A bound on the existence of the maximum jointly invariant set of input-coupled systems

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***Abstract*— We present a set-theoretical characterization of a bound on the maximal portion that an agent can cede of its input variable to another agent. By ceding control authority, agents can decompose coupling variables into public and private parts, which is of interest in situations of *partial* cooperation. In particular, sufficient conditions under which the non-existence of the maximum robust control invariant set is guaranteed are provided, expressed in terms of support functions and the dominant system eigenvalue. Finally, the results are illustrated via stable and unstable example systems with different coupling.**

I. INTRODUCTION

The increasing size and complexity of systems, justify the need for alternatives to centralized control approaches [1]. Non-centralized control methods partition the overall system into a set of dynamically coupled smaller subsystems, each of them assigned to a local controller or agent. Here, we distinguish distributed strategies, where agents share data through a communication network, from decentralized ones, where no data exchange occurs. In general, distributed strategies increase the demand for communication and computation [2], but the lack of coordination in decentralized schemes can compromise performance [3]. An intermediate solution is to dynamically adjust the control network so that only strongly coupled agents communicate [4]. That is, agents are clustered into time-varying coalitions to improve global performance, optimizing the communication resources; see, e.g., [5].

One of these methods of partial cooperation is the robust coalitional control scheme presented in [6], where agents can cede control authority on part of their input space to help neighbors. Thus, performance can be increased while minimizing the cooperation efforts and preserving privacy on local variables. Consequently, coupled variables are partitioned into a private part, which is locally controlled, and a public part that is ceded to other agents. The bounds on the latter can be negotiated, as the need arises, in a distributed fashion.

The objective of this article is to characterize the maximal portion of input space that agents can cede to their neighbors so that the global problem remains feasible in the previously mentioned variable decomposition method. To this end, we provide a bound on the portion that an agent can cede to other agents, which are expected to cede a part of its input, so that the state constraint set is reachable. This work is related to [7], which considers a constrained autonomous system subject to additive uncertainties bounded through a scaled disturbance set. In particular, the authors characterize the bounds of the scale of the disturbance set that renders impossible to find a robust positive invariant set, and therefore, to guarantee constraint satisfaction. Moreover, [8] proposes a set-theoretical analysis in the context of actuation attacks, setting a condition on the size of the attacker's input set so that the system cannot be robustly defended. Indeed, the framework of the latter is taken as a reference point for developing the conditions presented here. In contrast to these previous results, in this article we study a pair of input coupled subsystems working under the previously mentioned partially cooperative framework of [6], and provide limits of the joint shared portion of the input space for which the nonexistence of an invariant set is guaranteed. This way, agents can be provided with limits regarding the extent of their partial cooperation. This research is also connected with the results of [9], which introduces the concept of a distributed maximum robust positively invariant set, and can be of interest for works as [10], which propose a clustering method based on the volume of the robust positively invariant set that a coalition of agents can attain. Finally, we would like to remark that although our end goal is to characterize this set for a generic multi agent system, in this work we restrict our attention to the specific case of two agents, which will pave the way to the previously mentioned objective. Likewise, a two agent system allows a simpler and more effective representation to illustrate the results of our approach.

Notation: The set of natural numbers is $\mathbb{N}$. $A = \mathbf{0}$ is a matrix of zeros. $\lambda\Omega \triangleq \{\lambda x | x \in \Omega\}$ scales a set $\Omega \subset \mathbb{R}^n$ by a factor $\lambda \in \mathbb{R}$. $A\Omega$ represents the image of a set $\Omega \subset \mathbb{R}^n$ under the linear mapping $A : \mathbb{R}^n \to \mathbb{R}^p$, and $A^{-1}\Omega$ describes the preimage. For $\Omega, \Phi \subset \mathbb{R}^n$, the Minkowski set addition is $\Omega \oplus \Phi \triangleq \{x + y : x \in \Omega, y \in \Phi\}$, and $x \oplus \Omega = \{x\} \oplus \Omega$. The Minkowski set difference is $\Omega \ominus \Phi \triangleq \{x \in \mathbb{R}^n : \Phi + x \subset \Omega\}$. The support function of a non-empty set Ω in the direction of the vector η is $h_\Omega(\eta) \triangleq \sup\{\eta^\top x : x \in \Omega\}$. A C-set is a compact and convex set containing the origin, and a PC-set is a C-set containing the origin in its interior.

II. PROBLEM SETTING

We consider a discrete-time LTI system divided into a set $\mathcal{N} = \{1, 2\}$ of two input-coupled subsystems

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$$x_1(k+1) = A_{11}x_1(k) + B_{11}u_1(k) + B_{12}u_2(k),$$
$$x_2(k+1) = A_{22}x_2(k) + B_{22}u_2(k) + B_{21}u_1(k), \quad (1)$$

where $x_i(k) \in \mathbb{R}^{n_{x_i}}$ and $u_i(k) \in \mathbb{R}^{n_{u_i}}$ are the state and control input of subsystem $i \in \mathcal{N}$ at instant k, subject to the constraints $x_i(k) \in \mathbb{X}_i$ and $u_i(k) \in \mathbb{U}_i$, for all $k \geq 0$.

Assumption 1. The constraint sets $\mathbb{X}_i$ and $\mathbb{U}_i$ are PC-sets, and the input constraint set is assumed to be symmetrical about the origin, i.e., $\mathbb{U}_i = -\mathbb{U}_i$, for both subsystems $i \in \mathcal{N}$.

Remark 1. Equation (1) is satisfied naturally, e.g., by multiple industrial processes, traffic flow models, electrical networks, and irrigation canals. Likewise, the symmetry of Assumption 1 can be easily satisfied in situations where norm bounds are imposed performing a change of variable.

