Dynamics of multi-valued retarded p-Laplace equations driven by nonlinear colored noise ☆

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Abstract

This paper mainly considers the long-term behavior of p-Laplace equations with infinite delays driven by nonlinear colored noise. We firstly prove the existence of weak solutions to the equation, but the uniqueness of solutions cannot be guaranteed due to the lack of Lipschitz continuity conditions, and thus generate a multi-valued dynamical system. Moreover, the regularity of solutions is also proved. Then we prove the existence of a pullback attractor. Subsequently, the measurability of the pullback attractor and the multi-valued dynamical system are also proved.

Keywords: `pullback attractor; infinite delays; nonlinear colored noise; multi-valued dynamical system`
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1. Introduction

In this paper, we consider the existence of pullback random attractors for non-autonomous p-Laplace equations with infinite delay (representing the past history of variables) on a bounded domain $\mathcal{O} \subset \mathbb{R}^N$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u = f(t,x,u) + g(x, u(t-\varrho(t))) \\ \quad + \displaystyle\int_{-\infty}^{0} F(x,l,u(t+l))dl + J(t,x) + h(t,x,u)\zeta_\delta(\theta_t\omega), \ t > \tau, \quad x \in \mathcal{O}, \\ u(t,x) = 0, \ t > \tau, \quad x \in \partial\mathcal{O}, \\ u(\tau+s,x) = \varphi(s,x), \quad s \in (-\infty, 0], \quad x \in \mathcal{O}, \tau \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $p \geq 2, \lambda > 0$, ζ_δ is the colored noise with correlation time $\delta > 0$, and W is a scalar Wiener process on the classical Wiener space $(\Omega, \mathbb{P}, \mathcal{F}, \{\theta_t\}_{t\in\mathbb{R}})$. The nonlinear drift term f and the nonlinear diffusion term h are continuous functions but not necessarily Lipschitz continuous, and the delay term $g : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ and $F : \mathcal{O} \times \mathbb{R}_{-} \times \mathbb{R} \to \mathbb{R}$ are also non-Lipschitz continuous.

Since the sample paths of a Wiener process are nowhere differentiable, to solve this difficulty, we often use colored noise (see [23]) to approximate the Wiener process. Stochastic partial

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differential equations driven by nonlinear colored noise have been studied in many papers [9, 15, 11]. Regarding multi-valued random dynamical systems, there are also some related papers, such as [14, 10, 36].

The p-Laplace partial differential equation often appears in the physical studies about non-Newtonian fluid dynamics. It also occurs in descriptions of phenomena related to nonlinear elasticity, nonlinear filtering, or magnetic field distribution (see [22, 13]). The long-term dynamical behavior (especially the existence of pullback attractors) of the p-Laplace equation has been extensively studied, see e.g., [16, 27, 28, 25, 19, 20, 38, 35] in a general singled-valued random dynamical system with Lipschitz continuous conditions. In the absence of Lipschitz continuous condition, the uniqueness of the solution cannot be ensured, thus the long-time dynamics of the p-Laplace equation in multi-valued random dynamical systems are discussed, for instance, in [8, 10, 31, 14, 40, 34]. Note that none of the above papers possesses a delay term.

For infinite delay equations, choosing a suitable state space is more difficult than for bounded delay equations, see e.g., [17, 21]. The existence of attractors for models involving hereditary characteristics with infinite delays has been discussed extensively, such as [29, 26, 41, 42, 43] in singled-valued dynamical systems and [3, 4, 5, 6, 7, 30, 39] in multi-valued dynamical systems.

The existence of pullback attractors for the multi-valued process associated to differential equations with p-Laplace operators and infinite delays has been discussed in [30]. But as far as we know, there are very few papers on dynamics of multi-valued non-autonomous p-Laplace equations with infinite delays. Therefore, the topic of this paper is novel.

The main difficulties of this paper are: (i) The existence of weak solutions of equation (1.1). (ii) The measurability of multi-valued dynamical systems and random attractors and the asymptotic compactness of solutions. To solve these problems, we use the traditional Galerkin approximations technique, and according to the method in [4], we prove that $u \to \int_{-\infty}^{0} F(x, l, u(t + l))dl$ is continuous from $C_{\gamma,L^2(\mathcal{O})}$ into $L^2(\mathcal{O})$ as shown in Remark 2.3, in order to obtain the existence of weak solutions (see Theorem 2.8). To solve pullback asymptotically compactness of solutions, we will use the same technique as [30, Lemma 5.5]. The measurability of the pullback attractor will be deduced by proving the upper-semicontinuity of multi-valued functions, the closure of a graph on some subspaces of the probability space by using the methods in [4].

In fact, we try to prove the regularity of pullback random attractors for equation (1.1). For p-Laplacian equations with bounded delays, Sobolev's compactness theorem and Arzelà-Ascoli's theorem can be applied to prove the regularity of pullback attractors as in [25, 35]. But for the space considered in this paper (shown in (2.5)), we find that there is no embedding relationship between spaces $C_{\gamma,L^2(\mathcal{O})}$ and $C_{\gamma,W_0^{1,p}(\mathcal{O})}$. Therefore, we can only prove the regularity of the solution (see Theorem 2.9) by applying the method of [8].

In the next section, we prove the existence of weak solutions to equation (1.1), and that they generate a multi-valued dynamical system. Moreover, we also prove the regularity of solutions. Section 3 is dedicated to the existence of the pullback attractor, and proves the measurability of the pullback attractor and the multi-valued dynamical system. Therefore, the existence of pullback random attractors for the equation (1.1) is obtained.

2. Multivalued dynamical systems in $C_{\lambda,H}$

Let $H = L^2(\mathcal{O})$ equipped with the norm $\|\cdot\|$, and use $\|\cdot\|_s$ denote the norm in $L^s(\mathcal{O})$ (s can be any positive constant). We denote $W_0^{1,p}(\mathcal{O})$ by V, and the dual space $W^{-1,\hat{p}}(\mathcal{O})$ ($\hat{p}$ is the conjugate of p) of $W_0^{1,p}(\mathcal{O})$ by V^*. We also let $(\cdot,\cdot)$ denote the inner product in $L^2(\mathcal{O})$, and denote the duality product between V and V^* by $\langle\cdot,\cdot\rangle$. In addition, we have the usual chain of dense and compact embedding $V \subset H \subset V^*$.

For $p \geq 2$, we define the p-Laplacian operator $\Delta_p : V \to V^*$ by

$$\Delta_p u := -\text{div}(|\nabla u|^{p-2}\nabla u), \quad \langle \Delta_p u, v\rangle = \int_{\mathcal{O}} |\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \tag{2.1}$$

for all $u, v \in V$. Note that Δ_p is a monotone and hemicontinuous operator as in [32]. Moreover, by (2.1), for all $u, v \in V$

$$\begin{aligned}\langle \Delta_p u, v\rangle &= \int_{\mathcal{O}} |\nabla u|^{p-2}\nabla u \cdot \nabla v dx \\ &\leq \Big(\int_{\mathcal{O}} |\nabla u|^p dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathcal{O}} |\nabla v|^p dx\Big)^{\frac{1}{p}} = \|u\|_V^{p-1}\|v\|_V, \end{aligned}\tag{2.2}$$

and

$$\|\Delta_p u\|_{V^*} = \sup_{\|v\|_V \leq 1} \langle \Delta_p u, v\rangle \leq \|u\|_V^{p-1}. \tag{2.3}$$

Now, we consider the following non-autonomous random p-Laplace equation with infinite delays:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\Delta_p u - \lambda u + f(t,x,u) + g(x, u(t-\varrho(t))) + \displaystyle\int_{-\infty}^0 F(x,l,u(t+l))dl \\ \quad + J(t,x) + h(t,x,u)\zeta_\delta(\theta_t\omega),\ t > \tau, \quad x \in \mathcal{O}, \\ u(t,x) = 0,\ t > \tau, \quad x \in \partial\mathcal{O}, \\ u(\tau+s,x) = \varphi(s,x), \quad s \in (-\infty,0], \quad x \in \mathcal{O}, \tau \in \mathbb{R}. \end{cases} \tag{2.4}$$

Let X be a Hilbert space. To deal with the delay terms g and F in (2.4), we denote our phase space by

$$C_{\gamma,X} = \{w \in C((-\infty,0];X) : \lim_{\tau\to-\infty} e^{\gamma\tau}w(\tau) \ \text{ exists}\}, \tag{2.5}$$

where $\gamma > 0$ and we set $\|w\|_{C_{\gamma,X}} := \sup_{\tau\in(-\infty,0]} e^{\gamma\tau}\|w(\tau)\| < \infty$ for all $w \in C_{\gamma,X}$. From [4], we know that $C_{\gamma,X}$ is a separable Banach space. Consider $T > \tau$ and a function $u : (-\infty,T) \to X$, we can define for any $t \in [\tau,T)$ the mapping $u_t : (-\infty,0] \to X$ by $u_t(s) = u(t+s)$ for all $s \in (-\infty,0]$.

By [1, 15], we define a random variable $\zeta_\delta : \Omega \to \mathbb{R}$ by

$$\zeta_\delta(\omega) = \frac{1}{\delta}\int_{-\infty}^0 e^{\frac{s}{\delta}} dW(t,\omega),\ \text{for each } \delta > 0.$$

The process $z_\delta(t,\omega) = \zeta_\delta(\theta_t\omega)$ is called an Ornstein-Uhlenbeck process (i.e. the colored noise), which is a stationary Gaussian process with $\mathbb{E}(\zeta_\delta) = 0$ and is the unique stationary solution of the stochastic equation:

$$dz + \frac{1}{\delta}zdt = \frac{1}{\delta}dW.$$

By [14], there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset set (still denoted by) Ω of full measure such that for $\omega \in \Omega$,

$$\lim_{t\to\pm\infty}\frac{\omega(t)}{t} = 0, \qquad \lim_{t\to\pm\infty}\frac{|\zeta_\delta(\theta_t\omega)|}{t} = 0 \text{ for every } 0 < \delta \leq 1. \tag{2.6}$$

2.1. Assumptions

In order to achieve our final result, we need to impose the following assumptions:

H1. The external force fulfills $J(t,x) \in C(\mathbb{R}, H)$.

H2. $\varrho(\cdot) \in C^1(\mathbb{R}, [0,\rho])$ and $|\varrho'(\cdot)| \leq \rho^* < 1$.

H3. $f : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is continuous and for all $t, r \in \mathbb{R}, x \in \mathcal{O}$,

$$f(t,x,r)r \leq -\beta_1 |r|^q + \psi_1(t,x), \tag{2.7}$$

$$|f(t,x,r)| \leq \beta_2 |r|^{q-1} + \psi_2(t,x), \tag{2.8}$$

where $q > 2, \beta_1, \beta_2 > 0$, $\psi_1 \in L^{\infty}_{loc}(\mathbb{R}, L^1(\mathcal{O}) \cap L^{\infty}(\mathcal{O}))$, $\psi_2 \in L^2_{loc}(\mathbb{R}, L^2(\mathcal{O}))$.

H4. $g \in C(\mathbb{R}, \mathbb{R})$ and there is $\beta_3 > 0$ such that

$$|g(x,r)|^2 \leq \beta_3 |r|^2 + |\psi_3(x)|^2, \quad \forall r \in \mathbb{R}, \ x \in \mathcal{O}, \tag{2.9}$$

where $\psi_3 \in L^2(\mathcal{O})$.

H5. The nonlinear diffusion term $h : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that for all $t, r \in \mathbb{R}, x \in \mathcal{O}$,

$$|h(t,x,r)| \leq \psi_4(t,x)|r|^{\eta-1} + \psi_5(t,x), \tag{2.10}$$

where $2 \leq \eta < q, \psi_4 \in L^{\frac{2q-2}{q-\eta}}_{loc}(\mathbb{R}, L^{\frac{2q-2}{q-\eta}}(\mathcal{O}))$, $\psi_5 \in L^2_{loc}(\mathbb{R}, L^2(\mathcal{O}))$.

H6. $F : \mathcal{O} \times \mathbb{R}_- \times \mathbb{R} \to \mathbb{R}$ is continuous. There exist a scalar function $e^{-\gamma \cdot} m_1(\cdot) \in L^1((-\infty, 0], \mathbb{R})$, and a function $m_0(x, \cdot) \in L^1((-\infty,0], L^1(\mathcal{O}))$ such that F satisfies

$$|F(x,l,r)| \leq m_1(l)|r| + |m_0(x,l)|, \quad \forall x \in \mathcal{O}, l \in \mathbb{R}_-, r \in \mathbb{R}. \tag{2.11}$$

To simplify the calculation, we will denote

$$m_0 = \int_{-\infty}^0 \|m_0(\cdot, r)\|_1 dr, \tag{2.12}$$

$$m_1 = \int_{-\infty}^0 e^{-\gamma r} m_1(r) dr. \tag{2.13}$$

Remark 2.1. By (2.9), we can obtain that

$$\begin{aligned}\|g(\cdot, u(t-\varrho(t)))\|^2 &\leq \beta_3 \|u(t-\varrho(t))\|^2 + \|\psi_3(\cdot)\|^2 \\ &\leq \beta_3 e^{2\gamma\rho} \|u_t\|^2_{C_{\gamma,H}} + \|\psi_3(\cdot)\|^2.\end{aligned}$$

Remark 2.2. By (2.11), we can deduce that

$$\begin{aligned}&\Big\| \int_{-\infty}^0 F(\cdot, l, u(t+l)) dl \Big\|^2 \\ &\leq \int_{\mathcal{O}} (\int_{-\infty}^0 [m_1(l)|u(t+l)| + |m_0(x,l)|] dl)^2 dx \\ &\leq 2\int_{\mathcal{O}} (\int_{-\infty}^0 m_1(l)|u(t+l)| dl)^2 dx + 2\int_{\mathcal{O}} (\int_{-\infty}^0 |m_0(x,l)| dl)^2 dx \\ &\leq 2\int_{\mathcal{O}} (\sup_{l \leq 0} e^{\gamma l}|u_t(l)| \int_{-\infty}^0 e^{-\gamma l} m_1(l) dl)^2 dx + 2\int_{\mathcal{O}} (\int_{-\infty}^0 |m_0(x,l)| dl)^2 dx \\ &\leq 2m_1^2 \|u_t\|^2_{C_{\gamma,H}} + 2m_0^2.\end{aligned}$$

Remark 2.3. Given $n \in \mathbb{N}$. By **H6**, we know that if $\eta^n \to \eta$ in $C_{\gamma,H}$, then for all $l \leq 0$

$$F(x,l,\eta^n(l)) \to F(x,l,\eta(l)).$$

Thus, there exists a positive constant $C(M)$ such that, for any $l \in [-M,0]$,

$$\|F(\cdot,l,\eta^n(l)) - F(\cdot,l,\eta(l))\| \leq C(M).$$

Using Lebesgue's majorant theorem we have for any $M > 0$

$$\int_{-M}^{0} \|F(\cdot,l,\eta^n(l)) - F(\cdot,l,\eta(l))\| dl \to 0.$$

For any $\varepsilon > 0$ there exists an $M = M(\varepsilon) > 0$ such that

$$\begin{aligned}
&\int_{-\infty}^{-M} \|F(\cdot,l,\eta^n(l)) - F(\cdot,l,\eta(l))\| dl \\
&\leq \int_{-\infty}^{-M} \int_{\mathcal{O}} \big[m_1(l)(|\eta^n(l)| + |\eta(l)|) + 2|m_0(x,l)|\big] dx dl \\
&\leq \int_{-\infty}^{-M} m_1(l) e^{-\gamma l} \int_{\mathcal{O}} e^{\gamma l}(|\eta^n(l)| + |\eta(l)|) dx dl + 2\int_{-\infty}^{-M} \int_{\mathcal{O}} |m_0(x,l)| dx dl \\
&\leq (\|\eta^n\|_{C_{\gamma,H}} + \|\eta\|_{C_{\gamma,H}}) \int_{-\infty}^{-M} m_1(l) e^{-\gamma l} dl + 2\int_{-\infty}^{-M} \|m_0(x,l)\|_1 dx dl \leq \varepsilon.
\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that, for $n \geq N$,

$$\Big\| \int_{-\infty}^{0} F(\cdot,l,\eta^n(l)) dl - \int_{-\infty}^{0} F(\cdot,l,\eta(l)) dl \Big\| \leq 2\varepsilon,$$

which implies that $\eta \to \int_{-\infty}^{0} F(x,l,\eta) dl$ is continuous from $C_{\gamma,H}$ into H.

