Estimates of exponential convergence for solutions of stochastic nonlinear systems

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Abstract

This paper aims to analyze the behavior of the solutions of a stochastic perturbed system with respect to the solutions of the stochastic unperturbed system. To prove our stability results, we have derived a new Gronwall–type inequality instead of the Lyapunov techniques, which makes it easy to apply in practice and it can be considered as a more general tool in some situations. On the one hand, we present sufficient conditions ensuring the global practical uniform exponential stability of SDEs based on Gronwall's inequalities. On the other hand, we investigate the global practical uniform exponential stability with respect to a part of the variables of the stochastic perturbed system by using generalized Gronwall's inequalities. It turns out that, the proposed approach gives a better result comparing with the use of a Lyapunov function. A numerical example is presented to illustrate the applicability of our results.

Keywords: Stochastic differential equations, Gronwall's inequalities, Practical uniform exponential stability, Practical uniform exponential stability with respect to a part of the variables.
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1 Introduction

The stability of stochastic differential equations (SDEs) is a prevalent theme of the current study in Mathematics and its applications. Stochastic systems are utilized to model problems from the real world where some sort of randomness or noise is taken into consideration. Noise is used to destabilize a given stable system whereas it can also be used to stabilize a given unstable system or to make a given stable system even more stable. The asymptotic behaviors of solutions of some classes of stochastic systems have been studied by many authors (see [3], [18], [23], [27], [30], [31]-[33]).

Gronwall inequalities are crucial in the area of ordinary differential equations also in stochastic differential equations to prove results on existence, uniqueness, comparison, perturbation, boundedness, as well as stability of solutions.

The aim of this paper is twofold. In the first part, we establish some criteria for the global practical uniform exponential stability of a stochastic perturbed system of the form:

$$dx(t) = f(t, x)dt + g(t, x)dB(t),$$

which may be considered as a perturbation of the deterministic unperturbed system expressed as follows:

$$dx(t) = f(t, x)dt.$$

Different authors consider the stability problem the zero equilibrium point of stochastic differential equations, see ([14], [16], [19]-[21], [29]).

In the case where the origin is not necessarily an equilibrium point, we can investigate the asymptotic stability of solutions of stochastic systems in a small neighborhood of the origin in terms of convergence of solution in probability to a small ball. This property is defined as practical stability. The literature on practical stability is very extensive, see ([2]–[6], [11], [12], [22]) and references therein.

Some stochastic models cannot be proved to fulfill stability properties with respect to all the unknown variables of the system. However, it is very interesting in some practical situations to analyze if it is still possible to prove some stability properties with respect to some of the variables in the problem. Consequently, the second part of this paper is mainly devoted to establishing some criteria for the global practical uniform exponential stability with respect to part of the variables of a class of nonlinear stochastic perturbed system.

The method of Lyapunov is one of the most powerful tools to study the stability of stochastic dynamical systems, with the emergence of the second method of Lyapunov: Rymyanstev [25], Rymyanstev and Oziraner [26] developed the notion of stability with respect to a part of the variables, and many different authors are well developed the concept of partial stability within the method of Lyapunov, see [17, 24].

In [7, 8, 9, 10], the practical stability of stochastic differential systems is evaluated within the method of Lyapunov.

The qualitative behavior of the solutions of perturbed stochastic systems is generally analyzed by considering a Lyapunov function candidate for the unperturbed system and using it as a suitable Lyapunov function candidate for the stochastic system. Yet, unlike the linear case, the construction of proper Lyapunov function is a difficult task for nonlinear stochastic differential equations. This motivates us to enquire about the problem of stability of stochastic perturbed systems by using integral inequalities of Gronwall type under some restrictions on the perturbation term. The usual property of the solutions that can be deduced for such systems is ultimate boundedness. That means that the solutions remain in some neighborhoods of the origin after a sufficiently large time, (see [7]-[10]).

The novelty of our work is to analyze the asymptotic behavior of solutions to stochastic perturbed systems with respect to the solutions of the deterministic unperturbed system based on non–linear integral inequalities.

The organization of this paper is as follows. In Section 2, we prove a new non–linear integral inequality which will play a basic role in our study. In Section 3, we introduce some definitions and results about the global practical uniform exponential stability of stochastic perturbed system based on Gronwall's inequalities. Section 4 is devoted to establishing some criteria for global practical uniform exponential stability of the stochastic perturbed system with respect to a part of the variables based upon new non–linear integral inequality. In Section 5, a numerical example is exhibited to show the efficiency and accuracy of the method. Eventually, some conclusions are included in the last section.

2 Nonlinear integral inequalities

Gronwall–type's lemmas play a crucial role in the area of integral (and differential) equations. It is an essential tool to obtain different estimates in the theory of ordinary and stochastic differential equations. There exist different lemmas which carry the name of Gronwall's lemma.

The original lemma was first proposed by Thomas Hacon Gronwall [15]; see the following proposition:

Lemma 2.1. *Let $z : [\eta, \eta+h] \to \mathbb{R}$ be a continuous function that satisfies the following inequality*

$$0 \leq z(x) \leq \int_{\eta}^{x} (A + Bz(s))ds, \quad for \quad x \in [\eta, \eta+h],$$

where A, B are nonnegative constants. Then,

$$0 \leq z(x) \leq Ahe^{Bh}, \quad for \quad x \in [\eta, \eta+h].$$

Lemma 2.2. *[15] Let $u(t)$ be a continuous function defined on the interval $[t_0, t_1]$, satisfying the following inequality:*

$$u(t) \leq a + b\int_{t_0}^{t} u(s)ds,$$

where a, b are nonnegative constants. Then, for all $t \in [t_0, t_1]$,

$$u(t) \le a e^{b(t-t_0)}.$$

Bellman [1] extended the above inequality as follows:

