NONLINEAR DYNAMICAL ANALYSIS FOR GLOBALLY MODIFIED INCOMPRESSIBLE NON-NEWTONIAN FLUIDS

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Abstract. We present the global modification of the Ladyzhenskaya equations, for incompressible non-Newtonian fluids. This modification is through a cut-off function that multiplies the convective term of the equation and an additional artificial smoothing dissipation term as part of the viscous term of the equation. The goal of this work is the comparative analysis between the modified system and the non-modified system. Therefore, we show the existence and regularity of weak solutions, the existence of global attractors, the estimation of the fractal dimension of the global attractors, and finally, the relationship of the autonomous dynamics between the modified system and the non-modified system.

1. Introduction

An autonomous dynamical system is a pair $(\mathcal{X}, T(\cdot))$ formed by a metric space $(\mathcal{X}, d_{\mathcal{X}})$, also known as the phase space, and a family of operators parameterized in time $\{T(t) : \mathcal{X} \to \mathcal{X} : t \in \mathbb{R}_+\}$, called a semigroup, which satisfies:

- $T(0) = \mathrm{Id}_{\mathcal{X}}$, where $\mathrm{Id}_{\mathcal{X}}$ is the identity on $\mathcal{X}$;
- $T(t+s) = T(t) \circ T(s)$ for all $t, s \in \mathbb{R}_+$.

In addition, we can include the condition of continuity on the semigroup, which is

- $T(t) : \mathcal{X} \to \mathcal{X}$ is continuous for all $t \in \mathbb{R}_+$.

This last condition can be replaced by a weaker one, in the sense that we guarantee the uniqueness of convergence with respect to the semigroup, i.e. if $\{x_n\}_{n\mathbb{N}}$ is a sequence that converges to x, in $\mathcal{X}$, and if given $t \in \mathbb{R}$ we have that $T(t)x_n \to y_t$, in $\mathcal{X}$, when $n \to \infty$, then $y_t = T(t)x$. If the semigroup satisfies this last condition, we say that $\{T(t) : \mathcal{X} \to \mathcal{X} : t \in \mathbb{R}_+\}$ is a closed semigroup. For a more general theory of semigroups, such as global attractors associated with multi-valued semigroups, cf. [21], and for the non-autonomous case, cf. [4, 13].

One of the objectives of studying the asymptotic behavior of the solutions of dynamical systems, over long periods of time, is to concentrate all the dynamics of the solutions in some bounded sets of the phase space. In the context of fluid mechanics, to contain the velocity field in a bounded set, over long periods of time, allows us to understand the turbulence of the fluid, rather than on the transient behavior of the fluid flow cf. [24, Ch. 10]. Besides, mathematically it allows us to study the regularity of the solutions and the dissipative properties of the system, thus we can even understand the finite-dimensional structure of the set that attracts the solutions, e.g. [6, Sec. 7], or [2, 7, 8, 16, 17, 22] among many others. Therefore, the theory of dynamical systems is focused on finding a compact subset $\mathcal{A}$ from the phase space $\mathcal{X}$, with good properties of attraction and invariance in time with respect to the semigroup, that is,

(i) $\mathcal{A}$ is invariant, i.e. $T(t)\mathcal{A} = \mathcal{A}$, for all $t \in \mathbb{R}_+$,

(ii) $\mathcal{A}$ attracts each bounded subset $D \subset \mathcal{X}$, that is

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{X}}(T(t)D, \mathcal{A}) = 0,$$

where $\mathrm{dist}_{\mathcal{X}}$ is the Hausdorff semidistance on $\mathcal{X}$, e.g. [1, 4, 6, 24, 26].

The determination of this compact set $\mathcal{A}$, which we will call global attractor, is theoretically related to the dissipativity and asymptotic compactness of the dynamical system, that is

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- a dynamical system is dissipative if there exists a bounded subset $\mathcal{B}_0 \subset \mathcal{X}$ ($\mathcal{B}_0$ is also called an absorbing set) such that, for any bounded subset $D \subset \mathcal{X}$ there exists $t_0(D) > 0$ such that
$$T(t)D \subset \mathcal{B}_0, \quad \text{for all } t \geq t_0(D);$$
- a dynamical system is asymptotically compact if given any bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ and any sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $t_n \to \infty$ as $n \to \infty$, then the sequence $\{T(t_n)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathcal{X}$.

Before characterizing the global attractor, associated with a dissipative and asymptotically compact dynamical system, we introduce the so-called ω-limits sets. Given a bounded subset $D \subset \mathcal{X}$, the ω-limit set of D is defined as
$$\omega(D) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} T(s)D}^{\mathcal{X}}.$$

Observe that the ω-limit of a bounded set D, $\omega(D)$, concentrates whole the dynamic of each point of D, in the sense that it consists of all the limit points of the orbits of D, cf. [24, Ch. 10] i.e.
$$\omega(D) = \{y \in \mathcal{X} : \exists\, t_n \to \infty, \{x_n\} \subset D \text{ with } T(t_n)x_n \to y\}.$$

The ω-limit sets have good properties of compactness and invariance, cf. [24, Proposition 10.3], so they are potential candidates for global attractors. On the other hand, we know that given a bounded subset $D \subset \mathcal{X}$ it is attracted by its own ω-limit set whenever the semigroup be asymptotically compact, i.e. $\lim_{t\to+\infty} \operatorname{dist}_{\mathcal{X}}(T(t)D, \omega(D)) = 0$, cf. [4, Corollary 2.11], in general, it does not attract all the bounded sets of $\mathcal{X}$ in the sense of (ii), given above. Therefore, the set indicated to be the global attractor of the dynamical system $(\mathcal{X}, T(\cdot))$ is the ω-limit set of the absorbing set $\mathcal{B}_0$, i.e. $\mathcal{A} = \omega(\mathcal{B}_0)$. This one is summarized in the following theorem:

Theorem 1.1. *(cf.* [4, Corollary 2.21], [6, Theorem 2.5] *or* [24, Theorem 10.5]*) Let $(\mathcal{X}, T(\cdot))$ be a dynamical system that is dissipative and asymptotically compact, and let us denote by $\mathcal{B}_0$ the absorbing set associated to the dissipativity of the dynamical system. Then, there exists a global attractor $\mathcal{A} = \omega(\mathcal{B}_0)$.*

In this paper we study the autonomous dynamics of solutions of a perturbed parabolic system with nonlinear differential operator, which is physically associated with the flow of incompressible non-Newtonian fluids, e.g. [6, 13, 17]. This system can also be found in the literature as Ladyzhenskaya models cf. [9, 10, 11]. This type of study was carried out for the 3D-Navier-Stokes equations, cf. [5, 25], which mathematically represent a nonlinear parabolic system whose differential operator is linear, and physically it is related to the flow of incompressible Newtonian fluids, that is, the associated shear tensor, indicated by $\mathbb{S}$, is linear with respect to the symmetric gradient of the velocity field (Newton's law, e.g. [19, Ch. I]) i.e.
$$\mathbb{S}(Du) = 2\nu_0 Du,$$
where $Du = \frac{1}{2}(\nabla u + \nabla u^t)$ and the constant $\nu_0 > 0$ is called the viscosity of the fluid. When a fluid does not satisfy Newton's law, we say that the fluid is a non-Newtonian fluid. In this case, we can think on the shear tensor, $\mathbb{S} : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$, as a non-linear function with respect to the symmetric gradient of the velocity field of the fluid, for example, for some $p > 1$ we have the following shear tensors

(a) $\mathbb{S}^1(D) = 2\nu_0 |D|^p D$, $\quad (b)$ $\mathbb{S}^2(D) = 2\nu_0(1 + |D|^p)D$,
(c) $\mathbb{S}^3(D) = 2\nu_0(1 + |D|^2)^{p/2} D$, $\quad (d)$ $\mathbb{S}^{3+i}(D) = 2\nu_\infty D + \mathbb{S}^i(D)$, $i = 1, 2, 3$,

for all $D \in \mathbb{R}^{n^2}_{sym}$. Note that, fixed $p > 1$, these tensors satisfy
$$\begin{cases} \mathbb{S}(0) = 0, \\ (\mathbb{S}(D_1) - \mathbb{S}(D_2)) : (D_1 - D_2) \geq \nu_1 \big[1 + \mu(|D_1| + |D_2|)\big]^{p-2} |D_1 - D_2|^2, \\ |\mathbb{S}(D_1) - \mathbb{S}(D_2)| \leq c_1\nu_1 \big[1 + \mu(|D_1| + |D_2|)\big]^{p-2} |D_1 - D_2|, \end{cases} \tag{1.1}$$

where the positive constants ν_1 and ν_2 are the so-called generalized viscosities, and $\mu = (\nu_2/\nu_1)^{1/(p-2)}$ (with the convention that $\mu = 0$ for $p = 2$). Then, from (1.1) follows the p-coercivity and the $(p-1)$-growth condition for the shear tensor $\mathbb{S}$, i.e.

$$\mathbb{S}(D) : D \geq c_2(\nu_1|D|^2 + \nu_2|D|^p) \quad \text{and} \quad \begin{cases} |\mathbb{S}(D)| \leq c_3(1 + \mu|D|)^{p-1} \text{ for } p > 1 \\ \\ |\mathbb{S}(D)| \leq c_3(\nu_1|D| + \nu_2|D|^{p-1}) \text{ for } p \geq 2. \end{cases} \tag{1.2}$$

The pioneer regarding mathematical models for incompressible non-Newtonian fluids was Olga Ladyzhenskaya, who between 1969 and 1970 proposed three models cf. [9, 10, 11] or [12, sec. 5 Ch 2], also called variants of the Navier-Stokes equations, in general, these models are written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} \mathbb{S}(Du) + \operatorname{div}(u \otimes u) + \nabla P = f \quad \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u = 0 \quad \text{in } \Omega \times (0, \infty), \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$) is a bounded domain with smooth boundary, f is the external force, $u \otimes u = (u_i u_j)_{i,j=1}^n$ is the convective term, and P is the pressure. One of the models proposed by Ladyzhenskaya is for the shear tensor given by $\mathbb{S}^4$ in (d) cf. [11, 12, 19]. Another proposed model is when $\operatorname{div}\mathbb{S}(Du)$ is replaced by $(\nu + \nu_0\|\nabla u\|_{L^2}^2)\Delta u$, which was treated by Yang *et al.* in [27], proving the existence of attractors, in the pullback sense, with finite fractal dimension.

Consider the following Cauchy-Dirichlet conditions

$$\begin{cases} u(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.4}$$

The system (1.3) together with the conditions given in (1.4), will be indicated by **(LM)** *Ladyzhenskaya model for incompressible non-Newtonian fluids.* Regarding the existence of solutions of **(LM)**, in [19] is shown the existence of measure-valued solutions for $p > 2n/(n+2)$, weak solutions for $p \geq 1 + 2n/(n+2)$, and strong solutions for $p \geq (2+n)/2$. In [3] is explored the time regularity of the weak solutions of **(LM)**, when the external force is in some Nikolskii space, showing the uniqueness of weak solution for $p > 11/5$ and dimension $n = 3$. In [2] is shown that the global attractors associated to **(LM)** have finite fractal dimension, in the space of the square-integrable functions with divergence-free, for $p \geq 12/5$ with $n = 3$, and for $p \geq 2$ with $n = 2$, also see [6, 16]. On the other hand, the study of the asymptotic behavior for the non-autonomous case is carried out in [13], where is proved the existence and regularity of families of pullback attractors associated with the weak solutions of **(LM)** for $p \geq 12/5$ in dimension $n = 3$ and for $p > 2$ in dimension $n = 2$. More information on these types of results can be seen in [18, 22].

Now, by modifying the convective term, $\operatorname{div}(u \otimes u)$, of the system (1.3), we should expect that the solutions of this new system have higher regularity, both in time and space. This modification cannot be arbitrary, since the objective is the comparative analysis between the original system and the modified system. This idea is based on [5], where the convective term, $\operatorname{div}(u \otimes u)$, of the Navier-Stokes system, is modified by multiplying a cut-off function, and in this way controlling the polynomial growth of the solutions in spaces with higher regularity. This cut-off function, denoted by $F_N(\cdot)$, is defined as: given $N > 0$, $F_N : \mathbb{R}_+ \to (0, 1]$ is such that

$$F_N(s) = \min\left\{1, \frac{N}{s}\right\} \quad \text{for all} \quad s \in \mathbb{R}_+.$$

Another term that will be modified in the system (1.3) is the term related to the viscosity. This modification will be made from a small artificial viscosity that is associated with a sixth-order linear differential operator (cubic-Laplacian, Δ^3, cf. [22]). This will allow us to have information about the solutions when the power p, which accompanies the shear tensor $\mathbb{S}$, is varying in the semi-open interval $[2, 12/5)$ when the dimension is $n = 3$. Then, we call the modified

Ladyzhenskaya model, indicated by **(MLM)**, to the system given by

$$\begin{cases} \dfrac{\partial u}{\partial t} - N^{-1}\Delta^3 u - \operatorname{div} \mathbb{S}(Du) + F_N(\|u\|_{1,2})\operatorname{div}(u\otimes u) + \nabla P = f \quad \text{in } \Omega\times(0,\infty), \\ \operatorname{div} u = 0 \quad \text{in } \Omega\times(0,\infty), \\ u = \frac{\partial u}{\partial n} = \Delta u = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

where $\|u\|_{1,2} = |\nabla u|_2$ is the norm of u in the space $W_0^{1,2}(\Omega)^n$, and the shear tensor $\mathbb{S}$ satisfies (1.1) and (1.2).

As we have mentioned previously, for the case of the modified three-dimensional Navier-Stokes equations, we have the results given in [5], where the existence and uniqueness of a strong solution and the existence of global attractors are proved. In this case, the modification allows the use of the Stokes operator, applied in the solution, as a test function in the weak formulation and, in this way, higher regularity of the solutions is obtained in time and space. Regarding the uniqueness of the solutions, it can also be consulted [25].

The goal of this paper is to investigate the existence and regularity of the global attractors with finite fractal dimension, associated with the solutions of the modified system **(MLM)**, and show that, for sufficiently large powers p, associated with shear tensor $\mathbb{S}$, it is possible to obtain information of the global attractor associated with the non-modified system **(LM)**. The structure of the paper is as follows. Section 2 is devoted to briefly recalling the abstract functional setting of the problem **(LM)**, focusing on the existence, regularity, and uniqueness of the weak and strong solutions. In Section 3 we show the existence, and uniqueness of weak solutions of the problem **(MLM)**. For this purpose, we use some techniques established in [3, 5, 22]. Regarding uniqueness, this is a consequence of the continuous dependence on initial data and the parameter $N > 0$, that accompanies the cut-off function and the artificial smoothing dissipation. In Section 4, we show the regularity of the weak solutions of the problem **(MLM)**, by assuming the shear tensor has a potential. In Section 5, we explore the convergence of the weak solution of the problem **(MLM)**, as a sequence of the parameter $N > 0$. Moreover, the Galerkin sequence associated to the system **(MLM)**, converges to the Galerkin sequence of the system **(LM)**, when $N \to +\infty$, for $p \geq 11/5$ in dimension $n = 3$, and for $p \geq 2$ in dimension $n = 2$. Furthermore, we build a weak solution of problem **(LM)**, from the weak solutions of problem **(MLM)**. Section 6 is devoted to discussing the asymptotic behavior of weak solutions of **(MLM)** in L^2-norm, $W^{1,p}$-norm and $W^{3,2}$-norm, with $p \geq 2$. We treat the autonomous case, showing the existence and regularity of the global attractors. Finally, in Section 7 we investigate the fractal dimension of the global attractors, given in the previous section.

2. Important results

In this section, we recall some results on the existence, uniqueness, and regularity of weak solutions of **(LM)**. We will also state some properties of the cut-off function $F_N(\cdot)$.

Definition 2.1. *Throughout the manuscript we consider $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, an open bounded domain with regular boundary $\partial\Omega$. Let $p \in [1, +\infty)$. Thus, let us denote by*

$$\begin{cases} \mathcal{V} := \{\varphi \in C_c^\infty(\Omega)^n : \operatorname{div}\varphi = 0\}; \\ H = \text{ the closure of } \mathcal{V} \text{ in the } L^2(\Omega)^n - \text{ norm}; \\ V_p = \text{ the closure of } \mathcal{V} \text{ in the } W^{1,p}(\Omega)^n - \text{ norm}; \\ V^3 = \text{ the closure of } \mathcal{V} \text{ in the } W^{3,2}(\Omega)^n - \text{ norm.} \end{cases}$$

The spaces H, V_p, and V^3 are considered with topology of their corresponding closure.

It follows from Definition 2.1 and the Sobolev immersions that:

$$V^3 \hookrightarrow V_p \hookrightarrow\hookrightarrow H \equiv H^* \hookrightarrow V_p^* \hookrightarrow (V^3)^*.$$

For each space, their corresponding norm notations are the following: in H we will denote by $(\cdot,\cdot)$ the usual scalar product in $L^2(\Omega)^n$ and its norm by $|\cdot|_2$. The norm on V^3, denoted by

$\|\cdot\|_{3,2}$, will be $W^{3,2}(\Omega)^n$-norm, which comes from the scalar product $((\cdot,\cdot))$. The norm on V_p, denoted by $\|\cdot\|_{1,p}$, will be the L^p-norm of the gradient of an element (Poincaré inequality), i.e. $\|u\|_{1,p} = \|\nabla u\|_p$ for all $u \in V_p$. Finally, V_p^* denotes the topological dual of V_p, $\langle \cdot,\cdot \rangle$ the action among these spaces, and $\|\cdot\|_*$ the norm in V_p^*.

Theorem 2.2. *(Existence; cf.* [12, Théorème 5.1], [17, Theorem 2.14]*, or* [6, 9, 10, 11, 19]*). (i) Suppose $p \geq 1 + 2n/(n+2)$, $f \in L^{p'}_{loc}(\mathbb{R}; V_p^*)$ and $u_0 \in H$. Then, there exists at least one weak solution to problem* **(LM)***. (ii) If $p \geq (n+2)/2$, then the weak solution to* **(LM)** *is unique.*

Proposition 2.3. *(cf.* [6, Theorem 7.32]*) Consider $T > 0$, $u_0 \in H$ and $f \in L^2_{loc}(\mathbb{R}; L^2(\Omega)^n)$. Assume that $p > 2$ if $n = 2$ and $p > 12/5$ if $n = 3$. Then, any weak solution to problem* **(LM)** *associated to the initial condition u_0 satisfies*

$$u \in L^\infty(\varepsilon, T; V_p) \;\; \text{and} \;\; \frac{\partial u}{\partial t} \in L^2(\varepsilon, T; H)$$

for all $\varepsilon > 0$ such that $\varepsilon < T$. If $u_0 \in V_p$, then we can take $\varepsilon = 0$.

Recall the Korn inequality (cf. [19, Theorem 1.10, p. 196]), which relates the norms of the gradient and of the symmetrized gradient. Namely, for any $u \in W_0^{1,r}(\Omega)^n$, $1 < r < \infty$, there exists a constant $c(r) > 0$ such that

$$\|\nabla u\|_r \leq c(r)\|Du\|_r.$$

For short we denote $c_0 = c(2)$ and $\tilde{c}_0 = c(p)$. We also recall the Poincaré inequality

$$\lambda_1 |v|_2^2 \leq |\nabla v|_2^2 \quad \forall v \in V_2,$$

where λ_1 is the first eigenvalue of the Stokes operator with homogeneous Dirichlet boundary conditions.

