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ANALYSIS AND OPTIMAL CONTROL FOR CHEMOTAXIS-CONSUMPTION MODELS

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Doctoral thesis submitted in partial fulfillment of the requirements for the degree of Doctor in Mathematics at the University of Seville.

Supervisor: Francisco Manuel Guillén-González

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To my wife, Jessica.

Resumen

En esta tesis investigamos el siguiente modelo de quimiotaxis-consumo en dominios acotados de $\mathbb{R}^N$ $(N = 1, 2, 3)$:

$$\partial_t u - \Delta u = -\nabla \cdot (u \nabla v), \quad \partial_t v - \Delta v = -u^s v,$$

donde $s \geq 1$, dotado de condiciones de contorno aisladas y condiciones iniciales para (u, v), con u y v representando la densidad de células y la concentración de la señal química, respectivamente. Bajo hipótesis poco exigentes sobre la regularidad del dominio y a través de la convergencia de las soluciones de un modelo truncado adecuado, se establecen dos resultados principales: existencia de soluciones débiles uniformes en el tiempo en dominios 3D, y unicidad y regularidad en dominios 2D (o 1D). Utilizando la teoría desarrollada en este análisis teórico, proponemos y estudiamos un esquema discreto en tiempo implícito tipo Backward Euler para dicho modelo combinado con el uso de una variable auxiliar, probando existencia de solución, estimaciones *a priori* uniformes en el tiempo y convergencia hacia una solución débil (u, v) del modelo quimiotaxis-consumo. A continuación abordamos problemas de control óptimo sujetos al siguiente modelo de quimiotaxis-consumo controlado de forma bilineal en un dominio acotado $\Omega \subset \mathbb{R}^3$ durante un intervalo de tiempo $(0, T)$:

$$\partial_t u - \Delta u = -\nabla \cdot (u \nabla v), \quad \partial_t v - \Delta v = -u^s v + f v 1_{\Omega_c},$$

siendo f el control que actúa en un subdominio $\Omega_c \subset \Omega$. En primer lugar, abordamos un problema de control óptimo relacionado con las soluciones débiles del modelo de quimiotaxis-consumo controlado. Demostramos la existencia de soluciones débiles que satisfacen una desigualdad de energía, la existencia de control óptimo sujeto a controles acotados y discutimos la relación entre el problema de control considerado y otros dos relacionados que pueden ser de interés. A continuación estudiamos un problema de control óptimo sujeto a soluciones fuertes del citado modelo de quimiotaxis-consumo controlado. Demostramos un criterio de regularidad que nos permite obtener existencia y unicidad de soluciones fuertes globales en el tiempo, mostramos la existencia de una solución óptima global y, utilizando un teorema de multiplicadores de Lagrange, establecemos condiciones de optimalidad de primer orden para cualquier solución óptima local, probando existencia, unicidad y regularidad de los multiplicadores de Lagrange asociados. Finalmente, en el capítulo de conclusiones, discutimos una serie de posibles trabajos futuros relacionados con los resultados presentados en esta tesis.

Abstract

In this thesis we investigate the following chemotaxis-consumption model in bounded domains of $\mathbb{R}^N$ $(N = 1, 2, 3)$:

$$\partial_t u - \Delta u = -\nabla \cdot (u \nabla v), \quad \partial_t v - \Delta v = -u^s v,$$

where $s \geq 1$, endowed with isolated boundary conditions and initial conditions for (u, v), with u and v representing the cell density and chemical signal concentration, respectively. Under mild regularity assumptions on the domain and through the convergence of solutions of an adequate truncated model, two main results are established: existence of uniform in time weak solutions in 3D domains, and uniqueness and regularity in 2D (or 1D) domains. Using the theory developed in this theoretical analysis, we propose and study a Backward Euler implicit time discrete scheme combined with the use of an auxiliary variable for the aforementioned model, proving existence of solution, uniform in time *a priori* estimates and convergence towards a weak solution (u, v) of the chemotaxis-consumption model. In the sequel we approach optimal control problems subject to the following bilinear controlled chemotaxis-consumption model in a bounded domain $\Omega \subset \mathbb{R}^3$ during a time interval $(0, T)$:

$$\partial_t u - \Delta u = -\nabla \cdot (u \nabla v), \quad \partial_t v - \Delta v = -u^s v + f v 1_{\Omega_c},$$

with f being the control acting in a subdomain $\Omega_c \subset \Omega$. First, we approach an optimal control problem related to weak solutions of the controlled chemotaxis-consumption model. We prove the existence of weak solutions satisfying an energy inequality, the existence of optimal control subject to bounded controls and discuss the relation between the considered control problem and two other related ones that might be of interest. Next we study an optimal control problem subject to strong solutions of the aforementioned controlled chemotaxis-consumption model. We prove a regularity criterion that allows us to get existence and uniqueness of global-in-time strong solutions, we show the existence of a global optimal solution and, using a Lagrange multipliers theorem, we establish first order optimality conditions for any local optimal solution, proving existence, uniqueness and regularity of the associated Lagrange multipliers. Finally, in the conclusions chapter, we discuss a series of possible future works related to the results presented in this thesis.

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CONTENTS

INTRODUCTION

In microbiology, chemotaxis is understood as the directed migration of cells in response to a concentration gradient of a certain chemical substance, either toward attractant chemicals or away from repelents [53]. Chemotaxis plays an essential role in many biological processes such as wound healing, the immune cells migration, the migration of bacteria, among others. It is also an important factor in some undesired events such as tumor growth, cancer metastasis and inflamatory diseases, for example [46, 59]. In unicellular organisms, chemotaxis is frequently related to the search for nutrients and there are studies on its applications to the degradation of polluting substances [47, 48].

The introduction of one of the first mathematical models for chemotaxis is attributed Keller and Segel in two works from 1970 and 1971 [38, 39] which are also regarded by some authors as a development of the work of Patlak [50]. Since then, the research on this topic gave rise to different related models such as models with chemoattraction or chemorepulsion, combined with production or consumption of the chemical substance, with the presence of a logistic growth of the population of cells, models for angiogenesis, haptotaxis and so on, covering a wide variety of applications of very practical interest. From the mathematical point of view, the aforementioned models possess interesting and challenging features that attracted the attention of many authors along the years and make these models still relevant nowadays [4, 33, 34].

Among the various models for chemotaxis that were mentioned, in this thesis, we focus on a model that describes a situation where, inside a bounded and connected region of the d-dimensional space, with $d = 1, 2, 3$, the cells are attracted by the concentration gradient of a chemical substance that, in its turn, diffuses and is consumed by the cells. Let Ω be a bounded domain of $\mathbf{R}^d$ and Γ be its boundary. Let $u = u(t, x)$ and $v = v(t, x)$ be the density of cell population and the concentration of chemical substance, respectively, on $x \in \Omega$ and $t > 0$. This model is governed by the initial-boundary PDE problem

$$\begin{cases} \partial_t u - \Delta u = -\nabla \cdot (u \nabla v), \quad \partial_t v - \Delta v = -u^s v, \\ \partial_{\mathbf{n}} u|_\Gamma = \partial_{\mathbf{n}} v|_\Gamma = 0, \quad u(0) = u^0, \quad v(0) = v^0, \end{cases} \tag{1}$$

where $\nabla \cdot (u \nabla v)$ is the chemotaxis term and $u^s v$ is the consumption term, with $s \geq 1$. $\partial_{\mathbf{n}} u$ denotes the normal derivative of u on the boundary. We assume that the initial conditions u^0 and v^0 are nonnegative functions.

In the present thesis, we aim for contributing for three distinct areas: (i) the analysis, (ii) the numerical approximation and (iii) the optimal control theory related to the chemotaxis model (1). The controlled model consists of the chemotaxis-consumption model (1) with a bilinear control $f : (0,T) \times \Omega \to \mathbb{R}$, being $T > 0$ a fixed and finite final time, acting on the chemical equation:

$$\begin{cases} \partial_t u - \Delta u = -\nabla \cdot (u \nabla v), \quad \partial_t v - \Delta v = -u^s v + f v 1_{\Omega_c}, \\ \partial_{\mathbf{n}} u|_\Gamma = \partial_{\mathbf{n}} v|_\Gamma = 0, \quad u(0) = u^0, \quad v(0) = v^0, \end{cases} \tag{2}$$

where $\Omega_c \subset \Omega$ is the control domain, 1_{Ω_c} is its characteristic function.

In what follows, we introduce the reader to part of the available literature on the analysis, numerical approximation and optimal control theory of chemotaxis models, focusing on the chemotaxis-consumption models (1). After that, we remark the contributions of the present thesis, pointing out the chapter in which they are developed.

Background

Next we recall some developments in the chemotaxis model (1) for $s = 1$, beginning with the existence theory. In [56], existence of global weak solutions which become smooth after a sufficiently large period of time is proved in smooth and convex 3D domains. More recently, a parabolic-elliptic simplification of (1), for $s = 1$, is studied in [57], yielding results on the existence and long-time behavior of global classical solutions in d-dimensional smooth domains.

Still considering $s = 1$, there are some studies on the coupling of (1) with models for fluids, a setting in which the model without fluids can be regarded as a particular case. In [43], the author proves local existence of weak solutions for the chemotaxis-Navier-Stokes equations in 3D smooth domains, while in [12] the existence of global classical solutions is attained near constant states. In [60], considering smooth and convex domains, existence and uniqueness of a global classical solution for the chemotaxis-Navier-Stokes equations is proved in 2D and existence of global weak solutions which become smooth after a large enough period of time is proved for the chemotaxis-Stokes equations in 3D. In [35] the results of [60] on the existence of solution are extended to non-convex domains, but we observe that some estimates that were time-independent become time-dependent. In [61] the author studies the assymptotic behavior of the chemotaxis-Navier-Stokes equations in 2D domains with the chemotaxis and consumption terms generalized by using adequate functions depending on the chemical substance, proving the convergence towards constant states in the L^∞-norm. Finally, in [62] existence of global weak solutions for the chemotaxis-Navier-Stokes equations is established in 3D smooth and convex domains and in [63] the assymptotic behavior of these solutions is studied.

An interesting and challenging feature of chemotaxis models, both from theoretical and numerical point of view, is that the L^∞-norm of the cell density u may blow up in finite time. Some studies focus on the proof of the existence of blowing-up solutions, such as in [4, Theorem 3.3], for example, while others are dedicated to the proof of existence of uniformly bounded solutions. When it comes to the model (1), with $s = 1$, this question has been addressed for 2D smooth and convex domains, because existence and uniqueness of classical and uniformly bounded solutions is proved in [56]. On the other hand, as far as we know, this question remains open for 3D domains.

Studying conditions that could lead to no-blow-up results for (1), with $s = 1$, some researchers advanced under the assumption of adequate constraints relating the chemotaxis coefficient with $\|v^0\|_{L^\infty(\Omega)}$. On this subject, we refer the interested reader to [3] and [55], for the problem (1) with $s = 1$. In addition, we also have [17] and [18], where the authors extend these results to other related chemotaxis models with consumption.

For a exhaustive review on the analytical results on the model (1) (for $s = 1$) and some variants we refer the reader to the recent survey [40], which includes great part of the studies cited above.

We remark that all the aforementioned works are carried out using classical in time solution tools and therefore considering smooth coefficients and smooth domains. In this case, the available existence theory is not well suited neither to the numerical approximation of (1) nor to the study of optimal control problems subject to the controlled problem (2). Indeed, with respect to the numerical approximation, one usually employs a weak formulation of the problem posed in more general domains. The controlled problem (2), in its turn, contains the control term fv where $f = f(t, x)$ is usually a L^q-function and hence can be seen as a nonsmooth coefficient.

Regarding the numerical simulation of chemotaxis models, although it is a relevant and growing research topic, when we turn to problem (1), we still find a relatively small amount of studies on its numerical approximation. To the best of our knowledge, we can cite two studies, [13] and [27], about the numerical approximation of (1), both just for the case $s = 1$.

In [13] a chemotaxis-Navier-Stokes system is approached via Finite Elements (FE). In fact, by assuming the existence of a sufficiently regular solution, if the initial data of the scheme are small perturbations of the initial data of this regular solution, then optimal error estimates are deduced. The drawback of this result is that the existence of such a regular solution is not clear in general, especially when we consider polyhedral domains, which are broadly used in numerical simulations.

In [27], motivated by the treatment given to the chemorepulsion model with linear production in [28], several FE schemes are designed to approximate (1), with $s = 1$. The authors focus on FE schemes satisfying properties such as conservation of cells, discrete energy law and approximate positivity rather than convergence. In particular, they present a scheme satisfying a discrete energy law that, in 1D domains, yields decreasing energy. Numerical simulations are carried out to compare the performance of the different schemes.

One of the main difficulties of addressing issues concerning the convergence of numerical schemes towards weak solutions of (1) is probably the lack of energy (*a priori*) estimates over the solutions of the schemes. As we will see in Chapter 3, even if we consider only time discretizations of (1), the task of designing a convergent scheme is not straightforward. This could be attributed to the complex technique needed in order to obtain energy estimates through the cancellation of the chemoattraction and consumption effects.

Once we talked about the existence theory and the numerical approximation, we finally bring the growing topic of the optimal control theory of chemotaxis models. Some of the existing works are dedicated to the optimal control problem posed in 2D domains, where one is usually able to prove existence and uniqueness of a strong solution to the controlled model, which allows researchers to prove the existence of global optimal solution and to derive an optimality system, establishing existence and regularity of Lagrange multipliers for any local optimum. For more details on this kind of work, we refer the interested reader to the works on control problems in 2D domains related to: a Keller-Segel model [51]; a chemorepulsion-production model [29, 31]; a Keller-Segel logistic model [5]; a chemotaxis model with indirect consumption [65]; and a chemotaxis-haptotaxis model [54].

When we turn to optimal control problems related to chemotaxis models in 3D domains this analysis is more complex. In great part, this is because in 3D we have results of existence of weak solutions, however, in many cases, there is not any result on the existence and uniqueness of global in time strong solutions. To overcome this difficulty some authors introduce a regularity criterion, which is a mild additional regularity hypothesis on a weak solution, sufficient to conclude that this weak solution is actually the unique strong solution. For a motivated introduction of this kind of adaptation we refer the reader to [8], where the author studied an optimal control problem related to the Navier-Stokes equations in 3D domains. For a chemotaxis related work, we refer the reader to [30], where a regularity criterion is established and is used to study an optimal control problem related to a chemorepulsion-production model in $3D$ domains. The drawback of using a regularity criterion is that it is not clear if the admissible set is nonempty. In [30], the authors show that if $\Omega_c = \Omega$, that is, if the control acts in the whole domain, and the initial chemical density v^0 is strictly positive and separated from zero, then the admissible set is nonempty. To do that, the idea is to define the control f *a posteriori*, depending on a regular pair (u, v).

Concerning the optimal control theory of model (2), to the best of our knowledge, the closest related work is [44], where the authors prove a regularity criterion, which we compare to our regularity criterion in the end of this chapter, and apply it to a control problem subject to a chemotaxis-Navier-Stokes model. Again, one can prove that the admissible set is nonempty only for controls acting in the whole domain Ω.

Main Contributions

Accounting for what has been exposed so far, we felt the need of extending the existing theory about (1) to the weak framework, which is more suitable to the design of stable and convergent (time) numerical schemes and to the study of optimal control problems subject to (2). Then, in Chapter 2, we study the existence and regularity of solutions of (1) in a weak solution setting under mild regularity assumptions on Γ. The results of Chapter 2 have been published in [11].

For this purpose, we introduce a regularization process by using truncated models, each one depending on a truncation parameter $m \in \mathbb{N}$. These models are easier to analyze both from the theoretical and numerical point of views and we prove that the solutions of the truncated models converge to weak solutions of (1) as $m \to \infty$. As a consequence of this study, Chapter 2 gives the following main contributions:

(i) generalization of the model with the consumption term $u^s v$ for $s \geq 1$ (in previous works, only the term uv has been considered);

(ii) enlargement of the class of considered domains, maintaining the no blow-up effect in the 2D case and some weak time-independent estimates in 3D domains;

(iii) the introduction and analysis of a regularized model (see problem (2.1) below, in Chapter 2) for which we prove existence, uniqueness, regularity, positivity, a priori estimates and convergence towards the original model (1).

We would like to make a comment regarding the rigor of the calculations. We have observed that, in some papers on analysis of chemotaxis models, singular functions are taken as test function (as for instance $log(u)$, without taking care that one only has $u \geq 0$). In our opinion, it should be considered only as formal computations. Then, in this thesis, we have done a great effort in order to guarantee that all of our computations be rigorous. Similarly to [37] for a cross-diffusion model, in order to make rigorous computations, we rely on a regularization procedure (for instance, taking $log(u + \epsilon)$ as test function).

In the sequel, based on the analysis that was carried out in Chapter 2, Chapter 3 is devoted to propose a time discrete scheme convergent to (1). This convergence is valid in 3D domains and is based on energy estimates. To the best of our knowledge, excepting the case of 1D domains [27], there is no time discrete scheme for (1) possessing an energy inequality from which one can obtain estimates for the discrete solutions, yielding convergence.

Moreover, the proposed scheme preserves the properties of positivity and conservation of the population of cells. There is evidence that, concerning the numerical approximation of chemotaxis models, the preservation of the positivity could possibly enhance the performance of the numerical schemes, avoiding spurious oscillations [25].

In view of the relative low number of studies on the optimal control theory of the present chemotaxis-consumption model, in Chapters 4 and 5 we study optimal control problems subject to the controlled model (2).

In Chapter 4, we approach an optimal control problem related to (1) for which we are able to prove the existence of global optimal solution in the weak setting, that is, without using any regularity criterion or hypothesis over the admissible set. To achieve it, we introduce the concept of weak solutions of the controlled model (2) satisfying an energy inequality. Next, we consider an optimal control problem for which we prove existence of global optimal solution and, to conclude, we discuss the relation between this optimal control problem and two other related ones that might be of interest.

In this framework, it is not clear how to deduce some type of optimality system associated to local optimal solutions. This is mainly because it is not possible to prove the well-posedness of the linearized problem around a local optimal solution using only the available weak regularity. This question is addressed in Chapter 5.

Indeed, in Chapter 5, the optimal control problem is studied in a strong solution setting. First we introduce the appropriate concept of strong solution of the controlled problem (2), given the control f, and then prove a regularity criterion that allows us to get existence and uniqueness of global-in-time strong solutions. In the sequel, we show the existence of a global optimal solution. Finally, using a Lagrange multipliers theorem, we establish first order optimality conditions for any local optimal solution, proving existence, uniqueness and regularity of the associated Lagrange multipliers.

In addition to the contributions indicated in the outline of Chapter 5 above, we remark two other relevant side contributions. The first one is a sharp regularity criterion based on a generic bootstrap argument. Indeed, comparing with the available literature, in [44], for the case $s = 1$, the authors prove their regularity criterion under the hypothesis that $u \in L^{10/7}((0,T) \times \Omega)$ and $f \in L^4((0,T) \times \Omega)$ using a bootstrap procedure that was designed to this particular choice as well as to their particular chemotaxis model. In this setting, it is not clear if one can reach a similar result under the hypothesis that $u \in L^p((0,T) \times \Omega)$ and $f \in L^q((0,T) \times \Omega)$ with $p < 10/7$ or $q < 4$. In other words, it is not possible to identify the minimum possible values of p and q.

On the other hand, our regularity criterion is sharp in the sense that we prove it under the hypothesis that $u^s, f \in L^q((0,T) \times \Omega)$ for $q > 5/2$ and, at least using the techniques employed in this work, it is clear that it is not possible to reach the same conclusion if $q \leq 5/2$. This is done by means of a more generic bootstrap argument which does not depend on the particular $q > 5/2$ and could possibly be more easily adapted to other models.

The second side contribution is the mathematical analysis of a generic linear coupled system given in Theorem C.1. This result is applied to the linearized problem around a local optimum, in Subsection 5.4.2, and also to the (linear) Lagrange multiplier problem, in Subsection 5.4.3, to prove additional regularity for the Lagrange multiplier. Moreover, once the linear coupled system is written in a generic form, it can be used in optimal control problems related to other models.

The rest of the thesis is organized as follows. In Chapter 1 we present some preliminary results that will be used along the thesis. Chapter 2 is devoted to the theoretical analysis of the chemotaxis-consumption models (1) varying the power $s \geq 1$, where we prove existence of global weak solutions in 3D, existence and uniqueness of a more regular global solution in 2D is proved by means of a regularization procedure.

In Chapter 3, we propose an energy stable and convergent time discrete scheme based on the analysis of Chapter 2. Chapter 4 is devoted to the study of an optimal control problem related to weak solutions of (2). Using an adequate concept of weak solutions satisfying an energy inequality, we define an optimal control problem for which we are able to prove existence of solution and compare it to two other related optimal control problems. In Chapter 5, we study an optimal control problem related to (2) in the strong setting. We prove a regularity criterion that allows us to get existence and uniqueness of global-in-time strong solutions and use it to show the existence of a global optimal solution. Then, applying a Lagrange multipliers theorem, we establish first order optimality conditions for any local optimal solution, proving existence, uniqueness and regularity of the associated Lagrange multipliers.

Finally, we present the conclusions and perspectives for future work.

Chapter 1

PRELIMINARY RESULTS

In this section we present some technical tools which will be used in the rest of the thesis.

For $p \in [1, \infty]$, we denote by $L^p(\Omega)$, the usual Banach spaces of p-integrable Lebesgue-mensurable functions, with the norm $\|\cdot\|_{L^p(\Omega)}$. We denote by p' the conjugate exponent of p. We recall that $L^2(\Omega)$ is a Hilbert space with the inner product

$$(f, g) = \int_\Omega f(x)g(x)\ dx.$$

We also denote by $W^{l,p}(\Omega)$, with $l \in \mathbb{N}$, the usual Sobolev space, equipped with the usual norm $\|\cdot\|_{W^{l,p}(\Omega)}$; for $p = 2$, we denote $W^{l,2}(\Omega)$ by $H^l(\Omega)$, with norm $\|\cdot\|_{H^l(\Omega)}$.

If $X(\Omega)$ is a Banach space, then $L^p(0, T; X(\Omega))$ is the Bochner space with the norm

$$\|v\|_{L^p(0,T;X(\Omega))} = \left(\int_0^T \|v(t)\|^p_{X(\Omega)}\ dt \right)^{1/p}, \quad \|v\|_{L^\infty(0,T;X(\Omega))} = \operatorname*{ess\,sup}_{t \in (0,T)} \|v(t)\|_{X(\Omega)},$$

$C([0, T]; X(\Omega))$ is the Bochner space of functions defined in $[0, T]$ and continuous with values in $X(\Omega)$ and $C_w([0, T]; X(\Omega))$ is the Bochner space of functions defined in $[0, T]$ and weakly continuous with values in $X(\Omega)$. To simplify the notation, from now on, we denote the spaces $L^p(0, T, X(\Omega))$ by $L^p(X)$, $C([0, T]; X(\Omega))$ by $C(X)$ and $C_w([0, T]; X(\Omega))$ by $C_w(X)$, suppressing both the time interval $(0, T)$ and the domain Ω.

If $p = 2$ and X is a Hilbert space then $L^2(X)$ is a Hilbert space with the inner product

$$(u, v)_{L^2(X)} = \int_0^T (u(t), v(t))_X\ dt, \quad \forall u, v \in L^2(X),$$

where $(\cdot, \cdot)_X$ denotes the inner product of X.

Next we present some interpolation inequalities and other results which will be of frequent use in the article. Unless otherwise stated, we consider $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ to be an open, bounded and locally Lipschitz domain, whose boundary we denote by Γ.

Lemma 1.1. ([6, 58]) *We have the following interpolation inequalities: 1. Let* $1 \leq$

$p < q \leq \infty$, $\theta \in (0,1)$ and $r \in (p,q)$, with $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. If $f \in L^p(\Omega) \cap L^q(\Omega)$ then $f \in L^r(\Omega)$ and

$$\|f\|_{L^r(\Omega)} \leq \|f\|^{\theta}_{L^p(\Omega)} \|f\|^{1-\theta}_{L^q(\Omega)}.$$

2. There exist (different) constants $\beta > 0$ such that

(i) *if $N = 2$ then*

$$\|v\|_{L^4(\Omega)} \leq \beta \|v\|^{1/2}_{L^2(\Omega)} \|v\|^{1/2}_{H^1(\Omega)}, \forall v \in H^1(\Omega); \tag{1.1}$$

(ii) *if $N = 3$ then*

$$\|v\|_{L^4(\Omega)} \leq \beta \|v\|^{1/4}_{L^2(\Omega)} \|v\|^{3/4}_{H^1(\Omega)}, \forall v \in H^1(\Omega).$$

$$\|v\|_{L^3(\Omega)} \leq \beta \|v\|^{1/2}_{L^2(\Omega)} \|v\|^{1/2}_{H^1(\Omega)}, \forall v \in H^1(\Omega).$$

$$\|v\|_{L^{10/3}(\Omega)} \leq C \|v\|^{2/5}_{L^2(\Omega)} \|v\|^{3/5}_{H^1(\Omega)}, \ \forall v \in H^1(\Omega), \tag{1.2}$$

$$\|v\|_{L^{10}(\Omega)} \leq C \|v\|^{4/5}_{H^1(\Omega)} \|v\|^{1/5}_{H^2(\Omega)}, \ \forall v \in H^2(\Omega). \tag{1.3}$$

Lemma 1.2. ([16]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then the interpolation inequality*

$$\|w\|_{W^{\alpha,r}(\Omega)} \leq C \|w\|^{\lambda}_{W^{\beta,\tilde{p}}(\Omega)} \|w\|^{1-\lambda}_{W^{\gamma,\tilde{q}}(\Omega)} \tag{1.4}$$

holds for $0 \leq \alpha, \beta, \gamma, \lambda \leq 1$ and $1 < \tilde{p}, \tilde{q}, r < \infty$ such that $\alpha = \lambda\beta + (1-\lambda)\gamma$ and

$$\frac{1}{r} = \frac{\lambda}{\tilde{p}} + \frac{(1-\lambda)}{\tilde{q}}.$$

Lemma 1.3 (**Poincare's Inequality, [14]**). *There is a constant $C > 0$ such that*

$$\|v - \frac{1}{|\Omega|}\int_{\Omega} v(x)\, dx\|_{W^{1,p}(\Omega)} \leq C \|\nabla v\|_{L^p(\Omega)}, \ \forall\, v \in W^{1,p}(\Omega).$$

Lemma 1.4. ([30]) *Let $\Omega \subset \mathbb{R}^N$ and $p_1, q, p_2, \tilde{p}, \tilde{q} \geq 1$ be such that*

$$\frac{1}{\tilde{q}} = \frac{(1-\theta)}{q} + \theta\Big(\frac{1}{p_1} - \frac{r}{N}\Big), \ \text{and} \ \frac{1}{\tilde{p}} = \frac{\theta}{p_2}, \ \text{with } \theta \in [0,1] \text{ and } r > 0,$$

then $L^{\infty}(L^q) \cap L^{p_2}(W^{r,p_1}) \hookrightarrow L^{\tilde{p}}(L^{\tilde{q}})$.

Lemma 1.5. ([58]) *Let X and Y be two Banach spaces such that $X \subset Y$ with a continuous injection. If $\phi \in L^{\infty}(0,T;X)$ and $\phi \in C([0,T];Y)$, then $\phi \in C_w([0,T];X)$.*

Due to the nature of our problem, it is convenient to have information on the Poisson-Neumann problem

$$\begin{cases} -\Delta z + z &= f \quad \text{in } \Omega, \\ \partial_{\mathbf{n}} z|_\Gamma &= 0. \end{cases} \tag{1.5}$$

Then we give the following definition.

Definition 1.6. Let $z \in H^1(\Omega)$ is a weak solution of (1.5) with $f \in L^p(\Omega)$. If this implies that $z \in W^{2,p}(\Omega)$ with

$$\|z\|_{W^{2,p}(\Omega)} \leq C\|-\Delta z + z\|_{L^p(\Omega)},$$

then we say that the Poisson-Neumann problem (1.5) has the $W^{2,p}$-regularity. In the hilbertian case $p = 2$ we say H^2-regularity. □

Remark 1.7. According to Grisvard [20], if $f \in L^p(\Omega)$, $p \in [1, \infty]$, and the boundary Γ is at least $C^{1,1}$, then the Neumann problem (1.5) has the $W^{2,p}$-regularity for all $p \in [1, \infty]$. The aforementioned result is also true if Ω is a polygon, that is, a polyhedron in $\mathbb{R}^2$, or if Ω is convex and $p = 2$. For more regular domains, if $f \in W^{l,p}(\Omega)$ and the boundary Γ is at least $C^{l+1,1}$, then the solution of the Neumann problem (1.5) belongs to $W^{l+2,p}(\Omega)$. □

Lemma 1.8. *Let Ω be a Lipschitz domain such that the Poisson-Neumann problem* (1.5) *has the $W^{2,p}$-regularity. There is a constant $C > 0$ such that*

$$\|\nabla z\|_{W^{1,p}(\Omega)} \leq C\|\Delta z\|_{L^p(\Omega)}, \ \forall z \in W^{2,p}(\Omega) \textit{ such that } \partial_{\mathbf{n}} z|_\Gamma = 0. \tag{1.6}$$

Proof. Suppose that the result is false, that is, for each $n \in \mathbb{N}$ there is $z_n \in W^{2,p}(\Omega)$ with $\partial_{\mathbf{n}} z_n|_\Gamma = 0$ such that

$$\|\nabla z_n\|_{W^{1,p}(\Omega)} > n\|\Delta z_n\|_{L^p(\Omega)}. \tag{1.7}$$

Without loss of generality, we can take z_n such that

$$\int_\Omega z_n \, dx = 0 \ \text{ and } \ \|\nabla z_n\|_{W^{1,p}(\Omega)} = 1 \tag{1.8}$$

Accounting for (1.7), (1.8) and Lemma 1.3 we have $(z_n), (\nabla z_n)$ bounded in $W^{1,p}(\Omega)$ and

$$\Delta z_n \longrightarrow \Delta z = 0 \text{ strongly in } L^p(\Omega). \tag{1.9}$$

Using the $W^{2,p}$-regularity of the Poisson-Neumann problem (1.5) we have

$$\|z_n\|_{W^{2,p}(\Omega)} \leq C(\|\Delta z_n\|_{L^p(\Omega)} + \|z_n\|_{L^p(\Omega)})$$

and thus (z_n) is bounded in $W^{2,p}(\Omega)$. This allows us to conclude, using compactness results in Sobolev spaces, that there is $z \in W^{2,p}(\Omega)$ such that, up to a subsequence,

$$z_n \longrightarrow z \text{ weakly in } W^{2,p}(\Omega) \text{ and strongly in } W^{1,p}(\Omega), \tag{1.10}$$

Using again the $W^{2,p}$-regularity of the Poisson-Neumann problem (1.5) we have

$$\|z_i - z_j\|_{W^{2,p}(\Omega)} \leq C\|\Delta z_i - \Delta z_j\|_{L^p(\Omega)} + C\|z_i - z_j\|_{L^p(\Omega)}, \ \forall i,j \in \mathbb{N},$$

and accounting for (1.10) and (1.9) we conclude that

$$z_n \longrightarrow z \text{ strongly in } W^{2,p}(\Omega). \tag{1.11}$$

Now, considering the properties of each z_n and the convergences (1.9) and (1.11) we have

$$\Delta z = 0, \text{ with } \partial_{\mathbf{n}} z|_\Gamma = 0, \int_\Omega z_n \, dx = 0 \text{ and } \|\nabla z\|_{W^{1,p}(\Omega)} = 1.$$

But this is not possible because if z satisfies $\Delta z = 0$, $\partial_{\mathbf{n}} z|_\Gamma = 0$ and $\int_\Omega z \, dx = 0$, then we have $z \equiv 0$ and hence $\|\nabla z\|_{W^{1,p}(\Omega)} = 0$. Therefore we must have (1.6). ■

Lemma 1.9. ([15]) *Let $z^n, z^{n-1} \in L^\infty(\Omega)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then*

$$\int_\Omega \delta_t z^n \, f'(z^n) = \delta_t \int_\Omega f(z^n) \, dx + \frac{1}{2k} \int_\Omega f''(c^n(x))(z^n(x) - z^{n-1}(x))^2 \, dx,$$

where $c^n(x)$ is an intermediate point between $z^n(x)$ and $z^{n-1}(x)$. In particular, if f is convex then we have

$$\int_\Omega \delta_t z^n \, f'(z^n) \, dx \geq \delta_t \int_\Omega f(z^n) \, dx.$$

Lemma 1.10. *Let w_1 and w_2 be nonnegative real numbers. For each $s \geq 1$ we have*

$$|w_2^s - w_1^s| \leq s|\max\{w_1, w_2\}|^{s-1}|w_2 - w_1|.$$

Proof. Indeed, we have

$$|w_2^s - w_1^s| = s\left|\int_{w_1}^{w_2} r^{s-1} \, dr\right| \leq s|\max\{w_1, w_2\}|^{s-1}|w_2 - w_1|.$$

■

Using Lemma 1.10, we arrive at the following.

Lemma 1.11. *Let $p \in (1, \infty)$ and let $\{w_n\}$ be a sequence of nonnegative functions in $L^p(0,T;L^p(\Omega))$ such that $w_n \to w$ in $L^p(0,T;L^p(\Omega))$ as $n \to \infty$. Then, for every $r \in (1,p)$, $w_n^r \to w^r$ in $L^{p/r}(0,T;L^{p/r}(\Omega))$ as $n \to \infty$.*

Let X and Y be Banach spaces, we say that X is continuously injected in Y and denote it by $X \hookrightarrow Y$ if $X \subset Y$ and, moreover, there is a constant $C > 0$ such that

$$\|\varphi\|_Y \leq C\|\varphi\|_X, \ \forall \varphi \in X.$$

Lemma 1.12. ([6]) *Let B be a Banach space, let $\{w_n\}$ be a sequence in B and $w \in B$. Either if $w_n \to w$ weakly* or weakly in B then $\{w_n\}$ is bounded in B and $\|w\|_B \leq \liminf \|w_n\|_B$.*

Denote $Q = (0,T) \times \Omega$ and let X_p be the Banach space

$$X_p = \{v \in C([0,T]; W^{p-2/p}) \cap L^p(0,T; W^{2,p}) \mid \partial_t v \in L^p(Q)\}$$

endowed with the norm

$$\|v\|_{X_p} = \|v\|_{L^\infty(0,T;W^{p-2/p})} + \|v\|_{L^p(0,T;W^{2,p})} + \|\partial_t v\|_{L^p(Q)}.$$

Lemma 1.13. ([16]) *Let Ω be a bounded domain of $\mathbb{R}^N$ such that Γ is of class C^2. Let $p \in (1,3)$, $w^0 \in W^{2-2/p,p}(\Omega)$ and $h \in L^p(Q)$. Then the problem*

$$\begin{cases} \partial_t w - \Delta w = h \ \text{in } Q, \\ \partial_{\boldsymbol{n}} w|_\Gamma = 0 \ \text{on } (0,T) \times \Gamma, \\ w(0,x) = w^0(x) \ \text{in } \Omega, \end{cases}$$

has a unique solution w such that

$$w \in X_p.$$

Moreover, there is a positive constant $C = C(p,T,\Omega)$ such that

$$\begin{aligned} \|w\|_{X_p} = \|w\|_{C([0,T];W^{2-2/p,p}(\Omega))} + \|w\|_{L^p(0,T;W^{2,p}(\Omega))} + \|\partial_t w\|_{L^p(Q)} \\ \leq C(\|h\|_{L^p(Q)} + \|w^0\|_{W^{2-2/p,p}(\Omega)}). \end{aligned} \tag{1.12}$$

Lemma 1.14. (Compactness in Bochner spaces, [52]) *Let X, B and Y be Banach spaces, let*

$$F = \Big\{ f \in L^1(0,T;Y) \ \Big| \ \partial_t f \in L^1(0,T;Y) \Big\}.$$

Suppose that $X \subset B \subset Y$, with compact embedding $X \subset B$ and continuous embedding $B \subset Y$. We have:

1. *if the set F is bounded in $L^q(0,T;B) \cap L^1(0,T;X)$, for $1 < q \leq \infty$, and $\Big\{ \partial_t f, \ \forall f \in F \Big\}$ is bounded in $L^1(0,T;Y)$, then F is relatively compact in $L^p(0,T;B)$, for $1 \leq p < q$;*

2. *if F is bounded in $L^\infty(0,T;X)$ and $\Big\{ \partial_t f, \ \forall f \in F \Big\}$ is bounded in $L^r(0,T;Y)$, for some $r > 1$. Then F is relatively compact in $C([0,T];B)$.*

To define the regularized problems in Chapter 2, we use the following truncation of the identity function, from above and from below:

$$a_m(u) = \begin{cases} -1, & \text{if } u \leq -2, \\ C^2 \text{ extension}, & \text{if } u \in (-2, 0), \\ u, & \text{if } u \in [0, m], \\ C^2 \text{ extension}, & \text{if } u \in (m, m+2), \\ m+1, & \text{if } u \geq m+2. \end{cases} \tag{1.13}$$

Note that a_m is globally Lipschitz. Along the study of problem (1), for $s \geq 2$, we will need the following result.

Lemma 1.15. *Let $p \in (1, \infty)$. Let $\{w_m\}$ be a sequence of nonnegative functions which is uniformly bounded in $L^\infty(0, \infty; L^p(\Omega))$ with respect to m and defined for every $t \in (0, \infty)$. If there is $\alpha > 0$ such that*

$$\int_\Omega w_m(t, x)\ dx \geq \alpha, \ \forall t \in (0, \infty), \ \forall m \in \mathbb{N},$$

then there exist $\beta > 0$ and $m_0 \in \mathbb{N}$ large enough such that

$$\int_\Omega a_m(w_m(t, x))\ dx \geq \beta, \ \forall\ t \in (0, \infty), \ \forall m \geq m_0.$$

Proof. For every $t \in (0, \infty)$, let

$$S_m(t) = \left\{ x \in \Omega \ \middle|\ w_m(t, x) > m \right\}.$$

Then, for all $t \in (0, \infty)$, we have

$$\int_{S_m(t)} m^p\ dx + \int_{\Omega \setminus S_m(t)} a_m(w_m(t, x))^p\ dx \leq \int_\Omega a_m(w_m(t, x))^p\ dx \leq C_1(p).$$

This implies

$$|S_m(t)| = \int_{S_m(t)} dx \leq C_1(p) \left(\frac{1}{m} \right)^p. \tag{1.14}$$

We have

$$\begin{aligned} \int_\Omega a_m(w_m(t, x))\ dx &= \int_{\Omega \setminus S_m(t)} w_m(t, x)\ dx + \int_{S_m(t)} a_m(w_m(t, x))\ dx \\ &= \int_\Omega w_m(t, x)\ dx - \int_{S_m(t)} \Big(w_m(t, x) - a_m(w_m(t, x)) \Big)\ dx \\ &\geq \alpha - \int_{S_m(t)} \Big(w_m(t, x) - a_m(w_m(t, x)) \Big)\ dx. \end{aligned}$$

To finish the proof, we show that

$$\lim_{m \to \infty} \int_{S_m(t)} \Big(w_m(t, x) - a_m(w_m(t, x)) \Big)\ dx = 0,$$

uniformly with respect to $t \in (0, \infty)$. In fact, using Hölder's inequality and (1.14), we obtain

$$\begin{aligned}
&\int_{S_m(t)} \Big(w_m(t,x) - a_m(w_m(t,x)) \Big) \, dx \\
&\leq \left(\int_{S_m(t)} |w_m(t,x) - a_m(w_m(t,x))|^p \, dx \right)^{\frac{1}{p}} \left(\int_{S_m(t)} dx \right)^{\frac{p-1}{p}} \\
&\leq C_2(p) \left(\int_{S_m(t)} dx \right)^{\frac{p-1}{p}} \leq C(p) \left(\frac{1}{m} \right)^{p-1}.
\end{aligned}$$

for all $t \in (0, \infty)$. Therefore we have

$$\int_\Omega a_m(w_m(t,x)) \, dx \geq \alpha - C(p) \left(\frac{1}{m} \right)^{p-1}$$

and then we can choose m_0 large enough such that

$$\int_\Omega a_m(w_m(t,x)) \, dx \geq \beta = \frac{\alpha}{2},$$

completing the proof. ■

Lemma 1.16 (Gronwall's inequality). *Let f, g and h be nonnegative functions such that $f \in W^{1,1}(0,T)$ and $g, h \in L^1(0,T)$, for some $T > 0$. Let*

$$G(t) = \int_0^t g(r) \, dr \qquad \text{and} \qquad H(t) = \int_0^t h(r) \, dr.$$

If f is such that

$$\frac{d}{dt} f(t) \leq g(t) + h(t) f(t), \ \text{a.e. } t \in (0,T),$$

then

$$f(t) \leq \Big(G(t) + f(0) \Big) e^{H(t)}, \ \text{a.e. } t \in (0,T).$$

Lemma 1.17. *Let X and Y be Banach spaces. Let $S : X \longrightarrow Y$ be a continuous linear map. If $f \in L^1((0,\infty); X)$ then $Sf \in L^1((0,\infty); Y)$ and*

$$\int_0^\infty Sf \, dt = S \int_0^\infty f \, dt.$$

Proof. See the section about the Bochner's integral in the book of Yosida [64]. ■

We will apply this lemma for $S : L^2(\Omega) \longrightarrow \mathbb{R}$ given by $Sf = \int_\Omega f \, dx$.

Lemma 1.18. *If $w, w_t \in L^1_{loc}((0,\infty); L^1(\Omega))$ then $\dfrac{d}{dt}\left(\int_\Omega w(\cdot,x)\ dx\right) \in L^1_{loc}(0,\infty)$ and*

$$\frac{d}{dt}\left(\int_\Omega w(\cdot,x)\ dx\right) = \int_\Omega w_t(\cdot,x)\ dx.$$

Proof. We look at the integral

$$\int_0^\infty \int_\Omega w_t(t,x)\ dx\ \psi(t)\ dt,$$

for every $\psi \in C_c^\infty((0,\infty))$. Since the integral over Ω is a linear map and $\psi(t)$ is a real number, for each $t \in (0,\infty)$, we have

$$\int_\Omega w_t(t,x)\ dx\ \psi(t) = \int_\Omega w_t(t,x)\psi(t)\ dx,$$

for *a.e.* fixed t. Then we use lemma 1.17 with $X = L^1(\Omega)$, $Y = \mathbb{R}$ and $f = w_t\psi \in L^1((0,\infty); L^1(\Omega))$ to write

$$\int_0^\infty \int_\Omega w_t(t,x)\ dx\ \psi(t)\ dt = \int_0^\infty \int_\Omega w_t(t,x)\psi(t)\ dx\ dt = \int_\Omega \int_0^\infty w_t(t,x)\psi(t)\ dt\ dx$$
$$= -\int_\Omega \int_0^\infty w(t,x)\psi_t(t)\ dt\ dx = -\int_0^\infty \int_\Omega w(t,x)\ dx\ \psi_t(t)\ dt,$$

for all $\psi \in C_c^\infty((0,\infty))$. ■

Let $C_c^\infty(0,\infty; L^2(\Omega))$ denote the space of the infinitely differentiable functions defined in $[0,\infty)$ with range in $L^2(\Omega)$ and with compact support in $(0,\infty)$. Consider the space

$$H^1(0,\infty; L^2(\Omega)) = \left\{ w \in L^2(0,\infty; L^2(\Omega)) \ \Big|\ w_t \in L^2(0,\infty; L^2(\Omega)) \right\},$$

which is a Hilbert space with the norm

$$\|w\|_{H^1(0,\infty;L^2(\Omega))} = \left(\|w\|^2_{L^2(0,\infty;L^2(\Omega))} + \|w_t\|^2_{L^2(0,\infty;L^2(\Omega))} \right)^{1/2}.$$

Lemma 1.19. ([41]) *$C_c^\infty(0,\infty; L^2(\Omega))$ is dense in $H^1(0,\infty; L^2(\Omega))$.*

We end this section recalling the concept of positive and negative parts of a function. For $w \in L^p(\Omega)$, $1 \le p \le \infty$, the positive and negative parts of w are given by

$$w_+(x) = \max\{0, w(x)\} \text{ and } w_-(x) = \min\{0, w(x)\},$$

respectively. Then $w = w_+ + w_-$ and $|w| = w_+ - w_-$; besides, if $w \in H^1(\Omega)$ then $w_+, w_- \in H^1(\Omega)$ with

$$\nabla w_+(x) = \begin{cases} \nabla w, & \text{if } w(x) > 0, \\ 0, & \text{if } w(x) < 0, \end{cases} \quad \text{and} \quad \nabla w_-(x) = \begin{cases} \nabla w, & \text{if } w(x) < 0, \\ 0, & \text{if } w(x) > 0. \end{cases}$$

For more details on truncations applied to $H^1(\Omega)$ functions we suggest Gilbarg and Trudinger [19].