2.2. Existence of solutions in $C_{\gamma,H}$

In this section, we show the existence of weak solutions for the system (2.4). To that end, we assume that

$$8m_1^2 < \lambda^2, \tag{2.14}$$

and

$$2M_1 \leq \frac{\lambda}{8} \leq \frac{\gamma}{4} \quad \text{and} \quad \vartheta := \beta_1 - qM_1 > 0, \quad \text{where} \quad M_1 = \frac{\sqrt{\beta_3 e^{\lambda\rho}}}{\sqrt{1-\rho^*}} > 0. \tag{2.15}$$

Definition 2.4. Given $T > 0$, a function $u(\cdot,\tau,\omega,\varphi) \in C((-\infty,\tau+T);H) \cap L^p(\tau,\tau+T;V) \cap L^q(\tau,\tau+T;L^q(\mathcal{O}))$ is called a weak solution of (2.4) on $(\tau,\tau+T)$ with initial function $\varphi \in C_{\gamma,H}$, if for every $\eta \in V \cap L^q(\mathcal{O})$,

$$\begin{aligned}
&\frac{d}{dt}(u,\eta) + \langle \Delta_p u, \eta \rangle + \lambda(u,\eta) \\
&= \int_{\mathcal{O}} f(t,x,u)\eta dx + \int_{\mathcal{O}} g(x,u(t-\varrho(t)))\eta dx + \int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u(t+l)) dl\Big)\eta dx \\
&\quad + \int_{\mathcal{O}} J(t,x)\eta dx + \zeta_\delta(\theta_t\omega) \int_{\mathcal{O}} h(t,x,u)\eta dx,
\end{aligned} \tag{2.16}$$

in the sense of distributions.

It can be inferred from Definition 2.4 and **H3-H6** that $\frac{du}{dt} \in L^{\hat{p}}(\tau, \tau+T; V^*) + L^{\frac{q}{q-1}}(\tau, \tau+T; L^{\frac{q}{q-1}}(\mathcal{O}))$. By [33], we know that $u \in C([\tau, \tau+T]; H)$. Furthermore, for all $t \in [\tau, \tau+T]$,

$$\frac{d}{dt}\|u\|^2 + 2\|\nabla u\|_p^p + 2\lambda\|u\|^2 = 2\int_{\mathcal{O}} f(t,x,u)udx + 2\int_{\mathcal{O}} g(x, u(t-\varrho(t)))udx$$
$$+ 2\int_{\mathcal{O}} \Big(\int_{-\infty}^0 F(x,l,u(t+l))dl\Big)udx + 2\int_{\mathcal{O}} J(t,x)udx + 2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)udx. \quad (2.17)$$

In order to show the existence of a weak solution to system (2.4), we first need to establish a priori estimates for weak solutions to equation (2.4).

Lemma 2.5. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold. Let* $\tau \in \mathbb{R}$, $\omega \in \Omega, T > 0$, *and* u *be a weak solution of system* (2.4) *with initial condition* $\varphi \in C_{\gamma,H}$. *Then there exists* $c = c(M_1, \lambda, \gamma) > 0$ *such that, for all* $t \in [\tau, \tau+T]$,

$$\|u_t\|^2_{C_{\gamma,H}} \le ce^{(\frac{4m_1^2}{\lambda}-\lambda)(t-\tau)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4}{\lambda}\int_\tau^t e^{(\lambda-\frac{4m_1^2}{\lambda})(r-t)}\|J(r,\cdot)\|^2dr$$
$$+ c\int_\tau^t e^{(\lambda-\frac{4m_1^2}{\lambda})(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \quad (2.18)$$

Proof. By (2.17), **H3** and the Young inequality, we have

$$\frac{d}{dt}\|u\|^2 + 2\|\nabla u\|_p^p + \frac{7}{4}\lambda\|u\|^2$$
$$\le -2\beta_1\|u\|_q^q + 2\|\psi_1(t,\cdot)\| + 2\int_{\mathcal{O}} g(x, u(t-\varrho(t)))udx$$
$$+ 2\int_{\mathcal{O}} \Big(\int_{-\infty}^0 F(x,l,u(t+l))dl\Big)udx + \frac{4}{\lambda}\|J(t,\cdot)\|^2 + 2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)udx.$$

By **H5** and the Young inequality again, we have

$$2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)udx \le 2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} (\psi_4(t,x)|u|^\eta + \psi_5(t,x)|u|)dx$$
$$\le \beta_1\|u\|_q^q + c\|\psi_4(t,\cdot)\|^{\frac{q}{q-\eta}}_{\frac{q}{q-\eta}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-\eta}} + c\|\psi_5(t,\cdot)\|^{\frac{q}{q-1}}_{\frac{q}{q-1}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-1}}.$$

By Remark 2.2 and the Young inequality, for the infinite delay term we have

$$2\int_{\mathcal{O}} \Big(\int_{-\infty}^0 F(x,l,u(t+l))dl\Big)udx \le \frac{4m_1^2}{\lambda}\|u_t\|^2_{C_{\gamma,H}} + \frac{4m_0^2}{\lambda} + \frac{\lambda}{2}\|u\|^2. \quad (2.19)$$

Therefore, for all $t \in [\tau, \tau+T]$,

$$\frac{d}{dt}\|u\|^2 + 2\|u\|_V^p + \beta_1\|u\|_q^q + \frac{5}{4}\lambda\|u\|^2$$
$$\le 2\int_{\mathcal{O}} g(x, u(t-\varrho(t)))udx + \frac{4m_1^2}{\lambda}\|u_t\|^2_{C_{\gamma,H}} + \frac{4}{\lambda}\|J(t,\cdot)\|^2$$
$$+ c(1 + \|\psi_4(t,\cdot)\|^{\frac{q}{q-\eta}}_{\frac{q}{q-\eta}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-\eta}} + \|\psi_5(t,\cdot)\|^{\frac{q}{q-1}}_{\frac{q}{q-1}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-1}}). \quad (2.20)$$

Multiplying (2.20) by $e^{\lambda t}$ and integrating it over $t \in [\tau, \xi]$, we have for all $\xi \geq \tau$,

$$
\begin{aligned}
&e^{\lambda\xi}\|u(\xi)\|^2 + 2\int_\tau^\xi e^{\lambda r}\|u(r)\|_V^p dr + \beta_1\int_\tau^\xi e^{\lambda r}\|u(r)\|_q^q dr + \frac{\lambda}{4}\int_\tau^\xi e^{\lambda r}\|u(r)\|^2 dr \\
&\leq e^{\lambda\tau}\|u(\tau)\|^2 + 2\int_\tau^\xi e^{\lambda r}\int_{\mathcal{O}} g(x, u(r-\varrho(r)))u(r)dr + \frac{4m_1^2}{\lambda}\int_\tau^\xi e^{\lambda r}\|u_r\|^2_{C_{\gamma,H}} dr \\
&+ \frac{4}{\lambda}\int_\tau^\xi e^{\lambda r}\|J(r,\cdot)\|^2 dr + c\int_\tau^\xi e^{\lambda r}(|\zeta_\delta(\theta_r\omega)|^{\frac{q}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^{\frac{q}{q-1}} + 1)dr,
\end{aligned} \tag{2.21}
$$

where we used that $L^{\frac{2q-2}{q-\eta}}(\mathcal{O}) \subset L^{\frac{q}{q-\eta}}(\mathcal{O})$ and $L^2(\mathcal{O}) \subset L^{\frac{q}{q-1}}(\mathcal{O})$. For the finite delay term we have

$$
2\int_\tau^\xi e^{\lambda r}\int_{\mathcal{O}} g(x, u(r-\varrho(r)))u(r)dr \leq M_1\int_\tau^\xi e^{\lambda r}\|u(r)\|^2 dr + \frac{1}{M_1}\int_\tau^\xi e^{\lambda r}\|g(\cdot, u(r-\varrho(r)))\|^2 dr, \tag{2.22}
$$

where $M_1 = \frac{\sqrt{\beta_3 e^{\lambda\rho}}}{\sqrt{1-\rho^*}}$ is defined in (2.15). By **H4** and **H2**, we have

$$
\begin{aligned}
&\frac{1}{M_1}\int_\tau^\xi e^{\lambda r}\|g(\cdot, u(r-\varrho(r)))\|^2 dr \qquad (2.23)\\
&\leq \frac{\beta_3}{M_1}\int_\tau^\xi e^{\lambda r}\|u(r-\varrho(r))\|^2 dr + \frac{1}{M_1}\int_\tau^\xi e^{\lambda r}\|\psi_3(\cdot)\|^2 dr \\
&\leq \frac{\beta_3 e^{\lambda\rho}}{M_1(1-\rho^*)}\int_{\tau-\rho}^\xi e^{\lambda r}\|u(r)\|^2 dr + \frac{\|\psi_3(\cdot)\|^2}{M_1}\int_\tau^\xi e^{\lambda r} dr \\
&\leq \frac{\beta_3 e^{\lambda\rho}}{M_1(1-\rho^*)}\int_{-\rho}^0 e^{\lambda(r+\tau)-2\gamma r}e^{2\gamma r}\|u(\tau+r)\|^2 dr \\
&+ \frac{\beta_3 e^{\lambda\rho}}{M_1(1-\rho^*)}\int_\tau^\xi e^{\lambda r}\|u(r)\|^2 dr + \frac{\|\psi_3(\cdot)\|^2}{M_1}\int_\tau^\xi e^{\lambda r} dr \\
&\leq \frac{\beta_3 e^{\lambda\tau}e^{2\gamma\rho}}{M_1(1-\rho^*)(2\gamma-\lambda)}\|\varphi\|^2_{C_{\gamma,H}} + M_1\int_\tau^\xi e^{\lambda r}\|u(r)\|^2 dr + \frac{\|\psi_3(\cdot)\|^2}{M_1}\int_\tau^\xi e^{\lambda r} dr.
\end{aligned}
$$

It can be inferred from (2.22) and (2.23) that

$$
\begin{aligned}
&2\int_\tau^\xi e^{\lambda r}\int_{\mathcal{O}} g(x, u(r-\varrho(r)))u(r)dr \\
&\leq 2M_1\int_\tau^\xi e^{\lambda r}\|u(r)\|^2 dr + ce^{\lambda\tau}\|\varphi\|^2_{C_{\gamma,H}} + c\int_\tau^\xi e^{\lambda r} dr.
\end{aligned} \tag{2.24}
$$

By $\frac{q}{q-\eta} < \frac{2q-2}{q-\eta}$, $\frac{q}{q-1} < 2$ and $2M_1 \leq \frac{\lambda}{8}$ defined in (2.15), plugging (2.24) into (2.21), we have

$$
\begin{aligned}
&\|u(\xi)\|^2 + \int_\tau^\xi e^{\lambda(r-\xi)}(2\|u(r)\|_V^p + \beta_1\|u(r)\|_q^q + \frac{\lambda}{8}\|u(r)\|^2)dr \\
&\leq e^{\lambda(\tau-\xi)}\|u(\tau)\|^2 + ce^{\lambda(\tau-\xi)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_\tau^\xi e^{\lambda(r-\xi)}\|u_r\|^2_{C_{\gamma,H}} dr \\
&+ \frac{4}{\lambda}\int_\tau^\xi e^{\lambda(r-\xi)}\|J(r,\cdot)\|^2 dr + c\int_\tau^\xi e^{\lambda(r-\xi)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr.
\end{aligned} \tag{2.25}
$$

Then, multiplying (2.25) by $e^{2\gamma s}$, and replacing ξ by $t+s$, and taking the supremum in $s \in [\tau - t, 0]$, we obtain that

$$
\begin{aligned}
&\sup_{s\in[\tau-t,0]} e^{2\gamma s}\|u(t+s,\tau,\omega,\varphi)\|^2 \\
&\le \sup_{s\in[\tau-t,0]} e^{(2\gamma-\lambda)s}\Big[e^{\lambda(\tau-t)}\|u(\tau)\|^2 + ce^{\lambda(\tau-t)}\|\varphi\|^2_{C_{\gamma,H}} \\
&+ \frac{4m_1^2}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr + \frac{4}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr \\
&+ c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{q}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^{\frac{q}{q-1}} + 1)dr\Big] \qquad (2.26) \\
&\le e^{\lambda(\tau-t)}\|u(\tau)\|^2 + ce^{\lambda(\tau-t)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ \frac{4}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr + c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr,
\end{aligned}
$$

where we have used $\lambda \le 2\gamma$ defined in (2.15). For $s \in (-\infty, \tau - t]$, we consider

$$
\begin{aligned}
&\sup_{s\in(-\infty,\tau-t]} e^{2\gamma s}\|u(t+s,\tau,\omega,\varphi)\|^2 \\
&= \sup_{s\in(-\infty,\tau-t]} e^{2\gamma s}\|u_\tau(t+s-\tau,\tau,\omega,\varphi)\|^2 \\
&= \sup_{s\in(-\infty,\tau-t]} e^{2\gamma s}\|\varphi(t+s-\tau)\|^2 \\
&= \sup_{s\in(-\infty,\tau-t]} e^{-2\gamma(t-\tau)}e^{2\gamma(t+s-\tau)}\|\varphi(t+s-\tau)\|^2 \\
&= e^{-2\gamma(t-\tau)}\|\varphi\|^2_{C_{\gamma,H}} \le e^{-\lambda(t-\tau)}\|\varphi\|^2_{C_{\gamma,H}}. \qquad (2.27)
\end{aligned}
$$

Further

$$
\begin{aligned}
\|u_t(\cdot,\tau,\omega,\varphi)\|^2_{C_{\gamma,H}} \le \max\Big\{&\sup_{s\in(-\infty,\tau-t]} e^{2\gamma s}\|u(t+s,\tau,\omega,\varphi)\|^2, \\
&\sup_{s\in[\tau-t,0]} e^{2\gamma s}\|u(t+s,\tau,\omega,\varphi)\|^2\Big\}. \qquad (2.28)
\end{aligned}
$$

Using the fact that $\|u(\tau)\|^2 = \|\varphi(0)\|^2 \le \|\varphi\|^2_{C_{\gamma,H}}$, we deduce from (2.26)-(2.28) that for all $t \ge \tau$,

$$
\begin{aligned}
&\|u_t(\cdot,\tau,\omega,\varphi)\|^2_{C_{\gamma,H}} \le ce^{\lambda(\tau-t)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ \frac{4}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr + c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr, \qquad (2.29)
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&e^{\lambda t}\|u_t(\cdot,\tau,\omega,\varphi)\|^2_{C_{\gamma,H}} \le ce^{\lambda\tau}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_\tau^t e^{\lambda r}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ \frac{4}{\lambda}\int_\tau^t e^{\lambda r}\|J(r,\cdot)\|^2 dr + c\int_\tau^t e^{\lambda r}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \qquad (2.30)
\end{aligned}
$$

Hence, by (2.14) and using Gronwall's lemma we have

$$\|u_t(\cdot,\tau,\omega,\varphi)\|^2_{C_{\gamma,H}} \le ce^{(\frac{4m_1^2}{\lambda}-\lambda)(t-\tau)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4}{\lambda}\int_\tau^t e^{(\lambda-\frac{4m_1^2}{\lambda})(r-t)}\|J(r,\cdot)\|^2 dr$$
$$+ c\int_\tau^t e^{(\lambda-\frac{4m_1^2}{\lambda})(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr, \tag{2.31}$$

which completes the proof. □

Lemma 2.6. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold. Let* $\tau\in\mathbb{R}$, $\omega\in\Omega, T>0$, *and* B *be a bounded set of* $C_{\gamma,H}$. *Then, there exists* $c=c(M_1,\lambda,\gamma,B,T)>0$ *such that a weak solution* $u(\cdot)$ *of system* (2.4) *with initial condition* $\varphi\in B$ *satisfies*

$$\|u(t,\tau,\omega,\varphi)\|^2 \le e^{-\lambda(t-r)}\|u(r)\|^2 + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma$$
$$+ c\int_r^t e^{-\lambda(t-\sigma)}(|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2 + 1)d\sigma + c. \tag{2.32}$$

for all $\tau\le r\le t\le\tau+T$.

