zhang (1993) [41] the authors study localization properties of the (c-SFLM) with a standard admission policy: if a request for service arrives, and some service unit is idle, it is immediately dispatched to cover the demand. If all the service units are busy the request is lost. Our goal is to show that admission policies that can reject requests according to the group they belong to even if some service unit is idle, improve the global efficiency of the system while similar localization properties for the optimal location still hold.

Our model does not need to be applied only for emergency services. Moreover, it is clear that for emergency systems without a backup system, our kind of policy may be sometimes unacceptable. However, there are situations where this model must be naturally applied. Consider for instance, the case of helicopters used simultaneously for medical emergencies and to transport human organs to be transplanted. These helicopters are hardly used to carry slightly injured people to hospitals because they have to be prepared to be used to transport seriously injured people and/or human organs where rapidity is crucial. This is also the case of planes used to extinguish fires in mountains or national parks. The high cost of dispatching a plane makes the use of planes only acceptable when the magnitude of the fire is large enough thus justifying the cost involved, while for normal situations trucks are used.

These policies can also be applied in supercomputing centers. In a center where resources are limited and have to be shared among thousands of users who send tasks from different locations, it is important to distribute the services and to optimize the resources. This is for instance the case of the Regional Computing center in Andalusia (C.I.C.A.). After evaluating the needs and the costs, the parallel machine, an SPP 2000 X-class with 24 processors was configured as four "virtual machines" (called subcomplexes); the 24 processors were distributed as follows: four processors for interactive use, twelve processors for parallel computing, six processors for scalar computing, two processors for math computing. That way, there are groups of processors for specific computing. The users work in the interactive subcomplex and their batch jobs are sent to be processed in the other virtual computers. Even considering that at times, a virtual machine is not in use, the machine configuration is not modified. The cost of changing it would be too high (loss of jobs, the machine unavailable for a period of time), so that it is not worth making any change in the configuration. A different application of this approach to Computer Science can be found in Xu et al. (1992)[106]. Despite their importance these anticipated policies have been hardly considered in the literature. The reason is the mathematical difficulty to handle these models.

The chapter is organized as follows. In Section 5.1 the mathematical formulation of the model is presented. In Section 5.2 we derive localization results for the model. Section 5.3 is devoted to the development of an algorithm which allows us solve efficiently the problem in discrete spaces, in network and in continuous spaces with polyhedral norms. A procedure to obtain good approximated solutions in continuous spaces with any general norm is developed in Section 5.4. Section 5.5 includes an

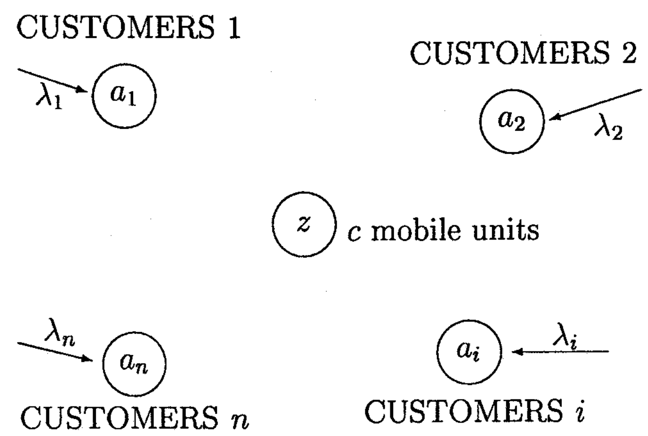


Figure 5.1: The problem.

example which shows the improvement of our model with respect to the c-SFLM model. Section 5.6 deals with an extension of the model, in which it is considered two service systems. Section 5.7 derives localization results and develops a branch and bound procedure to solve the new model.

5.1 The model

Consider the location of a demand-responsive service system in a metric space N in which c mobile service units are garaged at the same position. n classes of customers requesting service arrive at the system, being the arrival process of the different groups an independent homogeneous Poisson process (see Figure 5.1). For the ease of presentation, we consider in the formulation that the servers are dispatched to cover the requests. Nevertheless, there is no restriction to consider that the requests are sent to the servers. The standard admission policy consists of rejecting only those customers who arrive when all the service units are busy. Nevertheless, we consider an admission policy where customers can be discriminated according to their group so that they can be rejected even if the system has some idle service units. The introduction of this new admission policy is justified in order to enhance the global efficiency of the system.

The elements of our model are

1. $I := \{1, \dots, n\}$ is the index set of n classes of customers requesting service of the system.

2. $a_i \in N$ is the arrival point of the customers of type i.

3. $\lambda_i > 0$ is the Poisson arrival rate of customers of type i. Being the arrival process of the different groups independent.

4. $w_i \geq 0$ is the mean time of on-scene service of customers of type i.

5. $d(a_i, z)$ is the distance from the location a_i of the customer of type i to the facility garaged at z.

6. $\beta \geq 1$ is a travel factor (for instance $\beta = 2$ is a round trip factor).

7. $\hat{r_i} \geq 0$ is the cost of each lost customer of type i.

8. $\alpha > 0$ is the cost per unit travel time.

Using this notation, if the garage facility is located at z the expected service time of a customer of type i is $s_i(z) = w_i + \beta d(a_i, z)$, i.e. the mean time of on-scene service w_i plus a time proportional to the distance $d(a_i, z)$ times the travel factor β. Any accepted i-customer induces a cost to the system proportional to the travel time given by $r_i(z) = \alpha\beta d(a_i, z)$ and any lost i-customer induces a cost $\hat{r_i} \geq 0$. It is also assumed that we are considering a real loss system where not all the $\hat{r_i} = 0$. Otherwise, the optimal policy would be not to serve any customer.

To deal with this model we introduce two kinds of decision variables z and x_i, $i = 1, \dots, n$. Each x_i stands for the proportion of i-customers which are accepted when some server is idle and z is the location for the service units that we are looking for. The goal is to find the location z of the facility and the proportions $x = (x_1, \dots, x_n)$ of accepted customers that minimizes the expected total cost per unit time.

Obviously, given $z \in N$ and $x \in [0, 1]^n$ the system behaves as a $M/G/c/c$ queue with arrival rate $\lambda(x)$, mean service time $s(x,z)$ and throughput $\rho(x,z)$ given by

$$\begin{aligned} \lambda(x) &= \textstyle\sum_{i=1}^{n} \lambda_i x_i, \\ s(x,z) &= \textstyle\sum_{i=1}^{n} \frac{\lambda_i x_i}{\lambda(x)} s_i(z) \\ \rho(x,z) &= \lambda(x) s(x,z). \end{aligned}$$

Let $\Psi_c(x,z)$ denote the probability that not all the servers are busy. It is well-known (see e.g. Medhi (1991)[75]) that in our model

$$\Psi_c(x,z) = 1 - \frac{(\lambda(x)s(x,z))^c/c!}{\sum_{k=0}^{c}(\lambda(x)s(x,z))^k/k!}.$$

The expected total cost per unit time is given by the sum of the cost of the accepted plus the rejected customers. The expected number of i-customers per unit time that receive service is $\lambda_i x_i \Psi_c(x,z)$ and the expected number of rejected i-customers per unit time is $\lambda_i(1-x_i) + \lambda_i x_i(1-\Psi_c(x,z))$ then the expected cost per unit time is:

$$\Psi_c(x,z)\sum_{i=1}^{n} \lambda_i x_i (r_i(z) - \hat{r}_i) + \sum_{i=1}^{n} \lambda_i \hat{r}_i.$$

Since the term $\sum_{i=1}^{n} \lambda_i \hat{r}_i$ does not depend on the decision variables (x,z) one can consider equivalently the following objective function for the model:

$$F(x,z) = \Psi_c(x,z)(r(x,z) - \hat{r}(x))$$

where

$$\begin{aligned} r(x,z) &= \textstyle\sum_{i=1}^{n} \lambda_i r_i(z) x_i, \\ \hat{r}(x) &= \textstyle\sum_{i=1}^{n} \lambda_i \hat{r}_i x_i. \end{aligned}$$

Therefore the model is formulated as:

$$\min_{x\in[0,1]^n, z\in N} F(x,z) = \Psi_c(x,z)(r(x,z) - \hat{r}(x)). \tag{5.1}$$

Hereafter and by similarity with the c-Server-Single-Facility-Loss-Median model (c-SFLM) of Chiu and Larson (1985)[25] or Frenk, Labbé and Zhang (1993)[41] , we will call this problem the c-Server-Single-Facility-Loss-Median model with rejection (c-SFLMR).

5.2 Localization results for the c-SFLMR problem

In this section we derive localization results for the c-SFLMR. In order to prove these results we introduce the following auxiliary problem for each fixed $x \in [0,1]^n$.

$$\min_{z \in N} t(x,z) := \beta \sum_{i=1}^{n} \frac{\lambda_i x_i}{\lambda(x)} d(a_i, z). \tag{5.2}$$

It is worth noting that the objective function $t(x,z)$ of Problem (5.2) is the expected travel time for the feasible solution (x,z) of Problem (5.1). Therefore, Problem (5.2) is the well-known Weber or median problem on the metric space N with respect to the set of existing facilities given by $A = \{a_1, \ldots, a_n\}$ (see e.g. Hakimi (1964)[46] or Wesolowsky (1993)[104]). In addition, in order to simplify the notation we also introduce $w(x)$ the expected on-scene service time

$$w(x) = \sum_{i=1}^{n} \frac{\lambda_i x_i}{\lambda(x)} w_i.$$

The optimization Problem (5.1) we are interested in is different from (5.2). However, we prove in the next theorem that an optimal solution of (5.1) can be obtained among the minimizers of $t(x,z)$. This result extends the localization result for the c-SFLM, see Theorem 2 in Chiu & Larson (1985)[25] or Lemma 1.1 in Frenk, Labbé and Zhang (1993)[41].

