Stochastic 3D Globally Modified Navier-Stokes Equations: Weak Attractors, Invariant Measures and Large Deviations

Tomás Caraballo[a,b]*, Zhang Chen[c]†, Dandan Yang[c]‡

[a]Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, c/Tarfia s/n, 41012-Sevilla, Spain

[b]Department of Mathematics, Wenzhou University, Wenzhou 325035, China

[c]School of Mathematics, Shandong University, Jinan 250100, China

Abstract. This paper is mainly concerned with the asymptotic dynamics of *non-autonomous* stochastic 3D globally modified Navier-Stokes equations driven by *non-linear* noise. Based on the well-posedness of such equations, we first show the existence and uniqueness of weak pullback mean random attractors. Then we investigate the existence of (periodic) invariant measures, the zero-noise limit of periodic invariant measures and their limit as the modification parameter $N \to N_0 \in (0, +\infty)$. Furthermore, under weaker conditions, we obtain the existence of invariant measures as well as their limiting behaviors when the external term is independent of time. Finally, by using weak convergence method, we establish the large deviation principle for the solution processes.

Keywords. Stochastic 3D globally modified Navier-Stokes equations, weak mean attractor, periodic invariant measure, limit measure, large deviation.

AMS 2020 Mathematics Subject Classification. 60H15, 35B41, 37L40, 60F10.

Contents

*E-mail address: caraball@us.es

†E-mail address: zchen@sdu.edu.cn

‡E-mail address: danyang@mail.sdu.edu.cn

1 Introduction

In this paper, we investigate the dynamical behaviors of the following stochastic 3D globally modified Navier-Stokes equations:

$$\begin{cases} du - \nu \Delta u dt + F_N(\|u\|)\left[(u \cdot \nabla)u\right]dt + \nabla p dt = f(t)dt + \sqrt{\varepsilon}\sum_{k=1}^{\infty} G_k(u(t))dW_k(t) \text{ in } \mathcal{O} \times (\tau, \infty), \\ \text{div } u = 0 \text{ in } \mathcal{O} \times (\tau, \infty), \\ u = 0 \text{ on } \Gamma \times (\tau, \infty), \\ u(x, \tau) = u_0(x), \ x \in \mathcal{O}, \end{cases} \tag{1.1}$$

where $\nu, \varepsilon > 0$, u denotes the velocity field, $\mathcal{O} \subset \mathbb{R}^3$ is an open bounded set with regular boundary Γ, $F_N(\cdot) : (0, \infty) \to (0, 1]$ is defined as $F_N(r) = \min\left\{1, \frac{N}{r}\right\}$ where $r, N > 0$, p denotes the pressure, $f \in L^2_{loc}(\mathbb{R}; H)$ is an external force field, G_k is a nonlinear function which will be specified later, and $\{W_k(t)\}_{k\in\mathbb{N}}$ is a sequence of independent one-dimensional standard Brownian motions on some complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

Navier-Stokes equations have been studied by many experts because of their wide applications in fluids and combustion dynamics, manufacturing processes, etc. The existence, uniqueness and asymptotic behaviors of solutions of deterministic 2D Navier-Stokes equations have been studied in the literature (see, e.g., [42, 44] and the references therein). For stochastic 2D Navier-Stokes equations, the well-posedness, random attractors and invariant measures have been reported in [5, 7, 20, 35, 43, 58]. However, the uniqueness of weak solutions of 3D Navier-Stokes equations is still an open problem due to the nonlinear convection term. In order to overcome the difficulties caused by the nonlinear convection term, a class of 3D globally modified Navier-Stokes equations (GMNSE) was introduced in [10], and the existence and uniqueness of weak and strong solutions were proved. After that, the asymptotic behaviors of solutions of 3D GMNSE have also been investigated in [31, 36, 56] for deterministic cases and in [1, 23] for stochastic cases. In addition, we would like to mention another significant modification on the 3D Navier-Stokes equations called tamed 3D Navier-Stokes equations proposed by M. Röckner and X. Zhang in [39], where they investigated the existence and uniqueness of solutions. For stochastic tamed 3D Navier-Stokes equations, the existence of invariant measures and large deviations have been studied in [3, 40] and [24, 41], respectively. Inspired by the aforementioned works, this paper is devoted to the asymptotic dynamics of stochastic 3D globally modified Navier-Stokes equations.

The first aim of this paper is to study the existence and uniqueness of weak pullback mean random attractors for system (1.1). Recently, the well-posedness of stochastic 3D globally modified Navier-Stokes equations has been investigated in [1] where the nonlinear term G_k may not depend on the gradient. However, those results in [1] cannot be directly applied to such model when the nonlinear term G_k may depend on the gradient. To do that, based on Lemma 2.1 and a local monotonicity idea, we can investigate the existence and uniqueness of strong solutions (in the probabilistic sense) of (1.1), where the initial value u_0 belongs to $L^2(\Omega, \mathcal{F}_\tau; H)$. It is worth mentioning that those are different from [23] where the existence and uniqueness of strong solutions (in the sense of PDE) for stochastic globally modified Navier-Stokes equations were investigated under stronger conditions on u_0 and G_k. According to the well-posedness of (1.1), we can define a mean random dynamical system in the Bochner space $C([\tau, T], H)$ associated with (1.1). Then, by the uniform estimates of the solutions of (1.1) (see Lemma 3.1) and the theory of mean random dynamical systems in [47], the existence and uniqueness of weak pullback mean random attractors can be obtained (see Theorem 3.1). In addition, we would like to mention some studies about the pathwise random attractors [9, 26, 45, 55] and the weak mean random attractors [48, 49, 53, 54].

Another motivation of this paper is to investigate the invariant measures of *non-autonomous* stochastic 3D globally modified Navier-Stokes equations (1.1), which can provide some important information about long-term dynamics and be used to identify statistical equilibrium. In [20, 35], the existence of periodic invariant measures has been discussed for 2D stochastic Navier-Stokes equations with *linear* noise. But notice that the noise in (1.1) is *nonlinear*, which leads to those proofs in [20, 35] may not be applicable directly to (1.1). In order to obtain the existence of (periodic) invariant measures of non-autonomous stochastic system (1.1) with nonlinear noise, we need to show the weak compactness of distributions of a family of solutions. However in this case, the transition operator associated with (1.1) is inhomogeneous in time, which leads to the uniform estimates in space $L^2(\Omega, C([\tau, T], H)) \cap L^2(\Omega, L^2(\tau, T; V))$ are not enough to obtain such compactness of solutions. Because of this, we first establish the uniform estimates of solutions in a more regular space $L^2(\Omega, \mathcal{F}; V)$ when $u_0 \in L^2(\Omega, \mathcal{F}_\tau; V)$ (see Lemma 4.1). And then by the compactness method and the Prokhorov theorem, we obtain the existence of invariant measures (see Theorem 4.2). In addition, if f is periodic in t, we further show such an invariant measure is also periodic, which is different from [15, 18, 29, 32] where the periodic measures do not have invariant property. For the existence of invariant measures of autonomous systems, we refer the reader to [13, 46] for stochastic lattice systems and [6, 14, 30, 37, 52] for stochastic partial differential equations.

It is natural to wonder about the relationship of invariant measures between stochastic globally modified Navier-Stokes equations and the corresponding deterministic equations. And hence, we consider the limiting behaviors of periodic invariant measures of *non-autonomous* stochastic system (1.1) as the noise intensity $\varepsilon \to 0$, which also implies the zero-noise limit is observable as noise is non-negligible in real world. For the *autonomous* stochastic systems, the limits of invariant measures of time homogeneous transition semigroup have been investigated in [17, 33] for stochastic lattice systems and in [12, 16] for stochastic partial differential equations. However, for the non-autonomous case, the transition semigroup is no longer time homogeneous, and hence we need to establish the uniform estimates of solutions of (1.1) in $L^2(\Omega, \mathcal{F}; V)$ when $u_0 \in L^2(\Omega, \mathcal{F}_\tau; H)$ (see Lemma 4.2), which is crucial to construct a compact set to prove the tightness of the collection of all periodic invariant measures of (1.1) on space H as ε varies on a bounded interval. Then, based on the convergence of solutions of (1.1) with respect to noise intensity (see Lemma 4.5), we prove that every limit of a sequence of periodic invariant measures of (1.1) as $\varepsilon \to 0$ must be a periodic invariant measure of the corresponding deterministic equations (see Theorem 4.3). Moreover, we also show that limits of a family of periodic invariant measures are periodic invariant measures of the limiting equations as $N \to N_0 \in (0, \infty)$ (see Theorem 4.4), which will contribute to understanding the dynamical behaviors of stochastic 3D Navier-Stokes equations. In addition, under weaker conditions on the nonlinear term of G_k, we also study the existence of invariant measures of the autonomous version of (1.1) (see Theorem 4.5) as well as their limiting behaviors with respect to noise intensity ε (see Theorem 4.6) and modification parameter N (see Theorem 4.7).

The last goal of this paper is to establish the Wentzell-Freidlin large deviation principle (LDP) for stochastic 3D globally modified Navier-Stokes equations (1.1), which is useful to study the asymptotic behaviors of small probability on an exponential scale. Until now, the LDP has been extensively studied by many experts (see, e.g., [2] for stochastic ordinary differential equations, [50] for stochastic lattice systems, and [4, 11, 19, 25, 27, 28, 34, 38] for stochastic partial differential equations). Due to the nonlinear terms of different types of equations, the LDP for stochastic partial differential equations need to be dealt separately. To establish the LDP for stochastic 3D globally modified Navier-Stokes equations (1.1), we adopt the weak convergence method introduced in [8]. More precisely, we first need to obtain the compactness of K_M given in (5.11). The main difficulty lies in proving the convergence of solutions of the deterministic controlled equation (5.2) by using the weak convergence of h_n. Such convergence can be obtained by dealing with localized integral estimates of time increments as in [19, 25]. However, this method requires more assumptions on the nonlinear term G_k which may be independent of the gradient. In order to make those results be applied to the case that the nonlinear term G_k depends on the gradient, inspired by [34], we will use some truncation and approximation techniques to show the convergence of corresponding solutions (see Lemma 5.2 for more details). On the other hand, we also need to prove some convergence of solutions between the stochastic and deterministic controlled equations. To do that, we first establish the convergence of solutions of the above controlled equations with the same control (see Lemma 5.5). Then together with the convergence of solutions of deterministic controlled equation (Lemma 5.2) and the Skorokhod theorem, we can obtain that the solution of the stochastic controlled equation (5.30) converges to the solution of the deterministic controlled equation (5.2) in $C([0,T],H)\cap L^2(0,T;V)$ in distribution (see Theorem 5.2), which is different from the corresponding proof in [34] since the truncation technique is not used herein.

The paper is organized as follows. In Section 2, we discuss the existence and uniqueness of solutions of (1.1). Section 3 is concerned with the existence and uniqueness of weak mean random attractors. In Section 4, we investigate the existence of (periodic) invariant measures and the limiting behaviors of periodic invariant measures. In addition, we also study the existence of invariant measures as well as their limiting behaviors when the external term is independent of time. In the last section, we first recall some basic concepts and useful theory related to large deviation principle and Laplace principle, and then establish the large deviation principle of (1.1) by the weak convergence method.

2 Existence and uniqueness of solutions

In this section, we will show the well-posedness of system (1.1). We first introduce some abstract spaces which will be frequently used in the sequel.

$\mathcal{V}=\{\phi\in(\mathcal{C}_0^\infty(\mathcal{O}))^3:\nabla\cdot\phi=0\}$,

$H=$ closure of $\mathcal{V}$ in $(L^2(\mathcal{O}))^3$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$,

$V=$ closure of $\mathcal{V}$ in $(H_0^1(\mathcal{O}))^3$ with inner product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$,

$H'=$ dual space of H, $V'=$ dual space of V with norm $\|\cdot\|_{V'}$,

$\langle\cdot,\cdot\rangle$ denotes the dual pairing between V and V'.

It is clear that $V\subset H\equiv H'\subset V'$, where the injections are dense and compact.

Next, consider the operator $A:V\mapsto V'$ defined by $\langle Au,v\rangle=((u,v))$. And denote $D(A)=(H^2(\mathcal{O}))^3\cap V$. Then for any $u\in D(A)$, $Au=-P\triangle u$ is the Stokes operator, where P is the ortho-projector from $(L^2(\mathcal{O}))^3$ onto H. It follows from the classical spectral theorems that there exist a sequence $\{\lambda_j\}$ with $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_j\to\infty$ and a family of elements e_j of $D(A)$ which is orthonormal in H such that $Ae_j=\lambda_je_j$ for any $j=1,2,\cdots$. In this context, the Poincaré inequality reads

$$\lambda_1|u|^2 \leqslant \|u\|^2, \ \forall\ u \in V. \tag{2.1}$$

Then define a trilinear form $b(\cdot,\cdot,\cdot)$ as follows,

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall\ u,v,w \in V,$$

and set

$$\langle B(u,v), w\rangle = b(u,v,w).$$

For simplicity, define $B(u,u) = B(u)$ for any $u \in V$. Recall that the operator b satisfies

$$b(u,v,v) = 0, \qquad\qquad \forall\ u,v \in V, \tag{2.2}$$

and there exists a constant $C > 0$ depending only on $\mathcal{O}$ such that

$$|b(u,v,w)| \leqslant C\|u\|\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}, \qquad\qquad \forall\ u,v,w \in V, \tag{2.3}$$

$$|b(u,v,w)| \leqslant C\|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}\|v\||w|, \qquad\qquad \forall\ u \in D(A), v \in V, w \in H. \tag{2.4}$$

For any $u,v \in V$ and each $M,N > 0$,

$$|F_N(\|u\|) - F_N(\|v\|)| \leqslant \frac{1}{N} F_N(\|u\|) F_N(\|v\|) \|u - v\|, \tag{2.5}$$

$$|F_M(\|u\|) - F_N(\|u\|)| \leqslant \frac{|M-N|}{\|u\|}. \tag{2.6}$$

Furthermore, assume $G_k(\cdot)$, $k \geqslant 1$, is a sequence of mappings from V into H satisfying the following hypotheses:

$$\sum_{k=1}^{\infty} |G_k(u)|^2 \leqslant \alpha_1(1+|u|^2) + \alpha_2\|u\|^2, \ \forall\ u \in V, \tag{2.7}$$

$$\sum_{k=1}^{\infty} |G_k(u) - G_k(v)|^2 \leqslant \beta_1|u-v|^2 + \beta_2\|u-v\|^2, \ \forall\ u,v \in V. \tag{2.8}$$

Remark 2.1. *It is worth mentioning that such assumptions on the diffusion coefficient G_k are weaker than [1, (H3)] where the nonlinear term G_k may not depend on the gradient. Under the assumptions of (2.7) and (2.8), we can derive the uniform estimates of solutions with some complexity of calculation (see Theorem 2.1 for more details).*

For convenience, let $\ell^2 = \left\{ h = (h_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |h_k|^2 < \infty \right\}$. For every $k \in \mathbb{N}$, let $\mathcal{E}_k = (e_{k,j})_{j=1}^{\infty}$ with $e_{k,j} = 1$ for $j = k$, and $e_{k,j} = 0$ otherwise. Then $\{\mathcal{E}_k\}_{k=1}^{\infty}$ is an orthonormal basis of ℓ^2. Let I be the identity operator on ℓ^2 and W be the cylindrical Wiener process in ℓ^2 with covariance operator I given by $W(t) = \sum_{k=1}^{\infty} W_k(t)\mathcal{E}_k$ for any $t \geqslant \tau$, which is convergent in $L^2(\Omega, \mathcal{F}; C([\tau,T],U))$ for every $T > \tau$, where U is a separable Hilbert space such that the embedding $\ell^2 \hookrightarrow U$ is a Hilbert-Schmidt operator.

Given $u \in V$, define $G(u) : \ell^2 \to H$ by

$$G(u)(h) = \sum_{k=1}^{\infty} G_k(u) h_k, \ \ \forall\ h = (h_k)_{k=1}^{\infty} \in \ell^2. \tag{2.9}$$

It then follows from (2.7) that the series in (2.9) is convergent in H, Furthermore, the operator $G(u) : \ell^2 \to H$ is Hilbert-Schmidt, and

$$\|G(u)\|^2_{L(\ell^2,H)} \leqslant \|G(u)\|^2_{L_2(\ell^2,H)} = \sum_{k=1}^{\infty} |G_k(u)|^2 < \infty,$$

where $L(\ell^2, H)$ denotes the space of bounded linear operators from ℓ^2 to H with norm $\|\cdot\|_{L(\ell^2,H)}$, and $L_2(\ell^2, H)$ denotes the space of Hilbert-Schmidt operators from ℓ^2 to H with norm $\|\cdot\|_{L_2(\ell^2,H)}$.

With the above notation, problem (1.1) can be put into the form

$$\begin{cases} du + (\nu Au + F_N(\|u\|)B(u) - f(t))\, dt = \sqrt{\varepsilon} G(u) dW(t), \\ u(x,\tau) = u_0(x). \end{cases} \tag{2.10}$$

A solution of problem (2.10) will be considered in the following sense.

Definition 2.1. *Suppose $\tau \in \mathbb{R}$, $f \in L^2_{loc}(\mathbb{R}; H)$ and $u_0 \in L^2(\Omega, \mathcal{F}_\tau; H)$. Then, an H-valued $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$-adapted stochastic process $\{u(t)\}_{t\in[\tau,\infty)}$ is called a strong solution (in the probabilistic sense) of (2.10) on $[\tau,\infty)$ if, for any $T > \tau$, $u \in C([\tau,T], H) \cap L^2(\tau,T;V)$ $\mathbb{P}$-a.s. and it holds*

$$\begin{aligned} &(u(t), v) + \nu \int_\tau^t ((u(s), v))ds + \int_\tau^t F_N(\|u(s)\|)b(u(s), u(s), v)ds \\ &= (u_0, v) + \int_\tau^t (f(s), v)\, ds + \sqrt{\varepsilon} \int_\tau^t \big(v, G(u(s))dW(s)\big), \end{aligned} \tag{2.11}$$

$\mathbb{P}$-almost everywhere, for any $t > \tau$ and $v \in V$.

Lemma 2.1. *Let $0 < \varepsilon \leqslant \dfrac{\nu}{2\beta_2}$. Then, for any $u, v \in V$ the following estimates hold*

$$\begin{aligned} &2\big\langle -\nu Au + \nu Av - F_N(\|u\|)B(u) + F_N(\|v\|)B(v), u - v\big\rangle + \varepsilon \|G(u) - G(v)\|^2_{L_2(\ell^2,H)} \\ &\leqslant -\nu\|u - v\|^2 + \left(C_\nu N^4 + \frac{\nu\beta_1}{2\beta_2}\right) |u - v|^2. \end{aligned} \tag{2.12}$$

Proof. It follows from (2.5), (2.8) and Young's inequality that

$$\begin{aligned} &2\big\langle -\nu Au + \nu Av - F_N(\|u\|)B(u) + F_N(\|v\|)B(v), u - v\big\rangle + \varepsilon \|G(u) - G(v)\|^2_{L_2(\ell^2,H)} \\ &\leqslant -2\nu\|u - v\|^2 + F_N(\|u\|)|b(u - v, u, u - v)| + |(F_N(\|u\|) - F_N(\|v\|))\, b(v, u, u - v)| \\ &\quad + \varepsilon\beta_1|u - v|^2 + \varepsilon\beta_2\|u - v\|^2 \\ &\leqslant -\nu\|u - v\|^2 + \left(C_\nu N^4 + \frac{\nu\beta_1}{2\beta_2}\right) |u - v|^2. \end{aligned} \tag{2.13}$$

The proof is complete. □

Theorem 2.1. *Assume (2.7) and (2.8) hold. Let $\varepsilon_0 = \min\left\{\dfrac{\nu}{37\alpha_2}, \dfrac{\nu}{2\beta_2}\right\}$. Then, for every $\tau \in \mathbb{R}$, $N > 0$, $0 < \varepsilon \leqslant \varepsilon_0$ and $u_0 \in L^2(\Omega, \mathcal{F}_\tau; H)$, there exists a unique solution u of (2.10) in the sense of Definition 2.1. In addition, for every $T > \tau$, the solution u satisfies for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$,*

$$\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant T} \left(|u(r)|^2 + 2\nu \int_\tau^r \|u(s)\|^2 ds\right)\right] \leqslant C\left(1 + \mathbb{E}\left[|u_0|^2\right] + \int_\tau^T |f(s)|^2 ds\right), \tag{2.14}$$

where $C = C(\varepsilon_0, \nu, \alpha_2, \beta_2)$ is a positive constant independent of ε and N.