Proof. By (2.20) we have

$$\frac{d}{dt}\|u\|^2 + 2\|u\|^p_V + \beta_1\|u\|^q_q + \frac{5}{4}\lambda\|u\|^2$$
$$\le 2\int_{\mathcal{O}} g(x,u(t-\varrho(t)))udx + \frac{4m_1^2}{\lambda}\|u_t\|^2_{C_{\gamma,H}} + \frac{4}{\lambda}\|J(t,\cdot)\|^2$$
$$+ c(1+\|\psi_4(t,\cdot)\|^{\frac{q}{q-\eta}}_{\frac{q}{q-\eta}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-\eta}} + \|\psi_5(t,\cdot)\|^{\frac{q}{q-1}}_{\frac{q}{q-1}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-1}}). \tag{2.33}$$

Multiplying (2.33) by $e^{\lambda\sigma}$ and integrating it in $\sigma\in[r,t]$ with $\tau\le r\le t\le\tau+T$, we have for all $t\ge r$,

$$\|u(t)\|^2 + \int_r^t e^{-\lambda(t-\sigma)}(2\|u(\sigma)\|^p_V + \beta_1\|u(\sigma)\|^q_q + \frac{1}{4}\lambda\|u(\sigma)\|^2)d\sigma$$
$$\le e^{-\lambda(t-r)}\|u(r)\|^2 + 2\int_r^t e^{-\lambda(t-\sigma)}\int_{\mathcal{O}} g(x,u(\sigma-\varrho(\sigma)))u(\sigma)dxd\sigma$$
$$+ \frac{4m_1^2}{\lambda}\int_r^t e^{-\lambda(t-\sigma)}\|u_\sigma\|^2_{C_{\gamma,H}}d\sigma + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma$$
$$+ c\int_r^t e^{-\lambda(t-\sigma)}(1+|\zeta_\delta(\theta_\sigma\omega)|^{\frac{q}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^{\frac{q}{q-1}})d\sigma. \tag{2.34}$$

Similar to (2.22)-(2.23), we have

$$2\int_r^t e^{-\lambda(t-\sigma)}\int_{\mathcal{O}} g(x,u(\sigma-\varrho(\sigma)))u(\sigma)dxd\sigma \tag{2.35}$$
$$\le M_1\int_r^t e^{-\lambda(t-\sigma)}\|u(\sigma)\|^2 d\sigma + \frac{1}{M_1}\int_r^t e^{-\lambda(t-\sigma)}\|g(\cdot,u(\sigma-\varrho(\sigma)))\|^2 d\sigma$$
$$\le M_1\int_r^t e^{-\lambda(t-\sigma)}\|u(\sigma)\|^2 d\sigma + \frac{\beta_3 e^{-\lambda(t-\rho)}}{M_1(1-\rho^*)}\int_{r-\rho}^t e^{\lambda\sigma}\|u(\sigma)\|^2 d\sigma + c\int_r^t e^{-\lambda(t-\sigma)}d\sigma$$
$$\le 2M_1\int_r^t e^{-\lambda(t-\sigma)}\|u(\sigma)\|^2 d\sigma + \frac{\beta_3 e^{-\lambda(t-r)}e^{2\gamma\rho}}{M_1(1-\rho^*)(2\gamma-\lambda)}\|u_r\|^2_{C_{\gamma,H}} + c\int_r^t e^{-\lambda(t-\sigma)}d\sigma.$$

By $2M_1 \leq \frac{\lambda}{8}$ defined in (2.15), we have

$$\begin{aligned}
&\|u(t)\|^2 + \int_r^t e^{-\lambda(t-\sigma)}(2\|u(\sigma)\|_V^p + \beta_1\|u(\sigma)\|_q^q + \frac{1}{8}\lambda\|u(\sigma)\|^2)d\sigma \\
&\leq e^{-\lambda(t-r)}\|u(r)\|^2 + \frac{\beta_3 e^{2\gamma\rho}}{M_1(1-\rho^*)(2\gamma-\lambda)}e^{-\lambda(t-r)}\|u_r\|^2_{C_{\gamma,H}} \\
&\quad + \frac{4m_1^2}{\lambda}\int_r^t e^{-\lambda(t-\sigma)}\|u_\sigma\|^2_{C_{\gamma,H}}d\sigma + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma \\
&\quad + c\int_r^t e^{-\lambda(t-\sigma)}(1 + |\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2)d\sigma. \qquad (2.36)
\end{aligned}$$

By (2.6), **H1** and $\varphi \in B$, we can inferred from (2.18) in Lemma 2.5 that $\|u_t\|^2_{C_{\gamma,H}} \leq C_1 = C_1(B,T)$ for $t \in [\tau, \tau+T]$, we have

$$\begin{aligned}
&\|u(t)\|^2 + \int_r^t e^{-\lambda(t-\sigma)}(2\|u(\sigma)\|_V^p + \beta_1\|u(\sigma)\|_q^q + \frac{1}{8}\lambda\|u(\sigma)\|^2)d\sigma \\
&\leq e^{-\lambda(t-r)}\|u(r)\|^2 + cC_1 + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma \\
&\quad + c\int_r^t e^{-\lambda(t-\sigma)}(1 + |\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2)d\sigma. \qquad (2.37)
\end{aligned}$$

Then we conclude (2.32). □

We can draw the following results immediately from (2.37):

Corollary 2.7. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold. Let* $\tau \in \mathbb{R}$, $\omega \in \Omega, T > 0$, *and let* $B \subseteq C_{\gamma,H}$ *be a bounded set. Then, there exists* $c = c(M_1, \lambda, \gamma, B, T) > 0$ *such that a weak solution* $u(\cdot)$ *of system* (2.4) *with initial condition* $\varphi \in B$ *satisfies, for all* $\tau \leq r \leq t \leq \tau + T$,

$$\begin{aligned}
&\int_r^t (\|u(\sigma)\|_V^p + \|u(\sigma)\|_q^q + \|u(\sigma)\|^2)d\sigma \qquad (2.38) \\
&\leq c\|u(r)\|^2 + c\int_r^t \|J(\sigma,\cdot)\|^2 d\sigma + c\int_r^t (1 + |\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2)d\sigma + c.
\end{aligned}$$

Proof. Consider $C_2 = \min\{2, \beta_1, \frac{1}{8}\lambda\}$. By (2.37) we have, for all $\tau \leq r \leq t \leq \tau + T$,

$$\begin{aligned}
&e^{-\lambda(t-\tau)}C_2\int_r^t (\|u(\sigma)\|_V^p + \|u(\sigma)\|_q^q + \|u(\sigma)\|^2)d\sigma \\
&\leq C_2\int_r^t e^{-\lambda(t-\sigma)}(\|u(\sigma)\|_V^p + \|u(\sigma)\|_q^q + \|u(\sigma)\|^2)d\sigma \\
&\leq e^{-\lambda(t-r)}\|u(r)\|^2 + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma \\
&\quad + c\int_r^t e^{-\lambda(t-\sigma)}(1 + |\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2)d\sigma + c,
\end{aligned}$$

which implies (2.38). □

Theorem 2.8. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold and* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varphi \in C_{\gamma,H}$. *Then, equation* (2.4) *admits at least one weak solution.*

Proof. (1) First, we consider the Galerkin approximations to equation (2.4). It follows from [24] that $H_0^r(\mathcal{O}) \subset V \cap L^q(\mathcal{O})$ for $r \geq \max\{\frac{N(q-2)}{2q}, \frac{2p+N(p-2)}{2p}\}$. We consider a special basis of H consisting of elements $\{w_j\} \subset H_0^r(\mathcal{O})$, and denote by $W_n = span[w_1, \dots, w_n]$. Let the projector $P_n u = \sum_{j=1}^{n}(u, w_j)w_j$, then $\bigcup_{n\in\mathbb{N}} W_n$ is dense in $V \cap L^q(\mathcal{O})$.

For fixed $n \in \mathbb{N}$, consider $\widetilde{u}^n(t) = \sum_{j=1}^{n} \widetilde{\mu}_j^n(t)w_j$, where $\widetilde{\mu}_j^n$ are required to satisfy the following system:

$$\begin{aligned}
&\frac{d}{dt}(\widetilde{u}^n(t), w_j) + \langle \Delta_p \widetilde{u}^n(t), w_j\rangle + \lambda(\widetilde{u}^n(t), w_j)\\
&= (f(t,x,\widetilde{u}^n(t)), w_j) + (g(x, \widetilde{u}_t^n(-\varrho(t))), w_j) + \Big(\int_{-\infty}^{0} F(x,l,\widetilde{u}_t^n(l))dl, w_j\Big)\\
&\quad + (J(t,\cdot), w_j) + \zeta_\delta(\theta_t\omega)(h(t,x,\widetilde{u}^n(t)), w_j), \quad 1 \leq j \leq n,
\end{aligned} \tag{2.39}$$

where initial data is $\widetilde{u}^n(\tau+s) = P_n\varphi(s)$ for $s \in (-\infty, 0]$. It follows from [18, Theorem 1.1] the existence of local solutions for (2.39). Now, we show that solutions do exist in $[\tau, \tau+T]$ with $T > 0$.

(2) From (2.32) and setting $r = \tau$ in (2.38), we obtain for all $T > 0$,

$$\{\widetilde{u}^n\} \text{ is bounded in } L^\infty(\tau,\tau+T;H) \cap L^p(\tau,\tau+T;V) \cap L^q(\tau,\tau+T;L^q(\mathcal{O})).$$

By (2.10), the Hölder and Minkowski inequalities, we obtain

$$\begin{aligned}
&\int_\tau^t \int_{\mathcal{O}} \zeta_\delta(\theta_r\omega)h(t,x,\widetilde{u}^n(r))w\,dx\,dr\\
&\leq \int_\tau^t \Big(\int_{\mathcal{O}} \Big|\zeta_\delta(\theta_r\omega)\psi_4(r,x)|\widetilde{u}^n(r)|^{\eta-1} + \zeta_\delta(\theta_r\omega)\psi_5(r,x)\Big|^{\frac{q}{q-1}}dx\Big)^{\frac{q-1}{q}}dr \int_\tau^t \|w\|_q dr\\
&\leq 2^{\frac{1}{q}}\int_\tau^t \Big[\Big(\int_{\mathcal{O}} \big|\zeta_\delta(\theta_r\omega)\psi_4(r,x)\big|^{\frac{q}{q-1}}\big|\widetilde{u}^n(r)\big|^{\frac{q(\eta-1)}{q-1}}dx\Big)^{\frac{q-1}{q}}\\
&\quad + \Big(\int_{\mathcal{O}} \big|\zeta_\delta(\theta_r\omega)\psi_5(r,x)\big|^{\frac{q}{q-1}}dx\Big)^{\frac{q-1}{q}}\Big]dr \int_\tau^t \|w\|_q dr\\
&\leq c\int_\tau^t \Big[\Big(\Big(\int_{\mathcal{O}}|\zeta_\delta(\theta_r\omega)\psi_4(r,x)|^{\frac{q}{q-\eta}}dx\Big)^{\frac{q-\eta}{q-1}}\Big(\int_{\mathcal{O}}|\widetilde{u}^n(r)|^q dx\Big)^{\frac{\eta-1}{q-1}}\Big)^{\frac{q-1}{q}}\\
&\quad + \Big(\int_{\mathcal{O}}|\zeta_\delta(\theta_r\omega)\psi_5(r,x)|^{\frac{q}{q-1}}dx\Big)^{\frac{q-1}{q}}\Big]dr \int_\tau^t \|w\|_q dr\\
&\leq c\Big(\int_\tau^t \|\psi_4(r,\cdot)\|_{\frac{2q-2}{q-\eta}}dr + \int_\tau^t \|\widetilde{u}^n\|_q^{\eta-1}dr + \int_\tau^t \|\psi_5(r,\cdot)\|_2 dr\Big)\int_\tau^t \|w\|_q dr,
\end{aligned}$$

which together with (2.8) yields

$$\{f(t,x,\widetilde{u}^n)\} \text{ and } \{\zeta_\delta(\theta_t\omega)h(t,x,\widetilde{u}^n)\} \text{ are bounded in } L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O})).$$

Recall from Lemma 2.6 that

$$\|\widetilde{u}_t^n\|_{C_{\gamma,H}}^2 \leq C_1, \quad \forall t \in [\tau,\tau+T], \quad \varphi \in B \subset C_{\gamma,H}, \quad n \in \mathbb{N}. \tag{2.40}$$

Thus, by (2.40), Remark 2.1 and Remark 2.2, we have $\{g(x,\widetilde{u}^n_\cdot)\}$ is bounded in $L^2(\tau,\tau+T;H)$, and $\{\int_{-\infty}^0 F(x,l,\widetilde{u}^n_\cdot(l))dl\}$ is bounded in $L^2(\tau,\tau+T;H)$. Moreover, we can deduce that $\{\Delta_p\widetilde{u}^n\}$ is bounded in $L^{\hat{p}}(\tau,\tau+T;V^*)$ from (2.3). From the above, we know that $\{\frac{d\widetilde{u}^n}{dt}\}$ is bounded in $L^{\hat{p}}(\tau,\tau+T;H^{-r}(\mathcal{O}))$ by [8].