Definition 1.20. (Weak solution of (2)**)** Let $s \geq 1$, $q > 5/2$. Let $f \in L^q(Q)$ and let $u^0, v^0 \in W^{2-2/q,q}(\Omega)$ be non-negative functions. A pair (u, v) is called a weak solution of (2) if $u(t,x), v(t,x) \geq 0$ *a.e.* $(t,x) \in Q$, satisfying, for $s \in [1,2)$,

$$u \in L^{5s/(3+s)}(W^{1,5s/(3+s)}), \partial_t u \in L^{5s/(3+s)}((W^{1,5s/(4s-3)})'),$$

for $s \geq 2$,

$$u \in L^2(H^1),\ \partial_t u \in L^2((H^1)'),$$

for $s \geq 1$,

$$u \in L^\infty(L^s) \cap L^{5s/3}(Q),$$
$$v \in L^\infty(Q) \cap L^\infty(H^1) \cap L^4(W^{1,4}) \cap L^2(H^2),$$
$$\partial_t v \in L^{5/3}(Q),$$

and satisfying the initial conditions for (u, v), the u-equation of (2) and the boundary condition of u in the variational sense

$$(\partial_t u, \varphi) + (\nabla u, \nabla \varphi) = (u \nabla v, \nabla \varphi)\,, \tag{1.15}$$

for all $\varphi \in L^{5s/(4s-3)}(W^{1,5s/(4s-3)})$, the v-equation pointwisely (in fact, the v-equation is satisfied in $L^{5/3}(Q)$) and, since $\Delta v \in L^2(Q)$, the boundary condition of v in the sense of $H^{-1/2}(\Gamma)$. □

Definition 1.21. (Strong solution of (2)**)** Let $s \geq 1$, $q > 5/2$. Let $f \in L^q(Q)$ and let $u^0, v^0 \in W^{2-2/q,q}(\Omega)$ be non-negative functions. A pair (u, v) is called a strong solution of (2) if $u(t,x), v(t,x) \geq 0$ *a.e.* $(t,x) \in Q$, with regularity

$$(u, v) \in X_q \times X_q$$

and satisfying the initial and boundary conditions of (2), the u-equation and the v-equation of (2) pointwisely. □

Remark 1.22. Since $q > 5/2$, if (u, v) is a strong solution of (2) then, in particular, $u, v \in L^\infty(Q)$. Then, through a comparison argument we can prove that, for each fixed $f \in L^q(Q)$, the strong solution of (2) is unique. We refer the reader to the proof of uniqueness in 2D domains from Chapter 2 that, in view of the regularity of the strong solution, can be adapted to 3D domains. □

Chapter 2

UNIFORM IN TIME SOLUTIONS FOR CHEMOTAXIS WITH POTENTIAL CONSUMPTION MODELS

2.1 Main Results

To present the main results of this chapter, we introduce the following regularized problems, which depend on a truncation parameter $m \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \partial_t u_m - \Delta u_m = -\nabla \cdot (a_m(u_m)\nabla v_m), \quad \partial_t v_m - \Delta v_m = -a_m(u_m)^s v_m, \\ \partial_{\mathbf{n}} u_m|_\Gamma = \partial_{\mathbf{n}} v_m|_\Gamma = 0, \quad u_m(0) = u_m^0, \quad v_m(0) = v_m^0, \end{array} \right. \tag{2.1}$$

where $u_m^0 \geq 0$ and $v_m^0 \geq 0$ are suitable regular approximations of u^0 and v^0, respectively, and $a_m(\cdot)$ is the truncation of the identity function (bounded from above and from below) defined in (1.13).

With the objective of enlarging the class of considered domains, we state and demonstrate our results in terms of the regularity of the Poisson-Neumann (1.5) (see definition 1.6 in page 18), and, when necessary, in terms of the following technical hypothesis:

Hypothesis (H1). For each $z \in H^2(\Omega)$ such that $\partial_\eta z|_\Gamma = 0$ there is a sequence $\{\rho_n\} \subset C^2(\overline{\Omega})$ such that $\partial_\eta \rho_n|_\Gamma = 0$ and $\rho_n \to z$ in $H^2(\Omega)$.

Remark 2.1. In order to show that the Hypothesis (H1) is not too restrictive, we prove in Lemma A.1, in the Appendix A, that Hypothesis (H1) is satisfied if the Poisson-Neumann problem has the $W^{3,p}$-regularity (see definition 1.6 in page 18), for $p > N$. This is true, in particular, if Γ is $C^{2,1}$ (see [20]). □

Let us consider the average of u^0

$$u^* = \frac{1}{|\Omega|}\int_\Omega u^0(x)\ dx.$$

Now we highlight our main results in this work:

Theorem 2.2. ***(3D. Existence of global weak solutions)*** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that the Neumann problem (1.5) has the H^2-regularity (see definition 1.6 in page 18) and Hypothesis (H1) is satisfied. Let $u^0 \in L^{1+\varepsilon}(\Omega)$, for some $\varepsilon > 0$, if $s = 1$, and $u^0 \in L^s(\Omega)$, if $s > 1$, and $v^0 \in H^1(\Omega) \cap L^\infty(\Omega)$ be non-negative functions. Then there is a non-negative weak solution (u, v) of the original problem (1), for $s \geq 1$, obtained through a limit of non-negative solutions (u_m, v_m) of the regularized problems (2.1) as $m \to \infty$ and such that*

$$\begin{cases} \displaystyle\int_\Omega u(t,x)\ dx = \int_\Omega u^0(x)\ dx, \ \ a.e.\ t \in (0,\infty) \\ 0 \leq v(t,x) \leq \|v^0\|_{L^\infty(\Omega)}, \ \ a.e.\ (t,x) \in (0,\infty)\times\Omega, \end{cases} \tag{2.2}$$

$$\begin{cases} u \in L^\infty(0,\infty; L^s(\Omega)) \cap L^{5s/3}_{loc}([0,\infty); L^{5s/3}(\Omega)), \ \ if\ s \geq 1, \\ u^{s/2}\nabla v \in L^2(0,\infty; L^2(\Omega)), \ \ if\ s \geq 1, \end{cases}$$

$$\begin{cases} \nabla u \in L^2(0,\infty; L^s(\Omega)) \cap L^{\frac{5s}{3+s}}_{loc}([0,\infty); L^{\frac{5s}{3+s}}(\Omega)), \quad if\ s \in [1,2), \\ \nabla u \in L^2(0,\infty; L^2(\Omega)), \quad if\ s \geq 2, \end{cases}$$

$$\begin{cases} u\nabla v \in L^2(0,\infty; L^s(\Omega)), \quad if\ s \in [1,2), \\ u\nabla v \in L^2(0,\infty; L^2(\Omega)), \quad if\ s \geq 2 \end{cases}$$

and

$$v \in L^\infty(0,\infty; H^1(\Omega)) \cap L^2(0,\infty; H^2(\Omega)), \quad \nabla v \in L^4(0,\infty; L^4(\Omega)).$$

Remark 2.3. We remark that, from the regularities of u and v that are listed in Theorem 2.2, we can conclude that

$$\begin{cases} u_t \in L^2\Big(0,\infty; \big(W^{1,s/(s-1)}(\Omega)\big)'\Big), \quad \text{if } s \in [1,2), \\ u_t \in L^2\Big(0,\infty; \big(H^1(\Omega)\big)'\Big), \quad \text{if } s \geq 2 \end{cases}$$

and

$$v_t \in L^2(0,\infty; L^{3/2}(\Omega)).$$

Attending to the regularity of (u, v) given so far, one has that (u, v) satisfies the u-equation of (1) in a variational sense, while the v-equation is satisfied $a.e.$ in $(0,\infty)\times\Omega$. Moreover, the initial conditions have a sense because, thanks to the regularity of u, v, u_t and v_t, one has that (u, v) is weakly continuous from $[0,\infty)$ to $L^s(\Omega) \times H^1(\Omega)$, if $s \in [1,2]$, and $L^2(\Omega) \times H^1(\Omega)$, if $s \geq 2$ (see Lemma 1.5). ■

Remark 2.4. Note that, for $s \in [1,2]$, the regularity of the fluxes of the u-equation of (1), namely, self diffusion ∇u and chemotaxis $u\nabla v$, increase as s increases. When we consider $s > 2$, the regularity of ∇u and $u\nabla v$ do not increase as s increases anymore. On the other hand, the regularity of the function v is independent of s. ■

Theorem 2.5. ***(2D. Existence and uniqueness of global strong solution)*** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain such that the Neumann problem* (1.5) *has the $W^{2,3}$-regularity (see definition 1.6 in page 18) and Hypothesis (H1) is satisfied. Let $u^0 \in H^2(\Omega)$ and $v^0 \in H^2(\Omega)$ be such that $u^0 \geq 0$ and $v^0 \geq 0$. Then there is a unique non-negative solution (u, v) for the original problem* (1)*, for $s \geq 1$, satisfying* (2.2) *and*

$$u, v \in L^\infty(0, \infty; H^2(\Omega)), \quad (u - u^*), v \in L^2(0, \infty; W^{2,3}(\Omega))$$
$$\Delta u, \Delta v, u_t, v_t \in L^2(0, \infty; H^1(\Omega)).$$

In particular, u does not blow-up neither at finite nor infinite time, that is, $u \in L^\infty(0, \infty; L^\infty(\Omega))$ (recall that in Theorem 2.2 we already have $v \in L^\infty(0, \infty; L^\infty(\Omega))$). Consequently, there is $m_0 \in \mathbb{N}$ such that, for all $m \in [m_0, \infty)$, the solution of (2.1) *is also the solution of* (1)*, that is,*

$$(u_m, v_m) = (u, v) \ a.e. \ in \ (0, \infty) \times \Omega.$$

In this case, both equations of (1) are satisfied *a.e.* in $(t, x) \in (0, \infty) \times \Omega$.

We observe that in 3D domains it is not possible to state a complete result such as Theorem 2.5. This is due to the gap between the regularity of the solutions provided by the existence result and the regularity needed to prove uniqueness. Notice that v does not blow-up neither at finite nor infinite time. On the other hand, to the best of our knowledge, whether u may blow-up or not is an open problem.

The rest of the chapter is organized as follows. Section 2.2 is devoted to discuss the regularity's properties of the solutions of regularized problem (2.1). In Section 2.3 we deal with the proof of Theorem 2.2 and, finally, Section 2.4 is dedicated to the proof of Theorem 2.5.

2.2 The Regularized Problem

In this section we define and analyze the regularized problem (2.1), based on the truncation of the identity $a_m(u)$ given in (1.13). We remark the following properties

$$a_m(u) \leq u, \quad \forall u \geq 0,$$

$$|a_m(u)| \leq m, \quad |a'_m(u)|, |a''_m(u)| \leq C, \quad \forall u \in \mathbb{R}, \tag{2.3}$$

where $C > 0$ is a constant independent of $m \in \mathbb{N}$.

For each $m \in \mathbb{N}$, let (u_m, v_m) be the solution of (2.1) with initial data $u^0_m, v^0_m \in C^\infty(\overline{\Omega})$ with u^0_m and v^0_m being mollifier regularizations of proper extensions of u^0 and v^0 to $\mathbb{R}^N$. In fact, $u^0 \in L^p(\Omega)$ is extended by zero in $L^p(\mathbb{R}^N)$, while $v^0 \in$

$H^1(\Omega) \cap L^\infty(\Omega)$ is extended in the space $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. In particular, these regularizations have the following properties:

$$u_m^0 \geq 0, \quad \int_\Omega u_m^0 = \int_\Omega u^0, \quad u_m^0 \to u^0 \text{ strongly in } L^p(\Omega), \text{ as } m \to \infty, \tag{2.4}$$

for $p = 1 + \varepsilon$, for some $\varepsilon > 0$, if $s = 1$, and $p = s$, if $s > 1$, and

$$\operatorname{ess\,inf} v^0 \leq v_m^0 \leq \operatorname{ess\,sup} v^0, \quad v_m^0 \to v^0 \text{ strongly in } H^1(\Omega), \text{ as } m \to \infty. \tag{2.5}$$

2.2.1 Existence and uniqueness of problem (2.1)

We use the Galerkin's method based on the set of eigenfunctions $\{\varphi_j\}$ of the operator $(-\Delta + I)$ with Neumann homogeneous boundary condition. Unless otherwise stated, we will proceed under the assumption that Poisson-Neumann problem (1.5) has the H^2-regularity. Then $\{\varphi_j\}$ is a basis of $H^2(\Omega)$.

Let X^n be the finite $n-$dimensional space generated by the first n elements of the set $\{\varphi_j\}$. Then, we look for Galerkin solutions (u_n, v_n) of the form

$$u_n(t,x) = \sum_{j=1}^n g_j^n(t)\varphi_j(x) \qquad \text{and} \qquad v_n(t,x) = \sum_{j=1}^n h_j^n(t)\varphi_j(x)$$

such that

$$(\partial_t u_n, \varphi_i) + (\nabla u_n, \nabla \varphi_i) = (a_m(u_n)\nabla v_n, \nabla \varphi_i), \tag{2.6}$$

$$(\partial_t v_n, \varphi_i) - (\Delta v_n, \varphi_i) = -(a_m(u_n)^s v_n, \varphi_i), \tag{2.7}$$

$$u_n(0) = P_n(u_m^0), \quad v_n(0) = P_n(v_m^0), \tag{2.8}$$

for $i = 1, \ldots, n$, where $P_n(u_m^0)$ and $P_n(v_m^0)$ are orthogonal projections of u_m^0 and v_m^0 from $H^1(\Omega)$ into X^n. Since the application of the Galerkin's method is a very standard procedure, some details (such as the proof of existence of the Galerkin solutions, the obtaining of *a priori* estimates and the passage to the limit as $n \to \infty$) will be omitted here.

In order to obtain n-independent *a priori* estimates for (u_n, v_n), we test (2.6) by $u_n \in X^n$ and (2.7) by $v_n \in X^n$ and $-\Delta v_n \in X^n$. Then we can also test (2.7) by $\Delta^2 v_n \in X^n$ and (2.6) by $-\Delta u_n \in X^n$. Taking the truncation $a_m(\cdot)$ and its bounds (2.3) into account, it is not difficult to obtain the following *a priori* bounds (independent of n) for each final time $T > 0$:

$$\begin{gathered} (u_n, v_n)_n \text{ is bounded in } L^\infty(0,T; H^1(\Omega) \times H^2(\Omega)), \\ (\Delta u_n, \Delta v_n)_n, (\partial_t u_n, \partial_t v_n)_n \text{ are bounded in } L^2(0,T; L^2(\Omega) \times H^1(\Omega)). \end{gathered} \tag{2.9}$$

Therefore the Galerkin solution, (u_n, v_n), is defined up to infinity time.

Besides, if we assume that the Poisson-Neumann problem (1.5) has the $W^{2,3}$-regularity, then we have

$$(v_n)_n \text{ is bounded in } L^2(0,T;W^{2,3}(\Omega)). \tag{2.10}$$

We can also test (2.6) by $\Delta^2 u_n \in X^n$ and, using (2.10), we obtain the n-independent bounds

$$\begin{gathered}(u_n)_n \text{ is bounded in } L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;W^{2,3}(\Omega)),\\ (\partial_t u_n)_n, (\Delta u_n)_n \text{ are bounded in } L^2(0,T;H^1(\Omega)).\end{gathered} \tag{2.11}$$

Now, the *a priori* bounds (2.9), compactness results in the weak and weak* topologies (see [7]) and compactness results in Bochner spaces (see Lemma 1.14), for each $T > 0$, allow us to conclude that there exist limit functions u_m and v_m such that, up to a subsequence,

$$u_n \to u_m \text{ and } v_n \to v_m$$

weakly* in $L^\infty(0,T;H^1(\Omega))$, weakly in $L^2(0,T;H^2(\Omega))$ and strongly in $L^2(0,T;H^1(\Omega))$.

Then using these convergences and passing to the limit in the approximate system (2.6)-(2.8) it follows that (u_m, v_m) satisfies (2.1) *a.e.* in $(0,\infty)\times\Omega$. One can prove that the solution (u_m, v_m) is unique by straightforward calculations.

Thus, in this subsection, for each fixed $m \in \mathbb{N}$, we have proved the existence and uniqueness of (u_m, v_m), solution of (2.1), such that

$$\begin{gathered}u_m \in L^\infty_{loc}([0,\infty);H^1(\Omega)) \cap L^2_{loc}([0,\infty);H^2(\Omega)), \quad v_m \in L^\infty_{loc}([0,\infty);H^2(\Omega)),\\ \partial_t u_m \in L^2_{loc}([0,\infty);L^2(\Omega)), \quad \partial_t v_m, \Delta v_m \in L^2_{loc}([0,\infty);H^1(\Omega)).\end{gathered}$$

If we assume that the Poisson-Neumann problem (1.5) has the $W^{2,3}$-regularity, then it stems from the stronger n-independent bounds (2.10) and (2.11) that

$$\begin{gathered}u_m, v_m \in L^\infty_{loc}([0,\infty);H^2(\Omega)) \cap L^2_{loc}([0,\infty);W^{2,3}(\Omega)),\\ \partial_t u_m, \partial_t v_m, \Delta u_m, \Delta v_m \in L^2_{loc}([0,\infty);H^1(\Omega)).\end{gathered}$$

2.2.2 Regularity up to infinity time of problem (2.1)

Continuing the analysis, in the present subsection we prove the following main result.

Theorem 2.6. (Regularity up to infinity time of (2.1)**)** *Let u_m^0 and v_m^0 be approximations of u^0 and v^0 as defined in the beginning of Section 2.2. Under the assumption that the Poisson-Neumann problem* (1.5) *has the H^2-regularity, there is a unique solution (u_m, v_m) of* (2.1) *such that*

$$u_m(t,x) \geq 0, \ \textit{a.e. } (t,x) \in (0,\infty)\times\Omega,$$

$$\operatorname*{ess\,inf}_{x\in\Omega} v^0(x) e^{-m^s t} \leq v_m(t,x) \leq \|v^0\|_{L^\infty(\Omega)}, \ \textit{a.e. } (t,x) \in (0,\infty)\times\Omega,$$

with the following regularity:

$$\begin{aligned}(u_m - u^*) &\in L^\infty(0,\infty; H^1(\Omega)) \cap L^2(0,\infty; H^2(\Omega)),\\ v_m &\in L^\infty(0,\infty; H^2(\Omega)) \cap L^2(0,\infty; H^2(\Omega)),\\ \partial_t u_m \in L^2(0,\infty; L^2(\Omega)), &\quad \partial_t v_m, \Delta v_m \in L^2(0,\infty; H^1(\Omega)),\end{aligned} \tag{2.12}$$

where $u^* = \frac{1}{|\Omega|}\int_\Omega u^0(x)\,dx$. *Additionally, if we assume that the Poisson-Neumann problem* (1.5) *has the* $W^{2,3}$*-regularity, then*

$$u_m \in L^\infty(0,\infty; H^2(\Omega)), \quad \partial_t u_m, \Delta u_m \in L^2(0,\infty; H^1(\Omega)). \tag{2.13}$$

The proof of Theorem 2.6 will be carried out along the subsection in several steps. We begin with the proof of some pointwise estimates, in Lemma 2.7, and some direct m-independent estimates for the solution (u_m, v_m) of the regularized problem, in Lemma 2.8. Next we prove the weak regularity up to infinity time, in Lemma 2.9, and use it to finish the proof of Theorem 2.6.

Lemma 2.7. (Pointwise m-uniform estimates for (u_m, v_m))

1. *If* $u_m^0(x) \geq 0$ *a.e.* $x \in \Omega$ *then* $u_m(t,x) \geq 0$ *a.e.* $(t,x) \in (0,\infty) \times \Omega$*;*
2. *If* $v^0(x) \geq 0$ *a.e.* $x \in \Omega$ *and* $v^0 \in L^\infty(\Omega)$ *then*

$$\operatorname*{ess\,inf}_{x\in\Omega} \{v^0(x)\} \exp(-m^s t) \leq v_m(t,x) \leq \|v^0\|_{L^\infty(\Omega)} \ a.e.(t,x) \in (0,\infty) \times \Omega;$$

Proof. By testing the u_m-equation of (2.1) by $(u_m)_-$ and using that $|a_m((u_m)_-)| \leq |(u_m)_-|$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u_m)_-\|^2_{L^2(\Omega)} + \|\nabla(u_m)_-\|^2_{L^2(\Omega)} = \int_\Omega a_m((u_m)_-)\nabla v_m \cdot \nabla(u_m)_- \,dx\\ &\leq \|(u_m)_-\|_{L^3(\Omega)}\|\nabla v_m\|_{L^6(\Omega)}\|\nabla(u_m)_-\|_{L^2(\Omega)}\\ &\leq C\|\nabla v_m\|_{L^6(\Omega)}\Big(\|(u_m)_-\|_{L^2(\Omega)}\|\nabla(u_m)_-\|_{L^2(\Omega)} + C\|(u_m)_-\|^{1/2}_{L^2(\Omega)}\|\nabla(u_m)_-\|^{3/2}_{L^2(\Omega)}\Big).\end{aligned}$$

Hence, using Young's inequality and that $v_m \in L^\infty(0,T; W^{1,6}(\Omega))$ we can arrive at

$$\frac{1}{2}\frac{d}{dt}\|(u_m)_-\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla(u_m)_-\|^2_{L^2(\Omega)} \leq C\|(u_m)_-\|^2_{L^2(\Omega)}.$$

Note that $(u_m^0)_- = 0$, by hypothesis. Therefore, if we apply Gronwall's inequality (Lemma 1.16) we conclude that $(u_m)_-(t,x) = 0$, $a.e.(t,x) \in (0,T) \times \Omega$, for all $T > 0$, that is, $u_m(t,x) \geq 0$, $a.e.(t,x) \in (0,\infty) \times \Omega$.

In order to establish the positivity of v_m we define the function

$$V(t) = \min_{x\in\Omega}\{v_m^0(x)\} \exp(-m^s t).$$

Clearly, V is a sub-solution of the v_m-equation of (2.1), because $-a_m(u_m)^s \geq -m^s$. In fact, we have

$$V_t(t) - \Delta V(t) = -m^s V(t) \leq -a_m(u_m(t,x))^s V(t),$$

Comparing V and v_m we conclude that $v_m(t,x) \geq V(t)$ *a.e.* in $(0,\infty)\times\Omega$. The upper bound on v_m can be obtained an analogous argument, but now using the super-solution $V(t,x) = \|v_m^0\|_{L^\infty(\Omega)}$. This implies that

$$\operatorname*{ess\,inf}_{x\in\Omega}\{v_m^0(x)\}\exp(-m^s t) \leq v_m(t,x) \leq \|v_m^0\|_{L^\infty(\Omega)} \ a.e. (t,x) \in (0,\infty)\times\Omega,$$

and using (2.5) finally leads us to the desired result. ■

Lemma 2.8. (m-uniform estimates for (u_m, v_m))

1. *For every* $t \geq 0$,

$$\|u_m(t)\|_{L^1(\Omega)} = \int_\Omega u_m(t,x)\ dx = \|u_m^0\|_{L^1(\Omega)} = \|u^0\|_{L^1(\Omega)} = u^*|\Omega|;$$

2. *For every* $t > 0$,

$$\|v_m(t)\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla v_m(s)\|^2_{L^2(\Omega)}\ ds + \int_0^t \|a_m(u_m(s))^{s/2} v_m(s)\|^2_{L^2(\Omega)}\ ds \leq \|v^0\|^2_{L^2(\Omega)},$$

which allows us to conclude in particular that

$$\nabla v_m \text{ is bounded in } L^2(0,\infty;L^2(\Omega)), \text{ independently of } m \in \mathbb{N}. \tag{2.14}$$

Proof. Taking (2.4) and (2.5) into account, to prove the first item, we integrate the the u_m-equation of (2.1) and take into account that $u_m \geq 0$, thanks to Lemma 2.7. The second item can be proved by testing the the v_m-equation of (2.1) by $2v_m$. ■

Lemma 2.9. (Weak regularity of (u_m, v_m) up to infinity time) *For each fixed* $m \in \mathbb{N}$, *the following regularity at infinity time to* u_m *and* v_m *holds:*

$$(u_m - u^*), v_m \in L^\infty(0,\infty;L^2(\Omega)) \cap L^2(0,\infty;H^1(\Omega)).$$

Proof. From Lemma 2.8.1 we have $\int_\Omega (u_m(t) - u^*)\ dx = 0$, $\forall t \in [0,\infty)$, that is, $u_m - u^*$ is a null mean function. Besides, by testing the the u_m-equation of (2.1) by

$2u_m$ and using Lemma 2.8.2, we arrive at

$$\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u_m(s)\|^2_{L^2(\Omega)}\, ds \le \|u^0_m\|^2_{L^2(\Omega)} + m^2 \int_0^t \|\nabla v_m(s)\|^2_{L^2(\Omega)}\, ds$$
$$\le \|u^0_m\|^2_{L^2(\Omega)} + \frac{m^2}{2}\|v^0_m\|^2_{L^2(\Omega)}.$$

The latter allows us to conclude that

$$u_m \in L^\infty(0,\infty; L^2(\Omega)) \text{ and } \nabla u_m \in L^2(0,\infty; L^2(\Omega)).$$

Hence, using the Poincaré's type inequality of Lemma 1.3 we can prove that

$$u_m - u^* \in L^\infty(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; H^1(\Omega)). \tag{2.15}$$

Next, we use this fact and take v_m as a test function in the following reformulation of the v_m-equation of (2.1)

$$(v_m)_t - \Delta v_m + a_m(u^*)^s v_m = -(a_m(u_m)^s - a_m(u^*)^s)v_m,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_m(t)\|^2_{L^2(\Omega)} + \|\nabla v_m(t)\|^2_{L^2(\Omega)} + a_m(u^*)^s\|v_m(t)\|^2_{L^2(\Omega)}$$
$$= -\int_\Omega (a_m(u_m(t,x))^s - a_m(u^*)^s)v_m(t,x)^2\, dx.$$

Using Lemma 1.10, the right hand side can be estimated by

$$|\int_\Omega (a_m(u_m)^s - a_m(u^*)^s)v^2\, dx| \le \|v^0_m\|_{L^\infty(\Omega)} \int_\Omega |a_m(u_m)^s - a_m(u^*)^s| v_m\, dx$$
$$\le \|v^0\|_{L^\infty(\Omega)} \int_\Omega |a_m(u_m) + a_m(u^*)|^{s-1}|a_m(u_m) - a_m(u^*)| v\, dx$$
$$\le 2^{s-1}m^{s-1}\|v^0\|_{L^\infty(\Omega)}\|u_m - u^*\|_{L^2(\Omega)}\|v_m\|_{L^2(\Omega)}$$
$$\le C(u^*, v^0)m^{s-1}\|u_m - u^*\|^2_{L^2(\Omega)} + \frac{a_m(u^*)^s}{2}\|v_m\|^2_{L^2(\Omega)},$$

where $C(u^*, v^0) > 0$ is a constant (independent of t and x). Now, considering also the terms of the left hand side, one has

$$\frac{1}{2}\frac{d}{dt}\|v_m(t)\|^2_{L^2(\Omega)} + \|\nabla v_m(t)\|^2_{L^2(\Omega)} + a_m(u^*)\|v_m(t)\|^2_{L^2(\Omega)}$$
$$\le C(u^*, v^0)m^{s-1}\|u_m(t) - u^*\|^2_{L^2(\Omega)}.$$

Note that $a_m(u^*)$ is a fixed positive real number if $u^0 \neq 0$ and $\|u_m(t) - u^*\|^2_{L^2(\Omega)} \in L^1(0,\infty)$, because of (2.15). Hence we can conclude that $v_m \in L^2(0,\infty; L^2(\Omega))$ and, together with (2.14), we finally conclude that $v_m \in L^2(0,\infty; H^1(\Omega))$. ∎

The regularity given in Lemma 2.9 allows us to obtain the regularity (2.12). In fact, first we test the v_m-equation of (2.1) by $-\Delta v_m \in L^2_{loc}([0,\infty);H^1(\Omega))$. After some computations, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v_m\|^2_{L^2(\Omega)} + \|\Delta v_m\|^2_{L^2(\Omega)} + \|a_m(u_m)^{s/2}\nabla v_m(s)\|^2_{L^2(\Omega)}\\ \le C(m,\|v^0\|_{L^\infty(\Omega)})(\|\nabla u_m\|^2_{L^2(\Omega)} + \|\nabla v_m\|^2_{L^2(\Omega)}),\end{aligned}$$

and this allows us to conclude that

$$v_m \in L^\infty(0,\infty;H^1(\Omega)) \cap L^2(0,\infty;H^2(\Omega)) \tag{2.16}$$

because, according to Lemmas 2.8 and 2.9 we have $\|\nabla u_m\|^2_{L^2(\Omega)}$, $\|\nabla v_m\|^2_{L^2(\Omega)} \in L^1(0,\infty)$. Next we take the gradient of the v_m-equation of (2.1) and test the resulting equation by $-\nabla\Delta v_m \in L^2_{loc}([0,\infty);L^2(\Omega))$, obtaining

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta v_m\|^2_{L^2(\Omega)} + \|\nabla\Delta v_m\|^2_{L^2(\Omega)}\\ &= s\int_\Omega a'_m(u_m)a_m(u_m)^{s-1}v_m\nabla u_m\cdot\nabla\Delta v_m\,dx + \int_\Omega a_m(u_m)^s\nabla v_m\cdot\nabla\Delta v_m\,dx\\ &\le C(s,m,\|v^0\|_{L^\infty(\Omega)})(\|\nabla u_m\|^2_{L^2(\Omega)} + \|\nabla v_m\|^2_{L^2(\Omega)}) + \frac{1}{2}\|\nabla\Delta v_m\|^2_{L^2(\Omega)},\end{aligned}$$

which gives us

$$v_m \in L^\infty(0,\infty;H^2(\Omega)) \text{ and } \Delta v_m \in L^2(0,\infty;H^1(\Omega)). \tag{2.17}$$

From regularities (2.16) and (2.17), we can go back to the v_m-equation of (2.1) and conclude that

$$(v_m)_t \in L^2(0,\infty;H^1(\Omega)).$$

Now, we can test the u_m-equation of (2.1) by $-\Delta u_m \in L^2_{loc}([0,\infty);L^2(\Omega))$. Considering the bounds of the truncation $a_m(\cdot)$ given in (2.3), the interpolation inequality of Lemma 1.1-2 and inequality (1.6), we have

$$\begin{aligned}&\frac{d}{dt}\|\nabla u_m\|^2_{L^2(\Omega)} + \|\Delta u_m\|^2_{L^2(\Omega)}\\ &= \int_\Omega a'_m(u_m)\nabla u_m\cdot\nabla v_m\Delta u_m\,dx + \int_\Omega a_m(u_m)\Delta v_m\Delta u_m\,dx\\ &\le C\|\nabla u_m\|_{L^3(\Omega)}\|\nabla v_m\|_{L^6(\Omega)}\|\Delta u_m\|_{L^2(\Omega)} + m\|\Delta v_m\|_{L^2(\Omega)}\|\Delta u_m\|_{L^2(\Omega)}\\ &\le C\|\nabla u_m\|^{1/2}_{L^2(\Omega)}\|\nabla v_m\|_{L^6(\Omega)}\|\Delta u_m\|^{3/2}_{L^2(\Omega)} + m\|\Delta v\|_{L^2(\Omega)}\|\Delta u_m\|_{L^2(\Omega)}\\ &\le C\|\nabla v_m\|^4_{L^6(\Omega)}\|\nabla u_m\|^2_{L^2(\Omega)} + C(m)\|\Delta v_m\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta u_m\|^2_{L^2(\Omega)}.\end{aligned}$$

After absorbing the term $\frac{1}{2}\|\Delta u_m\|^2_{L^2(\Omega)}$, the other terms in the right hand side of the inequality belong to $L^1(0,\infty)$. Hence, integrating the last inequality with respect to t,

$$\nabla u_m \in L^\infty(0,\infty;L^2(\Omega)) \text{ and } \Delta u_m \in L^2(0;\infty;L^2(\Omega)),$$

finishing the proof of the regularity (2.12).

Finally, we consider the case in which we assume that the Poisson-Neumann problem (1.5) has the $W^{2,3}$-regularity. In this case, as observed in the end of Subsection 2.2.1, the regularity (2.13) already holds if we consider finite intervals $(0,T)$, for finite $T>0$, instead of $(0,\infty)$. Then we can take the gradient of the u-equation of (2.1) and test the resulting equation by $-\nabla\Delta u_m \in L^2_{loc}([0,\infty);L^2(\Omega))$, obtaining

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_m\|^2_{L^2(\Omega)} + \|\nabla\Delta u_m\|^2_{L^2(\Omega)} = \int_\Omega a_m(u_m)\nabla\Delta v_m\cdot\nabla\Delta u_m\,dx$$
$$+\int_\Omega a_m'(u_m)\Delta v_m\nabla u_m\cdot\nabla\Delta u_m\,dx + \int_\Omega a_m'(u_m)\nabla u_m\cdot D^2 v_m\nabla\Delta u_m\,dx$$
$$+\int_\Omega a_m'(u_m)\nabla v_m\cdot D^2 u_m\nabla\Delta u_m\,dx + \int_\Omega a_m''(u_m)\Big(\nabla u_m\cdot\nabla v_m\Big)\Big(\nabla u_m\cdot\nabla\Delta u_m\Big)\,dx.$$

We recall that, from (2.3), we have $|a_m(u_m)|\le m$ and $|a_m'(u_m)|, |a_m''(u_m)|\le C$, for some $C>0$. Then, using Hölder's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_m\|^2_{L^2(\Omega)} + \|\nabla\Delta u_m\|^2_{L^2(\Omega)} \le \Big(m\|\nabla\Delta v_m\|_{L^2(\Omega)}$$
$$+C\|\Delta v_m\|_{L^3(\Omega)}\|\nabla u_m\|_{L^6(\Omega)} + C\|D^2 v_m\|_{L^3(\Omega)}\|\nabla u_m\|_{L^6(\Omega)}$$
$$+C\|\nabla v_m\|_{L^6(\Omega)}\|D^2 u_m\|_{L^3(\Omega)} + C\|\nabla u_m\|^2_{L^6(\Omega)}\|\nabla v_m\|_{L^6(\Omega)}\Big)\|\nabla\Delta u_m\|_{L^2(\Omega)}$$

Using the continuous embedding $L^6(\Omega)\subset H^1(\Omega)$, the $W^{2,3}$-regularity, inequality (1.6), the interpolation inequality for the the L^3-norm (Lemma 1.1) in 3D domains and Young's inequality, we arrive at

$$\frac{d}{dt}\|\Delta u_m\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\Delta u_m\|^2_{L^2(\Omega)} \le Cm\|\nabla\Delta v_m\|^2_{L^2(\Omega)}$$
$$+C\Big[\|\Delta v_m\|^2_{L^3(\Omega)} + \|\Delta v_m\|^4_{L^2(\Omega)} + \|\Delta u_m\|^2_{L^2(\Omega)}\|\Delta v_m\|^2_{L^2(\Omega)}\Big]\|\Delta u_m\|^2_{L^2(\Omega)}.$$

Now notice that the first term of the right hand side of the last inequality belongs to $L^1(0,\infty)$ and, because of the regularity obtained so far, we have $C\Big[\|\Delta v_m\|^2_{L^3(\Omega)} + \|\Delta v_m\|^2_{L^2(\Omega)} + \|\Delta u_m\|^4_{L^2(\Omega)}\|\Delta v_m\|^2_{L^2(\Omega)}\Big] \in L^1(0,\infty)$. Therefore, using Grownwall's inequality (Lemma 1.16) we can conclude that

$$\Delta u_m \in L^\infty(0,\infty;L^2(\Omega))\cap L^2(0,\infty;H^1(\Omega))$$

and hence

$$u_m\in L^\infty(0,\infty;H^2(\Omega)).$$

Next we can go back to the u_m-equation of (2.1) and conclude that $(u_m)_t\in L^2(0,\infty;H^1(\Omega))$, finishing the proof of the regularity (2.13).

2.3 Proof of Theorem 2.2

In this section we will obtain m-independent estimates to (u_m, v_m), the solution of problem (2.1), in order to pass to the limit as $m \to \infty$ and prove the existence of solution to the original problem (1).

2.3.1 An energy law appears: formal computations

The basic idea to obtain additional *a priori* m-independent estimates is that the effects of the consumption and chemotaxis terms cancel. First of all, we present some formal calculations to illustrate how it works. Suppose (u, v) is a regular enough solution to the original problem (1) with $u, v > 0$. Consider the change o variable $z = \sqrt{v}$, then (1) can be rewritten as

$$\begin{aligned} \partial_t u - \Delta u &= -\nabla \cdot (u \nabla (z)^2) \\ \partial_t z - \Delta z - \frac{|\nabla z|^2}{z} &= -\frac{u^s z}{2} \\ \partial_{\mathbf{n}} u|_\Gamma &= \partial_{\mathbf{n}} z|_\Gamma = 0 \\ u(0) &= u^0, \quad z(0) = \sqrt{v^0}, \end{aligned} \tag{2.18}$$

We are going to obtain estimates for u and z and then extract estimates for v from the estimates of z.

For this, we consider a function $g(u)$ such that $g''(u) = u^{s-2}$. Formally, assuming $u, z > 0$ we can use

$$g'(u) = \begin{cases} \dfrac{u^{s-1}}{(s-1)} & , \text{ if } s > 1, \\ ln(u) & , \text{ if } s = 1. \end{cases} \tag{2.19}$$

as a test function in the u-equation of (2.18), obtaining

$$\frac{d}{dt} \int_\Omega g(u) \, dx + \int_\Omega g''(u) |\nabla u|^2 \, dx = \int_\Omega u g''(u) \nabla (z^2) \cdot \nabla u$$

and, since $u g''(u) = u^{s-1}$, we have

$$\frac{d}{dt} \int_\Omega g(u) \, dx + \int_\Omega u^{s-2} |\nabla u|^2 \, dx = \int_\Omega u^{s-1} \nabla (z^2) \cdot \nabla u \, dx = \frac{1}{s} \int_\Omega \nabla (z^2) \cdot \nabla (u^s) \, dx. \tag{2.20}$$

On the other hand, we can test the z-equation of (2.18) by $-\Delta z$. Then we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla z|^2 \, dx \quad &+ \int_\Omega |\Delta z|^2 \, dx + \int_\Omega \frac{|\nabla z|^2}{z} \Delta z \, dx + \frac{1}{2} \int_\Omega u^s |\nabla z|^2 \, dx \\ &= -\frac{1}{4} \int_\Omega \nabla (u^s) \cdot \nabla (z^2) \, dx. \end{aligned} \tag{2.21}$$

Hence, if we add (2.21) to $s/4$ times (2.20), then the two terms on the right hand side cancel each other and we obtain the time differential equation

$$\begin{aligned} \frac{d}{dt}\left[\frac{s}{4}\int_\Omega g(u)\,dx + \frac{1}{2}\int_\Omega |\nabla z|^2\,dx\right] + \frac{s}{4}\int_\Omega u^{s-2}|\nabla u|^2\,dx \\ +\frac{1}{2}\int_\Omega u^s|\nabla z|^2\,dx + \int_\Omega |\Delta z|^2\,dx + \int_\Omega \frac{|\nabla z|^2}{z}\Delta z\,dx = 0. \end{aligned} \tag{2.22}$$

The main idea now is to estimate from below the term

$$\int_\Omega |\Delta z|^2\,dx + \int_\Omega \frac{|\nabla z|^2}{z}\Delta z\,dx \tag{2.23}$$

(see Lemma 2.10 below).

2.3.2 Rigorous justification of the energy inequalities

In the sequel, we consider the regularized problem (2.1) and its solution (u_m, v_m) instead of the original problem (1) and (u, v). In this case, we have to deal with the truncation $a_m(\cdot)$ in the chemotaxis and consumption terms and with the fact that u_m is nonnegative, but not necessarily strictly positive (Lemma 2.7). In order to obtain time independent estimates, we will also need to separate the auxiliary variable z from zero, that's why we are going to consider the change of variables $z = \sqrt{v+\alpha}$, for $\alpha > 0$ to be chosen later, instead of $\sqrt{v}$. With this modification, we will obtain the corresponding version of (2.22). We will separate the cases $s = 1$, $s \in (1,2)$ and $s \geq 2$. Note that if $s = 1$ then $g'(u)$, given by (2.19), and $g''(u)$ have a singularity at $u = 0$. If $s \in (1,2)$, then only $g''(u)$ is singular at $u = 0$ and, if $s \geq 2$, then neither $g'(u)$ nor $g''(u)$ are singular.

Let us consider the variable $z_m(t,x) = \sqrt{v_m(t,x)+\alpha}$. Taking into account that the pair (u_m, v_m) is a strong solution of (2.1), on has by straightforward calculations that (u_m, z_m) satisfies the following equivalent problem:

$$\begin{aligned} \partial_t u_m - \Delta u_m &= -\nabla\cdot(a_m(u_m)\nabla(z_m)^2) \\ \partial_t z_m - \Delta z_m - \frac{|\nabla z_m|^2}{z_m} &= -\frac{1}{2}a_m(u_m)^s z_m + \frac{\alpha}{2}\frac{a_m(u_m)^s}{z_m} \\ \partial_{\mathbf{n}} u_m|_\Gamma &= \partial_{\mathbf{n}} z_m|_\Gamma = 0 \\ u_m(0) &= u^0, \quad z_m(0) = \sqrt{v^0+\alpha}. \end{aligned} \tag{2.24}$$

In the present subsection, we drop the m-subscript and write (u, z) for (u_m, z_m) to simplify the notation along the proofs of the forthcoming Lemmas. We remark that all the constants obtained in these Lemmas are independent of the parameter $m \in \mathbb{N}$ and this is why the energy inequalities proved in this section allow us to obtain m-independent bounds in the rest of the chapter.

We use the following lemma in order to estimate (2.23), whose proof can be found in the Appendix B.

Lemma 2.10. *Suppose that the Poisson-Neumann problem* (1.5) *has the* H^2*-regularity and assume that Hypothesis (H1) holds. Then there exist positive constants* $C_1, C_2 > 0$ *such that*

$$\int_\Omega |\Delta z|^2 \, dx + \int_\Omega \frac{|\nabla z|^2}{z} \Delta z \, dx \geq C_1 \Big(\int_\Omega |D^2 z|^2 \, dx + \int_\Omega \frac{|\nabla z|^4}{z^2} \, dx \Big) - C_2 \int_\Omega |\nabla z|^2 \, dx,$$

for all $z \in H^2(\Omega)$ *such that* $\partial_{\mathbf{n}} z|_\Gamma = 0$ *and* $z \geq \alpha$ *in* Ω*, for some* $\alpha > 0$*.*

Now we prove the following.

Lemma 2.11. *The solution* (u, z) *of* (2.24)*, satisfies the inequality*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla z\|^2_{L^2(\Omega)} + C_1 \Big(\int_\Omega |D^2 z|^2 \, dx + \int_\Omega \frac{|\nabla z|^4}{z^2} \, dx \Big) \\ &+ \frac{1}{2} \int_\Omega a_m(u)^s |\nabla z|^2 \, dx \leq \frac{s}{4} \int_\Omega a_m(u)^{s-1} \nabla(z^2) \cdot \nabla a_m(u) \, dx \\ &+ \frac{s}{2}\sqrt{\alpha} \int_\Omega a_m(u)^{s-1} |\nabla z| |\nabla a_m(u)| \, dx + C_2 \int_\Omega |\nabla z|^2 \, dx. \end{aligned}$$

Proof. We begin by testing the z-equation of (2.24) by $-\Delta z$. This gives us

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla z\|^2_{L^2(\Omega)} + \|\Delta z\|^2_{L^2(\Omega)} + \int_\Omega \frac{|\nabla z|^2}{z} \Delta z \, dx + \frac{1}{2} \int_\Omega a_m(u)^s |\nabla z|^2 \, dx \\ &\leq \frac{s}{4} \int_\Omega a_m(u)^{s-1} \nabla(z^2) \cdot \nabla a_m(u) \, dx + \frac{s}{2}\sqrt{\alpha} \int_\Omega a_m(u)^{s-1} |\nabla z| |\nabla a_m(u)| \, dx. \end{aligned}$$

Then, applying Lemma 2.10, we obtain the desired inequality. ■

Define, for $\sigma \geq 0$, the functions g_m and $g_{m,j}$, adequate regularizations of the function g that appears in the formal inequality (2.22), by

$$g_m(\sigma) = \int_0^\sigma g'_m(r) \, dr \quad \text{and} \quad g_{m,j}(\sigma) = \int_0^\sigma g'_{m,j}(r) \, dr, \tag{2.25}$$

where g'_m and $g'_{m,j}$ are defined for $r \geq 0$ and given by

$$g'_m(r) = \begin{cases} ln(a_m(r) + 1), & \text{if } s = 1, \\ \dfrac{a_m(r)^{s-1}}{s-1}, & \text{if } s > 1, \end{cases} \tag{2.26}$$

and

$$g'_{m,j}(r) = \frac{(a_m(r) + 1/j)^{s-1}}{s-1}, \text{ for } s \in (1,2) \text{ and } j \in \mathbb{N}. \tag{2.27}$$

In Lemmas 2.14, 2.15 and 2.16 below we will be interested in the terms

$$\int_\Omega u_t(t,x) \, g'_m(u(t,x)) \, dx \quad \text{and} \quad \int_\Omega u_t(t,x) \, g'_{m,j}(u(t,x)) \, dx,$$

recalling that $u(t,x)$ denotes the function $u_m(t,x)$ of the solution (u_m, z_m) of (2.24). We have the following results on these terms.

