Size of Graphs with High Girth

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Abstract

Let $n \geq 4$ be a positive integer and let $ex\,(\nu; \{C_3, \dots, C_n\})$ denote the maximum number of edges in a $\{C_3, \dots, C_n\}$-free simple graph of order ν. This paper gives the exact value of this function for all ν up to $\lfloor (16n-15)/5 \rfloor$. This result allows us to deduce all the different values of the girths that such extremal graphs can have.

Let $k \geq 0$ be an integer. For each $n \geq 2\log_2(k+2)$ there exists ν such that every extremal graph G with $e(G) - \nu(G) = k$ has minimal degree at most 2, and is obtained by adding vertices of degree 1 and/or by subdividing a graph or a multigraph H with $\delta(H) \geq 3$ and $e(H) - \nu(H) = k$.

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1 Introduction

For undefined terminology and notation we refer the reader to [3].

Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of a graph G, respectively. The order of G is denoted by $\nu(G)$ and the size by $e(G)$. The minimum length of a cycle contained in G is the *girth* $g(G)$ of G. If G does not contain a cycle we set $g(G) = \infty$. By C_n we denote the cycle of length n, $n \geq 3$.

By $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ we mean the maximum number of edges in a simple graph of order ν and girth at least $n+1$, and by $Ex(\nu; \{C_3, C_4, \ldots, C_n\})$ the set of all simple graphs of order ν, with $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ edges and girth at least $n + 1$.

It is well known that $ex(\nu; \{C_3\}) = \lfloor \nu^2/4 \rfloor$, and the extremal graph is the complete bipartite graph $K_{\lfloor \nu/2 \rfloor, \lceil \nu/2 \rceil}$. Therefore, we suppose in this paper that $n \geq 4$. The exact value of $ex(\nu; \{C_3, C_4\})$ for all ν up to 24 and constructive lower bounds for all ν up to 200 are given in [4]. The values of $ex(\nu; \{C_3, C_4\})$ for $25 \leq \nu \leq 30$ are determined in [5].

In [6] it is proved that $ex(2n + 2; \{C_3, C_4, \ldots, C_n\}) = 2n + 4$ for $n \geq 12$ and the authors affirmed that they believe to have a theorem similar to the previous one but for $\nu = 3n + 3$ whose proof is long and hard. No other exact value of $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ is until now published.

Due to the difficulty of finding the exact values of the extremal function, the researchers have also tried to find out the general properties of these extremal graphs. In this context, the authors of the paper [6] wonder if the girth of any extremal $\{C_3, C_4, \ldots, C_n\}$-free graph is always $n + 1$. The answer is not always affirmative and this problem has been studied in [1,2,4,5,6]. And, in [6] it is also suggested the study of the largest girth, $g_{max}(n)$, for a graph from $Ex(\nu; \{C_3, C_4, \ldots, C_n\})$, for all ν.

Considering the results above, we pretend to determine both the exact values of the extremal function and all the possible girths of the extremal graphs. In order to reach it, we have considered the difference between size and order of the extremal graphs. It has allowed us to find out the exact value of $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ for all $\nu \leq \lfloor (16n - 15)/5 \rfloor$ and for each $n \geq 4$. Furthermore, we have also determined all the different values of the girths that extremal graphs in this interval can have.

The question asked in [6] about $g_{max}(n)$ is solved in the Theorem 2.6 below. Analogously, we suggest a similar and more difficult problem. Let $n \geq 4$, $\nu \geq n + 1$ and let $g_{max}(n, \nu)$ denote the largest girth for a graph from $Ex(\nu; \{C_3, C_4, \ldots, C_n\})$. What is the largest value $g_{max}(n, \nu)$? Theorem 2.5

gives the answer for all $\nu \leq \lfloor(16n-15)/5\rfloor$.

We also show in the Theorem 2.2 below that, under certain conditions, the extremal grahs are similar one to another. That is, a great deal of them are obtained simply by subdividing the edges of a certain graph until the forbidden cycles are avoided and the girth becomes so large as desired.