Theorem 5.1 *Let $\bar{x}$ be a nonzero fixed vector in $[0,1]^n$. If $\bar{z}$ solves problem $\min_{z \in N} t(\bar{x}, z)$, then $\bar{z}$ solves the problem $\min_{z \in N} F(\bar{x}, z)$.*

Proof. We only need to show that $\frac{dF(\bar{x},z)}{dt(\bar{x},z)} \geq 0$, $\forall z \in N$. First, observe that

$$\frac{dF(\bar{x},z)}{dt(\bar{x},z)} = \alpha \Psi_c(\bar{x},z)\lambda(\bar{x}) + \frac{d\Psi_c(\bar{x},z)}{dt(\bar{x},z)}(r(\bar{x},z) - \hat{r}(\bar{x})).$$

Recall that $\rho(\bar{x},z) = \lambda(\bar{x})s(\bar{x},z) = \lambda(\bar{x})(w(\bar{x}) + t(\bar{x},z))$. Therefore, we can write the expression above as

$$\frac{dF(\bar{x},z)}{dt(\bar{x},z)} = \alpha \Psi_c(\bar{x},z)\lambda(\bar{x}) + \frac{d\Psi_c(\bar{x},z)}{d\rho(\bar{x},z)}\lambda(\bar{x})(\alpha(\rho(\bar{x},z) - \lambda(\bar{x})w(\bar{x})) - \hat{r}(\bar{x})).$$

Now, since $\frac{d(1-\Psi_c(\bar{x},z))}{d\rho(\bar{x},z)} \geq 0$ (see page 512 of the paper of Chiu and Larson(1985)[25]), thus $\frac{d\Psi_c(\bar{x},z)}{d\rho(\bar{x},z)} \leq 0$. Besides $\alpha \geq 0$ and $\hat{r}(\bar{x}) \geq 0$ then

$$\frac{dF(\bar{x},z)}{dt(\bar{x},z)} \geq \alpha\lambda(\bar{x})(\Psi_c(\bar{x},z) + \frac{d\Psi_c(\bar{x},z)}{d\rho(\bar{x},z)}\rho(\bar{x},z)).$$

Furthermore, by page 513 of the paper of Chiu and Larson(1985)[25]

$$\Psi_c(\bar{x},z) - \frac{d(1-\Psi_c(\bar{x},z))}{d\rho(\bar{x},z)}\rho(\bar{x},z) \geq 0,$$

then $\Psi_c(\bar{x},z) + \frac{d\Psi_c(\bar{x},z)}{d\rho(\bar{x},z)}\rho(\bar{x},z) \geq 0$. Hence, $\frac{dF(\bar{x},z)}{dt(\bar{x},z)} \geq 0$. □

Remark 5.1 . *It is possible that there exists $z^* \in \arg\min F(\bar{x},z)$ such that $z^* \notin \arg\min t(\bar{x},z)$, but there always exists $z_0 \in \arg\min F(\bar{x},z)$ such that $F(\bar{x},z_0) = F(\bar{x},z^*)$. Indeed, $F(\bar{x},z^*) \leq F(\bar{x},z)$ for all $z \in N$ and if $z^* \notin \arg\min t(\bar{x},z)$ there exists $z_0 \in N$ such that $t(\bar{x},z_0) < t(\bar{x},z^*)$. By the proof of Theorem 5.1 F is non-decreasing in $t(\bar{x},z)$ then, $F(\bar{x},z_0) \leq F(\bar{x},z^*)$. That implies, $F(\bar{x},z_0) = F(\bar{x},z^*)$.*

The above theorem implies that if there exists a finite dominating set D of candidates to be optimal solution for all the median problems (5.2) which is independent of the admission policy x, then D is also a finite dominating set for Problem (5.1). Hence, the resolution of Problem (5.1) reduces to solve a finite number of pure queuing problems:

$$\min_{x\in[0,1]^n} F(x,\bar{z}) \quad \text{for each } \bar{z} \in D$$

Each one of these problems determines the admission policy associated with a given location, $x^*(\bar{z})$. The optimal solution for Problem (5.1) is given by the location z^* and the policy $x^*(z^*)$ where the minimum of $F(x,z)$ is achieved.

In the next section we provide a solution procedure for the c-SFLMR problem in those frameworks where there exists a finite dominating set of candidates to be optimal location.

5.3 Solution procedure for the c-SFLMR problem

In this section we provide an algorithm to solve the c-SFLMR problem. This algorithm can be applied when there exists a finite dominating set D of candidates to optimal location. We show different frameworks where we can apply our algorithm as well as how to determine efficiently the set D.

If there exists a finite dominating set D of candidates to be optimal locations, Problem (5.1) can be solved by means of the following algorithm.

Algorithm 5.1

Input:

- Arrival points of the customers: $a_1, \cdots, a_n$.
- Poisson arrival rates of the customers: $\lambda_1, \cdots, \lambda_n$.
- Mean times of on-scene service of the customers: $w_1, \cdots, w_n$.
- Costs of lost customers: $\hat{r_1}, \cdots, \hat{r_n}$
- Parameters α, β and distance function

Output:

- Optimal objective function value:$=F^*$.
- Optimal solution:$=(x^*, z^*)$.

Steps:

- Obtain the finite dominating set D.
- For each fixed $\bar{z} \in D$, do:
  Solve the pure queuing problem with $\bar{z}$ fixed.
  Obtain the optimal admission policy $x^*(\bar{z})$ and the optimal value $F(x^*(\bar{z}), \bar{z}) := F^*(\bar{z})$.
- Obtain $\min_{z \in D} F^*(z) := F^*$, $z^* = \arg\min_{z \in D} F^*(z)$ and $x^* = x^*(z^*)$.

The two important tasks in the above algorithm are: 1) How do solve the pure queuing problem, i.e., the problem which results when the location z of the facility has been fixed, and 2) how to determine efficiently the finite dominating set D.

1) In Carrizosa et al. (1998)[20] procedures for solving the resulting pure queuing problem can be found. For the sake of completeness, we describe the nature of the optimal solution of these problems. Let v^i, $i = 0, \dots, n$, be the vectors with 1 in

the first i entries and 0 everywhere else. Let $p : [0, n] \to [0, 1]^n$ be the natural parameterization of the path through $v^0, \ldots, v^n$. As a consequence of theorems 3.4 and 3.5 in the above mentioned reference: (1) an optimal solution of $\min_{x \in [0,1]^n} F(x, \bar{z})$ is $x^* = p(t)$ for some $0 < t \leq n$; (2) for each $\bar{z}$ the function $F(p(t), \bar{z})$ is semilocally pseudoconvex on $[0, n]$; and (3) for $c = 1$ an optimal solution of $\min_{x \in [0,1]^n} F(x, \bar{z})$ can be always found in $\{(v^1, \bar{z}) \ldots, (v^n, \bar{z})\}$. It is worth noting that the order of the demand points in the natural parameterization is taken according with the increasing sequence of their modified service time $\frac{s_i(z)}{\bar{r}_i - r_i(z)}$ (see Assumptions A1 and A2 in Carrizosa et al. (1998)[20]). These properties enable the actual resolution of problem $\min_{x \in [0,1]^n} F(x, \bar{z})$ by solving one unimodal one-dimensional optimization problem per each location $\bar{z}$ fixed. Moreover, as shown in Hansen et al. (1991)[55], for the case $c = 1$ it is possible, using binary search techniques, to find an optimal solution for each $\bar{z}$ fixed with overall complexity $O(n)$.

2) In what follows we show different frameworks where we can apply Algorithm 5.1 and how to determine the finite dominating set D.

Algorithm 5.1 can be applied in a discrete space because the number of possible locations is finite and known in advance. A discrete space is the most obvious but not the only framework where we can apply our algorithm. The next theorem shows another case.

Theorem 5.2 *Let N be a network with the customers placed at its nodes. Then there exists a node which is an optimal location of Problem (5.1).*

Proof. Hakimi's Theorem [46] ensures that there exists a node which is an optimizer for any median problem in a network. Therefore, the result follows by applying the Theorem 5.1. □

The above theorem implies that we can solve the c-SFLMR problem in Networks, when the customers are placed at its nodes by using Algorithm 5.1. In this case the finite dominating set D is known in advance because it is the set of nodes of the network.

In a different framework we can still apply Algorithm 5.1. Let $P \subset \mathbb{R}^q$ be a compact, convex polytope, symmetric with respect to the origin and with non empty interior. P induces a norm γ_P in $I\!R^q$ given by $\gamma_P(x) = \inf\{\lambda > 0 : x \in \lambda P\}$. Consider N to be the space $I\!R^q$ with the metric induced by a polyhedral norm γ_P, i.e. $d(x,y) = \gamma_P(x-y)$. Let P^* denote the polar set of P and denote by $N(p)$ the normal cone to P^* at the point $p \in P^*$ (see Appendix A for the definition of polar set and for the definition of normal cone). Given a family $(p_1, \dots, p_n)$ of elements of P^* any set of the form $\cap_{i=1}^n (a_i + N(p_i))$ is called an elementary convex set (e.c.s.) (see Definition 2.3 in this memory). Any extremal point of any elementary convex set is called intersection point. Under these hypotheses the number of intersection points is finite and polynomially bounded (see Durier and Michelot (1985)[33] for further details).

Theorem 5.3 *Let N be the metric space $(I\!R^q, \gamma_P)$ and a finite set of customers located at $A = \{a_1, \dots, a_n\} \subset I\!R^q$. Then there exists an intersection point which is an optimal location of Problem (5.1).*

Proof. Theorem 3.1 in Durier & Michelot (1985)[33] proves that any median problem in $(I\!R^q, \gamma_P)$ has an optimal solution in the set of intersection points generated by the points in A. Therefore, by applying Theorem 5.1 the result follows. □

As consequence of above theorem, the set of intersection points is a finite dominating set for the location problem. Thus, once this set is calculated we can use Algorithm 5.1 to solve Problem (5.1). In the next subsection we show how to determine the finite dominating set in polyhedral-normed spaces.

5.3.1 Efficient determination of the finite dominating set D in polyhedral-normed spaces.

Let $\{b_j\}$ for $j = 1, \cdots, k$ be the set of extreme points of the polytope P and let $\{d_j^i\}$ for $j = 1, \cdots, k, \quad i = 1, \cdots, n$ be the set of translated fundamental directions defined by

$$d_j^i = \{x \in I\!R^2 \;:\; x = a_i + \mu b_j; \quad \mu \geq 0\}.$$

The determination of the intersection points D could be done in a first approach by computing the intersection for all pairs of translated fundamental directions.

Lemma 5.1 *The number of the intersection points is of order $O(n^2k^2)$ where, n is the number of different customers and k is the number of extreme points of the unit ball P of the polyhedral norm.*

Proof. For computing the number of intersections points, we have:
k extreme points (or k fundamental directions) and then, there exist $\frac{k}{2}$ lines with different slopes.
There are n arrival points of the customers.
Then, for two different types of customers the number of intersection points is $\frac{k}{2}(\frac{k}{2}-1)$ plus the two customers. Therefore, the number of the total intersections points is:
$\frac{k}{2}(\frac{k}{2}-1)(n-1)+\frac{k}{2}(\frac{k}{2}-1)(n-2)+\cdots+\frac{k}{2}(\frac{k}{2}-1)=(\frac{k^2}{4}-\frac{k}{2})\frac{n(n-1)}{2}+n$. Therefore, the overall complexity is $O(n^2k^2)$. □

However, it seems to be better to use an output-sensitive method. With the sweep-line technique of Bentley and Ottmann (1979)[13] it is possible to compute the intersection points of M line segments in the plane in $O((M+K)\log M)$ time, where K is the number of intersection points of the line segments. For this method it is essential, that the input consists of line segments. Half-lines are not feasible. Therefore we have to cut the translated fundamental directions. The question, which arises now, is the following:

> Which is the minimal length necessary for the segments to obtain at least all intersection points which can be candidate to optimal solutions?.