Proof. Based on Lemma 2.1, by slightly modifying the proof of [1, Theorem 3.1], one can obtain that for any $0 < \varepsilon \leqslant \varepsilon_0$, system (2.10) has a unique solution u in the sense of Definition 2.1.

Next, we derive the uniform estimates of solutions. Applying Itô's formula to (2.10), we have

$$d|u(t)|^2 + 2\nu\|u(t)\|^2 dt = 2(f(t), u(t))dt + \varepsilon \|G(u(t))\|^2_{L_2(\ell^2,H)}\, dt + 2\sqrt{\varepsilon}\big(u(t), G(u(t))dW(t)\big). \quad (2.15)$$

For every $T > \tau$ and each constant $R > 0$, define a stopping time

$$\tau_R = \inf\big\{t \geqslant \tau : \|u\|_{C([\tau,t],H)} + \|u\|_{L^2(\tau,t;V)} \geqslant R\big\} \wedge T. \quad (2.16)$$

As usual, the infimum of the empty set is taken to be $+\infty$. Then from (2.7) and Young's inequality, it follows that for any $t \in [\tau, T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}\left(|u(r)|^2 + 2\nu\int_\tau^r \|u(s)\|^2 ds\right)\right] \\
&\leqslant \mathbb{E}\left[|u_0|^2\right] + \int_\tau^T |f(s)|^2 ds + \int_\tau^t \mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s\wedge\tau_R}|u(r)|^2\right] ds + \varepsilon\mathbb{E}\left[\int_\tau^{t\wedge\tau_R}\|G(u(s))\|^2_{L_2(\ell^2,H)}\, ds\right] \\
&\quad + 2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}\left|\int_\tau^r \big(u(s), G(u(s))dW(s)\big)\right|\right]. \quad (2.17)
\end{aligned}$$

For the last term of (2.17), by the Young inequality and the Burkholder-Davies-Gundy (BDG) inequality, we have

$$\begin{aligned}
&2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}\left|\int_\tau^r \big(u(s), G(u(s))dW(s)\big)\right|\right] \\
&\leqslant 6\sqrt{\varepsilon}\mathbb{E}\left[\left(\int_\tau^{t\wedge\tau_R}\|G(u(s))\|^2_{L_2(\ell^2,H)}\,|u(s)|^2 ds\right)^{\frac{1}{2}}\right] \\
&\leqslant \frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}|u(r)|^2\right] + 36\varepsilon\mathbb{E}\left[\int_\tau^{t\wedge\tau_R}\|G(u(s))\|^2_{L_2(\ell^2,H)}\, ds\right], \quad (2.18)
\end{aligned}$$

which, together with (2.17), shows that for any $t \in [\tau, T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}\left(|u(r)|^2 + 2\nu\int_\tau^r \|u(s)\|^2 ds\right)\right] \\
&\leqslant \mathbb{E}\left[|u_0|^2\right] + \frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}|u(r)|^2\right] + \int_\tau^T |f(s)|^2 ds + \int_\tau^t \mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s\wedge\tau_R}|u(r)|^2\right] ds \\
&\quad + 37\varepsilon\mathbb{E}\left[\int_\tau^{t\wedge\tau_R}\|G(u(s))\|^2_{L_2(\ell^2,H)}\, ds\right]. \quad (2.19)
\end{aligned}$$

For the last term of (2.19), by (2.7), we have

$$37\varepsilon\mathbb{E}\left[\int_\tau^{t\wedge\tau_R}\|G(u(s))\|^2_{L_2(\ell^2,H)}\, ds\right] \leqslant 37\varepsilon\alpha_1\int_\tau^{t\wedge\tau_R}\big(1 + \mathbb{E}\left[|u(s)|^2\right]\big)\, ds + 37\varepsilon\alpha_2\int_\tau^{t\wedge\tau_R}\mathbb{E}\left[\|u(s)\|^2\right] ds.$$

Then by (2.19), we obtain for any $t \in [\tau, T]$ and $\varepsilon \in \left(0, \dfrac{\nu}{37\alpha_2}\right]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}\left(|u(r)|^2 + 2\nu\int_\tau^r \|u(s)\|^2 ds\right)\right] \\
&\leqslant \mathbb{E}\left[|u_0|^2\right] + \frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}|u(r)|^2\right] + \int_\tau^T |f(s)|^2 ds
\end{aligned}$$

$$+ (1+37\varepsilon\alpha_1)\int_\tau^t \mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s\wedge\tau_R}|u(r)|^2\right]ds$$

$$+ \nu\int_\tau^{t\wedge\tau_R}\mathbb{E}\left[\|u(s)\|^2\right]ds + 37\varepsilon\alpha_1(T-\tau), \tag{2.20}$$

which implies that

$$\nu\int_\tau^{t\wedge\tau_R}\mathbb{E}\left[\|u(s)\|^2\right]ds \leqslant \mathbb{E}\left[|u_0|^2\right] + \frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}|u(r)|^2\right] + \int_\tau^T|f(s)|^2ds$$
$$+ (1+37\varepsilon\alpha_1)\int_\tau^t \mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s\wedge\tau_R}|u(r)|^2\right]ds + 37\varepsilon\alpha_1(T-\tau). \tag{2.21}$$

Therefore, by (2.20) and (2.21), we find

$$\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}\left(|u(r)|^2 + 2\nu\int_\tau^r\|u(s)\|^2ds\right)\right]$$
$$\leqslant 2\mathbb{E}\left[|u_0|^2\right] + \frac{1}{2}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}|u(r)|^2\right] + 2\int_\tau^T|f(s)|^2ds$$
$$+ 2(1+37\varepsilon\alpha_1)\int_\tau^t \mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s\wedge\tau_R}|u(r)|^2\right]ds + 74\varepsilon\alpha_1(T-\tau). \tag{2.22}$$

From (2.22), it follows that

$$\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}|u(r)|^2\right] \leqslant 4\mathbb{E}\left[|u_0|^2\right] + 4\int_\tau^T|f(s)|^2ds$$
$$+ 4(1+37\varepsilon\alpha_1)\int_\tau^t \mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s\wedge\tau_R}|u(r)|^2\right]ds + 148\varepsilon\alpha_1(T-\tau),$$

which, along with Gronwall's inequality, implies

$$\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t\wedge\tau_R}|u(r)|^2\right] \leqslant 4\left(\mathbb{E}\left[|u_0|^2\right] + 37\varepsilon\alpha_1(T-\tau) + \int_\tau^T|f(s)|^2ds\right)e^{4(1+37\varepsilon\alpha_1)t}. \tag{2.23}$$

By (2.22) and (2.23), we can deduce that for any $\varepsilon\in[0,\varepsilon_0]$,

$$\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant T\wedge\tau_R}\left(|u(r)|^2 + 2\nu\int_\tau^r\|u(s)\|^2ds\right)\right] \leqslant C\left(1 + \mathbb{E}\left[|u_0|^2\right] + \int_\tau^T|f(s)|^2ds\right), \tag{2.24}$$

where $C>0$ is independent of ε, R, u_0 and f.

Note that $\lim_{R\to\infty}\tau_R = T$, a.s., since $u\in C([\tau,T],H)\cap L^2(\tau,T;V)$ $\mathbb{P}$-a.s.. Then by (2.24) and Fatou's lemma, we can obtain (2.14), and thus completes the proof. □

Remark 2.2. *Although the solutions of* (2.10) *depend on the parameter* ε *and* N*, we omit this dependence in this section since they are fixed from the beginning. Hence, we use* u *instead of using the notation* $u^{\varepsilon,N}$*. Similarly, we will use the notations* Φ *and* μ *instead of* $\Phi^{\varepsilon,N}$ *and* $\mu^{\varepsilon,N}$ *when no confusion may arise.*

3 Weak pullback mean random attractors

In this section, we first define a mean dynamical system for (2.10) based on Theorem 2.1. We then construct a weak pullback absorbing set and show the existence and uniqueness of weak pullback mean random attractors of (2.10).

Now, for any $t \geqslant 0$ and $\tau \in \mathbb{R}$, let $\Phi(t,\tau)$ be a map from $L^2(\Omega, \mathcal{F}_\tau; H)$ to $L^2(\Omega, \mathcal{F}_{\tau+t}; H)$ given by

$$\Phi(t,\tau)u_0 = u(t+\tau, \tau, u_0),$$

where u is the solution of (2.10) with initial value $u_0 \in L^2(\Omega, \mathcal{F}_\tau; H)$.

Based on the existence and uniqueness of solutions of (2.10), we can find Φ is a mean random dynamical system on $L^2(\Omega, \mathcal{F}; H)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.
Let $B = \{B(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau; H) : \tau \in \mathbb{R}\}$ be a family of nonempty bounded sets such that

$$\lim_{\tau\to-\infty} e^{\nu\lambda_1\tau} \|B(\tau)\|^2_{L^2(\Omega,\mathcal{F}_\tau;H)} = 0, \tag{3.1}$$

where $\|B(\tau)\|_{L^2(\Omega,\mathcal{F}_\tau;H)} = \sup\limits_{u\in B(\tau)} \|u\|_{L^2(\Omega,\mathcal{F}_\tau;H)}$. Denote by

$$\mathcal{D} = \left\{B = \{B(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau; H) : B(\tau) \neq \emptyset \text{ bounded}, \tau \in \mathbb{R}\} : B \text{ satisfies } (3.1)\right\}.$$

In order to investigate the existence and uniqueness of weak $\mathcal{D}$-pullback mean random attractors of Φ, we assume that the deterministic forcing term f satisfies

$$\int_{-\infty}^{\tau} e^{\nu\lambda_1 s}|f(s)|^2 ds < \infty, \ \forall\, \tau \in \mathbb{R}. \tag{3.2}$$

Next, we first derive uniform estimates on the solutions of (2.10).

Lemma 3.1. *Suppose (2.7), (2.8) and (3.2) hold. Let* $\varepsilon_0 = \min\left\{\frac{\nu\lambda_1}{4\alpha_1}, \frac{\nu}{37\alpha_2}, \frac{\nu}{2\beta_2}\right\}$. *Then, for every* $\tau \in \mathbb{R}$ *and* $B = \{B(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, *there exists* $T = T(\tau, B) > 0$ *such that for all* $t \geqslant T$, *and all* $0 < \varepsilon \leqslant \varepsilon_0$, *the corresponding solution* u *of (2.10) satisfies*

$$\mathbb{E}\left[|u(\tau, \tau - t, u_0)|^2\right] \leqslant 1 + \frac{2}{\nu\lambda_1} e^{-\nu\lambda_1\tau} \int_{-\infty}^{\tau} |f(s)|^2 e^{\nu\lambda_1 s} ds,$$

where $u_0 \in B(\tau - t)$.

Proof. By Itô's formula, we obtain from (2.10) that

$$\frac{d}{dt}\mathbb{E}\left[|u(t)|^2\right] + 2\nu\mathbb{E}\left[\|u(t)\|^2\right] = 2\mathbb{E}\left[(f(t), u(t))\right] + \varepsilon\mathbb{E}\left[\|G(u(t))\|^2_{L_2(\ell^2,H)}\right]. \tag{3.3}$$

For the last two terms of (3.3), by (2.7) and Young's inequality, we have

$$\begin{aligned}
&2\mathbb{E}\left[(f(t), u(t))\right] + \varepsilon\mathbb{E}\left[\|G(u(t))\|^2_{L_2(\ell^2,H)}\right] \\
&\leqslant \frac{2}{\nu\lambda_1}|f(t)|^2 + \frac{\nu\lambda_1}{2}\mathbb{E}\left[|u(t)|^2\right] + \varepsilon\alpha_1\mathbb{E}\left[\left(1 + |u(t)|^2\right)\right] + \varepsilon\alpha_2\mathbb{E}\left[\|u(t)\|^2\right].
\end{aligned} \tag{3.4}$$

By (2.1), (3.3) and (3.4), we derive that for any $0 < \varepsilon \leqslant \varepsilon_0$,

$$\frac{d}{dt}\mathbb{E}\left[|u(t)|^2\right] + \nu\lambda_1\mathbb{E}\left[|u(t)|^2\right] \leqslant \frac{2}{\nu\lambda_1}|f(t)|^2 + \frac{\nu\lambda_1}{4},$$

which, together with the Gronwall inequality, shows that

$$\mathbb{E}\left[|u(\tau, \tau - t, u_0)|^2\right] \leqslant \mathbb{E}\left[|u_0|^2\right] e^{-\nu\lambda_1 t} + \frac{2}{\nu\lambda_1} e^{-\nu\lambda_1\tau} \int_{\tau-t}^{\tau} |f(s)|^2 e^{\nu\lambda_1 s} ds + \frac{1}{4}. \tag{3.5}$$

Noting that $u_0 \in B(\tau - t)$ and $B \in \mathcal{D}$,

$$\begin{aligned}
\mathbb{E}\left[|u_0|^2\right] e^{-\nu\lambda_1 t} &= e^{-\nu\lambda_1\tau} e^{\nu\lambda_1(\tau-t)}\mathbb{E}\left[|u_0|^2\right] \\
&\leqslant e^{-\nu\lambda_1\tau} e^{\nu\lambda_1(\tau-t)} \|B(\tau - t)\|^2_{L^2(\Omega,\mathcal{F}_{\tau-t};H)} \to 0, \text{ as } t \to \infty.
\end{aligned}$$

Therefore, there exists $T = T(\tau, B) > 0$ such that for all $t \geqslant T$, $\mathbb{E}\left[|u_0|^2\right] e^{-\nu\lambda_1 t} \leqslant \frac{1}{2}$, which along with (3.5), concludes the proof. □

We are now in a position to show the existence of weak $\mathcal{D}$-pullback mean random attractors for Φ.

Theorem 3.1. *Suppose (2.7), (2.8) and (3.2) hold. Let* $\varepsilon_0 = \min\left\{\frac{\nu\lambda_1}{4\alpha_1}, \frac{\nu}{37\alpha_2}, \frac{\nu}{2\beta_2}\right\}$. *Then, for every* $0 < \varepsilon \leqslant \varepsilon_0$, *the mean random dynamical system* Φ *for (2.10) has a unique weak* $\mathcal{D}$*-pullback mean random attractor* $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ *in* $L^2(\Omega, \mathcal{F}; H)$ *over* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

Proof. Given $\tau \in \mathbb{R}$, denote by

$$K(\tau) = \left\{u \in L^2(\Omega, \mathcal{F}_\tau; H) : \mathbb{E}[|u|^2] \leqslant R(\tau)\right\},$$

where $R(\tau) = 1 + \frac{2}{\nu\lambda_1}e^{-\nu\lambda_1\tau}\int_{-\infty}^{\tau}|f(s)|^2 e^{\nu\lambda_1 s}ds$. Since $K(\tau)$ is a bounded closed convex subset of the reflexive Banach space $L^2(\Omega, \mathcal{F}_\tau; H)$ and hence is weakly compact in $L^2(\Omega, \mathcal{F}_\tau; H)$. By (3.2) we obtain

$$\lim_{\tau\to-\infty} e^{\nu\lambda_1\tau}\|K(\tau)\|^2_{L^2(\Omega,\mathcal{F}_\tau;H)} = \lim_{\tau\to-\infty} e^{\nu\lambda_1\tau}R(\tau) = 0,$$

which means $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$.

By Lemma 3.1, we derive that for any $\tau \in \mathbb{R}$ and $B = \{B(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, there exists $T = T(\tau, B) \geqslant 0$ such that for any $t \geqslant T$,

$$\Phi(t, \tau - t, B(\tau - t)) \subseteq K(\tau).$$

Therefore, K is a weakly compact $\mathcal{D}$-pullback absorbing set of Φ. It then follows from [46, Theorem 2.13] that Φ has a unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{A} \in \mathcal{D}$. □

4 Invariant measures for stochastic GMNSE

In this section, we will investigate the existence of invariant measures of (2.10) as well as their limiting behaviors. To that end, we first introduce the transition operator.

If $\phi : H \to \mathbb{R}$ is a bounded Borel function, then for $s \leqslant t$ and $u_0 \in H$, define a transition operator $P_{s,t}$ by $(P_{s,t}\phi)(u_0) = \mathbb{E}\left[\phi(u(t, s, u_0))\right]$, where $u(t, \tau, u_0)$ is the solution of (2.10) with initial condition $u_0 \in H$. Denote by $\mathcal{P}_r(H)$ and $\mathcal{B}(H)$ the sets of all probability measures and the Borel σ-algebra on H, respectively. And if $B \in \mathcal{B}(H)$, $s \leqslant t$ and $u_0 \in H$, we write $p(s, u_0; t, B) = (P_{s,t}\mathbb{1}_B)(u_0) = \mathbb{P}\left(\omega \in \Omega | u(t, s, u_0) \in B\right)$, where $\mathbb{1}_B$ is the characteristic function of B. We shall denote the adjoint operator $P^*_{s,t}$ of $P_{s,t}$ by $P^*_{s,t}\mu(B) = \int_H p(s, u_0; t, B)\mu(du_0)$ for any $\mu \in \mathcal{P}_r(H)$.

Recall that a mapping $t \in \mathbb{R} \mapsto \mu_t \in \mathcal{P}_r(H)$ satisfying $\int_H (P_{s,t}\psi)(u_0)\mu_s(du_0) = \int_H \psi(u_0)\mu_t(du_0)$ for every $s \leqslant t$ and bounded Borel function ψ, is called an invariant measure for $P_{s,t}$, and $\{\mu_t\}_{t\in\mathbb{R}}$ is also called an evolution system of measures (see [22] for more details). Given $\mathcal{T} > 0$, μ_t is called $\mathcal{T}$-periodic if $\mu_t = \mu_{t+\mathcal{T}}$ for any $t \in \mathbb{R}$. And, μ_t is called a $\mathcal{T}$-periodic invariant measure if it is invariant and $\mathcal{T}$-periodic.

When considering autonomous stochastic systems, we often write the operator $P_{0,t}$ as P_t for any $t \geqslant 0$. A probability measure $\mu \in \mathcal{P}_r(H)$ is called invariant with respect to the transition semigroup $\{P_t\}_{t\geqslant 0}$ if $\int_H (P_t\psi)(u_0)\mu(du_0) = \int_H \psi(u_0)\mu(du_0)$ for every $t > 0$ and bounded Borel function ψ.

4.1 Invariant measures as the external term f depends on t

In this subsection, we will discuss the existence and limiting behaviors of invariant measures of time inhomogeneous transition semigroup for the non-autonomous stochastic globally modified Navier-Stokes equations (2.10). To do that, we further assume: $G_k(\cdot)$, $k \geqslant 1$, is a sequence of mappings from

V into V satisfying that there exists $\alpha_3 > 0$ such that

$$\sum_{k=1}^{\infty} \|G_k(u)\|^2 \leqslant \alpha_3(1 + \|u\|^2), \ \forall \ u \in V. \tag{4.1}$$

In order to investigate the invariant measures of (2.10), in this subsection, we will use (4.1) instead of (2.7).

4.1.1 Existence of (periodic) invariant measures

The following result is concerned with the existence and uniqueness of strong solutions (in the sense of PDE) of (2.10), which can be proved by a similar technique to that one in [23, Theorem 3.2]. We therefore omit the details.

Theorem 4.1. *Suppose (2.8) and (4.1) hold. Then there exists $0 < \varepsilon_0 \leqslant 1$ such that for every $\tau \in \mathbb{R}$, $0 < \varepsilon \leqslant \varepsilon_0$ and $u_0 \in L^2(\Omega, \mathcal{F}_\tau; V)$, system (2.10) possesses a unique solution $u \in L^2\left(\Omega, C\left([\tau, T], V\right)\right) \cap L^2\left(\Omega, L^2(\tau, T; D(A))\right)$.*

Next, we provide some uniform estimates of the solutions to (2.10) in $L^2(\Omega, V)$, which will be helpful to prove the existence of invariant measures of non-autonomous system (2.10).