Lemma 2.3. *Let* $u(t)$ *and* $b(t)$ *be nonnegative continuous functions for* $t \in [t_0, t_1]$, *that satisfy*

$$u(t) \le a + \int_{t_0}^{t} b(s)u(s)ds, \quad t \in [t_0, t_1],$$

where $a \ge 0$ *is a constant. Then,*

$$u(t) \le a \exp\left(\int_{t_0}^{t} b(s)ds\right), \quad t \in [t_0, t_1].$$

Lemma 2.4. *[13] Let* $a(t), b(t), c(t), u(t)$ *be continuous functions for* $t \ge t_0$, *and* $b(t)$ *be nonnegative for* $t \ge t_0$, *suppose that*

$$u(t) \le a(t) + \int_{t_0}^{t} \left(b(s)u(s) + c(s)\right) ds, \quad t \ge t_0.$$

Then,

$$u(t) \le a(t) + \int_{t_0}^{t} \left(a(s)b(s) + c(s)\right) \exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds, \quad t \ge t_0.$$

Corollary 2.5. *[13] For* $a(t) \equiv a$, *we have*

$$u(t) \le a \exp\left(\int_{t_0}^{t} b(\tau)d\tau\right) + \int_{t_0}^{t} c(s) \exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds, \quad t \ge t_0.$$

Now, we introduce the following integral inequality of Gronwall type, which is a slight modification of the one given by [28].

Lemma 2.6. *Let* $u(t)$, $v(t)$, $\omega(t)$ *be nonnegative continuous functions for* $t \ge t_0$, *and suppose*

$$u(t) \le c + \int_{t_0}^{t} \left(u(s)v(s) + \omega(s)\right) ds,$$

where c is a positive constant. Then,

$$u(t) \le e^{\int_{t_0}^{t} v(s)ds} \left(c + \left(e^{\int_{t_0}^{t} \omega(s)ds} - 1\right)\right), \quad \forall t \ge t_0.$$

Now, we will generalize the previous lemma. Instead of taking c as a positive constant, we will consider it a nonnegative differentiable function.

Lemma 2.7. *Let $u(t)$, $v(t)$, $\omega(t)$ be nonnegative continuous functions for $t \geq t_0$, and $c(t)$ be a nonnegative differentiable function for $t \geq t_0$, suppose that*

$$u(t) \leq c(t) + \int_{t_0}^{t} (u(s)v(s) + \omega(s))\, ds.$$

Then,

$$u(t) \leq e^{\int_{t_0}^{t} v(s)ds} \left(c(t) + \left(e^{\int_{t_0}^{t} \omega(s)ds} - 1 \right) \right), \quad \forall t \geq t_0. \tag{2.1}$$

Proof. Noticing the inequality $e^z \geq z + 1$, we have $e^{\int_{t_0}^{t} \omega(s)ds} \geq \int_{t_0}^{t} \omega(s)ds + 1$.

Thus, we obtain

$$u(t) \leq c(t) + \left(e^{\int_{t_0}^{t} \omega(s)ds} - 1 \right) + \int_{t_0}^{t} u(s)v(s)ds. \tag{2.2}$$

That is, it follows that

$$u(t) \left(c(t) + \left(e^{\int_{t_0}^{t} \omega(s)ds} - 1 \right) + \int_{t_0}^{t} u(s)v(s)ds \right)^{-1} \leq 1.$$

Multiplying this product by $v(\cdot) \geq 0$ yields that

$$u(t)v(t) \left(c(t) + \left(e^{\int_{t_0}^{t} \omega(s)ds} - 1 \right) + \int_{t_0}^{t} u(s)v(s)ds \right)^{-1} \leq v(t).$$

Adding to each part the ensuing quantity:

$$\left(\dot{c}(t) + \omega(t) e^{\int_{t_0}^{t} \omega(s)ds} \right) \left(c(t) + \left(e^{\int_{t_0}^{t} \omega(t)ds} - 1 \right) \right)^{-1} \geq 0.$$

Then, we obtain

$$\begin{aligned} & u(t)v(t)(c(t) + (e^{\int_{t_0}^{t} \omega(s)ds} - 1) + \int_{t_0}^{t} u(s)v(s)ds)^{-1} \\ & + (\dot{c}(t) + \omega(t)e^{\int_{t_0}^{t} \omega(s)ds})(c(t) + (e^{\int_{t_0}^{t} \omega(s)ds} - 1))^{-1} \\ & \leq v(t) + (\dot{c}(t) + \omega(t)e^{\int_{t_0}^{t} \omega(s)ds})(c(t) + (e^{\int_{t_0}^{t} \omega(s)ds} - 1))^{-1}. \end{aligned}$$

Whence, it immediately follows that,

$$\left(u(t)v(t)+\dot{c}(t)+\omega(t)e^{\int_{t_0}^{t}\omega(s)ds}\right)\left(c(t)+\left(e^{\int_{t_0}^{t}\omega(s)ds}-1\right)+\int_{t_0}^{t}u(s)v(s)ds\right)^{-1}$$
$$\leq v(t)+\left(\dot{c}(t)+\omega(t)e^{\int_{t_0}^{t}\omega(s)ds}\right)\left(c(t)+\left(e^{\int_{t_0}^{t}\omega(s)ds}-1\right)\right)^{-1}.$$

Integrating both sides of the above inequality between t_0 and t,

$$\ln\left(c(t)+\left(e^{\int_{t_0}^{t}\omega(s)ds}-1\right)+\int_{t_0}^{t}u(s)v(s)ds\right)-\ln\left(c(t_0)\right)$$
$$\leq\int_{t_0}^{t}v(s)ds+\ln\left(c(t)+\left(e^{\int_{t_0}^{t}\omega(s)ds}-1\right)\right)-\ln\left(c(t_0)\right).$$

Converting this into exponential form and taking into account inequality (2.2), the inequality becomes (2.1). That is, one obtains

$$c(t)+\left(e^{\int_{t_0}^{t}\omega(s)ds}-1\right)+\int_{t_0}^{t}u(s)v(s)ds\leq e^{\int_{t_0}^{t}v(s)ds}\left(c(t)+\left(e^{\int_{t_0}^{t}\omega(s)ds}-1\right)\right).$$

Combining this result together with (2.2), we obtain the desired inequality (2.1). □

Remark 2.8. Notice that Lemma 2.6 is a particular case of Lemma 2.7.

3 Practical uniform exponential stability of stochastic perturbed system

The objective of this section is to state sufficient conditions for global practical exponential stability of stochastic perturbed systems based on generalized Gronwall's inequalities.