Let, $s_j^i := \{x \in \mathbb{R}^2 \ : \ x = a_i + \mu b_j; \quad 0 \leq \mu \leq 3r_A^P\}$ where $r_A^P = \max_i \gamma_P(a_i)$

We denote by G the graph generated by s_j^i, i. e., the graph whose edges are s_j^i and we denote by $B^P(0,r)$ the ball of the norm γ_P centered at 0 with radius r. We have the next result:

Lemma 5.2 *Any optimal location of Problem (5.1) can be obtained among the vertices of G.*

Proof. Let z^* be an optimal location and let x^* be an optimal admission policy. By Theorem 5.1 $F(x^*,z)$ is non-decreasing in $t(x^*,z)$, thus $t(x^*,z^*) \leq t(x^*,z) \quad \forall z \in N$.

Then, since $\beta > 0$, $\lambda_i > 0$, $x_i \geq 0$ $\forall i = 1, \cdots, n$ it does not exist z such that $d(a_i, z^*) > d(a_i, z)$ $\forall i = 1, \cdots, n$. Therefore, for all z there exists $a_i(z)$ such that $d(a_i(z), z) \geq d(a_i(z), z^*)$. Thus, $\gamma_P(z^*) \leq \gamma_P(z) + 2\gamma_P(a_i(z))$ for all $z \in N$. Now, considering, $z = 0$ we get $\gamma_P(z^*) \leq 2\max_i \gamma_P(a_i) = 2r_A^P$. Hence, $z^* \in B^P(0, 2r_A^P)$. In addition, $a_i \in B^P(0, r_A^P)$ for all $i = 1, \cdots, n$ and then $d(a_i, z^*) \leq 3r_A^P$ for all $i = 1, \cdots, n$. □

For $a_i \in A$ we can determine $\gamma_P(a_i)$ in $O(\log k)$ time and therefore, r_A^P in $O(n \log k)$. The totality of all segments s_j^i can be computed in $O(nk)$ time if r_A^P is known. To sum it up this means the computation of all the segments can be determined in $O(n \log k) + O(nk) = O(nk)$ time. Therefore, the set of vertices generated by the segments in G can be computed with the method of Bentley and Ottmann (1979)[13] in $O((nk+K)\log(nk))$ time, where K is the number of vertices. Notice that K is only in the worst case quadratic in nk.

Last, since the set of polyhedral or block norms is dense in the set of all norms in $\mathbb{R}^q$, we can use Theorem 5.3 to give good approximated solutions to any queuing-location Problem (5.1) in $\mathbb{R}^q$ for any norm. In the next section we prove some convergence results and we provide a procedure to obtain good approximated solutions to Problem (5.1) in a continuous space with any general norm.

5.4 A solution approach to the c-SFLMR problem in general normed spaces

Let N be a metric space $(\mathbb{R}^q, \gamma)$ with the metric induced by the norm $\gamma(x) = \inf\{\lambda > 0 : x \in \lambda B\}$, where B is a bounded, symmetric convex set (not necessarily polytopal), the interior of which contains the origin and consider $\{\gamma_{P^m}\}_{m \in \mathbb{N}}$ a sequence of polyhedral norms, so that their unit spheres are all inscribed in B and $P^m \subseteq P^{m+1} \subseteq B$, $\forall\, m \in \mathbb{N}$. Let $d(x,y) = \gamma(x-y)$ be the metric induced by the norm $\gamma(\cdot)$ and let $d^m(x,y) = \gamma_{P^m}(x-y)$ be the metric induced by the norm $\gamma_{P^m}(\cdot)$.

We denote:

$$\begin{aligned}
s^m(x,z) &= \sum_{i=1}^n \frac{\lambda_i x_i}{\lambda(x)}(w_i + \beta d^m(a_i,z))\\
\rho^m(x,z) &= \lambda(x)s^m(x,z),\\
\Psi_c^m(x,z) &= 1 - \frac{(\lambda(x)s^m(x,z))^c/c!}{\sum_{k=0}^c(\lambda(x)s^m(x,z))^k/k!},\\
r^m(x,z) &= \alpha\beta\sum_{i=1}^n \lambda_i d^m(a_i,z)x_i,\\
F^m(x,z) &= \Psi_c^m(x,z)(r^m(x,z) - \hat{r}(x)),\\
t^m(x,z) &= \beta\sum_{i=1}^n \frac{\lambda_i x_i}{\lambda(x)} d^m(a_i,z).
\end{aligned}$$

Lemma 5.3 *For all fixed (x,z), we have:*

1. $r(x,z) \leq r^m(x,z) \quad \forall m \in I\!N.$
2. $\rho(x,z) \leq \rho^m(x,z) \quad \forall m \in I\!N.$
3. $\Psi_c^m(x,z) \leq \Psi_c(x,z) \quad \forall m \in I\!N.$
4. $F(x,z) \leq F^{m+1}(x,z) \leq F^m(x,z).$

Proof. We know that $P^m \subseteq P^{m+1} \subseteq B$. That implies $\gamma(y) \leq \gamma_{P^{m+1}}(y) \leq \gamma_{P^m}(y)$. Then, for all fixed z, we have:

$$d(a_i,z) \leq d^{m+1}(a_i,z) \leq d^m(a_i,z) \quad \forall i = 1,\cdots,n. \tag{5.3}$$

Since $\alpha > 0, \quad \beta > 0, \quad \lambda_i > 0, \quad x_i \geq 0, \quad w_i \geq 0 \quad \forall i = 1,\cdots,n$, we obtain 1. and 2.

On the other hand, in the proof of Theorem 5.1 we prove that for all fixed x, $\Psi_c(x,z)$ is non-increasing in $\rho(x,z)$ because $\frac{d\Psi_c(x,z)}{d\rho(x,z)} \leq 0$. Then by 2. we have 3.

Finally, by (5.3) we have that $t(x,z) \leq t^{m+1}(x,z) \leq t^m(x,z)$ and we know by the proof of Theorem 5.1 that $F(x,z)$ is non-decreasing in $t(x,z)$. Thus, 4. is proved.

□

Let,

$$M(F^m) := \{(x,z) \in [0,1]^n \times N \ : \ (x,z) \in \arg\min F^m\} \qquad m \in I\!N,$$

$$M(F) := \{(x,z) \in [0,1]^n \times N \ : \ (x,z) \in \arg\min F\},$$

$r_A^1 = \max_{i\in I}\{\gamma_{P^1}(a_i)\}$ and denote by $B^m(0,r)$ the ball of the norm γ_{P^m} centered at 0 with radius r and by $B(0,r)$ the ball of the norm γ centered at 0 with radius r.

Lemma 5.4

$$\cup_{m\geq 1} M(F^m) \cup M(F) \subseteq [0,1]^n \times B(0, 2r_A^1)$$

Proof. Let $(x^*, z^*) \in M(F^m)$. Obviously, $x^* \in [0,1]^n$. On the other hand, by the proof of Lemma 5.2 we have that, $\gamma_{P^m}(z^*) \leq 2\max_i \gamma_{P^m}(a_i)$. Hence, for any norm $\gamma_{P^m}(\cdot)$ we have, $z^* \in B^m(0, 2\max_i \gamma_{P^m}(a_i)) \subseteq B^m(0, 2\max_i \gamma_{P^1}(a_i)) = B^m(0, 2r_A^1)$.

Finally, since $B^m(0, 2r_A^1) \subseteq B^{m+1}(0, 2r_A^1) \subseteq B(0, 2r_A^1)$, $z^* \in B(0, 2r_A^1)$ and the results follows. □

Definition 5.1 *Given two compact sets B, C the Hausdorff distance between B and C is*

$$d_H(B, C) = \max\{\max_{z\in B} d(z, C), \max_{y\in C} d(B, y)\}$$

where $d(z, C) = \min_{y\in C} d(z, y)$ being d the Euclidean distance.

Definition 5.2 *$\{\gamma_{P^m}(\cdot)\}_{m\geq 1}$ converges to $\gamma(\cdot)$ in the sense of Hausdorff if the unit ball B of $\gamma(\cdot)$ is the limit of the sequence of unit balls $\{P^m\}_{m\geq 1}$ of $\{\gamma_{P^m}(\cdot)\}_{m\geq 1}$, under the Hausdorff metric.*

Lemma 5.5 *Let K be a compact set in $\mathbb{R}^q$. If $\{\gamma_{P^m}(\cdot)\}_{m\geq 1}$ converges to $\gamma(\cdot)$ in the sense of Hausdorff then for all $\varepsilon > 0$ there exists m_0 such that for all $m \geq m_0$ $\max_{z\in K} |\gamma_{P^m}(z) - \gamma(z)| < \varepsilon$.*

The proof of this lemma is trivial and is therefore left out.