Lemma 4.1. *Suppose (2.8) and (4.1) hold. Then there exists $0 < \varepsilon_0 \leqslant 1$ such that for every $\tau \in \mathbb{R}$, $0 < \varepsilon \leqslant \varepsilon_0$ and $u_0 \in L^2(\Omega, \mathcal{F}_\tau; V)$, the solution of (2.10) satisfies*

$$\mathbb{E}\left[\|u(t,\tau,u_0)\|^2\right] \leqslant \widetilde{C}\left(1+N^4\right)\left(1 + e^{-\nu\lambda_1(t-\tau)}\mathbb{E}\left[\|u_0\|^2\right] + e^{-\nu\lambda_1 t}\int_\tau^t e^{\nu\lambda_1 s}|f(s)|^2 ds\right), \tag{4.2}$$

where $\widetilde{C} > 0$ is independent of u_0, τ, t, ε and N.

Proof. By Itô's formula and Young's inequality, we obtain that

$$\begin{aligned} &d(e^{\sigma t}|u(t)|^2) + 2\nu e^{\sigma t}\|u(t)\|^2 dt \\ &= \sigma e^{\sigma t}|u(t)|^2 dt + 2e^{\sigma t}\left(f(t), u(t)\right)dt + 2\sqrt{\varepsilon}e^{\sigma t}\big(u(t), G(u(t))dW(t)\big) + \varepsilon e^{\sigma t}\|G(u(t))\|^2_{L_2(\ell^2,H)}dt, \end{aligned} \tag{4.3}$$

where $\sigma \geqslant 0$ will be chosen later. Letting $0 < \varepsilon \leqslant \min\{1, \frac{\nu\lambda_1}{4\alpha_3}\}$ and taking expectation, it follows from (4.3) that for all $t \geqslant \tau$,

$$\begin{aligned} &e^{\sigma t}\mathbb{E}\left[|u(t)|^2\right] + 2\nu\int_\tau^t e^{\sigma s}\mathbb{E}\left[\|u(s)\|^2\right] ds \\ &\leqslant e^{\sigma\tau}\mathbb{E}\left[|u_0|^2\right] + \left(\frac{\nu\lambda_1}{2} + \sigma\right)\int_\tau^t e^{\sigma s}\mathbb{E}\left[|u(s)|^2\right] ds + \frac{4}{\nu\lambda_1}\int_\tau^t e^{\sigma s}|f(s)|^2 ds + \frac{\nu}{4}\int_\tau^t e^{\sigma s} ds, \end{aligned}$$

which shows that

$$\begin{aligned} &e^{\sigma t}\mathbb{E}\left[|u(t)|^2\right] + \frac{\nu}{2}\int_\tau^t e^{\sigma s}\mathbb{E}\left[\|u(s)\|^2\right] ds \\ &\leqslant e^{\sigma\tau}\mathbb{E}\left[|u_0|^2\right] + (-\nu\lambda_1 + \sigma)\int_\tau^t e^{\sigma s}\mathbb{E}\left[|u(s)|^2\right] ds + \frac{4}{\nu\lambda_1}\int_\tau^t e^{\sigma s}|f(s)|^2 ds + \frac{\nu}{4}\int_\tau^t e^{\sigma s} ds. \end{aligned} \tag{4.4}$$

Then taking $\sigma = \nu\lambda_1$, we deduce that

$$\mathbb{E}\left[|u(t)|^2\right] + \frac{\nu}{2}\int_\tau^t e^{\nu\lambda_1(s-t)}\mathbb{E}\left[\|u(s)\|^2\right] ds$$

$$\leqslant e^{-\nu\lambda_1(t-\tau)}\mathbb{E}\left[|u_0|^2\right] + \frac{4e^{-\nu\lambda_1 t}}{\nu\lambda_1}\int_\tau^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \frac{1}{4\lambda_1}. \tag{4.5}$$

By using Itô's formula again, we obtain

$$\begin{aligned}&de^{\sigma t}\|u(t)\|^2 + 2\nu e^{\sigma t}|Au(t)|^2 dt + 2e^{\sigma t}F_N(\|u(t)\|)b(u(t),u(t),Au(t))dt\\ &= \sigma e^{\sigma t}\|u(t)\|^2 dt + 2e^{\sigma t}\left(f(t),Au(t)\right)dt + 2\sqrt{\varepsilon}e^{\sigma t}\big(Au(t),G(u(t))dW(t)\big) + \varepsilon e^{\sigma t}\|G(u(t))\|^2_{L_2(\ell^2,V)}dt.\end{aligned}$$

It then follows from (4.1) that for any $t \geqslant \tau$,

$$\begin{aligned}&e^{\sigma t}\|u(t)\|^2 + 2\nu\int_\tau^t e^{\sigma s}|Au(s)|^2 ds + 2\int_\tau^t e^{\sigma s}F_N(\|u(s)\|)b(u(s),u(s),Au(s))ds\\ &\leqslant e^{\sigma\tau}\mathbb{E}\left[\|u_0\|^2\right] + (\sigma+\alpha_3)\int_\tau^t e^{\sigma s}\|u(s)\|^2 ds + \frac{\nu}{2}\int_\tau^t e^{\sigma s}|Au(s)|^2 ds + \frac{2}{\nu}\int_\tau^t e^{\sigma s}|f(s)|^2 ds\\ &\quad + \alpha_3\int_\tau^t e^{\sigma s}ds + 2\sqrt{\varepsilon}\int_\tau^t e^{\sigma s}\big(Au(s),G(u(s))dW(s)\big).\end{aligned} \tag{4.6}$$

For the third term on the left-hand side of (4.6), by (2.4) and Young's inequality, we have

$$2\left|\int_\tau^t e^{\sigma s}F_N(\|u(s)\|)b(u(s),u(s),Au(s))ds\right| \leqslant C_\nu N^4\int_\tau^t e^{\sigma s}\|u(s)\|^2 ds + \frac{\nu}{2}\int_\tau^t e^{\sigma s}|Au(s)|^2 ds, \tag{4.7}$$

which, together with (4.6) for $\sigma = \nu\lambda_1$, implies that

$$\begin{aligned}&e^{\nu\lambda_1 t}\mathbb{E}\left[\|u(t)\|^2\right] + \nu\int_\tau^t e^{\nu\lambda_1 s}\mathbb{E}\left[|Au(s)|^2\right]ds \qquad (4.8)\\ &\leqslant e^{\nu\lambda_1\tau}\mathbb{E}\left[\|u_0\|^2\right] + (\nu\lambda_1+\alpha_3+C_\nu N^4)\int_\tau^t e^{\nu\lambda_1 s}\mathbb{E}\left[\|u(s)\|^2\right]ds + \frac{2}{\nu}\int_\tau^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \frac{\alpha_3}{\nu\lambda_1}e^{\nu\lambda_1 t}.\end{aligned}$$

Therefore, by (4.5) and (4.8),

$$\begin{aligned}&\mathbb{E}\left[\|u(t)\|^2\right] + \nu\int_\tau^t e^{\nu\lambda_1(s-t)}\mathbb{E}\left[|Au(s)|^2\right]ds\\ &\leqslant e^{-\nu\lambda_1(t-\tau)}\mathbb{E}\left[\|u_0\|^2\right] + \frac{2(\nu\lambda_1+\alpha_3+C_\nu N^4)}{\nu}\left(e^{-\nu\lambda_1(t-\tau)}\mathbb{E}\left[|u_0|^2\right] + \frac{4e^{-\nu\lambda_1 t}}{\nu\lambda_1}\int_\tau^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \frac{1}{4\lambda_1}\right)\\ &\quad + \frac{2}{\nu}e^{-\nu\lambda_1 t}\int_\tau^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \frac{\alpha_3}{\nu\lambda_1}\\ &\leqslant \widetilde{C}\left(1+N^4\right)\left(1 + e^{-\nu\lambda_1(t-\tau)}\mathbb{E}\left[\|u_0\|^2\right] + e^{-\nu\lambda_1 t}\int_\tau^t e^{\nu\lambda_1 s}|f(s)|^2 ds\right),\end{aligned}$$

where $\widetilde{C} > 0$ is independent of u_0, τ, t, ε and N. The proof is finished. □

Theorem 4.2. *Suppose (2.8), (3.2) and (4.1) hold. Then there exists $0 < \varepsilon_0 \leqslant 1$ such that for every $t \in \mathbb{R}$, $0 < \varepsilon \leqslant \varepsilon_0$ and $N > 0$, system (2.10) has an invariant measure μ_t on H. In addition, if $f(t)$ is a $\mathcal{T}$-periodic function in t, then the invariant measure μ_t is $\mathcal{T}$-periodic.*

Proof. By a standard procedure, we can prove the Feller property of $P_{s,t}$, and the Markov property of the solutions of (2.10).

It follows from (4.2) and Chebyshev's inequality that for any $L > 0$ and $k > -n$,

$$\frac{1}{k+n}\int_{-n}^k \mathbb{P}(\|u(k,\tau,0)\| \geqslant L)d\tau \leqslant \frac{1}{L^2}\frac{1}{k+n}\int_{-n}^k \mathbb{E}\left[\|u(k,\tau,0)\|^2\right]d\tau \leqslant \frac{M_3}{L^2}, \tag{4.9}$$

where $M_3 = \widetilde{C}\left(1+N^4\right)\left(1+e^{-\nu\lambda_1 k}\int_{-\infty}^{k} e^{\nu\lambda_1 s}|f(s)|^2 ds\right) < \infty$. Noting that the embedding $V \hookrightarrow H$ is compact, (4.9) shows $\{\eta_{n,k}\}_{k>-n}$ is tight on H for each fixed k, where

$$\eta_{n,k} = \frac{1}{n+k}\int_{-n}^{k} p(\tau, 0; k, \cdot) d\tau.$$

Then by Prokhorov's theorem, we can deduce that there exists a probability measure η_k on H such that, up to a subsequence,

$$\eta_{n,k} \to \eta_k, \text{ weakly as } n \to \infty.$$

Letting $t \in \mathbb{R}$, we choose $k \in \mathbb{N}$ such that $t > -k$, and define $\mu_t := P^*_{k,t}\eta_k$. By [22, Theorem 3.1], one can verify that this definition is independent of the choice of k. Meanwhile, we can obtain that μ_t is an invariant measure on H for non-autonomous system (2.10).

Next, we will prove invariant measure μ_t is periodic. It is sufficient to show that for any $t \in \mathbb{R}$ and real-valued continuous and bounded functional ψ on H,

$$\int_H \psi(u)\mu_t(du) = \int_H \psi(u)\mu_{t+\mathcal{T}}(du). \tag{4.10}$$

Note that μ_t and $\mu_{t+\mathcal{T}}$ are invariant measures, then by the definition of invariant measure, we have that for any $s \leqslant t$,

$$\int_H \psi(u)\mu_t(du) = \int_H P_{s,t}\psi(u)\mu_s(du), \tag{4.11}$$

and

$$\int_H \psi(u)\mu_{t+\mathcal{T}}(du) = \int_H P_{s+\mathcal{T},t+\mathcal{T}}\psi(u)\mu_{s+\mathcal{T}}(du). \tag{4.12}$$

In addition, by a similar technique to the one in [20], we can deduce that the transition operator $P_{s,t}$ is $\mathcal{T}$-periodic, that is,

$$P_{s,t} = P_{s+\mathcal{T},t+\mathcal{T}}, \text{ for any } s \leqslant t. \tag{4.13}$$

Then, (4.10) follows from (4.11)-(4.13). The proof is complete. □

4.1.2 Zero-noise limit of periodic invariant measures

In this subsection, we are going to investigate the limiting behaviors of periodic invariant measures of time inhomogeneous transition semigroup for non-autonomous system (2.10) as the noise intensity $\varepsilon \to 0$. To that end, we first consider the regularity of solutions with initial data in $L^2(\Omega, \mathcal{F}_\tau; H)$. Hereafter, for any fixed $N > 0$, we write the solution of (2.10) as $u^\varepsilon(t, \tau, u_0)$ with initial value u_0 at initial time τ to highlight the dependence of solutions on the noise intensity ε.

Lemma 4.2. *Suppose (2.8), (3.2) and (4.1) hold. Then, there exists $0 < \varepsilon_0 \leqslant 1$ such that, for every $\tau \in \mathbb{R}$, $\delta > 0$, $0 < \varepsilon \leqslant \varepsilon_0$ and $u_0 \in L^2(\Omega, \mathcal{F}_\tau; H)$, the solution of (2.10) satisfies, for all $t > \tau + 1$,*

$$\begin{aligned}
&\mathbb{E}\left[\|u^\varepsilon(t,\tau,u_0)\|^2\right] \\
&\leqslant C\left(1+N^4\right)\left(e^{-\nu\lambda_1(t-\tau)}\mathbb{E}\left[|u_0|^2\right] + e^{-\nu\lambda_1 t}\int_{-\infty}^{t} e^{\nu\lambda_1 s}|f(s)|^2 ds + \int_{t-1}^{t}|f(s)|^2 ds + 1\right),
\end{aligned} \tag{4.14}$$

where $C > 0$ is independent of u_0, τ, t, ε and N.

Proof. Taking $\sigma = 0$ and $\tau = t-1$ in (4.4), we have

$$\mathbb{E}\left[|u^\varepsilon(t)|^2\right] + \frac{\nu}{2}\int_{t-1}^t \mathbb{E}\left[\|u^\varepsilon(s)\|^2\right] ds \leqslant \mathbb{E}\left[|u^\varepsilon(t-1)|^2\right] + \frac{4}{\nu\lambda_1}\int_{t-1}^t |f(s)|^2 ds + \frac{\nu}{4}, \tag{4.15}$$

which, together with (4.5), implies that

$$\begin{aligned}
&\mathbb{E}\left[|u^\varepsilon(t)|^2\right] + \frac{\nu}{2}\int_{t-1}^t \mathbb{E}\left[\|u^\varepsilon(s)\|^2\right] ds \\
&\leqslant e^{-\nu\lambda_1(t-1-\tau)}\mathbb{E}\left[|u_0|^2\right] + \frac{4e^{-\nu\lambda_1(t-1)}}{\nu\lambda_1}\int_\tau^{t-1} e^{\nu\lambda_1 s}|f(s)|^2 ds + \frac{4}{\nu\lambda_1}\int_{t-1}^t |f(s)|^2 ds + \frac{1}{4\lambda_1} + \frac{\nu}{4}.
\end{aligned} \tag{4.16}$$

From (4.6) and (4.7), it follows that for any $r \in [t-1, t]$,

$$\mathbb{E}\left[\|u^\varepsilon(t)\|^2\right] \leqslant \mathbb{E}\left[\|u^\varepsilon(r)\|^2\right] + (\alpha_3 + C_\nu N^4)\int_{t-1}^t \mathbb{E}\left[\|u^\varepsilon(s)\|^2\right] ds + \frac{2}{\nu}\int_{t-1}^t |f(s)|^2 ds + \alpha_3. \tag{4.17}$$

Then by (4.16) and (4.17), we have

$$\begin{aligned}
\mathbb{E}\left[\|u^\varepsilon(t)\|^2\right] \leqslant & C_1\left(1+N^4\right)\left(e^{-\nu\lambda_1(t-\tau)}\mathbb{E}\left[|u_0|^2\right] + e^{-\nu\lambda_1 t}\int_{-\infty}^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \int_{t-1}^t |f(s)|^2 ds + 1\right) \\
&+ \mathbb{E}\left[\|u^\varepsilon(r)\|^2\right] + \frac{2}{\nu}\int_{t-1}^t |f(s)|^2 ds + \alpha_3,
\end{aligned} \tag{4.18}$$

where $C_1 > 0$ is independent of u_0, τ, t, ε and N. Integrating in r over $(t-1, t)$, and using again (4.16), we can obtain the inequality (4.14), as desired. □

The next lemma is concerned with the convergence of solutions of (2.10) about noise intensity.

Lemma 4.3. *Suppose* (2.8) *and* (4.1) *hold. Then for every bounded subset* $E \subset H$, $\tau \in \mathbb{R}$, $T > 0$ *and* $L > 0$, *we have*

$$\lim_{\varepsilon\to 0}\sup_{u_0\in E}\mathbb{P}\left(\left\{\omega\in\Omega : \sup_{\tau\leqslant t\leqslant\tau+T}|u^\varepsilon(t,\tau,u_0) - u^0(t,\tau,u_0)| \geqslant L\right\}\right) = 0.$$

Proof. It follows from Itô's formula that

$$\begin{aligned}
&d|u^\varepsilon(t) - u^0(t)|^2 + 2\nu\|u^\varepsilon(t) - u^0(t)\|^2 dt \\
&= -2F_N(\|u^\varepsilon(t)\|)b\left(u^\varepsilon(t), u^\varepsilon(t), u^\varepsilon(t) - u^0(t)\right) dt + 2F_N(\|u^0(t)\|)b\left(u^0(t), u^0(t), u^\varepsilon(t) - u^0(t)\right) dt \\
&\quad + \varepsilon\|G(u^\varepsilon(t))\|^2_{L_2(\ell^2,H)} dt + 2\sqrt{\varepsilon}\left(u^\varepsilon(t) - u^0(t), G(u^\varepsilon(t))dW(t)\right).
\end{aligned} \tag{4.19}$$

For the first and second terms on the right-hand side of (4.19), by (2.5), we deduce that

$$\begin{aligned}
&\left|-2F_N(\|u^\varepsilon(t)\|)b\left(u^\varepsilon(t), u^\varepsilon(t), u^\varepsilon(t) - u^0(t)\right) + 2F_N(\|u^0(t)\|)b\left(u^0(t), u^0(t), u^\varepsilon(t) - u^0(t)\right)\right| \\
&\leqslant \frac{\nu}{2}\|u^\varepsilon(t) - u^0(t)\|^2 + C_\nu|u^\varepsilon(t) - u^0(t)|^2,
\end{aligned}$$

which, together with (2.1), (4.1) and (4.19), implies for any $t \in [\tau, \tau+T]$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|u^\varepsilon(r,\tau,u_0) - u^0(r,\tau,u_0)|^2\right] \leqslant & \frac{\nu}{2}\int_\tau^t \mathbb{E}\left[\|u^\varepsilon(s) - u^0(s)\|^2\right] ds + C_\nu\int_\tau^t \mathbb{E}\left[|u^\varepsilon(s) - u^0(s)|^2\right] ds \\
&+ \frac{\varepsilon\alpha_3}{\lambda_1}\int_\tau^t \left(1 + \mathbb{E}\left[\|u^\varepsilon(s)\|^2\right]\right) ds
\end{aligned}$$

$$+ 2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}\left|\int_{\tau}^{r}\left(u^{\varepsilon}(s)-u^{0}(s),G(u^{\varepsilon}(s))dW(s)\right)\right|\right]. \quad (4.20)$$

For the last term of (4.20), by (4.1) and BDG's inequality, we have

$$\begin{aligned}
&2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}\left|\int_{\tau}^{r}\left(u^{\varepsilon}(s)-u^{0}(s),G(u^{\varepsilon}(s))dW(s)\right)\right|\right]\\
&\leqslant \frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant s\leqslant t}|u^{\varepsilon}(s)-u^{0}(s)|^2\right]+\frac{36\varepsilon\alpha_3}{\lambda_1}\int_{\tau}^{t}\left(1+\mathbb{E}\left[\|u^{\varepsilon}(s)\|^2\right]\right)ds. \quad (4.21)
\end{aligned}$$

Therefore, by (4.4), (4.20) and (4.21), we find that, for any $t\in[\tau,\tau+T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|u^{\varepsilon}(r,\tau,u_0)-u^{0}(r,\tau,u_0)|^2\right]\\
\leqslant&\frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|u^{\varepsilon}(r,\tau,u_0)-u^{0}(r,\tau,u_0)|^2\right]\\
&+\frac{\nu}{2}\int_{\tau}^{t}\mathbb{E}\left[\|u^{\varepsilon}(s)-u^{0}(s)\|^2\right]ds+C_{\nu}\int_{\tau}^{t}\mathbb{E}\left[|u^{\varepsilon}(s)-u^{0}(s)|^2\right]ds\\
&+\frac{37\varepsilon\alpha_3T}{\lambda_1}+\frac{74\varepsilon\alpha_3}{\nu\lambda_1}\left(|u_0|^2+\frac{4\int_{\tau}^{\tau+T}|f(s)|^2ds}{\nu\lambda_1}+\frac{\nu T}{4}\right). \quad (4.22)
\end{aligned}$$