Consider the following nonlinear differential equation (DE):

$$\begin{cases} dx(t)=f(t,x(t))dt, \quad t\geq t_0\geq 0,\\ \\ x(t_0)=x_0, \quad t_0\geq 0, \end{cases} \tag{3.1}$$

with initial condition $x_0\in\mathbb{R},\ f:\mathbb{R}_+\times\mathbb{R}\longrightarrow\mathbb{R}$.

Assume that some parameters are excited or perturbed by Brownian motion, and the perturbed stochastic differential equation (SDE) has the following form:

$$dy(t)=f(t,y(t))dt+g(t,y(t))dB_t, \quad t\geq t_0\geq 0, \tag{3.2}$$

with the same initial conditions, $g : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}^{1\times m}$ and $B_t = (B_1(t), \cdots, B_m(t))^T$ is an m–dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We assume that both f and g satisfy the following conditions:

$$|f(t,y)|^2 + ||g(t,y)||^2 \le C_1(t)\big(1+|y|^2\big), \text{ for all } t \ge 0, \quad y \in \mathbb{R}, \tag{3.3}$$

$$|f(t,y) - f(t,\widetilde{y})| \vee ||g(t,y) - g(t,\widetilde{y})|| \le C_2(t)|y - \widetilde{y}|, \text{ for all } t \ge 0, \quad y, \widetilde{y} \in \mathbb{R}, \tag{3.4}$$

where $C_1(\cdot)$ and $C_2(\cdot)$ are non–negative functions.

Under the precedent assumptions there exists a unique global solution $x(t,t_0,x_0)$ of DE (3.1) corresponding to the initial condition $x_0 \in \mathbb{R}$ defined in an interval $[t_0, T)$, and a unique global solution $y(t,t_0,y_0)$ of SDE (3.2) corresponding to the initial condition $y_0 \in \mathbb{R}$ defined in an interval $[t_0, T)$. Or simply $x(t)$ and $y(t)$ to denote a solution to our systems and, as we will be interested in analyzing the asymptotic behavior of solutions, we assume $T = +\infty$, for extra details see [10].

Suppose that the origin $x = 0$ is an equilibrium point for the deterministic unperturbed system (3.1) and the perturbation g does not vanish at zero, that is $g(t,0) \ne 0, \ \forall t \ge 0$. Then, the stochastic perturbed system (3.2) does not possess the trivial solution $y(t,t_0,0) = 0$.

The study of the exponential stability of solutions of the stochastic perturbed system (3.2) leads to analyze the stability behavior of a ball centered at the origin:

$$B_r := \{y \in \mathbb{R} : |y| \le r\}, \ r > 0.$$

Definition 3.1.

i) The ball B_r is said to be almost surely globally uniformly exponentially stable, if there exists a pair of positive constants k and γ, such that for all $t_0 \in \mathbb{R}_+$, and all $y_0 \in \mathbb{R}$, the following inequality is satisfied:

$$|y(t,t_0,y_0)| \le k|y_0|e^{-\gamma(t-t_0)} + r, \quad \text{a.s.,} \quad \forall t \ge t_0 \ge 0.$$

ii) The stochastic perturbed system (3.2) is said to be almost surely globally practically uniformly exponentially stable, if there exists $r > 0$ such that B_r is almost surely globally uniformly exponentially stable.

Let us now state some assumptions, which we will impose later on:

($\mathcal{H}_1$) There exist a continuous nonnegative function $L(t)$ and a constant $0 < l < \infty$, such that $f(t,x)$ satisfies the following generalized Lipschitz condition:

$$|f(t,x) - f(t,y)| \leq L(t)|x-y|,$$

with

$$\int_0^{+\infty} L(s)ds \leq l < +\infty.$$

($\mathcal{H}_2$) There exist a continuous nonnegative function $\lambda(t)$, and a constant $0 < M < \infty$, such that

$$\|g(t,x)\| \leq \frac{\lambda(t)}{\sqrt{\ln(t)}}, \quad \forall x \in \mathbb{R},$$

with

$$\int_0^{+\infty} \lambda^2(s)ds \leq M < +\infty. \tag{3.5}$$

($\mathcal{H}_3$) There exist positive constants λ_1 and λ_2, such that

$$|x(t,t_0,x_0)| \leq \lambda_1 |x_0| e^{-\lambda_2(t-t_0)},$$

for all $t \geq t_0 \geq 0$, and all $x_0 \in \mathbb{R}$.

Remark 3.1. From assumption ($\mathcal{H}_3$), we infer that the trivial solution of the unperturbed system (3.1) is uniformly exponentially stable.

Now, we state and prove our first main result in this section.

Theorem 3.2. *Under assumptions ($\mathcal{H}_1$) − ($\mathcal{H}_3$), the stochastic perturbed system (3.2) is almost surely globally practically uniformly exponentially stable.*

To prove the previous theorem, we need to recall the following lemma.

Lemma 3.3. *[21] Let $g = (g_1, \cdots, g_m) \in L^2(\mathbb{R}_+, \mathbb{R}^{1\times m})$, T, α, β be any positive numbers. Then, for $t_0 \geq 0$,*

$$\mathbb{P}\left(\sup_{t_0 \leq t \leq T} \left[\int_{t_0}^t g(s)dB_s - \frac{\alpha}{2}\int_{t_0}^t ||g(s)||^2 ds \right] > \beta \right) \leq \exp(-\alpha\beta).$$

Proof of Theorem 3.2. Let $x(t) := x(t,t_0,x_0)$ be solution of (3.1) and $y(t) := y(t,t_0,y_0)$ be solution of (3.2). Thus, we have

$$x(t) = x_0 + \int_{t_0}^t f(s,x(s))ds,$$

and

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds + \int_{t_0}^{t} g(s, y(s))dB_s. \tag{3.6}$$

Let $n = 1, 2, \cdots$, by Lemma 3.3, we obtain

$$\mathbb{P}\left\{\sup_{t_0 \le t \le n}\left(\int_{t_0}^{t} g(s, y(s))dB_s - \frac{1}{2}\ln(n)\int_{t_0}^{t} ||g(s, y(s))||^2 ds\right) > 2\right\} \le \frac{1}{n^2}.$$