Let $\mathcal{K} = [0,1]^n \times B(0, 2r_A^1)$. We have the next result:

Lemma 5.6 *If P^m converges to B under the Hausdorff metric, then:*

1. *$r^m(x, z)$ uniformly converges to $r(x, z)$ in $\mathcal{K}$.*

2. *$\rho^m(x, z)$ uniformly converges to $\rho(x, z)$ in $\mathcal{K}$.*

3. *$\Psi_c^m(x, z)$ uniformly converges to $\Psi_c(x, z)$ in $\mathcal{K}$.*

Proof. By Lemma 5.5, for all δ_i there exists $m_0(i) \in \mathbb{N}$ such that for all $m \geq m_0(i)$ $\max_{z\in B(0,2r_A^1)} |d^m(a_i, z) - d(a_i, z)| < \delta_i$. Then, since $\alpha > 0$, $\beta > 0$, $\lambda_i > 0$, $0 \leq x_i \leq 1$ $\forall\, i = 1, \cdots, n$, we have:

$$|r^m(x, z) - r(x, z)| < \alpha\beta \sum_{i=1}^{n} \lambda_i \delta_i := \varepsilon_1 \quad \text{for any } (x, z) \in \mathcal{K} \tag{5.4}$$

$$|\rho^m(x,z) - \rho(x,z)| < \beta \sum_{i=1}^{n} \lambda_i \delta_i := \varepsilon_2 \quad \text{for any } (x,z) \in \mathcal{K} \tag{5.5}$$

In order to simplify, we denote by $\rho^m := \rho^m(x,z)$ and by $\rho := \rho(x,z)$. By Lemma 5.3 (case 3) and 5.3 (case 2 respectively) we have,

$$|\Psi_c^m(x,z) - \Psi_c(x,z)| = \frac{(\rho^m)^c/c!}{\sum_{k=0}^{c}(\rho^m)^k/k!} - \frac{\rho^c/c!}{\sum_{k=0}^{c}\rho^k/k!} \leq \frac{((\rho^m)^c - \rho^c)/c!}{\sum_{k=0}^{c}\rho^k/k!}$$

On the other hand, since $\rho \geq 0$, $\sum_{k=0}^{c}\rho^k/k! = 1 + \sum_{k=1}^{c}\rho^k/k! \geq 1$. Then,

$$|\Psi_c^m(x,z) - \Psi_c(x,z)| \leq ((\rho^m(x,z))^c - (\rho(x,z))^c)/c!.$$

Using the Binomial expansion of the function $((\rho^m - \rho) + \rho)^c$, we get:

$$(\rho^m)^c = \sum_{k=0}^{c} \binom{c}{k} \rho^{c-k}(\rho^m - \rho)^k, \tag{5.6}$$

where $\binom{c}{k} = \frac{c!}{(c-k)!k!}$. Then, by (5.5) and (5.6) we have:

$$|\Psi_c^m(x,z) - \Psi_c(x,z)| < 1/c!(\sum_{k=1}^{c} \binom{c}{k} \rho^{c-k}(x,z)\varepsilon_2^k)$$

Finally, $\rho(x,z) \leq \sum_{i=1}^{n}\lambda_i(w_i + 3r_A^1\beta)$ and

$$|\Psi_c^m(x,z) - \Psi_c(x,z)| \;<\; 1/c!(\sum_{k=1}^{c} \binom{c}{k} (\sum_{i=1}^{n}\lambda_i(w_i + 3r_A^1\beta))^{(c-k)}\varepsilon_2^k) := \varepsilon_3$$
$$\text{for any } (x,z) \in \mathcal{K}. \tag{5.7}$$

□

Theorem 5.4 *If P^m converges to B under the Hausdorff metric, then $F^m(x,z)$ uniformly converges to $F(x,z)$ in $\mathcal{K}$. Moreover, the sequence $\{F^m(x,z)\}_{m\in\mathbb{N}}$ is decreasing.*

Proof. By Lemma 5.3 we know that $F(x,z) \leq F^m(x,z)$ and $\Psi_c^m(x,z) \leq \Psi_c(x,z) \quad \forall m \in \mathbb{N}$. Then,

$$\begin{aligned}
|F^m(x,z) - F(x,z)| &= \Psi_c^m(x,z)r^m(x,z) - \Psi_c(x,z)r(x,z) + \hat{r}(x)(\Psi_c(x,z) - \Psi_c^m(x,z)) \\
&\leq \Psi_c(x,z)(r^m(x,z) - r(x,z)) + \hat{r}(x)(\Psi_c(x,z) - \Psi_c^m(x,z)) \\
&\leq (r^m(x,z) - r(x,z)) + \hat{r}(x)(\Psi_c(x,z) - \Psi_c^m(x,z))
\end{aligned}$$

because $0 \leq \Psi_c(x,z) \leq 1$.

On the other hand, $\hat{r}(x) \leq \sum_{i=1}^{n} \lambda_i \hat{r}_i$ Thus, by (5.4) and (5.7),

$$|F^m(x,z) - F(x,z)| < \varepsilon_1 + \varepsilon_3 \sum_{i=1}^{n} \lambda_i \hat{r}_i := \varepsilon \quad \text{for any } (x,z) \in \mathcal{K}.$$

□

Corollary 5.1 *In the metric space $(\mathbb{R}^q, \gamma)$ the error committed when solving Problem (5.1) by means of the approximated problems with norm γ_{p^m} is least than $\varepsilon_1 + \varepsilon_3 \sum_{i=1}^{n} \lambda_i \hat{r}_i$ where,*

$$\begin{aligned}
\varepsilon_1 &= \alpha\beta \sum_{i=1}^{n} \lambda_i \delta_i \\
\varepsilon_2 &= \beta \sum_{i=1}^{n} \lambda_i \delta_i \\
\varepsilon_3 &= 1/c![\sum_{k=1}^{c} \binom{c}{k} (\sum_{i=1}^{n} \lambda_i (w_i + 3r_A^1 \beta))^{(c-k)} \varepsilon_2^k] \\
\delta_i &= \max_{z \in B(0,2r_A^1)} |d^m(a_i,z) - d(a_i,z)|
\end{aligned}$$

The proof is a consequence of Lemma 5.6 and Theorem 5.4.

Theorem 5.5 *Let $\{(x^m, z^m)\}_{m \in \mathbb{N}}$ be a sequence such that $(x^m, z^m) \in M(F^m)$ then any accumulation point of $\{(x^m, z^m)\}_{m \in \mathbb{N}}$ belongs to $M(F)$.*

Proof. Since $F^m(x,z)$ is decreasing for all $m \in \mathbb{N}$, applying Theorem 2.46 in Attouch's book (1984)[3] we obtain that the sequence $\{F^m(x,z)\}_{m \in \mathbb{N}}$ is epi-convergent.

In addition, we get from proposition 2.48 in Attouch (1984)[3] that

$$\lim_{m \to \infty} \inf_{(x,z) \in [0,1]^n \times \mathbb{R}^q} F^m(x,z) = \inf_{(x,z) \in [0,1]^n \times \mathbb{R}^q} \lim_{m \to \infty} F^m(x,z) = \inf_{(x,z) \in [0,1]^n \times \mathbb{R}^q} F(x,z)$$

Since $[0,1]^n \times \mathbb{R}^q$ is a first countable space and $\{F^m(x,z)\}_{m\in\mathbb{N}}$ is epi-convergent, we get from Theorem 2.12 in the aforementioned book that any accumulation point of the sequence $\{(x^m,z^m)\}_{m\in\mathbb{N}}$ is an optimal solution of the problem with objective function F. □

Using these results, we can approximate up to a given accuracy the optimal solution of any problem. In fact, for a polyhedral norm with k travel directions the number of intersections points is, by Lemma 5.1, $O(k^2n^2)$. Therefore, this is the upper bound of the pure queue problems which need to be solved in the approximation process.

As an application of the above approximation scheme, we propose below a procedure to give an approximation for any l_p norm by a sequence of polyhedral norms, $\{\gamma^m\}_{m\in\mathbb{N}}$.

Let N be the metric space $(\mathbb{R}^2, ||\cdot||_p)$ with $p > 1$, where $||\cdot||_p$ is the l_p norm given by $||x||_p = (|x_1|^p + |x_2|^p)^{1/p}$. We denote by B_p the unit ball of $||\cdot||_p$, by P^m the unit ball of the polyhedral norm γ^m, being b_i^m for $i = 1,\cdots,2^{m+1}$ the extreme points of P^m and by Q_i^m for $i = 1,\cdots,2^{m+1}$ the cones generated by two consecutive extreme points of P^m.

Procedure

STEP 1. $(m = 1)$

Let $\gamma^1 := ||\cdot||_1$. Then, P^1 is the polytope whose extreme points are: $b_1^1 = (1,0)$, $b_2^1 = (0,1)$, $b_3^1 = (-1,0)$, $b_4^1 = (0,-1)$.

In each cone Q_i^1 for $i = 1,\cdots,4$ we evaluate the point of the unit ball B_p which has the maximum γ^1-norm value. Let $z(p)_i^1, i = 1,\cdots,4$ this point, thus the maximum error obtained in the approximation of the unit ball B_p by the unit ball P^1 is given by $E_p^1 = \max_{i=1,\cdots,4}\gamma^1(z(p)_i^1) - 1$.

STEP K. $(m = k)$

Let γ^k be the polyhedral norm induced by P^k, where P^k is the polytope whose extreme points are: $\{b_i^k\}_{i=1,\cdots,2^{k+1}} = \{b_i^{k-1}\}_{i=1,\cdots,2^k} \cup \{z(p)_i^{k-1}\}_{i=1,\cdots,2^k}$.

In each cone Q_i^k for $i = 1,\cdots,2^{k+1}$ we evaluate the point of the unit ball B_p

Polyhedral norms	l_p norms						
	$p = 3/2$	$p = 2$	$p = 5/2$	$p = 3$	$p = 7/2$	$p = 10$	$p = 20$
$m = 1$	0.259921	0.414213	0.515716	0.587401	0.640670	0.866066	0.931872
$m = 2$	0.063448	0.082392	0.087091	0.086498	0.083835	0.048078	0.02807
$m = 3$	0.014641	0.019591	0.021354	0.021821	0.021702	0.015062	0.009921
$m = 4$	0.003593	0.004838	0.005308	0.005458	0.00549	0.003964	0.002692
$m = 5$	0.000894	0.001206	0.001325	0.001364	0.001366	0.001003	0.000687
$m = 6$	0.000223	0.000301	0.000331	0.000341	0.000342	0.000251	0.000172
$m = 7$	5.58E-05	7.53E-05	8.28E-05	8.53E-05	8.54E-05	6.29E-05	4.32E-05
$m = 8$	1.39E-06	1.88E-05	2.07E-05	2.13E-05	2.13E-05	1.57E-05	1.08E-05
$m = 9$	3.48E-06	4.71E-06	5.17E-06	5.33E-06	5.34E-06	3.93E-06	2.70E-06
$m = 10$	8.72E-07	1.17E-06	1.29E-06	1.33E-06	1.33E-06	9.84E-07	6.75E-07
$m = 11$	2.18E-07	2.94E-07	3.23E-07	3.33E-07	3.34E-07	2.46E-07	1.69E-07
$m = 12$	5.45E-08	7.35E-08	8.08E-08	8.33E-08	8.34E-08	6.15E-08	4.22E-08
$m = 13$	1.36E-08	1.84E-08	2.02E-08	2.08E-08	2.09E-08	1.54E-08	1.05E-08
$m = 14$	3.41E-09	4.59E-09	5.05E-09	5.20E-09	5.21E-09	3.84E-09	2.64E-09
$m = 15$	8.52E-10	1.15E-09	1.26E-09	1.30E-09	1.30E-09	9.61E-10	6.61E-10
$m = 16$	2.12E-10	2.88E-10	3.15E-10	3.25E-10	3.25E-10	2.42E-10	1.64E-10
$m = 17$	5.45E-11	7.01E-11	7.57E-11	8.14E-11	7.76E-11	6.07E-11	5.29E-11
$m = 18$	1.32E-11	1.62E-11	1.42E-11	2.25E-11	1.98E-11	5.93E-12	4.12E-12