Hence, there exist a subsequence (relabeled the same) $\{\widetilde{u}^n\}$, an element $\widetilde{u}\in L^\infty(\tau,\tau+T;H)\cap L^p(\tau,\tau+T;V)\cap L^q(\tau,\tau+T;L^q(\mathcal{O}))$ with $\frac{d\widetilde{u}}{dt}\in L^{\hat{p}}(\tau,\tau+T;H^{-r}(\mathcal{O}))$, $\chi_1\in L^{\hat{p}}(\tau,\tau+T;V^*)$, $\chi_2\in L^2(\tau,\tau+T;H)$, $\chi_3\in L^2(\tau,\tau+T;H)$, $\chi_4\in L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O}))$, $\chi_5\in L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O}))$ such that, up to subsequence,

$$\begin{cases} \widetilde{u}^n\to\widetilde{u} \ \text{ weakly star in } \ L^\infty(\tau,\tau+T;H),\\ \widetilde{u}^n\to\widetilde{u} \ \text{ weakly in } \ L^p(\tau,\tau+T;V) \ \text{ and } \ L^q(\tau,\tau+T;L^q(\mathcal{O})),\\ \widetilde{u}^n\to\widetilde{u} \ \text{ strongly in } \ L^p(\tau,\tau+T;L^p(\mathcal{O})),\\ \Delta_p\widetilde{u}^n\to\chi_1 \ \text{ weakly in } \ L^{\hat{p}}(\tau,\tau+T;V^*),\\ g(x,\widetilde{u}^n_\cdot)\to\chi_2 \ \text{ weakly in } \ L^2(\tau,\tau+T;H),\\ \displaystyle\int_{-\infty}^0 F(x,l,\widetilde{u}^n_\cdot(l))dl\to\chi_3 \ \text{ weakly in } \ L^2(\tau,\tau+T;H),\\ \dfrac{d\widetilde{u}^n}{dt}\to\dfrac{d\widetilde{u}}{dt} \ \text{ weakly in } \ L^{\hat{p}}(\tau,\tau+T;H^{-r}(\mathcal{O})),\\ f(\cdot,x,\widetilde{u}^n)\to\chi_4 \ \text{ weakly in } \ L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O})),\\ \zeta_\delta(\theta.\omega)h(\cdot,x,\widetilde{u}^n)\to\chi_5 \ \text{ weakly in } \ L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O})), \end{cases} \tag{2.41}$$

for all $T>0$. From [24, Lemma 1.3], we can identify that $\chi_1=\Delta_p\widetilde{u}$, $\chi_4=f(\cdot,x,\widetilde{u})$ and $\chi_5=\zeta_\delta(\theta.\omega)h(\cdot,x,\widetilde{u})$.

By the compact embedding $H\hookrightarrow H^{-r}(\mathcal{O})$ and (2.40), we can infer from the Arzelà-Ascoli theorem that $\widetilde{u}^n\to\widetilde{u}$ in $C([\tau,\tau+T];H^{-r}(\mathcal{O}))$. Then, by (2.40) again, it is not difficult to prove that for any sequence $t_n\to t_0$ with $t_n,t_0\in[\tau,\tau+T]$,

$$\widetilde{u}^n(t_n)\to\widetilde{u}(t_0) \ \text{ weakly in } \ H. \tag{2.42}$$

In fact, we want to show that

$$\widetilde{u}^n(\cdot)\to\widetilde{u}(\cdot) \quad \text{in} \quad C([\tau,\tau+T];H). \tag{2.43}$$

By (2.41), passing to the limit in (2.39), we consider a solution $\widetilde{u}\in C([\tau,\tau+T];H)$ of a similar problem to (2.4), that is, for all $\eta\in V\cap L^q(\mathcal{O})$

$$\begin{aligned} &\frac{d}{dt}(\widetilde{u},\eta)+\langle\Delta_p\widetilde{u},\eta\rangle+\lambda(u,\eta)=(f(t,\cdot,\widetilde{u}),\eta)+(\chi_2,\eta)+(\chi_3,\eta)\\ &\quad+(J(t,\cdot),\eta)+(\zeta_\delta(\theta_t\omega)h(t,\cdot,\widetilde{u}),\eta), \end{aligned} \tag{2.44}$$

with the initial data $\widetilde{u}(\tau+s)=\varphi(s)$ for $s\in(-\infty,0]$. By (2.40), Remark 2.1 and Remark 2.2, for all $\tau\le r\le t\le\tau+T$,

$$\int_r^t\|\chi_2(\sigma)\|^2d\sigma\le\liminf_{n\to+\infty}\int_r^t\|g(\sigma,\widetilde{u}^n_\sigma)\|^2d\sigma\le c(t-r),$$

and

$$\int_r^t\|\chi_3(\sigma)\|^2d\sigma\le\liminf_{n\to+\infty}\int_r^t\Big\|\int_{-\infty}^0F(\sigma,l,\widetilde{u}^n_\sigma(l))dl\Big\|^2d\sigma\le c(t-r).$$

Therefore, by the same method as Lemma 2.6, $\widetilde{u}$ can also satisfies (2.32). We define functions $J_n, J : [\tau, \tau + t] \to \mathbb{R}$ by

$$J_n(t) = \|\widetilde{u}^n(t)\|^2 - c\int_\tau^t \|J(\sigma,\cdot)\|^2 d\sigma - c\int_\tau^t (|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2 + 1)d\sigma, \tag{2.45}$$

$$J(t) = \|\widetilde{u}(t)\|^2 - c\int_\tau^t \|J(\sigma,\cdot)\|^2 d\sigma - c\int_\tau^t (|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2 + 1)d\sigma. \tag{2.46}$$

where the c in (2.45) is the same as (2.46). It is clear that J_n and J are non-increasing and continuous functions. By (2.41) and [6, Lemma 11], we can deduce that

$$J_n(t) \to J(t), \quad \text{for a.e. } t \in [\tau, \tau + T]. \tag{2.47}$$

Then, we have

$$\limsup_{n\to+\infty} \|\widetilde{u}^n(t_n)\| \le \|\widetilde{u}(t_0)\|, \tag{2.48}$$

which together with (2.42) implies (2.43). By Remark 2.1 and (2.43), we can deduce that $\chi_2 = g(x, \widetilde{u}.)$.

By [29, Theorem 5], for the initial datum $\varphi \in C_{\gamma,H}$, we know that $P_n\varphi \to \varphi$ in $C_{\gamma,H}$. Indeed,

$$\begin{aligned}
&\sup_{s\le 0} e^{\gamma s}\|\widetilde{u}^n(t+s) - \widetilde{u}(t+s)\| \\
&\le \max\Big\{ \sup_{s\in[\tau-t,0]} e^{\gamma s}\|\widetilde{u}^n(t+s) - \widetilde{u}(t+s)\|, \\
&\qquad \sup_{s\in(-\infty,\tau-t]} e^{\gamma(\tau-t)}e^{\gamma(s+t-\tau)}\|\widetilde{u}^n_\tau(t+s-\tau) - \widetilde{u}_\tau(t+s-\tau)\|\Big\} \\
&\le \max\big\{ \sup_{s\in[\tau,t]} \|\widetilde{u}^n(s) - \widetilde{u}(s)\|, e^{\gamma(\tau-t)}\|P_n\varphi - \varphi\|_{C_{\gamma,H}}\big\} \to 0,
\end{aligned}$$

which implies that for all $t \in [\tau, \tau + T]$

$$\widetilde{u}^n_t \to \widetilde{u}_t \quad \text{in} \quad C_{\gamma,H}. \tag{2.49}$$

Therefore, by Remark 2.3, we deduce $\chi_3 = \int_{-\infty}^0 F(x, l, \widetilde{u}.(l))dl$. Finally, we can pass to the limit in (2.39), concluding that $\widetilde{u}$ is a solution of (2.4). □

2.3. Regularity of solutions

Now, we can show a regularity result for the solution of equation (2.4).

Theorem 2.9. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold and* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varphi \in C_{\gamma,H}$. *Then any weak solutions* u *to the equation* (2.4) *belongs to* $C_w((\tau, \tau + T]; V)$. *In particular, if* $\varphi(0) \in V \cap L^q(\mathcal{O})$, *then* $u \in C_w([\tau, \tau + T]; V)$.

In order to prove this result, we need the next lemma. In the proof of the latter, we will use the following Gronwall-type lemma for the estimate of the solutions in the regular space.

Lemma 2.10. *([37]) Let y, g and h be three nonnegative and locally integrable functions on $\mathbb{R}$, thus $\frac{dy}{dt}$ is also locally integrable and*

$$\frac{dy}{dt} + by(t) + g(t) \leq h(t), t \in \mathbb{R}. \tag{2.50}$$

Then, for every $t > \tau$ with $\tau \in \mathbb{R}$, one has

$$y(t) \leq \frac{1}{t-\tau} \int_\tau^t y(r) e^{b(r-t)} dr + \int_\tau^t h(r) e^{b(r-t)} dr. \tag{2.51}$$

In particular, if $b = 0$ then

$$y(t) \leq \frac{1}{t-\tau} \int_\tau^t y(r) dr + \int_\tau^t h(r) dr. \tag{2.52}$$

Lemma 2.11. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold. Let $\tau \in \mathbb{R}$, $\omega \in \Omega, T > 0$, and B be a bounded set of $C_{\gamma,H}$. Then there exists $c = c(M_1, \lambda, \gamma, B, T) > 0$ such that*

(1) A weak solution $u(\cdot)$ of system (2.4) *with initial condition $\varphi \in B$ satisfies*

$$\int_\tau^t \|u(r)\|_{2q-2}^{2q-2} dr \leq c \int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1) dr + c, \tag{2.53}$$

for all $t \in (\tau, \tau + T]$.

(2) A weak solution $u(\cdot)$ of system (2.4) *with initial condition $\varphi \in B$ and $\varphi(0) \in V \cap L^q(\mathcal{O})$ satisfies*

$$\int_\tau^t \|u(r)\|_{2q-2}^{2q-2} dr \leq c \int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1) dr + c, \tag{2.54}$$

for all $t \in [\tau, \tau + T]$.

Proof. Multiplying (2.4) by $|u|^{q-2}u$, and integrating over $\mathcal{O}$, we have

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{q}\|u\|_q^q) + \int_\mathcal{O} \Delta_p u(|u|^{q-2}u) dx + \lambda \|u\|_q^q \\
&= \int_\mathcal{O} f(t,x,u)(|u|^{q-2}u) dx + \int_\mathcal{O} g(x, u(t-\varrho(t)))(|u|^{q-2}u) dx \\
&+ \int_\mathcal{O} \big(\int_{-\infty}^0 F(x,l,u(t+l)) dl \big)(|u|^{q-2}u) dx + \int_\mathcal{O} J(t,x)(|u|^{q-2}u) dx \\
&+ \zeta_\delta(\theta_t\omega) \int_\mathcal{O} h(t,x,u)(|u|^{q-2}u) dx.
\end{aligned} \tag{2.55}$$

It is easy to check that for any $q > 2$,

$$\int_\mathcal{O} \Delta_p u(|u|^{q-2}u) dx \geq 0. \tag{2.56}$$

By (2.7) and the Young inequality, we deduce that

$$\begin{aligned}
\int_\mathcal{O} f(t,x,u)(|u|^{q-2}u) dx &\leq \int_\mathcal{O} [-\beta_1(|u|^q + |u|^p) + \psi_1(t,x)]|u|^{q-2} dx \\
&\leq -\beta_1 \|u\|_{2q-2}^{2q-2} - \beta_1 \|u\|_{p+q-2}^{p+q-2} + \int_\mathcal{O} \psi_1(t,x)|u|^{q-2} dx \\
&\leq -\beta_1 \|u\|_{2q-2}^{2q-2} + \frac{\lambda}{2}\|u\|_q^q + \frac{1}{2\lambda}\|\psi_1(t,\cdot)\|_{\frac{q}{2}}^{\frac{q}{2}}.
\end{aligned} \tag{2.57}$$

By Remark 2.2 and the Young inequality we have

$$\int_{\mathcal{O}}\Big(\int_{-\infty}^{0}F(x,l,u(t+l))dl\Big)(|u|^{q-2}u)dx\le\frac{2m_1^2}{\beta_1}\|u_t\|_{C_{\gamma,H}}^2+\frac{2m_0^2}{\beta_1}+\frac{\beta_1}{4}\|u\|_{2q-2}^{2q-2}.\tag{2.58}$$

By the Young inequality again,

$$\int_{\mathcal{O}}J(t,x)(|u|^{q-2}u)dx\le\frac{\beta_1}{4}\|u\|_{2q-2}^{2q-2}+\frac{1}{\beta_1}\|J(t,\cdot)\|^2.\tag{2.59}$$

Jointly with **H5**, we have

$$\begin{aligned}&\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}}h(t,x,u)(|u|^{q-2}u)dx\\&\le\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}}(\psi_4(t,x)|u|^{\eta-1}+\psi_5(t,x))(|u|^{q-2}u)dx\\&\le\frac{\beta_1}{4}\|u\|_{2q-2}^{2q-2}+c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{2q-2}{q-\eta}}^{\frac{2q-2}{q-\eta}}+c|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.\end{aligned}\tag{2.60}$$

For $q>2$, we substitute (2.56)-(2.60) into (2.55) to yield

$$\begin{aligned}&\frac{d}{dt}\|u\|_q^q+2\lambda\|u\|_q^q+\frac{\beta_1}{2}\|u\|_{2q-2}^{2q-2}\le q\int_{\mathcal{O}}g(x,u(t-\varrho(t)))(|u|^{q-2}u)dx+\frac{q}{2\lambda}\|\psi_1(t,\cdot)\|_{\frac{q}{2}}^{\frac{q}{2}}\\&+\frac{2qm_1^2}{\beta_1}\|u_t\|_{C_{\gamma,H}}^2+\frac{2qm_0^2}{\beta_1}+\frac{q}{\beta_1}\|J(t,\cdot)\|^2+c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{2q-2}{q-\eta}}^{\frac{2q-2}{q-\eta}}+c|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.\end{aligned}\tag{2.61}$$

Then, using (2.51) in Lemma 2.10 over the interval $[\tau,t]$, we have

$$\begin{aligned}&\|u(t)\|_q^q+\frac{\beta_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr\\&\le\frac{1}{\varepsilon}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_q^q dr+q\int_\tau^t e^{\lambda(r-t)}\int_{\mathcal{O}}g(x,u(r-\varrho(r)))(|u(r)|^{q-2}u(r))dxdr\\&+c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2dr+c\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2dr\\&+c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}}+|\zeta_\delta(\theta_r\omega)|^2+1)dr,\end{aligned}\tag{2.62}$$

where $\varepsilon\in(0,t-\tau)$. By the same method as (2.22)-(2.23), for all $\tau<t\le\tau+T$, we derive

$$\begin{aligned}&q\int_\tau^t e^{\lambda(r-t)}\int_{\mathcal{O}}g(x,u(r-\varrho(r)))(|u(r)|^{q-2}u(r))dxdr\\&\le\frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr+\frac{q}{2M_1}\int_\tau^t e^{\lambda(r-t)}\|g(x,u(r-\varrho(r)))\|^2dr\\&\le\frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr+\frac{q\beta_3e^{\lambda\rho}}{2M_1(1-\rho^*)}\int_{\tau-\rho}^{\tau}e^{\lambda(r-t)}\|u(r)\|^2dr\\&+\frac{q\beta_3e^{\lambda\rho}}{2M_1(1-\rho^*)}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^2dr+c\int_\tau^t e^{\lambda(r-t)}dr\\&\le\frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr+\frac{qM_1e^{2\gamma\rho}}{2\gamma-\lambda}e^{-\lambda(t-\tau)}\|u_\tau\|_{C_{\gamma,H}}^2\\&+\frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^2dr+c\int_\tau^t e^{\lambda(r-t)}dr,\end{aligned}\tag{2.63}$$

where $M_1 = \frac{\sqrt{\beta_3 e^{\lambda\rho}}}{\sqrt{1-\rho^*}}$. By (2.63) and let $r = \tau$ in (2.36), we have for all $t \in (\tau, \tau+T]$,

$$
\begin{aligned}
&\|u(t)\|_q^q + \frac{\vartheta}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr \\
&\le \frac{1}{\varepsilon}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_q^q dr + \frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^2 dr \\
&\quad + ce^{-\lambda(t-\tau)}\|u_\tau\|_{C_{\gamma,H}}^2 + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2 dr + c\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr \\
&\quad + c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr \\
&\le ce^{-\lambda(t-\tau)}\|u_\tau\|_{C_{\gamma,H}}^2 + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2 dr \\
&\quad + c\int_\tau^t e^{\lambda(r-t)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr,
\end{aligned} \tag{2.64}
$$

where $\vartheta = \beta_1 - qM_1 > 0$ is defined in (2.15). Since $\varphi \in B$, thus $\|u_r\|_{C_{\gamma,H}}^2 \le C_1 = C_1(B,T)$ for $r \in [\tau, \tau+T]$ as proved in Lemma 2.6. Thus for all $t \in (\tau, \tau+T]$,

$$
\int_\tau^t \|u(r)\|_{2q-2}^{2q-2}dr \le cC_1 + c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \tag{2.65}
$$

(2) Applying the general Gronwall inequality to (2.61), we have for all $t \in [\tau, \tau+T]$,

$$
\begin{aligned}
&\|u(t)\|_q^q + \frac{\beta_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr \\
&\le e^{-\lambda(t-\tau)}\|u_\tau\|_q^q dr + q\int_\tau^t e^{\lambda(r-t)}\int_{\mathcal{O}} g(x, u(r-\varrho(r)))(|u(r)|^{q-2}u(r))dxdr \\
&\quad + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2 dr + c\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr \\
&\quad + c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr,
\end{aligned} \tag{2.66}
$$

By (2.63) and similar to (2.64), we have

$$
\begin{aligned}
&\|u(t)\|_q^q + \frac{\vartheta}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr \\
&\le e^{-\lambda(t-\tau)}\|u_\tau\|_q^q dr + ce^{-\lambda(t-\tau)}\|u_\tau\|_{C_{\gamma,H}}^2 + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2 dr \\
&\quad + c\int_\tau^t e^{\lambda(r-t)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr.
\end{aligned} \tag{2.67}
$$

On account of $\varphi \in B$ and $\varphi(0) \in V \cap L^q(\mathcal{O})$, it follows that for all $t \in [\tau, \tau+T]$, (2.54) holds. □

Now we can finish the proof of Theorem 2.9.