Let us consider the public and private variable decomposition proposed in [6], where a local variable $u_i(k)$ is decomposed as $u_i(k) = u_i^{\text{pr}}(k) + u_{ij}^{\text{pu}}(k)$. The so-called private part of u_i, i.e., u_i^{pr}, is exclusively controlled by the agent i that *owns* it. On the other hand, u_{ij}^{pu} is the portion of u_i that agent i cedes to its neighboring agent j, as it is affected by u_i, i.e, $\{j \in \mathcal{N} : B_{ji} \neq \mathbf{0},\ j \neq i\}$. Moreover, we define different scale factors to limit the constraint sets of the partitioned variables. That is, $u_i^{\text{pr}}(k) \in \alpha_i(k)\mathbb{U}_i$, with $\alpha_i(k) \in [0,1]$, and $u_{ij}^{\text{pu}}(k) \in \alpha_{ij}(k)\mathbb{U}_i$, with $\alpha_{ij}(k) \in [0,1]$. To fulfill $u_i \in \mathbb{U}_i$, the following inequality must be met

$$\alpha_i(k) + \alpha_{ij}(k) \leq 1. \quad (2)$$

Note that each agent i will locally control the private part of its input variable u_i^{pr}, and the public part of its neighboring input u_{ji}^{pu}, as $\{B_{ij} \neq \mathbf{0},\ j \neq i\}$ with $i, j \in \mathcal{N}$. Variables that cannot be controlled locally by agent i are treated as disturbances, that is, the public part u_{ij}^{pu} of u_i, and the neighbor's private part u_j^{pr}. Therefore, considering the previous variable decomposition, we can express (1) as

$$\begin{aligned}
&x_1(k+1) = A_{11}x_1(k) + B_{11}u_1^{\text{pr}}(k) + B_{12}u_{21}^{\text{pu}}(k) + w_1(k),\\
&\text{with } w_1(k) = B_{11}u_{12}^{\text{pu}}(k) + B_{12}u_2^{\text{pr}}(k),\\
&x_2(k+1) = A_{22}x_2(k) + B_{22}u_2^{\text{pr}}(k) + B_{21}u_{12}^{\text{pu}}(k) + w_2(k),\\
&\text{with } w_2(k) = B_{22}u_{21}^{\text{pu}}(k) + B_{21}u_1^{\text{pr}}(k).
\end{aligned} \quad (3)$$

Consequently, the input variables controlled by agent 1 are constrained within $u_1^{\text{pr}} \in \alpha_1\mathbb{U}_1$, $u_{21}^{\text{pu}} \in \alpha_{21}\mathbb{U}_2$, while those controlled by agent 2 are constrained within $u_2^{\text{pr}} \in \alpha_2\mathbb{U}_2$, $u_{12}^{\text{pu}} \in \alpha_{12}\mathbb{U}_1$. Moreover, the bounds of the uncertainties induced by the interaction between subsystems $w_i(k) \in \mathbb{W}_i$, for both subsystems $i \in \mathcal{N}$, are defined as

$$\begin{aligned}
\mathbb{W}_1 &\triangleq B_{11}\alpha_{12}\mathbb{U}_1 \oplus B_{12}\alpha_2\mathbb{U}_2,\\
\mathbb{W}_2 &\triangleq B_{22}\alpha_{21}\mathbb{U}_2 \oplus B_{21}\alpha_1\mathbb{U}_1,
\end{aligned} \quad (4)$$

with $\mathbb{W}_1$ and $\mathbb{W}_2$ being C-sets, as they are mapped sets derived from $\mathbb{U}_1$ and $\mathbb{U}_2$ (see Assumption 1).

The objective of this work is to characterize a bound on the maximal portion α_{12} that agent 1 can cede to 2 of its input variable and vice versa based on the nonexistence of the jointly invariant set, for agents will not be able to find control laws of partial cooperation able to guarantee the infinitely reachability of their constraints sets $\{\mathbb{X}_1, \mathbb{X}_2\}$ in a robust manner.

III. DEFINITIONS AND PREVIOUS RESULTS

In this section, we will study some theoretic concepts and previous results for the reachability of a given set. To this end, let us consider a general linear system

$$x(k+1) = Ax(k) + Bv(k) + Ed(k), \quad (5)$$

subject to the constraints $x(k) \in \mathbb{X}$, $v(k) \in \mathbb{V}$ and $d(k) \in \mathbb{D}$. Our goal is to maintain the system within its state constraint set $\mathbb{X}$, applying an admissible input $v(k) \in \mathbb{V}$, regardless of the realization of the uncertainty.

A. Set invariance theory

Hereafter, we study the *reachability* [11]–[13] of the set $\mathbb{X}$. In this regard, a set Ω is said to be *infinitely reachable* if there exist a control law $v(\cdot)$ and some initial state $x(0) \in \Omega$ such that, for every possible realization of the uncertainty $d(k) \in \mathbb{D}$, the state satisfies $x(k) \in \Omega$, with $v(k) \in \mathbb{V}$. Similarly, a set Ω is said to be *strongly reachable* if there exists a control law $v(\cdot)$ such that, for all initial states $x(0) \in \Omega$ and for every possible realization of the uncertainty $d(k) \in \mathbb{D}$, the state satisfies $x(k) \in \Omega$, with $v(k) \in \mathbb{V}$. The latter is equivalent to saying that Ω is a *robust control invariant* (RCI) set. Both definitions are strongly related, as a set is said to be infinitely reachable if it contains an RCI set.