Similarly, by (4.19), we have

$$\begin{aligned}
2\nu\int_{\tau}^{t}\|u^{\varepsilon}(s)-u^{0}(s)\|^2ds\leqslant&\frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|u^{\varepsilon}(r,\tau,u_0)-u^{0}(r,\tau,u_0)|^2\right]\\
&+\frac{\nu}{2}\int_{\tau}^{t}\mathbb{E}\left[\|u^{\varepsilon}(s)-u^{0}(s)\|^2\right]ds+C_{\nu}\int_{\tau}^{t}\mathbb{E}\left[|u^{\varepsilon}(s)-u^{0}(s)|^2\right]ds\\
&+\frac{37\varepsilon\alpha_3T}{\lambda_1}+\frac{74\varepsilon\alpha_3}{\nu\lambda_1}\left(|u_0|^2+\frac{4\int_{\tau}^{\tau+T}|f(s)|^2ds}{\nu\lambda_1}+\frac{\nu T}{4}\right),
\end{aligned}$$

which, together with (4.22), shows that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|u^{\varepsilon}(r,\tau,u_0)-u^{0}(r,\tau,u_0)|^2\right]+2\nu\int_{\tau}^{t}\|u^{\varepsilon}(s)-u^{0}(s)\|^2ds\\
&\leqslant\frac{1}{2}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|u^{\varepsilon}(r,\tau,u_0)-u^{0}(r,\tau,u_0)|^2\right]+\nu\int_{\tau}^{t}\mathbb{E}\left[\|u^{\varepsilon}(s)-u^{0}(s)\|^2\right]ds\\
&\quad+2C_{\nu}\int_{\tau}^{t}\mathbb{E}\left[|u^{\varepsilon}(s)-u^{0}(s)|^2\right]ds+\frac{74\varepsilon\alpha_3T}{\lambda_1}+\frac{148\varepsilon\alpha_3}{\nu\lambda_1}\left(|u_0|^2+\frac{4\int_{\tau}^{\tau+T}|f(s)|^2ds}{\nu\lambda_1}+\frac{\nu T}{4}\right).
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|u^{\varepsilon}(r,\tau,u_0)-u^{0}(r,\tau,u_0)|^2\right]\\
&\leqslant 2C_{\nu}\int_{\tau}^{t}\mathbb{E}\left[|u^{\varepsilon}(s)-u^{0}(s)|^2\right]ds+\frac{74\varepsilon\alpha_3T}{\lambda_1}+\frac{148\varepsilon\alpha_3}{\nu\lambda_1}\left(|u_0|^2+\frac{4\int_{\tau}^{\tau+T}|f(s)|^2ds}{\nu\lambda_1}+\frac{\nu T}{4}\right).
\end{aligned}$$

Then, by Gronwall's inequality, we can obtain that for any $t\in[\tau,\tau+T]$,

$$\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|u^{\varepsilon}(t,\tau,u_0)-u^{0}(t,\tau,u_0)|^2\right]$$

$$\leqslant \frac{148}{\lambda_1}\left[\varepsilon\alpha_3 T + \frac{2\varepsilon\alpha_3}{\nu}\left(|u_0|^2 + \frac{4\int_\tau^{\tau+T}|f(s)|^2 ds}{\nu\lambda_1} + \frac{\nu T}{4}\right)\right] e^{4C_\nu t}. \tag{4.23}$$

By Chebyshev's inequality, we have

$$\begin{aligned}
&\sup_{u_0\in E}\mathbb{P}\left(\left\{\omega\in\Omega : \sup_{\tau\leqslant t\leqslant\tau+T}|u^\varepsilon(t,\tau,u_0) - u^0(t,\tau,u_0)| \geqslant L\right\}\right)\\
&\leqslant \frac{1}{L^2}\sup_{u_0\in E}\mathbb{E}\left[\sup_{\tau\leqslant t\leqslant\tau+T}|u^\varepsilon(t,\tau,u_0) - u^0(t,\tau,u_0)|^2\right],
\end{aligned}$$

which, together with (4.23), shows the desired result. □

Given $\varepsilon\in[0,\varepsilon_0]$ and $t\in\mathbb{R}$, let S_t^ε be the set of periodic invariant measures of non-autonomous system (2.10) on H corresponding to the parameter ε. It follows from Theorem 4.2 that the set S_t^ε is nonempty for every $\varepsilon\in[0,\varepsilon_0]$. The next theorem is concerned with the limiting behaviors of periodic invariant measures of non-autonomous system (2.10) with respect to noise intensity.

Theorem 4.3. *Suppose (2.8), (3.2) and (4.1) hold, and let $f(t)$ is a $\mathcal{T}$-periodic function in t. If $\varepsilon_n\to 0$ as $n\to\infty$, and $\mu_t^{\varepsilon_n}\in S_t^{\varepsilon_n}$, then there exist a subsequence $\{\varepsilon_{n_k}\}$ and a periodic invariant measure $\mu_t^0\in S_t^0$ such that $\mu_t^{\varepsilon_{n_k}}\to\mu_t^0$ weakly as $k\to\infty$.*

Proof. For any $L>0$, let $B_L=\{u\in V:\|u\|\leqslant L\}$, then B_L is a Borel subset of H. For any $t\in\mathbb{R}$ and $\varepsilon\in(0,\varepsilon_0]$, and given $\mu_t^\varepsilon\in S_t^\varepsilon$, by the definition of periodic invariant measure, we have for any $\tau\leqslant t$ and $m\in\mathbb{N}$,

$$\mu_t^\varepsilon(B_L) = \int_H \mathbb{P}\left(u^\varepsilon(t,\tau-m\mathcal{T},u_0)\in B_L\right)\mu^\varepsilon_{\tau-m\mathcal{T}}(du_0) = \int_H \mathbb{P}\left(u^\varepsilon(t,\tau-m\mathcal{T},u_0)\in B_L\right)\mu^\varepsilon_\tau(du_0).$$

Then, by Chebyshev's inequality, we can deduce that for any $\tau\leqslant t$, $m\in\mathbb{N}$ and $L>0$,

$$\begin{aligned}
\mu_t^\varepsilon(B_L) =& \liminf_{m\to\infty}\int_H \mathbb{P}(\|u^\varepsilon(t,\tau-m\mathcal{T},u_0)\|\leqslant L)\mu_\tau^\varepsilon(du_0)\\
=&1-\liminf_{m\to\infty}\int_H \mathbb{P}(\|u^\varepsilon(t,\tau-m\mathcal{T},u_0)\| > L)\mu_\tau^\varepsilon(du_0)\\
\geqslant&1-\frac{1}{L^2}\int_H \liminf_{m\to\infty}\mathbb{E}\left[\|u^\varepsilon(t,\tau-m\mathcal{T},u_0)\|^2\right]\mu_\tau^\varepsilon(du_0).
\end{aligned}\tag{4.24}$$

In addition, by (4.14), we can find that for every $\tau<t-1$ and $m\in\mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\left[\|u^\varepsilon(t,\tau-m\mathcal{T},u_0)\|^2\right]\\
&\leqslant C\left(1+N^4\right)\left(e^{-\nu\lambda_1(t+m\mathcal{T}-\tau)}\mathbb{E}\left[|u_0|^2\right] + e^{-\nu\lambda_1 t}\int_{-\infty}^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \int_{t-1}^t |f(s)|^2 ds + 1\right),
\end{aligned}\tag{4.25}$$

which, together with (3.2), implies that for any initial value $u_0\in H$, there exists a positive constant $M_0=M_0(\tau,t,u_0)$, which is independent of $\varepsilon\in(0,\varepsilon_0]$, such that the solution u^ε of (2.10) satisfies that, for any $m>M_0$,

$$\begin{aligned}
\mathbb{E}\left[\|u^\varepsilon(t,\tau-m\mathcal{T},u_0)\|^2\right] \leqslant& \widetilde{C}\left(1+N^4\right)\left(1+e^{-\nu\lambda_1 t}\int_{-\infty}^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \int_0^{\mathcal{T}}|f(s)|^2 ds\right)\\
=:&C(t,\mathcal{T},f,N)<\infty,
\end{aligned}\tag{4.26}$$

where $\widetilde{C}, C(t,\mathcal{T},f,N)>0$ are independent of u_0 and ε. From (4.24) and (4.26), it follows that

$$\mu_t^\varepsilon(B_L)\geqslant 1-\frac{1}{L^2}\int_H C(t,\mathcal{T},f,N)\mu_\tau^\varepsilon(du_0), \tag{4.27}$$

which, along with the fact that the set B_L is compact in H, implies that for each $t \in \mathbb{R}$, the union $\cup_{\varepsilon\in(0,\varepsilon_0]}\mu_t^\varepsilon$ is tight in the sense that for every $\epsilon > 0$, there exists a compact subset K in H such that

$$\mu_t^\varepsilon(K) > 1-\epsilon, \ \forall \ \mu_t^\varepsilon \in S_t^\varepsilon.$$

Therefore, we can obtain that $\{\mu_t^{\varepsilon_n}\}_{n=1}^\infty$ is tight. Consequently, there exist a probability measure μ_t^0 and a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that $\mu_t^{\varepsilon_{n_k}} \to \mu_t^0$ weakly as $k\to\infty$, where the choice of subsequence depends on t. It then follows from Lemma 4.3 and [51, Theorem 1.3] that μ_t^0 is an invariant measure of non-autonomous system (2.10) with $\varepsilon = 0$.

In order to show such invariant measure μ_t^0 is periodic, it is sufficient to prove that for any $t\in\mathbb{R}$ and real-valued continuous and bounded functional ψ on H,

$$\int_H \psi(u)\mu_t^0(du) = \int_H \psi(u)\mu_{t+\mathcal{T}}^0(du). \tag{4.28}$$

Indeed, by the definition of invariant measure, we have that for any $k\in\mathbb{N}$,

$$\int_H \psi(u)\mu_t^{\varepsilon_{n_k}}(du) = \int_H \psi(u)\mu_{t+\mathcal{T}}^{\varepsilon_{n_k}}(du). \tag{4.29}$$

Thanks to the weak convergence we obtain that there exists a subsequence of $\{\varepsilon_{n_k}\}_{k=1}^\infty$ (not relabeled) such that

$$\lim_{k\to\infty}\int_H \psi(u)\mu_t^{\varepsilon_{n_k}}(du) = \int_H \psi(u)\mu_t^0(du), \tag{4.30}$$

and a further subsequence of $\{\varepsilon_{n_k}\}_{k=1}^\infty$ (not relabeled) such that

$$\lim_{k\to\infty}\int_H \psi(u)\mu_{t+\mathcal{T}}^{\varepsilon_{n_k}}(du) = \int_H \psi(u)\mu_{t+\mathcal{T}}^0(du), \tag{4.31}$$

where the choices of subsequence depend on t and $\mathcal{T}$. Then, (4.28) follows from (4.29)-(4.31). The proof is complete. □

4.1.3 Limiting behaviors of periodic invariant measures with respect to N

In this subsection, the limiting behaviors of periodic invariant measures of (2.10) as $N \to N_0 \in (0,\infty)$ will be discussed. From now on, for any fixed $\varepsilon \in (0,\varepsilon_0]$, we write the solution of (2.10) as $u^N(t,\tau,u_0)$ with initial value u_0 at initial time τ to highlight the dependence of solutions on the parameter N. In what follows, we will show the convergence of solutions of (2.10) about N.

Lemma 4.4. *Suppose (2.8) and (4.1) hold. Then, for every bounded subset $K\subset H$, $\tau\in\mathbb{R}$, $T>0$, $N_0\in(0,\infty)$ and $L>0$, we have*

$$\lim_{N\to N_0}\sup_{u_0\in K}\mathbb{P}\left(\left\{\omega\in\Omega : \sup_{\tau\leqslant t\leqslant\tau+T}|u^N(t,\tau,u_0)-u^{N_0}(t,\tau,u_0)| \geqslant L\right\}\right) = 0. \tag{4.32}$$

Proof. Let $u^N(t) = u^N(t,\tau,u_0)$, $u^{N_0}(t) = u^{N_0}(t,\tau,u_0)$ and $w(t) = u^N(t) - u^{N_0}(t)$. Then, by Itô's formula, we have

$$\begin{aligned}&|w(t)|^2 + 2\nu\int_\tau^t \|w(s)\|^2 ds\\ &= -2\int_\tau^t \left(F_N(\|u^N(s)\|)b\left(u^N(s),u^N(s),w(s)\right) - F_{N_0}(\|u^{N_0}(s)\|)b\left(u^{N_0}(s),u^{N_0}(s),w(s)\right)\right)ds \end{aligned} \tag{4.33}$$

$$+ 2\sqrt{\varepsilon}\int_\tau^t \big(w(s),(G(u^N(s))-G(u^{N_0}(s)))dW(s)\big) + \varepsilon\int_\tau^t \|G(u^N(s))-G(u^{N_0}(s))\|^2_{L_2(\ell^2,H)}ds.$$

Using (2.3)-(2.6), we deduce that

$$\begin{aligned}
&\big|F_N(\|u^N(s)\|)b\left(u^N(s),u^N(s),w(s)\right) - F_{N_0}(\|u^{N_0}(s)\|)b\left(u^{N_0}(s),u^{N_0}(s),w(s)\right)\big|\\
&\leqslant N\|w(s)\|^{\frac{3}{2}}|w(s)|^{\frac{1}{2}} + \big|(F_N(\|u^N(s)\|)-F_{N_0}(\|u^N(s)\|))b\left(u^{N_0}(s),u^N(s),w(s)\right)\big|\\
&\quad + \big|(F_{N_0}(\|u^N(s)\|)-F_{N_0}(\|u^{N_0}(s)\|))b\left(u^{N_0}(s),u^N(s),w(s)\right)\big|\\
&\leqslant N\|w(s)\|^{\frac{3}{2}}|w(s)|^{\frac{1}{2}} + |N-N_0|\|u^{N_0}(s)\|\|w(s)\|^{\frac{1}{2}}|w(s)|^{\frac{1}{2}} + N_0\|w(s)\|^{\frac{3}{2}}|w(s)|^{\frac{1}{2}}\\
&\leqslant \frac{\nu}{2}\|w(s)\|^2 + C_\nu(N+N_0)^4|w(s)|^2 + C_\nu|w(s)|^2 + C_\nu(N-N_0)^2\|u^{N_0}(s)\|^2.
\end{aligned}\tag{4.34}$$

Noting $\varepsilon\in(0,\varepsilon_0]$, it follows from (2.8), (4.33) and (4.34) that, for any $t\in[\tau,\tau+T]$,

$$\begin{aligned}
&|w(t)|^2 + \nu\int_\tau^t \|w(s)\|^2ds\\
&\leqslant \left(C_\nu + C_\nu(N+N_0)^4\right)\int_\tau^t |w(s)|^2ds + C_\nu(N-N_0)^2\int_\tau^t \|u^{N_0}(s)\|^2ds\\
&\quad + 2\sqrt{\varepsilon}\int_\tau^t \big(w(s),(G(u^N(s))-G(u^{N_0}(s)))dW(s)\big) + \varepsilon\beta_1\int_\tau^t |u^N(s)-u^{N_0}(s)|^2ds,
\end{aligned}\tag{4.35}$$

which implies that, for any $t\in[\tau,\tau+T]$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|w(r)|^2\right] &\leqslant \left(C_\nu+\varepsilon\beta_1+C_\nu(N+N_0)^4\right)\int_\tau^t \mathbb{E}\left[|w(s)|^2\right]ds + C_\nu(N-N_0)^2\int_\tau^t \mathbb{E}\left[\|u^{N_0}(s)\|^2\right]ds\\
&\quad + 2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}\left|\int_\tau^r \big(w(s),(G(u^N(s))-G(u^{N_0}(s)))dW(s)\big)\right|\right].
\end{aligned}\tag{4.36}$$

For the last term of (4.36), by (2.8) and BDG's inequality, we have

$$\begin{aligned}
&2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}\left|\int_\tau^r \big(w(s),(G(u^N(s))-G(u^{N_0}(s)))dW(s)\big)\right|\right]\\
&\leqslant \frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant s\leqslant t}|w(s)|^2\right] + 36\varepsilon\beta_1\int_\tau^t \mathbb{E}\left[|w(s)|^2\right]ds + \frac{\nu}{2}\int_\tau^t \mathbb{E}\left[\|w(s)\|^2\right]ds,
\end{aligned}\tag{4.37}$$

which, together with (4.36), means that

$$\begin{aligned}
\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|w(r)|^2\right] &\leqslant \left(C_\nu+37\varepsilon\beta_1+C_\nu(N+N_0)^4\right)\int_\tau^t \mathbb{E}\left[|w(s)|^2\right]ds + \frac{\nu}{2}\int_\tau^t \mathbb{E}\left[\|w(s)\|^2\right]ds\\
&\quad + C_\nu(N-N_0)^2\int_\tau^t \mathbb{E}\left[\|u^{N_0}(s)\|^2\right]ds + \frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant s\leqslant t}|w(s)|^2\right].
\end{aligned}\tag{4.38}$$

From (4.35) and (4.37), it follows

$$\begin{aligned}
\nu\int_\tau^t \mathbb{E}\left[\|w(s)\|^2\right]ds &\leqslant \left(C_\nu+37\varepsilon\beta_1+C_\nu(N+N_0)^4\right)\int_\tau^t \mathbb{E}\left[|w(s)|^2\right]ds + \frac{\nu}{2}\int_\tau^t \mathbb{E}\left[\|w(s)\|^2\right]ds\\
&\quad + C_\nu(N-N_0)^2\int_\tau^t \mathbb{E}\left[\|u^{N_0}(s)\|^2\right]ds + \frac{1}{4}\mathbb{E}\left[\sup_{\tau\leqslant s\leqslant t}|w(s)|^2\right].
\end{aligned}\tag{4.39}$$

Therefore, by (4.38) and (4.39),

$$\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}|w(r)|^2\right] \leqslant 4\left(C_\nu+37\varepsilon\beta_1+C_\nu(N+N_0)^4\right)\int_\tau^t \mathbb{E}\left[|w(s)|^2\right]ds$$

$$+ 4C_\nu (N-N_0)^2 \int_\tau^t \mathbb{E}\left[\|u^{N_0}(s)\|^2\right] ds. \tag{4.40}$$

In addition, by (4.4), there exists $M_4 > 0$ such that, for any $t \in [\tau, \tau + T]$,

$$4C_\nu (N-N_0)^2 \int_\tau^t \mathbb{E}\left[\|u^{N_0}(s)\|^2\right] ds \leqslant M_4 (N-N_0)^2, \tag{4.41}$$

which, along with (4.40) and Gronwall's inequality, implies that

$$\mathbb{E}\left[\sup_{\tau \leqslant r \leqslant \tau+T} |w(r)|^2\right] \leqslant M_4 (N-N_0)^2 e^{4\left(C_\nu + 37\varepsilon\beta_1 + C_\nu (N+N_0)^4\right)T}. \tag{4.42}$$

Then, (4.32) follows from Chebyshev's inequality and (4.42) immediately. □

Given $N \in (0,\infty)$, let $\mathcal{S}_t^N$ be the set of periodic invariant measures of (2.10) on H corresponding to the parameter N. It follows from Theorem 4.2 that the set $\mathcal{S}_t^N$ is nonempty for every $N \in (0,\infty)$. Then we can obtain the following limit results for (2.10).