Theorem 2.12. *The weak time derivatives of $g_{m,j}(u)$ and $g_m(u)$ belong to $L^2(0,\infty;L^2(\Omega))$ and are given by*

$$\partial_t g_m(u) = g'_m(u)\ u_t \tag{2.28}$$

and

$$\partial_t g_{m,j}(u) = g'_{m,j}(u)\ u_t. \tag{2.29}$$

Proof. We are going to prove (2.29). The proof of (2.28) is analogous to the proof of (2.29). Because of Lemma 1.19, we know that there is a sequence $(u^n)_{n\in\mathbb{N}} \subset C_c^\infty(0,\infty;L^2(\Omega))$ that converges to u in the norm of $H^1((0,\infty);L^2(\Omega))$. For these functions u^n, which are very regular in the time variable, we can write

$$-\int_0^\infty g_{m,j}(u^n)\ \varphi'(t)\ dt = \int_0^\infty g'_{m,j}(u^n)u_t^n\ \varphi(t)\ dt = \int_0^\infty \frac{(a_m(u^n)+1/j)^{s-1}}{s-1}\ u_t^n\ \varphi(t)\ dt,$$

for all $\varphi \in C_c^\infty(0,\infty)$. Then we note that the convergence in the norm of $H^1((0,\infty);L^2(\Omega))$ is enough to pass to the limit as $n\to\infty$ in the first and in the last terms of the above equality, yielding

$$-\int_0^\infty g_{m,j}(u)\ \varphi'(t)\ dt = \int_0^\infty \frac{(a_m(u)+1/j)^{s-1}}{s-1}\ u_t\ \varphi(t)\ dt,\ \forall\varphi\in C_c^\infty(0,\infty).$$

By the definition of weak derivative, this is precisely (2.29), as we wanted to prove. ∎

Corollary 2.13. *The weak time derivatives $\frac{d}{dt}\int_\Omega g_m(u(t,x))\ dx$ and $\frac{d}{dt}\int_\Omega g_{m,j}(u(t,x))\ dx$ belong to $L^2(0,\infty)$ and are given by the expressions*

$$\frac{d}{dt}\int_\Omega g_m(u(t,x))\ dx = \int_\Omega u_t(t,x)g'_m(u(t,x))\ dx \tag{2.30}$$

and

$$\frac{d}{dt}\int_\Omega g_{m,j}(u(t,x))\ dx = \int_\Omega u_t(t,x)g'_{m,j}(u(t,x))\ dx. \tag{2.31}$$

Proof. We are going to prove (2.31). The proof of (2.30) is analogous to the proof of (2.31). Accounting for Lemma 1.18 and (2.29), we obtain

$$\frac{d}{dt}\int_\Omega g_{m,j}(u(t,x))\ dx = \int_\Omega \frac{d}{dt}g_{m,j}(u(t,x))\ dx = \int_\Omega u_t(t,x)\frac{(a_m(u(t,x))+1/j)^{s-1}}{s-1}\ dx.$$

Since $\frac{(a_m(u(t,x))+1/j)^{s-1}}{s-1}$ is pointwisely bounded and $u_t(t,x)\in L^2(0,\infty;L^2(\Omega))$ then

$$\frac{d}{dt}\int_\Omega g_{m,j}(u(t,x))\ dx \in L^2(0,\infty).$$

∎

Now, we are in position to prove an energy inequality associated to the formal inequality (2.22).

Lemma 2.14 **(Energy inequality for $s = 1$).** *The solution (u, z) of the problem (2.24) satisfies, for sufficiently small $\alpha > 0$,*

$$\frac{d}{dt}\left[\frac{1}{4}\int_\Omega g_m(u)\ dx + \frac{1}{2}\int_\Omega |\nabla z|^2\ dx\right] + C\int_\Omega |\nabla[a_m(u)+1]^{1/2}|^2\ dx + \frac{1}{4}\int_\Omega a_m(u)|\nabla z|^2\ dx$$
$$+C_1\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) \le C\int_\Omega |\nabla z|^2\ dx, \tag{2.32}$$

where $g_m(u)$ is given by (2.25).

Proof. In order to prove (2.32) we will use the cancellation effect mentioned in Subsection 2.3.1. Since now we are dealing with the regularized problem (instead of the original problem), we must pay attention to two technical difficulties that arise: the presence of the truncation $a_m(\cdot)$ in the chemotaxis and consumption terms, the fact that u is nonnegative, but not strictly positive, and now we consider $z = \sqrt{v+\alpha}$. For $s = 1$, this means that instead of using $g'(u) = ln(u)$ as a test function in the u_m-equation of (2.1), we must use $g'(a_m(u)+1) = ln(a_m(u)+1)$, in order to preserve the cancellation effect, avoid divisions by zero and invalid values for the argument of $ln(\cdot)$.

We begin by using $\varphi = ln(a_m(u)+1)$ in the u-equation of problem (2.1) to obtain

$$\frac{d}{dt}\int_\Omega g_m(u)\ dx + \int_\Omega \frac{a_m'(u)}{a_m(u)+1}|\nabla u|^2\ dx = \left(\frac{a_m(u)}{a_m(u)+1}\nabla(z^2), \nabla a_m(u)\right),$$

where

$$g_m(r) = \int_0^r ln(a_m(\theta)+1)\ d\theta$$

is a primitive of $ln(a_m(r)+1)$. Due to the regularity of u, u_t and the functions $g_m(r)$ and $g_m'(r)$, for $r \ge 0$, we can conclude that the weak derivative $\frac{d}{dt}\int_\Omega g_m(u)\ dx = \int_\Omega g_m'(u)\, u_t\ dx$ and belongs to $L^2(0,T)$.

Since $0 \le a_m'(u) \le C$, we have $(a_m'(u))^2 \le Ca_m'(u)$, and we can write

$$\int_\Omega \frac{a_m'(u)}{a_m(u)+1}|\nabla u|^2 dx \ge C\int_\Omega \frac{(a_m'(u))^2}{a_m(u)+1}|\nabla u|^2 dx$$
$$= C\int_\Omega \frac{|\nabla a_m(u)|^2}{a_m(u)+1}\ dx \ge C\int_\Omega |\nabla[a_m(u)+1]^{1/2}|^2 dx.$$

Hence, using that $\dfrac{1}{(a_m(u)+1)} \le \dfrac{1}{\sqrt{a_m(u)+1}}$, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega g_m(u)\ dx + C\int_\Omega |\nabla[a_m(u)+1]^{1/2}|^2\ dx &= 2\left(\frac{a_m(u)+1-1}{a_m(u)+1} z\nabla z, \nabla a_m(u)\right)\\
&= \big(\nabla(z^2), \nabla a_m(u)\big) - 2\left(z\nabla z, \frac{\nabla a_m(u)}{a_m(u)+1}\right)\\
&\le \big(\nabla(z^2), \nabla a_m(u)\big) + 2\sqrt{\|v^0\|_{L^\infty(\Omega)}+\alpha}\|\nabla z\|_{L^2(\Omega)}\|\nabla[a_m(u)+1]^{1/2}\|_{L^2(\Omega)},
\end{aligned}$$

Then we obtain

$$\frac{d}{dt}\int_\Omega g_m(u)\ dx + C\int_\Omega |\nabla[a_m(u)+1]^{1/2}|^2\ dx \le \big(\nabla(z^2), \nabla a_m(u)\big) + C\|\nabla z\|^2_{L^2(\Omega)} \tag{2.33}$$

Now, using Lemma 2.11 for $s=1$ we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla z\|^2_{L^2(\Omega)} + C_1\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) + \frac{1}{2}\int_\Omega a_m(u)|\nabla z|^2\ dx\\
&\quad\le \frac{1}{4}\int_\Omega \nabla(z^2)\cdot\nabla a_m(u)\ dx + \frac{\sqrt{\alpha}}{2}\int_\Omega |\nabla z||\nabla a_m(u)|\ dx + C_2\|\nabla z\|^2_{L^2(\Omega)}.
\end{aligned}$$

If we add the above inequality to 1/4 times (2.33), then the terms $\int_\Omega \nabla a_m(u)\cdot\nabla(z^2)\ dx$ cancel and we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{4}\int_\Omega g_m(u)\ dx + \frac{1}{2}\|\nabla z\|^2_{L^2(\Omega)}\Big] + C\int_\Omega |\nabla[a_m(u)+1]^{1/2}|^2\ dx\\
&+\frac{1}{2}\int_\Omega a_m(u)|\nabla z|^2\ dx + C_1\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big)\\
&\le \frac{\sqrt{\alpha}}{2}\int_\Omega |\nabla z||\nabla a_m(u)|\ dx + C_2\|\nabla z\|^2_{L^2(\Omega)}\\
&\le \int_\Omega \sqrt{\alpha}|\nabla[a_m(u)+1]^{1/2}||\sqrt{a_m(u)+1}||\nabla z|\ dx + C_2\|\nabla z\|^2_{L^2(\Omega)}.
\end{aligned} \tag{2.34}$$

We can deal with the first term in the right hand side of the inequality using Hölder's and Young's inequality,

$$\begin{aligned}
&\int_\Omega \sqrt{\alpha}|\nabla[a_m(u)+1]^{1/2}||\sqrt{a_m(u)+1}||\nabla z|\ dx\\
&\le \alpha\ C(\delta)\int_\Omega a_m(u)|\nabla z|^2\ dx + \delta\|\nabla[a_m(u)+1]^{1/2}\|^2_{L^2(\Omega)} + \alpha\ C(\delta)\int_\Omega |\nabla z|^2\ dx.
\end{aligned}$$

Therefore, we can first choose $\delta > 0$ and then $\alpha > 0$ sufficiently small in order to use the terms on the left hand side of inequality (2.34) to absorb the first two terms on the right hand side of the above inequality and finally obtain the desired inequality (2.32). ■

Lemma 2.15 (Energy inequality for $s \in (1,2)$). *The solution (u,z) of the problem (2.24) satisfies, for sufficiently small $\alpha > 0$,*

$$\frac{d}{dt}\left[\frac{s}{4}\int_\Omega g_m(u)\ dx + \frac{1}{2}\|\nabla z\|^2_{L^2(\Omega)}\right] + \frac{1}{4}\int_\Omega a_m(u)^s|\nabla z|^2\ dx$$
$$+C_1\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) \le C\int_\Omega |\nabla z|^2\ dx, \tag{2.35}$$

where $g_m(u)$ is given by (2.25).

Proof. Analogously to the case $s = 1$ (Lemma 2.14) in order to preserve the cancellation effect and avoid divisions by zero, for $s \in (1,2)$, instead of using $g'(u)$ as a test function in the u_m-equation of (2.1), we should consider the sequence $\{1/j\}_{j\in\mathbb{N}}$ and use $g'_{m,j}(u)$ given by (2.27). Due to the complexity of the procedures that are involved, we divide the proof in three main steps:

1. Obtain an inequality from the u_m-equation of problem (2.24);
2. Use this inequality and Lemma 2.11 to obtain the corresponding version of (2.22);
3. Pass to the limit as $j \to \infty$ to obtain (2.35).

STEP 1: By testing the u_m-equation of (2.24) by $g'_{m,j}(u) = (a_m(u)+1/j)^{s-1}/(s-1)$

$$\frac{d}{dt}\int_\Omega g_{m,j}(u)\ dx + \int_\Omega \frac{a'_m(u)}{(a_m(u)+1/j)^{2-s}}|\nabla u|^2\ dx = \left(\frac{a_m(u)}{(a_m(u)+1/j)^{2-s}}\nabla(z^2), \nabla a_m(u)\right),$$

where

$$g_{m,j}(r) = \int_0^r \frac{(a_m(\theta)+1/j)^{s-1}}{(s-1)}\ d\theta$$

is a primitive of $(a_m(r)+1/j)^{s-1}/(s-1)$. Since $0 \le a'_m(u) \le C$, we have $(a'_m(u))^2 \le Ca'_m(u)$, we can write

$$\int_\Omega \frac{a'_m(u)}{(a_m(u)+1/j)^{2-s}}|\nabla u|^2\ dx \ge C\int_\Omega \frac{(a'_m(u))^2}{(a_m(u)+1/j)^{2-s}}|\nabla u_m|^2\ dx \ge C\int_\Omega |\nabla[a_m(u)+1]^{s/2}|^2\ dx.$$

and hence we obtain

$$\frac{d}{dt}\int_\Omega g_{m,j}(u)\ dx + C\int_\Omega |\nabla[a_m(u)+1/j]^{s/2}|^2\ dx = \left(\frac{a_m(u)+1/j-1/j}{(a_m(u)+1/j)^{2-s}}\nabla(z^2), \nabla a_m(u)\right)$$
$$= \big((a_m(u)+1/j)^{s-1}\nabla(z^2), \nabla a_m(u)\big) - 2\left(z\nabla z, \left(\frac{1/j}{(a_m(u)+1/j)}\right)^{1-s/2}\frac{(1/j)^{s/2}\nabla a_m(u)}{(a_m(u)+1/j)^{1-s/2}}\right)$$
$$\le \big((a_m(u)+1/j)^{s-1}\nabla(z^2), \nabla a_m(u)\big) + \frac{4}{s}\sqrt{\|v^0\|_{L^\infty(\Omega)}+\alpha}(1/j)^{s/2}\|\nabla z\|_{L^2(\Omega)}\|\nabla[a_m(u)+1/j]^{s/2}\|_{L^2(\Omega)},$$

where in the last estimate we use $a_m(u) + 1/j \geq 1/j$. Then, using Young's inequality, we can absorb the term $\|\nabla[a_m(u)+1/j]^{s/2}\|_{L^2(\Omega)}$ obtaining

$$\begin{aligned} &\frac{d}{dt}\int_\Omega g_{m,j}(u)\ dx + C\int_\Omega |\nabla[a_m(u)+1/j]^{s/2}|^2\ dx \\ &\quad\leq \big((a_m(u)+1/j)^{s-1}\nabla(z^2), \nabla a_m(u)\big) + C(1/j)^s\|\nabla z\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.36}$$

STEP 2: We add the inequality of Lemma 2.11 to $s/4$ times (2.36), then we obtain

$$\begin{aligned} \frac{d}{dt}\Big[\frac{s}{4}\int_\Omega g_{m,j}(u)dx &+ \frac{1}{2}\|\nabla z\|^2_{L^2(\Omega)}\Big] + C\int_\Omega |\nabla[a_m(u)+1/j]^{s/2}|^2 dx \\ &+ C_1\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) + \frac{1}{2}\int_\Omega a_m(u)^s|\nabla z|^2\ dx \\ &\leq \frac{s}{2}\sqrt{\alpha}\int_\Omega a_m(u)^{s-1}|\nabla z||\nabla a_m(u)|\ dx + C\|\nabla z\|^2_{L^2(\Omega)} \\ &+ s\int_\Omega \Big[(a_m(u)+1/j)^{s-1} - a_m(u)^{s-1}\Big]\nabla a_m(u)\cdot z\nabla z\ dx \\ &\leq \int_\Omega \sqrt{\alpha}|\nabla[a_m(u)+1/j]^{s/2}||(a_m(u)+1/j)^{s/2}||\nabla z|\ dx + C\|\nabla z\|^2_{L^2(\Omega)} \\ &+ \frac{s}{4}\int_\Omega \Big[(a_m(u)+1/j)^{s-1} - a_m(u)^{s-1}\Big]\nabla a_m(u)\cdot\nabla(z^2)\ dx. \end{aligned}$$

Next, we deal with the first term in the right hand side of the previous inequality using Hölder's and Young's inequality,

$$\begin{aligned} &\int_\Omega \sqrt{\alpha}|\nabla[a_m(u)+1/j]^{s/2}||(a_m(u)+1/j)^{s/2}||\nabla z|\ dx \\ &\leq \sqrt{\alpha}\left[\int_\Omega (a_m(u)+1/j)^s|\nabla z|^2\ dx\right]^{1/2}\|\nabla[a_m(u)+1/j]^{s/2}\|_{L^2(\Omega)} \\ &\leq \alpha\ C(\delta)\int_\Omega (a_m(u)+1/j)^s|\nabla z|^2\ dx + \delta\|\nabla[a_m(u)+1/j]^{s/2}\|^2_{L^2(\Omega)}. \end{aligned}$$

Therefore, we first choose $\delta > 0$ and then $\alpha > 0$ sufficiently small to obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{s}{4}\int_\Omega g_{m,j}(u)\ dx + \frac{1}{2}\|\nabla z\|^2_{L^2(\Omega)}\right] + C\int_\Omega |\nabla[a_m(u)+1/j]^{s/2}|^2\ dx \\ &\quad + C_1\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) + \frac{1}{2}\int_\Omega \Big[a_m(u)^s - \frac{1}{2}(a_m(u)+1/j)^s\Big]|\nabla z|^2\ dx \\ &\leq C\|\nabla z\|^2_{L^2(\Omega)} + \frac{s}{4}\int_\Omega \Big[a_m(u)^{s-1} - (a_m(u)+1/j)^{s-1}\Big]\nabla a_m(u)\cdot\nabla(z^2)\ dx. \end{aligned}$$

In order to avoid problems with divisions by zero in the term $C\int_\Omega |\nabla[a_m(u)+1/j]^{s/2}|^2\,dx$ as we take the limit as $j\to\infty$, we use the fact that this term is nonnegative and write

$$\begin{aligned}&\frac{d}{dt}\left[\frac{s}{4}\int_\Omega g_{m,j}(u)\,dx+\frac{1}{2}\|\nabla z\|^2_{L^2(\Omega)}\right]+C_1\Big(\int_\Omega |D^2 z|^2\,dx+\int_\Omega \frac{|\nabla z|^4}{z^2}\,dx\Big)\\ &+\frac{1}{2}\int_\Omega\Big[a_m(u)^s-\frac{1}{2}(a_m(u)+1/j)^s\Big]|\nabla z|^2\,dx \qquad (2.37)\\ &\le C\|\nabla z\|^2_{L^2(\Omega)}+\frac{s}{4}\int_\Omega\Big[a_m(u)^{s-1}-(a_m(u)+1/j)^{s-1}\Big]\nabla a_m(u)\cdot\nabla(z^2)\,dx.\end{aligned}$$

STEP 3: Pass to the limit as $j\to\infty$.

Now we show that, passing to the limit as $j\to\infty$, we recover the cancellation of the chemotaxis and consumption terms. To deal with the passage to the limit as $j\to\infty$, we remind that $m\in\mathbf{N}$ is fixed and that the solution (u_m,z_m) of (2.24), denoted for simplicity as (u,z) in the present subsection, have the regularity

$$(u-u^*),z\in L^\infty(0,\infty;H^1(\Omega))\cap L^2(0,\infty;H^2(\Omega)),$$

$$u_t\in L^2(0,\infty;L^2(\Omega))\quad\text{and}\quad z_t\in L^2(0,\infty;H^1(\Omega)).$$

This means that there is a zero measure set $\mathcal{N}\subset(0,\infty)$ such that for any $t\in(0,\infty)\setminus\mathcal{N}$ we have

$$u_t(t,\cdot),\nabla z_t(t,\cdot),u(t,\cdot),z(t,\cdot),\nabla u(t,\cdot),\nabla z(t,\cdot),D^2z(t,\cdot)\in L^2(\Omega)$$

and, by Corollary 2.13, we have (2.31). Therefore each integral of the inequality (2.37) is well defined and (2.37) is satisfied for each $t\in(0,\infty)\setminus\mathcal{N}$.

We want to take to the limit as $j\to\infty$ in (2.37). We are going to do it term by term. Let $t\in(0,\infty)\setminus\mathcal{N}$ and let us first consider the term (2.31). We define the functions $f,F,f_j\in L^1(\Omega)$, for all $j\in\mathbb{N}$, by

$$f_j(x)=u_t(t,x)\frac{(a_m(u(t,x))+1/j)^{s-1}}{s-1},\qquad f(x)=u_t(t,x)\frac{a_m(u(t,x))^{s-1}}{s-1}$$

$$\text{and}\qquad F(x)=|f_1(x)|=|u_t(t,x)|\frac{(a_m(u(t,x))+1)^{s-1}}{s-1}.$$

Then, for almost every $x\in\Omega$, $f_j(x)\to f(x)$ as $j\to\infty$ with $|f_j(x)|\le F(x)$ for all $j\in\mathbb{N}$ and, by the Dominated Convergence Theorem, we conclude that $f_j\to f$ in $L^1(\Omega)$ as $j\to\infty$. This implies, in particular, that

$$\int_\Omega f_j\,dx\longrightarrow\int_\Omega f\,dx,\ \text{ as } j\to\infty. \qquad (2.38)$$

Therefore, using (2.31), (2.38) and then (2.30) we conclude that

$$\lim_{j\to\infty}\frac{d}{dt}\int_\Omega g_{m,j}(u(t,x))\;dx = \lim_{j\to\infty}\int_\Omega u_t(t,x)\frac{(a_m(u(t,x))+1/j)^{s-1}}{s-1}\;dx$$
$$=\int_\Omega u_t(t,x)\frac{a_m(u(t,x))^{s-1}}{s-1}\;dx = \frac{d}{dt}\int_\Omega g_m(u(t,x))\;dx,\text{ for each } t\in(0,\infty)\setminus\mathcal{N}.$$

We can follow this reasoning and take the limit as $j\to\infty$ in the other terms of the (2.37). Using the Dominated Convergence Theorem again we conclude that

$$\lim_{j\to\infty}\frac{1}{2}\int_\Omega\Big[a_m(u(t,x))^s-\frac{1}{2}(a_m(u(t,x))+1/j)^s\Big]|\nabla z(t,x)|^2\;dx = \frac{1}{4}\int_\Omega a_m(u(t,x))^s|\nabla z(t,x)|^2\;dx$$

and

$$\lim_{j\to\infty} s\int_\Omega\Big[a_m(u(t,x))^{s-1}-(a_m(u(t,x))+1/j)^{s-1}\Big]\nabla a_m(u(t,x))\cdot z(t,x)\nabla z(t,x)\;dx=0,$$

for each $t\in(0,\infty)\setminus\mathcal{N}$.

Then, since the limit preserves inequalities, after we take the limit as $j\to\infty$ in (2.37), we obtain

$$\frac{d}{dt}\Big[\frac{s}{4}\int_\Omega g_m(u(t,x))\;dx+\frac{1}{2}\|\nabla z(t,x)\|^2_{L^2(\Omega)}\Big]+\frac{1}{4}\int_\Omega a_m(u(t,x))^s|\nabla v(t,x)|^2\;dx$$
$$+C_1\Big(\int_\Omega|D^2z(t,x)|^2\;dx+\int_\Omega\frac{|\nabla z(t,x)|^4}{z(t,x)^2}\;dx\Big)\le C\|\nabla z\|^2_{L^2(\Omega)}$$

for all $t\in(0,\infty)\setminus\mathcal{N}$, which means that the inequality is valid for almost every $t\in(0,\infty)$. Therefore (2.35) holds. ■

Lemma 2.16 (Energy inequality for $s\ge 2$). *The solution (u,z) of the problem (2.24) satisfies, for sufficiently small $\alpha>0$,*

$$\frac{d}{dt}\Big[\frac{s}{4}\int_\Omega g_m(u)\;dx+\frac{1}{2}\|\nabla z\|^2_{L^2(\Omega)}\Big]+\int_\Omega|\nabla[a_m(u)]^{s/2}|^2dx+\frac{1}{4}\int_\Omega a_m(u)^s|\nabla z|^2\;dx$$
$$+C_1\Big(\int_\Omega|D^2z|^2\;dx+\int_\Omega\frac{|\nabla z|^4}{z^2}\;dx\Big)\le C\int_\Omega|\nabla z|^2\;dx, \tag{2.39}$$

where $g_m(u)$ is given by (2.25).

Proof. We test the u_m-equation of (2.24) by

$$g_m'(u)=\frac{(a_m(u))^{s-1}}{(s-1)}$$

and obtain

$$\frac{d}{dt}\int_\Omega g_m(u)\;dx+\int_\Omega(a_m(u))^{s-2}a_m'(u)|\nabla u|^2\;dx=\big(a_m(u)(a_m(u))^{s-2}\nabla(z^2),\nabla a_m(u)\big),$$

where

$$g_m(r) = \int_0^r \frac{(a_m(\theta))^{s-1}}{(s-1)}\ d\theta$$

is a primitive of $(a_m(r))^{s-1}/(s-1)$. Since $0 \leq a_m'(u) \leq C$, we have $(a_m'(u))^2 \leq Ca_m'(u)$, we can write

$$\int_\Omega a_m'(u)(a_m(u))^{s-2}|\nabla u|^2\ dx \geq C \int_\Omega (a_m'(u))^2(a_m(u))^{s-2}|\nabla u|^2\ dx \geq C \int_\Omega |\nabla(a_m(u))^{s/2}|^2\ dx.$$

Then we obtain

$$\frac{d}{dt}\int_\Omega g_m(u)\ dx + C\int_\Omega |\nabla[a_m(u)]^{s/2}|^2\ dx \leq \big(a_m(u)^{s-1}\nabla(z^2), \nabla a_m(u)\big). \tag{2.40}$$

If we add $s/4$ times (2.40) to the inequality of Lemma 2.11 then the term

$$\frac{s}{4}\int_\Omega a_m(u)^{s-1}\nabla a_m(u)\cdot\nabla(z^2)\ dx,$$

which appears in $s/4$ times (2.40) cancels with the term

$$-\frac{s}{4}\int_\Omega a_m(u)^{s-1}\nabla a_m(u)\cdot\nabla(z^2)\ dx,$$

which comes from the inequality Lemma 2.11 and we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{s}{4}\int_\Omega g_m(u)dx + \frac{1}{2}\|\nabla z\|^2_{L^2(\Omega)}\Big] + C\int_\Omega |\nabla[a_m(u)]^{s/2}|^2 dx + \frac{1}{2}\int_\Omega a_m(u)^s|\nabla z|^2\ dx \\
&+ C_1\Big(\int_\Omega |D^2 z|^2\ dx + \frac{1}{2}\int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) \leq \frac{s}{2}\sqrt{\alpha}\int_\Omega a_m(u)^{s-1}|\nabla z||\nabla a_m(u)|\ dx \\
&\leq \int_\Omega \sqrt{\alpha}|\nabla[a_m(u)]^{s/2}||a_m(u)^{s/2}||\nabla z|\ dx.
\end{aligned}$$

Next, we deal with the second term in the right hand side of the previous inequality using Hölder's and Young's inequality,

$$\begin{aligned}
&\int_\Omega \sqrt{\alpha}|\nabla[a_m(u)]^{s/2}||a_m(u)^{s/2}||\nabla z|\ dx \\
&\leq \alpha\ C(\delta)\int_\Omega a_m(u)^s|\nabla z|^2\ dx + \delta\|\nabla[a_m(u)]^{s/2}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore, choosing $\alpha, \delta > 0$ sufficiently small we finally obtain the desired inequality (2.39). ∎

The energy inequalities (2.32), (2.35) and (2.39) allow us to obtain m-independent estimates for the sequence $(v_m)_m$ that are valid up to infinity time in the next Subsection.

Remark 2.17. In the next subsection, the aforementioned m-independent estimates will be obtained upon integration of the energy inequalities (2.32), (2.35) and (2.39) with respect to the time variable. Therefore we find it appropriate to remark that, for each $T > 0$, we have $\int_\Omega g_m(u)\ dx \in L^2(0,T)$ and $\frac{d}{dt}\int_\Omega g_m(u)\ dx \in L^2(0,T)$ and it implies, in particular, that

$$\int_0^T \frac{d}{dt}\int_\Omega g_m(u(t,x))\ dx\ dt = \int_\Omega g_m(u(T,x))\ dx - \int_\Omega g_m(u(0,x))\ dx.$$

See [6]. □

2.3.3 m-independent estimates and passage to the limit as $m \to \infty$

Now we use again the notation (u_m, v_m) for the solution of the regularized problem (2.1), $z_m = \sqrt{v_m + \alpha}$ and (u, v) for the solution of the original problem (1). In this subsection we are going to obtain m-independent estimates for (u_m, v_m) that will allow us to pass to the limit in the problem (2.1) as $m \to \infty$ and prove the existence of solution to the original problem (1).

First, we obtain some m-independent bounds for ∇v_m that can be extracted from the energy inequalities (2.32), (2.35) and (2.39). Next, we prove m-independent bounds for (u_m, v_m) and pass to the limit in (2.1) as $m \to \infty$, considering the case $s \in [1, 2)$ and $s \geq 2$, respectively.

m-independent estimates for ∇v_m

Let us remind that, for $s \geq 1$, we have defined g'_m as

$$g'_m(u) = \begin{cases} ln(a_m(u) + 1) & \text{if } s = 1, \\ a_m(u)^{s-1}/(s-1) & \text{if } s > 1. \end{cases}$$

And let us define the energy

$$E_m(u_m, z_m)(t) = \frac{s}{4}\int_\Omega g_m(u_m(t,x))\ dx + \frac{1}{2}\int_\Omega |\nabla z_m(t,x)|^2\ dx. \tag{2.41}$$

We remark that, since $0 \leq v_m(t,x) \leq \|v^0\|_{L^\infty(\Omega)}$ *a.e.* $(t,x) \in (0,\infty) \times \Omega$, we have

$$0 < \sqrt{\alpha} \leq z_m(t,x) \leq \sqrt{\|v^0\|_{L^\infty(\Omega)} + \alpha}\ \ a.e.\ (t,x) \in (0,\infty) \times \Omega$$

and, by straightforward calculations, we can prove the following lemma.

Lemma 2.18. *There are $\beta_1, \beta_2 > 0$, depending on α, such that*

$$\beta_1|\nabla z_m(t,x)| \leq |\nabla v_m(t,x)| \leq \beta_2|\nabla z_m(t,x)| \tag{2.42}$$

and

$$\beta_1\Big(|\Delta z_m(t,x)| + |\nabla z_m(t,x)|^2\Big) \leq |\Delta v_m(t,x)| \leq \beta_2\Big(|\Delta z_m(t,x)| + |\nabla z_m(t,x)|^2\Big), \tag{2.43}$$

a.e. $(t,x) \in (0,\infty) \times \Omega$.

We will integrate the energy inequalities (2.32), (2.35) and (2.39) with respect to t, from 0 to some $T > 0$. We take into account that $\int_0^T \|\nabla z_m(t)\|^2_{L^2(\Omega)}\, dt$ is bounded, independently of T and m, because of (2.42) and (2.14), and we use the m-uniform bounds which stem from (2.4) and (2.5) on the initial data u_m^0 and v_m^0 in order to conclude that the energy given in (2.41) in time $t = 0$, $E_m(u_m, z_m)(0)$, is also bounded, independently of m. Thus we can conclude that

$$\nabla z_m \text{ is bounded in } L^\infty(0,\infty; L^2(\Omega)) \cap L^4(0,\infty; L^4(\Omega)),$$

$$a_m(u_m)^{s/2}\nabla z_m \text{ and } \Delta z_m \text{ are bounded in } L^2(0,\infty; L^2(\Omega)).$$

But using the fact that $z_m = \sqrt{v_m + \alpha}$, (2.42) and (2.43) we can conclude that

$$\nabla v_m \text{ is bounded in } L^\infty(0,\infty; L^2(\Omega)) \cap L^4(0,\infty; L^4(\Omega)), \tag{2.44}$$

$$a_m(u_m)^{s/2}\nabla v_m \text{ and } \Delta v_m \text{ are bounded in } L^2(0,\infty; L^2(\Omega)). \tag{2.45}$$

In particular, since $v_m(t) \in H^2(\Omega)$, for each $t \in (0,\infty)$, and $\frac{\partial}{\partial \eta} v_m|_\Gamma = 0$, it stems from (2.45), the H^2-regularity of the Poisson-Neumann problem (1.5) and (1.6) that

$$\nabla v_m \text{ is bounded in } L^2(0,\infty; H^1(\Omega)). \tag{2.46}$$

Using the results obtained until this point we analyze the existence of solutions of (1), first for $s \in [1,2)$ and then for $s \geq 2$.

m-independent estimates for (u_m, v_m) and passage to the limit for $s \in [1,2)$

Let

$$\forall r > 0, \qquad g'(r) = \begin{cases} ln(r) & \text{if } s = 1, \\ r^{s-1}/(s-1) & \text{if } s \in (1,2) \end{cases}$$

and let

$$g(r) = \int_0^r g'(\theta)\, d\theta = \begin{cases} rln(r) - r & \text{if } s = 1, \\ r^s/s(s-1) & \text{if } s \in (1,2). \end{cases}$$

Notice that $g''(r) = r^{s-2}$, $\forall r > 0$, in all cases.

We test the u_m-equation of (2.1) by $g'(u_m + 1)$ and obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega g(u_m+1)\, dx + \frac{4}{s^2}\int_\Omega |\nabla[u_m+1]^{s/2}|^2\, dx \\
&= \int_\Omega a_m(u_m)(u_m+1)^{s/2-1}\nabla v_m \cdot \nabla u_m\, (u_m+1)^{s/2-1}\, dx \\
&= \frac{2}{s}\int_\Omega \frac{a_m(u_m)^{1-s/2}}{(u_m+1)^{1-s/2}} a_m(u_m)^{s/2}\nabla v_m \cdot \nabla[u_m+1]^{s/2}\, dx \\
&\leq \frac{2}{s}\Big(\int_\Omega a_m(u_m)^s|\nabla v_m|^2\, dx\Big)^{1/2}\Big(\int_\Omega |\nabla[u_m+1]^{s/2}|^2\, dx\Big)^{1/2}
\end{aligned}$$

and thus we have

$$\frac{d}{dt}\int_\Omega g(u_m+1)\ dx + \frac{2}{s^2}\int_\Omega |\nabla[u_m+1]^{s/2}|^2\ dx \le \frac{1}{4}\int_\Omega a_m(u_m)^s|\nabla v_m|^2\ dx.$$

Integrating with respect to t from 0 to T, for any fixed $T\in(0,\infty)$, we obtain

$$\begin{aligned}&\int_\Omega g(u_m(T)+1)\ dx + \frac{2}{s^2}\int_0^T\int_\Omega |\nabla[u_m+1]^{s/2}|^2\ dx\ dt\\ &\le \frac{1}{4}\int_0^T\int_\Omega a_m(u_m)^s|\nabla v_m|^2\ dx\ dt + \int_\Omega g(u_m^0+1)\ dx.\end{aligned}$$

Then, because of Lemma 2.8.1, (2.4) and the definition of g and (2.45) we conclude that

$$(u_m+1)^{s/2} \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)), \tag{2.47}$$

in particular,

$$u_m \text{ is bounded in } L^\infty(0,\infty;L^s(\Omega)), \tag{2.48}$$

and

$$\nabla[u_m+1]^{s/2} \text{ is bounded in } L^2(0,\infty;L^2(\Omega)). \tag{2.49}$$

Consider the relation

$$\nabla u_m = \nabla(u_m+1) = \nabla\big((u_m+1)^{s/2}\big)^{2/s} = \frac{2}{s}(u_m+1)^{1-s/2}\ \nabla(u_m+1)^{s/2}. \tag{2.50}$$

Taking into account that we are considering $s\in[1,2)$, we can use (2.47) to obtain

$$(u_m+1)^{1-s/2} \text{ is bounded in } L^\infty(0,\infty;L^{2s/(2-s)}(\Omega))$$

and then (2.49) and (2.50) to conclude that

$$\nabla u_m \text{ is bounded in } L^2(0,\infty;L^s(\Omega)). \tag{2.51}$$

In conclusion, using (2.48), (2.51) and the Poincaré inequality for zero mean functions (Lemma 1.3),

$$u_m - u^* \text{ is bounded in } L^\infty(0,\infty;L^s(\Omega))\cap L^2(0,\infty;W^{1,s}(\Omega)). \tag{2.52}$$

Considering the chemotaxis term of the u_m-equation of (2.1), we can write $a_m(u_m)\nabla v_m$ as

$$a_m(u_m)\nabla v_m = a_m(u_m)^{1-s/2}a_m(u_m)^{s/2}\nabla v_m.$$

Then, we have $a_m(u_m)^{1-s/2}$ bounded in $L^\infty(0,\infty;L^{2s/(2-s)}(\Omega))$, because of (2.47), and $a_m(u_m)^{s/2}\nabla v_m$ bounded in $L^2(0,\infty;L^2(\Omega))$, because of (2.45), and hence we can conclude that

$$a_m(u_m)\nabla v_m \text{ is bounded in } L^2(0,\infty;L^s(\Omega)). \tag{2.53}$$

Then, if we consider the u_m-equation of (2.1), from (2.52) and (2.53) we conclude that

$$\partial_t u_m \text{ is bounded in } L^2\Big(0,\infty;\big(W^{1,s/(s-1)}(\Omega)\big)'\Big).$$

Now we turn to the v_m-equation, rewritten as

$$\partial_t v_m - \Delta v_m + a_m(u^*)^s v_m = -(a_m(u_m)^s - a_m(u^*)^s)v_m. \tag{2.54}$$

Analyzing the term on the right hand side of (2.54), we have

$$a_m(u_m)^s - a_m(u^*)^s \text{ is bounded in } L^2(0,\infty;L^{3/2}(\Omega)). \tag{2.55}$$

In fact, using Lemma 1.10, we obtain

$$|a_m(u_m)^s - a_m(u^*)^s| \le s|a_m(u_m) + a_m(u^*)|^{s-1}|u_m - u^*|.$$

Then, considering the m-uniform bound (2.52) and the Sobolev embedding $L^{3s/(3-s)}(\Omega)) \subset W^{1,s}(\Omega)$ we obtain (2.55).

With this information, now we can test (2.54) by v_m, obtaining

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_m\|^2_{L^2(\Omega)} + \|\nabla v_m\|^2_{L^2(\Omega)} + \frac{a_m(u^*)^s}{2}\|v_m\|^2_{L^2(\Omega)} &\le C\int_\Omega |a_m(u_m)^s - a_m(u^*)^s| v_m^2\,dx\\ &\le C\|v_m^0\|_{L^\infty(\Omega)}\|a_m(u_m)^s - a_m(u^*)^s\|_{L^{3/2}(\Omega)}\|v_m\|_{L^3(\Omega)}\\ &\le C(\delta)\|a_m(u_m)^s - a_m(u^*)^s\|^2_{L^{3/2}(\Omega)} + \delta\|v_m\|^2_{L^2(\Omega)} + \delta\|\nabla v_m\|^2_{L^2(\Omega)}.\end{aligned}$$

Note that if $u_0 \not\equiv 0$ then have $a_m(u^*) = u^* > 0$ all $m \ge u^*$. Hence, choosing $\delta > 0$ small enough, we can conclude that, for $m \ge u^*$, there is $\beta > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\|v_m\|^2_{L^2(\Omega)} + \beta\|\nabla v_m\|^2_{L^2(\Omega)} + \beta\|v_m\|^2_{L^2(\Omega)} \le C\|a_m(u_m)^s - a_m(u^*)^s\|^2_{L^{3/2}(\Omega)}.$$

Therefore, integrating with respect to t and using (2.55) we obtain

$$v_m \text{ is bounded in } L^2(0,\infty;L^2(\Omega)). \tag{2.56}$$

Hence, in view of (2.44), (2.45) and (2.56) we have

$$v_m \text{ is bounded in } L^2(0,\infty;H^2(\Omega)).$$

With the m-uniform bounds obtained so far we can obtain a m-uniform bound for the function $\partial_t v_m$ in $L^2(0,\infty;L^{3/2}(\Omega))$. In fact, going back to (2.54), reminding that v_m is uniformly bounded in $L^\infty(0,\infty;L^\infty(\Omega))$ with respect to m and considering (2.56) and (2.55), we conclude that

$$\partial_t v_m \text{ is bounded in } L^2(0,\infty;L^{3/2}(\Omega)).$$

Now we are going to obtain compactness for $\{u_m\}$ which are necessary in order to pass to the limit as $m \to \infty$ in the nonlinear terms of the equations of (2.1). Because of (2.47) and (2.49), we have that

$$(u_m+1)^{s/2} \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \text{ for every finite } T>0.$$

Using the Sobolev inequality $H^1(\Omega) \subset L^6(\Omega)$ and interpolation inequalities we obtain

$$(u_m)^{s/2} \text{ is bounded in } L^{10/3}(0,T;L^{10/3}(\Omega)),$$

which is equivalent to

$$u_m \text{ is bounded in } L^{5s/3}(0,T;L^{5s/3}(\Omega)). \tag{2.57}$$

By using (2.47) and (2.57) in (2.50) (remind that $s \in [1,2)$), we also have

$$u_m \text{ is bounded in } L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)).$$

We observe that $W^{1,5s/(3+s)}(\Omega) \subset L^q(\Omega)$, with continuous embedding for $q = 15s/(9-2s)$ and compact embedding for $q \in [1, 15s/(9-2s))$. Then, since $s \in [1,2)$, we have $5s/3 < 15s/(9-2s)$ and therefore the embedding $W^{1,5s/(3+s)}(\Omega) \subset L^{5s/3}(\Omega)$ is compact. Note also that $q = 5s/3 \geq 5/3 > 1$.

Now we can use Lemma 1.14 with

$$X = W^{1,5s/(3+s)}(\Omega), \qquad B = L^{5s/3}(\Omega), \qquad Y = \big(H^3(\Omega)\big)'$$

and $q = 5s/3$, to conclude that there is a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and a limit function u such that

$$u_m \longrightarrow u \text{ weakly in } L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)),\ \forall T>0,$$

and

$$u_m \longrightarrow u \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p \in [1,5s/3),\ \forall T>0. \tag{2.58}$$

Using the Dominated Convergence Theorem we can conclude from (2.58) that

$$a_m(u_m) \to u \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p \in (1,5s/3),\ \forall T>0. \tag{2.59}$$

It stems from the convergence (2.59) and Lemma 1.11 that

$$(a_m(u_m))^s \to u^s \text{ strongly in } L^q(0,T;L^q(\Omega)),\ \forall q \in (1,5/3),\ \forall T>0. \tag{2.60}$$

The convergence of v_m is better. There is a subsequence of $\{v_m\}$ (still denoted by $\{v_m\}$) and a limit function v such that

$$\begin{gathered}
v_m \to v \text{ weakly* in } L^\infty((0,\infty)\times\Omega)\cap L^\infty(0,\infty;H^1(\Omega)),\\
v_m \to v \text{ weakly in } L^2(0,\infty;H^2(\Omega)),\\
\nabla v_m \to \nabla v \text{ weakly in } L^4(0,\infty;L^4(\Omega)),\\
\text{and } \partial_t v_m \to \partial_t v \text{ weakly in } L^2(0,\infty;L^{3/2}(\Omega)).
\end{gathered} \tag{2.61}$$

Now we are going to use the weak and strong convergences obtained so far to pass to the limit as $m\to\infty$ in the equations of problem (2.1). We are going to identify the limits of the nonlinear terms related to chemotaxis and consumption,

$$a_m(u_m)\nabla v_m \text{ and } a_m(u_m)^s v_m,$$

respectively, with

$$u\nabla v \text{ and } u^s v.$$

In fact, considering the chemotaxis term, because of (2.59), (2.44) and (2.61), we can conclude that

$$a_m(u_m)\nabla v_m \longrightarrow u\nabla v \text{ weakly in } L^{20s/(5s+12)}(0,T;L^{20s/(5s+12)}(\Omega)),\ \forall T>0.$$

Considering now the consumption term, considering (2.60) and (2.61) we conclude that

$$a_m(u_m)^s v_m \longrightarrow u^s v \text{ weakly in } L^{5/3}(0,T;L^{5/3}(\Omega)),\ \forall T>0.$$

With these identifications and all previous convergences, it is possible to pass to the limit as $m\to\infty$ in each term of the equations of (2.1). In order to finish the proof of Theorem 2.2 we must obtain the regularity (up to infinite time) which is claimed for u.