The largest strongly reachable set of a given set Ω is denoted as the maximal robust control invariant set. The computation of this set stems from the *k-step robust constraint admissible* set $C_k(\Omega)$, which is the set of all initial states that can be kept within the set Ω for at least k steps, fulfilling the input constraints $v \in \mathbb{V}$, and for any realization of the disturbance $d \in \mathbb{D}$. It satisfies (i.) $C_{k+1}(\Omega) \subseteq C_k(\Omega)$, (ii.) $C_k(\Omega) = \cap_{n=0}^{k} C_n(\Omega)$, (iii.) $C_\infty(\Omega) := \lim_{k\to\infty} C_k(\Omega) = \cap_{n=0}^{\infty} C_n(\Omega)$, where $C_\infty(\Omega)$ corresponds to the maximal RCI set, which is finitely determined if, for a certain time step k, $C_{k+1}(\Omega) = C_k(\Omega) \neq \emptyset$, and therefore $C_\infty(\Omega) = C_k(\Omega)$. However, if $C_k(\Omega) = \emptyset$ for a given k, then $C_\infty(\Omega) = \emptyset$. In this regard, the set $C_k(\mathbb{X})$ can be computed as

$$C_{k+1}(\mathbb{X}) = Q\left(C_k(\mathbb{X})\right) \cap \mathbb{X}, \text{ with } C_0(\mathbb{X}) = \mathbb{X}, \quad (6)$$

where $Q(\cdot)$ is the robust one-step set [14]:

$$\begin{aligned}
Q(\Phi) &\triangleq \{x : \exists v \in \mathbb{V} \mid Ax + Bv \oplus E\mathbb{D} \in \Phi\}\\
&= (A)^{-1}\left((\Phi \ominus E\mathbb{D}) \oplus (-B\mathbb{V})\right).
\end{aligned} \quad (7)$$

The term $(A)^{-1}(\cdot)$ represents the preimage of the linear transformation $A(\cdot)$, and exists even if the matrix A is not invertible. Therefore, we can rewrite (6) as

$$C_{k+1}(\mathbb{X}) = (A)^{-1}\left((C_k(\mathbb{X}) \ominus E\mathbb{D}) \oplus (-B\mathbb{V})\right) \cap \mathbb{X}. \quad (8)$$

B. Closely-related known results

In this section, we recover some of the main results of [8], which have served as a starting point for the development of the results presented in this paper. The authors study the existence of the maximal RCI set by analyzing the sequence of sets $\{C_k(\mathbb{X})\}$, since when, for a given k, the set $C_k(\mathbb{X})$ is empty, it will be satisfied that $C_\infty(\mathbb{X}) = \emptyset$. More precisely, *Propositions 9, 12* and *13* of the aforementioned reference are considered:

- The set $C_k(\mathbb{X})$ is bounded as

$$C_k(\mathbb{X}) \subseteq \bigcap_{n=0}^{k} (A^n)^{-1} T_n(\mathbb{X}), \tag{9}$$

where

$$\begin{aligned} &T_{k+1}(\mathbb{X}) = (T_k(\mathbb{X}) \ominus A^k E\mathbb{D}) \oplus A^k(-B\mathbb{V}), \\ &\text{with } T_0(\mathbb{X}) = \mathbb{X}. \end{aligned} \tag{10}$$

- If $T_{k^*}(\mathbb{X}) = \emptyset$ for some $k^* > 0$, then $C_k(\mathbb{X}) = \emptyset$ for all $k \geq k^*$.
- If $S_{k^*}(\mathbb{X}) = \emptyset$ for some $k^* > 0$, then $T_k(\mathbb{X}) = \emptyset$ for all $k \geq k^*$, where

$$S_k(\mathbb{X}) \triangleq \mathbb{X} \oplus \left[\bigoplus_{n=0}^{k-2} A^n(-B\mathbb{V}) \right] \ominus \left[\bigoplus_{n=0}^{k-1} A^n E\mathbb{D} \right]. \tag{11}$$

In the framework of actuation attacks, where the attacker has a portion of the input available, i.e., $\mathbb{D} \triangleq \alpha\mathbb{V}$ for the system (5), the goal of the referenced paper was to determine, for any k^*, the smallest value of the scale factor α for which $C_{k^*}(\mathbb{X})$ is empty: $\alpha_{k^*} \triangleq \inf\{\alpha : C_{k^*}(\mathbb{X}) = \emptyset, \alpha \in [0, 1]\}$. This was achieved by determining a critical value of α, i.e., $\bar{\alpha}_{k^*}$, for which $S_k(\mathbb{X})$ is empty, which in turn guarantees $C_k(\mathbb{X})$ is empty.

Remark 2. The property $(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C)$ is used when deriving (9) using (8) and (6), which means that the Minkowski sum is not distributive under set intersections (see [8, Remark 10] for details). As a consequence, (9) does not hold with strict equality, and the theoretical bound will satisfy $\bar{\alpha}_{k^*} \geq \alpha_{k^*}$. Therefore, the admissible set $C_k(\mathbb{X})$ may be empty for a value of α below the value determined by the theoretical bound $\bar{\alpha}_{k^*}$, since the inclusion is not tight.

IV. LIMITS OF THE MAXIMUM SHARED PORTION OF THE INPUT SET

This section presents the main results of this work, where or extend the results of [8] to the distributed setting considered in this paper. In particular, we study two input-coupled subsystems and the limits of the portion of the input that both agents can cede to each other in a partially cooperative framework. That is, both agents must keep their states within their constraints $\{\mathbb{X}_1, \mathbb{X}_2\}$ for any possible realization of the input of the neighboring agent considering that they can use the entire portion of the input that has been ceded to them, satisfying the input constraints.