Lemma 2.4. *(cf.* [25, Lemma 2.1] *or* [5, Lemma 4]*) Let $N, M > 0$. Then, the cut-off functions associated to N, M, respectively, satisfy that*

$$(a)\; F_N(x)x \leq N; \quad (b)\; |F_N(x) - F_N(y)| \leq N\frac{|x-y|}{xy}; \quad (c)\; |F_N(x) - F_N(y)| \leq \frac{|x-y|}{y};$$
$$(d)\; |F_N(x) - F_N(y)| \leq \frac{1}{N}F_N(x)F_N(y)|x-y|; \;\; (d)\; |F_M(x) - F_N(y)| \leq \frac{|M-N|}{y} + \frac{|x-y|}{y};$$
$$(e)\; |F_M(x) - F_N(y)| \leq \frac{|M-N|}{y} + F_M(x)F_N(y)\frac{|x-y|}{M};$$

for all $x, y \in \mathbb{R}_+$.

3. Existence of Weak Solutions for the Modified Ladyzhenskaya Model (MLM)

In this section, we study the existence, uniqueness, and regularity of weak solutions of the modified Ladyzhenskaya model **(MLM)**. Regarding the existence of a weak solution, the Galerkin method is used. Moreover, we take the ideas of [22, Theorem 10], to estimate the derivative with respect to the time in the norm $L^2(\Omega)^n$, and the ideas of [5] to estimate the convective term $F_N(\|u\|_{1,2})\mathrm{div}(u \otimes u)$.

We know that the convective term of (1.3) is associated with the trilinear function $b(u,v,w)$ defined as

$$b(u,v,w) = \sum_{i,j=1}^n \int_\Omega u_j(x)\frac{\partial v_i}{\partial x_j}(x)w_i(x)dx.$$

For simplicity, we will omit the summation and indicate $b(u,v,w) = \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx$. Thus, cf. [5], the following inequalities and properties on $b(\cdot,\cdot,\cdot)$ hold:

$$(3.1)\quad \begin{aligned} &(1)\; b(u,v,w) = \int_\Omega (w \otimes u) : \nabla v dx; \quad (2)\; b(u,v,w) = -b(u,w,v); \quad (3)\; b(u,v,v) = 0; \\ &(4)\; |b(u,v,w)| \leq C_0\|u\|_6\|\nabla v\|_2|w|_2^{1/2}\|w\|_6^{1/2}; \quad (5)\; |b(u,v,u)| \leq C_0\|u\|_{\frac{2p}{p-1}}\|v\|_{1,p}; \end{aligned}$$

for all $u, v, w \in \mathcal{V}$ and $p > 1$. Emphasize that, the letter q will be used to denote the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q}$=1. Then, the inequality (5), given in (3.1), is written as $|b(u, v, u)| \leq \|u\|_{2q}\|v\|_{1,p}$.

Now, let us consider the following tri-parametric function

$$b_N(u, v, w) := F_N(\|v\|_{1,2})b(u, v, w) = F_N(\|v\|_{1,2}) \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx,$$

for any $u, v, w \in \mathcal{V}$. Thus, let us define $\langle B_N(u, v), w\rangle = b_N(u, v, w)$ for all $u, v, w \in V_2$. Observe that by (3.1) the operator $B_N(\cdot, \cdot)$ is well defined, cf. [5]. In particular, for $u = v$ we write $B_N(u) := B_N(u, u)$, i.e.

$$\langle B_N(u), w\rangle = F_N(\|u\|_{1,2}) \int_\Omega u_j \frac{\partial u_i}{\partial x_j} w_i dx.$$

In the same way, we define the operator $\mathbb{T} : V_p \to V_p^*$, associated with the shear tensor, by

$$\langle \mathbb{T}u, w\rangle = \int_\Omega \mathbb{S}(Du(x)) : Dw(x)dx,$$

for all $w \in V_p$. Observe that by (1.2) for $p > 1$, we have that $\langle \mathbb{T}u, w\rangle \leq c_3\|1 + \mu|Du|\|_p^{p-1}\|Dw\|_p$ for any $p > 1$. Then, the operator $\mathbb{T}$ is well defined.

Theorem 3.1. *Let us consider $N, M > 0$ and $u, v \in \mathcal{V}$. Then, putting $w = v - u$, the following inequality holds*

$$|\langle B_M(v) - B_N(u), w\rangle| \leq (1 + d)M\|w\|_{1,2}^{3/2}|w|_2^{1/2} + |M - N|\|v\|_{1,2}\|w\|_{1,2}^{1/2}|w|_2^{1/2}.$$

In particular, for $M = N$, we get $|\langle B_N(v) - B_N(u), w\rangle| \leq (1 + d)N\|w\|_{1,2}^{3/2}|w|_2^{1/2}$. Moreover, for the special case $M = N$, we also get

$$\langle B_N(v) - B_N(u), \varphi\rangle \leq \begin{cases} \left[N + 2\|u\|_{1,2}\right]\|v - u\|_{1,2}\|\varphi\|_{1,2} \\ \left[\|v\|_{1,2} + 2\|u\|_{1,2}\right]\|v - u\|_{1,2}\|\varphi\|_{1,2}, \end{cases}$$

for all $\varphi \in \mathcal{V}$.

Proof. Adding $\pm\langle B_M(u, v), w\rangle$ and $\pm F_N(\|u\|_{1,2})\langle B(u, v), w\rangle$, and taking into account that $b(u, w, w) = 0$, we deduce that

$$\begin{aligned}\langle B_M(v) - B_N(u), w\rangle &= \langle B_M(v), w\rangle - \langle B_M(u, v), w\rangle + \langle B_M(u, v), w\rangle - \langle B_N(u), w\rangle \\ &\quad + F_N(\|u\|_{1,2})\langle B(u, v), w\rangle - F_N(\|u\|_{1,2})\langle B(u, v), w\rangle \\ &= F_M(\|v\|_{1,2})\langle B(w, v), w\rangle + F_N(\|u\|_{1,2})\langle B_(u, w), w\rangle \\ &\quad + \left[F_M(\|v\|_{1,2}) - F_N(\|u\|_{1,2})\right]\langle B(u, v), w\rangle \\ &= F_M(\|v\|_{1,2})\langle B(w, v), w\rangle + \left[F_M(\|v\|_{1,2}) - F_N(\|u\|_{1,2})\right]\langle B(u, v), w\rangle \\ &=: \mathbb{I}_1 + \mathbb{I}_2.\end{aligned}$$

For $\mathbb{I}_1$: by the interpolation inequality for $n = 3$ (for $n = 2$ we use the Ladyzhenskaya inequality), we have $\|w\|_4 \leq d|w|_2^{1/4}\|w\|_{1,2}^{3/4}$ and applying the Young inequality, we have

$$\mathbb{I}_1 \leq F_M(\|v\|_{1,2}) \int_\Omega w_j \frac{\partial v_i}{\partial x_j} w_i dx \leq dM\|w\|_{1,2}^{3/2}|w|_2^{1/2}.$$

For $\mathbb{I}_2$: it follows from (3.1) and Lemma 2.4 that

$$\begin{aligned}\mathbb{I}_2 &\leq \left[F_M(\|v\|_{1,2}) - F_N(\|u\|_{1,2})\right]\|u\|_{1,2}\|v\|_{1,2}|w|_2^{\frac{1}{2}}\|w\|_{1,2}^{\frac{1}{2}} \\ &\leq \frac{F_M(\|v\|_{1,2})F_N(\|u\|_{1,2})}{N}\|u\|_{1,2}\|v\|_{1,2}|w|_2^{\frac{1}{2}}\|w\|_{1,2}^{\frac{3}{2}} + |M - N|\|v\|_{1,2}|w|_2^{1/2}\|w\|_{1,2}^{1/2} \\ &\leq M\|w\|_{1,2}^{3/2}|w|_2^{1/2} + |M - N|\|v\|_{1,2}\|w\|_{1,2}^{1/2}|w|_2^{1/2}.\end{aligned}$$

On the other hand, for $M = N$, let $\varphi \in \mathcal{V}$, then

$$\begin{aligned}\langle B_N(v) - B_N(u), \varphi\rangle &= F_N(\|v\|_{1,2})\langle B(v-u,v),\varphi\rangle + F_N(\|v\|_{1,2})\langle B(u,v-u),\varphi\rangle \\ &\quad + \Big[F_N(\|v\|_{1,2}) - F_N(\|u\|_{1,2})\Big]\langle B(u,u),\varphi\rangle \\ &\le \begin{cases} N\|v-u\|_4\|\varphi\|_4 + \|u\|_4\|v-u\|_{1,2}\|\varphi\|_4 + \|v-u\|_{1,2}\|u\|_4\|\varphi\|_4 \\ \|v-u\|_4\|v\|_{1,2}\|\varphi\|_4 + \|u\|_4\|v-u\|_{1,2}\|\varphi\|_4 + \|v-u\|_{1,2}\|u\|_4\|\varphi\|_4 \end{cases} \\ &\le \begin{cases} \Big[N + 2\|u\|_{1,2}\Big]\|v-u\|_{1,2}\|\varphi\|_{1,2} \\ \Big[\|v\|_{1,2} + 2\|u\|_{1,2}\Big]\|v-u\|_{1,2}\|\varphi\|_{1,2}. \end{cases}\end{aligned}$$

Therefore, we get

$$\langle B_N(v) - B_N(u), \varphi\rangle \le \begin{cases} \Big[N + 2\|u\|_{1,2}\Big]\|v-u\|_{1,2}\|\varphi\|_{1,2} \\ \Big[\|v\|_{1,2} + 2\|u\|_{1,2}\Big]\|v-u\|_{1,2}\|\varphi\|_{1,2}, \end{cases}$$

for all $\varphi \in \mathcal{V}$. □

Definition 3.2. *Let $p \ge 2$, $T \in (0,+\infty)$, $f \in L^q(0,T;V_p^*)$, and $u_0 \in H$. A weak solution to* **(MLM)** *is a function u such that*

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V^3) \cap L^p(0,T;V_p) \quad \text{and} \quad \frac{\partial u}{\partial t} \in \Big[L^2(0,T;V^3) \cap L^p(0,T;V_p)\Big]^*,$$

and also satisfies the following weak formulation

$$\Big\langle\Big\langle \frac{\partial u}{\partial t}(t), v\Big\rangle\Big\rangle_* + N^{-1}((u(t),v)) + \langle \mathbb{S}(Du(t)), Dv\rangle + \langle B_N(u(t)), v\rangle = \langle f(t), v\rangle, \tag{3.2}$$

for all $v \in V^3$, a.e. $t \in (0,T)$, and $u(0) = u_0$, where the first term $\langle\langle\cdot,\cdot\rangle\rangle_*$ expresses duality between $(V^3)^*$ and V^3.

Remark 3.3. *If u is a weak solution of* **(MLM)**, *then u has a continuous representative, i.e. $u \in C([0,T];H)$. Therefore u evaluated at initial time $t = 0$ makes sense in the definition of weak solution. Furthermore, the following energy equality holds*

$$|u(t)|_2^2 + \frac{2}{N}\int_s^t \|u(r)\|_{3,2}^2 dr + 2\int_s^t\int_\Omega \mathbb{S}(Du) : Dudxdr = |u(s)|_2^2 + 2\int_s^t \langle f(r), u(r)\rangle dr,$$

for all $0 \le s \le t$.

On the other hand, the weak formulation given in (3.2)*, is for elements $v \in V^3$ (test functions) and not for $v \in V^3 \cap V_p$, since we are working in dimension $n \in \{2,3\}$, and therefore $V^3 \hookrightarrow V_p$ for all $p > 1$.*

Now, we are going to prove the continuous dependence with respect to the initial data and to the parameter $N > 0$, associated with the artificial smoothing dissipation and the cut-off function, of the problem **(MLM)**. As a consequence, we obtain the uniqueness of the weak solutions.

Theorem 3.4. *(Continuous dependence) Let us consider $p \ge 2$, $f \in L^q(0,T;V_p^*)$, $N, M > 0$, and $u_0, v_0 \in H$. Let us denote by $u^N(t) = u^N(t;u_0)$ and $v^M(t) = v^M(t;v_0)$ the weak solutions of* **(MLM)** *corresponding to the parameter N and the initial value u_0 and to the parameter M and the initial condition v_0, respectively. Then, there exists a positive constant C_M such that for all $t \ge 0$, the following inequality holds*

$$|v^M(t) - u^N(t)|_2^2 \le e^{C_M t}\Big\{|v_0 - u_0|_2^2 + |M-N|^2\int_0^t \Big(\|v^M(s)\|_{1,2}^2 + \frac{1}{NM^2}\|u^N(s)\|_{3,2}^2\Big)ds\Big\}. \tag{3.3}$$

Proof. For simplicity we denote $u = u^N$ and $v = v^M$ and $w(t) := v(t) - u(t)$. Then, w satisfies

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + ((M^{-1}v - N^{-1}u, w)) + \langle \mathbb{T}(v) - \mathbb{T}(u), w\rangle + \langle B_M(v) - B_N(u), w\rangle = 0. \tag{3.4}$$

Then, it follows from Theorem 3.1, that

$$\begin{aligned}\langle B_M(v) - B_N(u), w\rangle &\leq (1+d)M\|w\|_{1,2}^{3/2}|w|_2^{1/2} + |M-N|\|v\|_{1,2}\|w\|_{1,2}^{1/2}|w|_2^{1/2}\\ &\leq \mathbf{I}_1 + \mathbf{I}_2.\end{aligned}$$

For $\mathbf{I}_1$: Applying the Young inequality

$$\mathbf{I}_1 = (1+d)M\|w\|_{1,2}^{3/2}|w|_2^{1/2} \leq \frac{\nu_1}{4c_0^2}\|w\|_{1,2}^2 + \frac{3^3c_0^6}{4\nu_1^3}(1+d)^4M^4|w|_2^2.$$

For $\mathbf{I}_2$: Applying the Young inequality

$$\begin{aligned}\mathbf{I}_2 = |M-N|\|v\|_{1,2}\|w\|_{1,2}^{1/2}|w|_2^{1/2} &\leq \frac{1}{2}|M-N|^2\|v\|_{1,2}^2 + \frac{1}{2}\|w\|_{1,2}|w|_2\\ &\leq \frac{1}{2}|M-N|^2\|v\|_{1,2}^2 + \frac{\nu_1}{4c_0^2}\|w\|_{1,2}^2 + \frac{c_0^2}{\nu_1}|w|_2^2.\end{aligned}$$

On the other hand, observe that

$$((M^{-1}v - N^{-1}u, w)) = M^{-1}((w,w)) + \frac{N-M}{NM}((u,w)).$$

Estimating the term $\frac{N-M}{NM}((u,w))$, we have

$$\Big|\frac{N-M}{NM}((u,w))\Big| \leq \frac{|N-M|^2}{2NM^2}\|u\|_{3,2}^2 + M^{-1}\|w\|_{3,2}^2. \tag{3.5}$$

Therefore, from $\mathbf{I}_1$, $\mathbf{I}_2$, the Korn inequality, (3.4), and (3.5), we deduce

$$\frac{d}{dt}|w|_2^2 \leq |M-N|^2\|v\|_{1,2}^2 + \frac{|N-M|^2}{NM^2}\|u\|_{3,2}^2 + C_M|w|_2^2,$$

where $C_M = \max\left\{\frac{c_0^2}{\nu_1}, \frac{3^3c_0^6}{4\nu_1^3}(1+d)^4M^4\right\}$.

Then, by the Gronwall inequality,

$$|v^M(t) - u^N(t)|_2^2 \leq e^{C_Mt}\Big\{|v_0-u_0|_2^2 + |M-N|^2\int_0^t\Big(\|v^M(s)\|_{1,2}^2 + \frac{1}{NM^2}\|u^N(s)\|_{3,2}^2\Big)ds\Big\},$$

for all $t \geq 0$. ☐

Remark 3.5. *It should be noted that, if we do not use the artificial smoothing dissipation, $N^{-1}\Delta^3u$, in* **(MLM)***, the continuous dependence on the initial data and the parameter $N > 0$, is maintained, obtaining the following inequality*

$$|v^M(t) - u^N(t)|_2^2 \leq e^{C_Mt}\Big\{|v_0-u_0|_2^2 + |M-N|^2\int_0^t\|v^M(s)\|_{1,2}^2ds\Big\},$$

for all $t \geq 0$.

Corollary 3.6. *Under the conditions of the previous theorem, if $(N, u_0) \to (M, v_0)$ in $\mathbb{R}_+ \times H$, then $u^N(\cdot, u_0) \to v^M(\cdot, v_0)$ in $C([0,T];H)$, for all $T > 0$.*

Proof. This follows directly from (3.3). ☐

Theorem 3.7. *(Existence) Let us consider $p \geq 2$, $T > 0$, $u_0 \in H$ and $f \in L^q(0,T;V_p^*)$. Then, there exists at least one weak solution of the problem* **(MLM)***.*

Proof. Let us consider the set $\{w_r\}_{r=1}^\infty \subset V^3$ formed by the eigenfunctions to problem

$$((w_r, v)) = \lambda_r(w_r, v) \quad \text{for all } v \in V^3,$$

which are orthonormal in H and orthogonal in V^3. If $v \in W^{3,2}(\Omega)^n$ then $\nabla v \in W^{2,2}(\Omega)^{n^2}$ and therefore $W^{2,2}(\Omega)^n \hookrightarrow L^\infty(\Omega)^n$ since $\frac{1}{2} - \frac{2}{n} < 0$, with $n \in \{2,3\}$. Consequently, for all $p > 1$ we have $\nabla v \in L^p(\Omega)^{n^2}$ and $V^3 \hookrightarrow V_p$.

Let us define the Galerkin approximation $u_m(t)=\sum_{r=1}^{m} y_r^m(t)w_r$, where the coefficients $y_r^m(t)$ solve the following system

$$\left\{\begin{array}{l} \dfrac{d}{dt}(u_m(t),w_j)+\dfrac{1}{N}((u_m(t),w_j))+\langle \mathbb{T}(u_m(t)),w_j\rangle+\langle B_N(u_m(t)),w_j\rangle=\langle f(t),w_j\rangle,\\ u_m(0)=P^m u_0, \end{array}\right. \tag{3.6}$$

where $1\le j\le m$, and P^m is the orthogonal projector of H onto the linear hull of the first m eigenvectors $\{w_j\}_{j=1}^m$ with

$$P^m u_0\to u_0 \ \ in \ \ L^2(\Omega)^n. \tag{3.7}$$

It follows from [19, Theorem 3.4, p. 287] that the system (3.6) has one solution $y^m(t)=(y_1^m(t),\dots,y_m^m(t))$ defined on the interval $[0,t_m)$ with $0<t_m\le T$. From a priori estimates we will deduce that $t_m=T$. In fact, we multiply the $j-$th equation of the Galerkin system (3.6) by $y_j^m(t)$ and add the equations. Thus, the result can be written in the form

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|_2^2+\frac{1}{N}\|u_m(t)\|_{3,2}^2+\int_\Omega \mathbb{S}(Du_m):Du_m dx=\langle f(t),u_m\rangle, \tag{3.8}$$

since $b_N(u,u,u)=0$.