Applying the well–known Borel–Cantelli lemma, we see that for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$, such that if $n \ge n_0$,

$$\int_{t_0}^{t} g(s, y(s))dB_s \le 2 + \frac{1}{2}\ln(n)\int_{t_0}^{t} ||g(s, y(s))||^2 ds, \quad \text{for all } t_0 \le t \le n.$$

Then, equality (3.6) becomes, for all $t_0 \le t \le n,\ n \ge n_0$,

$$y(t) \le y_0 + 2 + \int_{t_0}^{t} f(s, y(s))ds + \frac{1}{2}\ln(n)\int_{t_0}^{t} ||g(s, y(s))||^2 ds.$$

As a consequence, we obtain

$$|y(t) - x(t)| \le |y_0 - x_0| + 2 + \int_{t_0}^{t} |f(s, y(s)) - f(s, x(s))|ds + \frac{1}{2}\ln(n)\int_{t_0}^{t} ||g(s, y(s))||^2 ds,$$

for all $t_0 \le t \le n,\ n \ge n_0$.

Taking into account assumptions $(\mathcal{H}_1), (\mathcal{H}_2)$, and the fact that $x_0 \equiv y_0$, it follows that

$$\begin{aligned}
|y(t) - x(t)| &\le 2 + \int_{t_0}^{t} L(s)|y(s) - x(s)|ds + \frac{1}{2}\ln(n)\int_{t_0}^{t} \frac{\lambda^2(s)}{\ln(s)}ds, \\
&\le 2 + \int_{t_0}^{t} L(s)|y(s) - x(s)|ds + \frac{1}{2}\ln(n)\int_{t_0}^{t} \frac{\lambda^2(s)}{\ln(n)}ds, \\
&\le 2 + \int_{t_0}^{t} L(s)|y(s) - x(s)|ds + \frac{1}{2}\int_{t_0}^{t} \lambda^2(s)ds, \text{ for all } t_0 \le t \le n,\ n \ge n_0.
\end{aligned}$$

Applying the Gronwall Lemma 2.6, it follows that

$$\begin{aligned}
|y(t) - x(t)| &\le e^{\int_{t_0}^{t} L(s)ds}\left(2 + \left(\exp\left(\frac{1}{2}\int_{t_0}^{t} \lambda^2(s)ds\right) - 1\right)\right) \\
&\le e^{\int_{t_0}^{+\infty} L(s)ds}\left(2 + \left(\exp\left(\frac{1}{2}\int_{t_0}^{+\infty} \lambda^2(s)ds\right) - 1\right)\right) \\
&\le e^{l}\left(2 + \left(e^{\frac{M}{2}} - 1\right)\right), \quad \text{for all } t_0 \le t \le n,\ n \ge n_0.
\end{aligned}$$

Thus, we see that

$$|y(t) - x(t)| \leq 2e^l + \exp\left(l + \frac{M}{2}\right), \quad \text{a.s.}$$

Hence, from assumption $(\mathcal{H}_3)$, one can deduce that

$$\begin{aligned}|y(t)| &= |y(t) + x(t) - x(t)| \\ &\leq |x(t)| + |y(t) - x(t)| \\ &\leq \lambda_1 |y_0| e^{-\lambda_2(t-t_0)} + 2e^l + \exp\left(l + \frac{M}{2}\right).\end{aligned}$$

Then, we infer that for all $t \geq t_0 \geq 0$,

$$|y(t)| \leq \lambda_1 |y_0| e^{-\lambda_2(t-t_0)} + 2e^l + \exp\left(l + \frac{1}{2}M\right), \quad \text{a.s.}$$

Finally, we deduce that the stochastic perturbed system (3.2) is almost surely globally practically uniformly exponentially stable, with $r = 2e^l + \exp\left(l + \frac{1}{2}M\right)$. □

Remark 3.4. In [4], Caraballo et al. investigated the practical uniform exponential stability by using Lyapunov techniques for n–dimensional stochastic differential equation.

Indeed, they have considered the following n–dimensional stochastic differential equation (SDE):

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB_t, \quad t \geq 0, \tag{3.7}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$, $x = (x_1, \cdots, x_n)^T$ and $B_t = (B_1(t), \cdots, B_m(t))^T$ is an m–dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

f and g satisfy the following conditions:

$$||f(t,x)||^2 + ||g(t,x)||^2 \leq K_1\left(1 + ||x||^2\right), \text{ for all } t \geq 0, x \in \mathbb{R}^n, \tag{3.8}$$

$$||f(t,x) - f(t,y)|| \vee ||g(t,x) - g(t,y)|| \leq K_2||x - y||, \text{ for all } t \geq 0, x, y \in \mathbb{R}^n, \tag{3.9}$$

where K_1 and K_2 are given positive real constant, and they have obtained the following stability result by using the Lyapunov method.

Theorem 3.5. *Assume that there exist a function $V \in C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^n, {\mathbb{R}^*}_+)$ and constants $p \in \mathbb{N}^*$, $c_1 \geq 1$, $\varrho \geq c_1$, $\gamma \geq 0$ and $c_2 \in \mathbb{R}$, $c_3 \geq 0$ such that for all $t \geq t_0 \geq 0$, and $x \in \mathbb{R}^n$,*

1. $c_1||x||^p \leq V(t,x)$,
2. $LV(t,x) \leq c_2V(t,x) + \varrho$,

3. $||V_x(t,x)g(t,x)||^2 \geq c_3 V^2(t,x) + \gamma$.

Then,

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln \left(||x(t,t_0,x_0)|| - \left(\frac{\varrho}{c_1}\right)^{\frac{1}{p}} \right) \leq -\frac{(c_3 - 2(c_2+1))}{2}, \quad a.s., \ \text{for all } x_0 \in \mathbb{R}^n.$$

In particular, if $c_3 > 2(c_2+1)$*, then the solution of (3.7) is almost surely globally practically uniformly exponentially stable with* $r = \left(\frac{\varrho}{c_1}\right)^{\frac{1}{p}}$.