To this end, we will consider condition (2) with strict equality, that is, $\alpha_1 = 1 - \alpha_{12}$ and $\alpha_2 = 1 - \alpha_{21}$. For notational convenience, we define $\gamma \triangleq \alpha_{12}$ as the portion of u_1 that agent 1 cedes to 2, and $\beta \triangleq \alpha_{21}$ as the portion of u_2 that agent 2 cedes to 1. We can adapt the definitions of Section III to our particular setting (3) by identifying

$$\begin{aligned} B_1\mathbb{V}_1 &\triangleq (1-\gamma)B_{11}\mathbb{U}_1 \oplus \beta B_{12}\mathbb{U}_2, \\ E_1\mathbb{D}_1 &\triangleq \gamma B_{11}\mathbb{U}_1 \oplus (1-\beta)B_{12}\mathbb{U}_2, \\ B_2\mathbb{V}_2 &\triangleq (1-\beta)B_{22}\mathbb{U}_2 \oplus \gamma B_{21}\mathbb{U}_1, \\ E_2\mathbb{D}_2 &\triangleq \beta B_{22}\mathbb{U}_2 \oplus (1-\gamma)B_{12}\mathbb{U}_1. \end{aligned} \tag{12}$$

A. Bounds on γ and β

The definitions in Section III can be particularized for both subsystems (3), obtaining an expression of C_k, T_k, and S_k for each of them. In this regard, we will identify these particularized terms with the superindex ${}^{\gamma,\beta}$, as we want to obtain a bound on these scaling factors. In what follows, we proceed as in [8] to develop the conditions on $\{\gamma, \beta\}$ so that the constraint admissible sets $C_k^{\gamma,\beta}(\mathbb{X}_i)$ become empty; this ensures the non-existence of finitely determined maximal RCI sets $C_\infty^{\gamma,\beta}(\mathbb{X}_i)$ for both $i \in \mathcal{N}$. To this end, we will investigate when that the set $S_k^{\gamma,\beta}(\mathbb{X}_i)$ is empty for both $i \in \mathcal{N}$, where (11) can be adapted for each subsystem, with $\mathbb{X}_i$ the local state constraint set and $B_i\mathbb{V}_i$ and $E_i\mathbb{D}_i$ defined according to (12). Since we are in a partially cooperative framework in which constraint satisfaction is sought at the global system level, it is sufficient for one of the two sets $\{C_\infty^{\gamma,\beta}(\mathbb{X}_1), C_\infty^{\gamma,\beta}(\mathbb{X}_2)\}$ to be empty to no longer be able to guarantee constraint satisfaction. However, we develop the conditions in both scale factors, since they are symmetrical.

Remark 3. Note that each agent i shares a part of its control input u_i, but that its neighboring agent j has in turn ceded part of its input u_j to it. As a consequence, the value of the maximum portion that an agent can cede will depend on the value of what has been ceded to it by its neighbors.

Therefore, our main goal is to find, for every $k^* \in \mathbb{N}$, the smallest scale factors $\{\gamma_{k^*}, \beta_{k^*}\}$ such that

$$\begin{aligned} \gamma_{k^*} &\triangleq \inf\{\gamma : C_{k^*}^{\gamma,\beta}(\mathbb{X}_1) = \emptyset, \gamma \in [0,1], \beta \in [0,1]\}, \\ \beta_{k^*} &\triangleq \inf\{\beta : C_{k^*}^{\gamma,\beta}(\mathbb{X}_2) = \emptyset, \beta \in [0,1], \gamma \in [0,1]\}. \end{aligned} \tag{13}$$

Consequently, the portion an agent can cede to its neighbor $\{\gamma, \beta\}$ is upper bounded by the minimum value from which we can guarantee the non-existence of an invariant set, that is, if $\gamma > \gamma_{k^*}$ or $\beta > \beta_{k^*}$ for a certain value of k, $C_k^{\gamma,\beta}(\mathbb{X}_i) = \emptyset$. In this regard, let $\mathcal{F}_{k^*} \triangleq \{\gamma \in [0, \gamma_{k^*}), \beta \in [0, \beta_{k^*})\}$ be the set that contains all the scale factors $\{\gamma, \beta\}$ that have a value below the real bounds $\{\gamma_{k^*}, \beta_{k^*}\}$ for some $k^* \in \mathbb{N}$.

Assumption 2. The dominant eigenvalue of A_{ii} is real and positive for both $i \in \mathcal{N}$.

Remark 4. Assumption 2 is naturally fulfilled by some types of systems, such as positive systems (the Perron-Frobenius theorem guarantees that the largest eigenvalue is real), systems with triangular A matrices with real elements on their diagonal, and first-order systems.

In what follows, we refer to the dominant eigenvalue of the square matrix A_{ii} of subsystem $i \in \mathcal{N}$ as ρ_i. Moreover, the set $\mathcal{V}_i \triangleq \{v_i^1, \ldots, v_i^{n_v}, -v_i^1, \ldots, -v_i^{n_v}\}$ contains the n_v linearly independent eigenvectors related to ρ_i and their opposites. Also, let us define the following support function ratios for both subsystems $\{i,j\} \in \{1,2\}$ and $i \neq j$

$$H_{XU}^{ij} \triangleq \frac{h_{\mathbb{X}_i}(\bar{v}_i)}{h_{B_{ij}\mathbb{U}_j}(\bar{v}_i)},\ H_{BU}^{ij} \triangleq \frac{h_{B_{ii}\mathbb{U}_i}(\bar{v}_i)}{h_{B_{ij}\mathbb{U}_j}(\bar{v}_i)}. \tag{14}$$

where $\bar{v}_i$ is the eigenvector that minimizes

$$\bar{v}_i \triangleq \arg\min_{v_i \in \mathcal{V}} \frac{h_{\mathbb{X}_i}(v_i)}{h_{B_{ii}\mathbb{U}_i}(v_i)}. \tag{15}$$

Assumption 3. We can find at least a non-zero support of the mapped set $B_{ii}\mathbb{U}_i$ in one of the directions of $\mathcal{V}_i$. We also assume the support of $B_{ij}\mathbb{U}_j$ to be non-zero in the direction $\bar{v}_i$.