Theorem 4.4. *Suppose* (2.8), (3.2) *and* (4.1) *hold, and let $f(t)$ be a $\mathcal{T}$-periodic function in t. For any $N_0 \in (0,\infty)$, let $N_k \in (0, N_0]$ for all $k \in \mathbb{N}$ satisfying that $N_k \to N_0$ as $k \to \infty$. If $\mu_t^{N_k} \in \mathcal{S}_t^{N_k}$, then there exists a subsequence $\{N_{k_l}\}_{l=1}^\infty$ and a periodic invariant measure $\mu_t^{N_0} \in \mathcal{S}_t^{N_0}$ such that $\mu_t^{N_{k_l}} \to \mu_t^{N_0}$ weakly as $l \to \infty$.*

Proof. For any $L > 0$, let $B_L = \{u \in V : \|u\| \leqslant L\}$, then B_L is a Borel subset of H. For any $t \in \mathbb{R}$ and $N_k \in (0,\infty)$, and given $\mu_t^{N_k} \in \mathcal{S}_t^{N_k}$, by the definition of periodic invariant measure, we have for any $\tau \leqslant t$ and $m \in \mathbb{N}$,

$$\mu_t^{N_k}(B_L) = \int_H \mathbb{P}\left(u^{N_k}(t, \tau - m\mathcal{T}, u_0) \in B_L\right) \mu_{\tau - m\mathcal{T}}^{N_k}(du_0) = \int_H \mathbb{P}\left(u^{N_k}(t, \tau - m\mathcal{T}, u_0) \in B_L\right) \mu_\tau^{N_k}(du_0).$$

Similar to (4.24), by Chebyshev's inequality, we have that for any $\tau \leqslant t$, $m \in \mathbb{N}$ and $L > 0$,

$$\mu_t^{N_k}(B_L) \geqslant 1 - \frac{1}{L^2} \int_H \liminf_{m\to\infty} \mathbb{E}\left[\|u^{N_k}(t, \tau - m\mathcal{T}, u_0)\|^2\right] \mu_\tau^{N_k}(du_0). \tag{4.43}$$

And, by (4.14), we obtain that for every $\tau < t-1$ and $m \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{E}\left[\|u^{N_k}(t, \tau - m\mathcal{T}, u_0)\|^2\right] \\ &\leqslant C\left(1 + N_0^4\right)\left(e^{-\nu\lambda_1(t + m\mathcal{T} - \tau)} \mathbb{E}\left[|u_0|^2\right] + e^{-\nu\lambda_1 t}\int_{-\infty}^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \int_{t-1}^t |f(s)|^2 ds + 1\right), \end{aligned} \tag{4.44}$$

which, together with (3.2), implies that for any initial value $u_0 \in H$, there exists a positive constant $M_0 = M_0(\tau, t, u_0)$, which is independent of N_k, such that the solution u^{N_k} of (2.10) satisfies that for any $m > M_0$,

$$\begin{aligned} \mathbb{E}\left[\|u^{N_k}(t, \tau - m\mathcal{T}, u_0)\|^2\right] &\leqslant \widetilde{C}(1 + N_0^4)\left(1 + e^{-\nu\lambda_1 t}\int_{t,-\infty}^t e^{\nu\lambda_1 s}|f(s)|^2 ds + \int_0^{\mathcal{T}} |f(s)|^2 ds\right) \\ &=: C(t, \mathcal{T}, f, N_0), \end{aligned} \tag{4.45}$$

where $\widetilde{C}, C(t, \mathcal{T}, f, N_0) > 0$ are independent of u_0 and N_k. From (4.43) and (4.45), it follows that

$$\mu_t^{N_k}(B_L) \geqslant 1 - \frac{1}{L^2}\int_H C(t, \mathcal{T}, f, N_0) \mu_\tau^{N_k}(du_0).$$

Therefore, we can deduce that $\{\mu_t^{N_k}\}_{n=1}^\infty$ is tight.

Then, similar to the proof of Theorem 4.3, by Lemma 4.4, we can obtain that there exist a periodic invariant measure $\mu_t^{N_0}$ and a subsequence $\{N_{k_l}\}$ of $\{N_k\}$ such that $\mu_t^{N_{k_l}} \to \mu_t^{N_0}$ weakly as $l \to \infty$, where the choice of subsequence depends on t and $\mathcal{T}$. The proof is complete. □

4.2 Invariant measures as the external term f is independent of t

In this subsection, we will consider the case that the external term f is independent of t. In this case, the results in Sections 2 and 3 still hold, and the transition operator $\{P_{s,t}\}_{s\leqslant t}$ becomes homogeneous in time. Next, we will study the existence and limiting behaviors of invariant measures of autonomous stochastic system (2.10) under conditions weaker than (2.8) and (4.1) in Subsection 4.1. Such invariant measure may also be regarded as a periodic invariant measure with any period.

4.2.1 Existence of invariant measures

The existence of invariant measures of time homogeneous transition semigroup for autonomous system (2.10) is proved as below.

Theorem 4.5. *Suppose* (2.7) *and* (2.8) *hold. Then, there exists* $\varepsilon_0 > 0$ *such that, for every* $0 < \varepsilon \leqslant \varepsilon_0$ *and* $N > 0$*, system* (2.10) *has an invariant measure* μ *on* H*.*

Proof. Similarly, the Feller property of $\{P_{s,t}\}_{0\leqslant s\leqslant t}$, and the Markov property of the solutions to (2.10) can be obtained.

In addition, it follows from (3.3) and (3.4) that for any $0 < \varepsilon \leqslant \varepsilon_0$, $N > 0$ and $t \geqslant 0$,

$$\mathbb{E}\left[|u(t,0,u_0)|^2\right] + \nu \int_0^t \mathbb{E}\left[\|u(s,0,u_0)\|^2\right] ds \leqslant |u_0|^2 + \left(\frac{2|f|^2}{\nu\lambda_1} + \frac{\nu\lambda_1}{4}\right) t.$$

Then, by Chebyshev's inequality, we obtain that for any $L > 0$ and $t > 0$,

$$\frac{1}{t}\int_0^t \mathbb{P}(\|u(s,0,u_0)\| \geqslant L) ds \leqslant \frac{1}{L^2}\frac{1}{t}\int_0^t \mathbb{E}\left[\|u(s,0,u_0)\|^2\right] ds \leqslant \frac{1}{L^2}\left(\frac{|u_0|^2}{t} + \frac{2|f|^2}{\nu^2\lambda_1} + \frac{\lambda_1}{4}\right). \tag{4.46}$$

We can find that there exists a positive constant $k \in \mathbb{N}$ such that, for any $t \geqslant k$, $\dfrac{|u_0|^2}{t} < 1$. In addition, note that the embedding $V \hookrightarrow H$ is compact, thus (4.46) shows $\{\mu_n\}_{n=k}^{\infty}$ is tight, where

$$\mu_n = \frac{1}{n}\int_0^n p(0,u_0;s,\cdot) ds.$$

Then, by Prokhorov's theorem, there exists a probability measure μ on H such that, up to a subsequence, $\mu_n \to \mu$, as $n \to \infty$. By the argument of [48, Theorem 8.3], one can further verify that μ is an invariant measure of (2.10) on H. The proof is complete. □

4.2.2 Zero-noise limit of invariant measures

In this subsection, we investigate the limiting behaviors of invariant measures of (2.10) as the noise intensity $\varepsilon \to 0$. Hereafter, we denote by $u^{\varepsilon}(t,0,u_0)$ the solution of (2.10) with initial value u_0 at initial time 0 with respect to the noise intensity ε. Next, we will show the convergence of solutions of (2.10) with respect to noise intensity.

Lemma 4.5. *Suppose* (2.7) *and* (2.8) *hold. Then, for every bounded subset* $E \subset H$*,* $T > 0$ *and* $L > 0$*, we have*

$$\lim_{\varepsilon\to 0}\sup_{u_0\in E}\mathbb{P}\left(\left\{\omega\in\Omega : \sup_{0\leqslant t\leqslant T}|u^{\varepsilon}(t,0,u_0) - u^0(t,0,u_0)| \geqslant L\right\}\right) = 0.$$

Proof. The proof is similar to Lemma 4.3 and thus is omitted here. □

Given $\varepsilon \in [0, \varepsilon_0]$, let S^ε be the set of invariant measures of (2.10) on H corresponding to the parameter ε. From Theorem 4.5, it follows that the set S^ε is nonempty for every $\varepsilon \in [0, \varepsilon_0]$. The next theorem is concerned with the limiting behaviors of invariant measures of (2.10) with respect to noise intensity.

Theorem 4.6. *Suppose (2.7) and (2.8) hold. If $\varepsilon_n \to 0$ as $n \to \infty$, and $\mu^{\varepsilon_n} \in S^{\varepsilon_n}$, then there exists a subsequence $\{\varepsilon_{n_k}\}$ and an invariant measure $\mu^0 \in S^0$ such that $\mu^{\varepsilon_{n_k}} \to \mu^0$ weakly as $k \to \infty$.*

Proof. For any $\varepsilon \in (0, \varepsilon_0]$, given $\mu^\varepsilon \in S^\varepsilon$, by the definition of invariant measure, we can deduce that for any $s > 0$ and $L > 0$,

$$\mu^\varepsilon(\|u_0\| \geqslant L) = \int_H \mathbb{P}(\|u^\varepsilon(s,0,u_0)\| \geqslant L)\mu^\varepsilon(du_0).$$

Then by Fubini's theorem, Fatou's lemma and (4.46), we have

$$\begin{aligned}\mu^\varepsilon(\|u_0\| \geqslant L) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \mu^\varepsilon(\|u_0\| \geqslant L)ds \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t \int_H \mathbb{P}(\|u^\varepsilon(s,0,u_0)\| \geqslant L)\mu^\varepsilon(du_0)ds \\ &\leqslant \int_H \limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}(\|u^\varepsilon(s,0,u_0)\| \geqslant L)ds\mu^\varepsilon(du_0) \leqslant \frac{M_3}{L^2},\end{aligned}$$

which, along with the fact that $\{u^\varepsilon \in V : \|u^\varepsilon\| \leqslant L\}$ is compact in H, shows that the set $\cup_{\varepsilon\in(0,\varepsilon_0]}\mu^\varepsilon$ is tight. Therefore, we can obtain that $\{\mu^{\varepsilon_n}\}_{n=1}^\infty$ is tight. Consequently, there exist a probability measure μ^0 and a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that $\mu^{\varepsilon_{n_k}} \to \mu^0$ weakly as $k \to \infty$. It then follows from Lemma 4.5 and [33, Theorem 6.1] that μ^0 is an invariant measure of (2.10) for $\varepsilon = 0$. □

4.2.3 Limiting behaviors of invariant measures with respect to N

In this subsection, the limiting behaviors of invariant measures of (2.10) as $N \to N_0 \in (0, \infty)$ will be discussed. From now on, for any fixed $\varepsilon \in (0, \varepsilon_0]$, denote by $u^N(t, 0, u_0)$ the solution of (2.10) with initial value u_0 at initial time 0 with respect to parameter N. In what follows, we will show the convergence of solutions of (2.10) with respect to N.

Lemma 4.6. *Suppose (2.7) and (2.8) hold. Then, for every bounded subset $K \subset H$, $T > 0$, $N_0 \in (0, \infty)$ and $L > 0$, we have*

$$\lim_{N\to N_0} \sup_{u_0\in K} \mathbb{P}\left(\left\{\omega \in \Omega : \sup_{0\leqslant t\leqslant T} |u^N(t,0,u_0) - u^{N_0}(t,0,u_0)| \geqslant L\right\}\right) = 0.$$

Proof. The proof is similar to Lemma 4.4 and thus is omitted here. □

Given $N \in (0, \infty)$, let $\mathcal{S}^N$ be the set of invariant measures of (2.10) on H corresponding to the parameter N. It follows from Theorem 4.5 that the set $\mathcal{S}^N$ is nonempty for every $N \in (0, \infty)$. Then we can obtain the following limit results for (2.10).

Theorem 4.7. *Suppose (2.7) and (2.8) hold. Let $N_k, N_0 \in (0, \infty)$ for all $k \in \mathbb{N}$ satisfying that $N_k \to N_0$ as $k \to \infty$. If $\mu^{N_k} \in \mathcal{S}^{N_k}$, then there exists a subsequence $\{N_{k_l}\}_{l=1}^\infty$ and an invariant measure $\mu^{N_0} \in \mathcal{S}^{N_0}$ such that $\mu^{N_{k_l}} \to \mu^{N_0}$ weakly as $l \to \infty$.*

Proof. For any $N_k \in (0, \infty)$, choosing $\mu^{N_k} \in \mathcal{S}^{N_k}$, it follows from the definition of invariant measure, Fubini's theorem, Fatou's lemma and (4.46) that for any $L > 0$,

$$\mu^{N_k}(\|u_0\| \geqslant L) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mu^{N_k}(\|u_0\| \geqslant L)ds \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t \int_H \mathbb{P}(\|u^{N_k}(s,0,u_0)\| \geqslant L)\mu^{N_k}(du_0)ds$$

$$\leqslant \int_H \limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}(\|u^{N_k}(s,0,u_0)\| \geqslant L) ds \mu^{N_k}(du_0) \leqslant \frac{M_3}{L^2}.$$

Similar to the proof of Theorem 4.3, we can obtain that $\{\mu^{N_k}\}_{k=1}^\infty$ is tight. Therefore, there exist a probability measure μ^{N_0} and a subsequence $\{N_{k_l}\}$ of $\{N_k\}$ such that $\mu^{N_{k_l}} \to \mu^{N_0}$ weakly as $l \to \infty$. It then follows from Lemma 4.6 and [33, Theorem 6.1] that μ^{N_0} is an invariant measure of (2.10) for $N = N_0$. □

5 Large deviation principle of stochastic GMNSE

In this section, we will investigate the large deviation principle of (2.10) as $\varepsilon \to 0$ by using the weak convergence method.

5.1 Preliminaries of large deviation principle

In this subsection, we will first recall some definitions and results from large deviation theory.

Given $M > 0$, denote by

$$S_M = \left\{ h \in L^2(0,T;\ell^2) : \int_0^T \|h(t)\|_{\ell^2}^2 dt \leqslant M \right\}.$$

Then S_M is a polish space endowed with the weak topology. Throughout this paper, we always assume S_M is equipped with the weak topology, unless otherwise stated.

Let $\mathcal{A}$ be the space of all ℓ^2-valued stochastic processes h which are progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ and $\int_0^T \|h(t)\|_{\ell^2}^2 dt < \infty$ $\mathbb{P}$-almost surely. Denote by $\mathcal{A}_M = \{h \in \mathcal{A} : h(\omega) \in S_M$ for almost all $\omega \in \Omega\}$.

Let $\mathbb{S}$ be a polish space. For any $\varepsilon > 0$, let $\mathcal{G}^\varepsilon : C([0,T],U) \to \mathbb{S}$ be a measurable map. Denote by

$$u^\varepsilon = \mathcal{G}^\varepsilon(W), \quad \forall\, \varepsilon > 0. \tag{5.1}$$

Definition 5.1 (Rate function). *A function $I : \mathbb{S} \to [0,\infty]$ is called a rate function on $\mathbb{S}$ if it is lower semicontinuous in $\mathbb{S}$. A rate function I on $\mathbb{S}$ is said to be a good rate function on $\mathbb{S}$ if for every $0 \leqslant C < \infty$, the level set $\{x \in \mathbb{S} : I(x) \leqslant C\}$ is a compact subset of $\mathbb{S}$.*

Definition 5.2 (Large deviation principle). *The family $\{u^\varepsilon\}$ is said to satisfy the large deviation principle in $\mathbb{S}$ with a rate function $I : \mathbb{S} \to [0,\infty]$, if for every Borel subset A of $\mathbb{S}$,*

$$-\inf_{x\in A^\circ} I(x) \leqslant \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(u^\varepsilon \in A) \leqslant \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(u^\varepsilon \in A) \leqslant -\inf_{x\in\overline{A}} I(x),$$

where A° and $\overline{A}$ are the interior and the closure of A in $\mathbb{S}$, respectively.

Since $\mathbb{S}$ is a polish space, the Laplace principle and the large deviation principle are equivalent (see [8, Section 4]). Next, we recall the concept of Laplace principle.

Definition 5.3 (Laplace principle). *The family $\{u^\varepsilon\}$ is said to satisfy the Laplace principle in $\mathbb{S}$ with a rate function $I : \mathbb{S} \to [0,\infty]$ if for all bounded and continuous $\phi : \mathbb{S} \to \mathbb{R}$,*

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\left[e^{-\frac{1}{\varepsilon}\phi(u^\varepsilon)}\right] = -\inf_{x\in\mathbb{S}} \{\phi(x) + I(x)\}.$$

In order to prove the large deviation principle of u^ε, we will assume that the family $\{\mathcal{G}^\varepsilon\}$ fulfills the following conditions: there exists a measurable map $\mathcal{G}^0 : C([0,T],U) \to \mathbb{S}$ such that

(H1) For every $M > 0$, the set $\left\{\mathcal{G}^0\left(\int_0^\cdot h(t)dt\right) : h \in S_M\right\}$ is a compact subset of $\mathbb{S}$.

(H2) If $M > 0$ and $\{h^\varepsilon\} \subseteq \mathcal{A}_M$ such that $\{h^\varepsilon\}$ converges in distribution to h as S_M-valued random variables, then $\mathcal{G}^\varepsilon\left(W + \varepsilon^{-\frac{1}{2}}\int_0^\cdot h^\varepsilon(t)dt\right)$ converges in distribution to $\mathcal{G}^0\left(\int_0^\cdot h(t)dt\right)$.

We recall the following theorem proved by Budhiraja and Dupuis [8], which will be used to establish the large deviation of (2.10).

Theorem 5.1 ([8], Theorem 4.4)**.** *Under (H1)-(H2), the family $\{u^\varepsilon\}$, given by (5.1), satisfies the Laplace principle in $\mathbb{S}$ with rate function I defined for every $x \in \mathbb{S}$ by,*

$$I(x) = \inf\left\{\frac{1}{2}\int_0^T \|h(t)\|_{\ell^2}^2 dt : h \in L^2(0,T;\ell^2) \text{ such that } \mathcal{G}^0\left(\int_0^\cdot h(t)dt\right) = x\right\},$$

where the infimum over an empty set is taken to be ∞.

5.2 The large deviations result

In this subsection, we will establish the Laplace principle for (2.10) in $C([0,T],H) \cap L^2(0,T;V)$. Denote by $u^\varepsilon(t)$ the solution of (2.10) with initial value $u_0 \in H$ at initial time 0 with respect to the noise intensity ε. To that end, we first show $\mathcal{G}^0$ satisfies (H1).

Given a control $h \in L^2(0,T;\ell^2)$, the controlled equation is as follows:

$$du_h(t) + \nu A u_h(t)dt + F_N(\|u_h\|)B(u_h)dt = f(t)dt + G(u_h(t))h(t)dt \tag{5.2}$$

with initial condition

$$u_h(0) = u_0 \in H. \tag{5.3}$$

In what follows, we show the well-posedness of (5.2)-(5.3).

Lemma 5.1. *Suppose (2.7) and (2.8) hold. Then, for every $h \in L^2(0,T;\ell^2)$, problem (5.2)-(5.3) has a unique solution $u_h \in C([0,T],H) \cap L^2(0,T;V)$.*

Moreover, for any $|u_0| \vee \|h\|_{L^2(0,T;\ell^2)} \leqslant R$, with $R > 0$, the solution u_h satisfies that, for any $t \in [0,T]$,

$$|u_h(t)|^2 + \int_0^t \|u_h(s)\|^2 ds \leqslant c_1, \tag{5.4}$$

where $c_1 > 0$ depends on R and T.

Proof. To verify the existence and uniqueness of the solution of (5.2), we first assume $h \in L^\infty(0,T;\ell^2)$. Then, by (2.7), (2.8) and Young's inequality, we have for any $u, v \in C([0,T],H) \cap L^2(0,T;V)$,

$$\begin{aligned}&(-\nu A u - F_N(\|u\|)B(u) + f(t) + G(u)h(t), u)\\ &\leqslant -\frac{3\nu}{4}\|u\|^2 + \left(1 + \frac{\nu\alpha_1}{4\alpha_2} + \frac{\alpha_2}{\nu}\|h(t)\|_{\ell^2}^2\right)|u|^2 + \frac{1}{4}|f(t)|^2 + \frac{\nu\alpha_1}{4\alpha_2},\end{aligned}$$

and

$$(-\nu A u + \nu A v - F_N(\|u\|)B(u) + F_N(\|v\|)B(v) + G(u)h(t) - G(v)h(t), u - v)$$

$$\leqslant -\frac{\nu}{2}\|u-v\|^2 + \left(C_\nu + \sqrt{\beta_1}\|h(t)\|_{\ell^2} + \frac{\beta_2}{\nu}\|h(t)\|_{\ell^2}^2 \right)|u-v|^2.$$

Note $h \in L^\infty(0,T;\ell^2)$, and by using a similar technique to the one in [10, Theorem 7], we deduce that (5.2)-(5.3) possesses a unique solution $u_h \in C([0,T],H) \cap L^2(0,T;V)$.