From (2.47) and (2.49) there exists a subsequence of $\{(u_m+1)^{s/2}\}$, still denoted by $\{(u_m+1)^{s/2}\}$, and a limit function φ such that

$$\begin{gathered}
(u_m+1)^{s/2} \longrightarrow \varphi \ \text{ weakly* in } L^\infty(0,\infty;L^2(\Omega))\\
\nabla(u_m+1)^{s/2} \longrightarrow \nabla\varphi \ \text{ weakly in } L^2(0,\infty;L^2(\Omega)).
\end{gathered}$$

Then, using the strong convergence (2.58), the continuity of the function $u_m \mapsto f(u_m)=(u_m+1)^{s/2}$ and the Dominated Convergence Theorem, we prove that

$$\varphi=(u+1)^{s/2}.$$

Analogously, because of (2.45) we can conclude that, up to a subsequence, there is a limit function ϕ such that

$$a_m(u_m)^{s/2}\nabla v_m \longrightarrow \phi \text{ weakly in } L^2(0,\infty;L^2(\Omega)).$$

And using the convergences (2.60) and (2.61) we can conclude that

$$\phi = u^{s/2}\nabla v.$$

Therefore we have proved the global in time regularity

$$\begin{gathered}(u+1)^{s/2} \in L^\infty(0,\infty;L^2(\Omega)), \quad \nabla(u+1)^{s/2} \in L^2(0,\infty;L^2(\Omega)),\\ u^{s/2}\nabla v \in L^2(0,\infty;L^2(\Omega)).\end{gathered} \tag{2.62}$$

Considering (2.62) and proceeding as in the obtaining of (2.52) and (2.53) we conclude the global in time regularity

$$u \in L^\infty(0,\infty;L^s(\Omega)), \quad \nabla u,\ u\nabla v \in L^2(0,\infty;L^s(\Omega)),$$

finishing the proof of Theorem 2.2 in the case $s \in [1,2)$.

m-independent estimates for (u_m, v_m) and passage to the limit for $s \geq 2$

The procedure for the case $s \geq 2$ is slightly different. First we note that, integrating the energy inequality (2.39) from Lemma 2.16 with respect to t, we have

$$\nabla a_m(u_m)^{s/2} \text{ is bounded in } L^2(0,\infty;L^2(\Omega)). \tag{2.63}$$

We also remind that we defined $g_m'(r) = a_m(r)^{s-1}/(s-1)$, for $s \geq 2$. Then we have

$$a_m(r)^s = s\int_0^r a_m'(\theta)a_m(\theta)^{s-1}\,d\theta \leq Cs\int_0^r a_m(\theta)^{s-1}\,d\theta = Cs(s-1)g_m(r).$$

Therefore it also stems from integrating the energy inequality (2.39) with respect to t that

$$a_m(u_m)^{s/2} \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)). \tag{2.64}$$

From (2.64) and (2.63) we can conclude that

$$a_m(u_m)^{s/2} \text{ is bounded in } L^{10/3}(0,T;L^{10/3}(\Omega)),$$

that is,

$$a_m(u_m) \text{ is bounded in } L^{5s/3}(0,T;L^{5s/3}(\Omega)). \tag{2.65}$$

For each fixed $m \in \mathbb{N}$, consider the zero measure set $\mathcal{N} \subset (0,\infty)$ such that

$$u_m(t^*,\cdot), v_m(t^*,\cdot) \in H^1(\Omega),\ \forall t^* \in (0,\infty)\setminus\mathcal{N}.$$

Then, for each fixed $t^* \in (0,\infty)\setminus\mathcal{N}$, let us consider the sets

$$\{0 \leq u_m \leq 1\} = \Big\{x \in \Omega \;\Big|\; 0 \leq u_m(t^*,x) \leq 1\Big\}$$

and

$$\{u_m \geq 1\} = \Big\{x \in \Omega \;\Big|\; u_m(t^*, x) \geq 1\Big\}.$$

Now note that, since $s \geq 2$, we have

$$\begin{aligned}
&\int_\Omega a_m(u_m(t^*, x)^2 |\nabla v_m(t^*, x)|^2 \, dx \\
&\leq \int_{\{0 \leq u_m \leq 1\}} |\nabla v_m(t^*, x)|^2 \, dx + \int_{\{u_m \geq 1\}} a_m(u_m(t^*, x))^s |\nabla v_m(t^*, x)|^2 \, dx \\
&\leq \int_\Omega |\nabla v_m(t^*, x)|^2 \, dx + \int_\Omega a_m(u_m(t^*, x))^s |\nabla v_m(t^*, x)|^2 \, dx.
\end{aligned}$$

The last inequality is valid for all $t^* \in (0, \infty) \setminus \mathcal{N}$, then if we integrate in the variable t we obtain

$$\begin{aligned}
\int_0^\infty \int_\Omega a_m(u_m(t, x))^2 |\nabla v_m(t, x)|^2 \, dx \, dt \leq & \int_0^\infty \int_\Omega |\nabla v_m(t, x)|^2 \, dx \, dt \\
& + \int_0^\infty \int_\Omega a_m(u_m(t, x))^s |\nabla v_m(t, x)|^2 \, dx \, dt.
\end{aligned}$$

Therefore by (2.14) and (2.45) we can conclude that

$$a_m(u_m)\nabla v_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \tag{2.66}$$

Now we test the u_m-equation of problem (2.1) by u_m. This gives us

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)} &= \int_\Omega a_m(u_m)\nabla v_m \cdot \nabla u_m \, dx \\
&\leq \frac{1}{2}\int_\Omega a_m(u_m)^2 |\nabla v_m|^2 \, dx + \frac{1}{2}\|\nabla u_m\|^2_{L^2(\Omega)},
\end{aligned}$$

hence we have

$$\frac{d}{dt}\|u_m\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)} \leq \int_\Omega a_m(u_m)^2 |\nabla v_m|^2 \, dx.$$

Integrating with respect to t, we conclude from (2.66) that

$$u_m \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)) \tag{2.67}$$

and

$$\nabla u_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \tag{2.68}$$

Then, if we consider the u_m-equation of (2.1), by applying (2.68) and (2.66) we conclude that

$$\partial_t u_m \text{ is bounded in } L^2(0, \infty; (H^1(\Omega))'). \tag{2.69}$$

Let $(a_m(u_m)^s)^* = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega a_m(u_m)^s \, dx$, from (2.64) and (2.63), we can also conclude that

$$\nabla a_m(u_m)^s \text{ is bounded in } L^2(0, \infty; L^1(\Omega)).$$

In view of Lemma 1.3, the latter implies

$$a_m(u_m)^s - (a_m(u_m)^s)^* \text{ is bounded in } L^2(0,\infty; W^{1,1}(\Omega))$$

and, in particular, by the Sobolev embedding, we have

$$a_m(u_m)^s - (a_m(u_m)^s)^* \text{ is bounded in } L^2(0,\infty; L^{3/2}(\Omega)). \tag{2.70}$$

Now we consider the v_m-equation of (2.1) written as

$$\partial_t v_m - \Delta v_m + (a_m(u_m)^s)^* v_m = -(a_m(u_m)^s - (a_m(u_m)^s)^*)v_m. \tag{2.71}$$

Testing (2.71) by v_m and using Hölder's inequality we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_m\|^2_{L^2(\Omega)} + \|\nabla v_m\|^2_{L^2(\Omega)} + (a_m(u_m)^s)^*\|v_m\|^2_{L^2(\Omega)} \\
&\le C\|v_m^0\|_{L^\infty(\Omega)}\|a_m(u_m)^s - (a_m(u_m)^s)^*\|_{L^{3/2}(\Omega)}\|v_m\|_{L^3(\Omega)} \\
&\le C\|a_m(u_m)^s - (a_m(u_m)^s)^*\|^2_{L^{3/2}(\Omega)} + \delta\|v_m\|^2_{L^2(\Omega)} + \delta\|\nabla v_m\|^2_{L^2(\Omega)}.
\end{aligned}$$

In order to bound $(a_m(u_m)^s)^*$ from below, we will apply Lemma 1.15. Indeed,

$$(a_m(u_m)^s)^* \ge C\Big(\int_\Omega a_m(u_m)\,dx\Big)^s$$

and applying Lemma 1.15 (with $w_m = u_m$, $p = 2$, and using (2.67)) we conclude that there exist $\beta > 0$ and m_0 large enough such that $(a_m(u_m)^s)^* \ge \beta > 0$, *a.e.* $t \in (,\infty)$, for all $m \ge m_0$. Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_m\|^2_{L^2(\Omega)} + (1-\delta)\|\nabla v_m\|^2_{L^2(\Omega)} + (\beta-\delta)\|v_m\|^2_{L^2(\Omega)} \\
&\le C\|a_m(u_m)^s - (a_m(u_m)^s)^*\|^2_{L^{3/2}(\Omega)}.
\end{aligned}$$

Now, choosing δ small enough, integrating the last inequality with respect to t and using (2.70) we obtain

$$v_m \text{ is bounded in } L^2(0,\infty; L^2(\Omega)). \tag{2.72}$$

With the m-independent *a priori* bounds obtained so far we can also give an m-independent *a priori* bound for $\partial_t v_m$. In fact, if we consider again the equation (2.71), then the m-independent estimate in the L^∞-norm for v_m given by Lemma 2.7-2 and the m-independent *a priori* bounds (2.72), (2.70) and (2.45) allow us to conclude that

$$\partial_t v_m \text{ is bounded in } L^2(0,\infty; L^{3/2}(\Omega)).$$

Now, using (2.67), (2.68) and (2.69) we can conclude that there is a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, and a limit function u such that

$$\begin{aligned} u_m &\longrightarrow u \text{ weakly* in } L^\infty(0,\infty;L^2(\Omega)),\\ \nabla u_m &\longrightarrow \nabla u \text{ weakly in } L^2(0,\infty;L^2(\Omega)),\\ \partial_t u_m &\longrightarrow u \text{ weakly in } L^2\Big(0,\infty;\big(H^1(\Omega)\big)'\Big). \end{aligned}$$

By applying the compactness result Lemma 1.14, one has

$$u_m \longrightarrow u \text{ strongly in } L^2(0,T;L^2(\Omega)),\ \forall T>0.$$

Using the Dominated Convergence Theorem and (2.65) we can also prove that

$$a_m(u_m) \longrightarrow u \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p\in(1,5s/3),$$

and using Lemma 1.11,

$$a_m(u_m)^s \longrightarrow u^s \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p\in(1,5/3).$$

From the global in time estimate (2.64) we can conclude that, up to a subsequence,

$$a_m(u_m)\to u \text{ weakly* in } L^\infty(0,\infty;L^s(\Omega)),$$

hence, in particular,

$$u\in L^\infty(0,\infty;L^s(\Omega)).$$

For $s\geq 2$, if we consider the functions v_m, we have the same m-independent estimates that we had for $s\in[1,2)$. Then we have the same convergences given in (2.61).

Following the ideas of Subsection 2.3.3, we can identify the limits of $a_m(u_m)\nabla v_m$ and $a_m(u_m)^s v_m$ with $u\nabla v$ and $u^s v$, respectively.

This finishes the proof of existence of solution to the original problem (1) as a limit of solutions of the regularized problems (2.1) for $s\geq 2$.

2.4 Regularity and Uniqueness in 2D

In this section, we show that, for two dimensional domains, we can improve the results on the uniqueness and regularity of the solution of (1). The key point is the inequality (1.1), which allows us to improve the *a priori* estimates of u_m and then of v_m, where (u_m,v_m) is the solution of (2.1).

2.4.1 Uniqueness in 2D

Theorem 2.19. *In the two dimensional case, we have uniqueness of solution in the class of functions* (u, v) *such that*

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^{4s-4}(0,T;L^{4s-4}(\Omega)), \tag{2.73}$$

$$u \in L^4(0,T;L^{4+\epsilon}(\Omega)) \quad \text{if } s = 2 \tag{2.74}$$

and

$$v \in L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)). \tag{2.75}$$

Remark 2.20. The regularities (2.73) and (2.75) imply in particular that

$$u_t \in L^2(0,T;(H^1(\Omega))')$$

and therefore, the solution u can be taken as test function in the u-equation of (1).

The regularity $u \in L^{4s-4}(0,T;L^{4s-4}(\Omega))$ is an additional hypothesis only if $s > 2$, because in 2D domains, the regularity $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ implies $u \in L^4(0,T;L^4(\Omega))$. □

Proof of Theorem 2.19. Suppose (u_1, v_1) and (u_2, v_2) are two solutions of the original problem (1) with the regularity given in (2.73)-(2.75). Define $(u, v) = (u_2 - u_1, v_2 - v_1)$. Then (u, v) satisfies

$$(u_t(t), \varphi) + (\nabla u(t), \nabla\varphi) = (u(t)\nabla v_2(t), \nabla\varphi) + (u_1(t)\nabla v, \nabla\varphi)\,,\ \forall\varphi \in H^1(\Omega),\ a.e\ t \in (0,T), \tag{2.76}$$

and

$$v_t(t) - \Delta v(t) = -[(u_2(t))^s - (u_1(t))^s]v_2(t) - (u_1(t))^s v(t),\ a.e\ t \in (0,T), \tag{2.77}$$

with $(u(0), v(0)) = (0, 0)$. Note that we can conclude from (2.76) that u is a zero mean function.

Now we test (2.76) by u. We obtain, first using the interpolation inequality from Lemma 1.1-2, for 2D domains, and the Poincaré inequality for zero mean functions from Lemma 1.3, and after Young's inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \le \|u\|_{L^4(\Omega)}\|\nabla v_2\|_{L^4(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\
&+\|u_1\|_{L^4(\Omega)}\|\nabla v\|_{L^4(\Omega)}\|\nabla u\|_{L^2(\Omega)} \le C\|u\|^{1/2}_{L^2(\Omega)}\|\nabla v_2\|_{L^4(\Omega)}\|\nabla u\|^{3/2}_{L^2(\Omega)} \\
&+C\|u_1\|_{L^4(\Omega)}\|\nabla v\|^{1/2}_{L^2(\Omega)}\|\nabla v\|^{1/2}_{H^1(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\
&\le C(\delta)\|\nabla v_2\|^4_{L^4(\Omega)}\|u\|^2_{L^2(\Omega)} + C(\delta)\|u_1\|^4_{L^4(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} \\
&+\delta\|\nabla u\|^2_{L^2(\Omega)} + \delta\|\nabla v\|^2_{H^1(\Omega)},
\end{aligned}$$

for each $\delta > 0$. Then, accounting for (1.6), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} &\le C(\delta)\|\nabla v_2\|^4_{L^4(\Omega)}\|u\|^2_{L^2(\Omega)} \\ &+C(\delta)\|u_1\|^4_{L^4(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \delta\|\nabla u\|^2_{L^2(\Omega)} + \delta C\|\Delta v\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.78}$$

Next we test (2.77) by $v - \Delta v$. Taking into account that $v_2 \in L^\infty(\Omega)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^1(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \int_\Omega u_1^s v^2\ dx \\ &\le \int_\Omega |(u_2)^s - (u_1)^s||v_2||v - \Delta v|\ dx + \int_\Omega |u_1^{s/2} v||\Delta v|\ dx \\ &\le C(\delta)\|((u_2)^s - (u_1)^s)\|^2_{L^2(\Omega)} + C(\delta)\|(u_1)^s v\|^2_{L^2(\Omega)} + \delta\|v\|^2_{L^2(\Omega)} + 2\delta\|\Delta v\|^2_{L^2(\Omega)}, \end{aligned}$$

for each $\delta > 0$. We must estimate the terms $\|((u_2)^s - (u_1)^s)\|^2_{L^2(\Omega)}$ and $\|(u_1)^s v\|^2_{L^2(\Omega)}$. For the first of these two terms we use

$$|u_2^s - u_1^s| \le s|\max\{u_1, u_2\}|^{s-1}|u_2 - u_1| \le s|u_1 + u_2|^{s-1}|u_2 - u_1|,$$

and, considering Lemma 1.3 applied to zero mean function u, we find

$$\begin{aligned} &\|[(u_2)^s - (u_1)^s]\|^2_{L^2(\Omega)} = \int_\Omega |(u_2)^s - (u_1)^s|^2\ dx \\ &\le s^2 \int_\Omega |u_2 + u_1|^{2s-2}|u_2 - u_1|^2\ dx \le s^2\|u_2 + u_1\|^{2s-2}_{L^{4s-4}(\Omega)}\|u\|^2_{L^4(\Omega)} \\ &\le \|u_2 + u_1\|^{2s-2}_{L^{4s-4}(\Omega)}\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} \le \|u_2 + u_1\|^{4s-4}_{L^{4s-4}(\Omega)}\|u\|^2_{L^2(\Omega)} + \delta\|\nabla u\|^2_{L^2(\Omega)}. \end{aligned}$$

For the second term we have, for any $\varepsilon > 0$,

$$\|(u_1)^s v\|^2_{L^2(\Omega)} = \int_\Omega (u_1)^{2s} v^2\ dx \le \|u_1\|^{2s}_{L^{2s+\varepsilon}(\Omega)}\|v\|^2_{L^{(2s+\varepsilon)/\varepsilon}(\Omega)} \le C(\varepsilon)\|u_1\|^{2s}_{L^{2s+\varepsilon}(\Omega)}\|v\|^2_{H^1(\Omega)}.$$

Using the estimates of these two terms we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2_{H^1(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} &\le C(\delta)\|u_2 + u_1\|^{4s-4}_{L^{4s-4}(\Omega)}\|u\|^2_{L^2(\Omega)} \\ +\delta\|\nabla u\|^2_{L^2(\Omega)} &+ C(\delta,\varepsilon)\|u_1\|^{2s}_{L^{2s+\varepsilon}(\Omega)}\|v\|^2_{H^1(\Omega)} + \delta\|v\|^2_{L^2(\Omega)} + \delta\|\Delta v\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.79}$$

If we sum up (2.78) and (2.79) and choose $\delta > 0$ small enough so that the terms that are multiplied by δ on the right hand side can be absorbed by the corresponding nonnegative terms on the left hand side, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{H^1(\Omega)}\Big) &\le C\|\nabla v_2\|^4_{L^4(\Omega)}\|u\|^2_{L^2(\Omega)} \\ &+C\|u_1\|^4_{L^4(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + C\|u_2 + u_1\|^{4s-4}_{L^{4s-4}(\Omega)}\|u\|^2_{L^2(\Omega)} \\ &+C(\varepsilon)\|u_1\|^{2s}_{L^{2s+\varepsilon}(\Omega)}\|v\|^2_{H^1(\Omega)} + C\|v\|^2_{L^2(\Omega)} \end{aligned}$$

Now, taking into account that

$$\|v\|^2_{L^2(\Omega)}, \|\nabla v\|^2_{L^2(\Omega)} \leq \|v\|^2_{H^1(\Omega)}$$

and grouping the common factors, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{H^1(\Omega)}\Big) \leq C(\|\nabla v_2\|^4_{L^4(\Omega)} + \|u_2 + u_1\|^{4s-4}_{L^{4s-4}(\Omega)})\|u\|^2_{L^2(\Omega)} \\ &+(C\|u_1\|^4_{L^4(\Omega)} + C(\varepsilon)\|u_1\|^{2s}_{L^{2s+\varepsilon}(\Omega)} + C)\|v\|^2_{H^1(\Omega)}.\end{aligned} \tag{2.80}$$

Finally, we recall from the regularity hypotheses that we have, in particular,

$$u_1, u_2, \nabla v_1, \nabla v_2 \in L^4(0,T;L^4(\Omega)) \text{ and } u_1, u_2 \in L^{4s-4}(0,T;L^{4s-4}(\Omega)).$$

Therefore, it suffices to verify that there exists $\varepsilon > 0$ small enough such that

$$u_1, u_2 \in L^{2s}(0,T;L^{2s+\varepsilon}(\Omega)). \tag{2.81}$$

For $s \in [1,2)$, since $u_1, u_2 \in L^4(0,T;L^4(\Omega))$, then in particular one has (2.81). For $s = 2$, (2.81) is in fact the hypothesis (2.74). And for $s > 2$, hypothesis $u_1, u_2 \in L^{4s-4}(0,T;L^{4s-4}(\Omega))$ implies in particular that there is a $\varepsilon > 0$ such that (2.81) holds. Therefore, recalling that $u(0) = v(0) = 0$, we are able to apply Gronwall's inequality (Lemma 1.16) to (2.80) and conclude that $u = v = 0$, that is, $u_1 = u_2$ and $v_1 = v_2$. ■

2.4.2 Proof of Theorem 2.5

In the two dimensional case, we can study the regularity of the solution (u,v) of (1) for all $s \geq 1$ at the same time. These solutions can be obtained as a limit of the regularized solutions (u_m, v_m) of (2.1) as $m \to \infty$, considering initial data $(u^0, v^0) \in H^2(\Omega) \times H^2(\Omega)$. In this case, it is not necessary to regularize the initial data, taking directly $(u^0_m, v^0_m) = (u^0, v^0)$.

In order to prove that the solution (u,v) of (1) provided by Theorem 2.2 is in fact more regular, it suffices to prove the corresponding extra m-independent estimates for (u_m, v_m) in the spaces given in Theorem 2.5.

We take u_m^p, for any $1 \leq p < \infty$, as a test function in the u_m-equation to obtain

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega u_m^{p+1}(x)\ dx + p\int_\Omega u_m^{p-1}(x)|\nabla u_m(x)|^2\ dx \\
&\quad = p\int_\Omega a_m(u_m(x))\nabla v(x)\cdot\nabla u_m(x)u_m^{p-1}(x)\ dx \\
&\quad \leq p\int_\Omega u_m^p(x)|\nabla v_m(x)||\nabla u_m(x)|\ dx \\
&\quad \leq p\int_\Omega u_m^{p/2+1/2}(x)|\nabla v_m(x)|u_m^{p/2-1/2}|\nabla u_m(x)|\ dx \\
&\quad \leq p\|u_m^{p/2+1/2}\|_{L^4(\Omega)}\|\nabla v_m\|_{L^4(\Omega)}\Big(\int_\Omega u_m^{p-1}(x)|\nabla u_m(x)|^2\ dx\Big)^{1/2} \\
&\quad \leq Cp\|u_m^{p/2+1/2}\|_{L^2(\Omega)}\|\nabla v_m\|_{L^4(\Omega)}\Big(\int_\Omega u_m^{p-1}(x)|\nabla u_m(x)|^2\ dx\Big)^{1/2} \\
&\quad\ + Cp\sqrt{p+1}\|u_m^{p/2+1/2}\|_{L^2(\Omega)}^{1/2}\|\nabla v_m\|_{L^4(\Omega)}\Big(\int_\Omega u_m^{p-1}(x)|\nabla u_m(x)|^2\ dx\Big)^{3/4}.
\end{aligned}$$

By using Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega u_m^{p+1}(x)\ dx + \frac{p}{2}\int_\Omega u_m^{p-1}(x)|\nabla u_m(x)|^2\ dx \\
&\leq Cp\|\nabla v_m\|_{L^4(\Omega)}^2\int_\Omega u_m^{p+1}(x)\ dx + Cp(p+1)^2\|\nabla v_m\|_{L^4(\Omega)}^4\int_\Omega u_m^{p+1}(x)\ dx.
\end{aligned}$$

By estimates (2.44) and (2.46) we have

$$\|\nabla v_m\|_{L^4(\Omega)}^2, \|\nabla v_m\|_{L^4(\Omega)}^4 \text{ are bounded in } L^1(0,\infty),$$

then, Gronwall's inequality (Lemma 1.16) leads us to

$$u_m \text{ is bounded in } L^\infty(0,\infty;L^{p+1}(\Omega)),\ \text{for } 1 \leq p < \infty, \tag{2.82}$$

$$u_m^{(p-1)/2}\nabla u_m \text{ is bounded in } L^2(0,\infty;L^2(\Omega)),\ \text{for } 1 \leq p < \infty. \tag{2.83}$$

Remark 2.21. The m-independent bounds in (2.82) and (2.83) depend exponentially on $p(p+1)^2$. □

Then we recall from (2.12) that

$$\partial_t v_m, \Delta v_m \in L^2(0,\infty;H^1(\Omega)).$$

Hence, the following system is satisfied a.e. $(t,x) \in (0,\infty)\times\Omega$:

$$\nabla(\partial_t v_m) - \nabla\Delta v_m = -s\ a_m(u_m)^{s-1}\nabla a_m(u_m)v_m - a_m(u_m)^s\nabla v_m. \tag{2.84}$$

Since all the terms in this system are in $L^2(0,\infty;L^2(\Omega))$, we can take the inner product with $-\nabla\Delta v_m \in L^2(0,\infty;L^2(\Omega))$ and integrate over Ω. Using integration by parts, Hölder's and Young's inequalities and the estimate of v_m in $L^\infty(0,\infty;L^\infty(\Omega))$, we obtain

$$\frac{d}{dt}\|\Delta v_m\|^2_{L^2(\Omega)} + \|\nabla\Delta v_m\|^2_{L^2(\Omega)} \\ \leq C\|a_m(u_m)^{s-1}\nabla a_m(u_m)\|^2_{L^2(\Omega)} + C\|a_m(u_m)\|^{2s}_{L^{4s}(\Omega)}\|\nabla v_m\|^2_{L^4(\Omega)}. \tag{2.85}$$

Then, integrating (2.85) with respect to t and using (2.83) with $p = 2s-1$, (2.82) with $p = 4s-1$ and (2.44) we conclude that

$$v_m \text{ is bounded in } L^\infty(0,\infty; H^2(\Omega)), \\ \Delta v_m \text{ is bounded in } L^2(0,\infty; H^1(\Omega)). \tag{2.86}$$

Now we can use (2.84) to write

$$\nabla\partial_t v_m = \nabla\Delta v_m - s\ a_m(u_m)^{s-1}\nabla a_m(u_m) v_m - a_m(u_m)^s\nabla v_m.$$

Then, using the estimate in the L^∞-norm for v_m given by Lemma 2.7-2, (2.82) with $p = 4s-1$, (2.83) $p = 2s-1$ and (2.86), we also conclude that

$$\partial_t v_m \text{ is bounded in } L^2(0,\infty; H^1(\Omega)). \tag{2.87}$$

Then, because of the regularity of the solutions (u_m, v_m), we can use $-\Delta u_m \in L^2(0,\infty; L^2(\Omega))$ as a test function in the u_m-equation of (2.1),

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2_{L^2(\Omega)} + \|\Delta u_m\|^2_{L^2(\Omega)} \\ = \int_\Omega a_m(u_m)\Delta v_m\Delta u_m\ dx + \int_\Omega \nabla a_m(u_m)\cdot\nabla v_m\Delta u_m\ dx \\ \leq \|u_m\|_{L^4(\Omega)}\|\Delta v_m\|_{L^4(\Omega)}\|\Delta u_m\|_{L^2(\Omega)} + \|\nabla u_m\|_{L^4(\Omega)}\|\nabla v_m\|_{L^4(\Omega)}\|\Delta u_m\|_{L^2(\Omega)} \\ \leq \|u_m\|_{L^4(\Omega)}\|\Delta v_m\|_{L^4(\Omega)}\|\Delta u_m\|_{L^2(\Omega)} + C\|\nabla u_m\|^{1/2}_{L^2(\Omega)}\|\nabla v_m\|_{L^4(\Omega)}\|\Delta u_m\|^{3/2}_{L^2(\Omega)}.$$

Using Young's inequality we get

$$\frac{d}{dt}\|\nabla u_m\|^2_{L^2(\Omega)} + \|\Delta u_m\|^2_{L^2(\Omega)} \\ \leq C\|u_m\|^2_{L^4(\Omega)}\|\Delta v_m\|^2_{L^4(\Omega)} + C\|\nabla v_m\|^4_{L^4(\Omega)}\|\nabla u_m\|^2_{L^2(\Omega)}. \tag{2.88}$$

Therefore, using (2.44), (2.82) for $p = 3$, (2.86) and Gronwall's inequality (Lemma 1.16) in (2.88) we conclude that

$$\nabla u_m \text{ is bounded in } L^\infty(0,\infty; L^2(\Omega)). \tag{2.89}$$

$$\Delta u_m \text{ is bounded in } L^2(0,\infty; L^2(\Omega)). \tag{2.90}$$

Considering the $u_m$equation of (2.1),

$$\partial_t u_m - \Delta u_m = -a_m(u_m)\Delta v_m - \nabla a_m(u_m)\nabla v_m$$

and estimates (2.82), (2.86), (2.89) and (2.90) we obtain

$$\partial_t u_m \text{ is bounded in } L^2(0,\infty;L^2(\Omega)), \tag{2.91}$$

We can use (2.82), (2.89), (2.90), (2.86), (2.91) and (2.87) and compactness results in the weak*, weak and strong topologies and the uniqueness of the limit problem (1) for functions satisfying (2.73)-(2.75) to conclude that there is a unique limit (u, v) satisfying (1) *a.e.* in $(0,\infty) \times \Omega$.

Finally, if now we suppose that Ω has the $W^{2,3}$-regularity then we can prove better m-independent estimates for u_m. We would like to test the u_m-equation of (2.1) by $\Delta^2 u_m$, but we do not have enough regularity about $\Delta^2 u_m$. Instead of it, we argue as in (2.84), first we take the gradient of the u_m-equation of (2.1) and after we test the resulting equation by $\nabla\Delta u_m$.

Before doing this, we recall that, in case the Poisson-Neumann problem (1.5) has the $W^{2,3}$-regularity, the solution (u_m, v_m) of (2.1) have the regularity (2.13). Hence, if we take the gradient in the u_m-equation of (2.1) we obtain

$$\begin{aligned} \nabla(\partial_t u_m) - \nabla\Delta u_m \quad &= -a_m(u_m)\nabla\Delta v_m - \nabla a_m(u_m)\Delta v_m \\ &\quad - D^2 v_m \nabla a_m(u_m) - D^2 a_m(u_m)\nabla v_m. \end{aligned} \tag{2.92}$$

Now we test (2.92) by $\nabla\Delta u_m$. Using the m-uniform bounds obtained so far we can conclude that

$$\Delta u_m \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)) \cap L^2(0,\infty;H^1(\Omega)).$$

Finally, if we look at (2.92) again we can also conclude that

$$\partial_t u_m \text{ is bounded in } L^2(0,\infty;H^1(\Omega)).$$

This finishes the proof of Theorem 2.5.

Chapter 3

CONVERGENCE OF A TIME DISCRETE SCHEME FOR CHEMOTAXIS-CONSUMPTION MODELS

3.1 Main Results

As it was mentioned in the introduction, the design of the time discrete scheme is based on the analysis that was carried out in Chapter 2. In Chapter 2, it was convenient to rewrite (1) in terms of the variable $z = \sqrt{v+\alpha^2}$, because the test functions involved in obtaining a discrete energy law become simpler. Hence, in the present work we consider the following reformulation of (1)

$$\begin{cases} \partial_t u - \Delta u = -\nabla\cdot(u\nabla(z)^2), \quad \partial_t z - \dfrac{|\nabla z|^2}{z} - \Delta z = -\dfrac{1}{2}u^s\left(z - \dfrac{\alpha^2}{z}\right), \\ \partial_{\mathbf{n}} u|_\Gamma = \partial_{\mathbf{n}} z|_\Gamma = 0, \quad u(0)=u^0, \quad z(0)=\sqrt{v^0+\alpha^2}, \end{cases} \tag{3.1}$$

where $\alpha > 0$ is a fixed real number to be chosen later in Lemmas 3.7 and 3.8. Since it is proved in Chapter 2 that the v-equation of (1) is satisfied in the strong sense, with $v \in L^2(0,\infty;H^2(\Omega))$, one can check by straightforward calculations that (3.1) is equivalent to (1) if we use the change of variables $z=\sqrt{v+\alpha^2}$. We summarize this statement in the following lemma for further use.

Lemma 3.1. *Problems* (1) *and* (3.1) *are equivalent. More precisely,* (u,z) *is a weak-strong solution of* (3.1) *if, and only if,* (u,v) *is a weak-strong solution of* (1)*, with* $v = z^2-\alpha^2$.

For the time discretization we will divide the interval $[0,\infty)$ in subintervals denoted by $I_n = (t_{n-1},t_n)$, with $t_0 = 0$ and $t_n = t_{n-1}+k$, where $k>0$ is the length of the intervals I_n. If $\{z^n\}_n$ is a sequence of functions, then we use the notation

$$\delta_t z^n = \frac{z^n - z^{n-1}}{k}, \quad \forall n \geq 1, \tag{3.2}$$

for the discrete time derivative. We will also use the following upper truncation of u

$$T^m(u) = \begin{cases} u, & \text{if } u \leq m, \\ m, & \text{if } u \geq m. \end{cases}$$

In this chapter, we propose the following time discrete scheme:
Initialization: Let $u_m^0 = u^0 \in L^2(\Omega)$, $z_m^0 = \sqrt{v^0 + \alpha^2} \in L^\infty(\Omega)$ and $v_m^0 = v^0 \in L^\infty(\Omega)$.
Step n: Given $u_m^{n-1} \in L^2(\Omega)$, $z_m^{n-1} \in L^\infty(\Omega)$ and $v_m^{n-1} \in L^\infty(\Omega)$,

1. Find $(u_m^n, z_m^n) \in H^2(\Omega)^2$, satisfying the bounds

$$u_m^n(x) \geq 0 \quad \text{and} \quad \|z_m^{n-1}\|_{L^\infty(\Omega)} \geq z_m^n(x) \geq \alpha \quad \text{a.e. } x \in \Omega,$$

and the boundary-value problem

$$\begin{cases} \delta_t u_m^n - \Delta u_m^n = \nabla \cdot \Big(T^m(u_m^n) \nabla (z_m^n)^2 \Big), \\ \delta_t z_m^n - \dfrac{|\nabla z_m^n|^2}{z_m^n} - \Delta z_m^n = -\dfrac{1}{2} T^m(u_m^n)^s \left(z_m^n - \dfrac{\alpha^2}{z_m^n} \right), \\ \partial_\eta u_m^n \Big|_{\partial\Omega} = \partial_\eta z_m^n \Big|_{\partial\Omega} = 0. \end{cases} \tag{3.3}$$

2. Two variants for the approximation of v are possible (equally denoted), either depending on z_m^n or u_n^m:

- Find $v_m^n = v_m^n(z_m^n) \in H^2(\Omega)$ as

$$v_m^n = (z_m^n)^2 - \alpha^2. \tag{3.4}$$

- Find $v_m^n = v_m^n(u_m^n) \in H^2(\Omega)$ as the unique solution of the linear problem

$$\delta_t v_m^n - \Delta v_m^n + T^m(u_m^n)^s v_m^n = 0, \quad \partial_\eta v_m^n \Big|_{\partial\Omega} = 0. \tag{3.5}$$

Now, we are in position to present the main result that will be proved along the present work.

Theorem 3.2. *For each $n \in \mathbb{N}$, there exists at least one solution (u_m^n, z_m^n) of* (3.3)*, that jointly to v_m^n defined by* (3.4) *or* (3.5) *leads us to (u_m^n, v_m^n) satisfying*

$$u_m^n(x) \geq 0, \qquad \|v^0\|_{L^\infty(\Omega)} \geq v_m^n(x) \geq 0 \quad \text{a.e. } x \in \Omega.$$

In addition, up to a subsequence, (u_m^n, v_m^n) converges towards (u, v) a weak solution of (1) *as $(m, k) \to (\infty, 0)$.*

Remark 3.3. The number α is a sufficiently small positive real number that is chosen in Lemmas 3.7 and 3.8 independently of m and k. The convergence result given in Theorem 3.2 as $(m, k) \to (\infty, 0)$ is unconditional, that is, there is not any constraint over m and k as long as $m \to \infty$ and $k \to 0$. ■

In particular, we also prove the result on existence of weak solutions to (1) in 3D domains given in Theorem 2.2 of Chapter 2 but, this time, as a consequence of the convergence of the time discrete scheme introduced in this chapter.

In 2D domains, there exists a unique strong solution of (1), see Chapter 2. The proof is achieved through the obtaining of stronger m-independent estimates for the solution of an adequate truncated problem. Unfortunately, it is not clear how we could adapt these strong estimates for the time discrete scheme. Consequently, in 2D, the convergence of the whole sequence of solutions of the time discrete scheme towards the unique strong solution of (1) as $(m,k)\to(0,\infty)$ remains as an open problem.

In order to prove Theorem 3.2, the chapter is organized as follows. In Section 3.2 we establish the existence of solution (u_m^n, z_m^n) of the (u,z)-scheme (3.3), some pointwise estimates independent of (m,k,n) and an energy inequality for (u_m^n, z_m^n). In Section 3.3, starting from this energy inequality, we deduce additional *a priori* estimates for (u_m^n, z_m^n), independent of (m,k,n), that allow us to pass to the limit as $(m,k)\to(\infty,0)$, obtaining convergence of (3.3) towards the (u,z)-problem (3.1). Finally, in Section 3.4 we prove the convergence of (u_m^n, v_m^n), with v_m^n defined by (3.4) or (3.5), towards the (u,v)-problem (1).

3.2 Study of the (u,z)-Scheme (3.3)

3.2.1 Existence of solution of (3.3)

For simplicity, in the present subsection and in the following one we drop the m subscript and denote the solutions of (3.3) by (u^n, z^n). In Subsection 3.3 we go back to the notation (u_m^n, z_m^n).

Theorem 3.4. (Existence of solution to (3.3)) *Suppose $(u^{n-1}, z^{n-1})\in L^2(\Omega)\times L^\infty(\Omega)$ with $u^{n-1}(x)\geq 0$ and $z^{n-1}(x)\geq\alpha$ a.e. $x\in\Omega$. Then there is a solution (u^n,z^n) of (3.3) which satisfies $u_m^n(x)\geq 0$ and $\|z_m^{n-1}\|_{L^\infty(\Omega)}\geq z_m^n(x)\geq\alpha$ a.e. $x\in\Omega$.*

Proof. In order to avoid divisions by zero in some terms of (3.3) and obtain $u^n(x)\geq 0$ *a.e.* $x\in\Omega$, we define the lower truncation for z

$$T_\alpha(z)=\begin{cases}\alpha, & \text{if } z\leq\alpha,\\ z, & \text{if } z\geq\alpha,\end{cases}$$

and the lower-upper truncation for u

$$T_0^m(u)=\begin{cases}0, & \text{if } u\leq 0,\\ u, & \text{if } u\in[0,m],\\ m, & \text{if } u\geq m,\end{cases}$$

Then, we consider the auxiliary problem

$$\begin{cases} \delta_t u^n - \Delta u^n = \nabla \cdot \Big(T_0^m(u^n)\nabla(z^n)^2\Big), \\ \delta_t z^n - \dfrac{|\nabla z^n|^2}{T_\alpha(z^n)} - \Delta z^n = -\dfrac{1}{2}T_0^m(u^n)^s\left(z^n - \dfrac{\alpha^2}{T_\alpha(z^n)}\right), \end{cases} \tag{3.6}$$

with the same boundary and initial conditions of (3.3).

We prove the existence of a solution (u^n, z^n) to (3.6) via Leray-Schauder fixed point theorem [19]. Along this proof, we also have that any solution (u^n, z^n) of (3.6) satisfies $u^n(x) \geq 0$ and $\|z^{n-1}\|_{L^\infty(\Omega)} \geq z^n(x) \geq \alpha$ *a.e.* $x \in \Omega$, which implies that $T_\alpha(z^n) = z^n$, $T_0^m(u^n) = T^m(u^n)$ and therefore we conclude that (u^n, z^n) is also a solution of (3.3). Now we proceed with the proof of existence for (3.6) which is divided in three steps.

Step 1 (Definition of the compact mapping $\boldsymbol{S}$): For all $(\overline{u}, \overline{z}) \in W^{1,4}(\Omega)^2$, we define $(u, z) = S(\overline{u}, \overline{z}) \in H^2(\Omega)^2$ as the solution of

$$\frac{u}{k} - \Delta u = 2\nabla \cdot \Big(T_0^m(\overline{u})\,\overline{z}\,\nabla z\Big) + \frac{u^{n-1}}{k}, \tag{3.7}$$

$$\frac{z}{k} - \Delta z + \frac{1}{2}T_0^m(\overline{u})^s\left(z - \frac{\alpha^2}{T_\alpha(\overline{z})}\right) = \frac{|\nabla \overline{z}|^2}{T_\alpha(\overline{z})} + \frac{z^{n-1}}{k}. \tag{3.8}$$

We can use standard results on linear elliptic problems to conclude that $(u, z) = S(\overline{u}, \overline{z})$ is well defined. In fact, given $(\overline{u}, \overline{z})$, we begin by solving the z-equation (3.8). Since $0 < \frac{1}{k} + \frac{1}{2}T_0^m(\overline{u})^s \leq \frac{1}{k} + \frac{m^s}{2}$, we first prove the existence of a weak solution $z \in H^1(\Omega)$ by means of the Lax-Milgram Theorem and then we use the H^2-regularity of the Poisson-Neumann problem (1.5) to prove that $z \in H^2(\Omega)$. Once proved the existence of $z \in H^2(\Omega)$, we have $\nabla\cdot\Big(T_0^m(\overline{u})\,\overline{z}\,\nabla z\Big) \in L^2(\Omega)$ and therefore we are able to solve the u-equation (3.7) and, using again the H^2-regularity of the Poisson-Neumann problem (1.5), we obtain $u \in H^2(\Omega)$. Hence, $S(\overline{u}, \overline{z}) \in H^2(\Omega)^2$ and therefore S is a compact mapping defined in $W^{1,4}(\Omega)^2$.

Step 2 (Pointwise bounds for $\boldsymbol{u}$ and $\boldsymbol{z/\lambda}$ for any $\boldsymbol{(u,z) = \lambda S(u,z)}$): Let $\lambda \in [0,1]$. We will study the pairs (u, z) such that $(u, z) = \lambda S(u, z)$. If we consider $\lambda = 0$ then $S(u, z) \in H^2(\Omega)^2$ is well defined and $(u, z) = (0, 0)$. Once the case where $\lambda = 0$ is treated, we consider $\lambda \in (0, 1]$, therefore we can write $S(u, z) = (1/\lambda)(u, z)$ and we have (u, z) satisfying

$$\begin{cases} \dfrac{u}{k} - \Delta u = \nabla \cdot \Big(T_0^m(u)\nabla(z)^2\Big) + \lambda\dfrac{u^{n-1}}{k}, \\ \dfrac{1}{k}\dfrac{z}{\lambda} - \dfrac{|\nabla z|^2}{T_\alpha(z)} - \Delta\dfrac{z}{\lambda} = -\dfrac{1}{2}T_0^m(u)^s\left(\dfrac{z}{\lambda} - \dfrac{\alpha^2}{T_\alpha(z)}\right) + \dfrac{1}{k}z^{n-1}. \end{cases} \tag{3.9}$$

If we test the u-equation of (3.9) by the negative part of u defined, as $u_-(x) = \min\{0, u(x)\}$, we conclude that $u \geq 0$. Now let $c = \|z^{n-1}\|_{L^\infty(\Omega)}$. We rewrite the

z-equation of (3.9) as

$$\begin{aligned}&\frac{1}{k}(\frac{z}{\lambda}-c)-\lambda^2\frac{|\nabla(\frac{z}{\lambda}-c)|^2}{T_\alpha(z)}-\Delta(\frac{z}{\lambda}-c)\\&=-\frac{T_0^m(u)^s}{2}(\frac{z}{\lambda}-c)+\frac{T_0^m(u)^s}{2}(\frac{\alpha^2}{T_\alpha(z)}-c)+\frac{(z^{n-1}-c)}{k}.\end{aligned}\tag{3.10}$$

Now we test (3.10) by the positive part of $(\frac{z}{\lambda}-c)$, defined as

$$(\frac{z}{\lambda}-c)_+(x)=\max\{0,(\frac{z}{\lambda}-c)(x)\}\geq 0.$$

Since $c=\|z^{n-1}\|_{L^\infty(\Omega)}$, we have $z^{n-1}-c\leq 0$. Moreover, note that $(\frac{z}{\lambda}-c)_+\geq 0$ and $(\frac{z}{\lambda}-c)_+\neq 0\Longleftrightarrow z>\lambda c$. Then, an analysis taking account of the possible cases $\alpha<\lambda c$ and $\alpha\geq\lambda c$ leads us to

$$\frac{(\frac{z}{\lambda}-c)_+}{T_\alpha(z)}\leq\frac{1}{\lambda}.$$

Hence, reminding that $\lambda\in(0,1]$ and $T_\alpha(z)\geq\alpha$, if we test (3.10) by $(z/\lambda-c)_+$ we obtain

$$\begin{aligned}&\frac{1}{k}\|(\frac{z}{\lambda}-c)_+\|^2_{L^2(\Omega)}+\|\nabla(\frac{z}{\lambda}-c)_+\|^2_{L^2(\Omega)}\\&\leq\lambda\|\nabla(\frac{z}{\lambda}-c)_+\|^2_{L^2(\Omega)}+\frac{1}{2}T_0^m(u)^s(\alpha-c)(\frac{z}{\lambda}-c)_+.\end{aligned}$$

By hypothesis we have $z^{n-1}\geq\alpha$, which implies in particular $\alpha\leq c$ and then we can conclude that $\frac{z}{\lambda}\leq\|z^{n-1}\|_{L^\infty(\Omega)}$.