Considering the previous expressions, we can obtain, for any $k^* \in \mathbb{N}$, two theoretical bounds: $\bar{\gamma}_{k^*}$ so that $S_k^{\gamma,\beta}(\mathbb{X}_1) = \emptyset$, and $\bar{\beta}_{k^*}$ so that $S_k^{\gamma,\beta}(\mathbb{X}_2) = \emptyset$, if $k \geq k^*$.

Theorem 1. If, for some $k^* \in \mathbb{N}$, $\gamma > \bar{\gamma}_{k^*}$, with $\bar{\gamma}_{k^*} < 1$, then $C_k^{\gamma,\beta}(\mathbb{X}_1) = \emptyset$ for all $k \geq k^*$. The value of $\bar{\gamma}_{k^*}$ depends on the dominant eigenvalue as follows

- If $\rho_1 \neq 1$

$$\bar{\gamma}_{k^*} \triangleq \frac{1}{H_{BU}^{12}}\left[\frac{H_{XU}^{12}(1-\rho_1) + H_{BU}^{12}(1-\rho_1^{k^*-1}) + \rho_1^{k^*} - 1}{2 - \rho_1^{k^*-1} - \rho_1^{k^*}} + \beta\right], \tag{16}$$

- If $\rho_1 = 1$

$$\bar{\gamma}_{k^*} \triangleq \frac{1}{H_{BU}^{12}}\left[\frac{H_{XU}^{12} + H_{BU}^{12}(k^*-1) - k^*}{2k^*-1} + \beta\right]. \tag{17}$$

Proof 1. Let us consider (11) for the first subsystem, with $B_1\mathbb{V}_1$ and $E_1\mathbb{D}_1$ defined in (12)

$$\begin{aligned} S_k^{\gamma,\beta}(\mathbb{X}_1) = \mathbb{X}_1 \oplus \left[\bigoplus_{n=0}^{k-2} A_{11}^n(1-\gamma)B_{11}\mathbb{U}_1\right] \oplus \left[\bigoplus_{n=0}^{k-2} A_{11}^n \beta B_{12}\mathbb{U}_2\right] \\ \ominus \left[\bigoplus_{n=0}^{k-1} A_{11}^n \gamma B_{11}\mathbb{U}_1\right] \ominus \left[\bigoplus_{n=0}^{k-1} A_{11}^n (1-\beta) B_{12}\mathbb{U}_2\right]. \end{aligned}$$

Taking support functions in the direction of $\bar{v}_1$, and grouping the summations, we can express the previous as

$$\begin{aligned} h_{S_k^{\gamma,\beta}}(\bar{v}_1) \leq\ & h_{\mathbb{X}_1}(\bar{v}_1) + (1-2\gamma)\sum_{n=0}^{k-2} h_{A_{11}^n B_{11}\mathbb{U}_1}(\bar{v}_1) \\ & + (2\beta-1)\sum_{n=0}^{k-2} h_{A_{11}^n B_{12}\mathbb{U}_2}(\bar{v}_1) \\ & - \gamma h_{A_{11}^{k-1}B_{11}\mathbb{U}_1}(\bar{v}_1) - (1-\beta)h_{A_{11}^{k-1}B_{12}\mathbb{U}_2}(\bar{v}_1). \end{aligned}$$

By definition, $h_{M\Omega}(\eta) = h_\Omega(M^\top \eta)$ for a set Ω and matrix M. Since $\bar{v}_1$ is the eigenvector of A_{11} corresponding to the dominant eigenvalue ρ_1 it holds that $A_{11}^j \bar{v}_1 = \rho_1^j \bar{v}_1$. Also, as $\rho_1 > 0$, we can express $h_\Omega(\rho_1\eta) = \rho_1 h_\Omega(\eta)$.

$$\begin{aligned} h_{S_k^{\gamma,\beta}}(\bar{v}_1) \leq\ & h_{\mathbb{X}_1}(\bar{v}_1) + (1-2\gamma)h_{B_{11}\mathbb{U}_1}(\bar{v}_1)\sum_{n=0}^{k-2}\rho_1^n \\ & + (2\beta-1)h_{B_{12}\mathbb{U}_2}(\bar{v}_1)\sum_{n=0}^{k-2}\rho_1^n \\ & - \gamma h_{B_{11}\mathbb{U}_1}(\bar{v}_1)\rho_1^{k-1} - (1-\beta)h_{B_{12}\mathbb{U}_2}(\bar{v}_1)\rho_1^{k-1}. \end{aligned}$$

For $\rho_1 \neq 1$, the geometric series $\sum_{n=0}^{k-2}\rho_1^n = \frac{1-\rho_1^{k-1}}{1-\rho_1}$

$$\begin{aligned} h_{S_k^{\gamma,\beta}}(\bar{v}_1) \leq h_{\mathbb{X}_1}(\bar{v}_1) + \left[(1-2\gamma)\frac{1-\rho_1^{k-1}}{1-\rho_1} - \gamma\rho_1^{k-1}\right]h_{B_{11}\mathbb{U}_1}(\bar{v}_1) \\ + \left[(2\beta-1)\frac{1-\rho_1^{k-1}}{1-\rho_1} - (1-\beta)\rho_1^{k-1}\right]h_{B_{12}\mathbb{U}_2}(\bar{v}_1). \end{aligned}$$