By (1.2) and applying Korn inequality and Young inequality, we can find a positive constant $k_1>0$, such that

$$\frac{d}{dt}|u_m|_2^2+\frac{2}{N}\|u_m(t)\|_{3,2}^2+\frac{2c_2\nu_1}{c_0^2}|\nabla u_m|_2^2+\frac{c_2\nu_2}{\tilde{c}_0^p}\|\nabla u_m\|_p^p\le k_1\|f(t)\|_*^q$$

where $\frac{1}{p}+\frac{1}{q}=1$. Thus, integrating from 0 to t, we deduce

$$|u_m(t)|_2^2+\int_0^t\left[\frac{2}{N}\|u_m(s)\|_{3,2}^2+\frac{2c_2\nu_1}{c_0^2}\|u_m(s)\|_{1,2}^2+\frac{c_2\nu_2}{\tilde{c}_0^p}\|u_m(s)\|_{1,p}^p\right]ds\le|u_0|_2^2+k_1\int_0^t\|f(s)\|_*^q ds. \tag{3.9}$$

Then, we conclude that $t_m=T$, for all $m\in\mathbb{N}$, and the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $L^\infty(0,T;H), L^p(0,T;V_p)$ and $L^2(0,T;V^3)$.

Let us denote by $Y=L^p(0,T;V_p)\cap L^2(0,T;V^3)$. Thus, by definition, we know that

$$\Big\|\frac{\partial u_m}{\partial t}\Big\|_{Y^*}=\sup\left\{\Big\langle\frac{\partial u_m}{\partial t},\varphi\Big\rangle_{Y^*,Y}:\varphi\in Y \text{ with } \|\varphi\|_Y=1\right\}$$

Thus, it follows from the equation that

$$\begin{aligned}\Big\langle\frac{\partial u_m}{\partial t},\varphi\Big\rangle_{Y^*,Y}&=\int_0^T\Big(-\frac{1}{N}((u_m,\varphi))-(\mathbb{S}(Du_m),D\varphi)-\langle B_N(u),\varphi\rangle+\langle f,\varphi\rangle\Big)dt\\ &=:\mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3+\mathbb{I}_4.\end{aligned} \tag{3.10}$$

Estimating each term, we have that

$$\mathbb{I}_1\le N^{-1}\|u_m\|_{L^2(0,T;V^3)}\|\varphi\|_{L^2(0,T;V^3)},$$
$$\mathbb{I}_2\le c_3\int_0^T\|1+\mu|Du(t)|\|_p^{p-1}\|D\varphi(t)\|_p dt\le C_3(1+\|u\|_{L^p(0,T;V_p)}^{p-1})\|\varphi\|_{L^p(0,T;V_p)},$$
$$\mathbb{I}_3\le\int_0^T F_N(\|u\|_{1,2})\int_\Omega|u_m||\nabla u_m||\varphi|dxdt\le N\int_0^T\|u_m\|_4\|\varphi\|_4 dt\le N\|u_m\|_{L^2(0,T;V_2)}\|\varphi\|_{L^2(0,T;V_2)}$$
$$\mathbb{I}_4\le\|f\|_{L^q(0,T;V_p^*)}\|\varphi\|_{L^p(0,T;V_p)}.$$

Therefore, the sequence $\{\frac{\partial u_m}{\partial t}\}_{m=1}^\infty$ is bounded in Y^*. Then, by the compactness theorems, the Aubin-Lions Theorem and the monotonicity of the operator $\mathbb{T}$, it follows that there exists a subsequence of $\{u_m\}_{m=1}^{\infty}$ (not relabelled) such that

$$\begin{array}{lll} u_m \overset{*}{\rightharpoonup} u & \text{in} & L^\infty(0,T;H), \\ u_m \to u & \text{in} & L^r(0,T;H) \quad \text{for all } r\in(1,\infty) \\ u_m \to u & \text{in} & L^2(0,T;V^2) \\ \frac{\partial u_m}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \text{in} & \Big[L^p(0,T;V_p)\cap L^2(0,T;V^3)\Big]^*, \\ \mathbb{T}(u_m) \rightharpoonup \mathbb{T}(u) & \text{in} & L^q(0,T;V_p^*). \end{array} \tag{3.11}$$

To be able to introduce the limit in equation (3.6), it is necessary to have the following convergence

$$u_m(t) \to u(t) \text{ a.e. } t\in(0,T) \text{ in } V_2,$$

which is true for the convergences given in (3.11). Then, with this last convergence, by linearity, density, and following the same reasoning as [5, Theorem 7] we can prove that

$$\int_0^T \langle B_N(u_m(s)), v\rangle ds \overset{m\to\infty}{\to} \int_0^T \langle B_N(u(s)), v\rangle ds \quad \text{for all } v\in V^3. \tag{3.12}$$

Then, u is a weak solution of **(MLM)**. □

4. Regularity of weak solution

This section is devoted to study the regularity of the weak solutions of **(MLM)**. For this, we will assume that the shear tensor $\mathbb{S}:\mathbb{R}^{n^2}_{sym}\to\mathbb{R}^{n^2}_{sym}$ has a potential, e.g. [6], i.e. there exists $\Phi\in C^2(\mathbb{R}^{n\times n};\mathbb{R}_+)$ with

$$\begin{cases} \partial_B\Phi(B) = \mathbb{S}(B), \\ \partial_B^2\Phi(B):(C\otimes C) \geq \nu_1(1+\mu|B|)^{p-2}|C|^2, \\ |\partial_B^2\Phi(B)| \leq c_4\nu_1(1+\mu|B|)^{p-2}. \end{cases} \tag{4.1}$$

Observe that this means control from above and below for $\Phi(B)$ for any $B\in\mathbb{R}^{n\times n}$, namely

$$c_5\nu_1(1+\mu|B|)^{p-2}|B|^2 \leq \Phi(B) \leq c_6\nu_1(1+\mu|B|)^{p-2}|B|^2. \tag{4.2}$$

It should be noted that if the tensor $\mathbb{S}$ satisfies (4.1), then it also satisfies (1.1) and (1.2), cf. [19, Lemma 1.19].

Theorem 4.1. *(Regularity) Let us consider $p\geq 2$, $T>0$, $u_0\in H$ and $f\in L^2(0,T;L^2(\Omega)^n)$, and also suppose that the shear tensor $\mathbb{S}$ has a potential. Then, the weak solutions of the problem* **(MLM)** *have the following regularity*

$$u\in L^\infty(\varepsilon,T;V_p)\cap L^\infty(\varepsilon,T;V^3) \quad \text{and} \quad \frac{\partial u}{\partial t}\in L^2(\varepsilon,T;H),$$

for any $\delta>0$. If $u_0\in V^3$, then $\varepsilon=0$.

Proof. The demonstration will be carried out for $n=3$ and the calculations presented can be justified by using the Galerkin approximation of the weak solution. Thus, using $\frac{\partial u}{\partial t}$ as a test function in the weak formulation, we have

$$\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{2N}\frac{d}{dt}\|u\|_{3,2}^2 + \Big\langle \mathbb{T}(u), \frac{\partial u}{\partial t}\Big\rangle + F_N(\|u\|_{1,2})\int_\Omega u_j\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial t}dx = \Big(f, \frac{\partial u}{\partial t}\Big). \tag{4.3}$$

Since the shear tensor, $\mathbb{S}$, has a potential, we deduce that

$$\begin{aligned} \Big\langle \mathbb{S}(Du), D\Big(\frac{\partial u}{\partial t}\Big)\Big\rangle &= \int_\Omega \frac{\partial\Phi}{\partial D_{ij}}(Du)D_{ij}\Big(\frac{\partial u}{\partial t}\Big)dx = \int_\Omega \frac{\partial\Phi}{\partial D_{ij}}(Du)\frac{\partial}{\partial t}(D_{ij}(u))dx \\ &= \frac{d}{dt}\int_\Omega \Phi(Du)dx = \frac{d}{dt}\|\Phi(Du)\|_1. \end{aligned}$$

Now, from (4.3) and the Hölder inequality, it follows that

$$\frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{2N}\frac{d}{dt}\|u\|_{3,2}^2 + \frac{d}{dt}\|\Phi(Du)\|_1 \leq |f|_2^2 + F_N^2(\|u\|_{1,2})\int_\Omega |u|^2|\nabla u|^2dx. \tag{4.4}$$

Now, let us denote by $\Theta := \frac{1}{2N}\|u\|_{3,2}^2 + \|\Phi(Du)\|_1$. By the inequality (4.2), there exist positive constants k_8 and k_9 such that

$$k_8\|u\|_{1,p}^p \le \|\Phi(Du)\|_1 \le k_9(1+\|u\|_{1,p}^p), \tag{4.5}$$

which implies that there exist positive constants c_8 and c_9, that are independent of $N>0$, such that

$$c_8\Big(\frac{1}{2N}\|u\|_{3,2}^2 + \|u\|_{1,p}^p\Big) \le \Theta \le c_9\Big(1+\frac{1}{2N}\|u\|_{3,2}^2 + \|u\|_{1,p}^p\Big). \tag{4.6}$$

In what follows, we are going to identify three cases: $p\ge 3$, $12/5 \le p < 3$, and $2 \le p < 12/5$.

• $p \ge 3$: It follows from the embedding $W^{1,2}(\Omega)^3 \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)^3$, that

$$\frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{d\Theta}{dt} \le |f|_2^2 + \int_\Omega |u|^2|\nabla u|^2 dx \le |f|_2^2 + c_7 d^2\|u\|_{1,p}^2\|u\|_{1,2}^2.$$

Denoting by $\mathcal{U}(t) := 1+\Theta(t)$, we observe that $\frac{d\mathcal{U}}{dt} = \frac{d\Theta}{dt}$. Then, it follows from (4.6), that

$$\frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{d\mathcal{U}}{dt} \le |f|_2^2 + \widetilde{C}_2\mathcal{U}^{4/p}. \tag{4.7}$$

Now, if $4 \le p$, then $\mathcal{U}^{4/p} \le \mathcal{U}$, since $\mathcal{U}\ge 1$. Thus

$$\frac{d\mathcal{U}}{dt} \le |f|_2^2 + \widetilde{C}_2\mathcal{U}.$$

Integrating from s to t, we obtain that

$$\mathcal{U}(t) \le \mathcal{U}(s) + \int_0^T |f(\theta)|_2^2 d\theta + \widetilde{C}_2\int_0^T \mathcal{U}(\theta)d\theta.$$

Again integrating in $s\in[0,t]$, we get

$$\mathcal{U}(t) \le \frac{1}{t}\int_0^T \mathcal{U}(\theta)d\theta + \int_0^T |f(\theta)|_2^2 d\theta + \widetilde{C}_2\int_0^T \mathcal{U}(\theta)d\theta,$$

Thus, for $0<\varepsilon\le t$, we have that

$$\mathcal{U}(t) \le \frac{1}{\varepsilon}\int_0^T \mathcal{U}(\theta)d\theta + \int_0^T |f(\theta)|_2^2 d\theta + \widetilde{C}_2\int_0^T \mathcal{U}(\theta)d\theta,$$

for all $\varepsilon \le t \le T$. Note that everything on the right-hand is bounded.

Now, if $3 \le p < 4$, let us consider $\mu = (2p-4)/p$, then $\mu \in [\frac{2}{3},1)$. Multiplying (4.7) by $\mathcal{U}^{\mu-1}$, we obtain that

$$\mu\frac{d}{dt}(\mathcal{U}^\mu) \le |f|_2^2\mathcal{U}^{\mu-1} + \widetilde{C}_2\mathcal{U} \le |f|_2^2 + \widetilde{C}_2\mathcal{U},$$

since $\mathcal{U}^{\mu-1}\le 1$. Now, by integration, we get

$$\mu\mathcal{U}^\mu(t) \le \mu\mathcal{U}^\mu(s) + \int_s^t |f(\theta)|_2^2 d\theta + \widetilde{C}_2\int_s^t \mathcal{U}(\theta)d\theta \le \mu\mathcal{U}(s) + \int_s^t |f(\theta)|_2^2 d\theta + \widetilde{C}_2\int_s^t \mathcal{U}(\theta)d\theta,$$

since $\mathcal{U}\ge 1$ and $\mu \le 1$. Again by integration into s on $[0,t]$ and considering $0<\varepsilon\le t$, we have

$$\mu\mathcal{U}^\mu(t) \le \Big(\widetilde{C}_2 + \frac{\mu}{\varepsilon}\Big)\int_0^T \mathcal{U}(\theta)d\theta + \int_0^t |f(\theta)|_2^2 d\theta.$$

Thus, we conclude the boundedness of $\mathcal{U}(t)$, as before. Then, putting all the inequalities together, we arrive at that $u\in L^\infty(\varepsilon,T;V^3)\cap L^\infty(\varepsilon,T;V_p)$ and $\frac{\partial u}{\partial t}\in L^2(\varepsilon,T;H)$, for all $\varepsilon>0$ and $p\ge 3$.

• $12/5 \le p < 3$: By the interpolation inequality, we have

$$\|u\|_{2p/(p-2)} \le d\|u\|_{1,p}^{\frac{6}{5p-6}}|u|_2^{\frac{5p-12}{5p-6}},$$

where d is the constant of interpolation that depends on Ω and p. Then, the right-hand side of (4.4) can be estimated as

$$\|u\|_{1,p}^2\|u\|_{2p/(p-2)}^2 \le d^2\|u\|_{1,p}^p\|u\|_{1,p}^{p(16-5p)/(5p-6)}|u|_2^{2(5p-12)/(5p-6)}$$
$$\le \frac{d^2}{c_8}\Theta\|u\|_{1,p}^{p(16-5p)/(5p-6)}|u|_2^{2(5p-12)/(5p-6)}.$$

Thus, we have that

$$\frac{d\Theta}{dt} \le |f|_2^2 + \frac{c_7d^2}{c_8}\Theta\|u\|_{1,p}^{p(16-5p)/(5p-6)}|u|_2^{2(5p-12)/(5p-6)}.$$

Integrating, we get

$$\Theta(t) \le \Theta(s) + \int_0^t |f(\theta)|_2^2 d\theta + c_{10}\int_0^t \Theta(\theta)\|u(\theta)\|_{1,p}^{p(16-5p)/(5p-6)}|u(\theta)|_2^{2(5p-12)/(5p-6)}d\theta.$$

for all $s \in [0,t]$, where $c_{10} = \frac{c_7d^2}{c_8}$. Applying the Gronwall inequality

$$\Theta(t) \le \left(\Theta(s) + \int_0^t |f(\theta)|_2^2 d\theta\right) \times \exp\left(c_{10}\int_0^t \|u(\theta)\|_{1,p}^{\frac{p(16-5p)}{5p-6}}|u(\theta)|_2^{\frac{2(5p-12)}{5p-6}}d\theta\right),$$

for all $s \in [0,T]$. Now, given $\varepsilon > 0$ such that $\varepsilon \le T$ and integrating in s, from 0 to ε,

$$\Theta(t) \le \left(\frac{1}{\varepsilon}\int_0^t \Theta(s)ds + \int_0^t |f(\theta)|_2^2 d\theta\right) \times \exp\left(c_{10}\int_0^t \|u(\theta)\|_{1,p}^{\frac{p(16-5p)}{5p-6}}|u(\theta)|_2^{\frac{2(5p-12)}{5p-6}}d\theta\right),$$

for all $t \ge \varepsilon$. Observe that all terms of the right-hand side of the above inequality are bounded. In fact: we know that u belong to $L^2(0,T;V^3) \cap L^p(0,T;V_p) \cap L^\infty(0,T;H)$, and to conclude, only remains to prove that $\int_0^t \|u(r)\|_{1,p}^{\frac{p(16-5p)}{5p-6}}dr < \infty$ and $(5p-12)/(5p-6) \ge 0$, what is true since $p \ge 12/5$. It should be noted that, from (3.9), the right-hand side of this last inequality does not depend on N, only depends on $u_0 \in H$ and f.

- $2 \le p < 12/5$: In this case we can use the embedding $V^3 \hookrightarrow L^{2p/(p-2)}(\Omega)^n$. □

Remark 4.2. *It is easier to have an estimate for case $p \ge 3$ using the properties of the cut-off function. Indeed, by the Sobolev embedding $V_2 \hookrightarrow L^6(\Omega)^3$, we have that*

$$F_N^2(\|u\|_{1,2})\int_\Omega |u|^2|\nabla u|^2 dx \le F_N^2(\|u\|_{1,2})\|u\|_6^2\|u\|_{1,3}^2 \le N^2\|u\|_{1,3}^2 \le N^2\|u\|_{1,p}^2.$$

Thus, it follows from (4.6), that

$$\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|_2^2 + \frac{d\mathcal{U}}{dt} \le |f|_2^2 + \widetilde{C}_2\mathcal{U}^{2/p} \le |f|_2^2 + \widetilde{C}_2\mathcal{U}.$$

Since $p \ge 3$, everything continues analogously.

Theorem 4.3. *(Extra regularity) Let us consider $p \ge 2$, $T > 0$, $u_0 \in H$, $f \in W^{1,2}(0,T;L^2(\Omega)^n)$, and also suppose that the shear tensor $\mathbb{S}$ has a potential. Then, the derivative in time of the weak solutions, of the problem* **(MLM)***, have the following regularity*

$$\frac{\partial u}{\partial t} \in L^\infty(\varepsilon,T;H) \cap L^2(\varepsilon,T;V^3),$$

for all $\varepsilon > 0$.