Next we prove an inferior bound for z/λ. Considering the definition of T_α and $\frac{|\nabla z|^2}{T_\alpha(z)}+\frac{1}{k}z^{n-1}\geq\frac{\alpha}{k}$, which comes from the hypotheses of the theorem, we can use the z-equation of (3.9) to write

$$\frac{1}{k}(\frac{z}{\lambda}-\alpha)-\Delta(\frac{z}{\lambda}-\alpha)+\frac{1}{2}T_0^m(u^n)^s(\frac{z}{\lambda}-\alpha)\geq\frac{1}{2}T_0^m(u)^s(\frac{\alpha^2}{T_\alpha(z)}-\alpha).\tag{3.11}$$

Now we test (3.11) by the negative part of $(z-\lambda\alpha)$, $(z-\lambda\alpha)_-$. Note that $(z-\lambda\alpha)_-\leq 0$ and $(\frac{z}{\lambda}-\alpha)_-\neq 0\Longrightarrow z<\lambda\alpha\leq\alpha$. Therefore, testing (3.11) by $(\frac{z}{\lambda}-\alpha)_-$ and reminding that $\lambda\in(0,1]$, we obtain

$$\begin{aligned}&\frac{1}{k}\|(\frac{z}{\lambda}-\alpha)_-\|^2_{L^2(\Omega)}+\|\nabla(\frac{z}{\lambda}-\alpha)_-\|^2_{L^2(\Omega)}+\frac{1}{2}\|T_0^m(u)^{s/2}(\frac{z}{\lambda}-\alpha)_-\|^2_{L^2(\Omega)}\\&\leq\frac{1}{2}\int_\Omega T_0^m(u)^s(\frac{\alpha^2}{T_\alpha(z)}-\alpha)(\frac{z}{\lambda}-\alpha)_-\,dx=\frac{1}{2}\int_\Omega T_0^m(u)^s(\frac{\alpha^2}{\alpha}-\alpha)(\frac{z}{\lambda}-\alpha)_-\,dx\end{aligned}$$

Thus we conclude that $(\frac{z}{\lambda}-\alpha)_-=0$, that is, $\frac{z}{\lambda}\geq\alpha>0$.

Step 3 (λ-independent bounds for any $(u, z) = \lambda S(u, z)$): As we mentioned before, we consider $\lambda \in (0, 1]$ because if $\lambda = 0$ then we have $(u, z) = (0, 0)$. Because of the upper bound for z that we proved in the anterior step we have

$$\frac{z}{\lambda} \text{ is bounded in } L^\infty(\Omega), \tag{3.12}$$

independently of λ. Then we can integrate the z-equation of (3.9) and conclude that

$$\|\nabla z\|^2_{L^2(\Omega)} \le C(k, \alpha^2, m, \|z^{n-1}\|_{L^\infty(\Omega)}). \tag{3.13}$$

Since $z > 0$, we can multiply the z-equation of (3.9) by $\lambda\, T_\alpha(z)/z$. This gives us

$$\frac{T_\alpha(z)}{k} - \lambda\frac{|\nabla z|^2}{z} - \frac{T_\alpha(z)}{z}\Delta z = -\frac{1}{2}T_0^m(u)^s(T_\alpha(z) - \frac{\alpha^2}{z/\lambda}) + \frac{T_\alpha(z)}{z/\lambda}\frac{z^{n-1}}{k}. \tag{3.14}$$

Now we test (3.14) by $-\Delta z$ and, using that $T_\alpha(z)/z \ge 1$, $z/\lambda \ge \alpha$ and (3.12), we obtain

$$\begin{aligned}
&\frac{1}{k}\int_\Omega (T_\alpha)'(z)|\nabla z|^2\, dx + (1-\lambda)\int_\Omega |\Delta z|^2\, dx + \lambda\int_\Omega |\Delta z|^2\, dx + \lambda\int_\Omega \frac{|\nabla z|^2}{z}\Delta z\, dx\\
&\le \frac{1}{2}\int_\Omega |T_0^m(u)|^s|T_\alpha(z) + \frac{\alpha^2}{\alpha}|\,|\Delta z|\, dx + \int_\Omega |\frac{T_\alpha(z)}{\alpha\, k}z^{n-1}|\,|\Delta z|\, dx\\
&\le \frac{1}{\delta}C(m, \alpha^2, \|z^{n-1}\|_{L^\infty(\Omega)}) + \delta\|\Delta z\|^2_{L^2(\Omega)}.
\end{aligned}$$

Then, applying Lemma 2.10 and using (3.13) and the H^2-regularity we obtain

$$\begin{aligned}
&\frac{1}{k}\int_\Omega (T_\alpha)'(z)|\nabla z|^2\, dx + (1-\lambda)\int_\Omega |\Delta z|^2\, dx + C_1\lambda\Big(\int_\Omega |\Delta z|^2\, dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\, dx\Big)\\
&\le C_2\lambda\|\nabla z\|^2_{L^2(\Omega)} + \frac{1}{\delta}C(m, \alpha^2, \|z^{n-1}\|_{L^\infty(\Omega)}) + \delta\|\Delta z\|^2_{L^2(\Omega)}\\
&\le \frac{1}{\delta}C(k, m, \alpha^2, \|z^{n-1}\|_{L^\infty(\Omega)}) + \delta\|\Delta z\|^2_{L^2(\Omega)}.
\end{aligned}$$

Taking into account that $\int_\Omega (T_\alpha)'(z)|\nabla z|^2\, dx \ge 0$ and denoting $C = \min\{1, C_1\}$ we have $1 + (C_1 - 1)\lambda \ge C$ for all $\lambda \in [0, 1]$ and

$$C\int_\Omega |\Delta z|^2\, dx + C\lambda\int_\Omega \frac{|\nabla z|^4}{z^2}\, dx \le \frac{1}{\delta}C(m, \alpha^2, \|z^{n-1}\|_{L^\infty(\Omega)}) + \delta\|\Delta z\|^2_{L^2(\Omega)}$$

and therefore we can choose $\delta > 0$ small enough such that

$$\Delta z \text{ is bounded in } L^2(\Omega). \tag{3.15}$$

Because of the homogeneous Neumann boundary conditions, the H^2-regularity and the λ-uniform bounds (3.12) and (3.15), we can conclude that

$$z \text{ is bounded in } H^2(\Omega). \tag{3.16}$$

By Sobolev inequality in 3D domains, the latter implies that

$$z \text{ is bounded in } W^{1,4}(\Omega). \tag{3.17}$$

Considering the λ-independent bounds $0 \le z \le \|z^{n-1}\|_{L^\infty(\Omega)}$, $T_0^m(u) \le m$ and (3.16), we can test the u-equation (3.7) by u and prove that u is bounded in $H^1(\Omega)$. Using again the λ-independent bounds $0 \le z \le \|z^{n-1}\|_{L^\infty(\Omega)}$, $T_0^m(u) \le m$, (3.16) and (3.17), we have that the chemotaxis term $2\nabla \cdot \left(T_0^m(\overline{u})\overline{z}\nabla z\right)$ is bounded in $L^2(\Omega)$. Then we can test the u-equation (3.7) by $-\Delta u$, obtaining u is bounded in $H^2(\Omega)$, which implies, in particular, that

$$u \text{ is bounded in } W^{1,4}(\Omega). \tag{3.18}$$

With (3.17) and (3.18) we can finally conclude, using the Leray-Schauder fixed point theorem [19] that the auxiliary problem (3.6) has a solution (u, z). Because of the properties showed along the steps of the proof we can also conclude that $(u, z) \in H^2(\Omega)^2$, $u(x) \ge 0$ and $\|z^{n-1}\|_{L^\infty(\Omega)} \ge z(x) \ge \alpha$ *a.e.* $x \in \Omega$. Therefore we have $T_\alpha(z) = z$, $T_0^m(u) = T^m(u)$ and we conclude that the solution (u, z) of (3.6) is a solution of (3.3), finishing the proof of existence of solution. ■

3.2.2 First uniform in time estimates

The following direct estimates and the energy inequalities obtained in this subsection are valid for any solution (u^n, z^n) of (3.3) given by Theorem 3.4.

Lemma 3.5. ((m, k, n)-uniform estimates) *Let (u^n, z^n) be a solution of (3.3). Then we have*

1. $\displaystyle\int_\Omega u^n \, dx = \int_\Omega u^0 \, dx$, *for all $n \in \mathbb{N}$;*

2. $\displaystyle\|z^n\|^2_{L^2(\Omega)} + \sum_{j=1}^n \|z^j - z^{j-1}\|^2_{L^2(\Omega)} \le \|z^0\|^2_{L^2(\Omega)}$, *for all $n \in \mathbb{N}$;*

3. $\displaystyle k\sum_{j=1}^n \|\nabla z^j\|^2_{L^2(\Omega)} \le \frac{1}{4\alpha^2}\|v^0 + \alpha^2\|^2_{L^2(\Omega)}$, *for all $n \in \mathbb{N}$.*

Proof. The proof of item 1 is achieved by integrating the u^n-equation of (3.3).

For the items 2 and 3 we take the product of the z^n-equation of (3.3) by z^n. We obtain

$$\delta_t(z^n)^2 + \frac{1}{k}(z^n - z^{n-1})^2 - \Delta(z^n)^2 + T^m(u^n)^s((z^n)^2 - \alpha^2) = 0. \tag{3.19}$$

Since $((z^n)^2 - \alpha^2) \ge 0$ and $\displaystyle\int_\Omega \Delta(z^n)^2 \, dx = 0$, by integrating (3.19) we prove item 2. On the other hand, testing (3.19) by $k(z^n)^2$ leads us to

$$k\delta_t\|(z^n)^2\|^2_{L^2(\Omega)} + k\|\nabla(z^n)^2\|^2_{L^2(\Omega)} \le 0.$$

Then, summing up from $j = 1$ to n gives us

$$k\sum_{j=1}^{n} \|\nabla (z^j)^2\|^2_{L^2(\Omega)} \leq \|(z^0)^2\|^2_{L^2(\Omega)} = \|v^0 + \alpha^2\|^2_{L^2(\Omega)}.$$

Now using that $z^j \geq \alpha$ we have

$$\int_\Omega |\nabla z^j|^2 \, dx = \int_\Omega \frac{(z^j)^2}{(z^j)^2}|\nabla z^j|^2 \, dx \leq \int_\Omega \frac{1}{4\alpha^2}|\nabla (z^j)^2|^2 \, dx,$$

hence we obtain item 3. ■

3.2.3 Energy inequality

Now we turn to the energy inequalities given for $s \in [1, 2)$ in Lemma 3.7 and $s \geq 2$ in Lemma 3.8 below. We will need the following lemma.

Lemma 3.6. *Any solution (u^n, z^n) of (3.3), satisfies the inequality*

$$\begin{aligned}\frac{1}{2}\delta_t\|\nabla z^n\|^2_{L^2(\Omega)} + \frac{1}{2k}\|\nabla z^n - \nabla z^{n-1}\|^2_{L^2(\Omega)} + C_1\Big(\int_\Omega |D^2 z^n|^2 \, dx + \int_\Omega \frac{|\nabla z^n|^4}{(z^n)^2} \, dx\Big)\\ +\frac{1}{2}\int_\Omega T^m(u^n)^s|\nabla z^n|^2 \, dx \leq \frac{s}{4}\int_\Omega T^m(u^n)^{s-1}\nabla (z^n)^2 \cdot \nabla T^m(u^n) \, dx\\ +\frac{s\alpha}{2}\int_\Omega T^m(u^n)^{s-1}|\nabla z^n||\nabla T^m(u^n)| \, dx + C_2\int_\Omega |\nabla z^n|^2 \, dx.\end{aligned}$$

Proof. We begin by testing the z^n-equation of (3.3) by $-\Delta z^n$. This gives us

$$\begin{aligned}&\frac{1}{2}\delta_t\|\nabla z^n\|^2_{L^2(\Omega)} + \frac{1}{2k}\|\nabla z^n - \nabla z^{n-1}\|^2_{L^2(\Omega)} + \|\Delta z^n\|^2_{L^2(\Omega)}\\ &\quad + \int_\Omega \frac{|\nabla z^n|^2}{z^n}\Delta z^n \, dx + \frac{1}{2}\int_\Omega (1 + \frac{\alpha^2}{(z^n)^2})T^m(u^n)^s|\nabla z^n|^2 \, dx\\ &= \frac{s}{4}\int_\Omega T^m(u^n)^{s-1}\nabla (z^n)^2 \cdot \nabla T^m(u^n) \, dx + \frac{s}{2}\alpha^2\int_\Omega \frac{T^m(u^n)^{s-1}}{z^n}\nabla z^n \cdot \nabla T^m(u^n) \, dx.\end{aligned}$$

Then, estimating the last term on the left hand side from bellow, the last term on the right hand side from above and applying Lemma 2.10, we obtain the desired inequality. ■

Next we will obtain a local energy inequality for (u^n, z^n), first for $s \in [1, 2)$ and then for $s \geq 2$. We consider the function $f_m(r) = \displaystyle\int_0^r f'_m(\theta) \, d\theta$, where

$$f'_m(r) = \begin{cases} ln(T^m(r)), & \text{if } s = 1, \text{ for } r > 0,\\ \dfrac{T^m(r)^{s-1}}{(s-1)}, & \text{if } s > 1.\end{cases}$$

Lemma 3.7 (Energy inequality for $s \in [1,2)$). *Any solution (u^n, z^n) of the problem (3.3) satisfies, for sufficiently small $\alpha^2 > 0$,*

$$\begin{aligned} &\delta_t\Big[\frac{s}{4}\int_\Omega f_m(u^n)\ dx + \frac{1}{2}\|\nabla z^n\|^2_{L^2(\Omega)}\Big] \\ &+\frac{1}{2k}\|\nabla z^n - \nabla z^{n-1}\|^2_{L^2(\Omega)} + \frac{1}{4}\int_\Omega T^m(u^n)^s|\nabla z^n|^2\ dx \\ &+C_1\Big(\int_\Omega |D^2 z^n|^2\ dx + \int_\Omega \frac{|\nabla z^n|^4}{(z^n)^2}\ dx\Big) \le C\|\nabla z^n\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.20}$$

Proof. The proof follows the same ideas of the analogous result that was proved in Chapter 2 for the truncated model (2.1). We are going to show the main steps of the proof, calling the attention to the differences that appear due to the fact that we are considering the time discrete scheme (3.3). In fact, we begin by considering the sequence $\{1/j\}_{j\in\mathbb{N}}$, the function $f'_{m,j}(u^n) = f'_m(u^n + 1/j)$ and testing the u^n-equation of (3.3) by $f'_{m,j}(u^n)$. A difference appears in the treatment of the term of the discrete time derivative. Namely, accounting that $f_{m,j}$ is convex, we use Lemma 1.9 to obtain

$$\begin{aligned} &\delta_t \int_\Omega f_{m,j}(u^n)\ dx + \int_\Omega \frac{(T^m)'(u^n)}{(T^m(u^n)+1/j)^{2-s}}|\nabla u^n|^2\ dx \\ &\le \Big(\frac{T^m(u^n)}{(T^m(u^n)+1/j)^{2-s}}\nabla (z^n)^2, \nabla T^m(u^n)\Big). \end{aligned}$$

Then we can follow the ideas in Chapter 2 until the point that we reach the inequality

$$\begin{aligned} &\delta_t\Big[\frac{s}{4}\int_\Omega f_{m,j}(u^n)\ dx + \frac{1}{2}\|\nabla z^n\|^2_{L^2(\Omega)}\Big] \\ &+\frac{1}{2k}\|\nabla z^n - \nabla z^{n-1}\|^2_{L^2(\Omega)} + C_1\Big(\int_\Omega |D^2 z^n|^2\ dx + \int_\Omega \frac{|\nabla z^n|^4}{(z^n)^2}\ dx\Big) \\ &+\frac{1}{2}\int_\Omega\Big[T^m(u^n)^s - \frac{1}{2}(T^m(u^n)+1/j)^s\Big]|\nabla z^n|^2\ dx \le C\|\nabla z^n\|^2_{L^2(\Omega)} \\ &+\frac{s}{4}\int_\Omega\Big[T^m(u^n)^{s-1} - (T^m(u^n)+1/j)^{s-1}\Big]\nabla T^m(u^n)\cdot\nabla (z^n)^2\ dx. \end{aligned} \tag{3.21}$$

Finally we pass to the limit as $j \to \infty$ in (3.21). The presence of the discrete time derivative δ_t instead of ∂_t is what allows us to give an unified treatment for the case $s \in [1,2)$, differently from Chapter 2, where the cases $s = 1$ and $s \in (1,2)$ are separated. We proceed with the passage to the limit term by term. We detail the passage to the limit in the term which involves the discrete time derivative,

$$\delta_t \int_\Omega f_{m,j}(u^n(x))\ dx. \tag{3.22}$$

We define the functions $g_{m,j}, g_m, G \in L^1(\Omega)$ by $g_{m,j}(x) = \delta_t f_{m,j}(u^n(x))$, $g_m(x) = \delta_t f_m(u^n(x))$ and $G(x) = |g_{m,1}(x)|$. Then, for almost every $x \in \Omega$, $g_{m,j}(x) \to g_m(x)$ as $j \to \infty$ with $|g_{m,j}(x)| \le G(x)$ for all $j \in \mathbb{N}$. Therefore, using the Dominated

Convergence Theorem, we can conclude that

$$\lim_{j\to\infty}\delta_t\int_\Omega f_{m,j}(u^n(x))\ dx=\lim_{j\to\infty}\int_\Omega g_{m,j}(x)\ dx=\int_\Omega g_m(x)\ dx=\delta_t\int_\Omega f_m(u^n(x))\ dx.$$

For the other terms of (3.21), one can again follow Chapter 2, take the limit as $j\to\infty$ and obtain the desired result. ■

Lemma 3.8 (**Energy inequality for $s\geq 2$**). *The solution (u^n,z^n) of the problem* (3.3) *satisfies*

$$\begin{aligned}\delta_t\Big[\frac{s}{4}\int_\Omega f_m(u^n)\ dx+\frac{1}{2}\|\nabla z^n\|^2_{L^2(\Omega)}\Big]+\frac{1}{2k}\|\nabla z^n-\nabla z^{n-1}\|^2_{L^2(\Omega)}\\ +\int_\Omega|\nabla[T^m(u^n)]^{s/2}|^2dx+\frac{1}{4}\int_\Omega T^m(u^n)^s|\nabla z^n|^2\ dx\\ +C_1\Big(\int_\Omega|D^2z^n|^2\ dx+\int_\Omega\frac{|\nabla z^n|^4}{(z^n)^2}\ dx\Big)\leq C\|\nabla z^n\|^2_{L^2(\Omega)}.\end{aligned}\tag{3.23}$$

Proof. The proof follows the same ideas of the analogous result that was proved in Chapter 2 for the truncated model. Having in mind that we use Lemma 1.9 to treat the term which involves the discrete time derivative δ_t, we refer the reader to Chapter 2 for the details of the proof. ■

The energy inequalities (3.20) and (3.23) allow us to obtain (m,k,n)-independent estimates for the function z^n in the next subsection.

3.3 Energy Estimates and Passage to the Limit as $(m,k)\to(\infty,0)$

Now we use again the notation (u^n_m,z^n_m) for the solution of (3.3). We define the piecewise constant function $u^{k,r}_m$ and the locally linear and globally continuous function u^k_m by

$$\begin{aligned}u^{k,r}_m(t,x)=u^n_m(x)\text{ and}\\ u^k_m(t,x)=u^n_m(x)+\frac{(t-t_n)}{k}\big(u^n_m(x)-u^{n-1}_m(x)\big),\ \text{ if } t\in[t_{n-1},t_n).\end{aligned}\tag{3.24}$$

Analogously, we define the functions $z^{k,r}_m$, z^k_m, $v^{k,r}_m$ and v^k_m. With these functions we can rewrite (3.3) as the differential system, *a.e.* in $(t,x)\in(0,\infty)\times\Omega$,

$$\left\{\begin{aligned}&\partial_t u^k_m-\Delta u^{k,r}_m=\nabla\cdot\Big(T^m(u^{k,r}_m)\nabla(z^{k,r}_m)^2\Big),\\ &\partial_t z^k_m-\frac{|\nabla z^{k,r}_m|^2}{z^{k,r}_m}-\Delta z^{k,r}_m=-\frac{1}{2}T^m(u^{k,r}_m)^s\left(z^{k,r}_m-\frac{\alpha^2}{z^{k,r}_m}\right).\end{aligned}\right.\tag{3.25}$$

In this subsection we are going to prove (m,k)-independent estimates for $u^{k,r}_m$, $z^{k,r}_m$, u^k_m and z^k_m, which are also uniform in time, that will allow us to pass to the limit in (3.25).

First, in Subsection 3.3.1, we will obtain estimates for $\nabla z^{k,r}_m$ from the energy inequalities (3.20) and (3.23). Next we prove bounds for u^k_m and $u^{k,r}_m$ and pass to the limit in (3.25) as $(m,k) \to (\infty, 0)$, considering the cases $s \in [1,2)$ and $s \geq 2$, separately.

3.3.1 Estimates for $\nabla z_m^{k,r}$

Let us define the energy

$$E_m^n = \frac{s}{4}\int_\Omega f_m(u_m^n(t,x))\,dx + \frac{1}{2}\int_\Omega |\nabla z_m^n(t,x)|^2\,dx.$$

We use the regularity of the initial data u^0 and z^0 in order to conclude that the initial energy E_m^0, is also bounded, independently of (m,k). If we consider either (3.20) or (3.23), multiply it by k and sum from $j=1$ to n we can obtain

$$\begin{aligned}
&\frac{s}{4}\int_\Omega f_m(u_m^n)\,dx + \frac{1}{2}\|\nabla z_m^n\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{j=1}^n \|\nabla z^j - \nabla z^{j-1}\|^2_{L^2(\Omega)}\\
&+k\sum_{j=1}^n \int_\Omega T^m(u_m^j)^s |\nabla z_m^j|^2\,dx + C_1 k\sum_{j=1}^n\Big(\int_\Omega |D^2 z_m^j|^2\,dx + \int_\Omega \frac{|\nabla z_m^j|^4}{(z^j)^2}\,dx\Big) \qquad (3.26)\\
&\leq Ck\sum_{j=1}^n \|\nabla z_m^j\|^2_{L^2(\Omega)} + \frac{s}{4}\int_\Omega f_m(u_m^0)\,dx + \frac{1}{2}\|\nabla z_m^0\|^2_{L^2(\Omega)},
\end{aligned}$$

Thus, accounting that $z^j \geq \alpha$ and that Lemma 3.5.3 and (3.26) are valid for any $n \in \mathbb{N}$, we can conclude that

$$\nabla z_m^{k,r} \text{ is bounded in } L^\infty(0,\infty; L^2(\Omega)) \cap L^4(0,\infty; L^4(\Omega)), \qquad (3.27)$$

$$T^m(u_m^{k,r})^{s/2}\nabla z_m^{k,r} \text{ and } \Delta z_m^{k,r} \text{ are bounded in } L^2(0,\infty; L^2(\Omega)), \qquad (3.28)$$

$$\sum_{j=1}^\infty \|z^j - z^{j-1}\|^2_{H^1(\Omega)} \leq C. \qquad (3.29)$$

From (3.29), we can prove the following.

Lemma 3.9. *There is a positive constant C, independent of m and k, such that*

$$\|z_m^{k,r} - z_m^k\|^2_{L^2(0,\infty;H^1(\Omega))} \leq C\,k. \qquad (3.30)$$

Proof. From the definition of $z_m^{k,r}$ and z_m^k, we observe that

$$z_m^{k,r}(t) - z_m^k(t) = \frac{(t_n - t)}{k}(z^n - z^{n-1})$$

for $t \in (t_{n-1}, t_n)$. If $t \in (t_{j-1}, t_j)$ then $0 \leq t_j - t \leq k$ and hence

$$\begin{aligned}
\|z_m^{k,r} - z_m^k\|^2_{L^2(0,\infty;H^1(\Omega))} &= \sum_{j=1}^\infty \int_{t_{j-1}}^{t_j} \frac{(t_n - t)}{k}\|z^j - z^{j-1}\|^2_{H^1(\Omega)}\,dt\\
&\leq k\sum_{j=1}^\infty \|z^j - z^{j-1}\|^2_{H^1(\Omega)}.
\end{aligned}$$

Therefore, using (3.29) we conclude the proof. ■

In particular, since $z_m^{k,r} \in H^2(\Omega)$ and $\frac{\partial}{\partial \eta} z_m^n|_\Gamma = 0$, it stems from (3.28), the H^2-regularity of the Poisson-Neumann problem (1.5) and (1.6) that

$$\nabla z_m^{k,r} \text{ is bounded in } L^2(0,\infty; H^1(\Omega)). \tag{3.31}$$

Using the results obtained until this point we analyze the convergence towards problem (1), first for $s \in [1,2)$ and then for $s \geq 2$.

3.3.2 Estimates for $(u_m^{k,r}, z_m^{k,r})$ and passage to the limit for $s \in [1,2)$

Let

$$\forall r > 0, \qquad f'(r) = \begin{cases} ln(r) & \text{if } s = 1, \\ r^{s-1}/(s-1) & \text{if } s \in (1,2), \end{cases}$$

$$f(r) = \int_0^r f'(\theta)\, d\theta = \begin{cases} rln(r) - r & \text{if } s = 1, \\ r^s/s(s-1) & \text{if } s \in (1,2). \end{cases}$$

Notice that $f''(r) = r^{s-2}$, $\forall r > 0$, in all cases.

We test the u_m^n-equation of (3.3) by $f'(u_m^n + 1)$. Using Lemma 1.9 we obtain

$$\begin{aligned}
&\delta_t \int_\Omega f(u_m^n + 1)\, dx + \frac{1}{2k}\int_\Omega f''(c^n)(u_m^n - u_m^{n-1})^2\, dx + \frac{4}{s^2}\int_\Omega |\nabla [u_m^n + 1]^{s/2}|^2\, dx + \\
&= 2\int_\Omega T^m(u_m^n)(u_m^n + 1)^{s/2-1} z_m^n \nabla z_m^n \cdot \nabla u_m^n\, (u_m^n + 1)^{s/2-1}\, dx \\
&\leq \frac{4}{s}\int_\Omega \frac{T^m(u_m^n)^{1-s/2}}{(u_m^n + 1)^{1-s/2}} T^m(u_m^n)^{s/2} z_m^n \nabla z_m^n \cdot \nabla [u_m^n + 1]^{s/2}\, dx \\
&\leq \frac{4}{s}\|z^0\|_{L^\infty(\Omega)} \Big(\int_\Omega T^m(u_m^n)^s |\nabla z_m^n|^2\, dx\Big)^{1/2} \Big(\int_\Omega |\nabla [u_m^n + 1]^{s/2}|^2\, dx\Big)^{1/2}
\end{aligned}$$

and thus we have

$$\begin{aligned}
&\delta_t \int_\Omega f(u_m^n + 1)\, dx + \frac{1}{2k}\int_\Omega f''(c^n)(u_m^n - u_m^{n-1})^2\, dx \\
&+ \frac{2}{s^2}\int_\Omega |\nabla [u_m^n + 1]^{s/2}|^2\, dx \leq C\int_\Omega T^m(u_m^n)^s |\nabla z_m^n|^2\, dx.
\end{aligned}$$

Multiplying by k and summing up from 0 to n, for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\int_\Omega f(u_m^n + 1)\, dx + \frac{1}{2}\sum_{j=1}^n \int_\Omega f''(c^j)(u_m^j - u_m^{j-1})^2\, dx + k\sum_{j=1}^n \frac{2}{s^2}\int_\Omega |\nabla [u_m^j + 1]^{s/2}|^2\, dx \\
&\leq Ck\sum_{j=1}^n \int_\Omega T^m(u_m^j)^s |\nabla z_m^j|^2\, dx + \int_\Omega f(u_m^0 + 1)\, dx.
\end{aligned}$$

Then, from (3.28) and the definitions of f, $u_m^{k,r}$ and u_m^k we conclude that

$$\sum_{j=1}^{\infty} \int_{\Omega} f''(c^j)(u_m^j - u_m^{j-1})^2 \, dx \leq C, \tag{3.32}$$

$$(u_m^{k,r} + 1)^{s/2} \text{ is bounded in } L^{\infty}(0,\infty; L^2(\Omega)), \tag{3.33}$$

hence, in particular,

$$u_m^{k,r} \text{ is bounded in } L^{\infty}(0,\infty; L^s(\Omega)), \tag{3.34}$$

$$\nabla[u_m^{k,r} + 1]^{s/2} \text{ is bounded in } L^2(0,\infty; L^2(\Omega)). \tag{3.35}$$

With these bounds it is possible to prove the following.

Lemma 3.10. *There is a positive constant C, independent of m and k, such that*

$$\|u_m^{k,r} - u_m^k\|^2_{L^2(0,\infty;L^s(\Omega))} \leq C\, k. \tag{3.36}$$

Proof.

Step 1: We remind that in (3.32) we have $f''(c^j) = (c^j)^{s-2}$, where $s \in [1,2)$ and, for each j and for each x, c^j is a point between $(u_m^j(x)+1)$ and $(u_m^{j-1}(x)+1)$. Hence let us write $c^j(x)$ as $c^j(x) = \theta^j(x)(u_m^j(x)+1) + (1-\theta^j(x))(u_m^{j-1}(x)+1)$, where $\theta^j(x) \in [0,1]$. Since $u_m^j(x), u_m^{j-1}(x) \geq 0$ and $\theta^j(x), (1-\theta^j(x)) \in (0,1)$ we have

$$\begin{aligned} \frac{(u^j - u^{j-1})^2}{((u_m^j(x)+1) + (u_m^{j-1}(x)+1))^{2-s}} &\leq \frac{(u^j - u^{j-1})^2}{(c^j(x))^{2-s}} \\ &= f''(c^j(x))(u^j - u^{j-1})^2. \end{aligned} \tag{3.37}$$

Step 2: Now we estimate $|(u_m^j+1)^{s/2} - (u_m^{j-1}+1)^{s/2}|^2$ by $f''(c^j(x))(u^j - u^{j-1})^2$. If we consider the identity

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}, \ \forall a, b > 0,$$

and next apply Lemma 1.10, we have

$$
\begin{aligned}
|(u_m^j+1)^{s/2}-(u_m^{j-1}+1)^{s/2}| &= \frac{|(u_m^j+1)^s-(u_m^{j-1}+1)^s|}{(u_m^j+1)^{s/2}+(u_m^{j-1}+1)^{s/2}}\\
&\le s|u_m^j-u_m^{j-1}|\frac{\big[(u_m^j+1)+(u_m^{j-1}+1)\big]^{s-1}}{(u_m^j+1)^{s/2}+(u_m^{j-1}+1)^{s/2}}\\
&\le 2^{s/2}s|u_m^j-u_m^{j-1}|\frac{\big[(u_m^j+1)+(u_m^{j-1}+1)\big]^{s-1}}{\big[(u_m^j+1)+(u_m^{j-1}+1)\big]^{s/2}}\\
&\le 2^{s/2}s|u_m^j-u_m^{j-1}|\frac{1}{\big[(u_m^j+1)+(u_m^{j-1}+1)\big]^{1-s/2}}.
\end{aligned}
$$

Using (3.37)

$$
\begin{aligned}
|(u_m^j+1)^{s/2}-(u_m^{j-1}+1)^{s/2}|^2 &\le \frac{C(u_m^j-u_m^{j-1})^2}{\big[(u_m^j+1)+(u_m^{j-1}+1)\big]^{2-s}}\\
&\le Cf''(c^j)(u_m^j-u_m^{j-1})^2.
\end{aligned}
\tag{3.38}
$$

Step 3: Finally, we use (3.32) and (3.38) to prove (3.36). Considering the definition of $u_m^{k,r}$ and u_m^k and using Lemma 1.10, for $t\in(t^{n-1},t^n)$ we have

$$
\begin{aligned}
&|u_m^{k,r}(t)-u_m^k(t)|^s \le |u^n-u^{n-1}|^s \le |((u^n+1)^{s/2})^{2/s}-((u^{n-1}+1)^{s/2})^{2/s}|^s\\
&\le s|(u^n+1)^{s/2}-(u^{n-1}+1)^{s/2}|^s|(u^n+1)^{s/2}+(u^{n-1}+1)^{s/2}|^{2-s}.
\end{aligned}
$$

Integrating and using Hölder's inequality with the conjugate powers $2/s$ and $2/(2-s)$ we obtain

$$
\begin{aligned}
\int_\Omega |u_m^{k,r}(t)-u_m^k(t)|^s\,dx \le s\Big(\int_\Omega |(u^n+1)^{s/2}-(u^{n-1}+1)^{s/2}|^2\,dx\Big)^{s/2}\\
\times\Big(\int_\Omega |(u^n+1)^{s/2}+(u^{n-1}+1)^{s/2}|^2\,dx\Big)^{(2-s)/2}.
\end{aligned}
$$

Considering the (m,k)-uniform bound (3.33), then the latter implies that

$$
\|u_m^{k,r}(t)-u_m^k(t)\|_{L^s(\Omega)}^2 \le C\int_\Omega |(u^n+1)^{s/2}-(u^{n-1}+1)^{s/2}|^2\,dx.
$$

Then, using (3.38) we obtain

$$
\|u_m^{k,r}(t)-u_m^k(t)\|_{L^s(\Omega)}^2 \le C\int_\Omega f''(c^j)(u_m^n-u_m^{n-1})^2\,dx.
$$

Finally, if we integrate in t and use (3.32) we get

$$
\int_0^\infty \|u_m^{k,r}(t)-u_m^k(t)\|_{L^s(\Omega)}^2\,dt \le Ck\sum_{j=1}^\infty\int_\Omega f''(c^j)(u_m^j-u_m^{j-1})^2\,dx \le Ck,
$$

which conclude the proof. ■

Consider the relation

$$\nabla u_m^{k,r} = \nabla(u_m^{k,r}+1) = \nabla\big((u_m^{k,r}+1)^{s/2}\big)^{2/s} = \frac{2}{s}(u_m^{k,r}+1)^{1-s/2}\,\nabla(u_m^{k,r}+1)^{s/2}. \quad (3.39)$$

Taking into account that we are considering $s \in [1,2)$, we can use (3.33) to obtain

$$(u_m^{k,r}+1)^{1-s/2} \text{ is bounded in } L^\infty(0,\infty;L^{2s/(2-s)}(\Omega))$$

and then (3.35) and (3.39) to conclude that

$$\nabla u_m^{k,r} \text{ is bounded in } L^2(0,\infty;L^s(\Omega)). \quad (3.40)$$

In conclusion, using (3.34), (3.40) and the Poincare's inequality for zero mean functions (Lemma 1.3),

$$u_m^{k,r} - u^* \text{ is bounded in } L^\infty(0,\infty;L^s(\Omega)) \cap L^2(0,\infty;W^{1,s}(\Omega)). \quad (3.41)$$

Considering the chemotaxis term, we can write $T^m(u_m^{k,r})\nabla(z_m^{k,r})^2$ as

$$T^m(u_m^{k,r})\nabla(z_m^{k,r})^2 = 2T^m(u_m^{k,r})^{1-s/2}T^m(u_m^{k,r})^{s/2}z_m^{k,r}\nabla z_m^{k,r}.$$

Then, we have $T^m(u_m^{k,r})^{1-s/2}$ bounded in $L^\infty(0,\infty;L^{2s/(2-s)}(\Omega))$, because of (3.33), and $T^m(u_m^{k,r})^{s/2}z_m^{k,r}\nabla z_m^{k,r}$ bounded in $L^2(0,\infty;L^2(\Omega))$, because of (3.28), and hence we can conclude that

$$T^m(u_m^{k,r})\nabla(z_m^{k,r})^2 \text{ is bounded in } L^2(0,\infty;L^s(\Omega)). \quad (3.42)$$

Then, if we consider the u-equation of (3.25), from (3.41) and (3.42) we have

$$\partial_t u_m^k \text{ is bounded in } L^2\Big(0,\infty;\big(W^{1,s/(s-1)}(\Omega)\big)'\Big).$$

Now we turn to the z-equation of (3.25), rewritten as

$$\begin{aligned}&\partial_t z_m^k - \frac{|\nabla z_m^{k,r}|^2}{z_m^{k,r}} - \Delta z_m^{k,r} + T^m(u^*)^s(z_m^{k,r} - \frac{\alpha^2}{z_m^{k,r}})\\ &= -\frac{1}{2}(T^m(u_m^{k,r})^s - T^m(u^*)^s)(z_m^{k,r} - \frac{\alpha^2}{z_m^{k,r}}).\end{aligned} \quad (3.43)$$

Since $z_m^{k,r} \geq \alpha$, we have $z_m^{k,r} \geq z_m^{k,r} - \alpha^2/z_m^{k,r} \geq z_m^{k,r} - \alpha \geq 0$ and then we can write

$$\begin{aligned}&\partial_t z_m^k - \frac{|\nabla(z_m^{k,r}-\alpha)|^2}{z_m^{k,r}} - \Delta(z_m^{k,r}-\alpha) + T^m(u^*)^s(z_m^{k,r}-\alpha)\\ &\leq \frac{1}{2}|T^m(u_m^{k,r})^s - T^m(u^*)^s|z_m^{k,r}.\end{aligned} \quad (3.44)$$

Analyzing the term on the right hand side of (3.44), we have

$$T^m(u_m^{k,r})^s - T^m(u^*)^s \text{ is bounded in } L^2(0,\infty;L^{3/2}(\Omega)). \tag{3.45}$$

In fact, using Lemma 1.10, we get

$$|T^m(u_m^{k,r})^s - T^m(u^*)^s| \leq s^{3/2}|T^m(u_m^{k,r}) + T^m(u^*)|^{s-1}|u_m^{k,r} - u^*|$$

and therefore

$$\begin{aligned}&\int_\Omega |T^m(u_m^{k,r})^s - T^m(u^*)^s|^{3/2}\,dx\\ &\leq s^{3/2}\int_\Omega |T^m(u_m^{k,r}) + T^m(u^*)|^{3(s-1)/2}\,dx \int_\Omega |u_m^{k,r} - u^*|^{3/2}\,dx.\end{aligned}$$

We use Hölder's inequality with the conjugate exponents $2s/(3s-3)$ and $2s/(3-s)$ to obtain

$$\|T^m(u_m^{k,r})^s - T^m(u^*)^s\|^2_{L^{3/2}(\Omega)} \leq s^2\|T^m(u_m^{k,r}) + T^m(u^*)\|^{2(s-1)}_{L^s(\Omega)}\|u_m^{k,r} - u^*\|^2_{L^{3s/(3-s)}(\Omega)}.$$

Then, considering the (m,k)-uniform bound (3.41) and the Sobolev embedding $W^{1,s}(\Omega) \subset L^{3s/(3-s)}(\Omega))$ we obtain (3.45).

With this information, now we can test (3.44) by $k\ (z_m^{k,r} - \alpha)$, obtaining, for each time interval (t_{n-1}, t_n),

$$\begin{aligned}&\frac{1}{2}(\|z_m^n - \alpha\|^2_{L^2(\Omega)} - \|z_m^{n-1} - \alpha\|^2_{L^2(\Omega)})\\ &+ k\|\nabla(z_m^n - \alpha)\|^2_{L^2(\Omega)} + k\frac{T^m(u^*)^s}{2}\|z_m^n - \alpha\|^2_{L^2(\Omega)}\\ &\leq Ck\int_\Omega |T^m(u_m^n)^s - T^m(u^*)^s| z_m^n(z_m^n - \alpha)\,dx + \int_\Omega \frac{|\nabla z_m^n|^2}{z_m^n}(z_m^n - \alpha)\,dx\\ &\leq Ck\|z_m^0\|_{L^\infty(\Omega)}\|T^m(u_m^n)^s - T^m(u^*)^s\|_{L^{3/2}(\Omega)}\|z_m^n - \alpha\|_{L^3(\Omega)} + k\|\nabla z_m^n\|^2_{L^2(\Omega)}\\ &\leq C(\delta)k\|T^m(u_m^n)^s - T^m(u^*)^s\|^2_{L^{3/2}(\Omega)} + \delta k\|z_m^n - \alpha\|^2_{L^2(\Omega)} + (1+\delta)k\|\nabla z_m^n\|^2_{L^2(\Omega)}.\end{aligned}$$

Note that if $u_0 \not\equiv 0$ then have $T^m(u^*) = u^* > 0$, for all $m \geq u^*$. Hence, choosing $\delta > 0$ small enough, we can conclude that, for $m \geq u^*$, there is $\beta > 0$ such that

$$\begin{aligned}&\frac{1}{2}(\|z_m^n - \alpha\|^2_{L^2(\Omega)} - \|z_m^{n-1} - \alpha\|^2_{L^2(\Omega)}) + k\beta\|z_m^n - \alpha\|^2_{L^2(\Omega)}\\ &\leq Ck\|T^m(u_m)^s - T^m(u^*)^s\|^2_{L^{3/2}(\Omega)} + k\|\nabla z_m^n\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore, summing in n and using (3.45) and Lemma 3.5 we obtain

$$z_m^{k,r} - \alpha \text{ is bounded in } L^2(0,\infty;L^2(\Omega)). \tag{3.46}$$

Hence, in view of (3.27), (3.28) and (3.46) we have

$$z_m^{k,r} - \alpha \text{ is bounded in } L^2(0,\infty;H^2(\Omega)).$$

With the m-uniform bounds obtained so far we can derive a m-uniform bound for $\partial_t z_m^k$ in $L^2(0,\infty;L^{3/2}(\Omega))$. In fact, we notice that

$$0 \leq z_m^{k,r} - \frac{\alpha^2}{z_m^{k,r}} = \frac{(z_m^{k,r})^2 - \alpha^2}{z_m^{k,r}} \leq \frac{(z_m^{k,r})^2 - \alpha^2}{\alpha} \leq \frac{\|z^0\|_{L^\infty(\Omega)} + \alpha^2}{\alpha}(z_m^{k,r} - \alpha),$$

hence, in view of (3.46), we also have

$$z_m^{k,r} - \frac{\alpha^2}{z_m^{k,r}} \text{ is bounded in } L^2(0,\infty;L^2(\Omega)). \tag{3.47}$$

Therefore, going back to (3.43), reminding that $z_m^{k,r}$ is uniformly bounded in $L^\infty(0,\infty;L^\infty(\Omega))$ with respect to m and k and considering (3.47) and (3.45), we conclude that

$$\partial_t z_m^k \text{ is bounded in } L^2(0,\infty;L^{3/2}(\Omega)).$$

Now we are going to obtain compactness for $\{u_m^k\}$ and $\{u_m^{k,r}\}$, which is necessary in order to pass to the limit as $m \to \infty$ and $k \to 0$ in the nonlinear terms of the equations of (3.3). Because of (3.33) and (3.35), we have that

$$(u_m^{k,r} + 1)^{s/2} \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

for every finite $T > 0$. Using the Sobolev inequality $H^1(\Omega) \subset L^6(\Omega)$ and interpolation inequalities we obtain

$$(u_m^{k,r})^{s/2} \text{ is bounded in } L^{10/3}(0,T;L^{10/3}(\Omega)),$$

which is equivalent to

$$u_m^{k,r} \text{ is bounded in } L^{5s/3}(0,T;L^{5s/3}(\Omega)). \tag{3.48}$$

By using (3.33) and (3.48) in (3.39) (remind that $s \in [1,2)$), we also have

$$u_m^{k,r} \text{ is bounded in } L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)). \tag{3.49}$$

For any norm $\|\cdot\|$ we have

$$\|u_m^k(t)\| \leq \|u_m^k(t) - u_m^{k,r}(t)\| + \|u_m^{k,r}(t)\|,$$

$$\|u_m^k(t) - u_m^{k,r}(t)\| \leq \|u_m^n - u_m^{n-1}\| \leq \|u_m^n\| + \|u_m^{n-1}\|, \ \forall t \in (t_{n-1}, t_n),$$

because of (3.48) and (3.49) we can also conclude that

$$u_m^k \text{ is bounded in } L^{5s/3}(0,T;L^{5s/3}(\Omega)) \cap L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)). \tag{3.50}$$

We observe that $W^{1,5s/(3+s)}(\Omega) \subset L^q(\Omega)$, with continuous embedding for $q = 15s/(9-2s)$ and compact embedding for $q \in [1, 15s/(9-2s))$. Then, since $s \in [1,2)$, we have $5s/3 < 15s/(9-2s)$ and therefore the embedding $W^{1,5s/(3+s)}(\Omega) \subset L^{5s/3}(\Omega)$ is compact. Note also that $q = 5s/3 \geq 5/3 > 1$. Now we can use Lemma 1.14 with

$$X = W^{1,5s/(3+s)}(\Omega), \qquad B = L^{5s/3}(\Omega), \qquad Y = \left(H^3(\Omega)\right)'$$

to conclude that there is a subsequence of $\{u_m^k\}$ (still denoted by $\{u_m\}$) and a limit function u such that

$$\begin{gathered} u_m^k \longrightarrow u \text{ weakly in } L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)),\ \forall T > 0, \\ u_m^k \longrightarrow u \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p \in [1, 5s/3),\ \forall T > 0. \end{gathered}$$

We note that, because of the (m,k)-independent bounds obtained for $u_m^{k,r}$ and the convergence (3.36), we can conclude that these convergences are also valid if we replace u_m^k by $u_m^{k,r}$. Therefore we have

$$u_m^{k,r} \longrightarrow u \text{ weakly in } L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)),\ \forall T > 0,$$

$$u_m^{k,r} \longrightarrow u \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p \in [1, 5s/3),\ \forall T > 0. \tag{3.51}$$

Using the Dominated Convergence Theorem we can conclude from (3.51) that

$$T^m(u_m^{k,r}) \to u \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p \in [1, 5s/3),\ \forall T > 0. \tag{3.52}$$

It stems from the convergence (3.52) and Lemma 1.11 that

$$(T^m(u_m^{k,r}))^s \to u^s \text{ strongly in } L^q(0,T;L^q(\Omega)),\ \forall q \in [1, 5/3),\ \forall T > 0. \tag{3.53}$$

The compactness of $\{z_m^{k,r}\}$ is also necessary. Concerning the functions $z_m^{k,r}$ and z_m^k, if we consider the (m,k)-independent bounds derived so far, (3.30) and use the compactness result Lemma 1.14 then we conclude that there are subsequences of $\{z_m^{k,r}\}$ and $\{z_m^k\}$ (still denoted by $\{z_m^{k,r}\}$ and $\{z_m^k\}$) and a limit function z such that, for each $T > 0$,

$$\begin{gathered} z_m^{k,r} \to z \text{ weakly* in } L^\infty(0,\infty;L^\infty(\Omega)) \cap L^\infty(0,\infty;H^1(\Omega)), \\ z_m^{k,r} \to z \text{ weakly in } L^2(0,\infty;H^2(\Omega)), \\ z_m^{k,r} \to z \text{ strongly in } L^2(0,T;H^1(\Omega)) \cap L^p(0,T;L^p(\Omega)), p \in [1,\infty), \\ \nabla z_m^{k,r} \to \nabla z \text{ weakly in } L^4(0,\infty;L^4(\Omega)), \\ \text{and } \partial_t z_m^k \to \partial_t z \text{ weakly in } L^2(0,\infty;L^{3/2}(\Omega)). \end{gathered} \tag{3.54}$$

Taking the nonlinear terms of (3.25), where we have to pass to the limit as $(m,k) \to$

$(\infty, 0)$, it is convenient to consider two functions of $z_m^{k,r}$, namely $g_1(z_m^{k,r}) = (z_m^{k,r})^2$ and $g_2(z_m^{k,r}) = 1/z_m^{k,r}$. Since $0 < \alpha \leq z_m^{k,r} \leq \|z^0\|_{L^\infty(\Omega)}$, we can use the (m,k)-independent bounds derived so far and (3.54) to show that

$$\begin{aligned} 1/z_m^{k,r} \to 1/z \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \text{for each } T>0 \text{ and } p\in[1,\infty),\\ |\nabla z_m^{k,r}|^2 \to |\nabla z|^2 \text{ weakly in } L^2(0,\infty;L^2(\Omega)),\\ \nabla(z_m^{k,r})^2 \to \nabla(z)^2 \text{ weakly in } L^4(0,\infty;L^4(\Omega)). \end{aligned} \tag{3.55}$$

Now we are going to use the weak and strong convergences that we proved for $u_m^{k,r}$, u_m^k, $z_m^{k,r}$ and z_m^k to pass to the limit as $(m,k)\to(\infty,0)$ in the equations of problem (3.3). We are going to identify the limits of the nonlinear terms $T^m(u_m^{k,r})\nabla(z_m^{k,r})^2$, $\dfrac{|\nabla z_m^{k,r}|^2}{z_m^{k,r}}$, $T^m(u_m^{k,r})^s z_m^{k,r}$ and $\dfrac{T^m(u_m^{k,r})^s}{z_m^{k,r}}$, respectively, with $u\nabla(z)^2$, $\dfrac{|\nabla z|^2}{z}$, $u^s z$ and $\dfrac{u^s}{z}$. In fact, considering the chemotaxis term, because of (3.52), (3.27) and (3.55), we can conclude that

$$T^m(u_m^{k,r})\nabla(z_m^{k,r})^2 \longrightarrow u\nabla(z)^2 \text{ weakly in } L^{20s/(5s+12)}(0,T;L^{20s/(5s+12)}(\Omega)),$$

for each fixed $T>0$. Using (3.54) and (3.55) we can also conclude that

$$\frac{|\nabla z_m^{k,r}|^2}{z_m^{k,r}} \longrightarrow \frac{|\nabla z|^2}{z} \text{ weakly in } L^2(0,T;L^2(\Omega)),\ \text{for each fixed } T>0.$$

Regarding the consumption term, considering (3.53) and (3.54) we prove that

$$T^m(u_m^{k,r})^s z_m^{k,r} \longrightarrow u^s z \text{ weakly in } L^{5/3}(0,T;L^{5/3}(\Omega)),\ \text{for each fixed } T>0.$$

Finally, using (3.53) and (3.55) we obtain

$$\frac{T^m(u_m^{k,r})^s}{z_m^{k,r}} \longrightarrow \frac{u^s}{z} \text{ weakly in } L^{5/3}(0,T;L^{5/3}(\Omega)),\ \text{for each fixed } T>0.$$

With these identifications and all previous convergences, it is possible to pass to the limit as $(m,k)\to(\infty,0)$ in each term of the equations of (3.3). This finishes the proof of existence of a solution (u,z) of (3.1) for $s\in[1,2)$. In order to finish, we will prove the regularity (up to infinite time) of u.