Lyapunov's method is one of the most useful for investigating the stability of stochastic systems, without knowing the form of explicit solution of the system. However, constructing an appropriate Lyapunov function is still a challenging task. The novelty of our work is to develop the problem of the stability of perturbed stochastic systems on the basis of the explicit solution formed by generalized integral inequalities, however, we have to pay the price of considering only a differential equation instead of a differential system. We plan to investigate in future how this technique can be extended to differential systems.

Example 3.6. *Let consider the following stochastic system:*

$$dy(t) = \left(-ay(t) + y^2(t)\right) dt + \frac{e^{-\varsigma t}}{\sqrt{\ln(t)}} dB_t, \ \ a, \varsigma > 0. \tag{3.10}$$

where $y(t) \in \mathbb{R}$ *and with initial value* y_0.

The stochastic system (3.10) *can be regarded as a perturbed system of:*

$$dx(t) = \left(-ax(t) + x^2(t)\right) dt. \tag{3.11}$$

The unperturbed system (3.11) *is almost sure exponential stable, as shown in Fig.* 1*, for* $a = 2$.

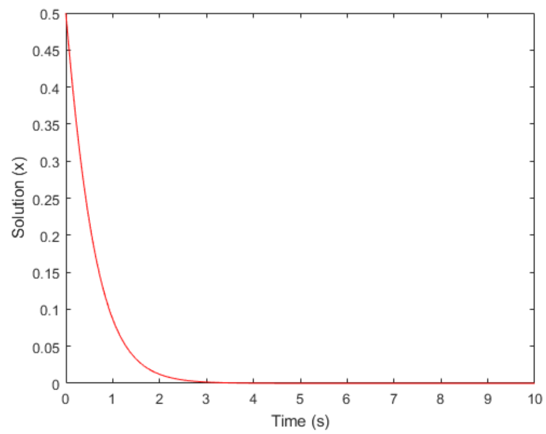

Figure 1: Time evolution of the solution of the unperturbed system (3.11)

On the other side, we have

$$|g(t,y)| = \frac{e^{-\varsigma t}}{\sqrt{\ln(t)}}.$$

Thus, the function in Theorem 3.2 become: $\lambda(t) = \exp(-\varsigma t)$, *which satisfies condition* (3.5). *Based on Theorem 3.2 the stochastic perturbed system* (3.10) *is almost sure globally practically uniformly exponentially stable, as we can see in Fig.* 2, *for* $\varsigma = 5$.

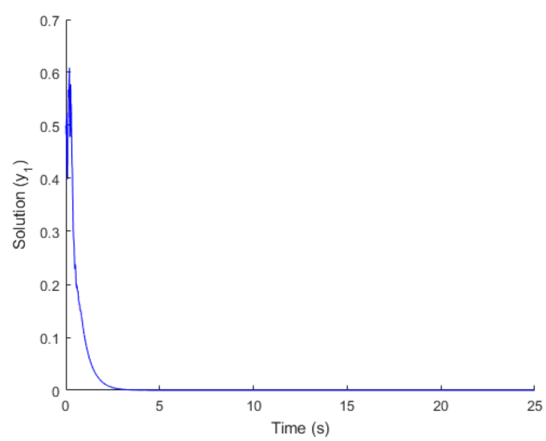

Figure 2: Time evolution of the state $y(t)$ of the solution of the stochastic system (3.10)

4 Practical uniform exponential stability of stochastic perturbed system with respect to a part of the variables

Different stochastic systems cannot be proved to fulfill stability properties for all the unknown variables. Though, it is very impressive in some practical problems to investigate whether it is even possible to prove some stability properties with respect to some of the variables in the problem. Consequently, in this section we will study the global practical uniform exponential stability with respect to a part of the variables for a class of stochastic perturbed systems. The principal mathematical technique employed is the use of generalized Gronwall's inequalities.

We consider the following stochastic system:

$$\begin{cases} dy_1(t) = f_1(t, y_1(t), y_2(t))dt + g_1(t, y_1(t), y_2(t))dB_t \\ \\ dy_2(t) = f_2(t, y_1(t), y_2(t))dt + g_2(t, y_1(t), y_2(t))dB_t, \end{cases} \tag{4.1}$$

with initial condition $y(t_0) = y_0 = (y_{10}, y_{20}) \in \mathbb{R} \times \mathbb{R}$,

- $f_1 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad g_1 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{1\times m}$.
- $f_2 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad g_2 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{1\times m}$.

Assume that both conditions (3.3) and (3.4) ensuring existence and uniqueness of solutions are satisfied, and $y(t, t_0, y_0) = (y_1(t, t_0, y_0), y_2(t, t_0, y_0))$ is the solution of the perturbed system (4.1).

The stochastic system (4.1) may be regarded as a perturbed system of the following:

$$\begin{cases} dx_1(t) = f_1(t, x_1(t), x_2(t))dt \\ \\ dx_2(t) = f_2(t, x_1(t), x_2(t))dt, \end{cases} \tag{4.2}$$

where $x := (x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and with the same initial condition $x(t_0) = y(t_0)$, and $x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0))$ is the solution of the perturbed system (4.1).

Suppose that $g(t, 0)$ is not necessarily zero. Now, we define the exponential stability with respect to a part of the variables of the stochastic perturbed system (4.1) when the origin is no longer an equilibrium point. In this case, we will study the stability with respect to a part of the variables of the stochastic perturbed system (4.1) in a small neighborhood of the origin in terms of convergence of solution in probability with respect to a part of the variables to the ball:

$$\mathcal{B}_r := \left\{ y \in \mathbb{R}^2 : ||y|| \leq r \right\}, \ r > 0.$$

Definition 4.1. i) The ball $\mathcal{B}_r$ is said to be almost surely globally uniformly exponentially stable with respect to y_1, if there exists a pair of positive constants k' and γ', such that for all $t_0 \in \mathbb{R}_+$, and all $y_0 \in \mathbb{R}^2$, the following inequality is fulfilled:

$$|y_1(t,t_0,y_0)| \leq k'||y_0||e^{-\gamma'(t-t_0)} + r, \quad \text{a.s.}, \ \forall t \geq t_0 \geq 0.$$

ii) The stochastic perturbed system (4.1) is said to be almost surely globally practically uniformly exponentially stable with respect to y_1, if there exists $r > 0$ such that $\mathcal{B}_r$ is almost surely globally uniformly exponentially stable with respect to y_1.