Proof. By differentiating the first equation in **(MLM)**, with respect to time, we get

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{N}\Delta^3\Big(\frac{\partial u}{\partial t}\Big) - \text{div}\Big(\partial_D^2\Phi(Du)D\Big(\frac{\partial u}{\partial t}\Big)\Big) + \frac{d}{dt}\Big(F_N(\|u\|_{1,2})\Big)B(u) +$$
$$+ F_N(\|u\|_{1,2})\frac{\partial}{\partial t}\Big(B(u)\Big) + \nabla\Big(\frac{\partial P}{\partial t}\Big) = \frac{\partial f}{\partial t},$$

where, by (3.1), we know that

$$B(u) = \text{div}[u \otimes u],$$

and therefore

$$\frac{\partial}{\partial t}\Big(B(u)\Big) = \operatorname{div}\Big[\frac{\partial u}{\partial t} \otimes u + u \otimes \frac{\partial u}{\partial t}\Big].$$

Multiplying the above equality by $\dfrac{\partial u}{\partial t}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{N}\Big\|\frac{\partial u}{\partial t}\Big\|_{3,2}^2 + \int_\Omega \partial_D^2\Phi(Du)D\Big(\frac{\partial u}{\partial t}\Big) : D\Big(\frac{\partial u}{\partial t}\Big)dx \\
&\qquad + \frac{d}{dt}\Big(F_N(\|u\|_{1,2})\Big)\int_\Omega (u\otimes u) : \nabla\frac{du}{dt}dx \\
&\qquad - F_N(\|u\|_{1,2})\int_\Omega \Big(\frac{\partial u}{\partial t} \otimes \frac{\partial u}{\partial t}\Big) : \nabla u dx \\
&\qquad = \Big(\frac{\partial f}{\partial t}, \frac{\partial u}{\partial t}\Big),
\end{aligned} \tag{4.8}$$

since, thanks to (3.1), we know that

$$\int_\Omega \Big(\frac{\partial u}{\partial t} \otimes u\Big) : \nabla\Big(\frac{\partial u}{\partial t}\Big)dx = 0,$$

$$\int_\Omega \Big(u \otimes \frac{\partial u}{\partial t}\Big) : \nabla\Big(\frac{\partial u}{\partial t}\Big)dx = -\int_\Omega \Big(\frac{\partial u}{\partial t} \otimes \frac{\partial u}{\partial t}\Big) : \nabla u dx.$$

On the other hand, using the properties of Φ given in (4.1), we obtain

$$\nu_1 \int_\Omega (1+\mu|Du|)^{p-2}\Big|D\Big(\frac{\partial u}{\partial t}\Big)\Big|^2 dx \le \int_\Omega \partial_D^2\Phi(Du)D\Big(\frac{\partial u}{\partial t}\Big) : D\Big(\frac{\partial u}{\partial t}\Big)dx.$$

With this and (4.8), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{N}\Big\|\frac{\partial u}{\partial t}\Big\|_{3,2}^2 + \nu_1\Big|D\Big(\frac{\partial u}{\partial t}\Big)\Big|_2^2 + \frac{d}{dt}\Big(F_N(\|u\|_{1,2})\Big)\int_\Omega (u\otimes u) : \nabla\frac{du}{dt}dx \\
&\le F_N(\|u\|_{1,2})\int_\Omega \frac{\partial u}{\partial t} \otimes \frac{\partial u}{\partial t} : \nabla u dx + \frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{2}\Big|\frac{\partial f}{\partial t}\Big|_2^2 \\
&\le F_N(\|u\|_{1,2})\int_\Omega \Big|\frac{\partial u}{\partial t}\Big|^2|\nabla u|dx + \frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{2}\Big|\frac{\partial f}{\partial t}\Big|_2^2 \\
&\le F_N(\|u\|_{1,2})\|u\|_{1,2}\Big\|\frac{\partial u}{\partial t}\Big\|_4^2 + \frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{2}\Big|\frac{\partial f}{\partial t}\Big|_2^2 \\
&\le N\Big\|\frac{\partial u}{\partial t}\Big\|_4^2 + \frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{2}\Big|\frac{\partial f}{\partial t}\Big|_2^2.
\end{aligned}$$

On the other hand, it is simple to show that $|F_N'(s)| \le \frac{N}{s^2}$. Thus, we deduce that

$$\pm\frac{d}{dt}\Big(F_N(\|u\|_{1,2})\Big) = \pm F_N'(\|u\|_{1,2})\frac{d}{dt}(\|u\|_{1,2}) \le |F_N'(\|u\|_{1,2})|\Big|\nabla\frac{\partial u}{\partial t}\Big|_2 \le \frac{N}{\|u\|_{1,2}^2}\Big|\nabla\frac{\partial u}{\partial t}\Big|_2.$$

With this last inequality, we obtain that

$$\begin{aligned}
\pm\frac{d}{dt}\Big(F_N(\|u\|_{1,2})\Big)\int_\Omega (u\otimes u) : \nabla\frac{du}{dt}dx &\le \Big|\frac{d}{dt}\Big(F_N(\|u\|_{1,2})\Big)\Big|\int_\Omega |u||\nabla u|\Big|\frac{\partial u}{\partial t}\Big|dx \\
&\le \frac{N}{\|u\|_{1,2}^2}\Big|\nabla\frac{\partial u}{\partial t}\Big|_2\|u\|_4\|u\|_{1,2}\Big\|\frac{\partial u}{\partial t}\Big\|_4 \\
&\le \tilde{c}_1 N\Big|\nabla\frac{\partial u}{\partial t}\Big|_2\Big\|\frac{\partial u}{\partial t}\Big\|_4,
\end{aligned}$$

where $\tilde{c}_1$ is the constant of the immersion $W^{1,2}(\Omega)^n \hookrightarrow L^4(\Omega)^n$. Then, putting all these estimates together we arrive at

$$\frac{1}{2}\frac{d}{dt}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{N}\Big\|\frac{\partial u}{\partial t}\Big\|_{3,2}^2 + \nu_1\Big|D\Big(\frac{\partial u}{\partial t}\Big)\Big|_2^2 \le \tilde{c}_1 N\Big|\nabla\frac{\partial u}{\partial t}\Big|_2\Big\|\frac{\partial u}{\partial t}\Big\|_4 + N\Big\|\frac{\partial u}{\partial t}\Big\|_4^2 + \frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{2}\Big|\frac{\partial f}{\partial t}\Big|_2^2.$$

By the interpolation inequality for $n = 3$ (for $n = 2$ we use the Ladyzhenskaya inequality), we know that $\|\frac{\partial u}{\partial t}\|_4 \le d|\frac{\partial u}{\partial t}|_2^{1/4}\|\frac{\partial u}{\partial t}\|_{1,2}^{3/4}$. Then, applying the Korn and the Young inequalities, there exist positive constants $\tilde{c}_2$ and $\tilde{c}_N$, such that

$$\frac{d}{dt}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{2}{N}\Big\|\frac{\partial u}{\partial t}\Big\|_{3,2}^2 + \tilde{c}_2\Big\|\frac{\partial u}{\partial t}\Big\|_{1,2}^2 \le \tilde{c}_N\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \Big|\frac{\partial f}{\partial t}\Big|_2^2.$$

Integrating from s to t, with $0 \le s \le t \le T$, we have

$$\begin{aligned}\Big|\frac{\partial u}{\partial t}(t)\Big|_2^2 + \frac{2}{N}\int_s^t \Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{3,2}^2 d\theta + \tilde{c}_2\int_s^t \Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{1,2}^2 d\theta &\le \Big|\frac{\partial u}{\partial t}(s)\Big|_2^2 + \tilde{c}_N\int_0^T \Big|\frac{\partial u}{\partial t}(\theta)\Big|_2^2 d\theta \\ &\quad + \int_0^T \Big|\frac{\partial f}{\partial t}(\theta)\Big|_2^2 d\theta.\end{aligned} \tag{4.9}$$

Integrating this last inequality in s, between 0 and t, we obtain

$$\Big|\frac{\partial u}{\partial t}(t)\Big|_2^2 \le \frac{1+\tilde{c}_N t}{t}\int_0^T \Big|\frac{\partial u}{\partial t}(s)\Big|_2^2 ds + \int_0^T \Big|\frac{\partial f}{\partial t}(\theta)\Big|_2^2 d\theta,$$

for all $0 < t \le T$. In particular, for any $\varepsilon > 0$, we have that

$$\Big|\frac{\partial u}{\partial t}(t)\Big|_2^2 \le \frac{1+\tilde{c}_N T}{\varepsilon}\int_0^T \Big|\frac{\partial u}{\partial t}(s)\Big|_2^2 ds + \int_0^T \Big|\frac{\partial f}{\partial t}(\theta)\Big|_2^2 d\theta,$$

for all $\varepsilon \le t \le T$. Thus, $\frac{\partial u}{\partial t} \in L^\infty(\varepsilon, T; H)$ for any $\varepsilon > 0$ small enough.

In the same way, it follows from (4.9), for any $\varepsilon > 0$,

$$\begin{aligned}\frac{2\varepsilon}{N}\int_\varepsilon^t \Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{3,2}^2 d\theta + \tilde{c}_2\varepsilon\int_\varepsilon^t \Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{1,2}^2 d\theta &\le \tilde{c}_2\int_\varepsilon^t\int_s^t \Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{1,2}^2 d\theta ds \\ &\le (1+\tilde{c}_N T)\int_0^T \Big|\frac{\partial u}{\partial t}(\theta)\Big|_2^2 d\theta + T\int_0^T \Big|\frac{\partial f}{\partial t}(\theta)\Big|_2^2 d\theta.\end{aligned}$$

Then, we have that

$$\int_\varepsilon^T \Big(\Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{3,2}^2 + \Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{1,2}^2\Big) d\theta \le \frac{1+\tilde{c}_N T}{\tilde{c}_{2,N}\varepsilon}\int_0^T \Big|\frac{\partial u}{\partial t}(\theta)\Big|_2^2 d\theta + \frac{T}{\tilde{c}_{2,N}\varepsilon}\int_0^T \Big|\frac{\partial f}{\partial t}(\theta)\Big|_2^2 d\theta, \tag{4.10}$$

for any $\varepsilon > 0$, where $\tilde{c}_{2,N} = \min\{2/N, \tilde{c}_2\}$. Then, we have that $\frac{\partial u}{\partial t}$ belong to $L^2(\varepsilon, T; V_2)$ and $L^2(\varepsilon, T; V^3)$ for any $\varepsilon > 0$. □

Remark 4.4. *With this last estimate, observe that, using $\frac{\partial u}{\partial t}$ as a test function in the weak formulation, we have*

$$\begin{aligned}F_N(\|u\|_{1,2})\int_\Omega u_j\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial t}dx &\le F_N(\|u\|_{1,2})\|u\|_4\|u\|_{1,2}\Big\|\frac{\partial u}{\partial t}\Big\|_4 \\ &\le N\|u\|_4\Big\|\frac{\partial u}{\partial t}\Big\|_4 \le N\|u\|_{1,2}\Big\|\frac{\partial u}{\partial t}\Big\|_{1,2}.\end{aligned}$$

Now, from (4.4) *and the previous inequalities, it follows that*

$$\frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|_2^2 + \frac{1}{2N}\frac{d}{dt}\|u\|_{3,2}^2 + \frac{d}{dt}\|\Phi(Du)\|_1 \le \frac{1}{2}|f(t)|_2^2 + \frac{N^2}{2}\|u(t)\|_{1,2}^2 + \frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|_{1,2}^2.$$

From this last inequality we obtain, in particular, that

$$\frac{1}{2N}\|u(t)\|_{3,2}^2+\|\Phi(Du(t))\|_1 \leq \frac{1}{2N}\|u(s)\|_{3,2}^2+\|\Phi(Du(s))\|_1 + \frac{1}{2}\int_s^t\Big(|f(\theta)|_2^2+N^2\|u(\theta)\|_{1,2}^2+\Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{1,2}^2\Big)d\theta,$$

for all $s\in[0,t]$. Thus, for $\varepsilon>0$, integrating in s, from ε to t,

$$\frac{1}{2N}\|u(t)\|_{3,2}^2+\|\Phi(Du(t))\|_1 \leq \frac{1}{(t-\varepsilon)}\int_\varepsilon^t\Big(\frac{1}{2N}\|u(s)\|_{3,2}^2+\|\Phi(Du(s))\|_1\Big)ds+\mathcal{M}(t,\varepsilon), \tag{4.11}$$

for all $t\in(\varepsilon,T]$, where

$$\mathcal{M}(t,\varepsilon)=\frac{1}{2}\int_0^t\Big(|f(\theta)|_2^2+N^2\|u(\theta)\|_{1,2}^2\Big)d\theta+\int_\varepsilon^t\Big\|\frac{\partial u}{\partial t}(\theta)\Big\|_{1,2}^2d\theta.$$

Observe that all terms on the right-hand side of the above inequality are bounded.

5. Convergence to Weak Solutions of Ladyzhenskaya Model

In this section, we explore the convergence of the weak solutions of problem **(MLM)**, as a sequence of the parameter N, of the cut-off function that accompanies the convective term and the artificial smoothing dissipation of the modified equation. This convergence will lead to a weak solution for system **(LM)** when $p\geq 1+2n/(n+2)$. Also, we will analyze the convergence of the Galerkin sequence given in (3.6), when the parameter N grows to infinity.

Theorem 5.1. *Let $p\geq 2$, $N>0$, $T>0$, $u_0\in H$, and $f\in L^q_{loc}(\mathbb{R};V_p^*)$. Consider $\{u_{0,m}\}_{m=1}^\infty\subset H$ such that $u_{0,m}\to u_0$ in H as $m\to\infty$. Thus, let $\{u_m\}_{m=1}^\infty$ be the sequence of weak solutions on $[0,T]$ such that $u_m(0)=u_{0,m}$ for each $m\in\mathbb{N}$. Then, there exists a subsequence of $\{u_m\}_{m=1}^\infty$ that converge to a certain function u, in the sense specified in* (3.11)*, such that u is again a solution to* **(MLM)** *on $[0,T]$.*

Proof. We know that, for each $m\in\mathbb{N}$, $u_m=u_m(\cdot;u_{0,m})$ satisfies

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|_2^2+N^{-1}\|u_m\|_{3,2}^2+\int_\Omega \mathbb{S}(Du_m):Du_m dx=\langle f(t),u_m\rangle,$$

since $b_N(u,u,u)=0$. Thus, by (1.2) and applying Korn inequality and Young inequality, we can find a positive constant $k_1>0$, such that

$$\frac{d}{dt}|u_m|_2^2+\frac{2}{N}\|u_m\|_{3,2}^2+\frac{2c_2\nu_1}{c_0^2}\|u_m\|_{1,2}^2+\frac{c_2\nu_2}{\tilde{c}_0^p}\|u_m\|_{1,p}^p\leq k_1\|f\|_*^q$$

where $\frac{1}{p}+\frac{1}{q}=1$. Thus, integrating from 0 to t, we deduce

$$|u_m(t)|_2^2+C_{p,N}\int_0^t\Big[\|u_m(s)\|_{3,2}^2+\|u_m(s)\|_{1,2}^2+\|u_m(s)\|_{1,p}^p\Big]ds\leq|u_0|_2^2+k_1\int_0^t\|f(s)\|_*^q ds,$$

where $C_{p,N}=\min\Big\{2N^{-1},\frac{2c_2\nu_1}{c_0^2},\frac{c_2\nu_2}{\tilde{c}_0^p}\Big\}$.

We obtain that $\{u_m\}_{m=1}^\infty$ is bounded in $L^\infty(0,T;H)$, $L^p(0,T;V_p)$, and $L^2(0,T;V^3)$, and the sequence $\{\frac{\partial u_m}{\partial t}\}_{m=1}^\infty$ is bounded in $\Big[L^p(0,T;V_p)\cap L^2(0,T;V^3)\Big]^*$. Then, by the compactness theorems, the Aubin-Lions Theorem and the monotonicity of the operator $\mathbb{T}$, it follows that there exists a subsequence of $\{u_m\}_{m=1}^\infty$ that converges to a certain function u, in the sense specified in (3.11). Thus, in the same way, as in the Theorem 3.7, we conclude the proof. □

Theorem 5.2. *Given $f\in L^q(0,T;V_p^*)$ and $u_0\in H$. We know that the Galerkin approximation of* **(MLM)** *depends on $m\in\mathbb{N}$ and $N\in\mathbb{R}_+$. Then fixed $m\in\mathbb{N}$, if $p\geq 1+2n/(n+2)$, there exists a subsequence that converges to the Galerkin approximation of* **(LM)** *on $[0,T]$, when $N\to\infty$.*

Proof. Let us consider the Galerkin approximation of the system (3.6), that is denoted by $u_m^N(t) = \sum_{r=1}^m y_r^N(t) w_r \in H^m := \text{Span}\{w_1, \dots, w_m\}$. On the other hand, it is simple to identify H^m by $\mathbb{R}^m$ and to identify $u_m^N(t) = (y_1^N(t), \dots, y_m^N(t)) \in \mathbb{R}^m$ with $t \in [\tau, T]$. Then, observe that

$$|u_m^N(t)|_2^2 = \sum_{r=1}^m \left[y_r^N(t)\right]^2,$$

since $(w_i, w_j) = \delta_{i,j}$ for any $i, j \in \mathbb{N}$. Therefore, we can identify the norm in H^m with the norm of $\mathbb{R}^m$. Thus, our goal is to fix the index $m \in \mathbb{N}$, and to study $\{u_m^N\}$ as a sequence of N.

It follows from (3.6) that u_m^N satisfies

$$\frac{1}{2}\frac{d}{dt}|u_m^N(t)|_2^2 + N^{-1}\|u_m^N\|_{3,2}^2 + \int_\Omega \mathbb{S}(Du_m^N) : Du_m^N dx = \langle f(t), u_m^N\rangle, \tag{5.1}$$

since $b(u, u, u) = 0$. Thus, applying the Korn and Young inequalities, we can find a positive constant $k_1 > 0$, such that

$$\frac{d}{dt}|u_m^N|_2^2 + 2N^{-1}\|u_m^N\|_{3,2}^2 + \frac{2c_2\nu_1}{c_0^2}\|u_m^N\|_{1,2}^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|u_m^N\|_{1,p}^p \le k_1\|f\|_*^q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$|u_m^N(t)|_2^2 + C_{p,N}\int_0^T \left(\|u_m(s)\|_{3,2}^2 + \|u_m^N(s)\|_2^{1,2} + \|u_m^N(s)\|_{1,p}^p\right) ds \le |u_0|_2^2 + k_1\int_0^T \|f(s)\|_*^q ds,$$

where $C_{p,N} = \min\left\{2N^{-1}, \frac{2c_2\nu_1}{c_0^2}, \frac{c_2\nu_2}{\tilde{c}_0^p}\right\}$. Then, we get

$$|u_m^N(t)|_2^2 \le |u_0|_2^2 + k_1\int_0^T \|f(s)\|_*^q ds. \tag{5.2}$$

It follows that the sequence $\{u_m^N\}_N$ is uniformly bounded with respect to N, also with respect to m, in $C([\tau, T]; \mathbb{R}^m)$.

On the other hand, in the same way as (3.10), is possible to show that the sequence $\left\{\frac{\partial u_m^N}{\partial t}\right\}$, is uniformly bounded, respect to $N > 0$, in $L^2(0, T; (V^3)^*)$, for $p \ge 1 + 2n/(n+2)$. Then, observe that

$$(u_m^N(t), \mathbf{v}) - (u_m^N(s), \mathbf{v}) = \int_s^t \left\langle\left\langle \frac{\partial u_m^N}{\partial t}(\theta), \mathbf{v}\right\rangle\right\rangle_* d\theta \le \int_s^t \left\|\frac{\partial u_m^N}{\partial t}(\theta)\right\|_{(V^3)^*}\|\mathbf{v}\|_{3,2} d\theta,$$

for all $\mathbf{v} \in V^3$. Thus, we derive

$$\|u_m^N(t) - u_m^N(s)\|_{(V^3)^*} \le \int_s^t \left\|\frac{\partial u_m^N}{\partial t}(\theta)\right\|_{(V^3)^*} d\theta \le |t-s|^{1/2}\left(\int_s^t \left\|\frac{\partial u_m^N}{\partial t}(\theta)\right\|_{(V^3)^*}^2 d\theta\right)^{1/2}. \tag{5.3}$$

Therefore, $\{u_m^N\}_N$ is an equicontinuous sequence in $C([0, T]; \mathbb{R}^m)$. It follows from Ascoli-Arzelà theorem that there exists a subsequence with respect to the super index N, again denoted by $\{u_m^N\}$, which converges uniformly in $C([0, T]; \mathbb{R}^m)$ to a function u_m^∞ that belongs to $C([0, T]; \mathbb{R}^m)$.