2 Results

Obviously, $ex\,(\nu+1;\{C_3,\ldots,C_n\}) \geq ex\,(\nu;\{C_3,\ldots,C_n\})+1$. Therefore, $ex(\nu;\{C_3,C_4,\ldots,C_n\})$ increases strictly but $ex(\nu;\{C_3,C_4,\ldots,C_n\})-\nu$ also increases but not strictly. It makes sense that our next purpose is to establish the existence, for some given positive integer k, of values of ν which satisfy $ex(\nu;\{C_3,C_4,\ldots,C_n\})-\nu=k$ and to determine all of them.

By $\mathcal{E}_k$ we denote the family of graphs, multigraphs and pseudographs G such that $e(G)-\nu(G)=k$ and $\delta(G)\geq 3$. Obviously $\mathcal{E}_k$ is finite for each k.

Definition 2.1 Let $n\geq 4$ and $k\geq 0$ be integers. We define
$\nu_k(n)=min\{\nu;\ ex(\nu;\{C_3,C_4,\ldots,C_n\})-\nu=k\}$

The number $\nu_k(n)$ does not always exist. For instance, in [4], it is proved that $ex(9;\{C_3,C_4\})=12$ and $ex(10;\{C_3,C_4\})=15$. Therefore it follows that $\nu_4(4)$ does not exist. But, we have proved:

Theorem 2.2 *Let $k\geq 0$ be an integer. For each $n\geq 2\log_2(k+2)$ there exists ν such that:*

(i) $ex(\nu;\{C_3,C_4,\ldots,C_n\})=\nu+k$.

(ii) Every graph of $Ex(\nu;\{C_3,C_4,\ldots,C_n\})$ has minimal degree at most 2.

(iii) Every graph of $Ex(\nu;\{C_3,C_4,\ldots,C_n\})$ is obtained by adding vertices of degree 1 and/or by subdividing a graph or a multigraph of $\mathcal{E}_k$

This result implies that part of the study of the extremal graphs can be changed to the determination of the graphs or multigraphs of $\mathcal{E}_k$, which allow to obtain the extremal graphs simply by subdividing its edges.

Theorem 2.3 *Let $n\geq 4$ be a integer. Then*

(i) $\nu_0(n)=n+1$

(ii) $\nu_1(n)=\lfloor 3n/2\rfloor+1$

(iii) $\nu_2(n)=2n$

$$(iv)\ \ \nu_3(n) = \begin{cases} \lceil 9n/4 \rceil & \textit{if } n \textit{ is even} \\ \lfloor 9n/4 \rfloor & \textit{if } n \textit{ is odd} \end{cases}$$

$$(v)\ \ \nu_4(n) = \begin{cases} \lceil (8n-2)/3 \rceil & \textit{if } n \neq 4 \textit{ is even} \\ \lfloor (8n-2)/3 \rfloor & \textit{if } n \textit{ is odd} \end{cases}$$

$$(vi)\ \ \nu_5(n) = \begin{cases} 3n-2 & \textit{if } n \neq 6 \\ 3n-1 & \textit{if } n = 6 \end{cases}$$

(vii) $\nu_6(4) = 12$, $\nu_6(5) = 14$, $\nu_6(6) = 19$, $\nu_6(7) = 21$,

$$\lceil (16n-14)/5 \rceil \leq \nu_6(n) \leq \begin{cases} \lceil (10n-5)/3 \rceil & \textit{if } n \geq 8 \textit{ is even} \\ \lfloor (10n-6)/3 \rfloor & \textit{if } n \geq 9 \textit{ is odd} \end{cases}$$

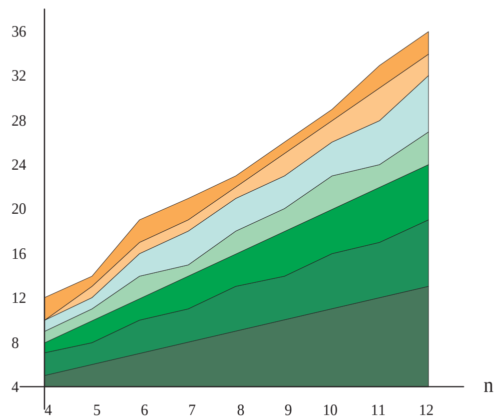

Fig. 1. *Exact values of* $ex(\nu; \{C_3, C_4, \ldots, C_n\})$.