Table 5.1: Maximum theoretical error of approximation (E_p^m)

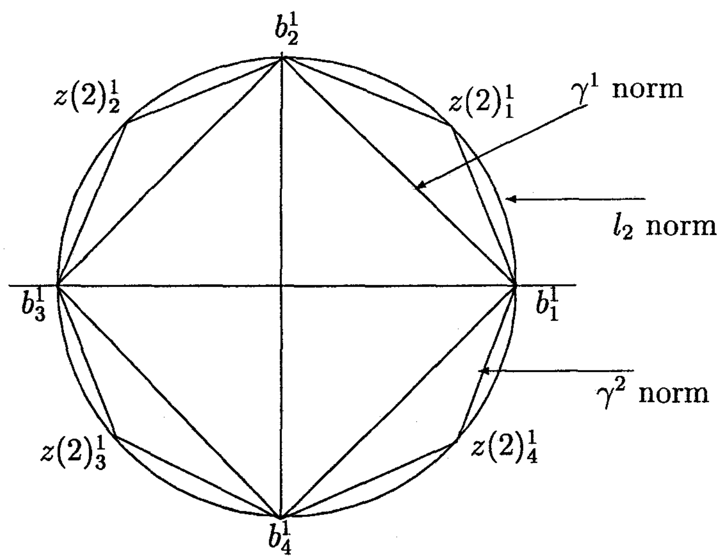


Figure 5.2: Approximation of the Euclidean norm by polyhedral norms.

which has the maximum γ^k-norm value. Let $z(p)_i^k$ for $i = 1, \cdots, 2^{k+1}$ this point, thus the maximum error obtained in the approximation of the unit ball B_p by the unit ball P^k is given by $E_p^k = \max_{i=1,\cdots,2^{k+1}} \gamma^k(z(p)_i^k) - 1$ (see Figure 5.2).

It can be proven that for the l_2 norm (Euclidean norm) the maximum approximation error of its unit ball by P^m is given by $E_2^m = \frac{1}{\cos\frac{\pi}{2^{m+1}}} - 1$ for all $m \in I\!N$. In Table 5.1 we show the error (E_p^m) obtained in the approximation of the unit ball of l_p-norms (for some values of $p > 1$) by the unit ball of the sequence of polyhedral norms γ^m above described. Once these values are known, $\delta_i = \max_{z \in B(0,2r_A^1)} |d^m(a_i, z) - d(a_i, z)| \leq 3r_A^1 E_k^m$ for all $i \in I$, where $r_A^1 = \max_{i \in I} ||a - i||_1$.

Example 5.1 Consider the 1-SFLMR problem on the Euclidean space $(I\!R^2, ||\cdot||_2)$ with 3 classes of customers located at the points:

Customer 1: $a_1 = (0,0)$
Customer 2: $a_2 = (\frac{1}{2\tan\frac{\pi}{2^4}}, \frac{1}{2})$
Customer 3: $a_3 = (-\frac{1}{2\tan\frac{\pi}{2^4}}, \frac{1}{2})$

The parameters of the problem are $\alpha = \beta = 1$, and the rest of values are shown in the following table:

i	λ_i	w_i	$\hat{r}_i$
1	0.5	0	2.6
2	0.35	0	5.1
3	0.15	0	5.25

In order to solve this problem we use the polyhedral norm γ^4 to obtain approximated solutions.

Intersection point	Policy						
	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	(1,1,1)
(0,0)	-1.3	–	–	–	-1.2301	–	-1.3569
(2.5136,0.5)	–	-1.785	–	–	–	-1.0366	-0.6051
(1.2071,0.5)	–	-09111	–	-0.9354	–	–	-0.8257
(0.7483,0.5)	-0.5862	–	–	-0.9754	–	–	-0.9054
(0.5,0.5)	-0.6992	–	–	-0.9846	–	–	-0.9409
(0.3341,0.5)	-0.7683	–	–	-0.9796	–	–	-0.9562
(0.2071,0.5)	-0.8101	–	–	-0.9659	–	–	-0.9599
(0.0994,0.5)	-0.8328	–	–	-0.9453	–	–	-0.95538
(0,0.5)	-0.84	–	–	–	-0.8976	–	-0.9436
(-0.0994,0.5)	-0.8328	–	–	–	-.09093	–	-0.9246
(-0.2071,0.5)	-0.8101	–	–	–	-0.9098	–	-0.8967
(-0.3341,0.5)	-0.7683	–	–	–	-0.8969	–	-0.8566
(-0.5,0.5)	-0.6992	–	–	–	-0.8648	–	-0.7977
(-0.7483,0.5)	–	–	-0.4132	–	-0.8005	–	-0.7057
(-1.2071,0.5)	–	–	-0.4945	–	-0.6695	–	-0.546
(-2.5136,0.5)	–	–	-0.7875	–	-0.3533	–	-0.2057

Table 5.2: Values of $F((x_1, x_2, x_3), z)$ in Example 5.1

By Theorem 5.3 we know that there exists an intersection point which is an optimal location of the 1-SFLMR problem on the metric space $(\mathbb{R}^2, \gamma^4)$. Furthermore, there is an optimal location in $\mathcal{C}(A)$ where, $\mathcal{C}(A)$ is the convex hull of $A = \{a_1, a_2, a_3, \}$, (see Proposition 4.2 in Durier and Michelot (1985)[33]). Hence, we have to solve 16 pure queuing problems. Once these problems are solved we obtain an approximated solution (x', z') of the 1-SFLMR problem on the Euclidean space $\mathbb{R}^2$. Besides, as $z' \in \mathcal{C}(A)$ we can ensure that, $\delta_1 = 0.0124$, $\delta_2 = \delta_3 = 0.0243$, where δ_i is given by, $\max_{z \in \mathcal{C}(A)} |d^4(a_i, z) - ||a_i - z||_2| < \delta_i$. Then, the maximum theoretical error is 0,0894.

In Table 5.2 we show the values of $F(x, z)$ for each intersection point and each evaluated policy. We observe that the optimal approximated solution is located at the point (2.5136,0.5) where the optimal policy is to accept only the customers of the type 2. Recall that the objective function is $F(x, z) + \sum_{i=1}^{4} \lambda_i \hat{r}_i$.

Example 5.2 Now, consider the 1-SFLMR problem on the Euclidean space $(\mathbb{R}^2, || \cdot ||_2)$ with 4 classes of customers located at the points:

Customer 1: $a_1 = (0, 0)$
Customer 2: $a_2 = (\frac{1}{2 \tan \frac{\pi}{2^4}}, \frac{1}{2})$
Customer 3: $a_3 = (-\frac{1}{2 \tan \frac{\pi}{2^4}}, \frac{1}{2})$
Customer 4: $a_4 = (0, 1)$

The parameters of the problem are $\alpha = \beta = 1$, and the rest of values are shown in the following table:

i	λ_i	w_i	$\hat{r}_i$
1	0.25	0	2.75
2	0.21	0	5.1
3	0.19	0	5.25
4	0.35	0	2.6

In order to solve this problem we use the polyhedral norm γ^4 to obtain approximated solutions.

In this case, the number of intersection points in the convex hull of $\{a_1, a_2, a_3, a_4\}$ is 101. The maximum theoretical error is 0,0801. The optimal approximated solution is located at the point (0,1) where the optimal policy is to accept the customers of types 1 and 4 and F((1,0,0,1),(0,1))=-1.078.

Example 5.3 Consider the 1-SFLMR problem on the Euclidean space $(\mathbb{R}^2, || \cdot ||_2)$ with 4 classes of customers located at the points:

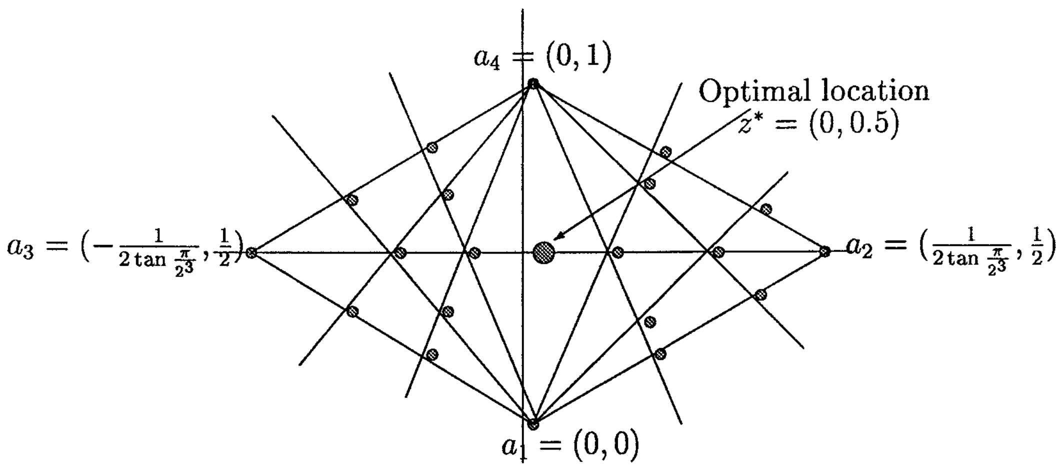


Figure 5.3: Example 5.3. Intersection points

Customer 1: $a_1 = (0, 0)$

Customer 2: $a_2 = (\frac{1}{2\tan\frac{\pi}{2^3}}, \frac{1}{2})$

Customer 3: $a_3 = (-\frac{1}{2\tan\frac{\pi}{2^3}}, \frac{1}{2})$

Customer 4: $a_4 = (0, 1)$

The parameters of the problem are $\alpha = \beta = 1$, and the rest of values are shown in the following table:

i	λ_i	w_i	$\hat{r}_i$
1	0.25	0	1.5
2	0.25	0	2.5
3	0.15	0	2.75
4	0.35	0	1.35

In order to solve this problem we use the polyhedral norm γ^3 to obtain approximated solutions.

For this example, the number of intersection points in the convex hull of $A = \{a_1, a_2, a_3, a_4\}$ is 21 (see Figure 5.3). Hence, we have to solve 21 pure queuing problems. The maximum theoretical error is 0,09889.