From (3.33) and (3.35) there exists a subsequence of $\{(u_m^k+1)^{s/2}\}$, still denoted by $\{(u_m^k+1)^{s/2}\}$, and a limit function φ such that

$$\begin{aligned} (u_m^{k,r}+1)^{s/2} \longrightarrow \varphi \ \text{ weakly* in } L^\infty(0,\infty;L^2(\Omega))\\ \nabla(u_m^{k,r}+1)^{s/2} \longrightarrow \nabla\varphi \ \text{ weakly in } L^2(0,\infty;L^2(\Omega)). \end{aligned}$$

Then, using the strong convergence (3.51), the continuity of the function $u_m^{k,r}\mapsto f(u_m^{k,r}) = (u_m^{k,r}+1)^{s/2}$ and the Dominated Convergence Theorem, we prove that $\varphi = (u+1)^{s/2}$.

Analogously, because of (3.28) we can conclude that, up to a subsequence, there is a limit function ϕ such that

$$T^m(u_m^{k,r})^{s/2}\nabla z_m^{k,r} \longrightarrow \phi \text{ weakly in } L^2(0,\infty;L^2(\Omega)).$$

And using the convergences (3.53) and (3.54) we can conclude that $\phi = u^{s/2}\nabla z$.

Therefore we have proved the global in time regularity

$$\begin{gathered}(u+1)^{s/2} \in L^\infty(0,\infty;L^2(\Omega)), \quad \nabla(u+1)^{s/2} \in L^2(0,\infty;L^2(\Omega)),\\ u^{s/2}\nabla z \in L^2(0,\infty;L^2(\Omega)).\end{gathered} \tag{3.56}$$

Considering (3.56) and proceeding as in the obtaining of (3.41) and (3.42) we conclude the global in time regularity $u \in L^\infty(0,\infty;L^s(\Omega))$, ∇u, $u\nabla(z)^2 \in L^2(0,\infty;L^s(\Omega))$. This finishes the proof that (u,z) is a weak solution of (3.1) and that $\{(u_m^{k,r}, z_m^{k,r})\}$ converges to (u,z) as $(m,k) \to (\infty,0)$ in the sense indicated in this section, for $s \in [1,2)$.

3.3.3 Estimates for $(u_m^{k,r}, z_m^{k,r})$ and passage to the limit for $s \geq 2$

The procedure for the case $s \geq 2$ is much more similar to the case $s \geq 2$ in Chapter 2. In the sequel, we highlight the main steps of the proof and refer the reader to Chapter 2 for details. First we note that, multiplying the energy inequality (3.23) from Lemma 3.8 by k and summing in n, for each $n \in \mathbb{N}$, we have

$$\nabla T^m(u_m^{k,r})^{s/2} \text{ is bounded in } L^2(0,\infty;L^2(\Omega)), \tag{3.57}$$

$$T^m(u_m^{k,r})^{s/2} \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)). \tag{3.58}$$

From (3.58) and (3.57) we can conclude that

$$T^m(u_m^{k,r}) \text{ is bounded in } L^{5s/3}(0,T;L^{5s/3}(\Omega)). \tag{3.59}$$

Analogously to Chapter 2 we use (3.28) and Lemma 3.5 to prove that

$$T^m(u_m^{k,r})\nabla(z_m^{k,r})^2 \text{ is bounded in } L^2(0,\infty;L^2(\Omega)). \tag{3.60}$$

Now we can test the u_m^n-equation of problem (3.3) by ku_m^n and, after bounding some terms, we sum the resulting inequality and use (3.60) to conclude that

$$u_m^{k,r} \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)), \tag{3.61}$$

$$\nabla u_m^{k,r} \text{ is bounded in } L^2(0,\infty;L^2(\Omega)), \tag{3.62}$$

$$\sum_{j=1}^{\infty} \|u^j - u^{j-1}\|_{L^2(\Omega)}^2 \leq C, \tag{3.63}$$

and then, analogously to (3.30), we prove that

$$\|u_m^{k,r} - u_m^k\|^2_{L^2(0,\infty;L^2(\Omega))} \le Ck. \tag{3.64}$$

Then, if we consider the u-equation of (3.25), by applying (3.62) and (3.60) we conclude that

$$\partial_t u_m^k \text{ is bounded in } L^2(0,\infty;(H^1(\Omega))'). \tag{3.65}$$

Using the (m,k)-independent bounds obtained so far, we can follow the ideas of Chapter 2 and subsection 3.3.2 in order to prove that

$$z_m^{k,r} - \alpha \text{ is bounded in } L^2(0,\infty;L^2(\Omega)), \tag{3.66}$$

$$\partial_t z_m^k \text{ is bounded in } L^2(0,\infty;L^{3/2}(\Omega)).$$

Now, using (3.61), (3.62), (3.64) and (3.65) we can conclude that there are subsequences of $\{u_m^{k,r}\}$ and $\{u_m^k\}$, still denoted by $\{u_m^{k,r}\}$ and $\{u_m^k\}$, and a limit function u such that

$$\begin{aligned} u_m^{k,r} &\longrightarrow u \text{ weakly* in } L^\infty(0,\infty;L^2(\Omega)),\\ \nabla u_m^{k,r} &\longrightarrow \nabla u \text{ weakly in } L^2(0,\infty;L^2(\Omega)),\\ \partial_t u_m^k &\longrightarrow u \text{ weakly in } L^2\Big(0,\infty;\big(H^1(\Omega)\big)'\Big). \end{aligned}$$

By applying the compactness result Lemma 1.14 to the sequence $\{u_m^k\}$ and using (3.64) we have

$$u_m^{k,r} \longrightarrow u \text{ strongly in } L^2(0,T;L^2(\Omega)),\ \forall T > 0.$$

Using the Dominated Convergence Theorem and (3.59) we can also prove that

$$T^m(u_m^{k,r}) \longrightarrow u \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p \in (1, 5s/3),$$

and using Lemma 1.11,

$$T^m(u_m^{k,r})^s \longrightarrow u^s \text{ strongly in } L^p(0,T;L^p(\Omega)),\ \forall p \in (1, 5/3).$$

From the global in time estimate (3.58) we can conclude that, up to a subsequence,

$$T^m(u_m^{k,r}) \to u \text{ weakly* in } L^\infty(0,\infty;L^s(\Omega)),$$

hence, in particular, $u \in L^\infty(0,\infty;L^s(\Omega))$.

For $s \ge 2$, if we consider the functions $z_m^{k,r}$ and z_m^k, we have the same (m,k)-independent estimates that we had for $s \in [1,2)$. Then we have the same convergences given in (3.54) and (3.55).

Following the ideas of Subsection 3.3.2, we can identify the limits of $T^m(u_m^{k,r})\nabla(z_m^{k,r})^2$, $|\nabla z_m^{k,r}|^2/z_m^{k,r}$, $T^m(u_m^{k,r})^s z_m^{k,r}$ and $T^m(u_m^{k,r})^s/z_m^{k,r}$ with $u\nabla(z)^2$, $|\nabla z|^2/z$, $u^s z$ and u^s/z, respectively.

This finishes the proof that (u, z) is a solution of (3.1) and that $\{(u_m^{k,r}, z_m^{k,r})\}$ converges to (u, z) as $(m, k) \to (\infty, 0)$ in the sense indicated in this section, for $s \geq 2$.

3.4 Convergence of $v_m^{k,r}$

Until this point, for any $s \geq 1$ fixed, we have proved that $(u_m^{k,r}, z_m^{k,r})$ converges to (u, z) a solution of (3.1) as $(m, k) \to (\infty, 0)$. Now, to conclude the proof of Theorem 3.2, we are going to prove that $(u_m^{k,r}, v_m^{k,r})$ converges to (u, v) a weak solution of (1) as $(m, k) \to (\infty, 0)$, where $v_m^{k,r}$ is given either by (3.4) or by (3.5). For simplicity, we consider $(u_m^{k,r}, z_m^{k,r})$ to be the subsequence which converge to the limit function (u, z).

3.4.1 $v_m^{k,r}$ given by (3.4)

In this case, it is enough to show that the sequence $v_m^{k,r} = (z_m^{k,r})^2 - \alpha^2$ converges to $v = z^2 - \alpha^2$ as $(m, k) \to (\infty, 0)$. Then, thanks to the equivalence of problems (3.1) and (1) (Lemma 3.1), we know that (u, z) is a solution of (3.1) if, and only if, (u, v) is a solution of (1), with $v = z^2 - \alpha^2$.

In fact, if we consider the (m, k)-uniform bounds obtained for $z_m^{k,r}$ (especially the pointwise estimates of Theorem 3.4, (3.27) and (3.28)) and the convergences listed in (3.54), we can prove by straightforward calculations that the sequence $v_m^{k,r} = (z_m^{k,r})^2 - \alpha^2$ converges to $v = z^2 - \alpha^2$ in the same senses indicated in (3.54). Hence, by Lemma 3.1, we have that (u, v) is a solution of (1).

Therefore, we conclude that $(u_m^{k,r}, v_m^{k,r})$ converges to (u, v), a solution of (1), as $(m, k) \to (\infty, 0)$.

3.4.2 $v_m^{k,r}$ given by (3.5)

We rewrite (3.5) as

$$\partial_t v_m^k - \Delta v_m^{k,r} + T^m(u_m^{k,r})^s v_m^{k,r} = 0. \tag{3.67}$$

Using the (m, k)-independent bounds obtained for $T^m(u_m^n)$ in Subsections 3.3.2 and 3.3.3 we can test (3.5) by v_m^n and by $-\Delta v_m^n$ and conclude that

$$v_m^{k,r} \text{ is bounded in } L^\infty(0, \infty; H^1(\Omega)), \tag{3.68}$$

$$\Delta v_m^{k,r} \text{ is bounded in } L^2(0, \infty; L^2(\Omega)), \tag{3.69}$$

$$\|v_m^k - v_m^{k,r}\|_{L^2(0,\infty;H^1(\Omega))} \leq C\, k. \tag{3.70}$$

Considering the ideas used in Subsections 3.3.2 and 3.3.3 to show that $z_m^{k,r} - \alpha$ is bounded in $L^2(0, \infty; H^2(\Omega))$ and that $\partial_t z_m^k$ is bounded in $L^2(0, \infty; L^{3/2}(\Omega))$, we prove that

$$v_m^{k,r} \text{ is bounded in } L^2(0, \infty; H^2(\Omega)), \tag{3.71}$$

$$\partial_t v_m^k \text{ is bounded in } L^2(0, \infty; L^{3/2}(\Omega)). \tag{3.72}$$

Then, using (3.68), (3.69), (3.70), (3.71) (3.72) and Lemma 1.14 we prove that there is a function v such that, up to a subsequence, for each fixed $T > 0$, we have

$$\begin{gathered} v_m^{k,r} \to v \text{ weakly* in } L^\infty(0,\infty;L^\infty(\Omega)) \cap L^\infty(0,\infty;H^1(\Omega)), \\ v_m^{k,r} \to v \text{ weakly in } L^2(0,\infty;H^2(\Omega)), \\ v_m^{k,r} \to v \text{ strongly in } L^2(0,T;H^1(\Omega)) \cap L^p(0,T;L^p(\Omega)), p \in [1,\infty), \\ \text{and } \partial_t v_m^k \to \partial_t v \text{ weakly in } L^2(0,\infty;L^{3/2}(\Omega)). \end{gathered} \tag{3.73}$$

Now, using the convergence obtained for $T^m(u_m^{k,r})$ in Subsections 3.3.2 and 3.3.3 and (3.73) we conclude that the limit function v is the unique solution of

$$\partial_t v - \Delta v + u^s v = 0,$$

where u is the function of the pair (u, z), fixed in the beginning of Subsection 3.4. Therefore, thanks to the uniqueness of the limit function v, we conclude that the whole sequence $v_m^{k,r}$ converges towards v as $(m, k) \to (\infty, 0)$. In addition, combining Lemma 3.1 and the uniqueness of the function v, given u, we deduce that $v = z^2 - \alpha^2$.

Thus we conclude that $\{(u_m^{k,r}, v_m^{k,r})\}$ converges to (u, v), a solution of (1), as $(m, k) \to (\infty, 0)$ in both cases, v_m^n given by (3.4) or (3.5), finishing the proof of Theorem 3.2.

Chapter 4

OPTIMAL CONTROL RELATED TO WEAK SOLUTIONS OF CHEMOTAXIS-CONSUMPTION MODELS

4.1 Main Results

Along this chapter, it will be necessary to impose the hypotheses:

$$\Omega \subset \mathbb{R}^3 \text{ is a bounded domain with boundary } \Gamma \text{ of class } C^{2,1},$$

$$Q = (0,T) \times \Omega,$$

$$f \in L^q(Q), \text{ for some } q > 5/2, \tag{4.1}$$

$$(u^0, v^0) \in L^p(\Omega) \times W^{2-2/q,q}(\Omega), \tag{4.2}$$

with $p = 1+\varepsilon$, for some $\varepsilon > 0$, if $s = 1$, and $p = s$, if $s > 1$.

Remark 4.1. In particular, Hypothesis (H1) is satisfied. □

Let us define the specific functional spaces appearing for the weak solution setting. For $s \in [1,2)$,

$$\mathcal{U} = \Big\{ u \in L^\infty(0,T;L^s(\Omega)) \,\cap\, L^{5s/3}(Q) \,\cap\, L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)); \\ \partial_t u \in L^{5s/(3+s)}(0,T;(W^{1,5s/(4s-3)}(\Omega))') \Big\},$$

for $s \geq 2$,

$$\mathcal{U} = \Big\{ u \in L^\infty(0,T;L^s(\Omega)) \,\cap\, L^{5s/3}(Q) \,\cap\, L^2(0,T;H^1(\Omega)); \\ \partial_t u \in L^2(0,T;H^1(\Omega))') \Big\},$$

and for $s \geq 1$,

$$\mathcal{V} = \Big\{ v \in L^\infty(Q) \cap L^\infty(0,T;H^1(\Omega)) \cap L^4(0,T;W^{1,4}(\Omega)) \cap L^2(0,T;H^2(\Omega)); \\ \partial_t v \in L^{5/3}(Q) \Big\}.$$

We also introduce the bounded convex set for the control

$$B_q(M) = \Big\{ f \in L^q(Q) \;\Big|\; \|f\|_{L^q(Q)} \leq M \Big\}.$$

Definition 4.2 (Weak Solution of the controlled problem (2)). A pair (u, v) is called a weak solution of (2) if $u(t,x), v(t,x) \geq 0$ *a.e.* $(t,x) \in Q$, with

$$u \in \mathcal{U}, \;\; v \in \mathcal{V}$$

and satisfying the initial conditions for (u, v), the u-equation of (2) and the boundary condition of u in the variational sense

$$(\partial_t u, \varphi) + (\nabla u, \nabla \varphi) = (u \nabla v, \nabla \varphi)\,,$$

for all $\varphi \in L^{5s/(4s-3)}(0,T;W^{1,5s/(4s-3)}(\Omega))$, the v-equation pointwisely (in fact, the v-equation is satisfied in $L^{5/3}(Q)$) and, since $\Delta v \in L^2(Q)$, the boundary condition of v in the sense of $H^{-1/2}(\Gamma)$. □

The proof of existence of weak solution to the controlled problem (2) is based in the treatment of the uncontrolled model (1) given in Chapter 2. This is because Chapter 2 is oriented to weak solutions and then it is well suited to extend (1) to a model in which we have a nonsmooth control f as coefficient in (2). An important step in Chapter 2 is the obtaining of an energy inequality using the change of variable from (u, v) to (u, z), with $z = \sqrt{v + \alpha^2}$, where $\alpha > 0$ is a sufficiently small but fixed real number chosen in Lemmas 4.12, 4.13 and 4.15 below, independently of (u, v) and f. Here, the energy inequality satisfied by the constructed weak solutions of (2) will also be written in terms of (u, z). In fact, we consider the energy

$$E(u,z)(t) = \frac{s}{4}\int_\Omega g(u(t,x))\;dx + \frac{1}{2}\int_\Omega |\nabla z(t,x)|^2\;dx,$$

where

$$g(u) = \begin{cases} (u+1)ln(u+1) - u, & \text{if } s = 1, \\ \dfrac{u^s}{s(s-1)}, & \text{if } s > 1. \end{cases}$$

We have the following result of existence of weak solutions to (2).

Theorem 4.3 (Existence of energy inequality weak solutions). *Given $f \in L^q(Q)$, there is a non-negative weak solution (u, v) of (2). Moreover, this weak solution*

(u,v) can be built satisfying the energy inequality

$$
\begin{aligned}
E(u,z)(t_2) + \beta\int_{t_1}^{t_2}\int_\Omega |\nabla[u+1]^{s/2}|^2\,dx\,dt + \frac{1}{4}\int_{t_1}^{t_2}\int_\Omega u^s|\nabla z|^2\,dx\,dt \\
+\beta\Big(\int_{t_1}^{t_2}\int_\Omega |D^2 z|^2\,dx\,dt + \int_{t_1}^{t_2}\int_\Omega \frac{|\nabla z|^4}{z^2}\,dx\,dt\Big) \\
\le E(u,z)(t_1) + \mathcal{K}(\|f\|_{L^q(Q)}, \|v_0\|_{W^{2-2/q,q}(\Omega)}),
\end{aligned}
\tag{4.3}
$$

for a.e. $t_1, t_2 \in [0,T]$, with $t_2 > t_1$; where $\mathcal{K}(\|f\|_{L^q(Q)}, \|v_0\|_{W^{2-2/q,q}(\Omega)})$ is a continuous and increasing function with respect to $\|f\|_{L^q(Q)}$ and $\beta > 0$ is a constant, independent of (u,v,f). Finally, in case $s > 1$, inequality (4.3) *also holds for all $t_2 \in (t_1, T]$.*

Remark 4.4. The existence of weak solutions satisfying an energy inequality is commonly seen, for instance for fluid models, and is used to prove either weak-strong uniqueness results [42] or large time behavior [45]. In this chapter, we use this "energy inequality weak solution setting" in order to prove the existence of a global optimal solution to an optimal control problem. To the best of our knowledge in chemotaxis PDE problems, this is the first time that the concept of weak solution with energy inequality is applied to this purpose. □

To highlight the main results of the present chapter we introduce the minimization problems. Consider the functional

$$
J : L^{5s/3}(Q) \times L^2(Q) \times L^q(Q) \longrightarrow \mathbb{R}
$$

given by

$$
\begin{aligned}
J(u,v,f) := \frac{3\gamma_u}{5s}\int_0^T \|u(t)-u_d(t)\|^{5s/3}_{L^{5s/3}(\Omega)}\,dt \\
+\frac{\gamma_v}{2}\int_0^T \|v(t)-v_d(t)\|^2_{L^2(\Omega)}\,dt + \frac{\gamma_f}{q}\int_0^T \|f(t)\|^q_{L^q(\Omega)}\,dt,
\end{aligned}
$$

where $(u_d, v_d) \in L^{5s/3}(Q) \times L^2(Q)$ represents the desired states and $\gamma_u, \gamma_v, \gamma_f > 0$ measure the costs of the states and control. In view of the existence result, Theorem 4.3, one could expect the following admissible sets

$$
\begin{aligned}
S^w_{ad} = \Big\{&(u,v,f) \in \mathcal{U}\times\mathcal{V}\times L^q(Q);\, (u,v) \text{ is a} \\
&\text{weak solution of (2) with control } f\Big\}
\end{aligned}
$$

or

$$
\begin{aligned}
S^E_{ad} = \Big\{&(u,v,f) \in \mathcal{U}\times\mathcal{V}\times L^q(Q);\, (u,v) \text{ is a weak solution of (2)} \\
&\text{with control } f \text{ and satisfies the energy inequality (4.3)}\Big\}
\end{aligned}
$$

with the corresponding minimization problems

$$
\left\{
\begin{array}{l}
min\ J(u,v,f) \\
\text{subject to } (u,v,f) \in S^w_{ad},
\end{array}
\right.
\tag{4.4}
$$

or

$$\begin{cases} min\ J(u,v,f) \\ \text{subject to } (u,v,f) \in S_{ad}^E. \end{cases} \tag{4.5}$$

Thanks to Theorem 4.3 we have that both S_{ad}^w and S_{ad}^E are nonempty sets. However, we are not able to prove that problem (4.4) or (4.5) has a solution, as we will analyze in Remark 4.7 and Subsection 4.3.1, respectively.

Hence, to prove existence of optimal control related to weak solutions of (2), we define the following admissible set for each $M > 0$:

$$\begin{aligned} S_{ad}^M = \Big\{ & (u,v,f) \in \mathcal{U} \times \mathcal{V} \times B_q(M); (u,v) \text{ is a weak solution of} \\ & \text{(2) with control } f \text{ and satisfies (4.3) changing the} \\ & \text{constant } \mathcal{K}(\|f\|_{L^q(Q)}, \|v_0\|_{W^{2-2/q,q}(\Omega)}) \text{ for } \mathcal{K}(M, \|v_0\|_{W^{2-2/q,q}(\Omega)}) \Big\} \end{aligned}$$

and the corresponding minimization problem

$$\begin{cases} min\ J(u,v,f) \\ \text{subject to } (u,v,f) \in S_{ad}^M. \end{cases} \tag{4.6}$$

Again, from Theorem 4.3, we have $S_{ad}^M \neq \emptyset$. But now, we are able to prove the following.

Theorem 4.5 (Existence of optimal control). *For each $M > 0$, the optimal control problem* (4.6) *has at least one global optimal solution, that is, there is $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^M$ such that*

$$J(\overline{u}, \overline{v}, \overline{f}) = \min_{(u,v,f) \in S_{ad}^M} J(u,v,f).$$

Moreover, we have the following relation between the three minimization problems (4.4), (4.5) and (4.6).

Theorem 4.6. *For*

$$M \geq \frac{q}{\gamma_f} \inf_{(u,v,f) \in S_{ad}^E} J(u,v,f),$$

we have the relations

$$\inf_{(u,v,f) \in S_{ad}^w} J(u,v,f) \leq \min_{(u,v,f) \in S_{ad}^M} J(u,v,f) \leq \inf_{(u,v,f) \in S_{ad}^E} J(u,v,f).$$

Remark 4.7. From Theorem 4.5, for each $M > 0$ there is $(u^M, v^M, f^M) \in S_{ad}^M$ such that

$$J(u^M, v^M, f^M) = \min_{(u,v,f) \in S_{ad}^M} J(u,v,f).$$

Let $M_2 > M_1 > 0$. Since $S_{ad}^{M_1} \subset S_{ad}^{M_2}$ then $J(u^M, v^M, f^M)$ decreases as M increases. Therefore, since $J(u^M, v^M, f^M)$ is bounded from below, there exists $\lim_{M \to \infty} J(u^M, v^M, f^M)$

and, accounting for Theorem 4.6, we have

$$\inf_{(u,v,f)\in S_{ad}^w} J(u,v,f) \leq \lim_{M\to\infty} J(u^M,v^M,f^M) \leq \inf_{(u,v,f)\in S_{ad}^E} J(u,v,f).$$

In particular, $\{(u^M,v^M,f^M)\}_M$ is bounded in $L^{5s/3}(Q)\times L^2(Q)\times L^q(Q)$ independently of M. Thus we conclude that there is $(u^\infty,v^\infty,f^\infty)\in L^{5s/3}(Q)\times L^2(Q)\times L^q(Q)$, defined as the weak limit in $L^{5s/3}(Q)\times L^2(Q)\times L^q(Q)$ of a subsequence of $\{(u^M,v^M,f^M)\}_M$. Then, the lower semicontinuity of J leads to

$$J(u^\infty,v^\infty,f^\infty) \leq \lim_{M\to\infty} J(u^M,v^M,f^M). \tag{4.7}$$

In our opinion, the proof or the refutation of the following two questions are interesting open problems:

1. $(u^\infty,v^\infty,f^\infty)\in S_{ad}^w$?
2. $\displaystyle\lim_{M\to\infty} J(u^M,v^M,f^M) = \inf_{(u,v,f)\in S_{ad}^w} J(u,v,f)$?

If item 2 were valid, then $\displaystyle\inf_{(u,v,f)\in S_{ad}^w} J(u,v,f)$ could be approximated by the minimums $\displaystyle\min_{(u,v,f)\in S_{ad}^M} J(u,v,f)$ as $M\to\infty$. Additionally, if items 1 and 2 were valid, then $(u^\infty,v^\infty,f^\infty)$ would be an optimal solution of (4.4). Indeed, from item 1 we have

$$\inf_{(u,v,f)\in S_{ad}^w} J(u,v,f) \leq J(u^\infty,v^\infty,f^\infty).$$

On the other hand, from item 2 and (4.7), we also have

$$J(u^\infty,v^\infty,f^\infty) \leq \inf_{(u,v,f)\in S_{ad}^w} J(u,v,f),$$

which allows us to conclude that

$$J(u^\infty,v^\infty,f^\infty) = \inf_{(u,v,f)\in S_{ad}^w} J(u,v,f),$$

that is, $(u^\infty,v^\infty,f^\infty)\in S_{ad}^w$ is a global optimal solution of (4.4). □

The rest of the chapter is organized as follows. The existence of weak solutions satisfying an energy inequality for the controlled problem given in Theorem 4.3 is established in Section 4.2 and in Section 4.3 we study the optimal control problem, proving Theorem 4.5 and Theorem 4.6.

4.2 Existence for the Bilinear Controlled Model

The existence of weak solutions of the uncontrolled problem (1) is proved in Chapter 2, by means of a sequence of truncated problems. Here, we prove the existence of the controlled problem (2) satisfying in addition the energy inequality (4.3), using the following controlled truncated problems:

$$\begin{cases} \partial_t u_m - \Delta u_m = -\nabla \cdot (T^m(u_m)\nabla v_m), \\ \partial_t v_m - \Delta v_m = -T^m(u_m)^s v_m + f v_m 1_{\Omega_c}, \\ \partial_{\mathbf{n}} u_m|_\Gamma = \partial_{\mathbf{n}} v_m|_\Gamma = 0, \quad u_m(0) = u_m^0, \quad v_m(0) = v^0, \end{cases} \tag{4.8}$$

for each $m \in \mathbb{N}$, with initial data u_m^0 and v^0 satisfying (4.2). In fact, u_m^0 is an adequate regularization of u^0 (see Chapter 2 for more details).

4.2.1 A L^∞ function that bounds v_m from above

In Chapter 2, where the uncontrolled model ($f \equiv 0$) is considered, a crucial step to prove the existence of a weak solution to (1) as a limit of solutions of the truncated models (4.8) was the obtaining of a m-independent bound for $\|v_m\|_{L^\infty(Q)}$. This remains essential now, with $f \not\equiv 0$. But, while in the case where $f \equiv 0$ this m-independent bound is obtained by straightforward calculations, it is not so obvious if we consider a control f such that $f_+ \not\equiv 0$.

The next result will help us build a function $w \in L^\infty$ in Theorem 4.9 below that will be an upper bound for v_m, for all m. Moreover, (4.10) will provide an estimate for $\|v_m\|_{L^\infty(Q)}$ in terms of the control f. We remark that it is because of this Lemma 4.8 that we need to assume that $f \in L^q(Q)$, for some $q > 5/2$.

Lemma 4.8. *Let Ω be a bounded domain of $\mathbb{R}^3$ such that Γ is of class C^2. Let $w^0 \in W^{2-2/q,q}(\Omega)$ and $\tilde f \in L^q(Q)$, for some $q > 5/2$. Then the problem*

$$\begin{cases} \partial_t w - \Delta w = \tilde f\, w, \quad \text{in } Q, \\ \partial_{\mathbf{n}} w|_\Gamma =, \quad \text{on } (0,T)\times\Gamma, \\ w(0,x) = w^0 \ \text{in } \Omega, \end{cases} \tag{4.9}$$

has a unique solution

$$w \in C([0,T]; W^{2-2/q,q}(\Omega)) \cap L^q(0,T; W^{2,q}(\Omega)),\ \partial_t w \in L^q(Q),$$

In particular, there is a positive constant $C(\|\tilde f\|_{L^q(Q)}, \|w^0\|_{W^{2-2/q,q}(\Omega)})$ such that

$$\|w\|_{L^\infty(Q)} \le C(\|\tilde f\|_{L^q(Q)}, \|w^0\|_{W^{2-2/q,q}(\Omega)}). \tag{4.10}$$

Proof. The key idea here is the injection $W^{2-2/q,q}(\Omega) \subset L^\infty(\Omega)$, the reason why we suppose that $q > 5/2$. The proof is divided in two steps.

Step 1 (Existence and uniqueness of problem (4.9)**):**

For any solution w of (4.9) such that

$$w \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),\ \partial_t w \in L^2(Q), \tag{4.11}$$

we have

$$\begin{aligned} &\|w(t)\|^p_{L^p(\Omega)} + \beta \textstyle\int_0^t \|\nabla[w(r)]^{p/2}\|^2_{L^2(\Omega)}\, dr \\ &\le \|w^0\|^p_{L^p(\Omega)} \exp\left(Cp^{5/2} \textstyle\int_0^t (\|\tilde f(r)\|^{5/2}_{L^{5/2}(\Omega)} + 1)\, dr\right), \end{aligned} \tag{4.12}$$

a.e. $t \in (0,T)$. In fact, we test the equation in (4.9) by pw^{p-1} and define $\tilde w := w^{p/2}$. Using inequality (1.2), we obtain

$$\begin{aligned} \frac{d}{dt}\|\tilde w(t)\|^2_{L^2(\Omega)} + \frac{4p(p-1)}{p^2}\int_0^t \|\nabla \tilde w\|^2_{L^2(\Omega)}\, dr &\le p\int_\Omega \tilde f\ \tilde w^2\, dx \\ &\le Cp\|\tilde f\|_{L^{5/2}(\Omega)}\|\tilde w\|^2_{L^{10/3}(\Omega)} \\ &\le Cp\|\tilde f\|_{L^{5/2}(\Omega)}\|\tilde w\|^{4/5}_{L^2(\Omega)}\|\tilde w\|^{6/5}_{H^1(\Omega)} \\ &\le C(\delta)p^{5/2}\|\tilde f\|^{5/2}_{L^{5/2}(\Omega)}\|\tilde w\|^2_{L^2(\Omega)} + \delta\|\tilde w\|^2_{L^2(\Omega)} + \delta\|\nabla\tilde w\|^2_{L^2(\Omega)}. \end{aligned}$$

Hence, choosing $\delta > 0$ small enough to absorb the last term in the right hand side and going back to the notation w we obtain

$$\frac{d}{dt}\|w(t)\|^p_{L^p(\Omega)} + \beta\int_0^t \|\nabla[w(r)]^{p/2}\|^2_{L^2(\Omega)}\, dr \le Cp^{5/2}(\|\tilde f\|^{5/2}_{L^{5/2}(\Omega)} + 1)\|w(t)\|^p_{L^p(\Omega)}$$

and Gronwall's Lemma leads us to (4.12).

For $\tilde f$ regular enough one can prove that (4.9) has a unique solution satisfying (4.11) by using Galerkin's method, for example. But accounting for the dependence of w on $\|\tilde f\|^{5/2}_{L^{5/2}(\Omega)}$ given by (4.12), we are actually able to prove that problem (4.9) has a unique strong solution satisfying (4.11) and (4.12) under a weaker assumption on the regularity of $\tilde f$. It is enough that $\tilde f \in L^{5/2}(Q)$, for instance. The uniqueness is proved by comparing two possibly distinct solutions of problem (4.9) and concluding that they are in fact the same solution.

Step 2 (Proof of the L^∞ estimate (4.10)):

Since $\tilde f \in L^q(Q)$ and $q > 5/2$, (4.12) implies that there are $\tilde q \in (5/2, q)$ and a positive constant $\tilde C(\|\tilde f\|_{L^q(Q)}, \|w^0\|_{W^{2-2/q,q}(\Omega)})$ such that $\tilde f w \in L^{\tilde q}(Q)$ with

$$\|\tilde f\ w\|_{L^{\tilde q}(Q)} \le \tilde C(\|\tilde f\|_{L^q(Q)}, \|w^0\|_{W^{2-2/q,q}(\Omega)}). \tag{4.13}$$

From (4.13) and (1.12) we can conclude,

$$\|w\|_{C([0,T];W^{2-2/\tilde q,\tilde q}(\Omega))} \le C(\|\tilde f\|_{L^q(Q)}, \|w^0\|_{W^{2-2/q,q}(\Omega)}).$$

But since $\tilde q > 5/2$, we have $C([0,T];W^{2-2/\tilde q,\tilde q}(\Omega)) \subset L^\infty(Q)$ and there is another positive constant $C(\|\tilde f\|_{L^q(Q)}, \|w^0\|_{W^{2-2/q,q}(\Omega)})$ such that

$$\|w\|_{L^\infty(Q)} \le C(\|\tilde f\|_{L^q(Q)}, \|w^0\|_{W^{2-2/q,q}(\Omega)}).$$

Finally, since $w \in L^\infty(Q)$, we have $\tilde{f}\ w \in L^q(Q)$ and therefore we can use Lemma 1.13 to conclude that

$$w \in C([0,T]; W^{2-2/q,q}(\Omega)) \cap L^q(0,T; W^{2,q}(\Omega)),\ \partial_t w \in L^q(Q).$$

■

4.2.2 Existence for the controlled truncated problem and the first uniform estimates

Theorem 4.9. *Given f satisfying* (4.1) *and* (u^0, v^0) *satisfying* (4.2), *there is a unique solution* (u_m, v_m) *of the truncated problem* (4.8) *with regularity*

$$\begin{gathered} u_m \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)),\ \partial_t u_m \in L^2(0,T; (H^1(\Omega))'), \\ v_m \in L^\infty(Q) \cap L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),\ \partial_t v_m \in L^2(Q), \end{gathered} \tag{4.14}$$

and satisfying

$$u_m(t,x), v_m(t,x) \geq 0,\ \text{a.e. } (t,x) \in Q, \tag{4.15}$$

$$\int_\Omega u_m(t,x)\ dx = \int_\Omega u_m^0(x)\ dx = \int_\Omega u^0(x)\ dx,\ \text{a.e. } t \in (0,T). \tag{4.16}$$

Moreover, there is a positive, continuous and increasing function of $\|f\|_{L^q(Q)}$,

$$\mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}),$$

also independent of m, such that

$$\|v_m\|_{L^\infty(Q)} + \|v_m\|_{L^2(0,T;H^1(\Omega))} \leq \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}). \tag{4.17}$$

Proof. Concerning the proof of existence and uniqueness of solution of (4.8), the truncation $T^m(\cdot)$ simplifies the treatment of the chemotaxis and consumption terms, $-\nabla \cdot (T^m(u_m)\nabla v_m)$ and $-T^m(u_m)^s v_m$, respectively. Then, one can deal with the control term, $f v_m$, likewise in the proof of existence of (4.9), in Lemma 4.8. The uniqueness is proved by comparing two possibly different solutions. Properties (4.15) and (4.16) can be proved following the ideas in Chapter 2. Finally, we prove (4.17), beginning by the estimate in the L^∞-norm. Using the already proved property (4.15) of v_m in the v_m-equation of (4.8), we obtain

$$\partial_t v_m - \Delta v_m \leq f_+\, v_m \quad \text{a.e. } (t,x) \in Q. \tag{4.18}$$

On the other hand, considering Lemma 4.8, as well as (4.9), with $\tilde{f} = f_+$ and $w^0 = v^0$, we have that w satisfies

$$\partial_t w - \Delta w = f_+\, w \quad \text{a.e. } (t,x) \in Q, \tag{4.19}$$

with $\partial_{\mathbf{n}} w|_\Gamma = 0$ and $w(0,x) = v^0(x)$. Subtracting (4.19) from (4.18) we conclude that $(v_m - w)$ satisfies

$$\begin{cases} \partial_t(v_m - w) - \Delta(v_m - w) \le f_+ \, (v_m - w) \quad a.e. \ (t,x) \in Q, \\ \partial_{\mathbf{n}}(v_m - w)|_\Gamma = 0, \quad (v_m - w)(0,x) = 0. \end{cases}$$

Multiplying the above inequality by $(v_m - w)_+$ and using (1.2) to estimate the right hand side term leads us to $(v_m - w)_+(t,x) = 0$ *a.e.* $(t,x) \in Q$, that is, $v_m(t,x) \le w(t,x)$ *a.e.* $(t,x) \in Q$. Then the bound in the L^∞-norm for v_m is a consequence of the estimate for w given in (4.10), with $\tilde f = f_+$ and $w^0 = v^0$. The bound of v_m in the norm of $L^2(0,T;H^1(\Omega))$ is obtained by testing the v_m-equation of (4.8) by v_m and estimating conveniently the term on the right hand side using Holder's inequality, the interpolation inequality (1.2) and Young's inequality. ■

4.2.3 Energy inequality

Analogously to Chapter 2, we consider the variable $z_m(t,x) = \sqrt{v_m(t,x) + \alpha^2}$ and the rewritten problem

$$\begin{aligned} \partial_t u_m - \Delta u_m \quad &= -\nabla \cdot (T^m(u_m)\nabla (z_m)^2) \\ \partial_t z_m - \Delta z_m - \frac{|\nabla z_m|^2}{z_m} \quad &= -\frac{1}{2} T^m(u_m)^s \left(z_m - \frac{\alpha^2}{z_m} \right) + \frac{1}{2} f \left(z_m - \frac{\alpha^2}{z_m} \right) 1_{\Omega_c} \\ \partial_{\mathbf{n}} u_m|_\Gamma \quad &= \partial_{\mathbf{n}} z_m|_\Gamma = 0 \\ u_m(0) \quad &= u_m^0, \quad z_m(0) = \sqrt{v^0 + \alpha^2}, \end{aligned} \tag{4.20}$$

which is equivalent to the controlled truncated problem (4.8). From the equivalence of (4.8) and (4.20) and from the results given in Lemma 4.9, we have the following.

Corollary 4.10. *Given f satisfying* (4.1)*, there is a unique solution* (u_m, z_m) *of problem* (4.20) *with regularity*

$$\begin{aligned} u_m \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \ \partial_t u_m \in L^2(0,T;(H^1(\Omega))'), \\ z_m \in L^\infty(Q) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \ \partial_t z_m \in L^2(Q), \end{aligned} \tag{4.21}$$

and satisfying the m-uniform estimates

$$u_m(t,x) \ge 0 \quad \text{and} \quad z_m(t,x) \ge \alpha, \ a.e. \ (t,x) \in Q, \tag{4.22}$$

$$\int_\Omega u_m(t,x) \, dx = \int_\Omega u_m^0(x) \, dx = \int_\Omega u^0(x) \, dx, \ a.e. \ t \in (0,T), \tag{4.23}$$

$$\|z_m\|_{L^\infty(Q)}, \|z_m\|_{L^2(0,T;H^1(\Omega))} \le \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}). \tag{4.24}$$

Using this change of variables, we obtain an energy inequality involving the control f. In this subsection, in order to simplify the notation, we drop the m subscript and denote the solution (u_m, z_m) of (4.20) by (u, z). We begin with the following lemma.