The next result makes it possible to determine $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ for each $\nu \leq \lfloor (16n-15)/5 \rfloor$ and every $n \geq 4$ (see Figure 1.)

Theorem 2.4 *Let* $n \geq 4$ *and* $k \geq 0$ *be integers. If* $\nu_k(n) \leq \nu < \nu_{k+1}(n)$, *then* $ex(\nu; \{C_3, C_4, \ldots, C_n\}) = \nu + k$

As a consequence of the knowledge of these exact values, we can deduce exactly the possible different girths of all extremal graphs.

Theorem 2.5 *Let* $n \geq 4$ *and* $\nu \geq n+1$ *be integers and let* $g_{max}(n, \nu)$ *denote the largest possible girth of all graphs from* $Ex(\nu; \{C_3, C_4, \ldots, C_n\})$. *Then*

i) $g_{max}(n, \nu)$ *has the following values:*

ν	$g_{max}(n,\nu)$
$\nu_0(n) \leq \nu < \nu_1(n)$	ν
$\nu_1(n) \leq \nu < \nu_2(n)$	$\lfloor (2\nu+2)/3 \rfloor$
$\nu_2(n) \leq \nu < \nu_3(n)$	$\lfloor (\nu+2)/2 \rfloor$
$\nu_3(n) \leq \nu < \nu_4(n)$	$\begin{cases} \lfloor (4\nu+12)/9 \rfloor & if \;\; 2\nu \neq 8 \;\; mod\; 9 \\ (4\nu+2)/9 & if \;\; 2\nu = 8 \;\; mod\; 9 \end{cases}$
$\nu_4(n) \leq \nu < \nu_5(n)$	$\begin{cases} \lfloor (3\nu+12)/8 \rfloor & if \;\; 3\nu \neq 12, 13 \;\; mod\; 16 \\ \lfloor (3\nu+4)/8 \rfloor & if \;\; 3\nu = 12, 13 \;\; mod\; 16 \end{cases}$
$\nu_5(n) \leq \nu < \nu_6(n)$	$\lfloor (\nu+5)/3 \rfloor$

ii) For every $\gamma \in [n+1,\ g_{max}(n,\nu)]$ it is possible to construct a graph G of $Ex(\nu;\{C_3,\ldots,C_n\})$ such that $g(G)=\gamma$.

Next, we answer the question suggested in [6] about the largest possible girth of all extremal graphs.

Theorem 2.6 *Let $n \geq 4$ be an integer.*

$$g_{max}(n) = \max\left\{g(G);\ G \in Ex(\nu;\{C_3,C_4,\ldots,C_n\}),\ \nu \in \mathbb{Z}^+\right\} = \lfloor 3n/2 \rfloor$$

References

[1] Balbuena, C., M. Cera, A. Diánez and P. García-Vázquez, *On the extremal graphs with high girth*, preprint submitted to Discrete Mathematics.

[2] Balbuena, C. and P. García-Vázquez, *On the minimum order of extremal graphs to have a prescribed girth*, SIAM J. Discrete Math., **21**(2007),251–257.

[3] Chartrand, G. and L. Lesniak, " Graphs and Digraphs," Third ed., Chapman and Hall, London, UK, 1996.

[4] Garnick, D. K. , Y. H. H. Kwong and F. Lazebnik, *Extremal graphs without three-cycles or four-cycles*, J. Graph Theory, **17** (1993), 633–645.

[5] Garnick, D. K. and N. A. Nieuwejaar, *Non-isomorphic extremal graphs without three-cycles or four-cycles*, JCMCC **12** (1992), 33–56.

[6] Lazebnik, F. and P. Wang, *On the estructure of extremal graphs of high girth*, J. Graph Theory **26** (1997), 147–153.