If $h_{S_{k^*}^{\gamma,\beta}}(\bar{v}_1) < 0 \rightarrow S_{k^*}^{\gamma,\beta}(\mathbb{X}_1) = \emptyset$, therefore, solving for γ

$$\gamma > \frac{1}{H_{BU}^{12}}\left[\frac{H_{XU}^{12}(1-\rho_1) + H_{BU}^{12}(1-\rho_1^{k^*-1}) + \rho_1^{k^*} - 1}{2-\rho_1^{k^*-1} - \rho_1^{k^*}} + \beta\right].$$

On the other hand, for $\rho_1 = 1$: $\sum_{n=0}^{k-2}\rho_1^n = k-1$

$$\begin{aligned} h_{S_k^{\gamma,\beta}}(\bar{v}_1) \leq h_{\mathbb{X}_1}(\bar{v}_1) + [(1-2\gamma)(k-1) - \gamma]h_{B_{11}\mathbb{U}_1}(\bar{v}_1) \\ + [(2\beta-1)(k-1) - 1 + \beta]h_{B_{12}\mathbb{U}_2}(\bar{v}_1). \end{aligned}$$

Therefore, if $h_{S_{k^*}^{\gamma,\beta}}(\bar{v}_1) < 0 \rightarrow S_{k^*}^{\gamma,\beta}(\mathbb{X}_1) = \emptyset$,

$$\gamma > \frac{1}{H_{BU}^{12}}\left[\frac{H_{XU}^{12} + H_{BU}^{12}(k^*-1) - k^*}{2k^*-1} + \beta\right].$$

■

The results for $\bar{\beta}_{k^*}$ are derived analogously based on Theorem 1, using the corresponding eigenvalue ρ_2 and ratios H_{BU}^{21}, H_{XU}^{21}, and substituting $\bar{\gamma}_{k^*} \rightarrow \bar{\beta}_{k^*}$ and $\beta \rightarrow \gamma$. Thus, if, for some $k^* \in \mathbb{N}$, $\beta > \bar{\beta}_{k^*}$, with $\bar{\beta}_{k^*} < 1$, then $C_k^{\gamma,\beta}(\mathbb{X}_2) = \emptyset$ for all $k \geq k^*$. In this sense, we define $\bar{\mathcal{F}}_{k^*} \triangleq \{\gamma \in [0, \bar{\gamma}_{k^*}), \beta \in [0, \bar{\beta}_{k^*})\}$ as the set that contains all the scale factors $\{\gamma, \beta\}$ that have a value below the theoretical limits for some $k^* \in \mathbb{N}$.

As noted in [8], the theoretical bounds are sufficient conditions but not necessary to determine the emptiness of the maximal RCI set, since they are derived from (9), which is not met with strict equality. Thus, according to Remark 2, it is satisfied that $\bar{\gamma}_{k^*} \geq \gamma_{k^*}$ and $\bar{\beta}_{k^*} \geq \beta_{k^*}$, which means that the set $\bar{\mathcal{F}}_{k^*}$ contains some $\{\gamma, \beta\}$ where $C_k^{\gamma,\beta}(\mathbb{X}_i) = \emptyset$, for both $i \in \mathcal{N}$, while for every pair $\{\gamma, \beta\} \in \mathcal{F}_{k^*}$, we have non-empty sets $C_k^{\gamma,\beta}(\mathbb{X}_i)$ for both $i \in \mathcal{N}$. However, the actual bounds on both scale factors $\{\gamma_{k^*}, \beta_{k^*}\}$ need to be calculated by exhaustive search over different values of $\{\beta, \gamma\}$, so that the definition of the set $C_k^{\gamma,\beta}(\mathbb{X}_i)$ (8) for a given value of k is empty, while for the theoretical boundaries simply we have to evaluate an expression.

We can obtain the theoretical bounds $\{\bar{\gamma}_\infty, \bar{\beta}_\infty\}$ for which $C_\infty^{\gamma,\beta} = \emptyset$ by taking limits when $k^* \rightarrow \infty$ of (16) and (17):

$$\bar{\gamma}_\infty \triangleq \begin{cases} \frac{1}{H_{BU}^{12}}\left[\frac{H_{XU}^{12}(1-\rho_1) + H_{BU}^{12} - 1}{2} + \beta\right] & \text{if } \rho_1 < 1 \\ \frac{1}{H_{BU}^{12}}\left[\frac{H_{BU}^{12} - \rho_1}{1+\rho_1} + \beta\right] & \text{if } \rho_1 \geq 1 \end{cases} \tag{18}$$

accordingly, $\bar{\beta}_\infty$ can be deduced analogously.

V. NUMERICAL EXAMPLES

In this section, we illustrate the results of Section IV through different case studies, where the subsystems are expressed according to (3). Accordingly, we define the matrices that we will use as example systems

$$\mathbf{E}_1 : A_{11} = A_{22} = \begin{bmatrix} 0.55 & 0.5 \\ 0 & 0.8 \end{bmatrix}, \ \mathbf{E}_2 : A_{11} = A_{22} = \begin{bmatrix} 1.5 & 0.1 \\ 0.2 & 1.2 \end{bmatrix},$$

$$\mathbf{E}_3 : A_{11} = A_{22} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{E}_4 : A_{11} = \{A_{ii}\}_{\mathbf{E}_1}, \ A_{22} = \{A_{ii}\}_{\mathbf{E}_2}.$$

First, we will study the case in which both subsystems are equal ($\mathbf{E}_1$ to $\mathbf{E}_3$) and therefore have the same dominant eigenvalues. Furthermore, we will consider $\mathbf{E}_4$, where the matrix of one subsystem is stable, while the other is unstable. For all cases, we consider the following matrices