Continuing, by definition of the eigenfunctions $\{w_r\} \subset V^3$, we have that

$$\begin{aligned}\|u_m^N(t)\|_{3,2}^2 &= \sum_{r=1}^m \left[y_r^N(t)\right]^2((w_r, w_r)) = \sum_{r=1}^m \left[y_r^N(t)\right]^2\lambda_r(w_r, w_r)\\ &\le \lambda_m\sum_{r=1}^m \left[y_r^N(t)\right]^2(w_r, w_r) = \lambda_m|u_m^N(t)|_2^2.\end{aligned}$$

Thus, we deduce that

$$\|u_m^N(t)\|_{3,2} \le \lambda_m^{1/2}|u_m^N(t)|_2, \quad \text{for all } N > 0.$$

Due to the immersion $V^3 \hookrightarrow V_p \hookrightarrow V_2$, we obtain the following relationship

$$\frac{1}{\|u_m^N(t)\|_{1,2}} \ge \frac{1}{\kappa_3\|u_m^N(t)\|_{3,2}} \ge \frac{1}{\lambda_m^{1/2}\kappa_3|u_m^N(t)|_2},$$

where κ_3 is a constant of the immersion $V^3 \hookrightarrow V_2$. On the other hand, observe that

$$1 \geq F_N(\|u_m^N(t)\|_{1,2}) = \min\left\{1, \frac{N}{\|u_m^N(t)\|_{1,2}}\right\} \geq \min\left\{1, \frac{N}{\kappa_3\|u_m^N(t)\|_{3,2}}\right\}$$
$$\geq \min\left\{1, \frac{N}{\lambda_m^{1/2}\kappa_3|u_m^N(t)|_2}\right\} \geq 1 \quad \text{if and only if} \quad N \geq \lambda_m^{1/2}\kappa_3|u_m^N(t)|_2.$$

Thus, by (5.2), if we choose $N_0 \geq \lambda_m^{1/2}\kappa_3\Big[|u_0|_2^2 + k_1\int_0^T \|f(s)\|_*^q ds\Big]^{1/2}$, we ensure that

$$F_N(\|u_m^N(t)\|_{1,2}) = 1, \quad \text{for all } N \geq N_0.$$

Note that the term $N^{-1}u_m^N \to 0$ as $N \to \infty$. Therefore, we conclude that $\{u_m^\infty\}_{m=1}^\infty$ is the Galerkin approximation for **(LM)**. ☐

Theorem 5.3. *Given $T > 0$, $u_0 \in H$, and $f \in L^q(0,T;V_p^*)$. Let $\{u_0^N\}_N \subset H$ be a sequence such that $u_0^N \to u_0$ in H. Let $u^N(t)$ be the weak solution to* **(MLM)** *associated to the initial data $u_0^N \in H$. Then, if $p \geq 1 + 2n/(n+2)$, there exists $u \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$ with $\frac{\partial u}{\partial t} \in L^q(0,T;V_p^*)$, such that (up to a subsequence)*

$$(5.4) \qquad \begin{cases} u^N \overset{*}{\rightharpoonup} u & \textit{in } L^\infty(0,T;H),\\ u^N \rightharpoonup u & \textit{in } L^p(0,T;V_p),\\ u^N \to u & \textit{in } L^2(0,T;H), \textit{ a.e. in } H \textit{ and } \Omega\times(0,T),\\ \frac{\partial u^N}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \textit{in } \Big[L^p(0,T;V_p)\cap L^2(0,T;V^3)\Big]^*. \end{cases}$$

Also, u is a weak solution to **(LM)**.

Proof. Let us identify by $u^N(t) = u^N(t;u_0^N)$ the weak solution to **(MLM)** associated the initial data $u_0^N \in H$. Therefore, our goal is to prove that $\{u^N\}$ converges to a weak solution of **(LM)** as a sequence of N.

It follows from (3.6) that u^N satisfies

$$\frac{1}{2}\frac{d}{dt}|u^N(t)|_2^2 + N^{-1}\|u^N\|_{3,2}^2 + \int_\Omega \mathbb{S}(Du^N) : Du^N dx = \langle f, u^N\rangle,$$

since $b(u,u,u) = 0$.

Applying the Korn and Young inequalities, we can find a positive constant $k_1 > 0$, such that

$$\frac{d}{dt}|u^N|_2^2 + C_{p,N}(\|u^N\|_{3,2}^2 + \|u^N\|_2^2 + \|u^N\|_{1,p}^p) \leq k_1\|f(t)\|_*^q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$(5.5) \quad |u^N(t)|_2^2 + C_{p,N}\int_0^T \Big(\|u^N(s)\|_{3,2}^2 + \|u^N(s)\|_2^2 + \|u^N(s)\|_{1,p}^p\Big)ds \leq |u_0^N|_2^2 + k_1\int_0^T \|f(s)\|_*^q ds,$$

where $C_{p,N} = \min\Big\{2N^{-1}, \frac{2c_2\nu_1}{c_0^2}, \frac{c_2\nu_2}{\tilde{c}_0^p}\Big\}$. Thus, we have

$$|u^N(t)|_2^2 \leq |u_0^N|_2^2 + k_1\int_0^T \|f(s)\|_*^q ds.$$

Then, it follows from the uniform boundedness principle that the sequence $\{u^N\}_N$ is uniformly bounded, with respect to N, in $L^\infty(0,T;H)$ and $L^p(0,T;V_p)$ and $\{N^{-1}u^N\}$ is uniformly bounded in $L^2(0,T;V^3)$. Therefore, there exists a subsequence of $\{u^N\}_N$ and $u \in L^\infty(0,T;H)\cap L^p(0,T;V_p)$ such that

$$(5.6) \qquad \begin{array}{lll} u^N \overset{*}{\rightharpoonup} u & \text{in} & L^\infty(0,T;H),\\ u^N \rightharpoonup u & \text{in} & L^p(0,T;V_p)\\ N^{-1}u^N \rightharpoonup 0 & \text{in} & L^2(0,T;V^3). \end{array}$$

Now, in the same way that (3.10), the sequence $\left\{\frac{\partial u^N}{\partial t}\right\}$ is uniformly bounded in $\Big[L^p(0,T;V_p)\cap L^2(0,T;V^3)\Big]^*$, with $p\geq 1+2n/(n+2)$. Then, we obtain

$$\begin{aligned} &u^N \to u \quad \text{in } \ L^2(0,T;H), \text{ a.e. in } \ H \text{ and } \Omega\times(0,T), \\ &\tfrac{\partial u^N}{\partial t} \rightharpoonup \tfrac{\partial u}{\partial t} \quad \text{in } \ \Big[L^p(0,T;V_p)\cap L^2(0,T;V^3)\Big]^*. \end{aligned} \tag{5.7}$$

Now, from (5.5) we have that $\int_0^T \|u^N(\theta)\|_{1,2}^2 d\theta$ is uniformly bounded. Then, from [5, Lemma 12] it follows that

$$F_N(\|u^N(s)\|_{1,2}) \overset{N\to\infty}{\to} 1 \quad \text{in } L^r(0,T;\mathbb{R}), \quad \text{for any } r\geq 1. \tag{5.8}$$

Continuing with the proof, reasoning as in the theorem of the existence of weak solutions to **(LM)**, we have

$$\int_0^t b(u^N(\theta),u^N(\theta),w)d\theta \to \int_0^t b(u(\theta),u(\theta),w)d\theta, \tag{5.9}$$

for all $t\in[0,T]$ and for all $w\in\mathcal{V}$. Thus, our purpose is to prove that

$$\int_0^t F_N(\|u^N(\theta)\|)b(u^N(\theta),u^N(\theta),w)d\theta \to \int_0^t b(u(\theta),u(\theta),w)d\theta, \tag{5.10}$$

for all $t\in[0,T]$ and for all $w\in\mathcal{V}$. Therefore, observe that

$$\begin{aligned} &\int_0^T \Big[F_N(\|u^N(\theta)\|)b(u^N(\theta),u^N(\theta),w) - b(u(\theta),u(\theta),w)\Big]d\theta = \\ &= \int_0^T \Big[F_N(\|u^N(\theta)\|)-1\Big]b(u^N(\theta),u^N(\theta),w)d\theta + \int_0^T \Big[b(u^N(\theta),u^N(\theta),w) - b(u(\theta),u(\theta),w)\Big]d\theta \\ &= \mathbb{I}_1+\mathbb{I}_2. \end{aligned}$$

Note that, by (5.9), we have that $\mathbb{I}_2\to 0$ as $N\to\infty$. Now, for $\mathbb{I}_1$

$$\begin{aligned} \mathbb{I}_1 &= \int_0^T \Big[F_N(\|u^N(\theta)\|)-1\Big]b(u^N(\theta),u^N(\theta),w)d\theta \\ &\leq \left\{\int_0^T \Big|F_N(\|u^N(\theta)\|)-1\Big|^p d\theta\right\}^{1/p}\left\{\int_0^T |b(u^N(\theta),u^N(\theta),w)|^q d\theta\right\}^{1/q} \\ &\leq \widetilde{C}(u^N)\|w\|_{1,p}\left\{\int_0^T \Big|F_N(\|u^N(\theta)\|)-1\Big|^p d\theta\right\}^{1/p}\left\{\int_0^T \Big[1+\|u^N\|_{1,p}^{p-1}\Big]^q d\theta\right\}^{1/q}. \end{aligned}$$

By the boundedness of the sequence $\{u^N\}$ in $L^\infty(\tau,T;H)$ and $L^p(\tau,T;V_p)$, it follows that $\widetilde{C}(u^N)<\infty$ and $\int_0^T\Big[1+\|u^N\|_{1,p}^{p-1}\Big]^q d\theta<\infty$, where $q=\frac{p}{p-1}$. By (5.8), $\int_0^T\Big|F_N(\|u^N(\theta)\|)-1\Big|^p d\theta\to 0$ as $N\to\infty$. Then, we have that $\mathbb{I}_1\to 0$ for all $w\in\mathcal{V}$. Therefore, we conclude that (5.10) holds.

Then, it follows from (5.6), (5.7) and (5.10) that

$$(u(t),w)+\int_0^t\langle\mathbb{T}(u(\theta)),w\rangle d\theta+\int_0^t b(u(\theta),u(\theta),w)d\theta = (u_0,w)+\int_0^t\langle f(\theta),w\rangle d\theta, \tag{5.11}$$

for all $t\in[0,T]$ and for all $w\in\mathcal{V}$. Then, by density argument, we conclude that u is a weak solution to **(LM)**. Furthermore, it follows from the variational formulation (5.11) that $\frac{\partial u}{\partial t}\in L^q(0,T;V_p^*)$. □

Lemma 5.4. *Given $T>0$, $u_0\in H$, and $f\in L^q(0,T;V_p^*)$. Let $\{u_0^N\}\subset H$ be a strongly convergent sequence to u_0 in H. Let $u^N(t)$ be the weak solution to* **(MLM)** *associated to the initial data $u_0^N\in H$. Then, if $p\geq 1+2n/(n+2)$, there exist a subsequence of $\{u^N\}$ (relabeled the seme) and u weak solution to* **(LM)***, such that*

$$u^N(s)\to u(s) \quad \textit{strongly in } H \textit{ for any } s\geq 0. \tag{5.12}$$

Proof. It follows from the Theorem 5.3 that there exists u solution of **(LM)**, with $u \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$ and $\frac{\partial u}{\partial t} \in L^q(0,T;V_p^*)$, such that the sequence of weak solutions $\{u^N\}$ of **(MLM)** (up to a subsequence) converges to u in the sense specified in (5.4).

Now, observe that $\{u^N\}$ is equicontinuous in $(V^3)^*$ on $[0,T]$ (in the same way that (5.3)) and that $\{u^N\}$ is bounded in $C([0,T];H)$. Therefore, by the Arzelà-Ascoli Theorem (up to a subsequence) we have that

$$u^N \to u \quad \text{strongly in } C([0,T];(V^3)^*).$$

Since $\{u^N\}$ is uniformly bounded in $C([\tau,T];H)$ we get

$$u^N(s) \rightharpoonup u(s) \quad \text{weakly in } H \text{ for any } 0 \le s \le T. \tag{5.13}$$

Now, since the estimate

$$|z(r)|_2^2 \le |z(s)|_2^2 + 2\int_s^r \langle f(\theta), z(\theta)\rangle d\theta, \quad 0 \le s \le r \le T.$$

holds for $z = u$ and $z = u^N$. Thus, the functions $J_N, J : [0,T] \to \mathbb{R}$ defined by

$$J_N(r) = |u^N(r)|_2^2 - 2\int_0^r \langle f(\theta), u^N(\theta)\rangle d\theta,$$
$$J(r) = |u(r)|_2^2 - 2\int_0^r \langle f(\theta), u(\theta)\rangle d\theta,$$

are non-increasing and continuous functions, and by (5.4), we have that

$$J_N(r) \to J(r) \quad \text{a.e. } r \in (0,T).$$

We affirm that $J_N(r) \to J(r)$ for any $r \in [0,T]$: Indeed, let us consider a fixed $t^* \in (0,T]$ and an increasing sequence $t_m \to t^*$ such that $\lim_{N\to\infty} J_N(t_m) = J(t_m)$ for all $m \ge 1$. Thus, given $\epsilon > 0$ there exist M, $K > 0$ such that

$$|J(t_m) - J(t^*)| \le \frac{\epsilon}{2} \quad \text{for } m \ge K, \quad \text{and} \quad |J_N(t_K) - J(t_K)| \le \frac{\epsilon}{2} \quad \text{for } N \ge M.$$

Since J_N is a non-increasing function, we have that

$$J_N(t^*) - J(t^*) \le |J_N(t_K) - J(t_K)| + |J(t_K) - J(t^*)| \le \epsilon$$

for all $N \ge M$. Then, we get $\limsup_{N\to\infty} J_N(t^*) \le J(t^*)$. Taking into account that

$$\int_0^{t^*} \langle f(\theta), u^N(\theta)\rangle d\theta \to \int_0^{t^*} \langle f(\theta), u(\theta)\rangle d\theta,$$

we deduce that $\limsup_{N\to\infty} |u^N(t^*)| \le |u(t^*)|$. Thus, it follows from (5.13), that

$$\limsup |u^N(t^*) - u(t^*)|_2^2 = \limsup |u^N(t^*)|_2^2 - 2\liminf (u^N(t^*), u(t^*)) + \limsup |u(t^*)|_2^2$$
$$\le |u(t^*)|_2^2 - 2|u(t^*)|_2^2 + |u(t^*)|_2^2 = 0.$$

Then, we conclude that (5.12) holds for all $s \in [0,T]$. □

6. Existence of Global Attractors

In this section, we study the dynamics of the solutions of the modified system **(MLM)**, showing the existence and regularity of global attractors. For this, let us consider $p \ge 2$ and the external force $f \in L^2(\Omega)^n$. Then, given $N > 0$, fixed, the existence of weak solutions is guaranteed thanks to Theorem 3.7. Thus, let us define the following single-valued map on H by

$$\mathcal{S}^N(\cdot) : \mathbb{R}_+ \times H \to H \quad \text{with} \quad \mathcal{S}^N(t)u_0 = u^N(t;u_0), \tag{6.1}$$

where $u^N(t) = u^N(t;u_0)$ is the unique weak solution to **(MLM)** associated to the initial condition $u_0 \in H$.

Thus, it follows from Theorem 3.4 and Theorem 3.7 that the map $\mathcal{S}^N(\cdot)$ is well defined and is a continuous semigroup on H, i.e.

(1) $\mathcal{S}^N(0) = Id_H$,
(2) $\mathcal{S}^N(t+s) = \mathcal{S}^N(t) \circ \mathcal{S}^N(s)$ for all $t \geq s \geq 0$,
(3) For all $t \in \mathbb{R}_+$, the mapping $\mathcal{S}^N(t) : H \to H$ is continuous.

Theorem 6.1. *Let $p \geq 2$, $f \in L^2(\Omega)^n$ and $u_0 \in H$. Let u^N be the unique weak solution of* **(MLM)** *associated to the initial condition u_0. Then u^N satisfies*

$$|u^N(t)|_2 \leq \mathcal{R}(t) \qquad \forall t \geq 0,$$

and

$$\frac{2}{N}\int_{t-1}^{t} \|u^N(s)\|_{3,2}^2 ds + \kappa_0 \int_{t-1}^{t+1} \|u^N(s)\|_{1,2}^2 ds + \kappa_p \int_{t-1}^{t+1} \|u^N(s)\|_{1,p}^p ds \leq \mathcal{R}^2(t-1) + c_{\lambda_1}^{-1}|f|_2^2,$$

for all $t \geq 1$, where $c_{\lambda_1} = \frac{c_2\nu_1\lambda_1}{c_0^2}$, $\kappa_0 = \frac{2c_2\nu_1}{c_0^2}$, $\kappa_p = \frac{2c_2\nu_2}{\tilde{c}_0^p}$ and $\mathcal{R}^2(t) = e^{-c_{\lambda_1}t}|u_0|_2^2 + c_{\lambda_1}^{-2}|f|_2^2$.

Proof. It follows from (3.6) that u^N satisfies

$$\frac{d}{dt}|u^N|_2^2 + \frac{2}{N}\|u^N\|_{3,2}^2 + \frac{c_2\nu_1\lambda_1}{c_0^2}|u^N|_2^2 + \frac{2c_2\nu_2}{\tilde{c}_0^p}\|u^N\|_{1,p}^p \leq \frac{c_0^2}{c_2\nu_1\lambda_1}|f|_2^2, \tag{6.2}$$

where λ_1 is the first eigenvalue of $-\Delta$. Denote by $c_{\lambda_1} = \frac{c_2\nu_1\lambda_1}{c_0^2}$. Then (6.2) reduces to

$$\frac{d}{dt}|u^N|_2^2 + c_{\lambda_1}|u|_2^2 + \frac{2}{N}\|u^N\|_{3,2}^2 + \frac{2c_2\nu_2}{\tilde{c}_0^p}\|u^N\|_{1,p}^p \leq c_{\lambda_1}^{-1}\|f\|_2^2.$$

Multiplying by $e^{c_{\lambda_1}t}$,

$$\frac{d}{dt}\Big(e^{c_{\lambda_1}t}|u^N|_2^2\Big) + \frac{e^{c_{\lambda_1}t}}{N}\|u^N\|_{3,2}^2 + \frac{2c_2\nu_2}{\tilde{c}_0^p}e^{c_{\lambda_1}t}\|u^N\|_{1,p}^p \leq c_{\lambda_1}^{-1}e^{c_{\lambda_1}t}|f|_2^2.$$

Therefore, we deduce that

$$|u^N(t)|_2^2 \leq e^{-c_{\lambda_1}t}|u_0|_2^2 + c_{\lambda_1}^{-2}|f|_2^2.$$

□

Corollary 6.2. *The semigroup $\mathcal{S}^N(\cdot) : \mathbb{R}_+ \times H \to H$ is dissipative.*

Proof. Let us consider $\mathcal{B}_H = \Big\{v \in H : |v|_2 \leq 1 + c_{\lambda_1}^{-2}|f|_2^2\Big\}$. Let D be a bounded subset of H. It follows from Theorem 6.1 that there exists $t_0(D) > 0$ such that

$$e^{-c_{\lambda_1}t}|u_0|_2^2 < 1,$$

for all $t \geq t_0(D)$ and uniformly for any $u_0 \in D$. Thus, we obtain the following inequality

$$|\mathcal{S}^N(t)u_0|_2 \leq \varrho_0 \qquad \text{for all } t \geq t_0(D) \text{ and } u_0 \in D,$$

where $\varrho_0^2 = 1 + c_{\lambda_1}^{-2}|f|_2^2$. Then we conclude that

$$\mathcal{S}^N(t)D \subset \mathcal{B}_H \quad \text{for all} \quad t \geq t_0(D).$$

□

Remark 6.3. *It follows from Theorem 6.1 and Corollary 6.2 that, given any bounded subset $D \subset H$, there exists $t_0 = t_0(D) \geq 1$ such that*

$$\int_{t-1}^{t+1} \|u(s)\|_{3,2}^2 ds + \int_{t-1}^{t+1} \|u(s)\|_{1,2}^2 ds + \int_{t-1}^{t+1} \|u(s)\|_{1,p}^p ds \leq \varrho_1 \qquad \textit{for all } t \geq t_0(D),$$

where $\varrho_1 = \frac{\varrho_0^2 + c_{\lambda_1}^{-1}|f|_2^2}{\min\{2N^{-1}, \kappa_0, \kappa_p\}}$.