Proof of Theorem 2.9. Given $T > 0$, let $u(\cdot,\tau,\omega,\varphi)$ be a weak solution of equation (2.4), for short denoted by u. Consider that problem

$$(P_u)\begin{cases} \dfrac{dy}{dt} = -\Delta_p y - \lambda y + f(t,x,u) + g(x,u(t-\varrho(t))) \\ \quad + \displaystyle\int_{-\infty}^{0} F(x,l,u(t+l))dl + J(t,x) + h(t,x,u)\zeta_\delta(\theta_t\omega),\ t>\tau, \quad x\in\mathcal{O}, \\ y(t,x)=0,\ t>\tau, \quad x\in\partial\mathcal{O}, \\ y(\tau+s,x)=u(\tau+s,x)=\varphi(s,x), \quad s\in(-\infty,0], \quad x\in\mathcal{O}, \tau\in\mathbb{R}, \end{cases} \tag{2.68}$$

possesses a local solution by [32]. Now, we will show the local solution is a global solution.

Recall from Lemma 2.6 that

$$\|u_t\|^2_{C_{\gamma,H}} \le C_1, \quad \forall t\in[\tau,\tau+T], \quad \varphi\in B\subset C_{\gamma,H}. \tag{2.69}$$

For fixed $n\in\mathbb{N}$, consider $\widehat{u}^n(t) = \sum\limits_{j=1}^{n} \widehat{\mu}^n_j(t) w_j$, where $\widehat{\mu}^n_j$ are required to satisfy the following system:

$$\begin{aligned} &\frac{d}{dt}(\widehat{u}^n(t),w_j) + \langle \Delta_p \widehat{u}^n(t), w_j\rangle + \lambda(\widehat{u}^n(t),w_j) = \int_{\mathcal{O}} f(t,x,u(t))w_j dx \\ &+ \int_{\mathcal{O}} g(x,u_t(-\varrho(t)))w_j dx + \int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u_t(l))dl\Big) w_j dx \\ &+ \int_{\mathcal{O}} J(t,\cdot)w_j dx + \zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u(t))w_j dx, \quad 1\le j\le n. \end{aligned} \tag{2.70}$$

Multiplying (2.70) by $\widehat{\mu}^n_j(t)$, summing from $j=1$ until n we have

$$\begin{aligned} &\frac{d}{dt}\|\widehat{u}^n(t)\|^2 + 2\|\nabla \widehat{u}^n(t)\|^p_p + 2\lambda\|\widehat{u}^n(t)\|^2 \\ &= 2\int_{\mathcal{O}} f(t,x,u(t))\widehat{u}^n(t)dx + 2\int_{\mathcal{O}} g(x,u(t-\varrho(t)))\widehat{u}^n(t)dx \\ &\quad + 2\int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u_t(l))dl\Big)\widehat{u}^n(t)dx + 2\int_{\mathcal{O}} J(t,x)\widehat{u}^n(t)dx \\ &\quad + 2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u(t))\widehat{u}^n(t)dx. \end{aligned} \tag{2.71}$$

By (2.8) and the Young inequality, we have

$$2\int_{\mathcal{O}} f(t,x,u)\widehat{u}^n(t)dx \le \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2 + \frac{16\beta_2^2}{\lambda}\|u\|^{2q-2}_{2q-2} + \frac{16}{\lambda}\|\psi_2(t,\cdot)\|^2. \tag{2.72}$$

By Remark 2.2, we obtain that

$$2\int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u(t+l))dl\Big)\widehat{u}^n(t)dx \le \frac{16m_1^2}{\lambda}\|u_t\|^2_{C_{\gamma,H}} + \frac{16m_0^2}{\lambda} + \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2. \tag{2.73}$$

By Remark 2.1, we have

$$2(g(x,u(t-\varrho(t))),\widehat{u}^n(t)) \le \frac{8\beta_3}{\lambda}e^{2\gamma\rho}\|u_t\|^2_{C_{\gamma,H}} + \frac{8}{\lambda}\|\psi_3(\cdot)\|^2 + \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2. \tag{2.74}$$

By the Young inequality again,

$$2\int_{\mathcal{O}} J(t,x)\widehat{u}^n(t)dx \leq \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2 + \frac{8}{\lambda}\|J(t,\cdot)\|^2. \tag{2.75}$$

Jointly with **H5**, we have

$$\begin{aligned}
&2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)\widehat{u}^n(t)dx \\
&\leq 2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}}(\psi_4(t,x)|u|^{\eta-1}+\psi_5(t,x))\widehat{u}^n(t)dx \\
&\leq \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2 + c\|u\|_{2q-2}^{2q-2} + c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{q-1}{q-\eta}}^{\frac{2q-2}{q-\eta}} + c|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2. \qquad (2.76)
\end{aligned}$$

Substituting (2.72)-(2.76) into (2.71),

$$\begin{aligned}
&\frac{d}{dt}\|\widehat{u}^n\|^2 + 2\|\nabla\widehat{u}^n(t)\|_p^p + \frac{11}{8}\lambda\|\widehat{u}^n\|^2 \qquad (2.77)\\
&\leq c\|u\|_{2q-2}^{2q-2} + \frac{16}{\lambda}\|\psi_2(t,\cdot)\|^2 + (\frac{16m_1^2}{\lambda} + \frac{8\beta_3}{\lambda}e^{2\gamma\rho})\|u_t\|_{C_{\gamma,H}}^2 + c \\
&+\frac{8}{\lambda}\|J(t,\cdot)\|^2 + c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{q-1}{q-\eta}}^{\frac{2q-2}{q-\eta}} + c|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.
\end{aligned}$$

Multiplying (2.77) by $e^{\lambda t}$, and integrating over (τ,t) with $t\in[\tau,\tau+T]$,

$$\begin{aligned}
&\|\widehat{u}^n(t)\|^2 + 2\int_\tau^t e^{\lambda(r-t)}\|\nabla\widehat{u}^n(r)\|_p^p dr + \frac{3}{8}\lambda\int_\tau^t e^{\lambda(r-t)}\|\widehat{u}^n(r)\|^2 dr \qquad (2.78)\\
&\leq e^{-\lambda(t-\tau)}\|\varphi\|_{C_{\gamma,H}}^2 + c\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2 dr \\
&+\frac{8}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr + c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr.
\end{aligned}$$

Using the same method as (2.26)-(2.28), and by (2.64) and (2.69) we have for all $t\in(\tau,\tau+T]$,

$$\begin{aligned}
\|\widehat{u}_t^n\|_{C_{\gamma,H}}^2 &\leq ce^{-\lambda(t-\tau)}\|\varphi\|_{C_{\gamma,H}}^2 + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2 dr \\
&\quad + c\int_\tau^t e^{\lambda(r-t)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr \\
&\leq c\int_\tau^t(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c. \qquad (2.79)
\end{aligned}$$

Hence we deduce the existence of global solution of equation (P_u) on $t\in[\tau+\varepsilon,+\infty)$ with $\varepsilon\in(0,t-\tau)$. Analogously, by (2.67), (2.69) and $\varphi(0)\in V\cap L^q(\mathcal{O})$, it follows the existence of the global solution of equation (P_u) on $t\in[\tau,+\infty)$.

Now we prove the uniqueness of the solution for equation (P_u). Taking the inner product of (2.68) with $\widehat{u}^n = y_1 - y_2$, we have

$$\frac{d}{dt}\|\widehat{u}^n\|^2 + 2\langle\Delta_p y_1 - \Delta_p y_2, \widehat{u}^n\rangle + 2\lambda\|\widehat{u}^n\|^2 = 0. \tag{2.80}$$

By [12, Lemma 2.1], we know the following inequality: for every $p \geq 2$, there exists $c > 0$ such that, for all $a_1, a_2 \in \mathbb{R}$,

$$(|a_1|^{p-2}a_1 - |a_2|^{p-2}a_2)(a_1 - a_2) \geq c|a_1 - a_2|^p. \tag{2.81}$$

The above fact yields

$$\langle \Delta_p y_1 - \Delta_p y_2, \widehat{u}^n \rangle = (|\nabla y_1|^{p-2}\nabla y_1 - |\nabla y_2|^{p-2}\nabla y_2, \nabla y_1 - \nabla y_2) \geq c\|\nabla \widehat{u}^n\|_p^p. \tag{2.82}$$

Thus

$$\frac{d}{dt}\|\widehat{u}^n\|^2 + 2c\|\nabla \widehat{u}^n\|_p^p + 2\lambda\|\widehat{u}^n\|^2 \leq 0. \tag{2.83}$$

Multiplying (2.83) by $e^{\lambda t}$, integrating over $t \in [\tau, \xi]$, we have for all $\xi \in [\tau, \tau + T]$,

$$\|\widehat{u}^n(\xi, \tau, \omega, \varphi)\|^2 \leq e^{-\lambda(\xi-\tau)}\|\widehat{u}^n(\tau, \tau, \omega, \varphi)\|^2. \tag{2.84}$$

Let $\xi = t + s$ with $s \leq 0$, then using the same method as (2.26)-(2.28), we have

$$\|\widehat{u}^n_t(\cdot, \tau, \omega, \varphi)\|^2_{C_{\gamma,H}} \leq e^{-\lambda(t-\tau)}\|\widehat{u}^n_\tau\|^2_{C_{\gamma,H}}, \tag{2.85}$$

which, together with $\widehat{u}^n_\tau = 0$, implies the uniqueness of solution to (P_u). Therefore, on account of u is a solution to equation (2.4), it follows that $y = u$.

(1) Let $C_3 = \min\{2, \frac{3}{8}\lambda\}$. Then we infer from (2.78) that, for all $t \in (\tau, \tau + T]$,

$$\begin{aligned}
&C_3 e^{-\lambda(t-\tau)}\int_\tau^t (\|\nabla \widehat{u}^n(r)\|_p^p dr + \|\widehat{u}^n(r)\|^2)dr \\
&\leq C_3 \int_\tau^t e^{\lambda(r-t)}(\|\nabla \widehat{u}^n(r)\|_p^p + \|\widehat{u}^n(r)\|^2)dr \\
&\leq e^{-\lambda(t-\tau)}\|\widehat{u}^n(\tau)\|^2 + c\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^{2q-2}_{2q-2}dr + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&\quad + c\int_\tau^t e^{\lambda(r-t)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr,
\end{aligned} \tag{2.86}$$

which means that by (2.64) and (2.69), for all $t \in (\tau, \tau + T]$,

$$\begin{aligned}
&\int_\tau^t (\|\nabla \widehat{u}^n(r)\|_p^p dr + \|\widehat{u}^n(r)\|^2)dr \\
&\leq c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c.
\end{aligned} \tag{2.87}$$

Similarly, by (2.67), (2.69) and $\varphi(0) \in V \cap L^q(\mathcal{O})$, for all $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\int_\tau^t (\|\nabla \widehat{u}^n(r)\|_p^p dr + \|\widehat{u}^n(r)\|^2)dr \\
&\leq c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c.
\end{aligned} \tag{2.88}$$

(2) Multiplying (2.70) by $\frac{d\widehat{\mu}_j^n(t)}{dt}$, summing from $j=1$ until n we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\nabla\widehat{u}^n(t)\|_p^p+\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2=-\lambda\int_{\mathcal{O}}\widehat{u}^n\frac{d\widehat{u}^n}{dt}dx+\int_{\mathcal{O}}f(t,x,u)\frac{d\widehat{u}^n}{dt}dx\\&+\int_{\mathcal{O}}g(x,u(t-\varrho(t)))\frac{d\widehat{u}^n}{dt}dx+\int_{\mathcal{O}}\big(\int_{-\infty}^0F(x,l,u(t+l))dl\big)\frac{d\widehat{u}^n}{dt}dx\\&+\int_{\mathcal{O}}J(t,x)\frac{d\widehat{u}^n}{dt}dx+\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}}h(t,x,u)\frac{d\widehat{u}^n}{dt}dx.\end{aligned}\tag{2.89}$$

By (2.8) and the Young inequality,

$$\int_{\mathcal{O}}f(t,x,u)\frac{d\widehat{u}^n}{dt}dx\le\frac{1}{4}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2+2\beta_2^2\|u\|_{2q-2}^{2q-2}+2\|\psi_2(t,\cdot)\|^2.\tag{2.90}$$

The Young inequality and (2.10) imply that

$$\begin{aligned}&-\lambda\int_{\mathcal{O}}\widehat{u}^n\frac{d\widehat{u}^n}{dt}dx+\int_{\mathcal{O}}J(t,x)\frac{d\widehat{u}^n}{dt}dx+\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}}h(t,x,u)\frac{d\widehat{u}^n}{dt}dx\\&\le\frac{1}{4}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2+3\lambda^2\|\widehat{u}^n\|^2+3\|J(t,\cdot)\|^2+6\int_{\mathcal{O}}|\zeta_\delta(\theta_t\omega)\psi_4(t,x)|^2|u|^{2\eta-2}dx\\&\quad+6|\zeta_\delta(\theta_t\omega)|^2\int_{\mathcal{O}}|\psi_5(t,x)|^2dx\\&\le\frac{1}{4}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2+3\lambda^2\|\widehat{u}^n\|^2+3\|J(t,\cdot)\|^2+\beta_2^2\|u\|_{2q-2}^{2q-2}\\&\quad+c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{2q-2}{q-\eta}}^{\frac{2q-2}{q-\eta}}+6|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.\end{aligned}\tag{2.91}$$

By Remark 2.1, we have

$$\int_{\mathcal{O}}g(x,u(t-\varrho(t)))\frac{d\widehat{u}^n}{dt}dx\le\frac{1}{8}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2+2\beta_3e^{2\gamma\rho}\|u_t\|_{C_{\gamma,H}}^2+2\|\psi_3(\cdot)\|^2.\tag{2.92}$$