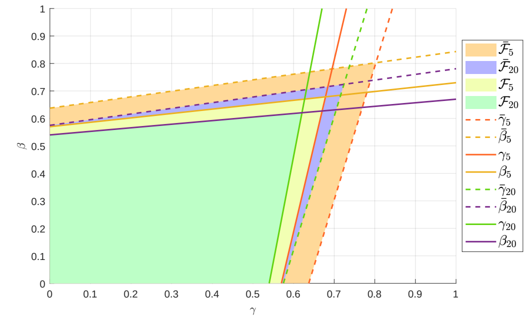
$$B_{11} = B_{22} = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}, \quad B_{12} = B_{21} = \begin{bmatrix} 0.15 \\ 0.05 \end{bmatrix}.$$

Figure 1 represents the real bounds $\{\gamma_{k^*}, \beta_{k^*}\}$ in solid lines, and the theoretical bounds $\{\bar{\gamma}_{k^*}, \bar{\beta}_{k^*}\}$ are plotted in dashed lines, for the step values $k^* = 5$ and $k^* = 20$. Accordingly, the real bounds $\{\gamma_{k^*}, \beta_{k^*}\}$ delimit the set $\mathcal{F}_{k^*}$, where we can guarantee finding a non-empty set $C_k^{\gamma,\beta}(\mathbb{X}_i)$ for both subsystems, represented in yellow for $k^* = 5$ and in green for $k^* = 20$. Moreover, the set $\bar{\mathcal{F}}_{k^*}$ is delimited by the theoretical bounds $\{\bar{\gamma}_{k^*}, \bar{\beta}_{k^*}\}$, and is represented in orange for $k^* = 5$ and in light-purple color for $k^* = 20$. In this regard, Figure 1 illustrates what is stated in Remark 2, as we can appreciate the differences on what the theoretical and real bound determine.

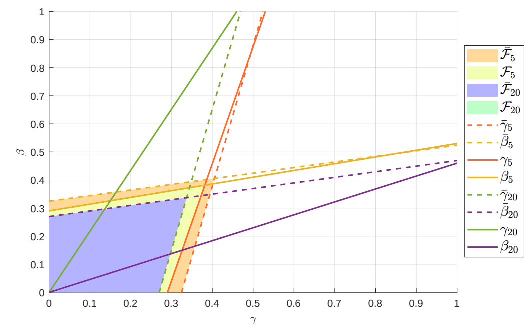
For stable subsystems, such as $\mathbf{E}_1$ (see Figure 1a), where the dominant eigenvalues of matrices A_{11} and A_{22} satisfy $\rho_1, \rho_2 < 1$, agents can cede a greater portion of its input to other agents guaranteeing that it will be possible to find a non-empty set $C_k^{\gamma,\beta}(\mathbb{X}_i)$. That is, the set $\mathcal{F}_{k^*}$ for the system $\mathbf{E}_1$ covers a wider range of $\{\gamma, \beta\}$ for which it is possible to guarantee a finite determined set $C_k^{\gamma,\beta}(\mathbb{X}_i)$, where the constraints will be satisfied for at least k^* steps compared to the set $\mathcal{F}_{k^*}$ for system $\mathbf{E}_2$ (see Figure 1b). In the latter, both subsystems are unstable as the dominant eigenvalues satisfy $\rho_1, \rho_2 > 1$. Therefore, agents are required to cede a smaller portion of their input to neighboring agents to guarantee constraint satisfaction. Also, it can be seen in Figure 1b that when we increase the number of steps to $k^* = 20$, we are not able to find a finitely determined $C_k^{\gamma,\beta}(\mathbb{X}_i)$ set, and therefore, we cannot guarantee constraint satisfaction.

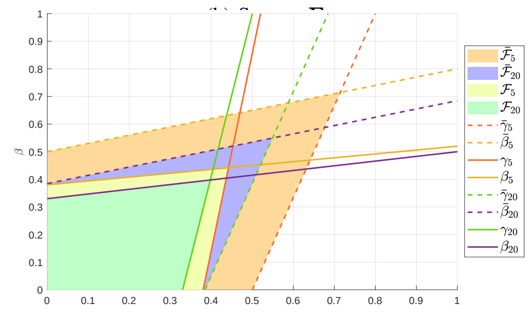
A combination of the above cases can be observed in Figure 1d, since $\mathbf{E}_4$ is composed of a stable subsystem 1 ($\rho_1 < 1$) and subsystem 2, which is unstable ($\rho_2 > 1$). For this case, we can observe that subsystem 1 is able to cede a greater portion of its input through the scale factor γ compared to subsystem 2. However, subsystem 2 will be able to cede a greater portion of its input β compared to the case where both systems are unstable (see Figure 1b), since it can receive a greater portion of its neighboring input through γ. Note, however, that the more a subsystem receives, the more it will be able to perform from its input applies to all cases.