Now, we can establish our main result in this section. In fact, we assume that the stochastic systems (4.1) and (4.2) satisfy the following assumptions:

$(\mathcal{H}'_1)$ There exist a continuous nonnegative function $L_1(t)$, and a constant $0 < l_1 < \infty$, such that $f_1(t,x_1,x_2)$ satisfies the following generalized Lipschitz condition on x_1:

$$|f_1(t,x_1,x_2) - f_1(t,y_1,y_2)| \leq L_1(t)|x_1 - y_1|,$$

with

$$\int_0^{+\infty} L_1(s)ds \leq l_1 < +\infty.$$

$(\mathcal{H}'_2)$ There exist continuous nonnegative functions $\varphi(t),\ \psi(t)$, and positive constants $M',\ \zeta$, such that

$$\|g(t,y_1,y_2)\|^2 \leq \frac{1}{\ln(t)}\left(\varphi(t)|y_1| + \psi(t)\right),$$

with

$$\int_0^{+\infty} \varphi(s)ds \leq M' < +\infty,$$

and

$$\int_0^{+\infty} \psi(s)ds \leq \zeta < +\infty.$$

$(\mathcal{H}'_3)$ There exist positive constants λ'_1 and λ'_2, such that

$$|x_1(t,t_0,x_0)| \leq \lambda'_1|x_0|e^{-\lambda'_2(t-t_0)},$$

for all $t \geq t_0 \geq 0,\ x_0 \in \mathbb{R}$.

Remark 4.1. From $(\mathcal{H}'_3)$ we deduce that the trivial solution of the deterministic unperturbed system (4.2) is exponentially stable with respect to x_1.

Theorem 4.2. *Under assumptions* $(\mathcal{H}_1') - (\mathcal{H}_3')$ *the stochastic perturbed system (4.1) is almost surely globally practically uniformly exponentially stable with respect to* y_1.

Proof Let $x_1(t) := x_1(t, t_0, x_0)$ and $y_1(t) := y_1(t, t_0, y_0)$. Thus, we have

$$x_1(t) = x_{10} + \int_{t_0}^{t} f_1(s, x_1(s), x_2(s))ds,$$

and

$$y_1(t) = y_{10} + \int_{t_0}^{t} f_1(s, y_1(s), y_2(s))ds + \int_{t_0}^{t} g_1(s, y_1(s), y_2(s))dB_s, \tag{4.3}$$

where $x_{10} = x_1(t_0, t_0, x_0)$ and $y_{10} = y_1(t_0, t_0, y_0)$.

Let $n = 1, 2, \cdots$, using Lemma 3.3, it yields that

$$\mathbb{P}\left\{\sup_{t_0 \le t \le n}\left(\int_{t_0}^{t} g_1(s, y_1(s), y_2(s))dB_s - \frac{1}{2}\ln(n)\int_{t_0}^{t} ||g_1(s, y_1(s), y_2(s))||^2 ds\right) > 2\right\} \le \frac{1}{n^2}.$$

Applying the Borel–Cantelli Lemma, then for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$, such that if $n \ge n_0$, we have

$$\int_{t_0}^{t} g_1(s, y_1(s), y_2(s))dB_s \le 2 + \frac{1}{2}\ln(n)\int_{t_0}^{t} ||g_1(s, y_1(s), y_2(s))||^2 ds, \quad \forall t_0 \le t \le n.$$

Then, equality (4.3) becomes

$$y_1(t) \le y_{10} + 2 + \int_{t_0}^{t} f_1(s, y_1(s), y_2(s))ds + \frac{1}{2}\ln(n)\int_{t_0}^{t} ||g_1(s, y_1(s), y_2(s))||^2 ds,$$

for all $t_0 \le t \le n$, $n \ge n_0$ almost surely.

Consequently, it follows that,

$$\begin{aligned}|y_1(t) - x_1(t)| \le |y_{10} - x_{10}| + 2\int_{t_0}^{t} |f_1(s, y_1(s), y_2(s)) - f_1(s, x_1(s), x_2(s))|ds \\ + \frac{1}{2}\ln(n)\int_{t_0}^{t} ||g_1(s, y_1(s), y_2(s))||^2 ds, \quad \forall t_0 \le t \le n,\ n \ge n_0.\end{aligned}$$

From conditions $(\mathcal{H}_1'), (\mathcal{H}_2')$, and the fact that $x_0 \equiv y_0$, one obtains

$$\begin{aligned}|y_1(t) - x_1(t)| &\le 2 + \int_{t_0}^{t} L_1(s)|y_1(s) - x_1(s)|ds + \frac{1}{2}\ln(n)\int_{t_0}^{t} \frac{1}{\ln(s)}\left(\varphi(s)|y_1(s)| + \psi(s)\right)ds, \\ &\le 2 + \int_{t_0}^{t} L_1(s)|y_1(s) - x_1(s)|ds + \frac{1}{2}\ln(n)\int_{t_0}^{t} \frac{1}{\ln(n)}\left(\varphi(s)|y_1(s)| + \psi(s)\right)ds, \\ &\le 2 + \int_{t_0}^{t} L_1(s)|y_1(s) - x_1(s)|ds + \frac{1}{2}\int_{t_0}^{t}\left(\varphi(s)|y_1(s)| + \psi(s)\right)ds,\ \forall t_0 \le t \le n, n \ge n_0.\end{aligned}$$