In Table 5.3 we show the values of $F(x, z)$ for each intersection point and each policy. In this case, we reflect only those policies included in the natural parameterization of the path through $v^0, \dots, v^4$ where, v^i, $i = 0, \dots, 4$ are the vectors with 1 in the first i entries and 0 everywhere else. Recall that the order of the customers in the natural parameterization is taken according with the increasing sequence of their modified service time $\frac{s_i(z)}{\bar{r}_i - r_i(z)}$ (see Assumptions A1 and A2 in Carrizosa et al. (1998)[20]). We observe that the optimal approximated solution is located at the point (0,0.5) where the optimal policy is to accept all the customers.

5.5 Efficiency of the c-SFLMR model

In conclusion, once a finite dominating set of candidates for optimal location exists the resolution of Problem (5.1) reduces to solve a finite number of unimodal one-dimensional optimization problems which can be solved (up to a given accuracy ε) by a wide variety of well-known methods (see e.g. Chapter 8 of Bazaraa & Shetty (1979)). Hence, Problem (5.1) can be efficiently solved for those cases covered by theorems 5.2 and 5.3. Besides, since the set of polyhedral norms is dense in the set of all norms in $\mathbb{R}^q$, we can give good approximated solutions to any queuing-location problem (5.1) in $\mathbb{R}^q$ for any norm, as shown in the above examples.

As we asserted in the introduction the c-SFLMR is an extension of the c-SFLM queuing-location problem, where both the facility location and coverage area (admission rates for each group of clients) are decision variables. This is a very interesting extension because, as shown in the next example, it reduces even in very easy examples the total cost of the model with respect to the classical acceptation policy improving the efficiency in the service use. This fact shows the conflict that exists between the overall coverage of a region (covering all the groups of customers) and the efficiency achieved when admission policies are allowed. An easy example illustrates this remark showing that the improvement of the c-SFLMR with respect to the c-SFLM can be even unbounded.

Example 5.4 Consider the 1-SFLMR problem on the network
$G = (\{1,2,3\}, \{(1,2),(1,3),(2,3)\})$ depicted in Figure 5.4. The customers are located at the nodes of the network, and the distances between nodes appear on the edges. The parameters of the problem are $\alpha = \beta = 1$, and the rest of values are shown in the following table:

Intersection point	Policy			
	(1,0,0,0)	(1,1,0,0)	(1,1,1,0)	(1,1,1,1)
(0,0)	-0.375	-0.4946	-0.5844	-0.5406
(0,0.5)	-0.2222	-0.4212	-0.5259	-0.6649
(0,1)	-0.4725	-0.5761	-0.6484	-0.5725
(1.2071,0.5)	-0.625	-0.5075	-0.4285	-0.4571
(0.7071,0.2929)	-0.4313	-0.5075	-0.4934	-0.5406
(0.3535,0.1464)	-0.2549	-0.5075	-0.5398	-0.5628
(0.7071,0.7071)	-0.43134	-0.4948	-0.4956	-0.5595
(0.5,0.5)	-0.3809	-0.4776	-0.5443	-0.5882
(0.2929,0.2929)	-0.2459	-0.4948	-0.5415	-0.6112
(0.3535,0.8535)	-0.2985	-0.5367	-0.5639	-0.5896
(0.2929,0.7071)	-0.286	-0.5206	-0.5741	-0.6145
(0.2071,0.5)	-0.2111	-0.4437	-0.5701	-0.6284
(-0.2071,0.5)	-0.2111	-0.3907	-0.5325	-0.6189
(-0.2929,0.7071)	-0.286	-0.4662	-0.5302	-0.5976
(-0.3535,0.8535)	-0.2985	-0.4813	-0.5031	-0.5798
(-0.2929,0.2929)	-0.2459	-0.4367	-0.4946	-0.5976
(-0.5,0.5)	-0.277	-0.3933	-0.4768	-0.5581
(-0.7071,0.7071)	-0.3064	-0.3972	-0.4133	-0.5157
(-0.3535,0.1464)	-0.2549	-0.4482	-0.4761	-0.5628
(-0.7071,0.2929)	-0.3064	-0.4047	-0.3929	-0.5157
(-1.2071,0.5)	-0.4125	-0.3473	-0.2498	-0.4571

Table 5.3: Values of $F(x, z)$ in the Example 5.3

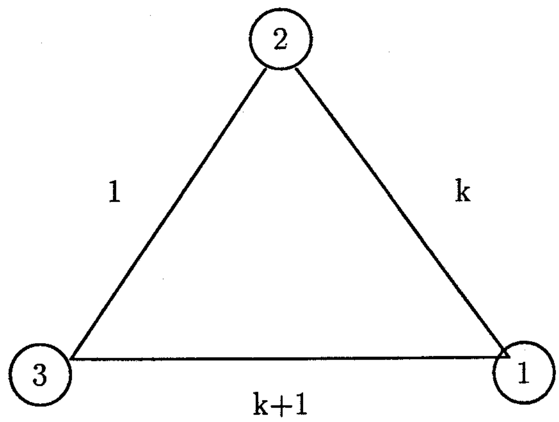


Figure 5.4: Network of Example 5.4

i	λ_i	w_i	$\hat{r}_i$
1	0.4	0	k^2
2	0.25	0	$k+1$
3	0.35	0	$k+2$

In order to solve the 1-SFLM, the results in Frenk, Labbé and Zhang (1993)[41] imply that an optimal location is obtained by finding the node z of G which minimizes the average service time $s((1,1,1),z)$. It is easily seen that the optimal location of the 1-SFLM problem is the node 2 for any $k \in N$.

Now, by Theorem 5.2 we know that a node of G is an optimal location of the 1-SFLMR problem. Hence, we have just to solve three pure queuing problems (one for each node) using the approach earlier described in this section. After some algebra one finds the optimal solution of our 1-SFLMR at the point $(x^*, z^*) = ((1,0,0),1)$ for all $k \geq 2$. Then, let us compare the costs associated with the optimal solutions of both problems. Since the total cost per unit time is $\hat{F}(x,z) = \sum_{i=1}^{3} \lambda_i \hat{r}_i (1 - x_i \Psi^1(x,z)) + \Psi^1(x,z) \sum_{i=1}^{3} \lambda_i d(a_i, z) x_i$ we obtain:

$$\begin{aligned} \hat{F}((1,1,1),2) = &\quad (0.4k^2 + 0.6k + 1.95)\frac{0.4k + 0.35}{0.4k + 1.35}, \\ \hat{F}((1,0,0),1) = &\quad 0.6k + 0.95, \end{aligned}$$

thus,

$$\lim_{k\to\infty} \frac{\hat{F}((1,1,1),2)}{\hat{F}((1,0,0),1)} = +\infty.$$

This shows that drastic reductions in costs can be obtained if admission policies are allowed. This simple example shows how in order to improve the use of the resources it is sometimes convenient not to cover the whole population. In this particular case, the economic behaviour of the system is optimized considering only the customers in node 1 .

Summarizing, this section we show how the use of anticipated policies improves the throughput of this queuing-location demand-responsive model. Nevertheless, this fact cannot be taken as a recommendation for the implementation of these policies in all the cases. These policies can only be implemented when they are used for emergency services provided that backup systems are available. On the other hand, if an anticipated policy is found to be optimal in a situation and a backup system does not exist it would be advisable to perform a trade off analysis. The goal of this analysis would be to determine whether or not to implement the backup system in order to globally save resources when combined with the anticipated policy. Assume that $(\hat{x}, \hat{z})$ is the optimal solution of Problem (5.1) and $(\mathbf{1}, z_1)$ the optimal solution for the c-SFLM model. The saving S achieved using $(\hat{x}, \hat{z})$ instead of $(\mathbf{1}, z_1)$ during a planning horizon of T time units would be $S := T(\hat{F}(\mathbf{1}, z_1) - \hat{F}(\hat{x}, \hat{z}))$. Therefore, any amount of money smaller than S could be invested either to implement or to improve the backup service and still the whole system would save money.

5.6 The model with two service systems

An extension of the c-SFLMR model can be to consider a new model with two service systems: a main service system and a secondary service system, so that, the rejected customers due to the admission policy in the main service system are accepted in the secondary service system if some server is idle.

We consider the location problem of two service systems with c servers each one, imbedded in a metric space N. We denote by $w_i^1 \geq 0$ the mean time of on-scene service of customers of type i, served in the main service system and by $w_i^2 \geq 0$ the mean time of on-scene service of customers of type i, served in the secondary service system. If the main service system is located at z^1 and the secondary service system at z^2, the expected service time of a customer of type i served in the main service system is $s_i^1(z^1) = w_i^1 + \beta d(a_i, z^1)$ and the expected service time of a customer of type i served in the secondary service system is $s_i^2(z^2) = w_i^2 + \beta d(a_i, z^2)$. Any accepted i-customer in the main service system (respectively, in the secondary service system) induces a cost to the system proportional to the travel time given by $r_i^1(z^1) = \alpha^1 \beta d(a_i, z^1)$ (respectively, $r_i^2(z^2) = \alpha^2 \beta d(a_i, z^2)$). If all the service units are busy in the main or in the secondary service system, any lost i-customer induces a cost $\hat{r}_i \geq 0$.

To deal with this model we introduce three kinds of decision variables z^1, z^2 and x_i, $i = 1, \dots, n$. z^1 is the location for the main service system, z^2 is the location for the secondary service system and each x_i stands for the proportion of i-customers which are accepted in the main service system when some server is idle. Then, $1 - x_i$ stands for the proportion of i-customers which are accepted in the secondary service system when some server is idle. The goal is to find the locations z^1 and z^2 of the two facilities and the proportions $x = (x_1, \dots, x_n)$ which minimizes the expected total cost per unit time.

Analysis of the model.