Lemma 4.11 (Chapter 2)**.** *The solution* (u, z) *of* (4.20)*, satisfies the inequality*

$$\frac{1}{2}\frac{d}{dt}\|\nabla z\|^2_{L^2(\Omega)} + C_1\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) + \frac{1}{2}\int_\Omega T^m(u)^s|\nabla z|^2\ dx$$
$$\leq \frac{s}{2}\alpha\int_\Omega T^m(u)^{s-1}|\nabla z||\nabla T^m(u)|\ dx + \mathcal{K}_1^2\|f\|^2_{L^2(\Omega)}$$
$$+C_2\|\nabla z\|^2_{L^2(\Omega)} + \frac{s}{4}\int_\Omega T^m(u)^{s-1}\nabla(z^2)\cdot\nabla T^m(u)\ dx.$$

Next we recall the function g_m defined by $g_m(r) = \displaystyle\int_0^r g_m'(\theta)\ d\theta$, where $g'(\theta)$ is defined for $\theta \geq 0$ by

$$g_m'(\theta) = \begin{cases} ln(T^m(\theta)+1), & \text{if } s = 1,\\ \dfrac{T^m(\theta)^{s-1}}{(s-1)}, & \text{if } s > 1,\end{cases}$$

and the energy

$$E_m(u,z)(t) = \frac{s}{4}\int_\Omega g_m(u(t,x))\ dx + \frac{1}{2}\int_\Omega |\nabla z(t,x)|^2\ dx. \tag{4.25}$$

Lemma 4.12 (**Energy inequality for $s = 1$**)**.** *The solution* (u, z) *of the problem* (4.20) *satisfies, for sufficiently small* $\alpha > 0$,

$$\begin{aligned}&\frac{d}{dt}E_m(u,z)(t) + \beta\int_\Omega |\nabla[T^m(u)+1]^{1/2}|^2\ dx + \frac{1}{4}\int_\Omega T^m(u)|\nabla z|^2\ dx\\ &+\beta\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) \leq C(\mathcal{K}^2)\|\nabla z\|^2_{L^2(\Omega)} + \mathcal{K}_1^2\|f\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.26}$$

Proof. We follow Chapter 2, pointing out the most relevant steps to deal with the control term and make explicit the dependence on $\mathcal{K}_1 = \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)})$, from Corollary 4.10. By testing the u-equation of problem (4.20) by $ln(T^m(u)+1)$, we obtain

$$\frac{d}{dt}\int_\Omega g_m(u)\ dx + \int_\Omega \frac{(T^m)'(u)}{T^m(u)+1}|\nabla u|^2\ dx = \left(\frac{T^m(u)}{T^m(u)+1}\nabla(z^2), \nabla T^m(u)\right).$$

Since $0 \leq (T^m)'(u) \leq C$, we have $((T^m)'(u))^2 \leq C(T^m)'(u)$, and we can write

$$\int_\Omega \frac{(T^m)'(u)}{T^m(u)+1}|\nabla u|^2\ dx \geq C\int_\Omega \frac{((T^m)'(u))^2}{T^m(u)+1}|\nabla u|^2\ dx \geq C\int_\Omega |\nabla[T^m(u)+1]^{1/2}|^2\ dx.$$

Hence, using that $\dfrac{1}{(T^m(u)+1)} \leq \dfrac{1}{\sqrt{T^m(u)+1}}$, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega g_m(u)\ dx + C\int_\Omega |\nabla[T^m(u)+1]^{1/2}|^2\ dx = 2\left(\frac{T^m(u)+1-1}{T^m(u)+1}z\nabla z, \nabla T^m(u)\right)\\ &= \big(\nabla(z^2), \nabla T^m(u)\big) - 2\left(z\nabla z, \frac{\nabla T^m(u)}{T^m(u)+1}\right)\\ &\leq \big(\nabla(z^2), \nabla T^m(u)\big) + 2\|z\|_{L^\infty(\Omega)}\|\nabla z\|_{L^2(\Omega)}\|\nabla[T^m(u)+1]^{1/2}\|_{L^2(\Omega)}.\end{aligned}$$

Using Young's inequality and (4.24), we arrive at

$$\begin{aligned} \frac{d}{dt}\int_\Omega g_m(u)\ dx + C\int_\Omega |\nabla[T^m(u)+1]^{1/2}|^2\ dx \\ \le \big(\nabla(z^2), \nabla T^m(u)\big) + \mathcal{K}_1^2\|\nabla z\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.27}$$

If we add the inequality of Lemma 4.11, for $s = 1$, to $1/4$ times (4.27), then the terms $\displaystyle\int_\Omega \nabla T^m(u)\cdot\nabla(z^2)\ dx$ cancel and we obtain

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{4}\int_\Omega g_m(u)\ dx + \frac{1}{2}\|\nabla z\|^2_{L^2(\Omega)}\Big] + C\int_\Omega |\nabla[T^m(u)+1]^{1/2}|^2\ dx \\ &\qquad + \frac{1}{2}\int_\Omega T^m(u)|\nabla z|^2\ dx + C_1\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) \\ &\quad \le \frac{\sqrt{\alpha}}{2}\int_\Omega |\nabla z||\nabla T^m(u)|\ dx + \mathcal{K}_1^2\|f\|^2_{L^2(\Omega)} + (C_2+\mathcal{K}_1^2)\|\nabla z\|^2_{L^2(\Omega)} \\ &\quad \le \int_\Omega \alpha|\nabla[T^m(u)+1]^{1/2}||\sqrt{T^m(u)+1}||\nabla z|\ dx \\ &\qquad + \mathcal{K}_1^2\|f\|^2_{L^2(\Omega)} + (C_2+\mathcal{K}_1^2)\|\nabla z\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.28}$$

We can deal with the first term in the right hand side of the inequality using Holder's and Young's inequality,

$$\begin{aligned} \int_\Omega \alpha|\nabla[T^m(u)+1]^{1/2}||\sqrt{T^m(u)+1}||\nabla z|\ dx \le \alpha^2\ C(\delta)\int_\Omega T^m(u)|\nabla z|^2\ dx \\ + \delta\|\nabla[T^m(u)+1]^{1/2}\|^2_{L^2(\Omega)} + \alpha^2\ C(\delta)\int_\Omega |\nabla z|^2\ dx. \end{aligned}$$

Therefore, we can first choose $\delta > 0$ and then $\alpha > 0$ sufficiently small in order to use the terms on the left hand side of inequality (4.28) to absorb the first two terms on the right hand side of the above inequality and finally obtain the desired inequality (4.26). ■

Now we obtain the energy inequalities for $s \in (1, 2)$ and for $s \ge 2$, respectively. Analogously to Lemma 4.12, we follow the ideas of Chapter 2, making the necessary changes in order to deal with the control term and to make explicit the dependence on the positive constant $\mathcal{K}_1 = \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)})$, from Corollary 4.10. Since these changes were covered in Lemma 4.12, next we will state the results, skipping their proofs.

Lemma 4.13 (Energy inequality for $s \in (1,2)$). *The solution (u, z) of the problem (4.20) satisfies, for sufficiently small $\alpha > 0$,*

$$\begin{aligned} \frac{d}{dt}E_m(u,z)(t) + \beta\int_\Omega |\nabla[T^m(u)+1]^{s/2}|^2\ dx + \frac{1}{4}\int_\Omega T^m(u)^s|\nabla z|^2\ dx \\ + \beta\Big(\int_\Omega |D^2 z|^2\ dx + \int_\Omega \frac{|\nabla z|^4}{z^2}\ dx\Big) \le C(\mathcal{K}_1^2)\|\nabla z\|^2_{L^2(\Omega)} + \mathcal{K}_1^2\|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.29}$$

Remark 4.14. In Chapter 2, when the authors prove the energy inequality for $s \in (1,2)$, the term $\int_\Omega |\nabla[T^m(u)+1/j]^{s/2}|^2 \, dx$ is estimated by

$$\int_\Omega |\nabla[T^m(u)+1/j]^{s/2}|^2 \, dx \geq 0, \text{ for all } j \in \mathbb{N},$$

but it can be estimated by

$$\int_\Omega |\nabla[T^m(u)+1/j]^{s/2}|^2 \, dx \geq \int_\Omega |\nabla[T^m(u)+1]^{s/2}|^2 \, dx, \text{ for all } j \in \mathbb{N},$$

instead, yielding (4.29). □

Lemma 4.15 (Energy inequality for $s \geq 2$). *The solution (u,z) of the problem (4.20) satisfies, for sufficiently small $\alpha > 0$,*

$$\begin{aligned} &\frac{d}{dt}E_m(u,z)(t) + \int_\Omega |\nabla[T^m(u)]^{s/2}|^2 dx + \frac{1}{4}\int_\Omega T^m(u)^s |\nabla z|^2 \, dx \\ &+\beta\Big(\int_\Omega |D^2 z|^2 \, dx + \int_\Omega \frac{|\nabla z|^4}{z^2} \, dx\Big) \leq C(\mathcal{K}_1^2)\|\nabla z\|^2_{L^2(\Omega)} + \mathcal{K}_1^2 \|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.30}$$

4.2.4 m-independent estimates and passage to the limit as $m \to \infty$

In the present subsection we go back to the notation (u_m, z_m) and (u_m, v_m) to the solution of problems (4.20) and (4.8), respectively.

m-independent estimates for ∇v_m

We will integrate the energy inequalities (4.26), (4.29) and (4.30) with respect to t, from 0 to $T > 0$. We take into account that, because of Corollary 4.10, we have the following bounds independently of m:

$$\nabla z_m \text{ is bounded in } L^2(Q)$$

and

$$0 < \alpha \leq z_m(t,x) \leq \mathcal{K}_1, \ \text{a.e. } (t,x) \in Q. \tag{4.31}$$

We also use the hypothesis (4.2) on the initial data u^0, v^0 to prove that the energy given in (4.25) at time $t = 0$, $E_m(u_m, z_m)(0)$, is also bounded, independently of m. Thus we conclude that

$$\nabla z_m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^4(Q),$$

$$T^m(u_m)^{s/2}\nabla z_m \text{ and } \Delta z_m \text{ are bounded in } L^2(Q).$$

But using the fact that $z_m = \sqrt{v_m + \alpha^2}$ and (4.31) we can conclude that

$$\nabla v_m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^4(Q), \tag{4.32}$$

$$T^m(u_m)^{s/2}\nabla v_m \text{ and } \Delta v_m \text{ are bounded in } L^2(Q). \tag{4.33}$$

Case $s \in [1, 2)$

First, following Chapter 2, to which we refer the reader that might be interested in more details, we prove the existence of weak solution (u, v) of problem (2). Next, to conclude the proof of Theorem 4.3, letting $z = \sqrt{v + \alpha^2}$, we prove the energy inequality (4.3).

Existence of weak solution to (2)**:**

In order to prove the existence of a weak solution (u, v) to (2), first we obtain m-independent estimates for the solutions (u_m, v_m) of (4.8) and then we use compactness results in weak*, weak and strong topologies to pass to the limit as $m \to \infty$.

Let

$$g'(r) = \begin{cases} ln(r) & \text{if } s = 1, \\ r^{s-1}/(s-1) & \text{if } s \in (1, 2), \end{cases} \quad \forall r > 0.$$

and let

$$g(r) = \int_0^r g'(\theta)\, d\theta = \begin{cases} rln(r) - (r-1) & \text{if } s = 1, \\ r^s/s(s-1) & \text{if } s \in (1, 2). \end{cases}$$

Notice that $g''(r) = r^{s-2}$, $\forall r > 0$, in all cases.

We test the u_m-equation of (4.8) by $g'(u_m + 1)$ and obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega g(u_m+1)\, dx &+ \frac{4}{s^2}\int_\Omega |\nabla[u_m+1]^{s/2}|^2\, dx \\ &= \int_\Omega T^m(u_m)(u_m+1)^{s/2-1}\nabla v_m \cdot \nabla u_m\, (u_m+1)^{s/2-1}\, dx \\ &= \frac{2}{s}\int_\Omega \frac{T^m(u_m)^{1-s/2}}{(u_m+1)^{1-s/2}} T^m(u_m)^{s/2}\nabla v_m \cdot \nabla[u_m+1]^{s/2}\, dx \\ &\le \frac{2}{s}\Big(\int_\Omega T^m(u_m)^s|\nabla v_m|^2\, dx\Big)^{1/2}\Big(\int_\Omega |\nabla[u_m+1]^{s/2}|^2\, dx\Big)^{1/2} \end{aligned}$$

and thus we have

$$\frac{d}{dt}\int_\Omega g(u_m+1)\, dx + \frac{2}{s^2}\int_\Omega |\nabla[u_m+1]^{s/2}|^2\, dx \le \frac{1}{4}\int_\Omega T^m(u_m)^s|\nabla v_m|^2\, dx.$$

Integrating with respect to t from 0 to T we obtain

$$\begin{aligned} &\int_\Omega g(u_m(T)+1)\, dx + \frac{2}{s^2}\int_0^T\int_\Omega |\nabla[u_m+1]^{s/2}|^2\, dx\, dt \\ &\le \frac{1}{4}\int_0^T\int_\Omega T^m(u_m)^s|\nabla v_m|^2\, dx\, dt + \int_\Omega g(u^0+1)\, dx. \end{aligned}$$

Then, because of (4.17), (4.2) and the definition of g and (4.33) we conclude that

$$(u_m+1)^{s/2} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.34}$$

Using the Sobolev inequality $H^1(\Omega) \subset L^6(\Omega)$ and interpolation inequalities we obtain

$$(u_m)^{s/2} \text{ is bounded in } L^{10/3}(Q).$$

The latter and (4.34) imply that

$$u_m \text{ is bounded in } L^\infty(0,T;L^s(\Omega)) \cap L^{5s/3}(Q). \tag{4.35}$$

From (4.35) we can conclude, using the v_m-equation of (4.8) that

$$\partial_t v_m \text{ is bounded in } L^{5/3}(Q).$$

Reminding that $s \in [1,2)$, if we use (4.34) and (4.35) in the relation

$$\nabla u_m = \nabla(u_m+1) = \nabla\big((u_m+1)^{s/2}\big)^{2/s} = \frac{2}{s}(u_m+1)^{1-s/2}\ \nabla(u_m+1)^{s/2}.$$

then we also have

$$u_m \text{ is bounded in } L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)). \tag{4.36}$$

Considering the chemotaxis term of the u_m-equation of (4.8), we can write $T^m(u_m)\nabla v_m$ as

$$T^m(u_m)\nabla v_m = T^m(u_m)^{1-s/2}T^m(u_m)^{s/2}\nabla v_m.$$

Then, we have $T^m(u_m)^{1-s/2}$ bounded in $L^{10s/(6-3s)}(Q)$, because of (4.34), and $T^m(u_m)^{s/2}\nabla v_m$ bounded in $L^2(Q)$, because of (4.33), and hence we can conclude that

$$T^m(u_m)\nabla v_m \text{ is bounded in } L^{5s/(3+s)}(Q). \tag{4.37}$$

Then, if we consider the u_m-equation of (4.8), from (4.36) and (4.37) we conclude that

$$\partial_t u_m \text{ is bounded in } L^{5s/(3+s)}(0,T;(W^{1,5s/(4s-3)}(\Omega))'). \tag{4.38}$$

Now we are going to obtain compactness for $\{u_m\}$ which is necessary to pass to the limit as $m \to \infty$ in the nonlinear terms of the equations of (4.8).

We observe that $W^{1,5s/(3+s)}(\Omega) \subset L^q(\Omega)$, with continuous embedding for $q = 15s/(9-2s)$ and compact embedding for $q \in [1, 15s/(9-2s))$. Then, since $s \in [1,2)$, we have $5s/3 < 15s/(9-2s)$ and therefore the embedding $W^{1,5s/(3+s)}(\Omega) \subset L^{5s/3}(\Omega)$ is compact. Note also that $q = 5s/3 \geq 5/3 > 1$.

Now we can use Lemma 1.14 with

$$X = W^{1,5s/(3+s)}(\Omega), \quad B = L^{5s/3}(\Omega), \quad Y = \big(W^{1,5s/(4s-3)}(\Omega)\big)'$$

and $q = 5s/3$, to conclude that there is a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and a limit function u such that

$$u_m \longrightarrow u \text{ weakly in } L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)),$$

and

$$u_m \longrightarrow u \text{ strongly in } L^p(Q),\ \forall p \in [1, 5s/3). \tag{4.39}$$

Using the Dominated Convergence Theorem we can conclude from (4.39) that

$$T^m(u_m) \to u \text{ strongly in } L^p(Q),\ \forall p \in [1, 5s/3). \tag{4.40}$$

It stems from the convergence (4.40) and Lemma 1.11 that

$$(T^m(u_m))^s \to u^s \text{ strongly in } L^q(Q),\ \forall q \in [1, 5/3). \tag{4.41}$$

The convergence of v_m is better. There is a subsequence of $\{v_m\}$ (still denoted by $\{v_m\}$) and a limit function v such that

$$\begin{gathered} v_m \to v \text{ weakly* in } L^\infty(Q) \cap L^\infty(0,T;H^1(\Omega)), \\ v_m \to v \text{ weakly in } L^2(0,T;H^2(\Omega)), \\ \nabla v_m \to \nabla v \text{ weakly in } L^4(Q), \\ \text{and } \partial_t v_m \to \partial_t v \text{ weakly in } L^{5/3}(Q). \end{gathered} \tag{4.42}$$

Now we are going to use the weak and strong convergences obtained so far to pass to the limit as $m \to \infty$ in the equations of problem (4.8). We are going to identify the limits of the nonlinear terms related to chemotaxis and consumption,

$$T^m(u_m)\nabla v_m \text{ and } T^m(u_m)^s v_m,$$

respectively, with

$$u\nabla v \text{ and } u^s v.$$

In fact, considering the chemotaxis term, from (4.40), (4.32) and (4.42), we can conclude that

$$T^m(u_m)\nabla v_m \longrightarrow u\nabla v \text{ weakly in } L^{20s/(5s+12)}(Q).$$

Considering now the consumption term, considering (4.41) and (4.42) we conclude that

$$T^m(u_m)^s v_m \longrightarrow u^s v \text{ weakly in } L^{5/3}(Q).$$

With these identifications and all previous convergences, it is possible to pass to the limit as $m \to \infty$ in each term of the equations of (4.8).

Energy inequality (4.3)**:**

In order to finish we must prove the energy inequality (4.3). First we obtain an integral inequality for the solution (u_m, z_m) of (4.20), where we remind that $z_m = \sqrt{v_m + \alpha^2}$, for small enough but fixed $\alpha > 0$, being (u_m, v_m) the solution of (4.8). According to Lemmas (4.12) and (4.13), (u_m, z_m) satisfies

$$\begin{aligned} E_m(u_m, z_m)(t_2) + \beta \int_{t_1}^{t_2} \int_\Omega |\nabla[T^m(u_m) + 1]^{s/2}|^2 \, dx \, dt \\ + \frac{1}{4} \int_{t_1}^{t_2} \int_\Omega T^m(u_m)^s |\nabla z_m|^2 \, dx \, dt \\ + \beta \Big(\int_{t_1}^{t_2} \int_\Omega |D^2 z_m|^2 \, dx \, dt + \int_{t_1}^{t_2} \int_\Omega \frac{|\nabla z_m|^4}{z_m^2} \, dx \, dt \Big) \\ \leq E_m(u_m, z_m)(t_1) + C(\mathcal{K}_1^2) \int_{t_1}^{t_2} \|\nabla z_m\|^2_{L^2(\Omega)} \, dt + \mathcal{K}_1^2 \int_{t_1}^{t_2} \|f\|^2_{L^2(\Omega)} \, dt, \end{aligned} \tag{4.43}$$

where E_m is given by (4.25). Next we collect some convergences that can be obtained from the m-independent bounds and the weak and strong convergences proved so far and that will be useful to pass to the limit in the inequality (4.43). Recalling that we denote $z = \sqrt{v + \alpha^2}$, we have, in particular,

$$\begin{gathered} (u_m + 1)^{s/2} \longrightarrow (u + 1)^{s/2} \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \\ (u_m + 1)^{s/2} \longrightarrow (u + 1)^{s/2} \text{ weakly in } L^2(0, T; H^1(\Omega)), \\ \nabla T^m(u_m)^{s/2} \longrightarrow \nabla u^{s/2} \text{ weakly in } L^2(Q), \\ T^m(u_m)^{s/2} \nabla z_m \longrightarrow u^{s/2} \nabla z \text{ weakly in } L^2(Q), \\ \frac{\nabla z_m}{\sqrt{z_m}} \longrightarrow \frac{\nabla z}{\sqrt{z}} \text{ weakly in } L^4(Q), \\ D^2 z_m \longrightarrow D^2 z \text{ weakly in } L^2(Q), \\ \nabla z_m \longrightarrow \nabla z \text{ strongly in } L^2(Q). \end{gathered} \tag{4.44}$$

Recalling that we are dealing with the case $s \in [1, 2)$, let

$$E(u, z)(t) = \frac{s}{4} \int_\Omega g(u(t, x)) \, dx + \frac{1}{2} \int_\Omega |\nabla z(t, x)|^2 \, dx,$$

where

$$g(u) = \begin{cases} (u + 1) ln(u + 1) - u, & \text{if } s = 1, \\ \dfrac{u^s}{s(s-1)}, & \text{if } s \in (1, 2). \end{cases}$$

Then the following convergence will be also necessary.

Lemma 4.16. *$E_m(u_m, z_m) \longrightarrow E(u, z)$ in $L^1(0, T)$.*

Proof. From (4.44) we have that $\nabla z_m \to \nabla z$ in $L^2(Q)$ which, in particular, leads us to

$$\int_\Omega |\nabla z_m(t, x)|^2 \, dx \longrightarrow \int_\Omega |\nabla z(t, x)|^2 \, dx \text{ in } L^1(0, T).$$

Then, it remains to prove that

$$\int_\Omega g_m(u_m) \, dx \longrightarrow \int_\Omega g(u) \, dx \text{ in } L^1(0, T). \tag{4.45}$$

We begin by rewriting $g_m(u_m) - g(u)$ as

$$g_m(u_m) - g(u) = g_m(u_m) - g_m(u) + g_m(u) - g(u). \tag{4.46}$$

For the first difference in (4.46), $g_m(u_m) - g_m(u)$, using that g'_m and g' are monotone increasing functions and that $g'_m(r) \leq g'(r)$ for all $r \geq 0$, we have

$$\begin{aligned} |g_m(u_m) - g_m(u)| &= |\int_u^{u_m} g'_m(\theta)\, d\theta| \\ &\leq |u_m - u|(g'_m(u_m) + g'_m(u)) \\ &\leq |u_m - u|(g'(u_m) + g'(u)). \end{aligned}$$

Then, for $s = 1$, we have

$$|g_m(u_m) - g_m(u)| \leq C|u_m - u|(ln(u_m + 1) + ln(u + 1))$$

and, for $s \in (1, 2)$, we have

$$|g_m(u_m) - g_m(u)| \leq C|u_m - u|(|u_m|^{s-1} + |u|^{s-1}).$$

Considering the case $s \in (1, 2)$, since from (4.35) we have $(|u_m|^{s-1} + |u|^{s-1})$ bounded in $L^{5s/(3s-3)}(Q)$ and, from (4.39), we have $|u_m - u| \to 0$ in $L^{5s/(2s+3)}(Q)$, we conclude that

$$g_m(u_m) - g_m(u) \longrightarrow 0 \text{ in } L^1(Q). \tag{4.47}$$

Considering now the case $s = 1$, from (4.35) we have $ln(u_m + 1) + ln(u + 1)$ bounded in $L^p(Q)$, for all $p \in [1, \infty)$. Then, analogously to the case $s \in (1, 2)$, we use (4.39) and obtain (4.47) also for $s = 1$. From (4.47) we conclude, in particular, that

$$\int_\Omega g_m(u_m)\, dx - \int_\Omega g_m(u)\, dx \longrightarrow 0 \text{ in } L^1(0, T). \tag{4.48}$$

For the second difference in (4.46), $g_m(u) - g(u)$, we use the Dominated Convergence Theorem. In fact, we write

$$g_m(u) - g(u) = \int_0^u (g'_m(\theta) - g'(\theta))\, d\theta.$$

Using this expression one can verify that

$$g_m(u) - g(u) \longrightarrow 0,\ a.e.\ (t, x) \in Q.$$

Next we note that $g_m(u) - g(u)$ is bounded by $2g(u) \in L^1(Q)$. Therefore we conclude, by using the Dominated Convergence Theorem, that $g_m(u) - g(u) \to 0$ in $L^1(Q)$ and, in particular

$$\int_\Omega g_m(u)\, dx - \int_\Omega g(u)\, dx \longrightarrow 0 \text{ in } L^1(0, T). \tag{4.49}$$

With (4.48) and (4.49) we obtain (4.45), finishing the proof. ∎

Lemma 4.17. *For $s \geq 1$ we have $v \in C_w([0,T];H^1(\Omega))$ and $u \in C_w([0,T];L^s(\Omega))$.*

Proof. For any $s \geq 1$, we have $v \in L^\infty(Q) \subset L^{5/3}(Q)$ and $v_t \in L^{5/3}(Q)$, which implies that $v \in C([0,T];L^{5/3}(\Omega))$ and, in particular, $v \in C_w([0,T];L^{5/3}(\Omega))$. Since we also have $v \in L^\infty(0,T;H^1(\Omega))$, we use Lemma 1.5 to conclude that $v \in C_w([0,T];H^1(\Omega))$. Considering u we have $u \in L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)) \subset L^{5s/(3+s)}(0,T;(W^{1,5s/(4s-3)}(\Omega))')$ and $\partial_t u \in L^{5s/(3+s)}(0,T;(W^{1,5s/(4s-3)}(\Omega))')$,
which implies that $u \in C([0,T];(W^{1,5s/(4s-3)}(\Omega))')$ and, in particular,
$u \in C_w([0,T];(W^{1,5s/(4s-3)}(\Omega))')$. Since from (4.44), $u \in L^\infty(0,T;L^s(\Omega))$, we conclude, using Lemma 1.5, that $u \in C_w([0,T];L^s(\Omega))$. ∎

Next we pass to the limit in inequality (4.43). Because of Lemma 4.16, we conclude that, up to a subsequence,

$$E_m(u_m,z_m)(t) \longrightarrow E(u,z)(t) \ a.e.\ t \in [0,T]. \tag{4.50}$$

Therefore, using the convergences (4.44), the weak lower semicontinuity of the norm (Lemma 1.12) and the almost everywhere pointwise convergence (4.50), we are able to pass to the limit in (4.43) and conclude that, for *a.e.* $t_1,t_2 \in [0,T]$, with $t_2 > t_1$, we have

$$\begin{aligned}E(u,z)(t_2) + \beta\int_{t_1}^{t_2}\int_\Omega |\nabla[u+1]^{s/2}|^2\,dx\,dt + \frac{1}{4}\int_{t_1}^{t_2}\int_\Omega u^s|\nabla z|^2\,dx\,dt\\ +\beta\Big(\int_{t_1}^{t_2}\int_\Omega |D^2 z|^2\,dx\,dt + \int_{t_1}^{t_2}\int_\Omega \frac{|\nabla z|^4}{z^2}\,dx\,dt\Big)\\ \leq E(u,z)(t_1) + C(\mathcal{K}_1^2)\int_{t_1}^{t_2}\|\nabla z\|^2_{L^2(\Omega)}\,dt + \mathcal{K}_1^2\int_{t_1}^{t_2}\|f\|^2_{L^2(\Omega)}\,dt.\end{aligned}$$

Accounting for the m-independent bound for ∇z_m given in (4.24) and the strong convergence of ∇z_m to ∇z given in (4.44), we have $\|\nabla z\|_{L^2(Q)} \leq \mathcal{K}_1$. And since $\mathcal{K}_1 = \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)})$, we conclude that there is other constant $\mathcal{K} = \mathcal{K}(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)})$, which is increasing and continuous with respect to $\|f\|_{L^q(Q)}$, such that, for *a.e.* $t_1,t_2 \in [0,T]$, with $t_2 > t_1$, we have

$$\begin{aligned}E(u,z)(t_2) + \beta\int_{t_1}^{t_2}\int_\Omega |\nabla[u+1]^{s/2}|^2\,dx\,dt + \frac{1}{4}\int_{t_1}^{t_2}\int_\Omega u^s|\nabla z|^2\,dx\,dt\\ +\beta\Big(\int_{t_1}^{t_2}\int_\Omega |D^2 z|^2\,dx\,dt + \int_{t_1}^{t_2}\int_\Omega \frac{|\nabla z|^4}{z^2}\,dx\,dt\Big)\\ \leq E(u,z)(t_1) + \mathcal{K}(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}).\end{aligned}\tag{4.51}$$

To finish, we consider the case $s > 1$. For simplicity, let us write (4.51) in terms of the energy $E(u, z)(\cdot)$ and the dissipative term

$$D(u, z)(t_1, t_2) = \beta \int_{t_1}^{t_2} \int_\Omega |\nabla [u + 1]^{s/2}|^2 \, dx \, dt + \frac{1}{4} \int_{t_1}^{t_2} \int_\Omega u^s |\nabla z|^2 \, dx \, dt \\ + \beta \Big(\int_{t_1}^{t_2} \int_\Omega |D^2 z|^2 \, dx \, dt + \int_{t_1}^{t_2} \int_\Omega \frac{|\nabla z|^4}{z^2} \, dx \, dt \Big),$$

for *a.e.* $t_1, t_2 \in [0, T]$, with $t_2 > t_1$, as

$$E(u, z)(t_2) + D(u, z)(t_1, t_2) \leq E(u, z)(t_1) + \mathcal{K}(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}).$$

Now, let $t_2 \in (t_1, T]$ and let $\{t_2^n\}_n$ be a sequence such that $t_2^n \to t_2$ as $n \to \infty$ and such that, for all n, we have

$$E(u, z)(t_2^n) + D(u, z)(t_1, t_2^n) \leq E(u, z)(t_1) + \mathcal{K}(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}). \tag{4.52}$$

If we take the lim inf in both sides of (4.52), we obtain

$$\liminf_{n \to \infty} E(u, z)(t_2^n) + D(u, z)(t_1, t_2) \leq E(u, z)(t_1) + \mathcal{K}(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}).$$

Then, from the definition of $E(u, z)$ for $s > 1$ and Lemma 1.5, we have

$$E(u, z)(t_2) \leq \liminf_{n \to \infty} E(u, z)(t_2^n)$$

and therefore we conclude that, for $s > 1$, the energy inequality (4.51) is satisfied for *a.e.* $t_1 \in [0, T]$, and for all $t_2 \in (t_1, T]$.

Case $s \geq 2$

Existence of weak solution to (2)**:**

The procedure for the case $s \geq 2$ is slightly different. First we note that, integrating the energy inequality (4.30) from Lemma 4.15 with respect to t, we have

$$\nabla T^m(u_m)^{s/2} \text{ is bounded in } L^2(Q). \tag{4.53}$$

We also remind that we defined $g_m'(r) = T^m(r)^{s-1}/(s-1)$, for $s \geq 2$. Then we have

$$T^m(r)^s = s \int_0^r (T^m)'(\theta) T^m(\theta)^{s-1} \, d\theta \leq Cs \int_0^r T^m(\theta)^{s-1} \, d\theta = Cs(s-1) g_m(r).$$

Therefore it also stems from integrating the energy inequality (4.30) with respect to t that

$$T^m(u_m)^{s/2} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)). \tag{4.54}$$

From (4.54) and (4.53) we can conclude that

$$T^m(u_m)^{s/2} \text{ is bounded in } L^{10/3}(Q),$$

that is,

$$T^m(u_m) \text{ is bounded in } L^{5s/3}(Q). \tag{4.55}$$

For each fixed $m \in \mathbb{N}$, consider the zero measure set $\mathcal{N} \subset (0,T)$ such that

$$u_m(t^*,\cdot), v_m(t^*,\cdot) \in H^1(\Omega),\ \forall t^* \in (0,T) \setminus \mathcal{N}.$$

Then, for each fixed $t^* \in (0,T) \setminus \mathcal{N}$, let us consider the sets

$$\{0 \le u_m \le 1\} = \Big\{x \in \Omega \ \Big|\ 0 \le u_m(t^*,x) \le 1\Big\}$$

and

$$\{u_m \ge 1\} = \Big\{x \in \Omega \ \Big|\ u_m(t^*,x) \ge 1\Big\}.$$

Now note that, since $s \ge 2$, we have

$$\begin{aligned}
&\int_\Omega T^m(u_m(t^*,x)^2 |\nabla v_m(t^*,x)|^2\ dx \\
&\le \int_{\{0\le u_m \le 1\}} |\nabla v_m(t^*,x)|^2\ dx + \int_{\{u_m \ge 1\}} T^m(u_m(t^*,x))^s |\nabla v_m(t^*,x)|^2\ dx \\
&\le \int_\Omega |\nabla v_m(t^*,x)|^2\ dx + \int_\Omega T^m(u_m(t^*,x))^s |\nabla v_m(t^*,x)|^2\ dx.
\end{aligned}$$

The last inequality is valid for all $t^* \in (0,T) \setminus \mathcal{N}$, then if we integrate in the variable t we obtain

$$\begin{aligned}
\int_0^\infty \int_\Omega T^m(u_m(t,x)^2 |\nabla v_m(t,x)|^2\ dx\ dt \le& \int_0^\infty \int_\Omega |\nabla v_m(t,x)|^2\ dx\ dt \\
&+ \int_0^\infty \int_\Omega T^m(u_m(t,x))^s |\nabla v_m(t,x)|^2\ dx\ dt.
\end{aligned}$$

Therefore by (4.17) and (4.33) we can conclude that

$$T^m(u_m)\nabla v_m \text{ is bounded in } L^2(Q). \tag{4.56}$$

Now we test the u_m-equation of problem (4.8) by u_m. This gives us

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)} &= \int_\Omega T^m(u_m)\nabla v_m \cdot \nabla u_m\ dx \\
&\le \frac{1}{2}\int_\Omega T^m(u_m)^2 |\nabla v_m|^2\ dx + \frac{1}{2}\|\nabla u_m\|^2_{L^2(\Omega)},
\end{aligned}$$

hence we have

$$\frac{d}{dt}\|u_m\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)} \le \int_\Omega T^m(u_m)^2 |\nabla v_m|^2\ dx.$$

Integrating with respect to t, we conclude from (4.56) that

$$u_m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \tag{4.57}$$

and

$$\nabla u_m \text{ is bounded in } L^2(Q). \tag{4.58}$$

Then, if we consider the u_m-equation of (4.8), by applying (4.58) and (4.56) we conclude that

$$\partial_t u_m \text{ is bounded in } L^2(0,T;(H^1(\Omega))'). \tag{4.59}$$

Considering (4.33), (4.17) and (4.55) we conclude from the v_m-equation of (4.8) that

$$\partial_t v_m \text{ is bounded in } L^2(0,T;L^{3/2}(\Omega)).$$

Now, using (4.57), (4.58) and (4.59) we can conclude that there is a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, and a limit function u such that

$$\begin{aligned} u_m &\longrightarrow u \text{ weakly* in } L^\infty(0,T;L^2(\Omega)),\\ \nabla u_m &\longrightarrow \nabla u \text{ weakly in } L^2(Q),\\ \partial_t u_m &\longrightarrow u \text{ weakly in } L^2\Big(0,\infty;\big(H^1(\Omega)\big)'\Big). \end{aligned}$$

By applying the compactness result Lemma 1.14, one has

$$u_m \longrightarrow u \text{ strongly in } L^2(Q).$$

Using the Dominated Convergence Theorem and (4.55) we can also prove that

$$T^m(u_m) \longrightarrow u \text{ strongly in } L^p(Q),\ \forall p \in (1, 5s/3),$$

and using Lemma 1.11,

$$T^m(u_m)^s \longrightarrow u^s \text{ strongly in } L^p(Q),\ \forall p \in (1, 5/3).$$

From the global in time estimate (4.54) we can conclude that, up to a subsequence,

$$T^m(u_m) \to u \text{ weakly* in } L^\infty(0,T;L^s(\Omega)),$$

hence, in particular,

$$u \in L^\infty(0,T;L^s(\Omega)).$$

For $s \geq 2$, if we consider the functions v_m, we have the same m-independent estimates that we had for $s \in [1,2)$. Then we have the same convergences given in (4.42).

Following the ideas of Subsection 4.2.4, for $s \in [1,2)$, we can identify the limits of $T^m(u_m)\nabla v_m$ and $T^m(u_m)^s v_m$ with $u\nabla v$ and $u^s v$, respectively.

Energy inequality (4.3):

One can reach it by following the reasoning used in Subsection 4.2.4 for $s \in [1, 2)$.

4.3 Existence of an Optimal Control

In the present section we first prove Theorem 4.5, in Subsection 4.3.1, establishing the existence of solution to the minimization problem (4.6). Afterwards, we prove Theorem 4.6 in Subsection 4.3.2.

4.3.1 Proof of Theorem 4.5

Since the functional J in (4.6) is nonnegative,

$$J_{inf} := \inf_{(u,v,f)\in S_{ad}^M} J(u, v, f) \geq 0$$

is well defined and there is a minimizing sequence $\{(u_n, v_n, f_n)\} \subset S_{ad}^M$ such that

$$\lim_{n\to\infty} J(u_n, v_n, f_n) = J_{inf}. \tag{4.60}$$

Next we prove that there is $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^M$, that will be defined as the limit of a subsequence of $\{(u_n, v_n, f_n)\}_n$, such that $J(\overline{u}, \overline{v}, \overline{f}) = J_{inf}$.

Since $(u_n, v_n, f_n) \in S_{ad}^M$, then

$$\begin{cases} (\partial_t u_n, \varphi) + (\nabla u_n, \nabla\varphi) = (u_n \nabla v_n, \nabla\varphi) \\ \partial_t v_n - \Delta v_n = -u_n^s v_n + f_n v_n 1_{\Omega_c}, \\ \partial_{\mathbf{n}} u_n|_\Gamma = \partial_{\mathbf{n}} v_n|_\Gamma = 0, \quad u_n(0) = u^0, \quad v_n(0) = v^0, \end{cases} \tag{4.61}$$

for every $\varphi \in L^{5s/(4s-3)}(0, T; W^{1,5s/(4s-3)}(\Omega))$. Denoting $z_n = \sqrt{v_n + \alpha^2}$, we have

$$\begin{aligned} E(u_n, z_n)(t_2) + \beta \int_{t_1}^{t_2}\int_\Omega |\nabla[u_n + 1]^{s/2}|^2 \,dx\,dt \\ + \frac{1}{4}\int_{t_1}^{t_2}\int_\Omega u_n^s |\nabla z_n|^2 \,dx\,dt + \beta\Big(\int_{t_1}^{t_2}\int_\Omega |D^2 z_n|^2 \,dx\,dt \\ + \int_{t_1}^{t_2}\int_\Omega \frac{|\nabla z_n|^4}{z^2} \,dx\,dt\Big) \leq E(u_n, z_n)(t_1) + \mathcal{K}(M, \|v_0\|_{W^{2-2/q,q}(\Omega)}). \end{aligned} \tag{4.62}$$

Since $(u_n, v_n, f_n) \in S_{ad}^M$, we have

$$\|f_n\|_{L^q(Q)} \leq M. \tag{4.63}$$

Then, comparing v_n with the solution w_n of (4.9), with $\tilde{f} = f_n$ and $w^0 = v^0$, yields $0 \leq v_n(t, x) \leq w_n(t, x)$ *a.e.* $(t, x) \in Q$. From Lemma 4.8 and (4.63), we obtain $\|v_n\|_{L^\infty(Q)} \leq \|w_n\|_{L^\infty(Q)} \leq \mathcal{K}_1(M)$ and, in particular, we conclude that there is a constant $C(M) > 0$ such that

$$0 < \alpha \leq z_n(t,x) \leq C(M), \ a.e. \ (t,x) \in Q. \tag{4.64}$$

With the energy inequality (4.62) and the pointwise bound (4.64), we are able to get the same estimates of Subsection 4.2.4 and pass to the limit as $n \to \infty$. In fact, from (4.62), we conclude that, for $s \geq 1$, we have the following bounds independently of n:

$$\begin{gathered}\nabla z_n \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^4(Q),\\ u_n^{s/2}\nabla z_n \text{ and } D^2 z_n \text{ are bounded in } L^2(Q).\end{gathered}$$

But using the fact that $z_n = \sqrt{v_n + \alpha^2}$ and (4.64) we can conclude that

$$\begin{gathered}\nabla v_n \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^4(Q),\\ u_n^{s/2}\nabla v_n \text{ and } \Delta v_n \text{ are bounded in } L^2(Q).\end{gathered}$$

From (4.62) (and by testing the u_n-equation of (4.61) by $\varphi = 1$, in the case $s = 1$) we also have

$$\begin{gathered}\nabla[u_n + 1]^{s/2} \text{ is bounded in } L^2(Q),\\ (u_n + 1)^{s/2} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)).\end{gathered}$$

Afterwards, using some ideas of Subsection 4.2.4 we conclude that for $s \in [1,2)$ we have

$$\begin{gathered}u_n \text{ is bounded in } L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)),\\ \partial_t u_n \text{ is bounded in } L^{5s/(3+s)}(0,T;(W^{1,5s/(4s-3)}(\Omega))'),\end{gathered}$$

for $s \geq 2$ we have

$$\begin{gathered}u_n \text{ is bounded in } L^2(0,T;H^1(\Omega)),\\ \partial_t u_n \text{ is bounded in } L^2(0,T;(H^1(\Omega))'),\end{gathered}$$

and, for $s \geq 1$, we have

$$\begin{gathered}u_n \text{ is bounded in } L^\infty(0,T;L^s(\Omega)) \cap L^{5s/3}(Q),\\ v_n \text{ is bounded in } L^\infty(Q) \cap L^4(0,T;W^{1,4}(\Omega)) \cap L^2(0,T;H^2(\Omega)),\\ \partial_t v_n \text{ is bounded in } L^{5/3}(Q).\end{gathered}$$

In view of these n-uniform bounds we follow the reasoning of Subsections 4.2.4 and 4.2.4 and conclude that, up to a subsequence, there is $(\overline{u}, \overline{v}, \overline{f})$ such that, if $s \in [1,2)$, we have

$$\begin{gathered}u_n \longrightarrow \overline{u} \text{ weakly in } L^{5s/3}(Q) \cap L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega)),\\ u_n \longrightarrow \overline{u} \text{ strongly in } L^p(Q), \ \forall p \in [1, 5s/3)\\ \text{and } \partial_t u_n \to \partial_t \overline{u} \text{ weakly in } L^{5s/(3+s)}(0,T;(W^{1,5s/(4s-3)}(\Omega))'),\end{gathered} \tag{4.65}$$

for $s \geq 2$ we have

$$\begin{aligned} u_n &\longrightarrow \overline{u} \text{ weakly in } L^{5s/3}(Q) \cap L^2(0,T;H^1(\Omega)), \\ u_n &\longrightarrow \overline{u} \text{ strongly in } L^p(Q),\ \forall p \in [1, 5s/3) \\ \text{and } \partial_t u_n &\to \partial_t \overline{u} \text{ weakly in } L^2(0,T;(H^1(\Omega))'), \end{aligned} \tag{4.66}$$

and, for $s \geq 1$,

$$\begin{aligned} v_n &\to \overline{v} \text{ weakly* in } L^\infty(Q) \cap L^\infty(0,T;H^1(\Omega)), \\ v_n &\to \overline{v} \text{ weakly in } L^4(0,T;W^{1,4}(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ \text{and } \partial_t v_n &\to \partial_t \overline{v} \text{ weakly in } L^{5/3}(Q) \end{aligned} \tag{4.67}$$

and

$$f_n \to \overline{f} \text{ weakly in } L^q(Q). \tag{4.68}$$

With these convergences we can pass to the limit as $n \to \infty$ in (4.61) and conclude that $(\overline{u}, \overline{v})$ is a weak solution of (2) with control $\overline{f}$.

Now we are going to prove that $(\overline{u}, \overline{v})$ satisfies the energy inequality (4.62) and therefore $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^M$. Let $\overline{z} = \sqrt{\overline{v} + \alpha^2}$, if we follow the ideas of Subsection 4.2.4 (for the cases $s \in [1,2)$ and $s \geq 2$) we conclude the convergences

$$\begin{aligned} (u_n+1)^{s/2} &\longrightarrow (\overline{u}+1)^{s/2} \text{ weakly* in } L^\infty(0,T;L^2(\Omega)), \\ (u_n+1)^{s/2} &\longrightarrow (\overline{u}+1)^{s/2} \text{ weakly in } L^2(0,T;H^1(\Omega)), \\ u_n^{s/2} \nabla z_n &\longrightarrow \nabla \overline{u}^{s/2} \nabla \overline{z} \text{ weakly in } L^2(Q), \\ \frac{\nabla z_n}{\sqrt{z_n}} &\longrightarrow \frac{\nabla \overline{z}}{\sqrt{\overline{z}}} \text{ weakly in } L^4(Q), \\ D^2 z_n &\longrightarrow D^2 \overline{z} \text{ weakly in } L^2(Q), \\ \nabla z_n &\longrightarrow \nabla \overline{z} \text{ strongly in } L^2(Q), \end{aligned} \tag{4.69}$$

$$E(u_n, z_n)(t) \to E(\overline{u}, \overline{z})(t) \ a.e.\ t \in (0,T), \tag{4.70}$$

and the weak continuity regularity

$$\overline{v} \in C_w([0,T];H^1(\Omega)) \text{ and } \overline{u} \in C_w([0,T];L^s(\Omega)), \text{ for } s \geq 1. \tag{4.71}$$

Therefore, using (4.69), (4.70) and (4.71) we are able to pass to the limit in the energy inequality (4.62) and conclude that $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^M$.