Thus,

$$|y_1(t)-x_1(t)| \le 2+\int_{t_0}^{t}\left(L_1(s)|y_1(s)-x_1(s)|+\omega(s)\right)ds, \quad \forall t_0\le t\le n,\ n\ge n_0,$$

where $\omega(s)=\dfrac{1}{2}\left(\varphi(s)|y_1(s)|+\psi(s)\right).$

Applying Corollary 2.5, we obtain

$$\begin{aligned}
|y_1(t)-x_1(t)| &\le 2e^{\int_{t_0}^{t}L_1(\tau)d\tau}ds+\int_{t_0}^{t}\omega(s)e^{\int_s^t L_1(\tau)d\tau}ds\\
&\le 2e^{\int_{t_0}^{+\infty}L_1(\tau)d\tau}ds+\int_{t_0}^{t}\omega(s)e^{\int_s^{+\infty} L_1(\tau)d\tau}ds\\
&\le 2e^{l_1}+e^{l_1}\int_{t_0}^{t}\omega(s)ds,\ \ t_0\le t\le n,\ n\ge n_0, \quad \text{a.s.}
\end{aligned}$$

Combining the last inequality with assumption $(\mathcal{H}_3')$, it follows that

$$\begin{aligned}
|y_1(t)| &= |y_1(t)\pm x_1(t)|\\
&\le |x_1(t)|+|y_1(t)-x_1(t)|\\
&\le \lambda_1'\|x_0\|e^{-\lambda_2'(t-t_0)}+2e^{l_1}+e^{l_1}\int_{t_0}^{t}(\varphi(s)|y_1(s)|+\psi(s))ds.
\end{aligned}$$

Since $x_0\equiv y_0$, then we obtain

$$|y_1(t)|\le \lambda_1'\|y_0\|e^{-\lambda_2'(t-t_0)}+2e^{l_1}+e^{l_1}\int_{t_0}^{t}(\varphi(s)|y_1(s)|+\psi(s))ds,\ \text{a.s.}$$

Thanks now to Gronwall's lemma (Lemma 2.7), it yields that

$$\begin{aligned}
|y_1(t)| &\le \exp\left(e^{l_1}\int_{t_0}^{t}\varphi(s)ds\right)\left(\lambda_1'\|y_0\|e^{-\lambda_2'(t-t_0)}+2e^{l_1}+e^{e^{l_1}\int_{t_0}^{t}\psi(s)ds}-1\right)\\
&\le \exp\left(e^{l_1}\int_{t_0}^{+\infty}\varphi(s)ds\right)\left(\lambda_1'\|y_0\|e^{-\lambda_2'(t-t_0)}+2e^{l_1}+e^{e^{l_1}\int_{t_0}^{+\infty}\psi(s)ds}-1\right)\\
&\le \exp\left(e^{l_1}M'\right)\left(\lambda_1'\|y_0\|e^{-\lambda_2'(t-t_0)}+2e^{l_1}+e^{e^{l_1}\zeta}\right).
\end{aligned}$$

That is, we obtain

$$|y_1(t)|\le \lambda_1'\exp\left(e^{l_1}M'\right)\|y_0\|e^{-\lambda_2'(t-t_0)}+\exp\left(e^{l_1}M'\right)\left(2e^{l_1}+e^{e^{l_1}\zeta}\right), \quad \text{a.s.}$$

Eventually, we conclude that the stochastic perturbed system (4.1) is globally practically uniformly exponentially stable with respect to y_1, with $r=\exp\left(e^{l_1}M'\right)\left(2e^{l_1}+e^{e^{l_1}\zeta}\right)$. □

Remark 4.3. Here we should mention that in our paper [7] we have established sufficient conditions ensuring the practical exponential stability with respect to a part of the variables by using Lyapunov techniques. Indeed, we have considered the n–dimensional stochastic differential equation (3.7), where both functions f and g satisfy conditions (3.8) and (3.9), and we have obtained the following stability result by using the Lyapunov method.

Theorem 4.4. *[7] Assume that there exist a function $V \in C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ and constants $p \in \mathbb{N}^*$, $\beta_1 \geq 1$, $\gamma \geq \beta_1$, $\zeta \geq 0$ and $\beta_2 \in \mathbb{R}$, $\beta_3 \geq 0$ such that for all $t \geq t_0 \geq 0$, and all $x = (y, z) \in \mathbb{R}^n$,*

1. $\beta_1 ||y||^p \leq V(t, x)$,
2. $LV(t, x) \leq \beta_2 V(t, x) + \gamma$,
3. $||V_x(t, x) g(t, x)||^2 \geq \varrho(t) V^2(t, x) + \zeta$,

where $\varrho(t)$ is a continuous nonnegative function with

$$\limsup_{t \to +\infty} \left(\frac{\int_0^t \varrho(s) ds}{t} \right) \leq \beta_3.$$

Furthermore, we suppose that $z(t, t_0, x_0)$ is globally uniformly bounded in probability.

Then,

$$\limsup_{t \to +\infty} \frac{1}{t} \ln \left(||y(t, t_0, x_0)|| - \left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}} \right) \leq -\frac{(\beta_3 + \sigma) - 2(\beta_2 + 1)}{2}, \quad a.s., \text{ for all } x_0 \in \mathbb{R}^n,$$

where σ is a positive constant.

In particular, if $(\beta_3 + \sigma) > 2(\beta_2 + 1)$, then the system (3.7) is said to be almost surely globally practically uniformly exponentially stable with respect to y, with $r = \left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}}$.

Example 4.5. *Consider the following stochastic system:*

$$\begin{cases} dy_1(t) = (-3y_1(t) + y_1^3(t))dt + \dfrac{1}{\sqrt{\ln(t)}\sqrt{ch(t)}} \dfrac{\sqrt{y_1^2 + y_2^2}}{1 + (y_1^2 + y_2^2)^{\frac{1}{4}}} + \dfrac{e^{-t}}{\sqrt{\ln(t)}} dB_t \\ dy_2(t) = 3\cos(t) y_2(t) dt, \end{cases} \tag{4.4}$$

where $y(t) = (y_1(t), y_2(t))^T \in \mathbb{R}^2$ and with initial value $y_0 = (y_{10}, y_{20})$.

The stochastic system (4.4) can be regarded as a stochastic perturbed system of the following system:

$$\begin{cases} dx_1(t) = \left(-3x_1(t) + x_1^3(t)\right) dt \\ dx_2(t) = 3\cos(t)x_2(t)dt, \end{cases} \tag{4.5}$$

where $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$, *with the same initial value* $x_0 \equiv y_0$.