Obviously, given $z^1 \in N$ and $x \in [0,1]^n$ the main service system behaves as a $M/G/c/c$ queue with arrival rate $\lambda^1(x)$, mean service time $s^1(x, z^1)$ and throughput $\rho^1(x, z^1)$ given by

$$\begin{aligned} \lambda^1(x) &= \textstyle\sum_{i=1}^n \lambda_i x_i, \\ s^1(x, z^1) &= \textstyle\sum_{i=1}^n \frac{\lambda_i x_i}{\lambda^1(x)} s_i^1(z^1) \\ \rho^1(x, z^1) &= \lambda^1(x) s^1(x, z^1). \end{aligned}$$

Let $\Psi_c^1(x, z^1)$ denote the probability that not all the servers in the main service system are busy. It is well-known (see e.g. Medhi (1991)[75]) that in our model

$$\Psi_c^1(x, z^1) = 1 - \frac{(\lambda^1(x) s^1(x, z^1))^c / c!}{\sum_{k=0}^c (\lambda^1(x) s^1(x, z^1))^k / k!}.$$

Similarly, given $z^2 \in N$ and $x \in [0,1]^n$ the secondary service system behaves as a $M/G/c/c$ queue with arrival rate $\lambda^2(x)$, mean service time $s^2(x,z^2)$ and throughput $\rho^2(x,z^2)$ given by

$$\begin{aligned} \lambda^2(x) = & \sum_{i=1}^n \lambda_i(1-x_i) = 1 - \lambda^1(x), \\ s^2(x,z^2) = & \sum_{i=1}^n \frac{\lambda_i(1-x_i)}{\lambda^2(x)} s_i^2(z^2) \\ \rho^2(x,z^2) = & \lambda^2(x)s^2(x,z^2). \end{aligned}$$

Let $\Psi_c^2(x,z^2)$ denote the probability that not all the servers in the secondary service system are busy. Then,

$$\Psi_c^2(x,z^2) = 1 - \frac{(\lambda^2(x)s^2(x,z^2))^c/c!}{\sum_{k=0}^c(\lambda^2(x)s^2(x,z^2))^k/k!}.$$

The expected total cost per unit time is given by the sum of the cost of the accepted plus the rejected customers. The expected number of i-customers per unit time which receive service in the main service system is $\lambda_i x_i \Psi_c^1(x,z^1)$ and the expected number of i-customers per unit time which receive service in the secondary service system is $\lambda_i(1-x_i)\Psi_c^2(x,z^2)$. The expected number of rejected i-customers per unit time in both, main and secondary service system is $\lambda_i x_i(1-\Psi_c^1(x,z^1)) + \lambda_i(1-x_i)(1-\Psi_c^2(x,z^2))$ then the expected cost per unit time is:

$$\Psi_c^1(x,z^1)\sum_{i=1}^n \lambda_i x_i(r_i^1(z^1) - \hat{r}_i) + \Psi_c^2(x,z^2)\sum_{i=1}^n \lambda_i(1-x_i)(r_i^2(z^2) - \hat{r}_i) + \sum_{i=1}^n \lambda_i\hat{r}_i.$$

Since the term $\sum_{i=1}^n \lambda_i\hat{r}_i$ does not depend on the decision variables (x,z^1,z^2) one can consider equivalently the following objective function for the model:

$$\begin{aligned} F(x,z^1,z^2) &= F^1(x,z^1) + F^2(x,z^2) \\ &= [\Psi_c^1(x,z^1)(r^1(x,z^1) - \hat{r}^1(x))] + [\Psi_c^2(x,z^2)(r^2(x,z^2) - \hat{r}^2(x))] \end{aligned}$$

where

$$\begin{aligned} r^1(x,z^1) = & \sum_{i=1}^n \lambda_i r_i^1(z^1)x_i, \\ r^2(x,z^2) = & \sum_{i=1}^n \lambda_i r_i^2(z^2)(1-x_i), \\ \hat{r}^1(x) = & \sum_{i=1}^n \lambda_i\hat{r}_i x_i, \\ \hat{r}^2(x) = & \sum_{i=1}^n \lambda_i\hat{r}_i(1-x_i). \end{aligned}$$

Therefore the model is formulated as:

$$\min_{x\in[0,1]^n, z^1\in N, z^2\in N} F(x,z^1,z^2) = F^1(x,z^1) + F^2(x,z^2) \tag{5.8}$$

where

$$\begin{aligned} F^1(x,z^1) &= \Psi_c^1(x,z^1)(r^1(x,z^1) - \hat{r}^1(x)) \\ F^2(x,z^2) &= \Psi_c^2(x,z^2)(r^2(x,z^2) - \hat{r}^2(x)) \end{aligned}$$

We will call this problem the c-Server-Two-Facilities-Loss-Median model with rejection (c-TFLMR). The function F is sum of two fractional functions where each one of them has no good structural properties: 1) they are ratio of non-linear functions; 2)they do not exhibit neither convexity nor monotonicity properties and; 3) they are not differentiable.

These reasons lead us to avoid the use of standard optimization methods and to exploit some properties of the problem in its location face. We will prove that there exist a finite set of candidates for optimal solutions in the location variables. Then, we will need to solve the allocation subproblem for each location candidate.

5.7 Localization results and solution procedure for the c-TFLMR

In this section we derive localization results for the c-TFLMR. For each fixed $\bar{x} \in [0,1]^n$, we observe that the c-TFLMR problem can be separated into two subproblems:

$$\min_{z^1\in N} F^1(\bar{x},z^1) = \Psi_c^1(\bar{x},z^1)(r^1(\bar{x},z^1) - \hat{r}^1(\bar{x})). \tag{5.9}$$

and,

$$\min_{z^2\in N} F^2(\bar{x},z^2) = \Psi_c^2(\bar{x},z^2)(r^2(\bar{x},z^2) - \hat{r}^2(\bar{x})). \tag{5.10}$$

Every one of the these subproblems is a c-SFLMR problem for a fixed $\bar{x} \in [0,1]^n$ and they can be solved independently. Indeed, as we did in Section 5.2 we can introduce the two following auxiliary problems for each fixed $\bar{x} \in [0,1]^n$.

$$\min_{z^1 \in N} t^1(\bar{x}, z^1) := \beta \sum_{i=1}^{n} \frac{\lambda_i \bar{x}_i}{\lambda^1(\bar{x})} d(a_i, z^1), \tag{5.11}$$

$$\min_{z^2 \in N} t^2(\bar{x}, z^2) := \beta \sum_{i=1}^{n} \frac{\lambda_i (1 - \bar{x}_i)}{\lambda^2(\bar{x})} d(a_i, z^2), \tag{5.12}$$

where the objective functions $t^1(x, z^1)$ and $t^2(x, z^2)$ are respectively the expected travel time for the feasible solution $(\bar{x}, z^1)$ of Problem (5.9) and for the feasible solution $(\bar{x}, z^2)$ of Problem (5.10). Therefore, both, Problem (5.11) and Problem (5.12) are a Weber or median problem on the metric space N with respect to the set of existing facilities given by $A = \{a_1, \dots, a_n\}$. As we did in Section 5.2, we prove in the next theorem that an optimal solution of (5.8) can be obtained among the minimizers of $t^1(x, z^1)$ and $t^2(x, z^2)$.

Theorem 5.6 *Let $\bar{x}$ be a nonzero vector in $[0,1]^n$. If $\bar{z}^1$ solves problem $\min_{z^1 \in N} t^1(\bar{x}, z^1)$ and $\bar{z}^2$ solves problem $\min_{z^2 \in N} t^2(\bar{x}, z^2)$, then $(\bar{z^1}, \bar{z^2})$, solves the problem $\min_{z^1 \in N, z^2 \in N} F(\bar{x}, z^1, z^2)$.*

Proof. Once $F(\bar{x}, z^1, z^2)$ can be separated into $F^1(\bar{x}, z^1)$ and $F^2(\bar{x}, z^2)$ the proof runs parallel to the proof of Theorem 5.1. □

The above theorem implies that if there exists a finite dominating set D of candidates to be optimal solution for all the median problems (5.11) and (5.12) which is independent of the admission policy x, then D is also a finite dominating set for the Problem (5.8). We can use this result to give a procedure to solve Problem (5.8). We introduce the following problem for each fixed $\bar{z}^1$ and $\bar{z}^2$:

$$\min_{x \in [0,1]^n} F(x, \bar{z}^1, \bar{z}^2) = F^1(x, \bar{z}^1) + F^2(x, \bar{z}^2) \tag{5.13}$$

Problem (5.13) is a nonlinear programming problem with linear constraints and differentiable but non convex objective function. The objective function $F(x, \bar{z}^1, \bar{z}^2)$ is sum of two fractional non-linear functions. These functions are well-known to be extremely hard to optimize. There exist several methods adapted to solve these kinds of problems when the two terms of the sum are the ratio of linear functions (see e.g. Schaible (1995)[93]). However, as far as we know no methods have been proposed in the literature for the general case. This fact leads us to develop "ad hoc" optimization heuristics for the variable x minimization phase.

A first strategy is to take advantage of the differentiable nature of F in x. This property allows to apply methods of feasible directions to find stationary points which at least are local optima. This approach can be combined with bounds in a branch and bound procedure. Indeed, we can obtain lower and upper bounds for this problem. Indeed, if x_1^* is the optimal solution of the pure queuing problem, $\min_{x\in[0,1]^n} F^1(x,\bar{z}^1)$ and x_2^* is the optimal solution of the pure queuing problem, $\min_{x\in[0,1]^n} F^2(x,\bar{z}^2)$ we obtain the following inequalities:

$$F^1(x_1^*,\bar{z}^1)+F^2(x_2^*,\bar{z}^2)\leq \min_{x\in[0,1]^n} F(x,\bar{z}^1,\bar{z}^2)\leq \min\{F(x_1^*,\bar{z}^1,\bar{z}^2),F(x_2^*,\bar{z}^1,\bar{z}^2)\}$$

Then, $\underline{F}(\bar{z}^1,\bar{z}^2):=F^1(x_1^*,\bar{z}^1)+F^2(x_2^*,\bar{z}^2)$ is a lower bound for Problem (5.13) and $\bar{F}(\bar{z}^1,\bar{z}^2):=\min\{F(x_1^*,\bar{z}^1,\bar{z}^2),F(x_2^*,\bar{z}^1,\bar{z}^2)\}$ is a upper bound for Problem (5.13).

Procedure:

Initialization.

Let $D^*:=\emptyset$

For each $(z^1,z^2)\in D$ obtain $\underline{F}(z^1,z^2)$ and $\bar{F}(z^1,z^2)$.

If $\bar{F}(z_i^1,z_j^2)\leq \underline{F}(z_k^1,z_l^2)$ for some $(z_i^1,z_j^2)\in D$ and some $(z_k^1,z_l^2)\in D$ then, let $D:=D\setminus\{(z_k^1,z_l^2)\}$.

Steps.

Repeat

Let $(z_0^1,z_0^2)\in \arg\min_{(z^1,z^2)\in D}\underline{F}(z^1,z^2)$

Solve Problem (5.13) for (z_0^1,z_0^2) fixed. Let $F^*(z_0^1,z_0^2)$ the optimal objective value obtained.

Let $D:=D\setminus\{(z_0^1,z_0^2)\}$ and $D^*:=D^*\cup\{(z_0^1,z_0^2)\}$.