Similar to (2.73), we obtain

$$\int_{\mathcal{O}}\Big(\int_{-\infty}^0F(x,l,u(t+l))dl\Big)\frac{d\widehat{u}^n}{dt}dx\le2m_1^2\|u_t\|_{C_{\gamma,H}}^2+2m_0^2+\frac{1}{4}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2.\tag{2.93}$$

Then, plugging (2.90)-(2.93) into (2.89),

$$\begin{aligned}&\frac{d}{dt}\|\nabla\widehat{u}^n(t)\|_p^p+\frac{p}{8}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2\\&\le3\lambda^2p\|\widehat{u}^n\|^2+3p\beta_2^2\|u\|_{2q-2}^{2q-2}+2p(\|\psi_2(t,\cdot)\|^2+\|\psi_3(\cdot)\|^2)+(2pm_1^2+2p\beta_3e^{2\gamma\rho})\|u_t\|_{C_{\gamma,H}}^2\\&\quad+2pm_0^2+3p\|J(t,\cdot)\|^2+c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{2q-2}{q-\eta}}^{\frac{2q-2}{q-\eta}}+6p|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.\end{aligned}\tag{2.94}$$

Applying (2.52) in Lemma 2.10 to (2.94) over the interval $(\tau,t]$,

$$\begin{aligned}&\|\nabla\widehat{u}^n(t)\|_p^p+\frac{p}{8}\int_\tau^t\Big\|\frac{d\widehat{u}^n}{dr}\Big\|^2dr\qquad(2.95)\\&\le\frac{1}{\varepsilon}\int_\tau^t\|\nabla\widehat{u}^n(r)\|_p^pdr+3\lambda^2p\int_\tau^t\|\widehat{u}^n(r)\|^2dr+2pm_1^2\int_\tau^t\|u_r\|_{C_{\gamma,H}}^2dr\\&\quad+c\int_\tau^t\|u(r)\|_{2q-2}^{2q-2}dr+c\int_\tau^t(\|J(r,\cdot)\|^2+|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}}+|\zeta_\delta(\theta_r\omega)|^2+1)dr,\end{aligned}$$

where $\varepsilon \in (0, t-\tau)$. By (2.53), (2.69) and (2.87), we have

$$\|\nabla \widehat{u}^n(t)\|_p^p \leq c \int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c, \tag{2.96}$$

for all $\tau + \varepsilon \leq t \leq \tau + T$ with $\varepsilon \in (0, t-\tau)$. Therefore, we can deduce that $\{\widehat{u}^n\}$ is bounded in $L^\infty(\tau+\varepsilon, \tau+T; V)$. Then, by the uniqueness of solution to (P_u) and $u \in C([\tau, \tau+T]; H)$, it follows that $u \in C_w((\tau, \tau+T]; V)$ by [8, Theorem 4].

(3) Integrating (2.94) in $r \in [\tau, t]$, with $\tau \leq t \leq \tau + T$, we have

$$\begin{aligned} &\|\nabla \widehat{u}^n(t)\|_p^p + \frac{p}{8}\int_\tau^t \Big\|\frac{d\widehat{u}^n}{dr}\Big\|^2 dr \\ &\leq \|\widehat{u}^n(\tau)\|_V^p + c\int_\tau^t \|\widehat{u}^n(r)\|^2 dr + c\int_\tau^t (\|u_r\|_{C_{\gamma,H}}^2 + \|u(r)\|_{2q-2}^{2q-2})dr \\ &\quad + c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \end{aligned} \tag{2.97}$$

Similar to (2), by (2.54), (2.69), (2.88) and $\widehat{u}^n(\tau) = \varphi(0) \in V \cap L^q(\Omega)$, we obtain that

$$\|\nabla \widehat{u}^n(t)\|_p^p \leq c \int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c, \tag{2.98}$$

for all $\tau \leq t \leq \tau + T$. As [8, Theorem 4], we have $u \in C_w([\tau, \tau+T]; V)$. □

2.4. Generation of a multi-valued cocycle in $C_{\gamma,H}$

Denote by $\mathcal{C}(X)$ the collection of all nonempty closed subsets of X.

Definition 2.12. ([4, 34]) A multi-valued mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{\gamma,H} \to \mathcal{C}(C_{\gamma,H})$ is called a strict multi-valued non-autonomous dynamical system on $C_{\gamma,H}$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta\}_{t\in\mathbb{R}})$ if for all $t, s \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $\varphi \in C_{\gamma,H}$, the following conditions (i)-(ii) are satisfied:
(i) $\Phi(0, \tau, \omega, \cdot) = I_{C_{\gamma,H}}$;
(ii) $\Phi(t+s, \tau, \omega, \varphi) = \Phi(t, \tau+s, \theta_s\omega, \Phi(s, \tau, \omega, \varphi))$.

Now, we define a multi-valued mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{\gamma,H} \to \mathcal{C}(C_{\gamma,H})$ by

$$\Phi(t, \tau, \omega, \varphi) = \{u_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \varphi) : u \text{ is a solution of } (2.4)\} \tag{2.99}$$

for every $(t, \tau, \omega, u_\tau) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{\gamma,H}$.

Lemma 2.13. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold and $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varphi \in C_{\gamma,H}$. The mapping $\Phi(t, \tau, \omega, \varphi)$ in* (2.99) *is a multi-valued cocycle on $C_{\gamma,H}$ over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.*

Proof. By the same argument as in [4, Lemma 5.1], the cocycle property (ii) in Definition 2.12 of Φ can be proved. Lemma 2.6 implies that the set $\Phi(t, \tau, \omega, \varphi)$ is nonempty. Moreover, we are able to verify $\Phi(t, \tau, \omega, \varphi)$ has compact values by using Theorem 2.8. Therefore, we complete the proof in the sense of Definition 2.12. □

Lemma 2.14. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold and $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varphi \in C_{\gamma,H}$. The mapping $\Phi(t, \tau, \omega, \cdot) : C_{\gamma,H} \to \mathcal{C}(C_{\gamma,H})$ is upper-semicontinuous.*

Proof. Given $T > 0, n \in \mathbb{N}$, let $\tau \in \mathbb{R}, \omega \in \Omega, \varphi^n, \varphi^0 \in C_{\gamma,H}$ such that $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$. Meanwhile, let η^n such that $\eta^n \in \Phi(t,\tau,\omega,\varphi^n)$, that is,

$$\eta^n = u_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,\varphi^n).$$

As $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$, without loss of generality, we can assume that

$$\|\varphi^n\|^2_{C_{\gamma,H}} \leq 1 + 2\|\varphi^0\|^2_{C_{\gamma,H}}, \quad \forall n \in \mathbb{N}.$$

Arguing as in the proof of Lemma 2.6 and Corollary 2.7, η^n is bounded in $L^\infty(\tau,\tau+T;H) \cap L^p(\tau,\tau+T;V) \cap L^q(\tau,\tau+T;L^q(\mathcal{O}))$. Similar to the proof of Theorem 2.8, we can ensure that there exist $\eta^0 \in \Phi(t,\tau,\omega,\varphi^0)$ and a subsequence of η^n (still denote the same) such that $\eta^n \to \eta^0$ in $C_{\gamma,H}$ for all $t \in [\tau,\tau+T]$. As T is arbitrary, it follows that Φ is upper-semicontinuous. □

3. Existence of random attractors in $C_{\lambda,H}$

In this part, we need to establish the existence of $\mathfrak{D}$-pullback attractor of Φ, where $\mathfrak{D}$ is the universe of all tempered time-sample sets $\mathcal{D} = \{\mathcal{D}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that $\mathcal{D}(\tau,\omega)$ is a nonempty bounded subset of $C_{\gamma,H}$ and

$$\lim_{t\to-\infty} e^{\frac{\lambda}{2}t}\|\mathcal{D}(\tau+t,\theta_t\omega)\|^2_{C_{\gamma,H}} = 0, \quad \forall \mathcal{D} \in \mathfrak{D}, \tau \in \mathbb{R}, \omega \in \Omega. \tag{3.1}$$

Consider a number α satisfying

$$\alpha \in (0, \lambda - \frac{4m_1^2}{\lambda}). \tag{3.2}$$

We state now an assumption.

H7. The non-autonomous term $J \in L^2_{loc}(\mathbb{R},H)$ satisfies: For every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^0 e^{\alpha r}\|J(r+\tau,\cdot)\|^2 dr < \infty, \tag{3.3}$$

and for every positive constant c,

$$\lim_{t\to-\infty} e^{ct}\int_{-\infty}^0 e^{\alpha r}\|J(r+t,\cdot)\|^2 dr = 0. \tag{3.4}$$

3.1. Existence of pullback attractors in $C_{\lambda,H}$

Lemma 3.1. *Suppose that* **H1-H7**, (2.14)-(2.15) *hold. For each $(\tau,\omega,\mathcal{D}) \in \mathbb{R}\times\Omega\times\mathfrak{D}$, there exists $T = T(\tau,\omega,\mathcal{D},\delta) > 0$ such that the solution of* (2.4) *satisfies*

$$\|u_\tau(\cdot,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{C_{\gamma,H}} \leq c_0 R(\tau,\omega), \tag{3.5}$$

for all $t \geq T$ and $u_{\tau-t} \in \mathcal{D}(\tau-t,\theta_{-t}\omega)$, where c_0 is a constant and

$$R(\tau,\omega) = \int_{-\infty}^0 e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \tag{3.6}$$

Proof. Using $\tau - t$ instead of τ and $\theta_{-\tau}\omega$ instead of ω in (2.25), we have for all $\xi \in [\tau - t, \tau]$,

$$\begin{aligned}
&\|u(\xi)\|^2 + C_2 \int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}(\|u(r)\|_V^p + \|u(r)\|_q^q + \|u(r)\|^2)dr \\
&\leq ce^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ c\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr,
\end{aligned} \tag{3.7}$$

where $\|u(\tau-t)\|^2 \leq \|u_{\tau-t}\|^2_{C_{\gamma,H}}$ and $C_2 = \min\{2, \beta_1, \frac{1}{8}\lambda\}$ is defined in Corollary 2.7. Multiplying (3.7) by $e^{2\gamma s}$, and replacing ξ by $\xi+s$, then taking the supremum in $s \in [\tau-t-\xi, 0]$, we obtain that

$$\begin{aligned}
&\sup_{s\in[\tau-t-\xi,0]} e^{2\gamma s}\|u(\xi+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\
&\leq \sup_{s\in[\tau-t-\xi,0]} e^{(2\gamma-\lambda)s}\Big[ce^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ c\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr\Big] \\
&\leq ce^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ c\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr,
\end{aligned} \tag{3.8}$$

where we have used $\lambda \leq 2\gamma$ defined in (2.15). For $s \in (-\infty, \tau - t - \xi]$, we consider

$$\begin{aligned}
&\sup_{s\in(-\infty,\tau-t-\xi]} e^{2\gamma s}\|u(\xi+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\
&= \sup_{s\in(-\infty,\tau-t-\xi]} e^{-2\gamma(\xi-\tau+t)}e^{2\gamma(t+\xi+s-\tau)}\|u_{\tau-t}(t+\xi+s-\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\
&= e^{-2\gamma(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} \leq e^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}}.
\end{aligned} \tag{3.9}$$

Therefore, similar to (2.28) we have for all $\xi \in [\tau - t, \tau]$,

$$\begin{aligned}
&\|u_\xi(\cdot,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{C_{\gamma,H}} \\
&\leq ce^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}e^{-\lambda\xi}\int_{\tau-t}^{\xi} e^{\lambda r}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ ce^{-\lambda\xi}\int_{\tau-t}^{\xi} e^{\lambda r}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr.
\end{aligned} \tag{3.10}$$

Then, using the Gronwall lemma we have, for all $\xi \in [\tau - t, \tau]$,

$$\begin{aligned}
\|u_\xi(\cdot,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{C_{\gamma,H}} &\leq ce^{(\frac{4m_1^2}{\lambda}-\lambda)(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + ce^{(\frac{4m_1^2}{\lambda}-\lambda)\xi}\int_{\tau-t}^{\xi} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r,\cdot)\|^2 \\
&+ |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr \\
&\leq ce^{(\frac{4m_1^2}{\lambda}-\lambda)(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + ce^{(\frac{4m_1^2}{\lambda}-\lambda)(\xi-\tau)}\int_{-t}^{\xi-\tau} e^{(\lambda-\frac{4m_1^2}{\lambda})r} \\
&\times (\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr.
\end{aligned} \tag{3.11}$$

Let $\xi = \tau$, we have

$$\|u_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{C_{\gamma,H}} \le ce^{(\frac{4m_1^2}{\lambda} - \lambda)t} \|u_{\tau-t}\|^2_{C_{\gamma,H}} + cR(\tau, \omega), \tag{3.12}$$

where $R(\tau, \omega)$ is defined in (3.6). Since $u_{\tau-t} \in \mathcal{D}(\tau - t, \theta_{-t}\omega)$ and $\mathcal{D} \in \mathfrak{D}$, we see from (2.14) and (3.1) that

$$ce^{(\frac{4m_1^2}{\lambda} - \lambda)t} \|u_{\tau-t}\|^2_{C_{\gamma,H}} \le e^{-\frac{1}{2}\lambda t} \|\mathcal{D}(\tau - t, \theta_{-t}\omega)\|^2_{C_{\gamma,H}} \to 0, \text{ as } t \to \infty. \tag{3.13}$$

Thus, we complete the proof. □

Proposition 3.2. *Suppose that* **H1-H7**, (2.14)-(2.15) *hold. Then, the multi-valued cocycle* Φ *has a closed* $\mathfrak{D}$*-pullback absorbing set* $\mathcal{K} \in \mathfrak{D}$*, given by*

$$\mathcal{K}(\tau, \omega) = \{w \in C_{\gamma,H} : \|w\|^2_{C_{\gamma,H}} \le c_0 R(\tau, \omega)\}, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega. \tag{3.14}$$

where $R(\tau, \omega)$ *is defined in* (3.6).

Proof. By (3.5), we know that for $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $T = T(\tau, \omega, \mathcal{D}, \delta) > 0$ such that, for all $t \ge T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{D}(\tau - t, \theta_{-t}\omega)) \subseteq \mathcal{K}(\tau, \omega). \tag{3.15}$$

Now, we show that $\mathcal{K} \in \mathfrak{D}$. Let $\alpha_0 = \min\{\lambda - \frac{4m_1^2}{\lambda} - \alpha, \frac{\lambda}{4}\}$, by (3.6) and (3.14) we have, for $t \le 0$,

$$\begin{aligned}
&e^{\frac{\lambda}{2}t} R(\tau + t, \theta_t\omega) \\
&= e^{\frac{\lambda}{2}t} \int_{-\infty}^0 e^{(\lambda - \frac{4m_1^2}{\lambda})r} (\|J(r + \tau + t, \cdot)\|^2 + |\zeta_\delta(\theta_{r+t}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r+t}\omega)|^2 + 1) dr \\
&= e^{-\frac{\lambda}{2}\tau} e^{\frac{\lambda}{2}(\tau+t)} \int_{-\infty}^0 e^{(\lambda - \frac{4m_1^2}{\lambda})r} \|J(r + \tau + t, \cdot)\|^2 dr \\
&\quad + e^{\frac{\lambda}{4}t} \int_{-\infty}^0 e^{\alpha r + (\lambda - \frac{4m_1^2}{\lambda} - \alpha)r + \frac{\lambda}{4}t} (|\zeta_\delta(\theta_{r+t}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r+t}\omega)|^2 + 1) dr \\
&\le e^{-\frac{\lambda}{2}\tau} e^{\frac{\lambda}{2}(\tau+t)} \int_{-\infty}^0 e^{(\lambda - \frac{4m_1^2}{\lambda})r} \|J(r + \tau + t, \cdot)\|^2 dr \\
&\quad + e^{\frac{\lambda}{4}t} \int_{-\infty}^0 e^{\alpha r + \alpha_0(r+t)} (|\zeta_\delta(\theta_{r+t}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r+t}\omega)|^2 + 1) dr.
\end{aligned} \tag{3.16}$$

By the same method as [38, Lemma 3.4], we can deduce that there exist $b_1 = b_1(\omega, \delta) > 0$ and $b_2 = b_2(\omega, \delta) > 0$ such that, for all $r, t \le 0$

$$0 < e^{\alpha_0(r+t)} |\zeta_\delta(\theta_{r+t}\omega)|^{\frac{2q-2}{q-\eta}} < b_1, \quad 0 < e^{\alpha_0(r+t)} |\zeta_\delta(\theta_{r+t}\omega)|^2 < b_2. \tag{3.17}$$

Hence, by (3.4) and (3.17), it follows from (3.16) that, for $t \le 0$,

$$\lim_{t \to -\infty} e^{\frac{\lambda}{2}t} R(\tau + t, \theta_t\omega) = 0. \tag{3.18}$$

Therefore $\mathcal{K} \in \mathfrak{D}$ as desired. □

Now we establish the $\mathfrak{D}$-pullback asymptotic compactness of Φ.