Theorem 6.4. *Consider $p \geq 2$, $N > 0$ and $f \in L^2(\Omega)^n$. Then, given a bounded subset $D \subset H$, there exist $t_0 = t_0(D) > 0$ and positive constants $\varrho_2, \varrho_3, \varrho_4$ and ϱ_5, which depend on p, N and the norm $|f|_2$, such that any weak solution $u^N(t) = \mathcal{S}^N(t)u \in \mathcal{S}^N(t)D$, satisfies*

$$\begin{cases} \displaystyle\int_{t-1}^{t+1} \left|\frac{\partial u^N}{\partial t}(s)\right|_2^2 ds \leq \varrho_2; \\ \displaystyle\left|\frac{\partial u^N}{\partial t}(t)\right|_2 \leq \varrho_3; \\ \displaystyle\int_t^{t+1} \left\|\frac{\partial u^N}{\partial t}(\theta)\right\|_{1,2}^2 d\theta \leq \varrho_4; \\ \|u^N(t)\|_{3,2}^2 + \|u^N(t)\|_{1,p}^p \leq \varrho_5, \end{cases} \tag{6.3}$$

for all $t \geq t_0(D) + 1$.

Proof. It follows from Theorem 4.1 and Remark 6.3, that there exist $\varrho_2 > 0$ and $t_0 = t_0(D) > 1$ such that

$$\int_{t-1}^{t+1} \left|\frac{\partial u^N}{\partial t}(s)\right|_2^2 ds \leq \varrho_2 \quad \forall t \geq t_0(D) + 1. \tag{6.4}$$

On the other hand, by (4.9), we obtain that

$$\left|\frac{\partial u^N}{\partial t}(t)\right|_2^2 \leq \left|\frac{\partial u^N}{\partial t}(s)\right|_2^2 + \tilde{c}_2 N^2 \int_s^t \left|\frac{\partial u_m}{\partial t}(\theta)\right|_2^2 d\theta.$$

Therefore, integrating in s, from $t-1$ to t, we get

$$\left|\frac{\partial u^N}{\partial t}(t)\right|_2^2 \leq [1 + \tilde{c}_2 N^2] \int_{t-1}^t \left|\frac{\partial u^N}{\partial t}(\theta)\right|_2^2 d\theta.$$

Thus, it follows from (6.4), that

$$\left|\frac{\partial u^N}{\partial t}(t)\right|_2 \leq \varrho_3 \quad \forall t \geq t_0(D) + 1, \tag{6.5}$$

where $\varrho_3^2 = [1 + \tilde{c}_2 N^2]\varrho_2$. Again, by (4.9) we obtain that

$$\tilde{c}_2 \int_s^{t+1} \left\|\frac{\partial u^N}{\partial t}(\theta)\right\|_{1,2}^2 d\theta ds \leq \left|\frac{\partial u^N}{\partial t}(s)\right|_2^2 + \tilde{c}_2 N^2 \int_s^{t+1} \left|\frac{\partial u^N}{\partial t}(\theta)\right|_2^2 d\theta.$$

Now, integrating in s, from $t-1$ to $t+1$, we have

$$\int_t^{t+1} \left\|\frac{\partial u^N}{\partial t}(\theta)\right\|_{1,2}^2 d\theta \leq \int_{t-1}^{t+1}\int_s^{t+1} \left\|\frac{\partial u^N}{\partial t}(\theta)\right\|_{1,2}^2 d\theta ds \leq \frac{1 + 2\tilde{c}_2 N^2}{\tilde{c}_2} \int_{t-1}^{t+1} \left|\frac{\partial u^N}{\partial t}(\theta)\right|_2^2 d\theta.$$

Thus, it follows from (6.4) that

$$\int_t^{t+1} \left\|\frac{\partial u^N}{\partial t}(\theta)\right\|_{1,2}^2 d\theta \leq \varrho_4 \quad \forall t \geq t_0(D) + 1, \tag{6.6}$$

where $\varrho_4 = \frac{(1+2\tilde{c}_2 N^2)\varrho_2}{\tilde{c}_2}$. On the other hand, in the same way that (4.11), we obtain that

$$\begin{aligned} \frac{1}{2N}\|u^N(t)\|_{3,2}^2 + \|\Phi(Du^N(t))\|_1 \leq \int_{t-1}^t \Big(\frac{1}{2N}\|u^N(s)\|_{3,2}^2 + \|\Phi(Du^N(s))\|_1\Big) ds \\ + \frac{1}{2}\int_{t-1}^t \Big(|f|_2^2 + \|u^N(\theta)\|_{1,2}^2 + \left\|\frac{\partial u^N}{\partial t}(\theta)\right\|_{1,2}^2\Big) d\theta. \end{aligned}$$

Thus it follows from (4.5), Theorem 6.1, Remark 6.3, and (6.6) that

$$\|u^N(t)\|_{3,2}^2 + \|u^N(t)\|_{1,p}^p \leq \varrho_5 \quad \forall t \geq t_0(D) + 1, \tag{6.7}$$

where $\varrho_5 = \max\{\frac{1}{2N}, \frac{c_9}{c_8}, \frac{1}{2c_8}\}\Big(1 + |f|_2^2 + \varrho_1^2 + \varrho_4\Big)$. □

Remark 6.5. *If we denote by $\mathcal{B}_{V_p} = \{v \in V_p : \|v\|_{1,p} \leq \varrho_5\}$. Then, it follows from Theorem 6.4 that given a bonded subset $D \subset H$, there exists $t_0 = t_0(D) > 1$, such that*

$$\mathcal{S}^N(t)D \subset \mathcal{B}_{V_p} \quad \forall t \geq t_0(D).$$

Corollary 6.6. *The semigroup $\mathcal{S}^N(\cdot) : \mathbb{R}_+ \times H \to H$ is asymptotically compact.*

Proof. Let us consider the bounded sequence $\{x_m\}$ in H and $t_m \to \infty$. We will prove that the sequence $\mathcal{S}(t_m)x_m$ has a limit point in H. In fact, by Remark 6.5 there exists $m_0 \in \mathbb{N}$ such that the sequence $\{\mathcal{S}^N(t_m)x_m\}$ is contained in $\mathcal{B}_{V_p}$ for all $m \geq m_0$. Since $\mathcal{B}_{V_p}$ is a compact subset of H. Therefore the sequence $\{\mathcal{S}^N(t_m)x_m\}$ has a convergent subsequence at H. □

Theorem 6.7. *Let $p \geq 2$, $f \in L^2(\Omega)^n$ and $N > 0$. Then, the dynamical system $\big(\mathcal{S}^N(\cdot), H\big)$ has a global attractor $\mathcal{A}_H^N$. Moreover the set-valued mapping $N \longmapsto \mathcal{A}_H^N$ is upper semi-continuous, i.e.*

$$\operatorname{dist}_H\left(\mathcal{A}_H^M, \mathcal{A}_H^N\right) \to 0 \quad as \quad M \to N, \tag{6.8}$$

where dist_H *is the Hausdorff semidistance on* H.

Proof. The asymptotic compactness of semigroup $\mathcal{S}^N(\cdot)$ is given by Collorary 6.6 and the existence of an absorbent set follows from Collorary 6.2. Therefore, the existence of a global attractor for the semigroup $\mathcal{S}^N(\cdot)$ follows from Theorem 1.1. Finally, (6.8) follows from (3.3). □

6.1. **Regularity of the global attractor.** Now we study the dynamical behavior of the solutions of system **(MLM)** in the Hilbert space V^3, in the same way, it can be studied for the spaces V_2, V_p.

Proposition 6.8. *(cf.* [1, Theorem 1.6, pg. 21]*) Let X and Y be Banach spaces, with X reflexive and $X \hookrightarrow Y$. If $u \in L^\infty(\tau, T; X) \cap C_w([\tau, T], Y)$, then $u \in C_w([\tau, T], X)$ and $u(t)$ belongs to X for all $t \in [\tau, T]$.*

For this purpose, we are going to restrict the semigroup $\mathcal{S}^N(\cdot)$, defined by (6.1), to the Hilbert space V^3. Namely, by Theorem 5.2, all solutions to problem **(MLM)**, $u^N(t) = u^N(t; u_0)$, belong to the space $L^\infty(\varepsilon, t; V^3) \cap C([0, t], H)$, for any $\varepsilon > 0$ and for all $t \geq \varepsilon$. Then, it follows from Proposition 6.8 that the single-valued map $\mathcal{S}^N(\cdot)$ is well defined on V^3. Thus,

$$\mathcal{S}^N(t)u_0 = u^N(t; u_0) \quad \text{with} \quad \mathcal{S}^N(\cdot) : \mathbb{R}_+ \times V^3 \to V^3, \tag{6.9}$$

where $u^N(t) = u^N(t; u_0)$ is the unique solution to (3.6) associated to the initial condition $u_0 \in V^3$. Moreover, $\mathcal{S}^N(\cdot)$ is a semigroup on V^3, i.e.

(1) $\mathcal{S}^N(0) = \mathrm{Id}_{V^3}$,
(2) $\mathcal{S}^N(t + s) = \mathcal{S}^N(t) \circ \mathcal{S}^N(s)$ for all $t \geq s \geq 0$.

Theorem 6.9. *The semigroup $\mathcal{S}^N(\cdot) : \mathbb{R}_+ \times V^3 \to V^3$ is a closed semigroup on V^3.*

Proof. Let us consider the sequence $\{u_m\}$ that converges to u in V^3, and suppose that $\mathcal{S}^N(t)u_m \to v$ in V^3. Then, by Theorem 3.4 we know that $\mathcal{S}^N(t)u_m \to \mathcal{S}^N(t)u$ in H. Therefore, from the uniqueness of the limit it follows that $v = \mathcal{S}^N(t)u$. □

Corollary 6.10. *The semigroup $\mathcal{S}^N(\cdot) : \mathbb{R}_+ \times V^3 \to V^3$ is dissipative.*

Proof. Let us consider $\mathcal{B}_{V^3} = \{u \in V^3 : \|v\|_{3,2}^2 \leq \varrho_5\}$. Let D be a bounded subset of V^3. It follows from Theorem 6.4 that there exists $t_0 = t_0(D) > 0$ such that

$$\mathcal{S}^N(t)D \subset \mathcal{B}_{V^3} \quad \text{for all} \quad t \geq t_0(D).$$

□

Lemma 6.11. *The semigroup $\mathcal{S}^N(\cdot) : \mathbb{R}_+ \times V^3 \to V^3$ is asymptotically compact.*

Proof. Given a bounded sequence $\{u_m\}$ in V^3 and $t_m \to +\infty$ as $m \to \infty$. We will prove that the sequence $\{\mathcal{S}^N(t_m)u_m\}$ is relatively compact in V^3. Let us denote by $u_m = \mathcal{S}^N(t_m)u_m$. Observe that, by inequality (6.5) the sequence $\{\frac{\partial u_m}{\partial t}(t_m)\}$ is bounded in H, i.e.

$$\Big|\frac{\partial u_m}{\partial t}(t_m)\Big|_2 \le \varrho_3. \tag{6.10}$$

Besides this, by (6.7) the sequence $\{u_m\}$ is bounded in V_2, V_p and V^3.

Since the semigroup $\mathcal{S}^N(\cdot)$ is asymptotically compact in H, without loss of generality we can suppose that the sequence $\{u_m\}$ is a Cauchy sequence in H. Thus, from the p-coercivity of $\mathbb{S}$ given in (1.1), the Korn inequality and the Theorem 3.1, we have that

$$\begin{aligned}\frac{1}{N}\|u_m - u_k\|_{3,2}^2 + \frac{c_2\nu_1}{c_0^2}\|u_m - u_k\|_{1,2}^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|u_m - u_k\|_{1,p}^p \le \frac{1}{N}\|u_m - u_k\|_{3,2}^2 \\ + (\mathbb{S}(Du_m) - \mathbb{S}(Du_k), Du_m - Du_k) \\ = -\Big(\frac{\partial u_m}{\partial t} - \frac{\partial u_k}{\partial t}, u_m - u_k\Big) - \Big\langle B_N(u_m, u_m) - B_N(u_k, u_k), u_m - u_k\Big\rangle \\ \le \Big|\frac{\partial u_m}{\partial t} - \frac{\partial u_k}{\partial t}\Big|_2 |u_m - u_k|_2 + (1+d)N\|u_m - u_k\|_{1,2}^{3/2}|u_m - u_k|_2^{1/2} \\ \le 2\varrho_3|u_m - u_k|_2 + 2^{3/2}(1+d)N\varrho_5^3|u_m - u_k|_2^{1/2}.\end{aligned}$$

Therefore, the sequence $\{u_m\}$ is also a Cauchy sequence in V^3, so we conclude that the semigroup $\mathcal{S}^N(\cdot) : \mathbb{R}_+ \times V^3 \to V^3$ is asymptotically compact in V^3 (it is also in V_2 and V_p). □

Theorem 6.12. *Let $p \ge 2$, $f \in L^2(\Omega)^n$ and $N > 0$. Then the dynamical system $(\mathcal{S}^N(\cdot), V^3)$ ($(\mathcal{S}^N(\cdot), V_p)$ and $(\mathcal{S}^N(\cdot), V_2)$) has a global attractor $\mathcal{A}^N_{V^3}$ in V^3 ($\mathcal{A}^N_{V_p}$ in V_p and $\mathcal{A}^N_{V_2}$ in V_2). Moreover*

$$\mathcal{A}^N_{V^3} = \mathcal{A}^N_{V_p} = \mathcal{A}^N_{V_2} = \mathcal{A}^N_H. \tag{6.11}$$

Proof. The asymptotic compactness of semigroup $\mathcal{S}^N(\cdot)$ is given by Lemma 6.11 and the existence of an absorbing set follows from Collorary 6.10. Therefore, the existence of a global attractor for semigroup $\mathcal{S}^N(\cdot)$ in V^3, V_p and V_2 follows from Theorem 1.1. The equality given in (6.11) follows from the uniqueness of global attractors. □

7. Finite Fractal Dimension

Our aim in this last section is to prove that the global attractors, associated with the dynamical system $(\mathcal{S}^N(\cdot), H)$, have finite fractal dimension in the Hilbert spaces H, V^3 and the Banach space V_p for any $N > 0$ and $p \ge 2$, fixed. For this we will use the ℓ-trajectories method, cf. [15, Lemma 1.3] or [6].

Definition 7.1. *Let $\mathcal{X}$ be a metric space and $\mathscr{C}$ a compact subset of $\mathcal{X}$. The fractal dimension of $\mathscr{C}$ (also called the "upper box-counting dimension", e.g.* [4]*[Ch. 4]) is defined by*

$$d_f^{\mathcal{X}}(\mathscr{C}) = \limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}^{\mathcal{X}}[\mathscr{C}; \varepsilon]}{-\log \varepsilon},$$

where $\mathcal{N}^{\mathcal{X}}[\mathscr{C}; \varepsilon]$ is the minimum number of balls of radius ε, centered at some point of $\mathscr{C}$, that cover $\mathscr{C}$.

Lemma 7.2. *(cf.* [15, Lemma 1.3]*) Let $\mathcal{X}$, $\mathcal{Y}$ be two normed spaces such that $\mathcal{Y} \hookrightarrow\hookrightarrow \mathcal{X}$ and $\mathscr{C} \subset \mathcal{X}$ be bounded. Assume that there exists a mapping $\mathscr{L} : \mathcal{X} \to \mathcal{Y}$ such that $\mathscr{C} \subset \mathscr{L}\mathscr{C}$ and*

$$\|\mathscr{L}(x) - \mathscr{L}(y)\|_{\mathcal{Y}} \le \kappa \|x - y\|_{\mathcal{X}} \quad \forall x, y \in \mathscr{C},$$

where $\kappa > 0$. Then, the fractal dimension of $\mathscr{C}$ is finite and

$$d_f^{\mathcal{X}}(\mathscr{C}) \le \frac{\log \mathcal{N}_{1/4\kappa}}{\log 2},$$

where $\mathcal{N}_{1/4\kappa} := \mathcal{N}^{\mathcal{X}}[B_{\mathcal{Y}}(0,1); 1/4\kappa]$, *and* $B_{\mathcal{Y}}(0,1)$ *is the unit ball in* $\mathcal{Y}$.

Proposition 7.3. *(cf.* [4, Lemma 4.2]*) Let* $\mathcal{X}$, $\mathcal{Y}$ *be two normed spaces. Consider* $\mathscr{C} \subset \mathcal{X}$, *and* $f : \mathscr{C} \to \mathcal{Y}$ *is Hölder continuous with exponent* θ $(0 < \theta \le 1)$, *i.e. there exists an* $L > 0$ *such that*

$$\|f(x) - f(y)\|_{\mathcal{Y}} \le L\|x - y\|_{\mathcal{X}}^{\theta},$$

for all $x, y \in \mathscr{C}$. *Then* $d_f^{\mathcal{Y}}(f(\mathscr{C})) \le d_f^{\mathcal{X}}(\mathscr{C})/\theta$.

Theorem 7.4. *(Mañé Theorem cf.* [20, Lemma 1.1]*) If* $\mathcal{X}$ *is a Banach space on* $\mathbb{R}$, $K = \bigcup_{n=1}^{\infty} K_n$ *with* K_n *being a compact set for all* $n \in \mathbb{N}$, $d_{\mathcal{H}}(K - K) < \infty$, *where* $d_{\mathcal{H}}$ *is the Hausdorff dimension, and* Y *is a subspace of* $\mathcal{X}$ *with* $d_{\mathcal{H}}(K - K) + 1 < d_f(Y) < \infty$, *then the set* $\{P \in \mathcal{P}(\mathcal{X}, Y) : P|_K$ *is injective* $\}$ *is residual in* $\mathcal{P}(\mathcal{X}, Y)$.