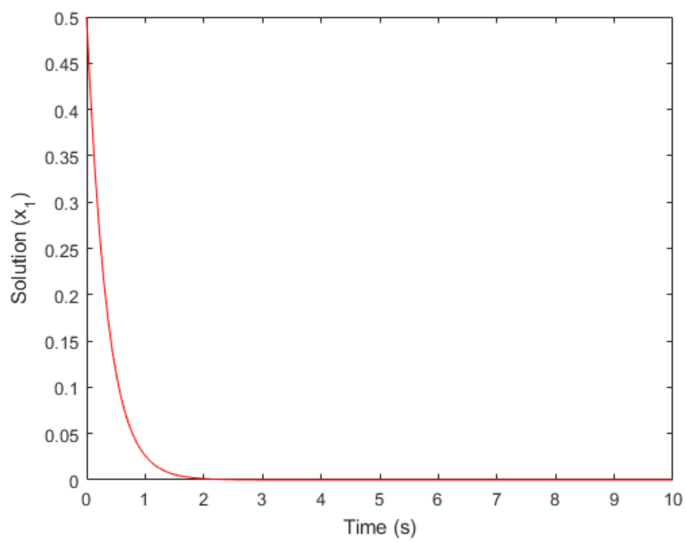

Figure 3: Time evolution of the component $x_1(t)$ of the solution of stochastic unperturbed system(4.5)

Note that the previous Figure 4.5 shows that the stochastic system (4.4) *is almost sure exponential stable with respect to* x_1.

The solution of the sub–system with respect to the variable x_2 *is globally uniformly bounded with probability one. In fact for all* $\iota > 0$ *(independent of* t_0*), all* $t \geq t_0 \geq 0$*, and all* $x_{20} \in \mathbb{R}$ *with* $|x_{20}| \leq \iota$*, we have* $||x_2(t)|| \leq \iota e^{3\sin(t)}$*, a.s.*

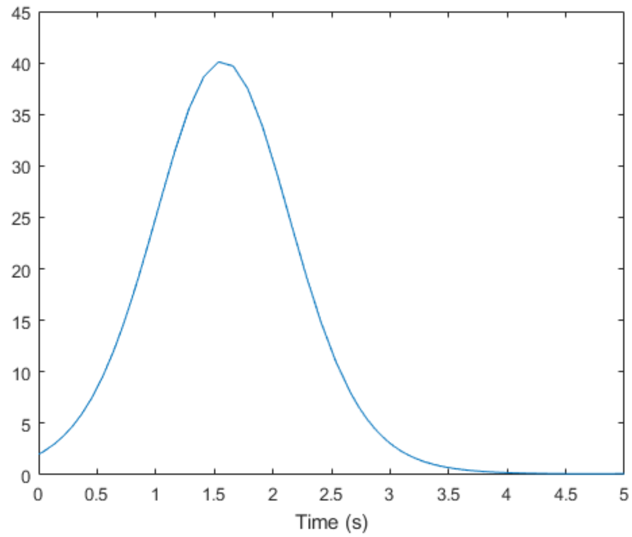

Figure 4: Time evolution of the components $y_2(t)$ of the solution of stochastic perturbed system (4.4)

On the other side, we have

$$||g_1(t,y)||^2 = \left(\frac{1}{\sqrt{\ln(t)}\sqrt{ch(t)}}\frac{\sqrt{y_1^2+y_2^2}}{1+(y_1^2+y_2^2)^{\frac{1}{4}}}\right)^2 + \frac{e^{-2t}}{\ln(t)}$$
$$\leq \frac{1}{ch(t)}||(y_1,y_2)|| + \frac{e^{-2t}}{\ln(t)}.$$

Thus, the functions in Theorem 4.2 become:

$$\varphi(t) = \frac{1}{ch(t)}, \quad \psi(t) = e^{-2t}.$$

It is obvious that,

$$\int_0^{+\infty} \varphi(t) < +\infty, \quad \int_0^{+\infty} \psi(t) \leq +\infty.$$

Finally, all assumptions of Theorem 4.2 are fulfilled. That is, the stochastic perturbed system (4.4) is almost surely globally practically uniformly exponentially stable with respect to (y_1, y_2)*, as we can see in Figure 5.*

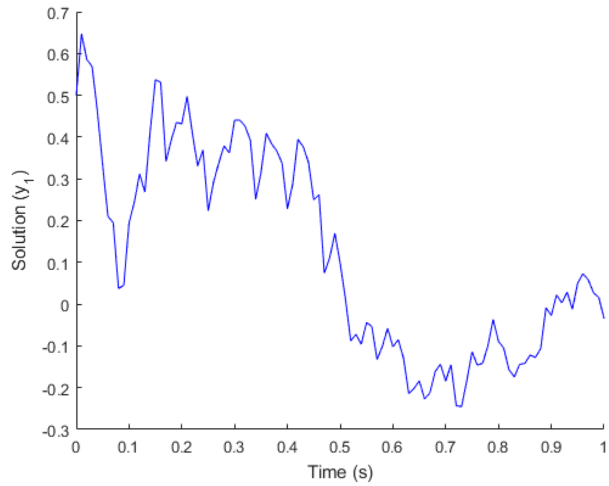

Figure 5: Time evolution of the components $(y_1(t), y_2(t))$ of the solution of stochastic perturbed system (4.4)

Remark 4.6. Note that, we cannot apply Theorem 3.2 to obtain the practical stability in all variables for the above example because the solution of the sub-system with respect to the variable y_2 is globally uniformly bounded with probability one but not attractive.

Remark 4.7. We cannot proved the practical uniform exponential stability of the stochastic perturbed system (4.4) with respect to the variable y_1 by using Theorem 4.4. Indeed, Assumption 3 of Theorem 4.4 is not satisfied. Our new Theorem 4.2 allows us to prove the partial uniform practical exponential stability of system (4.4).

5 Conclusion

In this paper, we tackle the practical exponential stability and practical exponential stability with respect to a part of the variables of stochastic perturbed systems. The principal technical tool for deriving our stability results is a generalized Gronwall inequality. A numerical example has been introduced to prove the accuracy of our developed methods.

Conflict of interest

The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest, or non financial interest in the subject matter or materials discussed in this manuscript.

Data availability statement

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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