Lemma 3.3. *Suppose that* **H1-H7**, (2.14)-(2.15) *hold. Then* Φ *is* $\mathfrak{D}$*-pullback asymptotically compact in* $C_{\gamma,H}$.

Proof. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\mathcal{D} \in \mathfrak{D}$, and $x^n \in \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{\tau-t_n})$ with $u_{\tau-t_n} \in \mathcal{D}(\tau - t_n, \theta_{-t_n}\omega)$ and $t_n \to \infty$. Then

$$x^n(s) = u^n_\tau(s, \tau - t_n, \theta_{-\tau}\omega, u_{\tau-t_n}), \quad \forall s \in (-\infty, 0], \tag{3.19}$$

where u^n is a solution of (2.4). We need to show $\{u^n_\tau(s, \tau-t_n, \theta_{-\tau}\omega, u_{\tau-t_n})\}_{n=1}^\infty$ has a convergent subsequence in $C_{\gamma,H}$.

(1) We will verify that there exists $W \in C([-T, 0]; H)$ and a subsequence of $\{x^n\}$ (not relabeled) such that $x^n \to W$ in $C([-T, 0]; H)$ for every $T > 0$.

Let T be a positive integer, similar to (3.11) and (3.13) in Lemma 3.1, there exist $n_0 = n_0(\tau, \omega, \mathcal{D}) \geq 1$ and $c_T = c_0 e^{(\lambda - \frac{4m_1^2}{\lambda})T} > 0$ such that, for all $n \geq n_0$ and $t_n \geq T$,

$$\|u^n_\xi\|^2_{C_{\gamma,H}} \leq c_T R(\tau, \omega), \quad \forall \xi \in [\tau - T, \tau], \tag{3.20}$$

where $R(\tau, \omega)$ is defined in (3.6) and then

$$\|u^n(\xi)\|^2 \leq c_T R(\tau, \omega), \quad \forall \xi \in [\tau - T, \tau],\ n \geq n_0. \tag{3.21}$$

Consider

$$Y^n(\xi) = u^n(\xi - T), \quad \forall \xi \in [\tau, \tau + T]. \tag{3.22}$$

By (3.21), we have

$$\|Y^n(\xi)\|^2 \leq c_T R(\tau, \omega), \quad \forall \xi \in [\tau, \tau + T],\ n \geq n_0. \tag{3.23}$$

Thus, for fixed T, $\{Y^n\}$ is bounded in $L^\infty(\tau, \tau + T; H)$. Note that Y^n is a solution of the following system:

$$\begin{aligned} \frac{dY^n}{d\xi} &= -\Delta_p Y^n - \lambda Y^n + \widetilde{f}(\xi, x, Y^n) + \widetilde{g}(x, Y^n(\xi - \varrho(\xi))) \\ &+ \int_{-\infty}^0 \widetilde{F}(x, l, Y^n(\xi + l))dl + \widetilde{J}(\xi, x) + \widetilde{h}(\xi, x, Y^n)\zeta_\delta(\theta_\xi\omega), \quad \forall \xi \in [\tau, \tau + T], \end{aligned} \tag{3.24}$$

with initial data $Y^n_\tau = u^n_{\tau-T}$. It can be inferred from (3.20) that

$$\|Y^n_\tau\|^2_{C_{\gamma,H}} \leq c_T R(\tau, \omega), \quad \forall n \geq n_0. \tag{3.25}$$

From (3.22) and (3.24), we consider for all $\xi \in [\tau, \tau + T]$,

$$\begin{aligned} &\widetilde{f}(\xi, x, Y^n) = f(\xi - T, x, u^n),\ \widetilde{g}(x, Y^n(\xi - \varrho(\xi))) = g(x, u^n(\xi - T - \varrho(\xi - T))) \\ &\widetilde{F}(x, l, Y^n(\xi + l)) = F(x, l, u^n(\xi - T + l)),\ \widetilde{h}(\xi, x, Y^n) = h(\xi - T, x, u^n). \end{aligned} \tag{3.26}$$

Therefore, using the same method as (3.7), we have for all $r \in [\tau - T, \tau]$,

$$
\begin{aligned}
& C_2 e^{-\lambda T} \int_{\tau-T}^{\tau} (\|u^n(r)\|_V^p + \|u^n(r)\|_q^q) dr \\
& \leq C_2 \int_{\tau-T}^{\tau} e^{\lambda(r-\tau)} (\|u^n(r)\|_V^p + \|u^n(r)\|_q^q) dr \\
& \leq c e^{-\lambda T} \|u^n_{\tau-T}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda} \int_{\tau-T}^{\tau} e^{\lambda(r-\tau)} \|u^n_r\|^2_{C_{\gamma,H}} dr \\
& + c \int_{\tau-T}^{\tau} e^{\lambda(r-\tau)} (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1) dr \\
& \leq c e^{-\lambda T} \|u^n_{\tau-T}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda} \int_{\tau-T}^{\tau} e^{\lambda(r-\tau)} \|u^n_r\|^2_{C_{\gamma,H}} dr \\
& + c \int_{-\infty}^{0} e^{\lambda r} (\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1) dr. \qquad (3.27)
\end{aligned}
$$

Plugging (3.20) into (3.27) and thanks to $\lambda > \lambda - \frac{4m_1^2}{\lambda}$, we have

$$\int_{\tau-T}^{\tau} (\|u(r)\|_V^p + \|u(r)\|_q^q) dr \leq c_0 e^{\lambda T} R(\tau,\omega), \qquad (3.28)$$

which means that $\{Y^n\}$ is bounded in $L^p(\tau,\tau+T;V)$ and $L^q(\tau,\tau+T;L^q(\mathcal{O}))$ in view of (3.22). In addition, we are able to show $\{\widetilde{f}(\cdot,x,Y^n)\}$ and $\{\zeta_\delta(\theta_t\omega)\widetilde{h}(\cdot,x,Y^n)\}$ are bounded in $L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O}))$. Owing to the above estimates, there exists $Y \in L^\infty(\tau,\tau+T;H) \cap L^p(\tau,\tau+T;V) \cap L^q(\tau,\tau+T;L^q(\mathcal{O}))$ such that

$$\begin{cases} Y^n \to Y \quad \text{weakly star in} \quad L^\infty(\tau,\tau+T;H), \\ Y^n \to Y \quad \text{weakly in} \quad L^p(\tau,\tau+T;V) \quad \text{and} \quad L^q(\tau,\tau+T;L^q(\mathcal{O})). \end{cases} \qquad (3.29)$$

By Remark 2.1, Remark 2.2 and (3.20), we deduce that $\widetilde{g}(x,Y^n_\cdot)$ is bounded in $L^2(\tau,\tau+T;H)$ and $\left\{\int_{-\infty}^0 \widetilde{F}(x,l,Y^n_\cdot(l))\right\}$ is bounded in $L^2(\tau,\tau+T;H)$. Similar to the proof of Theorem 2.8, we can come to this conclusion

$$Y^n \to Y \quad \text{in} \quad C([\tau,\tau+T];H). \qquad (3.30)$$

Let $W(s) := Y(s+\tau+T)$ for $s \in [-T,0]$. By (3.29), then using the diagonal technique we can obtain that there exist a function $W \in C((-\infty,0],H)$ and a subsequence of $\{n\}$ (relabeled the same) such that $x^n = u^n_\tau \to W$ in $C([-T,0];H)$ on every interval $[-T,0]$. Thus, by (3.20) we have for any $T > 0$,

$$\lim_{n\to\infty} \sup_{s\in[-T,0]} e^{2\gamma s}\|u^n_\tau(s)\|^2 = \sup_{s\in[-T,0]} e^{2\gamma s}\|W(s)\|^2 \leq c_T R(\tau,\omega), \qquad (3.31)$$

that is, $W \in C_{\gamma,H}$ and

$$\|W(s)\|^2_{C_{\gamma,H}} \leq c_T R(\tau,\omega), \quad \forall s \in [-T,0], \quad \text{for any} \quad T > 0. \qquad (3.32)$$

(2) We will prove that $x^n \to W$ in $C_{\gamma,H}$. To that end, we consider for every $\varepsilon > 0$ there exists n_ε such that, for all $n \geq n_\varepsilon$

$$\sup_{s\in(-\infty,0]} e^{2\gamma s}\|u^n_\tau(s) - W(s)\|^2 \leq \varepsilon. \qquad (3.33)$$

Due to $\lambda - \frac{4m_1^2}{\lambda} < \lambda < 2\gamma$, for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$c_0 e^{-[2\gamma-(\lambda-\frac{4m_1^2}{\lambda})]T_\varepsilon} R(\tau,\omega) \le \frac{\varepsilon}{2}. \tag{3.34}$$

By (3.11) we have

$$\|u_\tau^n(s)\|^2 \le c_0 e^{-(\lambda-\frac{4m_1^2}{\lambda})s} R(\tau,\omega). \tag{3.35}$$

Then, (3.34) and (3.35) yield

$$\sup_{s\le -T_\varepsilon} e^{2\gamma s}\|u_\tau^n(s)\|^2 \le c_0 \sup_{s\le -T_\varepsilon} e^{[2\gamma-(\lambda-\frac{4m_1^2}{\lambda})]s} R(\tau,\omega) \le \frac{\varepsilon}{2}. \tag{3.36}$$

Given $k \ge 0$, by (3.32) and (3.34), we have for all $s \in [-(T_\varepsilon + k + 1), -(T_\varepsilon + k)]$

$$\begin{aligned} e^{2\gamma s}\|W(s)\|^2 &\le c_0 e^{-2\gamma(T_\varepsilon+k)} e^{(\lambda-\frac{4m_1^2}{\lambda})(T_\varepsilon+k+1)} R(\tau,\omega) \\ &\le c_0 e^{\lambda-\frac{4m_1^2}{\lambda}} e^{-[2\gamma-(\lambda-\frac{4m_1^2}{\lambda})](T_\varepsilon+k)} R(\tau,\omega) \le \frac{\varepsilon}{2}, \end{aligned} \tag{3.37}$$

which means that

$$\sup_{s\le -T_\varepsilon} e^{2\gamma s}\|W(s)\|^2 \le \frac{\varepsilon}{2}. \tag{3.38}$$

It can be inferred from (3.36) and (3.38) that for all $n \ge n_\varepsilon$

$$\sup_{s\in(-\infty,-T_\varepsilon]} e^{2\gamma s}\|u_\tau^n(s) - W(s)\|^2 \le \varepsilon. \tag{3.39}$$

From (1), the convergence of $u_\tau^n(\cdot)$ to W is true in compact intervals. Therefore, together with (3.39) we conclude (3.33). □

Then, we prove the existence of a pullback attractor in $C_{\lambda,H}$. Let us first recall the definition.

Definition 3.4. [34] $\mathcal{A} \in \mathfrak{D}$ is called a $\mathfrak{D}$-pullback attractor for Φ if the following conditions (i)-(iii) are satisfied: for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,
(i) $\mathcal{A}(\tau,\omega)$ is compact in $C_{\lambda,H}$;
(ii) $\mathcal{A}(\tau,\omega)$ is strictly invariant, i.e.

$$\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{A}(\tau - t, \theta_{-t}\omega)) = \mathcal{A}(\tau,\omega), \ \forall t \ge 0.$$

(iii) $\mathcal{A}(\tau,\omega)$ is pullback attracting, that is, for each $\mathcal{D} \in \mathfrak{D}$,

$$\lim_{t\to+\infty} \mathrm{dist}_{C_{\lambda,H}}(\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{D}(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0.$$

Theorem 3.5. *Suppose that* **H1-H7**, (2.14)-(2.15) *hold. Then the multi-valued cocycle* Φ *generated from the p-Laplace equation* (2.4) *has a unique* $\mathfrak{D}$*-pullback attractor* $\mathcal{A} \in \mathfrak{D}$.

Proof. Given $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$. We can obtain the multi-valued cocycle Φ is upper semi-continuous by the result in Lemma 2.14. Proposition 3.2 yields the existence of a closed $\mathfrak{D}$-pullback absorbing set $\mathcal{K} \in \mathfrak{D}$. Lemma 3.3 gives the asymptotic compactness of the multi-valued cocycle Φ. Thus we come to this conclusion in view of [4, Theorem 3.4]. □

3.2. Measurability of the pullback attractor

We recall (see [1]) that Ω, a subspace of $C(\mathbb{R},\mathbb{R})$, can be equipped with the Fréchet metric

$$d(\omega_1,\omega_2)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\sup_{t\in[-n,n]}|\omega_1(t)-\omega_2(t)|}{1+\sup_{t\in[-n,n]}|\omega_1(t)-\omega_2(t)|},\ \forall\omega_1,\omega_2\in\Omega,$$

and $\mathcal{F}$ is the Borel σ-algebra $\mathcal{B}(\Omega)$ with respect to the metric.

For $m\in\mathbb{N}$, we introduce the subset of Ω as

$$\Omega_m:=\Big\{\omega\in\Omega:|\omega(t)|\le|t|,\quad\forall|t|\ge m\Big\}. \tag{3.40}$$

Lemma 3.6. *([14]) Let $\delta\in(0,1]$ and $\Omega_m\subseteq\Omega$ given by* (3.40) *for $m\in\mathbb{N}$.*

(i) If $\omega^n\to\omega$ with $\omega^n,\omega\in\Omega_m$, then $\zeta_\delta(\theta_t\omega^n)\to\zeta_\delta(\theta_t\omega)$ uniformly for t in any compact interval of $\mathbb{R}$ as $n\to\infty$.

(ii) Ω_m is a closed subset of Ω and $\Omega=\bigcup\limits_{m=1}^{\infty}\Omega_m$.

(iii) Given $\omega\in\Omega_m$, we have for all $t\le-m$,

$$|\zeta_\delta(\theta_t\omega)|\le\frac{2}{\delta}|t|+1. \tag{3.41}$$

Lemma 3.7. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold. Let $T>0,M>0$, $\tau\in\mathbb{R}$, $\omega^n\to\omega$ with $\omega^n,\omega\in\Omega_m$. Then there exists $c=c(\delta,\tau,T,M,\omega)>0$ such that the solutions of* (2.4) *satisfy*

$$\|u_t(\cdot,\tau,\theta_{-\tau}\omega^n,\varphi)\|^2_{C_{\gamma,H}}\le c, \tag{3.42}$$

and

$$\int_\tau^t e^{-\lambda(t-r)}(\|u(r,\tau,\theta_{-\tau}\omega^n,\varphi)\|_V^p+\|u(r,\tau,\theta_{-\tau}\omega^n,\varphi)\|_q^q)dr\le c, \tag{3.43}$$

for all $n\in\mathbb{N},t\in[\tau,\tau+T]$ and the initial condition $\varphi\in C_{\gamma,H}$ with $\|\varphi\|_{C_{\gamma,H}}\le M$.