Before showing the finitude of the fractal dimension of the attractors given in the Theorem 6.12, we will show the following lemma that will allow us to estimate the fractal dimension via the ℓ-trajectories method. Moreover, we will indicate by u' the partial derivative of u with respect to t, i.e. $\frac{\partial u}{\partial t}$, since it is an easier notation to manipulate.

Lemma 7.5. *Let* $p \ge 2$, $N > 0$, *and* $f \in L^2(\Omega)^n$. *Given* $T > 0$ *and two weak solutions* u *and* v *of* **(MLM)**. *Then, for any* $t > 0$ *we have that*

$$\int_t^{T+t} \Big[\|u(s) - v(s)\|_{3,2}^2 + \|u(s) - v(s)\|_{1,2}^2 + I^2(u(s), v(s))\Big] ds \le \frac{c_9(t + T, N)}{\min\{t, T\}} \int_0^T |u(s) - v(s)|_2^2 ds,$$

and

$$\|u'(t + \cdot) - v'(t + \cdot)\|_{L^2(0,T;(V^3)^*)}^2 \le \int_t^{T+t} \Big[\frac{1}{N}\|u(s) - v(s)\|_{3,2}^2 + k_1[u(s), v(s)] I^2(u(s), v(s)) + k_2[u(s), v(s)]\|u(s) - v(s)\|_{1,2}^2\Big] ds,$$

where $k_1[u, v] = \Big[\int_\Omega \Big(\nu_1 + \nu_2(|Du| + |Dv|)\Big)^p dx\Big]^{\frac{p-2}{p}}$ *and* $k_2[u, v] = \big(N + 2\|v\|_{1,2}\big)^2$.

Proof. Denote by $I^2(u, v) := \int_\Omega \Big(\nu_1 + \nu_2(|Du| + |Dv|)^{p-2}\Big)|Dw|^2 dx$, where $w := u - v$. Thus, we know that w satisfies

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + \frac{1}{N}\|w\|_{3,2}^2 + \langle \mathbb{T}(u) - \mathbb{T}(v), w\rangle + \langle B_N(u) - B_N(v), w\rangle = 0. \tag{7.1}$$

It follows from (1.1), and the Korn inequality that, there is $c_7 > 0$ such that

$$c_7\|\nabla w\|_2^2 + \frac{1}{2} I^2(u, v) \le \langle \mathbb{T}(u) - \mathbb{T}(v), w\rangle.$$

Then, we deduce that

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + \frac{1}{N}\|w\|_{3,2}^2 + c_7\|\nabla w\|_2^2 + \frac{1}{2} I^2(u, v) \le |\langle B_N(u) - B_N(v), w\rangle|. \tag{7.2}$$

Now, it follows from Theorem 3.1 that there exists $c_8(N) > 0$, such that

$$|\langle B_N(u) - B_N(v), w\rangle| \le \frac{c_7}{2}\|w\|_{1,2}^2 + \frac{c_8(N)}{2}|w|_2^2.$$

Thus, substituting in (7.2), we obtain that

$$\frac{d}{dt}|w|_2^2 + \frac{2}{N}\|w\|_{3,2}^2 + c_7\|w\|_{1,2}^2 + I^2(u, v) \le c_8(N)|w|_2^2. \tag{7.3}$$

Applying the Gronwall inequality, we get

$$|w(\tau)|_2^2 \le e^{c_8(N)(\tau - s)}|w(s)|_2^2, \quad \text{for all } \tau \ge s. \tag{7.4}$$

On the one hand, for $T > 0$ and $s \in [0, T]$, integrating (7.3) from s to $T + t$, we have

$$\int_s^{T+t} \Big[\frac{2}{N}\|w(\tau)\|_{3,2}^2 + c_7\|w(\tau)\|_{1,2}^2 + I^2(u(\tau), v(\tau))\Big] d\tau \le |w(s)|_2^2 + c_8(N)\int_s^{T+t} |w(\tau)|_2^2 d\tau.$$

Thus using (7.4), we deduce that

$$\min\{t, T\}\int_t^{T+t} \Big[\|w(\tau)\|_{3,2}^2 + \|w(\tau)\|_{1,2}^2 + I^2(u(\tau), v(\tau))\Big] d\tau \le c_9(t + T, N)\int_0^T |w(s)|_2^2 ds,$$

where $c_9(t + T, N) = (1 + c_8(N)e^{c_8(N)(T+t)})/\min\{2/N, c_7, 1\}$.

Now, let us consider $\varphi \in L^2(0, T; (V^3)^*)$ with $\|\varphi\|_{L^2(0,T;(V^3)^*)} \le 1$. For simplicity, in the following estimates, we will omit the time variable, which is $t + s$ with $s \in (0, T)$. Then, by (1.1) it follows that

$$\begin{aligned}\int_0^T\int_\Omega \Big(\mathbb{S}(Du) - \mathbb{S}(Dv)\Big) : \nabla\varphi dx ds &\le \int_0^T\int_\Omega |\mathbb{S}(Du) - \mathbb{S}(Dv)||\nabla\varphi| dx ds\\
&\le \int_0^T I(u, v)\Big(\int_\Omega \Big(\nu_1 + \nu_2(|Du| + |Dv|)\Big)^p dx\Big)^{\frac{p-2}{2p}} \|\nabla\varphi\|_p ds\\
&\le \Big[\int_0^T I^2(u, v)\Big(\int_\Omega \Big(\nu_1 + \nu_2(|Du| + |Dv|)\Big)^p dx\Big)^{\frac{p-2}{p}} ds\Big]^{\frac12}\\
&= \Big(\int_0^T k_1[u(s), v(s)]I^2(u(s), v(s)) ds\Big)^{\frac12},\end{aligned}$$

where $k_1[u, v] = \Big[\int_\Omega \Big(\nu_1 + \nu_2(|Du| + |Dv|)\Big)^p dx\Big]^{\frac{p-2}{p}}$.

On the other hand, it follows from Theorem 3.1, that

$$\begin{aligned}\int_0^T \langle B_N u - B_N v, \varphi\rangle ds &\le \Big(\int_0^T \Big[N + 2\|v\|_{1,2}\Big]^2 \|w\|_{1,2}^2 ds\Big)^{\frac12}\Big(\int_0^T \|\varphi\|_{1,2}^2\Big)^{\frac12}\\
&\le \Big(\int_0^T k_2[u(s), v(s)]\|w(s)\|_{1,2}^2 ds\Big)^{\frac12},\end{aligned}$$

where $k_2[u, v] = [N + 2\|v\|_{1,2}]^2$. And finally

$$\int_0^T ((w(s), \varphi(s))) ds \le \Big(\int_0^T \|w(s)\|_{3,2}^2 ds\Big)^{1/2}\Big(\int_0^T \|\varphi(s)\|_{V^3}^2 ds\Big)^{1/2}.$$

Putting all the inequalities together, we arrive at

$$\|w'\|_{L^2(0,T;(V^3)^*)}^2 \le \frac{1}{N}\int_0^T \|w(s)\|_{3,2}^2 ds + \int_0^T k_1[u, v]I^2(u, v) ds + \int_0^T k_2[u, v]\|w\|_{1,2}^2 ds.$$

□

Remark 7.6. *Consider the set $\mathcal{B}_{0,H} = \{u_0 \in H : |u_0|_2 \le \eta\}$, where η is any positive real number with $\eta^2 > c_{\lambda_1}^{-2}|f|_2^2$. It is easy to prove that $\mathcal{B}_{0,H}$ is an absorbing set for the dynamical system $(\mathcal{S}^N(\cdot), H)$, and also $\mathcal{B}_{0,H}$ is a positively invariant set, i.e. $\mathcal{S}^N(t)\mathcal{B}_{0,H} \subset \mathcal{B}_{0,H}$ for all $t > 0$. From Theorem 6.4, there exists $t_0(\mathcal{B}_{0,H}) > 0$ such that $\|\mathcal{S}^N(t)u_0\|_{1,2} \le \varrho_5$, for all $t \ge t_0(\mathcal{B}_{0,H}) + 1$ and for any $u_0 \in \mathcal{B}_{0,H}$. Therefore, by Lemma 7.5 we have, in particular, that*

$$\begin{aligned}&\int_t^{T+t} \Big[\|u(s) - v(s)\|_{3,2}^2 + \|u(s) - v(s)\|_{1,2}^2 + I^2(u(s), v(s))\Big] ds\\
&\qquad\le \frac{c_9(T + t, N)}{\min\{T, t_0(\mathcal{B}_{0,H})\}}\int_0^T |u(s) - v(s)|_2^2 ds,\end{aligned}$$

for all $t \geq t_0(\mathcal{B}_{0,H})+1$*. And, again by Lemma 7.5 we have*

$$\|u'(t+\cdot)-v'(t+\cdot)\|_{L^2(0,T;(V^3)^*)} \leq c_{10}(T+t,N)\|u(\cdot)-v(\cdot)\|_{L^2(0,T;H)}, \tag{7.5}$$

for all $t \geq t_0(\mathcal{B}_{0,H})+1$*, where* $c_{10}(T+t,T)^2 = \varrho_6(N)c_9(N,t)/\min\{T,t_0(\mathcal{B}_{0,H})\}$*, and* $\varrho_6(N)$ *is such that* $\max\{1/N, k_1[u(t),v(t)], k_2[u(t),v(t)]\} \leq \varrho_6(N)$ *for all* $t \geq t_0(\mathcal{B}_{0,H})+1$*.*

As we mentioned at the beginning of this section, to estimate the fractal dimension of the attractors, given in Theorem 6.7 and Theorem 6.12, we will use the ℓ-trajectories method. Let us fix $N>0$ and $p\geq 2$, thus given $\ell>0$, by ℓ-trajectory we mean any solution on the interval $[0,\ell]$, and we will denote by $H_{\ell,N}$ the set of the ℓ-trajectories, i.e.

$$H_{\ell,N} := \{\gamma : [0,\ell] \to H : \gamma \text{ is a solution to } \textbf{(MLM)}\}.$$

On the other hand, let us denote by $\mathbf{H}_\ell := L^2(0,\ell;H)$ and $\mathbf{Z}_\ell := \{u \in L^2(0,\ell;V_2) : u' \in L^2(0,\ell;(V^3)^*)\}$. The set $H_{\ell,N}$ is equipped with the topology of $\mathbf{H}_\ell$. Note that $H_{\ell,N} \subset C([0,\ell];H)$, then it makes sense to evaluate at each point. Thus, thanks to Theorem 3.4, we have that for $\gamma \in H_{\ell,N}$ and $T>\ell$ there exists a unique solution u to **(MLM)** on $[0,T]$ such that $u|_{[0,\ell]}=\gamma$.

Now, for any $N>0$, we can define the map $\mathscr{L}_N : \mathbb{R}_+ \times H_{\ell,N} \to H_{\ell,N}$ by

$$\Big[\mathscr{L}_N(t)\gamma\Big](s) = u(t+s) \qquad s\in[0,\ell], \tag{7.6}$$

where u is the solution on $[0,\ell+t]$ such that $u|_{[0,\ell]}=\gamma$.

On the other hand, let us define the set of ℓ-trajectories starting at any point of $\mathcal{B}_{0,H}=\{u_0 \in H : |u_0|_2 \leq \eta\}$, which will be denoted by $\mathcal{B}_{H_{\ell,N}}$, i.e.

$$\mathcal{B}_{H_{\ell,N}} := \{\gamma \in H_{\ell,N} : \gamma(0) \in \mathcal{B}_{0,H}\}.$$

Observe that the set $\mathcal{B}_{H_{\ell,N}}$ is a closed set in $H_{\ell,N}$. Indeed, let us consider $\{\gamma_m\}$ a sequence in $\mathcal{B}_{H_{\ell,N}}$. Then, $\{\gamma_m(0)\}$ is a bounded sequence in $\mathcal{B}_{0,H}$. It follows from Theorem 5.1 that, there exist $\gamma:[0,\ell]\to H$ and a subsequence of $\{\gamma_m\}$ (relabeled the same) such that $\gamma_m \to \gamma$ in $H_{\ell,N}$. Then γ is again a solution on $[0,\ell]$. Observe that $\gamma(0)\in\mathcal{B}_{0,H}$, because $\mathcal{B}_{0,H}$ is closed in H. Therefore $\gamma \in \mathcal{B}_{H_{\ell,N}}$.

Theorem 7.7. *The map* $\mathscr{L}_N : \mathbb{R}_+ \times H_{\ell,N} \to H_{\ell,N}$ *defined in* (7.6) *is a semigroup. Besides, the following statements are fulfilled:*

(1) *For all* $t\in\mathbb{R}_+$*, the mapping* $\mathscr{L}_N(t) : H_{\ell,N} \to H_{\ell,N}$ *is continuous;*
(2) *There exits* $\tau>0$ *such that* $\overline{\mathscr{L}_N(\tau)\mathcal{B}_{H_{\ell,N}}}^{\mathbf{H}_\ell} \subset \mathcal{B}_{H_{\ell,N}}$*.*

Proof. For (1), consider $t\in\mathbb{R}_+$ and $\gamma_1,\gamma_2\in H_\ell$. By (7.4)

$$\begin{aligned}\|\mathscr{L}_N(t)\gamma_1 - \mathscr{L}_N(t)\gamma_2\|^2_{\mathbf{H}_\ell} &= \int_0^\ell |u_1(t+s)+u_2(t+s)|_2^2 ds \\ &\leq e^{c_8(N)t}\int_0^\ell |u_1(s)-u_2(s)|_2^2 ds = e^{c_8(N)t}\|\gamma_1-\gamma_2\|^2_{\mathbf{H}_\ell}.\end{aligned}$$

For (2), consider any $t>0$ and $\gamma \in \overline{\mathscr{L}_N(t)\mathcal{B}_{H_{\ell,N}}}^{\mathbf{H}_\ell}$. Then, there exists a sequence $\{\gamma_m\}\subset\mathcal{B}_{H_{\ell,N}}$ such that $\mathscr{L}_N(t)\gamma_m\to\gamma$ in $H_{\ell,N}$. From Theorem 5.1, the function γ is also solution on $[0,\ell]$ of **(MLM)**. Therefore, if for some $t>0$ the sequence $\mathscr{L}_N(t)\gamma_m$ belongs to $\mathcal{B}_{H_{\ell,N}}$ for all m, then $\gamma\in\mathcal{B}_{H_{\ell,N}}$ because $\mathcal{B}_{H_{\ell,N}}$ is a closed set. Thus, it follows from Corollary 6.2 and Theorem 6.4 that there exists $t_0(\mathcal{B}_{0,H})>0$ such that $\Big[\mathscr{L}^N(t)\gamma_m\Big](s)\in\mathcal{B}_{0,H}$ for all $t\geq t_0(\mathcal{B}_{0,H})+1$, for all m and for all $s\in[0,\ell]$, which is equivalent to $\mathscr{L}_N(t)\gamma_m\in\mathcal{B}_{H_{\ell,N}}$ for all $t\geq t_0(\mathcal{B}_{0,H})+1$ and for all m. Therefore, as mentioned above, choosing $\tau = t_0(\mathcal{B}_{0,H})+1$ we have that $\overline{\mathscr{L}_N(\tau)\mathcal{B}_{H_{\ell,N}}}^{\mathbf{H}_\ell}\subset \mathcal{B}_{H_{\ell,N}}$. □

Let us denote by $\mathcal{B}^1_{H_{\ell,N}} := \overline{\mathscr{L}_N(\tau)\mathcal{B}_{H_{\ell,N}}}^{\mathbf{H}_\ell}$ with $\tau = t_0(\mathcal{B}_{0,H}) + 1$. Thus, let us restrict the map $\mathscr{L}_N(\cdot)$ to the set $\mathcal{B}^1_{H_{\ell,N}}$. Then, by previous theorem, $\mathscr{L}_N : \mathbb{R}_+ \times \mathcal{B}^1_{H_{\ell,N}} \to \mathcal{B}^1_{H_{\ell,N}}$ is well defined. Indeed, for $t \in \mathbb{R}_+$ we deduce the positive invariance of $\mathcal{B}_{0,H}$, that is

$$\mathscr{L}_N(t)\mathcal{B}^1_{H_{\ell,N}} = \mathscr{L}_N(t)\Big(\overline{\mathscr{L}_N(\tau)\mathcal{B}_{H_{\ell,N}}}^{\mathbf{H}_\ell}\Big) \subset \overline{\mathscr{L}_N(t+\tau)\mathcal{B}_{H_{\ell,N}}}^{\mathbf{H}_\ell} \subset \overline{\mathscr{L}_N(\tau)\mathcal{B}_{H_{\ell,N}}}^{\mathbf{H}_\ell} = \mathcal{B}^1_{H_{\ell,N}}.$$

Theorem 7.8. *Let $p \geq 2$ and $N > 0$. The pair $(\mathscr{L}^N(\cdot), \mathcal{B}^1_{H_{\ell,N}})$ is a dynamical system. In addition, it possesses a global attractor $\mathcal{A}^N_\ell$ with finite fractal dimension.*

Proof. By the Aubin-Lions theorem we know that $\mathbf{Z}_\ell \hookrightarrow\hookrightarrow \mathbf{H}_\ell$. Now, we are going to show that $\mathscr{L}_N(\tau) : \mathbf{H}_\ell \to \mathbf{Z}_\ell$, with $\tau = t_0(\mathcal{B}_{0,H}) + 1$. Indeed, it follows from Lemma 7.5 and Remark 7.6 for $T = \ell$ and $\tau = t_0(\mathcal{B}_{0,H}) + 1$, that

$$\|\mathscr{L}_N(\tau)\gamma_1 - \mathscr{L}_N(\tau)\gamma_2\|_{L^2(0,\ell;V_2)} \leq \widehat{c}_9(\ell+\tau, N)\|\gamma_1 - \gamma_2\|_{\mathbf{H}_\ell},$$

and

$$\|\mathscr{L}'_N(\tau)\gamma_1 - \mathscr{L}'_N(\tau)\gamma_2\|_{L^2(0,\ell;(V^3)^*)} \leq c_{10}(\ell+\tau, N)\|\gamma_1 - \gamma_2\|_{L^2(0,\ell;H)},$$

for all $\gamma_1, \gamma_2 \in \mathcal{B}^1_{H_{\ell,N}}$. The first conclusion is that $\mathscr{L}_N(\tau)\mathcal{B}^1_{H_{\ell,N}} \subset \mathbf{Z}_\ell$. Then, the dynamical system $(\mathscr{L}_N(\cdot), \mathcal{B}^1_{H_{\ell,N}})$ is asymptotically compact. Thus, it follows from Theorem 1.1 that the dynamical system $(\mathscr{L}_N(\cdot), \mathcal{B}^1_{H_{\ell,N}})$ has a global attractor, which we denote by $\mathcal{A}^N_\ell$. The second conclusion is that there exists $c_{11}(N) > 0$ such that

$$\|\mathscr{L}_N(\tau)\gamma_1 - \mathscr{L}_N(\tau)\gamma_2\|_{\mathbf{Z}_\ell} \leq c_{11}(N)\|\gamma_1 - \gamma_2\|_{\mathbf{H}_\ell},$$

for all $\gamma_1, \gamma_2 \in \mathcal{B}^1_{H_{\ell,N}}$. It follows from Lemma 7.2 that

$$d_f^{\mathbf{H}_\ell}(\mathcal{A}^N_\ell) \leq \frac{\log \mathcal{N}_{1/4c_{11}(N)}}{\log 2}. \tag{7.7}$$

□

Theorem 7.9. *Given $p \geq 2$ and $N > 0$, let us define the function $\mathbf{e} : H_{\ell,N} \to H$ by $\mathbf{e}(\varphi) = \varphi(\ell)$. Then, the following statements hold:*
(1) $\mathbf{e} : H_{\ell,N} \to H$ is continuous ; (2) $\mathcal{A}^N_H = \mathbf{e}(\mathcal{A}^N_\ell)$; and (3) $d_f^H(\mathcal{A}^N_H) \leq d_f^{\mathbf{H}_\ell}(\mathcal{A}^N_\ell)$.