For general $h \in L^2(0,T;\ell^2)$, we can find a sequence $h_n \in L^\infty(0,T;\ell^2)$ such that

$$h_n \to h \text{ strongly in } L^2\left(0,T;\ell^2\right).$$

Let u_n be the unique solution of (5.2)-(5.3) when we replace h by h_n. Next, we will show $\{u_n\}$ is a Cauchy sequence in $C([0,T],H) \cap L^2(0,T;V)$. By (2.7), (2.8) and (5.2), we have

$$\begin{aligned}
&\frac{d}{dt}|u_n(t)-u_m(t)|^2 \\
&= 2\left(-\nu A u_n(t) + \nu A u_m(t) - F_N(\|u_n(t)\|)B(u_n(t)) + F_N(\|u_m(t)\|)B(u_m(t)), u_n(t)-u_m(t)\right) \\
&\quad + 2\left((G(u_n(t)) - G(u_m(t)))h_n(t), u_n(t)-u_m(t)\right) + 2\left(G(u_m(t))(h_n(t)-h_m(t)), u_n(t)-u_m(t)\right) \\
&\leqslant -\nu\|u_n(t)-u_m(t)\|^2 + \left(C_\nu + 2\sqrt{\beta_1}\|h_n(t)\|_{\ell^2} + \frac{4\beta_2}{\nu}\|h_n(t)\|_{\ell^2}^2 \right)|u_n(t)-u_m(t)|^2 \\
&\quad + \left(\alpha_1(1+|u_m(t)|^2) + \alpha_2\|u_m(t)\|^2\right)|u_n(t)-u_m(t)|^2 + \|h_n(t)-h_m(t)\|_{\ell^2}^2,
\end{aligned} \tag{5.5}$$

which, together with Gronwall's inequality, implies that, for any $t \in [0,T]$,

$$|u_n(t)-u_m(t)|^2 \leqslant e^{\int_0^T M_1\left(1+\|h_n(s)\|_{\ell^2}+\|h_n(s)\|_{\ell^2}^2+|u_m(s)|^2+\|u_m(s)\|^2\right)ds} \int_0^T \|h_n(s)-h_m(s)\|_{\ell^2}^2 ds, \tag{5.6}$$

where $M_1 = \max\left\{C_\nu + \alpha_1, 2\sqrt{\beta_1}, \frac{4\beta_2}{\nu}, \alpha_2\right\}$. Similar to (5.5), we have

$$\begin{aligned}
\frac{d}{dt}|u_n(t)|^2 =& 2\left(-\nu A u_n(t) + f(t), u_n(t)\right) + 2\left(G(u_n(t))h_n(t), u_n(t)\right) \\
\leqslant& -\nu\|u_n(t)\|^2 + |u_n(t)|^2 + |f(t)|^2 + \frac{\nu\alpha_1}{\alpha_2}\left(1+|u_n(t)|^2\right) + \frac{\alpha_2}{\nu}\|h_n(t)\|_{\ell^2}^2|u_n(t)|^2.
\end{aligned} \tag{5.7}$$

From (5.7) and Gronwall's inequality, it follows that, for any $t \in [0,T]$,

$$|u_n(t)|^2 \leqslant |u_0|^2 e^{\int_0^T\left(1+\frac{\nu\alpha_1}{\alpha_2}+\frac{\alpha_2}{\nu}\|h_n(s)\|_{\ell^2}^2\right)ds} + e^{\int_0^T\left(1+\frac{\nu\alpha_1}{\alpha_2}+\frac{\alpha_2}{\nu}\|h_n(s)\|_{\ell^2}^2\right)ds} \int_0^T \left(\frac{\nu\alpha_1}{\alpha_2} + |f(s)|^2\right) ds. \tag{5.8}$$

By (5.7), (5.8), the boundedness of h_n in $L^2(0,T;\ell^2)$ and the fact that $f \in L^2_{loc}(\mathbb{R};H)$, we find that there exists $M_2 > 0$, independent of n, such that

$$\sup_{0\leqslant t\leqslant T}|u_n(t)|^2 + \int_0^T \|u_n(s)\|^2 ds < M_2 < \infty,$$

which, together with (5.5) and (5.6), implies that $\{u_n\}$ is a Cauchy sequence in $C([0,T],H) \cap L^2(0,T;V)$, and we denote the limit by u_h.

Then by using the standard monotonicity argument (e.g. [57, Theorem 30.A]), one can show that u_h is the solution of (5.2)-(5.3) corresponding to h.

In addition, similar to (5.7) and (5.8), (5.4) can be obtained. The proof is complete. □

Define $\mathcal{G}^0 : C([0,T],U) \to C([0,T],H)$ by, for every $\zeta \in C([0,T],U)$,

$$\mathcal{G}^0(\zeta) = \begin{cases} u_h, \text{ if } \zeta = \int_0^\cdot h(t)dt \text{ for some } h \in L^2(0,T;\ell^2), \\ 0, \quad \text{otherwise,} \end{cases} \tag{5.9}$$

where u_h is the solution of (5.2)-(5.3). It follows from Lemma 5.1 that the mapping $\mathcal{G}^0$ is well-defined.

Next, we will show $\mathcal{G}^0$ satisfies (H1).

Lemma 5.2. *Suppose* (2.7) *and* (2.8) *hold. Then, for every $M > 0$, the set*

$$K_M = \left\{ \mathcal{G}^0 \left(\int_0^{\cdot} h(t)dt \right) : h \in S_M \right\} \tag{5.10}$$

is a compact subset of $C([0,T], H) \cap L^2(0,T;V)$.

Proof. Form (5.9) and (5.10), it follows that

$$K_M = \left\{ u_h : h \in L^2(0,T;\ell^2), \int_0^T \|h(t)\|_{\ell^2}^2 dt \leqslant M \right\}, \tag{5.11}$$

where u_h is the solution of (5.2)-(5.3). In order to prove the compactness of K_M, we need to show that for any sequence $\{u_{h_n}\}_{n=1}^{\infty} \subseteq K_M$, there exists a convergent subsequence of $\{u_{h_n}\}_{n=1}^{\infty}$. Indeed, given $\{u_{h_n}\}_{n=1}^{\infty} \subseteq K_M$, by (5.11), we find that $h_n \in L^2(0,T;\ell^2)$ and $\int_0^T \|h_n(t)\|_{\ell^2}^2 dt \leqslant M$, which imply that there exist $h \in S_M$ and a subsequence of $\{h_n\}$, still denote as $\{h_n\}$, which converges weakly to a limit h in $L^2(0,T;\ell^2)$. Next, we will show that $u_{h_n} \to u_h$ strongly in $C([0,T], H) \cap L^2(0,T;V)$ as $n \to \infty$.

Since $\int_0^T \|h_n(t)\|_{\ell^2}^2 dt \leqslant M$, by Lemma 5.1, there exists $c_2 = c_2(T, M) > 0$ such that for any $n \in \mathbb{N}$,

$$\sup_{0 \leqslant t \leqslant T} |u_{h_n}(t)|^2 + \sup_{0 \leqslant t \leqslant T} |u_h(t)|^2 + \int_0^T \|u_{h_n}(t)\|^2 dt + \int_0^T \|u_h(t)\|^2 dt \leqslant c_2. \tag{5.12}$$

Similar to (5.5), by (2.8) and (5.2), we have

$$\begin{aligned}
&|u_{h_n}(t) - u_h(t)|^2 + \nu \int_0^t \|u_{h_n}(s) - u_h(s)\|^2 ds \\
&\leqslant \int_0^t \left(C_\nu + 2\sqrt{\beta_1} \|h_n(s)\|_{\ell^2} + \frac{4\beta_2}{\nu} \|h_n(s)\|_{\ell^2}^2 \right) |u_{h_n}(s) - u_h(s)|^2 ds \\
&\quad + 2 \int_0^t \left(G(u_h(s))(h_n(s) - h(s)), u_{h_n}(s) - u_h(s) \right) ds.
\end{aligned} \tag{5.13}$$

For any fixed $m \geqslant 1$, define

$$\varphi_n^m(t) = \int_0^t P_m G(u_h(s)) \left(h_n(s) - h(s) \right) ds,$$

where P_m is the orthogonal projection onto the subspace of H spanned by $\{e_1, e_2, \cdots, e_m\}$, and e_j is the orthonormal eigenfunctions of the operator A (see Section 2 for more details). Then, we can deduce the compactness of $H_m \hookrightarrow V$ embedding. In addition, it follows from (2.7) and (5.12) that

$$\sup_{0 \leqslant t \leqslant T} \|\varphi_n^m(t)\|_{H_m} \leqslant \left(\int_0^T \left(\alpha_1 + \alpha_1 |u_h(s)|^2 + \alpha_2 \|u_h(s)\|^2 \right) ds \right)^{\frac{1}{2}} \left(\int_0^T \|h_n(s) - h(s)\|_{\ell^2}^2 \, ds \right)^{\frac{1}{2}}, \tag{5.14}$$

which, together with the boundedness of $\{h_n\}$ in $L^2(0,T;\ell^2)$, implies $\varphi_n^m \in C([0,T], H_m)$.

Next, we will show $\varphi_n^m \to 0$ in $C([0,T], H) \cap L^2(0,T;V)$ as $n \to \infty$. From (5.14) and the boundedness of $\{h_n\}$ in $L^2(0,T;\ell^2)$, it follows that $\{\varphi_n^m\}_{n=1}^{\infty}$ is equicontinuous on $[0,T]$. In addition, thanks to the compactness of $H_m \hookrightarrow V$ embedding, we can deduce that there exists a subsequence of $\{\varphi_n^m\}_{n=1}^{\infty}$, still denote as $\{\varphi_n^m\}_{n=1}^\infty$, such that $\varphi_n^m \to 0$ in $C([0,T],V)$ by the Arzelà-Ascoli theorem, which implies that for any $m\in\mathbb{N}$,

$$\varphi_n^m \to 0 \text{ in } C([0,T],H)\cap L^2(0,T;V) \text{ as } n\to\infty. \tag{5.15}$$

As in Lemma 5.1, we first assume $h_n, h \in L^\infty(0,T;\ell^2)$. It then follows from (2.7) and (5.12) that

$$\begin{aligned}\int_0^T \|\frac{d}{dt}\varphi_n^m(t)\|_{V^*}^2 dt \leqslant & \int_0^T \|P_m G(u_h(t))(h_n(t)-h(t))\|_{V^*}^2 dt\\ \leqslant & C\int_0^T \left(1+|u_h(t)|^2+\|u_h(t)\|^2\right) dt <\infty,\end{aligned}$$

which shows $\frac{d}{dt}\varphi_n^m(t)$ is an element in $L^2(0,T;V^*)$.

From [57, Proposition 23.23], it follows that

$$\begin{aligned}(u_{h_n}(t)-u_h(t),\varphi_n^m(t)) = & \int_0^t {}_{V^*}\left\langle \frac{d}{ds}(u_{h_n}(s)-u_h(s)),\varphi_n^m(s)\right\rangle_V ds\\ & +\int_0^t {}_{V^*}\left\langle \frac{d}{ds}\varphi_n^m(s),u_{h_n}(s)-u_h(s)\right\rangle_V ds.\end{aligned} \tag{5.16}$$

Therefore, by (5.2) and (5.16),

$$\begin{aligned}&\int_0^t (P_m G(u_h(s))(h_n(s)-h(s)),u_{h_n}(s)-u_h(s))\, ds\\ &= (u_{h_n}(t)-u_h(t),\varphi_n^m(t)) - \int_0^t {}_{V^*}\left\langle \frac{d}{ds}(u_{h_n}(s)-u_h(s)),\varphi_n^m(s)\right\rangle_V ds\\ &= (u_{h_n}(t)-u_h(t),\varphi_n^m(t)) + \nu\int_0^t {}_{V^*}\langle Au_{h_n}(s)-Au_h(s),\varphi_n^m(s)\rangle_V\, ds\\ &\quad +\int_0^t {}_{V^*}\langle F_N(\|u_{h_n}(s)\|)B(u_{h_n}(s))-F_N(\|u_h(s)\|)B(u_h(s)),\varphi_n^m(s)\rangle_V\, ds\\ &\quad -\int_0^t (G(u_{h_n}(s))h_n(s)-G(u_h(s))h(s),\varphi_n^m(s))\, ds\\ &:= I_1+I_2+I_3+I_4.\end{aligned} \tag{5.17}$$

By (2.5), (2.8), (5.12), Young's inequality and Hölder's inequality, we deduce for any $t\in[0,T]$,

$$\begin{aligned}I_1 \leqslant & \frac14|u_{h_n}(t)-u_h(t)|^2+|\varphi_n^m(t)|^2,\\ I_2 \leqslant & \nu\left(\int_0^t \|Au_{h_n}(s)-Au_h(s)\|_{V^*}^2 ds\right)^{\frac12}\left(\int_0^t \|\varphi_n^m(s)\|^2 ds\right)^{\frac12}\\ \leqslant & \sqrt{2}\nu\left(\int_0^t (\|u_{h_n}(s)\|^2+\|u_h(s)\|^2) ds\right)^{\frac12}\left(\int_0^t \|\varphi_n^m(s)\|^2 ds\right)^{\frac12}\\ \leqslant & \sqrt{2c_2}\nu\left(\int_0^t \|\varphi_n^m(s)\|^2 ds\right)^{\frac12},\\ I_3 \leqslant & CN\left(\int_0^t \|u_{h_n}(s)-u_h(s)\|^2 ds\right)^{\frac12}\left(\int_0^t \|\varphi_n^m(s)\|^2 ds\right)^{\frac12} \leqslant \sqrt{2c_2}CN\left(\int_0^t \|\varphi_n^m(s)\|^2 ds\right)^{\frac12},\\ I_4 \leqslant & \int_0^t |G(u_{h_n}(s))h_n(s)-G(u_h(s))h(s)|\,|\varphi_n^m(s)|\, ds\\ \leqslant & \sup_{0\leqslant s\leqslant t}|\varphi_n^m(s)|\left(\int_0^t \left(\alpha_1(1+|u_{h_n}(s)|^2)+\alpha_2\|u_{h_n}(s)\|^2\right) ds\right)^{\frac12}\left(\int_0^t \|h_n(s)\|_{\ell^2}^2 ds\right)^{\frac12}\end{aligned}$$

$$+\sup_{0\leqslant s\leqslant t}|\varphi_n^m(s)|\left(\int_0^t\left(\alpha_1(1+|u_h(s)|^2)+\alpha_2\|u_h(s)\|^2\right)ds\right)^{\frac{1}{2}}\left(\int_0^t\|h(s)\|_{\ell^2}^2ds\right)^{\frac{1}{2}}. \tag{5.18}$$

It follows from (5.13), (5.17) and (5.18) that there exists a constant $c_3>0$ such that for any $t\in[0,T]$,

$$\begin{aligned}&\frac{1}{2}|u_{h_n}(t)-u_h(t)|^2+\nu\int_0^t\|u_{h_n}(s)-u_h(s)\|^2ds\\&\leqslant\int_0^t\left(C_\nu+2\sqrt{\beta_1}\|h_n(s)\|_{\ell^2}+\frac{4\beta_2}{\nu}\|h_n(s)\|_{\ell^2}^2\right)|u_{h_n}(s)-u_h(s)|^2ds\\&\quad+c_3\left(\sup_{0\leqslant s\leqslant t}|\varphi_n^m(s)|+\left(\int_0^t\|\varphi_n^m(s)\|^2ds\right)^{\frac{1}{2}}\right)\\&\quad+\int_0^t\|(I-P_m)G(u_h(s))\|_{L_2(\ell^2,H)}^2ds+\int_0^t\|h_n(s)-h(s)\|_{\ell^2}^2|u_{h_n}(s)-u_h(s)|^2ds.\end{aligned}\tag{5.19}$$

Then, by Gronwall's inequality, we deduce that for any $t\in[0,T]$,

$$\begin{aligned}|u_{h_n}(t)-u_h(t)|^2\leqslant&2c_3\left(\sup_{0\leqslant s\leqslant T}|\varphi_n^m(s)|+\left(\int_0^T\|\varphi_n^m(s)\|^2ds\right)^{\frac{1}{2}}\right)e^{c_4\int_0^T(1+\|h_n(s)\|_{\ell^2}+\|h_n(s)\|_{\ell^2}^2+\|h(s)\|_{\ell^2}^2)ds}\\&+2e^{c_4\int_0^T(1+\|h_n(s)\|_{\ell^2}+\|h_n(s)\|_{\ell^2}^2+\|h(s)\|_{\ell^2}^2)ds}\int_0^T\|(I-P_m)G(u_h(s))\|_{L_2(\ell^2,H)}^2ds.\end{aligned}\tag{5.20}$$

Noting that $G(\cdot)$ is a Hilbert-Schmidt operator, by the dominated convergence theorem, we have

$$\int_0^T\|(I-P_m)G(u_h(s))\|_{L_2(\ell^2,H)}^2ds\to0,\quad\text{as } m\to\infty,$$

which, together with (5.15), (5.19), (5.20) and the boundedness of $\{h_n\}$ in $L^2(0,T;\ell^2)$, implies $u_{h_n}\to u_h$ in $C([0,T],H)\cap L^2(0,T;V)$ as $n\to\infty$.

Then, similar to the proof of Lemma 5.1, we can deduce such convergence also holds for the case $h_n,h\in L^2(0,T;\ell^2)$. Hence the proof is complete. □

In order to prove condition (H2), suppose $0\leqslant\varepsilon\leqslant\min\left\{\frac{\nu}{37\alpha_2},\frac{\nu}{2\beta_2}\right\}=:\varepsilon_0$. It then follows from the well-posedness of (2.10) that there exists a Borel-measurable mapping $\mathcal{G}^\varepsilon:C([0,T],U)\to C([0,T],H)\cap L^2(0,T;V)$ such that

$$u^\varepsilon=\mathcal{G}^\varepsilon(W),\quad\mathbb{P}\text{-almost surely.}$$

In the sequel, we will establish some property of $\mathcal{G}^\varepsilon$, which will be useful to verify (H2). To do that, the following version of Gronwall's inequality in [19] is needed.

Lemma 5.3 ([19], Lemma A.1)**.** *Let X,Y,I and φ be non-negative processes and Z be a non-negative integrable random variable. Assume that I is non-decreasing and there exist non-negative constants $C,\alpha,\beta,\gamma,\delta$ with the following properties*

$$\int_0^T\varphi(s)ds\leqslant C\ a.s.,\quad2\beta e^C\leqslant1,\quad2\delta e^C\leqslant\alpha,\tag{5.21}$$

and such that, for $0\leqslant t\leqslant T$,

$$X(t)+\alpha Y(t)\leqslant Z+\int_0^t\varphi(r)X(r)dr+I(t),\quad a.s.,$$

$$\mathbb{E}[I(t)] \leqslant \beta\mathbb{E}[X(t)] + \gamma \int_0^t \mathbb{E}[X(s)]ds + \delta\mathbb{E}[Y(t)] + C',$$

where $C' > 0$ is a constant. If $X \in L^\infty([0,T]\times\Omega)$, then we have for any $t \in [0,T]$,

$$\mathbb{E}[X(t) + \alpha Y(t)] \leqslant 2\,\exp\left(C + 2t\gamma e^C\right)\left(\mathbb{E}[Z] + C'\right).$$

Lemma 5.4. *Suppose that (2.7) and (2.8) hold, and $h \in \mathcal{A}_M$ with $M > 0$. Then, $u_h^\varepsilon := \mathcal{G}^\varepsilon(W + \varepsilon^{-\frac{1}{2}}\int_0^\cdot h(t)dt)$ is the unique solution of*

$$du_h^\varepsilon + \nu A u_h^\varepsilon dt + F_N(\|u_h^\varepsilon\|)B(u_h^\varepsilon)dt = \left(f(t) + G(u_h^\varepsilon)h\right)dt + \sqrt{\varepsilon}G(u_h^\varepsilon)dW(t), \tag{5.22}$$

with initial value $u_h^\varepsilon(0) = u_0 \in H$.