Proof. Since $\omega,\omega^n\in\Omega_m$, by Lemma 3.6 (i) there exists $N=N(\delta,T,\tau,\omega)\ge1$ such that, for all $n\ge N$ and $r\in[\tau,\tau+T]$,

$$|\zeta_\delta(\theta_{r-\tau}\omega^n)|\le|\zeta_\delta(\theta_{r-\tau}\omega)|+1,$$

which, with the continuity of $\zeta_\delta(\theta_s\omega)$ in s, imply that there exists $c_1=c_1(\delta,T,\tau,\omega)>0$ such that, for all $r\in[\tau,\tau+T]$,

$$|\zeta_\delta(\theta_{r-\tau}\omega)|\le c_1,\quad\text{so}\quad|\zeta_\delta(\theta_{r-\tau}\omega^n)|\le1+c_1. \tag{3.44}$$

Replacing ω in (2.18) with $\theta_{-\tau}\omega^n$, and plugging (3.44) into (2.18) imply (3.42). Analogously, replace ω by $\theta_{-\tau}\omega^n$ and let $r=\tau$ in (2.36). Then, by (3.44), we can also figure out (3.43). □

Now, we will show the multi-valued cocycle Φ is random in the sense of the following definition:

Definition 3.8. ([4]) A multi-valued cocycle Φ is said to be **random** if

$$\Phi(\cdot,\tau,\cdot,\cdot):\mathbb{R}^+\times\Omega\times C_{\gamma,H}\to\mathcal{C}(C_{\gamma,H})\ \text{ is }\ \mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(C_{\gamma,H})\text{-measurable},$$

that is, for any open set O in $C_{\gamma,H}$, the set $\{(t,\omega,x)\in\mathbb{R}^+\times\Omega\times C_{\gamma,H}:\ \Phi(t,\tau,\omega,x)\cap O\neq\emptyset\}$ belongs to $\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(C_{\gamma,H})$.

Lemma 3.9. *Suppose that* **H1-H6**, (2.14)-(2.15) *hold. Then, for every* $\tau \in \mathbb{R}$*, the mapping*

$$(t,\omega,\varphi) \to \Phi(t,\tau,\omega,\varphi)$$

is $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(C_{\gamma,H})$*-measurable.*

Proof. (1) Given $n \in \mathbb{N}$, $t^0 > 0$, let $t^n \to t^0$, $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$ and $\varphi^n, \varphi^0 \in C_{\gamma,H}$, $\omega^n \to \omega^0$ with $\omega^n, \omega^0 \in \Omega_m$. By [4, Lemma 2.5], we need to verify the above mapping is upper-semicontinuous, that is, we will show that for any sequence $\chi^n \in \Phi(t^n,\tau,\omega^n,\varphi^n)$, there exists a subsequence χ^{n_k} converging to some $\chi^0 \in \Phi(t^0,\tau,\omega^0,\varphi^0)$ in $C_{\gamma,H}$.

We assume that, for all $n \in \mathbb{N}$,

$$0 \le t^n \le 1 + t^0 \qquad \text{and} \qquad \|\varphi^n\|^2_{C_{\gamma,H}} \le 1 + 2\|\varphi^0\|^2_{C_{\gamma,H}}. \tag{3.45}$$

By the definition of Φ in (2.99), then $\chi^n = u^n_{\tau+t^n}(\cdot,\tau,\theta_{-\tau}\omega^n,\varphi^n)$. On account of (3.42) in Lemma 3.7, we have the sequence $\{u^n(\cdot,\tau,\theta_{-\tau}\omega^n,\varphi^n)\}$ is bounded in $L^\infty(\tau,\tau+1+t^0;H)$. By similar reasons to the ones in (3.43), $\{u^n(\cdot,\tau,\theta_{-\tau}\omega^n,\varphi^n)\}$ is bounded in $L^p(\tau,\tau+1+t^0;V)$ and $L^q(\tau,\tau+1+t^0;L^q(\mathcal{O}))$. Similar to the proof of Theorem 2.8, there exist a subsequence $\{u^{n_k}(\cdot,\tau,\theta_{-\tau}\omega^{n_k},\varphi^{n_k})\}$ and $u^0(\cdot,\tau,\theta_{-\tau}\omega^0,\varphi^0) \in L^\infty(\tau,\tau+1+t^0;H) \cap L^p(\tau,\tau+1+t^0;V) \cap L^q(\tau,\tau+1+t^0;L^q(\mathcal{O}))$ such that

$$u^{n_k}(\cdot) \to u^0(\cdot) \quad \text{in} \quad C([\tau,\tau+1+t^0];H). \tag{3.46}$$

Thus, for a given $\varepsilon > 0$, we have

$$\begin{aligned}
&\sup_{s\in[-1-t^0,0]} e^{\gamma s}\|u^{n_k}(\tau+t^{n_k}+s,\tau,\theta_{-\tau}\omega^{n_k},\varphi^{n_k}) - u^0(\tau+t^0+s,\tau,\theta_{-\tau}\omega^0,\varphi^0)\| \\
&\le \sup_{s\in[-1-t^0,0]} e^{\gamma s}\|u^{n_k}(\tau+t^{n_k}+s,\tau,\theta_{-\tau}\omega^{n_k},\varphi^{n_k}) - u^0(\tau+t^{n_k}+s,\tau,\theta_{-\tau}\omega^0,\varphi^0)\| \\
&\quad + \sup_{s\in[-1-t^0,0]} e^{\gamma s}\|u^0(\tau+t^{n_k}+s,\tau,\theta_{-\tau}\omega^0,\varphi^0) - u^0(\tau+t^0+s,\tau,\theta_{-\tau}\omega^0,\varphi^0)\| \\
&\le \frac{\varepsilon}{4}.
\end{aligned} \tag{3.47}$$

For $s \in (-\infty,-1-t^0]$, we have

$$\begin{aligned}
&\sup_{s\in(-\infty,-1-t^0]} e^{\gamma s}\|u^{n_k}(\tau+t^{n_k}+s) - u^0(\tau+t^0+s)\| \\
&\le \sup_{s\in(-\infty,-1-t^0]} e^{\gamma s}\|u^{n_k}(\tau+t^{n_k}+s) - u^0(\tau+t^{n_k}+s)\| \\
&\quad + \sup_{s\in(-\infty,-1-t^0]} e^{\gamma s}\|u^0(\tau+t^{n_k}+s) - u^0(\tau+t^0+s)\| \\
&\le e^{-\gamma t^{n_k}}\|\varphi^{n_k} - \varphi^0\|_{C_{\gamma,H}} + \sup_{s\in(-\infty,-1-t^0]} e^{\gamma s}\|\varphi^0(t^{n_k}+s) - \varphi^0(t^0+s)\|.
\end{aligned} \tag{3.48}$$

By the fact that $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$, for large k we obtain

$$e^{-\gamma t^{n_k}}\|\varphi^{n_k} - \varphi^0\|_{C_{\gamma,H}} \le \frac{\varepsilon}{4}. \tag{3.49}$$

Owing to $\varphi^0 \in C_{\gamma,H}$, there exists $\lim_{s\to-\infty} e^{\gamma s}\varphi^0(s) = \varphi \in H$. Thus consider $T > 1+t^0$ we have

$$\begin{aligned}
&\sup_{s\in(-\infty,-T]} e^{\gamma s}\|\varphi^0(t^{n_k}+s) - \varphi^0(t^0+s)\| \\
&\le \sup_{s\in(-\infty,-T]} e^{\gamma s}\|\varphi^0(t^{n_k}+s) - \varphi\| + \sup_{s\in(-\infty,-T]} e^{\gamma s}\|\varphi - \varphi^0(t^0+s)\| \le \frac{\varepsilon}{4}.
\end{aligned} \tag{3.50}$$

For $s \in [-T, -1-t^0]$, we have

$$\sup_{s\in[-T,-1-t^0]} e^{\gamma s}\|\varphi^0(t^{n_k}+s)-\varphi^0(t^0+s)\| \le \frac{\varepsilon}{4}, \tag{3.51}$$

for large k. It can be inferred from (3.47)-(3.51) that $\chi^{n_k} \to \chi^0$ in $C_{\gamma,H}$ as $k \to \infty$. Then, $\chi^0 = u^0_{\tau+t^0}(\cdot, \tau, \theta_{-\tau}\omega^0, \varphi^0) \in C_{\gamma,H}$.

By the continuity of f, g, h (see **H3-H5**), Remark 2.3 and Lemma 3.6 (i), we know that u^0 is a solution of equation (2.4), then $\chi^0 \in \Phi(t^0, \tau, \omega^0, \varphi^0)$.

(2) Due to (1), we can obtain that the mapping $(t, \omega, \varphi) \to \Phi(t, \tau, \omega, \varphi)$ is $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_{\Omega_m} \times \mathcal{B}(C_{\gamma,H})$-measurable, where $\mathcal{F}_{\Omega_m}$ is the trace σ-algebra of $\mathcal{F}$ with respect to Ω_m. Since Ω_m is a closed subset of Ω by Lemma 3.6 (ii), then $\Omega_m \in \mathcal{F}$, which implies that $\mathcal{F}_{\Omega_m} \subset \mathcal{F}$. Then, together with $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$ by Lemma 3.6 (ii), we complete the proof. □

Recall that the graph of a set-valued map $\omega \mapsto F(\omega) : \Omega \to 2^X$ is defined by

$$\mathrm{Gr}(F) = \{(\omega, x) \in \Omega \times X : \ x \in F(\omega)\}.$$

Lemma 3.10. *Suppose that* **H1-H7***, (2.14)-(2.15) hold and let $m \in \mathbb{N}$, $\tau \in \mathbb{R}$. Then, for any $t \ge 0$, the map $\Omega_m \ni \omega \to \Phi(t, \tau, \omega, \mathcal{K}(\tau, \omega))$ is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{F}_{\Omega_m}$. In addition, $\Phi(t, \tau, \omega, \mathcal{K}(\tau, \omega))$ is closed.*

Proof. Based on the above definition of graph, by [2], we have to verify the graph of the map $\omega \to \Phi(t, \tau, \omega, \mathcal{K}(\tau, \omega))$ is closed in $\Omega_m \times C_{\gamma,H}$. Let $\omega^n \to \omega^0$ in Ω_m and $\chi^n \to \chi^0$ in $C_{\gamma,H}$, where $\chi^n \in \Phi(t, \tau, \omega^n, \mathcal{K}(\tau, \omega^n)$. Thus, we only need to prove that $\chi^0 \in \Phi(t, \tau, \omega^0, \mathcal{K}(\tau, \omega^0)$.

On account of $\chi^n \in \Phi(t, \tau, \omega^n, \varphi^n)$ and $\varphi^n \in \mathcal{K}(\tau, \omega^n)$, we have

$$\chi^n(s) = u^n_\tau(t+s, \tau, \theta_{-\tau}\omega^n, \varphi^n), \ \forall s \le -t,$$

where u^n is a solution of (2.4). Then $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$.

By Lemma 3.6 (i), there exists $c_m > 0$ such that, for all $n \in \mathbb{N}$ and $r \in [-m, 0]$,

$$|\zeta_\delta(\theta_r\omega^n)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^n)|^2 \le c_m. \tag{3.52}$$

In view of **H7** and (3.52), applying the Lebesgue theorem, we have

$$\begin{aligned} &\lim_{n\to\infty}\int_{-m}^{0} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega^n)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^n)|^2 + 1)dr \\ &= \int_{-m}^{0} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega^0)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^0)|^2 + 1)dr. \end{aligned} \tag{3.53}$$

Since $\omega^n \in \Omega_m$, by Lemma 3.6 (i), there exists $c_\delta > 0$ such that, for all $n \in \mathbb{N}$ and $r \in (-\infty, -m]$,

$$|\zeta_\delta(\theta_r\omega^n)|^{\frac{2q-2}{q-\eta}} \le c_\delta(|r|^{\frac{2q-2}{q-\eta}} + 1) \ \text{ and } \ |\zeta_\delta(\theta_r\omega^n)|^2 \le c_\delta(|r|^2 + 1). \tag{3.54}$$

In view of **H7** and (3.54), applying the Lebesgue theorem, we have

$$\begin{aligned} &\lim_{n\to\infty}\int_{-\infty}^{-m} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega^n)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^n)|^2 + 1)dr \\ &= \int_{-\infty}^{-m} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega^0)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^0)|^2 + 1)dr. \end{aligned} \tag{3.55}$$

Thus, $R(\tau,\omega^n)\to R(\tau,\omega^0)$.

Since $\varphi^n\in\mathcal{K}(\tau,\omega^n)$, we deduce from Proposition 3.2 that $\|\varphi^n\|^2_{C_{\gamma,H}}\le c_0R(\tau,\omega^n)$. Hence, $\|\varphi^0\|^2_{C_{\gamma,H}}\le c_0R(\tau,\omega^0)$ and $\varphi^0\in\mathcal{K}(\tau,\omega^0)$. Using a similar argument as the proof of Lemma 3.9, we deduce that $\chi^0\in\Phi(t,\tau,\omega^0,\varphi^0)\subset\Phi(t,\tau,\omega^0,\mathcal{K}(\tau,\omega^0)$. □

A pullback attractor $\mathcal{A}$ is **random** with respect to the $\mathbb{P}$-completion $\overline{\mathcal{F}}$ of the σ-algebra $\mathcal{F}$, that is,

$$\{\omega\in\Omega:\mathcal{A}(\tau,\omega)\cap O\neq\emptyset\}\in\overline{\mathcal{F}},$$

for any open set $O\subset C_{\gamma,H}$ and $\tau\in\mathbb{R}$.

Theorem 3.11. *Suppose that* **H1-H7**, (2.14)-(2.15) *hold. Then, the multi-valued random non-autonomous dynamical system* Φ *generated from the p-Laplace equation* (2.4) *has a* $\mathfrak{D}$*-pullback random attractor* $\mathcal{A}$ *over* $(\Omega,\overline{\mathcal{F}},\overline{\mathbb{P}})$.

Proof. By Lemma 3.10, the map $\omega\to\Phi(t,\tau,\omega,\mathcal{K}(\tau,\omega))$ is measurable w.r.t. the $\mathbb{P}$-completion of $\mathcal{F}_{\Omega_m}$, that is

$$C_m:=\{\omega\in\Omega_m:\ \Phi(t,\tau,\omega,\mathcal{K}(\tau,\omega))\cap O\neq\emptyset\}\in\overline{\mathcal{F}}_{\Omega_m}.$$

By Lemma 3.6 (ii), we have

$$\{\omega\in\Omega:\ \Phi(t,\tau,\omega,\mathcal{K}(\tau,\omega))\cap O\neq\emptyset\}=\bigcup_{m=1}^{\infty}C_m\in\overline{\mathcal{F}},$$

together with the closedness of the graph proved in Lemma 3.10 and Lemma 3.9, we can deduce the conclusion according to [4, Theorem 3.5]. □

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Data availability

The authors declare that all the data related to the research carried out in this paper are included in the paper.

Competing interests

The authors declare that they have no competing interests.

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