If $F^*(z_0^1,z_0^2)\leq \underline{F}(z_k^1,z_l^2)$ for some $(z_k^1,z_l^2)\in D$ then, let $D:=D\setminus\{(z_k^1,z_l^2)\}$.

until $D=\emptyset$.

Output.

The optimal objective value for Problem (5.8) is given by $\min_{(z^1,z^2)\in D^*} F^*(z^1,z^2)$.

As we saw in Section 5.3 this procedure can be applied in the next cases where there exists a finite dominating set D of candidates to be optimal solution for all the median problems:

- In discrete problems.
- In network problems when the customers are placed at the nodes of the network.
- In continuous problems with polyhedral norms. For this kind of problems, there exists an intersection point which is an optimal location of the median problem
- In continuous problems with any norm. In this case, since the set of polyhedral norms is dense in the set of all norms in $\mathbb{R}^q$ we can give good approximated solutions as we have described in Section 5.4. It is straightforward to prove that for Problem (5.8) the error committed in the approximation is least than $2(\varepsilon_1 + \varepsilon_3 \sum_{i=1}^{n} \lambda_i \hat{r}_i)$ where ε_1 and ε_3 are established in Corollary 5.1.

Appendix A

Appendix

In this appendix we include several concepts of convex analysis (the interested readers can find further details in the books of Hiriart-Urruty & Lemaréchal (1993)[60], or Rockafellar (1970)[91]) and we give a brief description of the subgradient method (further details can be found in the papers of Held et al. (1974) [57] or Fisher (1981) [39]).

Concepts of convex analysis

- Let C be a nonempty convex set of $\mathbb{R}^n$.
  - We say that $x \in C$ is an *extreme point* of C if there are not two different points x_1 and x_2 in C such that $x = \alpha x_1 + (1-\alpha)x_2\,, \quad 0 < \alpha < 1$. The set of extreme points of C is denoted by *ext* C.
  - We say that $d \in \mathbb{R}^n$ is a *direction* of C if $x + \lambda d \in C \quad \forall x \in C, \quad \forall \lambda > 0$. The set of directions of C is called the *asymptotic cone* and it is denoted by C_∞ (see [60] to check that if $\varphi(\cdot)$ is the semigauge of C, C_∞ is given by (2.2)). If C is bounded, $C_\infty = \{0\}$.
  - $d \in \mathbb{R}^n$ is an *extreme direction* of C if d is a direction of C and there are not two different directions in C, d_1 and d_2 with $d_1 \neq \mu d_2 \quad \forall \mu > 0$ such that $d = \lambda_1 d_1 + \lambda_2 d_2\,, \quad \lambda_1, \lambda_2 > 0$.
    We denote the set of extreme directions of C by C_∞^{ext}.

- The *polar set* of C, C^* is

$$C^* = \{s \in I\!R^n \ : < s, c > \leq 1 \text{ for all } c \in C\} \tag{A.1}$$

C^* is a closed convex set in $I\!R^n$, containing the origin. If C is a closed convex set containing the origin then $C^{**} = C$.

- A *cone* K is a set such that the half-line $\{\alpha x \ : \ \alpha > 0\}$ is entirely contained in K whenever $x \in K$. A *convex cone* is, of course, a cone which is convex.

- The *polar cone* of a nonempty convex cone K is

$$K^o = \{d \in I\!R^n \ : < d, x > \leq 0 \text{ for all } x \in K\} \tag{A.2}$$

K^o is a closed convex cone.

- The direction $s \in I\!R^n$ is said to be *normal* to a set C at x if $< s, c - x > \leq 0$ for all $c \in C$. The set of all such directions is denoted by $N_C(x)$, the *normal cone* to C at x.

- Let S be a set of points of $I\!R^n$. We define the *convex hull*, *conv*(S), of S to be the smallest convex set in $I\!R^n$ such that $S \subset$ *conv*(S). The convex hull of S is the set of all convex combinations of elements of S.

- A *conical combination* of elements $d_1, \cdots, d_k$ in $I\!R^n$ is an element of the form $\sum_{i=1}^{k} \lambda_i d_i$ with $\lambda_i \geq 0 \ \forall i = 1, \cdots, k$.

- Let S be a nonempty set in $I\!R^n$. We define the *conical hull*, *cone*(S), of S to be the smallest convex cone in $I\!R^n$ such that $S \subset$ *cone*(S). The conical hull of S is the set of all conical combinations of elements of S.

- A function $\varphi : I\!R^n \longrightarrow I\!R \cup \{+\infty\}$ is said to be *sublinear* if it is convex and positively homogeneous.

- Given C a nonempty set in $\mathbb{R}^n$, the *support function* of C, $\sigma_C : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is given by

$$\sigma_C(x) = \sup\{< c, x >: \ c \in C\} \tag{A.3}$$

A support function is closed and sublinear.

- Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ a convex function. The function φ^* defined by

$$\varphi^*(s) = \sup_{x \in \mathbb{R}^n} \{< s, x > -\varphi(x) \ : \ x \in dom(\varphi)\}$$

is said to be the *conjugate* of φ. φ^* is closed and convex.

- The *subdifferential* $\partial\varphi(x)$ of a convex function φ at x is the set of $s \in \mathbb{R}^n$ verifying

$$\varphi(y) \geq \varphi(x) + < s, y - x > \qquad \forall y \in \mathbb{R}^n. \tag{A.4}$$

Note here that $\partial\varphi(x)$ is empty if $x \notin dom(\varphi)$.

Each element in the subdifferential set is named a *subgradient* of φ at x.

- *Theorem VI.4.3.1 in Hiriart-Urruty & Lemaréchal book[60].*

Let $\begin{array}{rcl} F : \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ x & \longrightarrow & (f_1(x), \cdots, f_m(x)), \quad f_i \text{ convex } \forall i = 1, \cdots, m \end{array}$

and $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ convex and increasing componentwise. For all $x \in \mathbb{R}^n$,

$$\partial(g \circ F)(x) = \{\sum_{i=1}^{m} \rho_i s_i \ : \ (\rho_1, \cdots, \rho_m) \in \partial g(F(x)), \ s_i \in \partial f_i(x) \ \forall i\}$$

The subgradient method.

The subgradient method is one of the most popular methods used to optimize lagrangean duals. Assume that we have the following (mixed) integer programming problem:

$$\begin{array}{lll} (P) & \min & cx \\ & \text{s.t.} & Ax \geq b \\ & & Dx \leq e \\ & & x \geq 0 \text{ and integral,} \end{array}$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $D \in \mathbb{R}^{p \times n}$ and $e \in \mathbb{R}^p$.

Let $(LR(\lambda))$ be a Lagrangean relaxation of Problem (P) formulated as:

$$
\begin{array}{lll}
(LR(\lambda)) & \min & cx + \lambda(b - Ax) \\
 & \text{s.t.} & Dx \leq e \\
 & & x \geq 0 \text{ and integral,}
\end{array}
$$

where $\lambda = (\lambda_1, \cdots, \lambda_n)$ is a vector of Lagrange multipliers.

We denote by $v(\cdot)$ the optimal objective value of Problem $(\cdot)$. Then, the Lagrangean dual of Problem (P) with respect to the constraint set $Ax \geq b$ is given by,

$$(DL) \qquad \max_{\lambda} v(LR(\lambda))$$

In order to solve Problem (DL), it is worth nothing that for each fixed multiplier λ the function $v(LR(\lambda))$ is a piecewise linear, concave function because it can be written as a pointwise minimum of affine-linear functions. This property guarantees that any local maximum is also global. Moreover, this function is continuous and we can obtain the subdifferential set of $v(LR(\lambda))$ at any point with the appropriate changes made for concave, rather than convex functions.

Let x_λ^* be an optimal solution of Problem $(LR(\lambda))$, then a subgradient of $v(LR(\lambda))$ at λ is given by $b - Ax_\lambda^*$. If the set of optimal solution of Problem $(LR(\lambda))$ is a singleton, then $b - Ax_\lambda^*$ is the gradient. It is well-known that λ^* is an optimal solution of Problem (DL) if and only if 0 is a subgradient of $v(LR(\lambda))$ at λ^* (see Fisher (1981) [39]).

These properties allow us to use the subgradient method to solve the dual problem.

The subgradient method is an adaptation of the gradient method in which the gradient is replaced by any subgradient. This method is an iterative procedure which start with some initial multipliers λ_0. In each iteration k, and once the multipliers λ_k are known, it solves the relaxed problem $LR(\lambda_k)$. The multipliers of the next iteration are obtained by means of

$$\lambda_{k+1} = \max\left\{0, \lambda_k + t_k(b - Ax_{\lambda_k}^*)\right\}$$

where t_k is a scalar which gives the length of the step and $x^*_{\lambda_k}$ is an optimal solution of Problem $LR(\lambda_k)$. Some theoretical results relating to an appropriate choice of step size for the subgradient algorithm have been given. The most general result which assures the global convergence for this method (see Poljak (1967)[87]) is,

$$\lim_{k \to \infty} t_k = 0, \qquad \sum_{k=0}^{\infty} t_k = \infty$$

Held et al. (1974)[57] propose the following choice of t_k:

$$t_k = u_k \frac{v(DL) - v(LR(\lambda_k))}{||b - Ax^*_{\lambda_k}||^2}$$

where $||\cdot||$ denotes the Euclidean norm, u_k is a scalar satisfying $0 < u_k \leq 2$ and $v(DL)$ is the optimal objective value of problem (DL). However, an upper bound on $v(DL)$ is used in practice instead of the recommended but unknown optimal objective value of problem (DL). This upper bound is frequently obtained by applying a heuristic to Problem (P). Often the sequence u_k is determined by setting $u_0 = 2$ for $2n$ iterations (where n is a measure of the problem size) and successively halving both, u_k and the number of iterations whenever $v(LR(\lambda_k))$ fails to increase in the fixed number of iterations. The number of iterations is halved until this number reaches some threshold value n_0, then u_k is halved whenever $v(LR(\lambda_k))$ fails to increase in n_0 iterations. Observe that such a procedure would involve checking if the null vector belongs to the subgradient set at the current iteration point but it is not easy to check this. In practice, the method is stopped when the resulting t_k is sufficiently small or after performing a specified number of iterations.

This rule has performed well empirically, even though it is not guaranteed to satisfy the sufficient condition given above to achieve convergence.

On the other hand, the election of the initial set of multipliers is crucial because the quality of the initial solution depends very much of this choice. Usually $\lambda_0 = 0$ is the most natural choice but in some cases one can do better. The multiperiod two-echelon multicommodity capacitated plant location problem studied in Chapter 3 is a good example.

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