Moreover, for any fixed $M > 0$, there exists a positive constant $\widetilde{\varepsilon}_0 := \widetilde{\varepsilon}_0(M) \leqslant \varepsilon_0$ such that for any $h \in \mathcal{A}_M$ and $u_0 \in H$ with $|u_0| \leqslant R$ $(R > 0)$, the solution u_h^ε satisfies, for any $0 \leqslant \varepsilon \leqslant \widetilde{\varepsilon}_0$,

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|u_h^\varepsilon(t)|^2 + \int_0^T \|u_h^\varepsilon(t)\|^2 dt\right] \leqslant C_1, \tag{5.23}$$

where $C_1 > 0$ depends on M, R and T.

Proof. From the Girsanov theorem ([21, Appendix A.1.]), it follows that (5.22), with initial value $u_h^\varepsilon(0) = u_0 \in H$, has a unique solution $u_h^\varepsilon \in C([0,T], H) \cap L^2(0,T;V)$ $\mathbb{P}$-almost surely.

By (5.22) and Itô's formula, we have that, for any $t \in [0,T]$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
&|u_h^\varepsilon(t)|^2 + 2\nu\int_0^t \|u_h^\varepsilon(s)\|^2 ds \\
&\leqslant |u_0|^2 + \int_0^t |f(s)|^2 ds + \int_0^t |u_h^\varepsilon(s)|^2 ds + 2\int_0^t \left(G(u_h^\varepsilon(s))h(s), u_h^\varepsilon(s)\right) ds \\
&\quad + 2\sqrt{\varepsilon}\int_0^t \left(u_h^\varepsilon(s), G(u_h^\varepsilon(s))dW(s)\right) + \varepsilon\int_0^t \|G(u_h^\varepsilon(s))\|_{L_2(\ell^2,H)}^2 ds.
\end{aligned} \tag{5.24}$$

For the fourth term on the right-hand side of (5.24), by (2.7), we find

$$\begin{aligned}
&2\int_0^t \left(G(u_h^\varepsilon(s))h(s), u_h^\varepsilon(s)\right) ds \leqslant 2\int_0^t \|G(u_h^\varepsilon(s))\|_{L_2(\ell^2,H)} \|h(s)\|_{\ell^2}|u_h^\varepsilon(s)|ds \\
&\leqslant 2\int_0^t \left(\alpha_1\left(1 + |u_h^\varepsilon(s)|^2\right) + \alpha_2\|u_h^\varepsilon(s)\|^2\right)^{\frac{1}{2}} \|h(s)\|_{\ell^2}|u_h^\varepsilon(s)|ds \\
&\leqslant \frac{\nu}{4}\int_0^t \|u_h^\varepsilon(s)\|^2 ds + \frac{\nu\alpha_1}{4\alpha_2}\int_0^t \left(1 + |u_h^\varepsilon(s)|^2\right) ds + \frac{4\alpha_2}{\nu}\int_0^t \|h(s)\|_{\ell^2}^2|u_h^\varepsilon(s)|^2 ds.
\end{aligned} \tag{5.25}$$

For the last term of (5.24), by (2.7), we derive for any $\varepsilon \in \left(0, \frac{\nu}{4\alpha_2}\right)$,

$$\varepsilon\int_0^t \|G(u_h^\varepsilon(s))\|_{L_2(\ell^2,H)}^2 ds \leqslant \frac{\nu\alpha_1}{4\alpha_2}\int_0^t (1 + |u_h^\varepsilon(s)|^2)ds + \frac{\nu}{4}\int_0^t \|u_h^\varepsilon(s)\|^2 ds. \tag{5.26}$$

Then by (5.24)-(5.26), we deduce for any $t \in [0,T]$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
|u_h^\varepsilon(t)|^2 + \nu\int_0^t \|u_h^\varepsilon(s)\|^2 ds \leqslant& |u_0|^2 + \int_0^t |f(s)|^2 ds + \int_0^t \left(1 + \frac{\nu\alpha_1}{2\alpha_2} + \frac{4\alpha_2}{\nu}\|h(s)\|_{\ell^2}^2\right)|u_h^\varepsilon(s)|^2 ds \\
&+ \frac{\nu\alpha_1 T}{2\alpha_2} + 2\sqrt{\varepsilon}\int_0^t \left(u_h^\varepsilon(s), G(u_h^\varepsilon(s))dW(s)\right).
\end{aligned} \tag{5.27}$$

For a fixed constant $L>0$, define a stopping time $\tau_L^\varepsilon = \inf\left\{0\leqslant t\leqslant T : \|u_h^\varepsilon\|_{C([0,t],H)} + \|u_h^\varepsilon\|_{L^2(0,t;V)} \geqslant L\right\}$.

From (5.27), it follows that, for any $t\in[0,T]$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
&\sup_{0\leqslant r\leqslant t\wedge\tau_L^\varepsilon}\left(|u_h^\varepsilon(r)|^2+\nu\int_0^r\|u_h^\varepsilon(s)\|^2ds\right)\\
&\leqslant |u_0|^2+\int_0^t|f(s)|^2ds+\int_0^t\left(1+\frac{\nu\alpha_1}{2\alpha_2}+\frac{4\alpha_2}{\nu}\|h(s)\|_{\ell^2}^2\right)|u_h^\varepsilon(s)|^2ds\\
&\quad+\frac{\nu\alpha_1 T}{2\alpha_2}+2\sqrt{\varepsilon}\sup_{0\leqslant r\leqslant t\wedge\tau_L^\varepsilon}\left|\int_0^r\left(u_h^\varepsilon(s),G(u_h^\varepsilon(s))dW(s)\right)\right|,
\end{aligned}$$

which shows that

$$\begin{aligned}
&\sup_{0\leqslant r\leqslant t\wedge\tau_L^\varepsilon}|u_h^\varepsilon(r)|^2+\nu\int_0^{t\wedge\tau_L^\varepsilon}\|u_h^\varepsilon(s)\|^2ds\\
&\leqslant 2|u_0|^2+2\int_0^t|f(s)|^2ds+2\int_0^t\left(1+\frac{\nu\alpha_1}{2\alpha_2}+\frac{4\alpha_2}{\nu}\|h(s)\|_{\ell^2}^2\right)\sup_{0\leqslant r\leqslant s\wedge\tau_L^\varepsilon}|u_h^\varepsilon(r)|^2ds\\
&\quad+\frac{\nu\alpha_1 T}{\alpha_2}+4\sqrt{\varepsilon}\sup_{0\leqslant r\leqslant t\wedge\tau_L^\varepsilon}\left|\int_0^r\left(u_h^\varepsilon(s),G(u_h^\varepsilon(s))dW(s)\right)\right|,
\end{aligned}\tag{5.28}$$

Similar to (2.18), by (2.7), we have for any $t\in[0,T]$,

$$\begin{aligned}
&4\sqrt{\varepsilon}\mathbb{E}\left[\sup_{0\leqslant r\leqslant t\wedge\tau_L^\varepsilon}\left|\int_0^r\left(u_h^\varepsilon(s),G(u_h^\varepsilon(s))dW(s)\right)\right|\right]\\
&\leqslant\sqrt{\varepsilon}\mathbb{E}\left[\sup_{0\leqslant r\leqslant t\wedge\tau_L^\varepsilon}|u_h^\varepsilon(r)|^2\right]+36\sqrt{\varepsilon}\alpha_1\int_0^t\mathbb{E}\left[\sup_{0\leqslant r\leqslant s\wedge\tau_L^\varepsilon}|u_h^\varepsilon(r)|^2\right]ds\\
&\quad+36\sqrt{\varepsilon}\alpha_2\int_0^{t\wedge\tau_L^\varepsilon}\mathbb{E}\left[\|u_h^\varepsilon(s)\|^2\right]ds+36\sqrt{\varepsilon}\alpha_1 T.
\end{aligned}\tag{5.29}$$

Thus we can apply Lemma 5.3 to (5.28) and (5.29) for

$$X(t)=\sup_{0\leqslant r\leqslant t\wedge\tau_L^\varepsilon}|u_h^\varepsilon(r)|^2,\ Y(t)=\int_0^{t\wedge\tau_L^\varepsilon}\|u_h^\varepsilon(s)\|^2ds,\ I(t)=4\sqrt{\varepsilon}\sup_{0\leqslant r\leqslant t\wedge\tau_L^\varepsilon}\left|\int_0^r\left(u_h^\varepsilon(s),G(u_h^\varepsilon(s))dW(s)\right)\right|,$$

$$\alpha=\nu,\quad \beta=\sqrt{\varepsilon},\quad \gamma=36\sqrt{\varepsilon}\alpha_1,\quad \delta=36\sqrt{\varepsilon}\alpha_2,\quad Z=2|u_0|^2+2\int_0^T|f(s)|^2ds+\frac{\nu\alpha_1 T}{\alpha_2},$$

$$\varphi(s)=2+\frac{\nu\alpha_1}{\alpha_2}+\frac{8\alpha_2}{\nu}\|h(s)\|_{\ell^2}^2,\quad C'=36\sqrt{\varepsilon}\alpha_1 T.$$

Therefore, we have

$$\int_0^T\varphi(s)ds=\int_0^T\left(2+\frac{\nu\alpha_1}{\alpha_2}+\frac{8\alpha_2}{\nu}\|h(s)\|_{\ell^2}^2\right)ds\leqslant 2T+\frac{\nu\alpha_1 T}{\alpha_2}+\frac{8\alpha_2 M}{\nu}:=C,\quad a.s..$$

By choosing ε small enough, we obtain $2\beta e^C=2\sqrt{\varepsilon}e^C\leqslant 1$ and $2\delta e^C=72\sqrt{\varepsilon}e^C\leqslant 1$. Consequently, by Lemma 5.3, there exists $\widetilde{\varepsilon}_0>0$, such that, for any $0\leqslant\varepsilon\leqslant\widetilde{\varepsilon}_0$,

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T\wedge\tau_L^\varepsilon}|u_h^\varepsilon(t)|^2+\int_0^{T\wedge\tau_L^\varepsilon}\|u_h^\varepsilon(t)\|^2dt\right]\leqslant C_1,$$

where C_1 is independent of L.

In addition, since $\lim_{L\to\infty}\tau_L^\varepsilon=T$, by Fatou's lemma, we can obtain (5.23), as desired. □

The following lemma is devoted to the convergence of $\mathcal{G}^\varepsilon$, which is necessary to prove (H2).

Lemma 5.5. *Suppose* (2.7) *and* (2.8) *hold, and* $\{h^\varepsilon\} \subseteq \mathcal{A}_M$ *with* $M > 0$*. Then,*

$$\lim_{\varepsilon\to 0}\left(\mathcal{G}^\varepsilon\left(W + \varepsilon^{-\frac{1}{2}}\int_0^\cdot h^\varepsilon(t)dt\right) - \mathcal{G}^0\left(\int_0^\cdot h^\varepsilon(t)dt\right)\right) = 0 \text{ in probability in } C([0,T],H)\cap L^2(0,T;V).$$

Proof. Let $u^\varepsilon_{h^\varepsilon} = \mathcal{G}^\varepsilon\left(W + \varepsilon^{-\frac{1}{2}}\int_0^\cdot h^\varepsilon(t)dt\right)$. It follows from Lemma 5.4 that $u^\varepsilon_{h^\varepsilon}$ is the solution of

$$du^\varepsilon_{h^\varepsilon}(t) + \nu A u^\varepsilon_{h^\varepsilon}(t)dt + F_N(\|u^\varepsilon_{h^\varepsilon}\|)B(u^\varepsilon_{h^\varepsilon})dt = \left(f(t) + G(u^\varepsilon_{h^\varepsilon}(t))h^\varepsilon(t)\right)dt + \sqrt{\varepsilon}G(u^\varepsilon_{h^\varepsilon}(t))dW, \quad (5.30)$$

with initial condition $u^\varepsilon_{h^\varepsilon}(0) = u_0 \in H$. Let $u_{h^\varepsilon} = \mathcal{G}^0\left(\int_0^\cdot h^\varepsilon(t)dt\right)$, then u_{h^ε} is the unique solution of

$$\frac{d}{dt}u_{h^\varepsilon}(t) + \nu A u_{h^\varepsilon}(t) + F_N(\|u_{h^\varepsilon}(t)\|)B(u_{h^\varepsilon}(t)) = f(t) + G(u_{h^\varepsilon}(t))h^\varepsilon(t), \quad (5.31)$$

with initial value $u_{h^\varepsilon}(0) = u_0 \in H$. Therefore, we only need to show $u^\varepsilon_{h^\varepsilon} - u_{h^\varepsilon}$ converges in probability to zero in $C([0,T],H)\cap L^2(0,T;V)$ as $\varepsilon \to 0$. By (5.30) and (5.31), we have

$$\begin{aligned}
&d\left(u^\varepsilon_{h^\varepsilon}(t) - u_{h^\varepsilon}(t)\right) + \nu A\left(u^\varepsilon_{h^\varepsilon}(t) - u_{h^\varepsilon}(t)\right)dt + \left(F_N(\|u^\varepsilon_{h^\varepsilon}(t)\|)B(u^\varepsilon_{h^\varepsilon}(t)) - F_N(\|u_{h^\varepsilon}(t)\|)B(u_{h^\varepsilon}(t))\right)dt \\
&= \big(G(u^\varepsilon_{h^\varepsilon}(t))h^\varepsilon(t) - G(u_{h^\varepsilon}(t))h^\varepsilon(t)\big)dt + \sqrt{\varepsilon}G(u^\varepsilon_{h^\varepsilon}(t))dW(t),
\end{aligned}$$

with initial condition $u^\varepsilon_{h^\varepsilon}(0) - u_{h^\varepsilon}(0) = 0$. Then, thanks to Itô's formula, we obtain

$$\begin{aligned}
&|u^\varepsilon_{h^\varepsilon}(t) - u_{h^\varepsilon}(t)|^2 + 2\nu\int_0^t \|u^\varepsilon_{h^\varepsilon}(s) - u_{h^\varepsilon}(s)\|^2 ds \\
&= -2\int_0^t \big(F_N(\|u^\varepsilon_{h^\varepsilon}(s)\|)B(u^\varepsilon_{h^\varepsilon}(s)) - F_N(\|u_{h^\varepsilon}(s)\|)B(u_{h^\varepsilon}(s)), u^\varepsilon_{h^\varepsilon}(s) - u_{h^\varepsilon}(s)\big)ds \\
&\quad + 2\int_0^t \big(G(u^\varepsilon_{h^\varepsilon}(s))h^\varepsilon(s) - G(u_{h^\varepsilon}(s))h^\varepsilon(s), u^\varepsilon_{h^\varepsilon}(s) - u_{h^\varepsilon}(s)\big)ds \\
&\quad + 2\sqrt{\varepsilon}\int_0^t \big(u^\varepsilon_{h^\varepsilon}(s) - u_{h^\varepsilon}(s), G(u^\varepsilon_{h^\varepsilon}(s))dW(s)\big) + \varepsilon\int_0^t \|G(u^\varepsilon_{h^\varepsilon}(s))\|^2_{L_2(\ell^2,H)}ds. \quad (5.32)
\end{aligned}$$

For a fixed constant $L > 0$, define a stopping time

$$\tau_\varepsilon = \inf\left\{0 \leqslant t \leqslant T : \|u^\varepsilon_{h^\varepsilon}\|_{C([0,t],H)} + \|u^\varepsilon_{h^\varepsilon}\|_{L^2(0,t;V)} \geqslant L\right\}. \quad (5.33)$$

It follows from (5.32) and (5.33) that, for any $t \in [0,T]$,

$$\begin{aligned}
&\sup_{0\leqslant r\leqslant t}\left(|u^\varepsilon_{h^\varepsilon}(r\wedge\tau_\varepsilon) - u_{h^\varepsilon}(r\wedge\tau_\varepsilon)|^2 + 2\nu\int_0^{r\wedge\tau_\varepsilon}\|u^\varepsilon_{h^\varepsilon}(s) - u_{h^\varepsilon}(s)\|^2 ds\right) \\
&\leqslant 2\int_0^{t\wedge\tau_\varepsilon}\left|\big(F_N(\|u^\varepsilon_{h^\varepsilon}(s)\|)B(u^\varepsilon_{h^\varepsilon}(s)) - F_N(\|u_{h^\varepsilon}(s)\|)B(u_{h^\varepsilon}(s)), u^\varepsilon_{h^\varepsilon}(s) - u_{h^\varepsilon}(s)\big)\right|ds \\
&\quad + 2\int_0^{t\wedge\tau_\varepsilon}\left|\big(G(u^\varepsilon_{h^\varepsilon}(s))h^\varepsilon(s) - G(u_{h^\varepsilon}(s))h^\varepsilon(s), u^\varepsilon_{h^\varepsilon}(s) - u_{h^\varepsilon}(s)\big)\right|ds \\
&\quad + 2\sqrt{\varepsilon}\sup_{0\leqslant r\leqslant t}\left|\int_0^{r\wedge\tau_\varepsilon}\big(u^\varepsilon_{h^\varepsilon}(s) - u_{h^\varepsilon}(s), G(u^\varepsilon_{h^\varepsilon}(s))dW(s)\big)\right| + \varepsilon\int_0^{t\wedge\tau_\varepsilon}\|G(u^\varepsilon_{h^\varepsilon}(s))\|^2_{L_2(\ell^2,H)}ds. \quad (5.34)
\end{aligned}$$

In addition, by Lemma 5.1, there exists $C_2 = C_2(M,T) > 0$ such that, $\mathbb{P}$-almost surely,

$$\sup_{\varepsilon\in[0,\varepsilon_0]}\left(\|u_{h^\varepsilon}\|_{C([0,T],H)} + \int_0^T \|u_{h^\varepsilon}(s)\|^2 ds\right) \leqslant C_2. \quad (5.35)$$

For the first term on the right-hand side of (5.34), by (2.5), we can deduce that

$$\begin{aligned}
&2\int_0^{t\wedge\tau_\varepsilon}\left|\left(F_N(\|u^\varepsilon_{h^\varepsilon}(s)\|)B(u^\varepsilon_{h^\varepsilon}(s))-F_N(\|u_{h^\varepsilon}(s)\|)B(u_{h^\varepsilon}(s)),u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s)\right)\right|ds\\
&\leqslant\frac{\nu}{2}\int_0^{t\wedge\tau_\varepsilon}\|u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s)\|^2ds+C_\nu\int_0^{t\wedge\tau_\varepsilon}|u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s)|^2ds. \qquad (5.36)
\end{aligned}$$

As for the second term on the right-hand side of (5.34), by (2.8) and Young's inequality, we derive

$$\begin{aligned}
&2\int_0^{t\wedge\tau_\varepsilon}\left|\left(G(u^\varepsilon_{h^\varepsilon}(s))h^\varepsilon(s)-G(u_{h^\varepsilon}(s))h^\varepsilon(s),u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s)\right)\right|ds \qquad (5.37)\\
&\leqslant\int_0^{t\wedge\tau_\varepsilon}\left(2\sqrt{\beta_1}\|h^\varepsilon(s)\|_{\ell^2}+\frac{2\beta_2}{\nu}\|h^\varepsilon(s)\|^2_{\ell^2}\right)|u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s)|^2ds+\frac{\nu}{2}\int_0^{t\wedge\tau_\varepsilon}\|u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s)\|^2ds.
\end{aligned}$$

From (5.34), (5.36) and (5.37), it follows that, for any $t\in[0,T]$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
&\sup_{0\leqslant r\leqslant t}|u^\varepsilon_{h^\varepsilon}(r\wedge\tau_\varepsilon)-u_{h^\varepsilon}(r\wedge\tau_\varepsilon)|^2+\nu\int_0^{t\wedge\tau_\varepsilon}\|u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s)\|^2\,ds\\
&\leqslant\int_0^t\left(C_\nu+2\sqrt{\beta_1}\|h^\varepsilon(s)\|_{\ell^2}+\frac{2\beta_2}{\nu}\|h^\varepsilon(s)\|^2_{\ell^2}\right)\sup_{0\leqslant r\leqslant s}|u^\varepsilon_{h^\varepsilon}(r\wedge\tau_\varepsilon)-u_{h^\varepsilon}(r\wedge\tau_\varepsilon)|^2ds\\
&\quad+4\sqrt{\varepsilon}\sup_{0\leqslant r\leqslant t}\left|\int_0^{r\wedge\tau_\varepsilon}\left(u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s),G(u^\varepsilon_{h^\varepsilon}(s))dW(s)\right)\right|+2\varepsilon\int_0^{t\wedge\tau_\varepsilon}\|G(u^\varepsilon_{h^\varepsilon}(s))\|^2_{L_2(\ell^2,H)}ds, \qquad (5.38)
\end{aligned}$$

which, together with Gronwall's inequality, shows that for any $t\in[0,T]$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
&\sup_{0\leqslant r\leqslant t}|u^\varepsilon_{h^\varepsilon}(r\wedge\tau_\varepsilon)-u_{h^\varepsilon}(r\wedge\tau_\varepsilon)|^2\\
&\leqslant 4\sqrt{\varepsilon}e^{C_\nu T+2\sqrt{\beta_1 MT}+\frac{2\beta_2M}{\nu}}\sup_{0\leqslant r\leqslant T}\left|\int_0^{r\wedge\tau_\varepsilon}\left(u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s),G(u^\varepsilon_{h^\varepsilon}(s))dW(s)\right)\right|\\
&\quad+2\varepsilon e^{C_\nu T+2\sqrt{\beta_1 MT}+\frac{2\beta_2M}{\nu}}\int_0^{T\wedge\tau_\varepsilon}\|G(u^\varepsilon_{h^\varepsilon}(s))\|^2_{L_2(\ell^2,H)}ds. \qquad (5.39)
\end{aligned}$$