In addition, Figure 1 introduces a normalized index of the degree of partial cooperation that can occur in a system in which this method is applied. For systems with an area of $\mathcal{F}_{k^*}$ close to one would indicate that cooperation can be maximum a priori. On the other hand, systems with an area

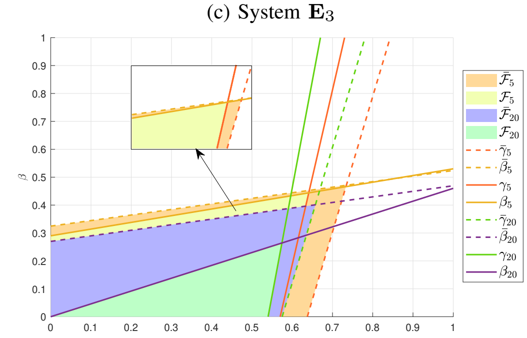


(a) System $\mathbf{E}_1$





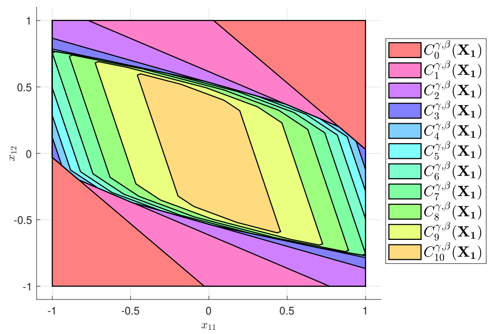
(c) System $\mathbf{E}_3$



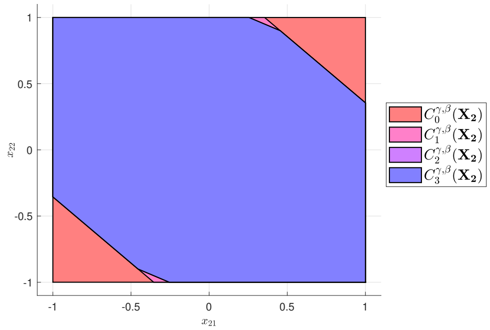
(d) System $\mathbf{E}_4$

Fig. 1. Comparison of the theoretical $\{\bar{\gamma}_{k^*}, \bar{\beta}_{k^*}\}$ and the real bounds $\{\gamma_{k^*}, \beta_{k^*}\}$ for systems $\mathbf{E}_1$ to $\mathbf{E}_4$ for a value of $k^* = 5$ and $k^* = 20$.

of $\mathcal{F}_{k^*}$ close to zero would indicate that they cannot afford to cede too much to their neighbors because otherwise they will not be able to find a jointly invariant set. Moreover, Figure 1 leads us to study the asymmetrical case where an agent cedes control authority to an agent that needs it, but does not receive anything in return. That is, setting γ or β equal to zero and looking at the maximum value that an agent could provide to its neighbor. For instance, when one subsystem is under-actuated or under-powered, while the other has a



(a) Subsystem 1 of $\mathbf{E}_3$



(b) Subsystem 2 of $\mathbf{E}_3$

Fig. 2. Representation of the k-steps constraint admissible sets, $C_k^{\gamma,\beta}(\mathbb{X}_i)$, for system $\mathbf{E}_3$ for $k < 20$.

surfeit of actuation capability. In this case, there is a trade-off between the improvement afforded to the receiving agent and the deterioration experienced by the providing agent.

Furthermore, Figure 2 illustrates the computation of $C_k^{\gamma,\beta}(\mathbb{X}_i)$ through (8). To this end, we have calculated these sets for the integrator of system $\mathbf{E}_3$, until $k = 20$ and using as scale factors $\gamma = 0.35$, $\beta = 0.1$. As can be seen in Figure 1c, these selected values satisfy $\{\gamma = 0.35,\ \beta = 0.1\} \in \mathcal{F}_5$, and $\{\gamma = 0.35,\ \beta = 0.1\} \notin \mathcal{F}_{20}$. In this sense, for subsystem 1 we can identify $C_{11}^{\gamma,\beta}(\mathbb{X}_1) = \emptyset$, which means that for these values of the scale factors, it is impossible to find a finitely determined set $C_\infty^{\gamma,\beta}(\mathbb{X}_1)$. Conversely, subsystem 2 converges in two steps to its maximal RCI set, since $C_3^{\gamma,\beta}(\mathbb{X}_2) = C_2^{\gamma,\beta}(\mathbb{X}_2)$, and therefore $C_\infty^{\gamma,\beta}(\mathbb{X}_2) = C_2^{\gamma,\beta}(\mathbb{X}_2)$. Let k^c denote the time step in which convergence to $C_\infty^{\gamma,\beta}(\mathbb{X}_i)$ is achieved, that is, $k^c = 2$ for subsystem 2. Consequently, in Figure 2b we have only represented the set $C_k^{\gamma,\beta}(\mathbb{X}_2)$ for the first 3 time steps, as for a given time step $k > k_c$ the set $C_k^{\gamma,\beta}(\mathbb{X}_i) = C_\infty^{\gamma,\beta}(\mathbb{X}_i)$. However, since we are within a partially cooperative approach where we want constraints to be satisfied at the global system level if for a given subsystem (subsystem 1 in this case) there is no finite maximal RCI set, we cannot guarantee constraint satisfaction.

VI. Conclusions

This article considers a decomposition method in which coupling variables are partitioned into multiple versions sharing a common constraint space, and are distributed among local agents. In this regard, we characterize a bound on the maximal size of the portion that an agent can cede of its input variable, considering that other agents will behave in an analogous manner so that we can compute a finitely determined jointly robust control invariant set for each subsystem. This bound has been characterized analytically and it is simple to evaluate. However, our results are based on several assumptions that might limit their applicability. Firstly, the system is required to be linear, although the extension to a nonlinear case might be achieved leveraging the robust nature of the approach to account for modeling errors. Also, the largest eigenvalue of the system is required to be real, but this condition is fulfilled naturally by some type of systems. Likewise, constraints must be expressed symmetrically with respect to the origin –which can be attained for most practical situations with a change of variable– and subsystem dynamics must be expressed using only input coupling.

Future work will address the relaxation of these requirements and the extension towards a multi-agent scenario, since the different interaction possibilities of such a scenario require careful consideration, making the analysis and derivation of the corresponding expressions more complex.

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