Proof. For (1): by the inequality given in (7.4) we have

$$|\mathbf{e}(\varphi_1) - \mathbf{e}(\varphi_2)|_2^2 = |\varphi_1(\ell) - \varphi_2(\ell)|_2^2 \leq e^{c_8(N)(\ell - s)}|\varphi_1(s) - \varphi_2(s)|_2^2,$$

for all $s \in [0, \ell]$. Then, integrating in s we obtain

$$|\mathbf{e}(\varphi_1) - \mathbf{e}(\varphi_2)|_2^2 \leq \frac{e^{c_8(N)\ell}}{\ell}\|\varphi_1 - \varphi_2\|^2_{L^2(0,\ell;H)}. \tag{7.8}$$

For (2): it is enough to show that $\mathbf{e}(\mathcal{A}^N_\ell)$ is a global attractor in H. Indeed, by the continuity of $\mathbf{e}(\cdot)$ and the compactness of global attractor $\mathcal{A}^N_\ell$ it follows that $\mathbf{e}(\mathcal{A}^N_\ell)$ is a compact set in H. By the invariance of $\mathcal{A}^N_\ell$, we have that

$$\mathcal{S}^N(t)(\mathbf{e}(\mathcal{A}^N_\ell)) = \Big[\mathscr{L}_N(t)\mathcal{A}^N_\ell\Big](\ell) = \Big[\mathscr{L}_N(\ell)\mathcal{A}^N_\ell\Big](0) = \mathbf{e}(\mathcal{A}^N_\ell).$$

Then, the set $\mathbf{e}(\mathcal{A}^N_\ell)$ is invariant on $\mathcal{S}^N(\cdot)$. Finally, the property of attracting bounded sets is a consequence of the fact that $\mathcal{A}^N_\ell$ is a global attractor. Then, it follows from the uniqueness of the global attractor that $\mathcal{A}^N_H = \mathbf{e}(\mathcal{A}^N_\ell)$. The item (3) is an immediate consequence of the estimate given in (7.7), inequality (7.8) and Proposition 7.3 for $\mathcal{X} = \mathcal{H}_\ell$, $\mathcal{Y} = H$ and $\theta = 1$. □

Corollary 7.10. *Let $p \geq 2$ and $N > 0$. Then, given $t > 0$, there exist $\kappa(t, N) > 0$ such that*

$$\|u(t) - v(t)\|^2_{3,2} + \|u(t) - v(t)\|^2_{1,2} + \|u(t) - v(t)\|^p_{1,p} \leq \kappa(t, N)|u_0 - v_0|_2,$$

for all $u_0, v_0 \in \mathcal{A}^N_H$. Moreover, it holds

$$d_f^{V^3}(\mathcal{A}^N_H) \leq 2 \cdot d_f^H(\mathcal{A}^N_H) \quad \text{and} \quad d_f^{V_p}(\mathcal{A}^N_H) \leq p \cdot d_f^H(\mathcal{A}^N_H).$$

Proof. Let us denote by $u(t)=\mathcal{S}^N(t)u_0$, $v(t)=\mathcal{S}^N(t)v_0$. Then, by (7.4) we have

$$|\mathcal{S}^N(t)v_0-\mathcal{S}^N(t)u_0|_2^2\le e^{c_8(N)t}|v_0-u_0|_2^2, \tag{7.9}$$

for all $t>0$. Thus, it follows from p-coercivity of $\mathbb{S}$, the Young inequality, and the Theorem 3.1, that

$$\begin{aligned}\widehat{C}_{p,N}\Big(\|u-v\|_{3,2}^2+\|u-v\|_{1,2}^2+\|u-v\|_{1,p}^p\Big)&\le\frac{1}{N}\|u-v\|_{3,2}^2+\Big(\mathbb{S}(Du)-\mathbb{S}(Dv),Du-Dv\Big)\\&=-\Big(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t},u-v\Big)-\Big\langle B_N(u)-B_N(v),u-v\Big\rangle\\&\le\Big|\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t}\Big|_2|u-v|_2+\frac{c_2\nu_1}{2c_0^2}\|u-v\|_{1,2}^2+C_N|u-v|_2^2,\end{aligned}$$

where $\widehat{C}_{p,N}=\min\Big\{\frac{1}{N},\frac{c_2\nu_1}{c_0^2},\frac{c_2\nu_2}{\tilde{c}_0^p}\Big\}$. Thus, by Theorem 6.4 and the invariance of the global attractor $\mathcal{A}_H^N$, we have that $|\mathcal{S}^N(t)u_0|_2\le\varrho_0$ and $\Big|\frac{d}{dt}[\mathcal{S}^N(t)u_0]\Big|_2\le\varrho_3$ for all $u_0\in\mathcal{A}_H^N$. Then, by (7.9), we get

$$\|u(t)-v(t)\|_{3,2}^2+\|u(t)-v(t)\|_{1,2}^2+\|u(t)-v(t)\|_{1,p}^p\le\kappa(t,N)|u_0-v_0|_2,$$

where $\kappa(t,N)=(2\varrho_3+2\varrho_0C_N)\exp\{\frac{c_8(N)}{2}t\}/\min\Big\{\frac{1}{N},\frac{c_2\nu_1}{2c_0^2},\frac{c_2\nu_2}{2\tilde{c}_0^p}\Big\}$, for any $u_0,v_0\in\mathcal{A}_H^N$ and for all $t>0$. ☐

Remark 7.11. *It is possible to uniformly bound, relative to $N>0$, the fractal dimension of the attractors $\mathcal{A}_H^N\subset\mathcal{B}_H\subset H$, when $p\ge 12/5$ if $n=3$ or $p>2$ if $n=2$. For this purpose, using the estimates given in Theorem 4.1 and using the fact that $|F_N(\cdot)|\le 1$, we obtain that, there exists $t_0(\mathcal{B}_H)>0$ such that the weak solutions of system* **(MLM)** *satisfy*

$$\|u^N(t)\|_{1,2}+\|u^N(t)\|_{1,p}\le\varrho_7\quad\text{for all}\quad t\ge t_0(\mathcal{B}_H), \tag{7.10}$$

where t_0 and ϱ_7 depends only on p,ν_1,ν_2, $|f|_2$ and on the constants related to the tensor $\mathbb{S}$.

Theorem 7.12. *Under assumptions of the Remark 7.11, the fractal dimension of global attractors $\{\mathcal{A}_H^N\}_{N>0}$ is uniformly bounded.*

Proof. Let us denote by $w:=v-u$. Thus, using the item (c) of the Lemma 2.4, and applying the Young inequality, we get

$$\begin{aligned}|\langle B_N(v)-B_N(u),w\rangle|&\le|\langle B(w,v),w\rangle|+\Big(F_N(\|v\|_{1,2})-F_N(\|u\|_{1,2})\Big)|\langle B(u,v),w\rangle|\\&\le\|v\|_{1,p}\|w\|_{2p'}^2+\|v\|_{1,2}|w|_2^{\frac{1}{2}}\|w\|_{1,2}^{\frac{3}{2}}.\end{aligned}$$

From interpolation inequality, for $n=3$, yields

$$\|w\|_{2p'}^2\le\widehat{C}|w|_2^{\frac{2p-3}{p}}\|w\|_{1,2}^{\frac{3}{p}}\quad\text{and}\quad\Big(\text{for } n=2\quad\|w\|_{2p'}^2\le\widehat{C}|w|_2^{\frac{2(p-1)}{p}}\|w\|_{1,2}^{\frac{2}{p}}\Big).$$

Then (for $n=3$), by (7.10) we have

$$|\langle B_N(v)-B_N(u),w\rangle|\le\varrho_7\widehat{C}|w|_2^{\frac{2p-3}{p}}\|w\|_{1,2}^{\frac{3}{p}}+2\varrho_7^2|w|_2^{\frac{1}{2}}\|w\|_{1,2}^{\frac{1}{2}}\le\frac{c_7}{2}\|w\|_{1,2}^2+\frac{c_{12}}{2}|w|_2^2,$$

for all $t\ge t_0(\mathcal{B}_H)$, where c_{12}, independent of N. Thus, it follows from(7.2) and (7.10), that

$$\frac{d}{dt}|w|_2^2+\frac{2}{N}\|w\|_{3,2}^2+c_7\|w\|_{1,2}^2+I^2(u,v)\le c_{12}|w|_2^2. \tag{7.11}$$

Applying the Gronwall lemma, we deduce

$$|w(\tau)|_2^2\le e^{c_{12}(\tau-s)}|w(s)|_2^2,\quad\text{for all }\tau\ge s.$$

Now, for $s\in[0,\ell]$, integrating (7.11) from s to $\ell+t$, we have

$$\frac{2}{N}\int_s^{t+\ell}\|w(\tau)\|_{3,2}^2d\tau+c_7\int_s^{\ell+t}\|w(\tau)\|_{1,2}^2d\tau+\int_s^{\ell+t}I^2(u(\tau),v(\tau))d\tau\le|w(s)|_2^2+c_{12}\int_s^{\ell+t}|w(\tau)|_2^2d\tau.$$

Thus, we derive that

$$\frac{2}{N}\int_t^{t+\ell}\|w(\tau)\|_{3,2}^2d\tau+\int_t^{\ell+t}\|w(\tau)\|_{1,2}^2d\tau+c_7\int_t^{\ell+t}I^2(u(\tau),v(\tau))d\tau\le c_{13}(t)\int_0^\ell |w(s)|_2^2ds,$$

for all $t\ge t_0(\mathcal{B}_H)$, where $c_{13}(t)=1+c_{12}e^{c_{12}(\ell+t)}$.

On the other hand, consider $\varphi\in L^2(0,T;V^3)$ with $\|\varphi\|_{L^2(0,T;V^3)}\le 1$. For simplicity in the following estimates, we will omit the time variable, which is $t+s$ with $s\in(0,\ell)$. Then, by (1.1), the Young inequality and (7.10), it follows that

$$\begin{aligned}\int_0^\ell\int_\Omega\Big(\mathbb{S}(Du)-\mathbb{S}(Dv)\Big):\nabla\varphi dxds&\le\int_0^\ell\int_\Omega|\mathbb{S}(Du)-\mathbb{S}(Dv)||\nabla\varphi|dxds\\&\le\Big(\int_0^\ell I^2(u,v)ds\Big)^{\frac12}\int_0^\ell\int_\Omega\Big(\nu_1+\nu_2(|Du|+|Dv|)\Big)^p dxds\\&\le c_{14}\Big(\int_0^\ell I^2(u,v)ds\Big)^{\frac12},\end{aligned}$$

where c_{14} independent of N. Next, it follows from Theorem 3.1, that

$$\int_0^\ell\langle B_Nu-B_Nv,\varphi\rangle ds\le\Big(\int_0^\ell\Big[\|u\|_{1,2}+2\|v\|_{1,2}\Big]^2\|u-v\|_{1,2}^2ds\Big)^{\frac12}\Big(\int_0^\ell\|\varphi\|_{1,2}^2\Big)^{\frac12}.$$

Putting all the inequalities together, we obtain that

$$\begin{aligned}\|u'(t+\cdot)-v'(t+\cdot)\|_{L^2(0,\ell;(V^3)^*)}^2&\le c_{15}\Big[\frac1N\int_0^\ell\|u(u)-v(s)\|_{3,2}^2ds+\int_0^\ell I^2(u(s),v(s))ds\\&\qquad+\int_0^\ell\Big(\|u(s)\|_{1,2}+2\|v(s)\|_{1,2}\Big)^2\|u(s)-v(s)\|_{1,2}^2ds\Big]\\&\le c_{16}(t)\int_0^\ell|u(s)-v(s)|_2^2ds,\end{aligned}$$

for all $t\ge t_0(\mathcal{B}_H)$, where $c_{16}(t)=c_{15}c_{13}(t)\max\{1,9\varrho_7^2\}/t$. Therefore, applying Lemma 7.2 for $t=t^*=t_0(\mathcal{B}_H)$, $\mathcal{X}=L^2(0,\ell;H)$, $\mathcal{Y}=L^2(0,\ell;(V^3)^*)$, we obtain that

$$d_f^H(\mathcal{A}_H^N)\le d_f^{\mathbf{H}_\ell}(\mathcal{A}_\ell^N)\le\frac{\log\mathcal{N}_{1/4c_{16}}}{\log 2}\quad\text{for all}\quad N>0.$$

□

Remark 7.13. *It follows from Corollary 7.10 that*

$$d_f^{V^3}(\mathcal{A}_H^N)\le 2\cdot d_f^H(\mathcal{A}_H^N)\quad\textit{and}\quad d_f^{V_p}(\mathcal{A}_H^N)\le p\cdot d_f^H(\mathcal{A}_H^N).$$

Then, the fractal dimension of $\mathcal{A}_H^N$ is uniformly bounded in V_p and V^3, with respect to $N>0$, when $p\ge 12/5$ if $n=3$, and $p>2$ if $n=2$.

Theorem 7.14. *Let $p\ge 12/5$ if $n=3$ or $p>2$ if $n=2$ and $f\in L^2(\Omega)^n$. Then, the following statements hold:*

(a) *the set $\mathcal{K}=\bigcup_{N=1}^\infty\mathcal{A}_H^N$ is precompact in H;*

(b) *let $\{x_N\}$ be a sequence such that $x_N\in\mathcal{A}_H^N$ for each $N\in\mathbb{N}$ and let $\mathcal{A}_H$ be the global attractor associated with the dynamical system $(\mathcal{S}(\cdot),H)$ of* **(LM)***. Then, there exist a subsequence of $\{x_N\}$, denoted by $\{x_{N_k}\}$, and $x_0\in\mathcal{A}_H$ such that $x_{N_k}\to x_0$ in H, as $k\to\infty$. In particular, the sequence of global attractors $\{\mathcal{A}_H^N\}_{N\in\mathbb{N}}$ is upper semicontinuous, i.e. $\lim_{N\to\infty}\mathrm{dist}_H(\mathcal{A}_H^N,\mathcal{A}_H)=0$;*

(d) *there exists a finite dimensional subspace $Y\subset H$ such that $\{P\in\mathcal{P}(H;Y):P|_{\mathcal{K}}$ is $1-1\}$ is residual.*

Proof. (a) By (7.10) we know that $\|\mathcal{S}^N(t^*)x_0\|_{1,p} \le \varrho_7$ for all $x_0 \in \mathcal{A}_H^N$ and for all $N > 0$ and some $t^* > 0$. Then, since $V_p \hookrightarrow\hookrightarrow H$, we obtain that $\mathcal{K} = \bigcup_{N=1}^{\infty} \mathcal{A}_H^N$ is a precompact set in H.

(b) Since $\{x_N\} \subset \mathcal{K}$ and $\overline{\mathcal{K}}$ is a compact set in H, there exist a subsequence $\{x_{N_k}\}$ and $x_0 \in H$ such that $x_{N_k} \to x_0$ in H as $k \to \infty$. It only remains to show that $x_0 \in \mathcal{A}_H$. Indeed, let us denote by $u(t) = \mathcal{S}(t)x_0$, $u_{N_k}(t) = \mathcal{S}^{N_k}(t)x_{N_k}$. Then, it follows from the Theorem 5.3 and Lemma 5.4 that, except from a subsequence, $u_{N_k}(t) \to u(t)$ in H for all $t \ge 0$ as $k \to \infty$, and also $u(t) = \mathcal{S}(t)x_0$ is solution of the problem **(LM)**.

Now, let us build a global solution that goes through x_0. Let us consider a family $\{\varphi_{N_k}\}_{k\in\mathbb{N}}$ such that each function $\varphi_{N_k} : \mathbb{R} \to H$ is a global bounded solution of the dynamical system $(\mathcal{S}^{N_k}(\cdot), H)$ with $\varphi_{N_k}(0) = x_{N_k}$ for each $k \in \mathbb{N}$. Now, let us do $N_k^0 = N_k$, with $k \in \mathbb{N}$, thus for $j = 1$, let $\{N_k^1\}$ be a subsequence of $\{N_k^0\}$ such that there exists $x_{-1} \in H$ with $\varphi_{N_k^1}(-1) \to x_{-1}$ as $k \to \infty$ (this is possible since $\varphi_{N_k^1}(-1)$ belongs to $\overline{\mathcal{K}}$). In the same way, for each $j \in \mathbb{N}^*$, there exist a subsequence $\{N_k^j\}$ of $\{N_k^{j-1}\}$ and $x_{-j} \in H$ such that $\varphi_{N_k^j}(-j) \to x_{-j}$ as $k \to \infty$.

With the same reasoning as above, for $j - 1 \le s \le j$, we deduce that

$$\varphi_{N_k^j}(-j) = \mathcal{S}^{N_k^j}(s)\varphi_{N_k^j}(-j-s) \to x_{-j} = \mathcal{S}(s)x_{-j-s}.$$

Let us define

$$\varphi_0(t) = \begin{cases} \mathcal{S}(t)x_0 & \text{for } t \ge 0; \\ \mathcal{S}(t+j)x_0 & \text{for } -j \le t < -j+1, \;\; j \in \mathbb{N}^*. \end{cases}$$

Thus, $\varphi_0 : \mathbb{R} \to H$ is a bounded global solution of $(\mathcal{S}(\cdot), H)$, then $x_0 \in \mathcal{A}_H$.
For (d), it follows from [23, Proposition 2.8], that

$$d_{\mathcal{H}}(\mathcal{K} - \mathcal{K}) = d_{\mathcal{H}}\Big(\bigcup_{i,j=1}^{\infty} (\mathcal{A}_H^i - \mathcal{A}_H^j)\Big) = \sup_{i,j} d_{\mathcal{H}}(\mathcal{A}_H^i - \mathcal{A}_H^j) \le 2\frac{\log \mathcal{N}_{1/4c_{16}}}{\log 2}.$$

Let $Y \subset H$ a finite dimensional subspace, with $d_f(Y) = M$, where M is the smallest natural number such that $M > 1 + 2\frac{\log \mathcal{N}_{1/4c_{16}}}{\log 2}$. Then, by the Mañe Theorem 7.4 the set $\{P \in \mathcal{P}(H;Y) : P|_{\mathcal{K}} \text{ is } 1-1\}$ is residual. ☐

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