For the last term of (5.39), by (2.7) and (5.33), we have

$$\begin{aligned}
&\lim_{\varepsilon\to0}2\varepsilon e^{C_\nu T+2\sqrt{\beta_1 MT}+\frac{2\beta_2M}{\nu}}\int_0^{T\wedge\tau_\varepsilon}\|G(u^\varepsilon_{h^\varepsilon}(s))\|^2_{L_2(\ell^2,H)}ds\\
&\leqslant\lim_{\varepsilon\to0}2\varepsilon e^{C_\nu T+2\sqrt{\beta_1 MT}+\frac{2\beta_2M}{\nu}}\int_0^{T\wedge\tau_\varepsilon}\left[\alpha_1\left(1+|u^\varepsilon_{h^\varepsilon}(s)|^2\right)+\alpha_2\|u^\varepsilon_{h^\varepsilon}(s)\|^2\right]ds\\
&=0,\quad \mathbb{P}\text{-almost surely}. \qquad (5.40)
\end{aligned}$$

For the first term on the right-hand side of (5.39), by (5.33), (5.35) and (5.40), we deduce

$$\begin{aligned}
&2\lim_{\varepsilon\to0}\sqrt{\varepsilon}\mathbb{E}\left[\sup_{0\leqslant r\leqslant T}\left|\int_0^{r\wedge\tau_\varepsilon}\left(u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s),G(u^\varepsilon_{h^\varepsilon}(s))dW(s)\right)\right|^2\right]\\
&\leqslant 16\lim_{\varepsilon\to0}\varepsilon\mathbb{E}\left[\int_0^{T\wedge\tau_\varepsilon}\|u^\varepsilon_{h^\varepsilon}(s)-u_{h^\varepsilon}(s)\|^2\|G(u^\varepsilon_{h^\varepsilon}(s))\|^2_{L_2(H,\ell^2)}ds\right]\\
&\leqslant 16(L+C_2)^2\lim_{\varepsilon\to0}\varepsilon\mathbb{E}\left[\int_0^T\|G(u^\varepsilon_{h^\varepsilon}(s))\|^2_{L_2(\ell^2,H)}ds\right]=0. \qquad (5.41)
\end{aligned}$$

It then follows from (5.39)-(5.41) that

$$\lim_{\varepsilon\to0}\sup_{0\leqslant r\leqslant T}|u^\varepsilon_{h^\varepsilon}(r\wedge\tau_\varepsilon)-u_{h^\varepsilon}(r\wedge\tau_\varepsilon)|^2=0\ \text{ in probability}. \qquad (5.42)$$

Similarly, by (5.38) and (5.40)-(5.42), we can also deduce that

$$\lim_{\varepsilon\to 0}\int_0^T \|u^\varepsilon_{h^\varepsilon}(r\wedge\tau_\varepsilon)-u_{h^\varepsilon}(r\wedge\tau_\varepsilon)\|^2\,dr=0 \ \text{ in probability.} \tag{5.43}$$

By (5.23) and (5.33), we have for any $0<\varepsilon\leqslant\widetilde{\varepsilon}_0$,

$$\begin{aligned}\mathbb{P}(\tau_\varepsilon<T)=&\mathbb{P}\left(\|u^\varepsilon_{h^\varepsilon}\|_{C([0,T],H)}+\|u^\varepsilon_{h^\varepsilon}\|_{L^2(0,T;V)}>L\right)\\ \leqslant&\mathbb{P}\left(\|u^\varepsilon_{h^\varepsilon}\|_{C([0,T],H)}>\frac{L}{2}\right)+\mathbb{P}\left(\|u^\varepsilon_{h^\varepsilon}\|_{L^2(0,T;V)}>\frac{L}{2}\right)\\ \leqslant&\frac{1}{4L^2}\mathbb{E}\left[\sup_{0\leqslant r\leqslant T}|u^\varepsilon_{h^\varepsilon}(r)|^2\right]+\frac{1}{4L^2}\mathbb{E}\left[\int_0^T\|u^\varepsilon_{h^\varepsilon}(s)\|^2ds\right]\leqslant\frac{C_1}{2L^2},\end{aligned} \tag{5.44}$$

where $\widetilde{\varepsilon}_0, C_1>0$ are given in Lemma 5.4. From (5.44), it follows that for any $\epsilon>0$,

$$\begin{aligned}&\mathbb{P}\left(\|u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon}\|_{C([0,T],H)}+\|u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon}\|_{L^2(0,T;V)}>\epsilon\right)\\ &\leqslant\mathbb{P}\left(\|u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon}\|_{C([0,T],H)}+\|u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon}\|_{L^2(0,T;V)}>\epsilon,\tau_\varepsilon=T\right)\\ &\quad+\mathbb{P}\left(\|u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon}\|_{C([0,T],H)}+\|u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon}\|_{L^2(0,T;V)}>\epsilon,\tau_\varepsilon<T\right)\\ &\leqslant\mathbb{P}\left(\|u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon}\|_{C([0,T],H)}+\|u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon}\|_{L^2(0,T;V)}>\epsilon,\tau_\varepsilon=T\right)+\frac{C_1}{2L^2}.\end{aligned} \tag{5.45}$$

First taking the limit as $\varepsilon\to 0$, and then as $M\to\infty$, it follows from (5.42), (5.43) and (5.45) that

$$\lim_{\varepsilon\to 0}(u^\varepsilon_{h^\varepsilon}-u_{h^\varepsilon})=0 \ \text{ in } C([0,T],H)\cap L^2(0,T;V) \text{ in probability.} \tag{5.46}$$

The proof is complete. □

The main result of this paper is given below.

Theorem 5.2. *Suppose* (2.7) *and* (2.8) *hold, and u^ε is the solution of* (2.10)*. Then the family $\{u^\varepsilon\}$, as $\varepsilon\to 0$, satisfies the large deviation principle in $C([0,T],H)\cap L^2(0,T;V)$ with the good rate function given by*

$$I(\varphi)=\inf\left\{\frac{1}{2}\int_0^T\|h(t)\|^2_{\ell^2}dt : h\in L^2(0,T;\ell^2),\ u_h=\varphi\right\}, \tag{5.47}$$

where $\varphi\in C([0,T],H)\cap L^2(0,T;V)$, u_h is the solution of (5.2)-(5.3)*, and the infimum of the empty set is taken to be ∞.*

Proof. It follows from Lemma 5.2 that (H1) holds. It remains to prove that $\mathcal{G}^\varepsilon$ and $\mathcal{G}^0$ satisfy (H2). Let $\{h^\varepsilon\}$ be a sequence in $\mathcal{A}_M$ $(M>0)$, which converges in distribution to h as S_M-valued random variables. In addition, by Skorokhod's theorem, there exist a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ and S_M-valued random variables $\widetilde{h^\varepsilon}$ and $\widetilde{h}$ on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ such that $\widetilde{h^\varepsilon}\to\widetilde{h}$ in S_M, almost surely, which is equipped with the weak topology, where $\widetilde{h^\varepsilon}$ and $\widetilde{h}$ have the same distribution laws as h^ε and h respectively. Let $u_h=\mathcal{G}^0\left(\int_0^\cdot h(t)dt\right)$, then u_h is the solution to (5.2)-(5.3). From Lemma 5.2, it follows that $u_{\widetilde{h^\varepsilon}}$ converges to $u_{\widetilde{h}}$ in distribution in $C([0,T],H)\cap L^2(0,T;V)$. Therefore, we have

$$u_{h^\varepsilon}\to u_h,\ \text{ in } C([0,T],H)\cap L^2(0,T;V) \text{ in distribution,}$$

which, together with (5.46), implies that

$$u^\varepsilon_{h^\varepsilon}\to u_h,\ \text{ in } C([0,T],H)\cap L^2(0,T;V) \text{ in distribution,}$$

then (H2) holds. Therefore, by Theorem 5.1, the result is completely proved. □

Acknowledgements

The work is partially supported by the NNSF of China (11471190, 11971260), the SDNSF (ZR2014AM002), the Spanish Ministerio de Ciencia e Innovación under project PID2021-122991NB-C21, and Junta de Andalucía (Spain) under project P18-FR-4509. The authors are grateful to the editor and referees for their very valuable suggestions and comments.

Declarations

Competing interests The authors have not disclosed any competing interests.

References

[1] C.T. Anh, N.V. Thanh and P.T. Tuyet, Asymptotic behaviour of solutions to stochastic three-dimensional globally modified Navier-Stokes equations, *Stochastics*, **95** (2023) 997-1021.

[2] J. Bao and C. Yuan, Large deviations for neutral functional SDEs with jumps, *Stochastics*, **87** (2015) 48-70.

[3] Z. Brzeźniak and G. Dhariwal, Stochastic tamed Navier-Stokes equations on $\mathbb{R}^3$: The existence and the uniqueness of solutions and the existence of an invariant measure, *J. Math. Fluid Mech.*, **22** (2020) 1-54.

[4] Z. Brzeźniak, B. Goldys and T. Jegaraj, Large deviations and transitions between equilibria for stochastic Landau-Lifshitz-Gilbert equation, *Arch. Ration. Mech. Anal.*, **226** (2017) 497-558.

[5] Z. Brzeźniak, E. Motyl and M. Ondrejat, Invariant measure for the stochastic Navier-Stokes equations in unbounded 2D domains, *Ann. Probab.*, **45** (2017) 3145-3201.

[6] Z. Brzeźniak, M. Ondreját and J. Seidler, Invariant measures for stochastic nonlinear beam and wave equations, *J. Differ. Equ.*, **260** (2016) 4157-4179.

[7] Z. Brzeźniak, X. Peng and J. Zhai, Well-posedness and large deviations for 2D Stochastic Navier-Stokes equations with jumps, *J. Eur. Math. Soc.*, **25** (2023) 3093-3176.

[8] A. Budhiraja and P. Dupuis, A variational representation for positive functionals of infinite dimensional Brownian motion, *Probab. Math. Statist.*, **20** (2000) 39-61.

[9] T. Caraballo, B. Guo, N.H. Tuan and R. Wang, Asymptotically autonomous robustness of random attractors for a class of weakly dissipative stochastic wave equations on unbounded domains, *Proc. Roy. Soc. Edinburgh Sect. A*, **151** (2021) 1700-1730.

[10] T. Caraballo, J. Real and P.E. Kloeden, Unique strong solutions and V-attractors of a three dimensional system of globally modified Navier-Stokes equations, *Adv. Nonlinear Stud.*, **6** (2006) 411-436.

[11] S. Cerrai and M. Röckner, Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term, *Ann. Probab.*, **32** (2004) 1100-1139.

[12] L. Chen, Z. Dong, J. Jiang and J. Zhai, On limiting behavior of stationary measures for stochastic evolution systems with small noise intensity, *Sci. China Math.*, **63** (2020) 1463-1504.

[13] Z. Chen, X. Li and B. Wang, Invariant measures of stochastic delay lattice systems, *Discret. Contin. Dyn. Syst. Ser. B*, **26** (2021) 3235-3269.

[14] Z. Chen and B. Wang, Invariant measures of fractional stochastic delay reaction-diffusion equations on unbounded domains, *Nonlinearity*, **34** (2021) 3969-4016.

[15] Z. Chen and B. Wang, Existence, exponential mixing and convergence of periodic measures of fractional stochastic delay reaction-diffusion equations on $\mathbb{R}^n$, *J. Differ. Equ.*, **336** (2022) 505-564.

[16] Z. Chen and B. Wang, Limit measures and ergodicity of fractional stochastic reaction-diffusion equations on unbounded domains, *Stoch. Dyn.*, **22** (2022) 2140012.

[17] Z. Chen and B. Wang, Limit measures of stochastic Schrödinger lattice systems, *Proc. Amer. Math. Soc.*, **150** (2022) 1669-1684.

[18] Z. Chen, D. Yang and S. Zhong, Weak mean attractor and periodic measure for stochastic lattice systems driven by Lévy noises, *Stoch. Anal. Appl.*, **41** (2023) 509-544.

[19] I. Chueshov and A. Millet, Stochastic 2D hydrodynamical type systems: well posedness and large deviations, *Appl. Math. Optim.*, **61** (2010) 379-420.

[20] G. Da Prato and A. Debussche, 2D stochastic Navier-Stokes equations with a time-periodic forcing term, *J. Dyn. Differ. Equ.*, **20** (2008) 301-335.

[21] G. Da Prato, F. Flandoli, E. Priola and M. Röckner, Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift, *Ann. Probab.*, **41** (2013) 3306-3344.

[22] G. Da Prato and M. Röckner, A note on evolution systems of measures for time-dependent stochastic differential equations, *Progr. Probab.*, **59** (2009) 115-122.

[23] G. Deugoue and T.T. Medjo, The stochastic 3D globally modified Navier-Stokes equations: Existence, uniqueness and asymptotic behavior, *Commun. Pure Appl. Anal.*, **17** (2018) 2593-2621.

[24] Z. Dong and R. Zhang, 3D tamed Navier-Stokes equations driven by multiplicative Lévy noise: Existence, uniqueness and large deviations, *J. Math. Anal. Appl.*, **492** (2020) 124404.

[25] J. Duan and A. Millet, Large deviations for the Boussinesq equations under random influences, *Stoch. Process. Appl.*, **119** (2009) 2052-2081.

[26] B. Gess, W. Liu and A. Schenke, Random attractors for locally monotone stochastic partial differential equations, *J. Differ. Equ.*, **269** (2020) 3414-3455.

[27] W. Hong, S. Li and W. Liu, Freidlin-Wentzell type large deviation principle for multiscale locally monotone SPDEs, *SIAM J. Math. Anal.*, **53** (2021) 6517-6561.

[28] W. Hu, M. Salins and K. Spiliopoulos, Large deviations and averaging for systems of slow-fast stochastic reaction-diffusion equations, *Stoch. Partial Differ. Equ. Anal. Comput.*, **7** (2019) 808-874.

[29] J. Kim, Periodic and invariant measures for stochastic wave equations, *Electron. J. Differential Equations*, **2004** (2004) 1-30.

[30] J. Kim, On the stochastic Benjamin-Ono equation, *J. Differ. Equ.*, **228** (2006) 737-768.

[31] P.E. Kloeden, J.A. Langa and J. Real, Pullback V-attractors of the 3-dimensional globally modified Navier-Stokes equations, *Commun. Pure Appl. Anal.,* **6** (2007) 937-955.

[32] D. Li, B. Wang and X. Wang, Periodic measures of stochastic delay lattice systems, *J. Differ. Equ.*, **272** (2021) 74-104.

[33] D. Li, B. Wang and X. Wang, Limiting behavior of invariant measures of stochastic delay lattice systems, *J. Dyn. Differ. Equ.*, **34** (2022) 1453-1487.

[34] W. Liu, Large deviations for stochastic evolution equations with small multiplicative noise, *Appl. Math. Optim.*, **61** (2010) 27-56.

[35] R. Liu and K. Lu, Statistical properties of 2D stochastic Navier-Stokes equations with time-periodic forcing and degenerate stochastic forcing, arXiv: 2105.00598, 2021.

[36] P. Marín-Rubio, A.M. Márquez-Durán and J. Real, Pullback attractors for globally modified Navier-Stokes equations with infinite delays, *Discrete Contin. Dyn. Syst.*, **31** (2011) 779-796.

[37] O. Misiats, O. Stanzhytskyi and N.K. Yip, Existence and uniqueness of invariant measures for stochastic reaction-diffusion equations in unbounded domains, *J. Theor. Probab.*, **29** (2016) 996-1026.

[38] M.T. Mohan, Well posedness, large deviations and ergodicity of the stochastic 2D Oldroyd model of order one, *Stoch. Process. Appl.*, **130** (2020) 4513-4562.

[39] M. Röckner and X. Zhang, Tamed 3D Navier-Stokes equation: Existence, uniqueness and regularity, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **12** (2009) 525-549.

[40] M. Röckner and X. Zhang, Stochastic tamed 3D Navier-Stokes equations: Existence, uniqueness and ergodicity, *Probab. Theory Related Fields*, **145** (2009) 211-267.

[41] M. Röckner, T. Zhang and X. Zhang, Large deviations for stochastic tamed 3D Navier-Stokes equations, *Appl. Math. Optim.*, **61** (2010) 267-285.

[42] G.R. Sell and C. You, *Dynamics of Evolutionary Equations*, Applied Mathematical Sciences, 143. Springer-Verlag, New York, 2002.

[43] S.S. Sritharan and P. Sundar, Large deviations for the two-dimensional Navier-Stokes equations with multiplicative noise, *Stoch. Proce. Appl.*, **116** (2006) 1636-1659.

[44] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.

[45] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differ. Equ.*, **253** (2012) 1544-1583.

[46] B. Wang, Dynamics of stochastic reaction-diffusion lattice systems driven by nonlinear noise, *J. Math. Anal. Appl.*, **477** (2019) 104-132.

[47] B. Wang, Weak pullback attractors for mean random dynamical systems in Bochner spaces, *J. Dyn. Differ. Equ.*, **31** (2019) 2177-2204.

[48] B. Wang, Dynamics of fractional stochastic reaction-diffusion equations on unbounded domains driven by nonlinear noise, *J. Differ. Equ.*, **268** (2019) 1-59.

[49] B. Wang, Weak pullback attractors for stochastic Navier-Stokes equations with nonlinear diffusion terms, *Proc. Amer. Math. Soc.*, **147** (2019) 1627-1638.

[50] B. Wang, Large deviation principles of stochastic reaction-diffusion lattice systems, arXiv: 2305.06510, 2023

[51] R. Wang, T. Caraballo and N.H. Tuan, Asymptotic stability of evolution systems of probability measures for nonautonomous stochastic systems: Theoretical results and applications, *Proc. Amer. Math. Soc.*, **151** (2023) 2449-2458.

[52] R. Wang, B. Guo and B. Wang, Well-posedness and dynamics of fractional FitzHugh-Nagumo systems on $\mathbb{R}^n$ driven by nonlinear noise, *Sci. China Math.*, **64** (2021) 2395-2436.

[53] X. Wang, P.E. Kloeden and X. Han, Stochastic dynamics of a neural field lattice model with state dependent nonlinear noise, *Nonlinear Differ. Equ. Appl.*, **28** (2021) 1-31.

[54] R. Wang and B. Wang, Random dynamics of p-Laplacian lattice systems driven by infinite-dimensional nonlinear noise, *Stoch. Process. Appl.*, **130** (2020) 7431-7462.

[55] J. Xu and T. Caraballo, Long time behavior of stochastic nonlocal partial differential equations and Wong-Zakai approximations, *SIAM J. Math. Anal.*, **54** (2022) 2792-2844.

[56] D. Yang, Z. Chen and T. Caraballo, Dynamics of a globally modified Navier-Stokes model with double delay, *Z. Angew. Math. Phys.*, **73** (2022) 1-32.

[57] E. Zeidler, *Nonlinear Functional Analysis and its Applications, II/A,B, Nonlinear Monotone Operators*, Springer, New York, 1990.

[58] C. Zhao, J. Wang and T. Caraballo, Invariant sample measures and random Liouville type theorem for the two-dimensional stochastic Navier-Stokes equations, *J. Differ. Equ.*, **317** (2022) 474-494.