Finally, we prove that the infimum is attained in $(\overline{u}, \overline{v}, \overline{f})$. Since $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^M$, we have $J_{inf} \leq J(\overline{u}, \overline{v}, \overline{f})$. On the other hand, considering again Lemma 1.12, the functional J is weakly lower semicontinuous and then

$$J(\overline{u}, \overline{v}, \overline{f}) \leq \liminf_{n \to \infty} J(u^n, v^n, f^n) = J_{inf}.$$

Therefore we conclude that there is at least one $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^M$ such that $J(\overline{u}, \overline{v}, \overline{f}) = J_{inf}$, as we wanted to prove.

4.3.2 Proof of Theorem 4.6

Since the functional J in (4.5) is nonnegative,

$$J_{inf} := \inf_{(u,v,f)\in S_{ad}^E} J(u,v,f) \geq 0$$

is well defined and there is a sequence $\{(u_n, v_n, f_n)\} \subset S_{ad}^E$ such that

$$\lim_{n\to\infty} J(u_n, v_n, f_n) = J_{inf}. \tag{4.72}$$

Since $(u_n, v_n, f_n) \in S_{ad}^E$, in particular it satisfies the system (4.61) and the energy inequality

$$\begin{aligned} E(u_n, z_n)(t_2) + \beta\int_{t_1}^{t_2}\int_\Omega |\nabla[u_n+1]^{s/2}|^2 dx\ dt + \frac{1}{4}\int_{t_1}^{t_2}\int_\Omega u_n^s|\nabla z_n|^2 dx\ dt \\ +\beta\Big(\int_{t_1}^{t_2}\int_\Omega |D^2 z_n|^2\ dx\ dt + \int_{t_1}^{t_2}\int_\Omega \frac{|\nabla z_n|^4}{z^2}\ dx\ dt\Big) \\ \leq E(u_n, z_n)(t_1) + \mathcal{K}(\|f_n\|_{L^q(Q)}, \|v_0\|_{W^{2-2/q,q}(\Omega)}). \end{aligned} \tag{4.73}$$

Following the proof of Theorem 4.5, in Subsection 4.3.1, but this time using (4.73), we conclude that there is a continuous and increasing function of $\|f_n\|_{L^q(Q)}$, let us denote it by $C(\|f_n\|_{L^q(Q)}) > 0$, such that

$$\begin{aligned} \|u_n\|_{L^{5s/3}(Q)}, \|u_n\|_{L^{5s/(3+s)}(0,T;W^{1,5s/(3+s)}(\Omega))} \leq C(\|f_n\|_{L^q(Q)}), \\ \|\partial_t u_n\|_{L^{5s/(3+s)}(0,T;(W^{1,5s/(4s-3)}(\Omega))')} \leq C(\|f_n\|_{L^q(Q)}), \end{aligned} \tag{4.74}$$

for $s \in [1, 2)$,

$$\|u_n\|_{L^{5s/3}(Q)}, \|u_n\|_{L^2(0,T;H^1(\Omega))}, \|\partial_t u_n\|_{L^2(0,T;(H^1(\Omega))')} \leq C(\|f_n\|_{L^q(Q)}), \tag{4.75}$$

for $s \geq 2$ and

$$\begin{aligned} \|v_n\|_{L^\infty(Q)}, \|v_n\|_{L^4(0,T;W^{1,4}(\Omega))}, \|v_n\|_{L^2(0,T;H^2(\Omega))} \leq C(\|f_n\|_{L^q(Q)}) \\ \|\partial_t v_n\|_{L^{5/3}(Q)} \leq C(\|f_n\|_{L^q(Q)}) \end{aligned} \tag{4.76}$$

for $s \geq 1$. From the definition of the functional J and (4.72) we also conclude that

$$f_n \text{ is bounded in } L^q(Q).$$

Analogously to Subsection 4.3.1, from the latter and (4.73) we prove that there is $(\overline{u}, \overline{v}, \overline{f})$ such that, up to a subsequence, we have the convergences (4.65), (4.66), (4.67) and (4.68). These convergences allow us to conclude that $(\overline{u}, \overline{v})$ is a weak solution of (2) with control $\overline{f}$. Because of the weak lower semicontinuity of the norm (Lemma 1.12)

$$J(\overline{u}, \overline{v}, \overline{f}) \leq \liminf J(u_n, v_n, f_n) = J_{inf}. \tag{4.77}$$

However, we are not able to prove the $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^E$ and then we can not guarantee that $J_{inf} = J(\overline{u}, \overline{v}, \overline{f})$. In fact, following the ideas of Subsection 4.3.1, we are able pass to take the lim inf in (4.73) and, using that the map $r \in \mathbb{R}_+ \mapsto \mathcal{K}(r, \|v_0\|_{W^{2-2/q,q}(\Omega)})$ is continuous and therefore

$$\liminf_{n\to\infty} \mathcal{K}(\|f_n\|_{L^q(Q)}, \|v_0\|_{W^{2-2/q,q}(\Omega)}) \leq \mathcal{K}(\liminf_{n\to\infty} \|f_n\|_{L^q(Q)}, \|v_0\|_{W^{2-2/q,q}(\Omega)}),$$

we obtain

$$\begin{aligned} E(\overline{u}, \overline{z})(t_2) + \beta \int_{t_1}^{t_2} \int_\Omega |\nabla[\overline{u}+1]^{s/2}|^2 \, dx \, dt + \frac{1}{4} \int_{t_1}^{t_2} \int_\Omega \overline{u}^s |\nabla \overline{z}|^2 \, dx \, dt \\ + \beta \Big(\int_{t_1}^{t_2} \int_\Omega |D^2 \overline{z}|^2 \, dx \, dt + \int_{t_1}^{t_2} \int_\Omega \frac{|\nabla \overline{z}|^4}{z^2} \, dx \, dt \Big) \\ \leq E(\overline{u}, \overline{z})(t_1) + \mathcal{K}(\liminf_{n\to\infty} \|f_n\|_{L^q(Q)}, \|v_0\|_{W^{2-2/q,q}(\Omega)}). \end{aligned} \tag{4.78}$$

Since by Lemma 1.12 we have $\|\overline{f}\|_{L^q(Q)} \leq \liminf_{n\to\infty} \|f_n\|_{L^q(Q)}$, it is not clear that $(\overline{u}, \overline{z})$ satisfies (4.3), that is, we can not guarantee that $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^E$.

On the other hand, we can prove that for $M \geq \frac{q}{\gamma_f} J_{inf}$ we have $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^M$. Indeed, because of the convergence (4.68), the weak lower semicontinuity of the norm (see Lemma 1.12) and (4.72) we have

$$\|\overline{f}\|_{L^q(Q)} \leq \liminf_{n\to\infty} \|f_n\|_{L^q(Q)} \leq \frac{q}{\gamma_f} \liminf_{n\to\infty} J(u_n, v_n, f_n) = \frac{q}{\gamma_f} J_{inf}. \tag{4.79}$$

Then, taking $M \geq \frac{q}{\gamma_f} J_{inf}$, we have $\|\overline{f}\|_{L^q(Q)} \leq M$. Moreover, from (4.78) and (4.79) we also have $(\overline{u}, \overline{z})$ satisfying

$$\begin{aligned} E(\overline{u}, \overline{z})(t_2) + \beta \int_{t_1}^{t_2} \int_\Omega |\nabla[\overline{u}+1]^{s/2}|^2 \, dx \, dt + \frac{1}{4} \int_{t_1}^{t_2} \int_\Omega \overline{u}^s |\nabla \overline{z}|^2 \, dx \, dt \\ + \beta \Big(\int_{t_1}^{t_2} \int_\Omega |D^2 \overline{z}|^2 \, dx \, dt + \int_{t_1}^{t_2} \int_\Omega \frac{|\nabla \overline{z}|^4}{z^2} \, dx \, dt \Big) \\ \leq E(\overline{u}, \overline{z})(t_1) + \mathcal{K}(M, \|v_0\|_{W^{2-2/q,q}(\Omega)}), \end{aligned}$$

which allows us to conclude that $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}^M$.

Hence we have

$$\inf_{(u,v,f)\in S_{ad}^M} J(u, v, f) \leq J(\overline{u}, \overline{v}, \overline{f}),$$

and using (4.77) we finally conclude that

$$\inf_{(u,v,f)\in S_{ad}^M} J(u, v, f) \leq \inf_{(u,v,f)\in S_{ad}^E} J(u, v, f),$$

as we wanted to prove.

Chapter 5

AN OPTIMAL CONTROL PROBLEM SUBJECT TO STRONG SOLUTIONS OF CHEMOTAXIS-CONSUMPTION MODELS

5.1 Main Results

The first main contribution of this chapter is to give a regularity criterion that, under a mild additional regularity hypothesis over the u-component of a weak solution (see Definition 1.20 below) of the controlled problem (2), allows us to conclude that it is actually the unique strong solution (see Definition 1.21 below) of (2). In this result it is also established the continuous dependence in the strong regularity (see relation (5.1) below), which is essential to prove the existence of global optimal solution.

We have the following result.

Theorem 5.1 (Regularity criterion). *Let (u, v) be a weak solution of problem* (2) *with $f \in L^q(Q)$, for some $q > 5/2$. If, additionally, we suppose that*

$$u^s \in L^q(Q),$$

then $(u, v) \in X_q \times X_q$ and is the unique strong solution of problem (2). *Moreover, there is $\mathcal{K} = \mathcal{K}(\|u^s\|_{L^q(Q)}, \|f\|_{L^q(Q)}) > 0$, where $\mathcal{K}(\cdot,\cdot)$ is a continuous and increasing function with respect to each entry, $\|u^s\|_{L^q(Q)}$ and $\|f\|_{L^q}$, such that*

$$\|(u, v)\|_{X_q \times X_q} \leq \mathcal{K}(\|u^s\|_{L^q(Q)}, \|f\|_{L^q(Q)}). \tag{5.1}$$

Remark 5.2. Following the proof of Theorem 5.1 we observe that the power $5/2$ is critical in the sense that the result is proved for any $q > 5/2$ and, at least using the techniques employed in this proof, it is not possible to reach the same conclusion if

$q \leq 5/2$. We also recall that the hypothesis $q > 5/2$ is essential in Lemma 4.8 and therefore in the proof of existence of weak solutions to (2) with control $f \in L^q(Q)$. Moreover, if $q > 5/2$ then $X_q \hookrightarrow L^\infty(Q)$. Then Theorem 5.1 also gives an additional regularity hypothesis over a weak solution of the controlled problem (2) such that the cell density u does not blow up at finite time. □

The second main contribution is the existence of optimal solution to the following optimal control problem. Let $\mathcal{F}$ be a closed and convex subset of $L^q(Q)$, for a given $q > 5/2$. Consider the cost functional

$$J : L^{sq}(Q) \times L^2(Q) \times \mathcal{F} \longrightarrow \mathbb{R}$$

given by

$$\begin{aligned} J(u,v,f) &:= \frac{\gamma_u}{sq}\int_0^T \|u(t)-u_d(t)\|_{L^{sq}}^{sq}\, dt \\ +\frac{\gamma_v}{2}\int_0^T &\|v(t)-v_d(t)\|_{L^2}^2\, dt + \frac{\gamma_f}{q}\int_0^T \|f(t)\|_{L^q(\Omega_c)}^q\, dt, \end{aligned} \tag{5.2}$$

where $(u_d, v_d) \in L^{sq}(Q) \times L^2(Q)$ represents the desired states and the parameters $\gamma_u, \gamma_v, \gamma_f \geq 0$ measure the costs of the states and control. In addition, we will consider γ_u, γ_f satisfying

$$\begin{gathered} \gamma_u > 0 \quad \text{and} \\ \gamma_f > 0 \quad \text{or} \quad \mathcal{F} \text{ is bounded in } L^q(Q). \end{gathered} \tag{5.3}$$

We are going to minimize $J(u,v,f)$ subject to the admissible set of the triples (u,v,f) satisfying the controlled problem (2) in the strong setting, that is

$$\begin{aligned} S_{ad} = \ & \{(u,v,f) \in X_q \times X_q \times \mathcal{F}; (u,v) \text{ is the} \\ & \text{strong solution of (2) with control } f\}. \end{aligned}$$

Then, the following minimization problem is considered:

$$\left\{ \begin{array}{l} min\ J(u,v,f) \\ \text{subject to } (u,v,f) \in S_{ad}. \end{array} \right. \tag{5.4}$$

Since, given $f \in \mathcal{F}$, one can not assure the existence of a strong solution (u,v) associated to f, we are going to assume that

$$S_{ad} \neq \emptyset. \tag{5.5}$$

Remark 5.3. Analogously to [30] and [44], if $\Omega_c = \Omega$, that is, if the control acts in the whole domain, then (5.5) holds. In addition, when Ω is a 2D domain then (5.5) also holds. Indeed, from Theorem 2.5 we have the existence and uniqueness of weak solution (u,v) with $u \in L^\infty(Q)$, of the uncontrolled problem, that is (2) with $f = 0$. Since (u,v) and $f = 0$ satisfy the hypotheses of Theorem 5.1, we conclude that $(u,v) \in X_q \times X_q$ is the strong solution of (2) with $f \equiv 0$. In particular, $(u,v,0) \in S_{ad}$ and hence $S_{ad} \neq \emptyset$. □

Theorem 5.4 (Existence of optimal control). *Assume $S_{ad} \neq \emptyset$. Then the optimal control problem* (5.4) *has at least one global optimal solution* $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$.

The third main contribution of this chapter is the existence and uniqueness of Lagrange multipliers associated to any local optimal solution of (5.4). Let $q' = q/(q-1)$, the conjugate exponent of q, let $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$ be a local optimal solution of (5.4) and consider the following Lagrange multiplier problem for (λ, η) associated to $(\overline{u}, \overline{v}, \overline{f})$:

$$\begin{cases} -\partial_t \lambda - \Delta\lambda - \nabla\overline{v} \cdot \nabla\lambda + s\overline{u}^{s-1}\overline{v}\eta = g_\lambda, \\ -\partial_t \eta - \Delta\eta + \overline{u}^s \eta - \overline{f}\eta\, 1_{\Omega_c} + \nabla \cdot (\overline{u}\nabla\lambda) = g_\eta, \\ \partial_{\boldsymbol{n}}\lambda|_\Gamma = \partial_{\boldsymbol{n}}\eta|_\Gamma = 0,\ \lambda(T,x) = \eta(T,x) = 0, \end{cases} \tag{5.6}$$

where

$$g_\lambda = \gamma_u sgn(\overline{u} - u_d)|\overline{u} - u_d|^{sq-1}, \quad g_\eta = \gamma_v(\overline{v} - v_d). \tag{5.7}$$

Definition 5.5. (Very weak solution of (5.6)) Let $s \geq 1$, $q > 5/2$ and $f \in L^q(Q)$. A pair $(\lambda, \eta) \in L^{p'}(Q) \times L^{p'}(Q)$ is called a very weak solution of (5.6) if (λ, η) satisfies (5.6) in the sense of the dual space of $X_p \times X_p$, that is, the following variational formulation holds for any $U, V \in X_q$ with $\partial_{\boldsymbol{n}} U|_\Gamma = \partial_{\boldsymbol{n}} V|_\Gamma = 0$ and $U(0) = V(0) = 0$:

$$\begin{aligned} &\int_0^T\int_\Omega \lambda\Big(\partial_t U - \Delta U + \nabla\cdot(U\nabla\overline{v})\Big)\ dx\ dt + \int_0^T\int_\Omega s\overline{u}^{s-1}\overline{v}\eta\ U\ dx\, dt \\ &= \int_0^T\int_\Omega g_\lambda U\ dx\, dt, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\int_0^T\int_\Omega \eta\Big(\partial_t V - \Delta V + \overline{u}^s V - \overline{f}V 1_{\Omega_c}\Big)\ dx\, dt + \int_0^T\int_\Omega \lambda\ \nabla\cdot(\overline{u}\nabla V)\ dx\, dt \\ &= \int_0^T\int_\Omega g_\eta\ V\ dx\, dt. \end{aligned} \tag{5.9}$$

□

Theorem 5.6 (Existence of Lagrange multipliers). *Let $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$ be a local optimal solution of* (5.4)*. Then there exists a unique Lagrange multiplier $(\lambda, \eta) \in L^{q'}(Q) \times L^{q'}(Q)$ which is a very weak solution of the optimality system* (5.6) *and satisfies the optimality condition*

$$\int_0^T\int_{\Omega_c} (\gamma_f sgn(\overline{f})|\overline{f}|^{q-1} + \overline{v}\eta)(f - \overline{f})\ dx\ dt \geq 0, \quad \forall f \in \mathcal{F}. \tag{5.10}$$

Remark 5.7. If $\gamma_f > 0$ and there is no convex constraint on the control, that is $\mathcal{F} = L^q(Q)$, then (5.10) is equivalent to

$$\gamma_f sgn(\overline{f})|\overline{f}|^{q-1} + \overline{v}\eta = 0.$$

Since $\overline{v} \geq 0$, we conclude that

$$\overline{f} = -sgn(\eta)\left(\frac{1}{\gamma_f}\overline{v}|\eta|\right)^{1/(q-1)}.$$

□

Remark 5.8. The key to establish the existence of a Lagrange multiplier is to prove the existence of solution to the linearized problem given in (5.38) below. To help with this proof, in the Appendix C, we provide a result of existence of solution to an adequate generic parabolic linear system. This result is also useful in the study of the regularity of the Lagrange multiplier provided by Theorem 5.6 that is carried out in Theorem 5.9 below. □

Finally, we establish a result on the additional regularity of the Lagrange multiplier (λ, η) given by Theorem 5.6 depending on the L^p regularity of the right hand side term g_λ.

Theorem 5.9. *Let* $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$ *be a local optimal of problem* (5.4). *We have:*

1. *if* $g_\lambda \in L^p(Q)$, *for* $p \in [10/9, 10/7)$, *then the Lagrange multiplier* $(\lambda, \eta) \in L^2(Q) \times L^2(Q)$ *and satisfies* (5.6) *in the very weak sense;*

2. *if* $g_\lambda \in L^p(Q)$, *for* $p \in [10/7, 2]$, *then the Lagrange multiplier* $(\lambda, \eta) \in X_p \times X_p$ *and satisfies* (5.6) *in the strong sense.*

Remark 5.10. Since we consider $v_d \in L^2(Q)$, which implies $g_\eta \in L^2(Q)$, the previous analysis for $p > 2$ does not seem to lead to more relevant conclusions. □

Remark 5.11. To guarantee that the terms of the functional J given in (5.2) make sense it is enough that $u_d \in L^{\tilde{q}}(Q)$, with $\tilde{q} \geq sq$, and $v_d \in L^2(Q)$. With this regularity, $g_\eta \in L^2(Q)$ and $g_\lambda \in L^p(Q)$, where $p = p(s, q, \tilde{q}) = \tilde{q}/(sq - 1)$. Hence the regularity of g_λ depends on $s \geq 1$, $q > 5/2$ and $\tilde{q} \geq sq$, and is decreasing with respect to s with $p(s, q, \tilde{q}) \to 1$ as $s \to \infty$. For instance if $\tilde{q} = sq$, we have $p = sq/(sq - 1)$. In this case, since $s \geq 1$ and $q > 5/2$, then $p \in (1, 5/3)$. Let us fix $q > 5/2$ close to $5/2$ and vary the values of s. Then, if $s \in [1, 10/3q]$ we are in item 2 of Theorem 5.9, and if $s \in (10/3q, 10/q]$ we are in item 1 of Theorem 5.9. But, if $s > 10/q$ then $p \in (1, 10/9)$ and we only have the existence and uniqueness given in Theorem 5.6 because, in this case, additional regularity for the Lagrange multiplier is not clear. □

The rest of the chapter is organized as follows. In Section 5.2 we prove Theorem 5.1 after establishing some preliminary results. In Section 5.3 we prove Theorem 5.4. In Section 5.4 we prove Theorem 5.6 and study the additional regularity of the Lagrange multiplier demonstrating Theorem 5.9.

5.2 Regularity Criterion

The main objective of the present section is to prove Theorem 5.1. To do it, we first introduce and prove a series of useful results.

Lemma 5.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then we have:*

1. *$X_p \hookrightarrow L^{5p/(5-2p)}(Q)$, if $p \in [1, 5/2)$;*

2. *$X_p \hookrightarrow L^\infty(L^q)$, for all $q \in [1, \infty)$, if $p = 5/2$;*

3. *$X_p \hookrightarrow L^\infty(Q)$ if $p > 5/2$.*

Proof. By definition of X_p, if $w \in X_p$ then we have $w \in C(W^{2-2/p,p}) \cap L^p(W^{2,p})$. If $p \in [1, 5/2)$, this implies

$$w \in C(W^{2-2/p,p}) \cap L^p(W^{2,p}) \hookrightarrow L^\infty(L^{3p/(5-2p)}) \cap L^p(W^{2,p}).$$

Then, using Lemma 1.4 yields the desired result. For $p = 5/2$ we use the continuous injection $W^{2-2/p,p}(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in [1, \infty)$, and for $p > 5/2$, the continuous injection $W^{2-2/p,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. ■

Lemma 5.13. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $p \in (1, 5)$. If $w \in X_p$ then $\nabla w \in L^{5p/(5-p)}(Q)$. Moreover, there is a constant $C > 0$ such that,*

$$\|\nabla w\|_{L^{5p/(5-p)}(Q)} \leq C \|w\|_{X_p}, \forall w \in X_p.$$

Proof. Suppose $p \in [2, 5)$. Since $w \in X_p$, we have by definition that

$$\nabla w \in L^\infty(W^{1-2/p,p}) \cap L^p(W^{1,p}) \hookrightarrow L^\infty(L^{3p/(5-p)}) \cap L^p(W^{1,p})$$

and using Lemma 1.4 we conclude that

$$\nabla w \in L^{5p/(5-p)}(Q).$$

Now we must deal with the case $p \in (1, 2)$. In this case we have, from the definition of X_p,

$$w \in L^\infty(W^{2-2/p,p}) \cap L^p(W^{2,p}(\Omega)).$$

Then we have

$$D^{2-2/p} w \in L^\infty(L^p) \cap L^p(W^{2/p,p})$$

and this implies that

$$D^{2-2/p} w \in L^\infty(L^p) \cap L^p(W^{\beta,3p/(1+\beta p)}), \text{ for any } \beta \in (1, 2/p).$$

Now, using Lemma 1.2 with

$$\alpha = \frac{2}{p} - 1,\ \beta = \beta,\ \tilde{p} = \frac{3p}{1+\beta p},\ \gamma = 0 \text{ and } \tilde{q} = p$$

we obtain

$$\lambda = \frac{2}{p} - 1 \text{ and } r = \frac{3\beta p^2}{-\beta p^2 + (5\beta + 2)p - 4}$$

with

$$\|D^{2-2/p}w\|^r_{W^{2/p-1,r}(\Omega)} \leq C\|D^{2-2/p}w\|^{(2/p-1)r}_{W^{\beta,3p/(1+\beta p)}(\Omega)}\|D^{2-2/p}w\|^{(2-2/p)r}_{L^p(\Omega)}.$$

For the right hand side of this inequality to be integrable, we must choose β such that $(2/p-1)r = p$. Therefore, choosing

$$\beta = \frac{10-5p}{5p-p^2}$$

we conclude that $\nabla w \in L^r(Q)$, with

$$r = \frac{10p-5p^2}{p^2-7p+10} = \frac{5p}{5-p}.$$

■

Lemma 5.14. *Let (u, v) be a weak solution of (2). Suppose $u \in L^p(Q)$, for some $p > 5/3$, and $v \in X_q$, for some $q > 5/2$. We conclude that $u \in X_{pq/(p+q)}$ and that there is $C = C(\|u\|_{L^p(Q)}, \|\nabla u\|_{L^{5/4}(Q)}, \|v\|_{X_q}) > 0$, which is continuous and increasing with respect to each entry, $\|u\|_{L^p(Q)}$, $\|\nabla u\|_{L^{5/4}(Q)}$ and $\|v\|_{X_q}$, such that*

$$\|u\|_{X_{pq/(p+q)}} \leq C(\|u\|_{L^p(Q)}, \|\nabla u\|_{L^{5/4}(Q)}, \|v\|_{X_q}). \tag{5.11}$$

The result is also valid for $p = \infty$ and, in this case, we conclude that $u \in X_q$ with

$$\|u\|_{X_q} \leq C(\|u\|_{L^\infty(Q)}, \|\nabla u\|_{L^{5/4}(Q)}, \|v\|_{X_q}). \tag{5.12}$$

Proof. The basic idea of the proof is a bootstrapping in the u-equation of (2) that allows one to arrive at the desired regularity in a finite number of iterations. We are going to consider the case $p < \infty$ and, with small adaptations, one can follow the same reasoning to prove the result for $p = \infty$. Also, we are going to prove that $u \in X_{pq/(p+q)}$. The proofs of (5.11) and (5.12) come from the fact that all the results used along this proof, such as Lemmas 5.13 and 1.13, for example, give us continuous injections. Indeed, since the number of steps of the procedure of gaining regularity, to be presented in what follows, is finite, one can follow the estimates furnished by Lemmas 5.13 and 1.13 each time they are applied and, at the end, conclude (5.11) and (5.12). Bearing that in mind, we proceed with the proof of $u \in X_{pq/(p+q)}$, for finite p.

Using Lemma 5.13 for v we conclude that $\nabla v \in L^{5q/(5-q)}(Q)$. Since $q > 5/2$ we have, in particular, that

$$\text{there is } \beta > 1 \text{ such that } \nabla v \in L^{5\beta}(Q). \tag{5.13}$$

And since (u, v) is a weak solution of (2) we have, in particular,

$$\nabla u \in L^{5/4}(Q).$$

By hypothesis and by the definition of X_q we have

$$u \in L^p(Q), \text{ with } p > 5/3 \text{ and } \Delta v \in L^q(Q) \text{ with } q > 5/2.$$

Considering these regularities, we have the u-equation satisfied in the strong sense

$$\partial_t u - \Delta u = -u\Delta v - \nabla u \cdot \nabla v \tag{5.14}$$

with

$$u\Delta v \in L^{pq/(p+q)}(Q), \text{ with } \frac{pq}{p+q} > 1, \tag{5.15}$$

and

$$\nabla u \cdot \nabla v \in L^{r_0}, \text{ with } r_0 = \frac{5\beta}{4\beta+1} > 1. \tag{5.16}$$

Hence, using Lemma 1.13 for (5.14) we conclude that

$$u \in X_r, \text{ with } r = \min\left\{r_0, \frac{pq}{p+q}\right\} > 1.$$

If $r_0 \geq \frac{pq}{p+q}$ then $r = \frac{pq}{p+q}$ and the proof is finished. Therefore let us treat the case in which $r_0 < \frac{pq}{p+q}$. Since for $u\Delta v$ we already have (5.15), we focus on enhancing the regularity of the term $\nabla u \cdot \nabla v$.

In this case, we have $u \in X_{r_0}$. Using Lemma 5.13 we obtain

$$\nabla u \in L^{5r_0/(5-r_0)}(Q)$$

Considering this regularity and (5.13), where $\beta > 1$, and (5.16), where $r_0 > 1$, the new regularity of $\nabla u \cdot \nabla v$ is $L^\gamma(Q)$, with

$$\gamma = \frac{5\beta}{5\beta - (\beta-1)r_0}\, r_0 > \frac{5\beta}{4\beta+1}\, r_0.$$

Define $\alpha = 5\beta/(4\beta+1)$. Note that $\alpha = r_0 > 1$ and $\gamma > \alpha r_0$. Then, let us define $r_1 = \alpha r_0$. Since $\alpha > 1$, we have $r_1 > r_0 > 1$. Now, if $r_1 < \frac{pq}{p+q}$ then, from Lemma 1.13, we have $u \in X_{r_1}$. Proceeding by induction, if we have $\nabla u \cdot \nabla v \in L^{r_{n-1}}$, with $r_{n-1} = \alpha^{n-1} r_0 < \frac{pq}{p+q}$, then we have $u \in X_{r_{n-1}}$ and, by Lemma 5.13 we obtain

$$\nabla u \in L^{5r_{n-1}/(5-r_{n-1})}(Q).$$

And using again (5.13), where $\beta > 1$, and (5.16), where $r_0 > 1$, the new regularity of $\nabla u \cdot \nabla v$ is $L^\gamma(Q)$, with

$$\gamma = \frac{5\beta}{5\beta - (\beta-1)r_{n-1}}\, r_{n-1} > \frac{5\beta}{4\beta+1}\, r_{n-1} = \alpha r_{n-1} = \alpha^n r_0.$$

Therefore we can define $r_n = \alpha^n r_0$ and again using Lemma 1.13 for (5.14) we conclude that

$$u \in X_r, \text{ with } r = \min\Big\{\alpha^n r_0, \frac{pq}{p+q}\Big\}.$$

Since $\alpha > 1$, there is an index n_0 such that $\alpha^{n_0} r_0 < \frac{pq}{p+q}$ but $\alpha^{n_0+1} r_0 \geq \frac{pq}{p+q}$. Therefore we conclude, using the result proved by induction, that in fact we have $u \in X_{pq/(p+q)}$, as we wanted to prove. ■

Theorem 5.15. *Let (u, v) be a weak solution of (2) given $f \in L^q(Q)$, $q > 5/2$. If, additionally, $u^s \in L^q(Q)$ then $v \in X_q$ and $u \in L^\infty(Q)$. This implies, in particular, that $\nabla v \in L^{5q/(5-q)}(Q) \hookrightarrow L^5(Q)$, $u \in X_q$ and that (u, v) is the unique strong solution of (2). Moreover, there is $C = C(\|u^s\|_{L^q(Q)}, \|f\|_{L^q(Q)}, \|\nabla u\|_{L^{5/4}(Q)}) > 0$, which is continuous and increasing with respect to each entry, $\|u^s\|_{L^q(Q)}$, $\|f\|_{L^q}$ and $\|\nabla u\|_{L^{5/4}(Q)}$, such that*

$$\|(u, v)\|_{X_q \times X_q} \leq C(\|u^s\|_{L^q(Q)}, \|f\|_{L^q(Q)}, \|\nabla u\|_{L^{5/4}(Q)}). \tag{5.17}$$

Proof. Analogously to Lemma 5.14, we are going to prove that $(u, v) \in X_q \times X_q$ and, since the number of steps of the procedure of gaining regularity, to be presented in what follows, is finite, the proof of (5.17) is a consequence of the estimates furnished by Lemmas 1.13, 5.12 and 5.14.

Considering the regularity $v \in L^\infty(Q)$ given by the regularity of the weak solution (u, v) of (2), if we have $u^s, f \in L^q(Q)$ then, by applying Lemma 1.13 to the v-equation of (2), we conclude that

$$v \in X_q. \tag{5.18}$$

Now denote $p_0 = sq > 5/2$. Then we have $u \in L^{p_0}(Q)$, $\Delta v \in L^q(Q)$ and we can apply Lemma 5.14 to conclude that

$$u \in X_{qp_0/(q+p_0)}.$$

At least in this first iterations, we assume that we are in the case in which we have $qp_0/(q + p_0) < 5/2$. Then, if now we apply Lemma 5.12 we obtain

$$u \in L^r(Q),$$

with

$$r = \frac{5q}{5q + 5p_0 - 2qp_0}\, p_0.$$

Since $q > 5/2$ we can say that that there is $\alpha > 1$ such that $q = 5\alpha/2$. Then using the fact that $p_0 \geq q$, we obtain

$$\begin{aligned} r &= \frac{5q}{5q - 5(\alpha-1)p_0}\, p_0 = \frac{q}{q - (\alpha-1)p_0}\, p_0 = \left(1 + \frac{(\alpha-1)p_0}{q - (\alpha-1)p_0}\right) p_0 \\ &> \left(1 + \frac{(\alpha-1)p_0}{q}\right) p_0 \geq (1 + \alpha - 1)p_0 = \alpha p_0 \end{aligned}$$

Define $p_1 = \alpha p_0$. Since $r \geq p_1$ we have, in particular $u \in L^{p_1}(Q)$. Proceeding by induction, if we have $u \in L^{p_{n-1}}(Q)$, with $p_{n-1} = \alpha^{n-1} p_0 \geq q$ satisfying $qp_{n-1}/(q+p_{n-1}) \leq 5/2$ then we can apply Lemmas 5.14 and 5.12 and conclude that

$$u \in L^{p_n}(Q), \text{ with } p_n = \alpha^n p_0.$$

As a consequence, we can apply Lemma 5.14 and obtain

$$u \in X_{qp_n/(q+p_n)}.$$

Since $\alpha > 1$, $p_n = \alpha^n p_0 = \alpha^n sq$ grows as n increases in such a way that there is an index n_0 such that

$$qp_{n_0-1}/(q+p_{n_0-1}) \leq 5/2$$

but applying the result proved by induction we conclude that

$$u \in X_{qp_{n_0}/(q+p_{n_0})}, \text{ with } \frac{qp_{n_0}}{(q+p_{n_0})} > 5/2.$$

Hence, applying Lemma 5.12 we obtain

$$u \in L^\infty(Q).$$

Finally, once we have $u \in L^\infty(Q)$, we use (5.18) and Lemma 5.14 to conclude that

$$u \in X_q,$$

finishing the proof. ■

Now let $z = \sqrt{v+\alpha^2}$, for some $\alpha > 0$, and consider the energy

$$E(u,z)(t) = \frac{s}{4}\int_\Omega g(u(t,x))\, dx + \frac{1}{2}\int_\Omega |\nabla z(t,x)|^2\, dx,$$

where

$$g(u) = \begin{cases} (u+1)ln(u+1) - u, & \text{if } s = 1, \\ \dfrac{u^s}{s(s-1)}, & \text{if } s > 1. \end{cases}$$

We have the following.

Lemma 5.16. *Let (u,v) be the strong solution of (2) given $f \in L^q(Q)$, with $q > 5/2$, let α be a positive real number and $z = \sqrt{v+\alpha^2}$. There is $\alpha_0 > 0$, independent of (u,v,f), such that if $0 < \alpha \leq \alpha_0$ then (u,v) satisfies*

$$0 < \alpha \leq z(t,x) \leq \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}) \tag{5.19}$$

and the energy inequality

$$\begin{aligned} E(u,z)(t_2) + \beta \int_{t_1}^{t_2}\int_\Omega |\nabla [u+1]^{s/2}|^2 \, dx \, dt + \frac{1}{4}\int_{t_1}^{t_2}\int_\Omega u^s |\nabla z|^2 \, dx \, dt \\ + \beta \Big(\int_{t_1}^{t_2}\int_\Omega |D^2 z|^2 \, dx \, dt + \int_{t_1}^{t_2}\int_\Omega \frac{|\nabla z|^4}{z^2} \, dx \, dt \Big) \\ \le E(u,z)(t_1) + \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}), \end{aligned} \tag{5.20}$$

for a.e. $t_1, t_2 \in [0,T]$, *with* $t_2 > t_1$; *where* $\mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)})$ *is a continuous and increasing function with respect to* $\|f\|_{L^q(Q)}$ *and* $\beta > 0$ *is a constant, independent of* (u, v, f).

Sketch of the proof. In Chapter 4, given $f \in L^q(Q)$, the existence of weak solutions of (2) satisfying the pointwise bound (5.19) and the energy inequality (5.20) is proved. In the present lemma, we state that the unique strong solution of (2) satisfies (5.19) and (5.20). For the proof of this statement we refer the reader to Chapter 4, where the authors first prove the pointwise bound and the energy inequality for the solution of a truncated problem, depending on a parameter m, and then pass to the limit as $m \to \infty$, proving (5.19) and (5.20) for the weak solution of (2) obtained through this limit. In the present case, due to the strong regularity, the ideas of Chapter 4 can be applied directly to the strong solution (u, v) of (2), yielding the desired result. ■

Now the idea is to eliminate the dependence on ∇u in (5.17). Using the results developed in the present section we are finally in position of proving Theorem 5.1.

5.2.1 Proof of Theorem 5.1

Let (u, v) be a weak solution of (2) given $f \in L^q(Q)$, $q > 5/2$, satisfying, additionally, $u^s \in L^q(Q)$. It stems from Theorem 5.15 that $(u, v) \in X_q \times X_q$ is the strong solution of (2) satisfying (5.17). Therefore it suffices to prove that $\|\nabla u\|_{L^{5/4}(Q)}$ can be estimated in terms of $\|f\|_{L^q(Q)}$ to obtain (5.1), finishing the proof. We analyze the cases $s \in [1, 2)$ and $s \ge 2$ separately.

Case $s \in [1, 2)$: From (5.20) with $t_1 = 0$ (and by integrating the u-equation of (2), in the case $s = 1$), we have

$$\begin{aligned} \|(u+1)^{s/2}\|_{L^\infty(L^2)} &\le E(u,z)(0) + \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}), \\ \|\nabla [u+1]^{s/2}\|_{L^2(Q)} &\le E(u,z)(0) + \mathcal{K}_1(\|f\|_{L^q(Q)}, \|v^0\|_{W^{2-2/q,q}(\Omega)}). \end{aligned} \tag{5.21}$$

This implies that there exists $C_1 = C_1(\|f\|_{L^q(Q)}) > 0$ (C_1 also depends on (u^0, v^0), but since the initial data are fixed we omit it from now on) which is continuous and increasing with respect to $\|f\|_{L^q(Q)}$ and such that

$$\|(u+1)^{s/2}\|_{L^\infty(L^2) \cap L^2(H^1)} \le C_2(\|f\|_{L^q(Q)}).$$

In particular, by interpolation we have

$$\|(u+1)\|_{L^{5s/3}(Q)} \leq C_3(\|f\|_{L^q(Q)}). \tag{5.22}$$

Now, reminding that $s \in [1,2)$, if we use (5.21), (5.22) and the relation

$$\nabla u = \nabla(u+1) = \nabla\big((u+1)^{s/2}\big)^{2/s} = \frac{2}{s}(u+1)^{1-s/2}\ \nabla(u+1)^{s/2},$$

we conclude that there exists $C_4 = C_4(\|f\|_{L^q(Q)}) > 0$ which is continuous and increasing with respect to $\|f\|_{L^q(Q)}$ satisfying

$$\|\nabla u\|_{L^{5s/(3+s)}(Q)} \leq C_4(\|f\|_{L^q(Q)}).$$

Since $s \geq 1$, we have $5s/(3+s) \geq 5/4$ and this implies, in particular, that there is $C > 0$ such that

$$\|\nabla u\|_{L^{5/4}(Q)} \leq C\ C_4(\|f\|_{L^q(Q)}). \tag{5.23}$$

Therefore, using (5.23) in (5.17) we conclude (5.1).

Case $s \geq 2$: From (5.19) and (5.20) with $t_1 = 0$ we conclude, in particular, that there is $C_1 = C_1(\|f\|_{L^q(Q)}) > 0$ such that

$$\|z\|_{L^\infty(Q)}, \|\nabla z\|_{L^\infty(L^2)}, \|u^{s/2}\nabla z\|_{L^2(Q)} \leq C_1(\|f\|_{L^q(Q)}). \tag{5.24}$$

Now, let us consider the sets

$$\{0 \leq u \leq 1\} = \Big\{(t,x) \in \Omega \ \Big|\ 0 \leq u(t,x) \leq 1\Big\}$$

and

$$\{u \geq 1\} = \Big\{(t,x) \in \Omega \ \Big|\ u(t,x) \geq 1\Big\}.$$

Note that, since $s \geq 2$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega u(t,x)^2 |\nabla z(t,x)|^2 dx\ dt \\
&\leq \int_0^T \int_{\{0\leq u\leq 1\}} |\nabla z(t,x)|^2 dx\ dt + \int_0^T \int_{\{u\geq 1\}} u(t,x))^s |\nabla z(t,x)|^2 dx\ dt \\
&\leq \int_0^T \int_\Omega |\nabla z(t,x)|^2\ dx + \int_\Omega u(t,x))^s |\nabla z(t,x)|^2\ dx.
\end{aligned}$$

Thus, by (5.24) we conclude that there is $C > 0$ such that

$$\|u^{s/2}\nabla z\|_{L^2(Q)} \leq C\ C_1(\|f\|_{L^q(Q)}). \tag{5.25}$$

Now we test the u-equation of (2) by u and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} &= 2\int_\Omega uz\nabla z \cdot \nabla u\, dx \\ &\leq C\|z\|^2_{L^\infty(Q)}\int_\Omega u^2|\nabla z|^2 dx + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq C\|z\|^2_{L^\infty(Q)}\int_\Omega u^2|\nabla v|^2 dx.$$

Integrating with respect to t, we conclude from (5.24) that there is $C_2(\|f\|_{L^q(Q)}) > 0$ such that

$$\|\nabla u\|_{L^2(Q)} \leq C_2(\|f\|_{L^q(Q)}).$$

This implies, in particular, that we have (5.23) also for the case $s \geq 2$ and therefore, using again (5.23) in (5.17) leads us to (5.1).

5.3 Existence of Global Optimal Solution

From (5.5) and since the functional J in (5.4) is nonnegative,

$$J_{inf} := \inf_{(u,v,f)\in S_{ad}} J(u,v,f) \geq 0$$

is well defined and there is a minimizing sequence $\{(u_n, v_n, f_n)\} \subset S_{ad}$ satisfying

$$\begin{cases} \partial_t u_n - \Delta u_n = -\nabla\cdot(u_n\nabla v_n), \quad \partial_t v_n - \Delta v_n = -u_n^s v_n + f_n v_n 1_{\Omega_c}, \\ \partial_{\boldsymbol{n}} u_n|_\Gamma = \partial_{\boldsymbol{n}} v_n|_\Gamma = 0, \quad u_n(0) = u^0, \quad v_n(0) = v^0, \end{cases} \tag{5.26}$$

and

$$\lim_{n\to\infty} J(u_n, v_n, f_n) = J_{inf}.$$

Next we prove that there is $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$, defined as the limit of a subsequence of $\{(u_n, v_n, f_n)\}_n$, such that $J(\overline{u}, \overline{v}, \overline{f}) = J_{inf}$.

In fact, from the definition of J and the hypothesis (5.3), we have

$$\begin{aligned}&\{u_n^s\}_n \text{ is bounded in } L^q(Q),\\ &\{f_n\}_n \text{ is bounded in } L^q(Q).\end{aligned} \tag{5.27}$$

Since $(u_n, v_n, f_n) \in S_{ad}$, (u_n, v_n) is the strong solution of (2) with control f_n. Then, from (5.27) and Theorem 5.1 we obtain

$$\{u_n\}_n \text{ and } \{v_n\}_n \text{ are bounded in } X_q. \tag{5.28}$$

We recall that since $\mathcal{F}$ is a closed and convex subset of $L^q(Q)$ then $\mathcal{F}$ is also weakly closed in $L^q(Q)$. Therefore, accounting for the n-independent bounds (5.27) and (5.28) we conclude that there exists $(\overline{u}, \overline{v}, \overline{f}) \in X_q \times X_q \times \mathcal{F}$ such that, up to a subsequence, we have the weak convergences as $n \to +\infty$:

$$\begin{gathered}(u_n, v_n) \to (\overline{u}, \overline{v}) \text{ weakly* in } L^\infty(W^{2-2/q,q}) \times L^\infty(W^{2-2/q,q}),\\ (u_n, v_n) \to (\overline{u}, \overline{v}) \text{ weakly in } L^q(W^{2,q}) \times L^q(W^{2,q}),\\ (\partial_t u_n, \partial_t v_n) \to (\partial_t \overline{u}, \partial_t \overline{v}) \text{ weakly in } L^q(Q) \times L^q(Q),\\ f_n \to \overline{f} \text{ weakly in } L^q(Q).\end{gathered} \tag{5.29}$$

Since $q > 5/2$, we have $W^{2-2/q,q}$ compactly embedded in $C^0(\overline{\Omega})$ and $5q/(5-q) > 2q$, hence, from Lemma 1.14 we also have the strong convergences:

$$\begin{gathered}(u_n, v_n) \to (\overline{u}, \overline{v}) \text{ strongly in } C(\overline{Q}) \times C(\overline{Q}),\\ (\nabla u_n, \nabla v_n) \to (\nabla \overline{u}, \nabla \overline{v}) \text{ strongly in } L^{2q}(Q) \times L^{2q}(Q).\end{gathered} \tag{5.30}$$

From the above strong convergence we conclude that $\overline{u}(0) = u^0$ and $\overline{v}(0) = v^0$. With the convergences (5.29) and (5.30) we pass to the limit in the nonlinear terms of (5.26) and prove that

$$\begin{gathered}\nabla u_n \cdot \nabla v_n + u_n \Delta v_n \to \nabla \overline{u} \cdot \nabla \overline{v} + \overline{u} \Delta \overline{v} \text{ weakly in } L^q(Q),\\ u_n^s v_n \to \overline{u}^s\, \overline{v} \text{ strongly in } C(\overline{Q}),\\ f_n v_n 1_{\Omega_c} \to \overline{f}\overline{v} 1_{\Omega_c} \text{ weakly in } L^q(Q).\end{gathered}$$

Since, passing to the limit in the linear terms of (5.26) is rather standard, we prove that $(\overline{u}, \overline{v}) \in X_q \times X_q$ is the strong solution of (2) with control $\overline{f} \in \mathcal{F}$, that is, $(\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$. Hence, we have, in particular,

$$\inf_{(u,v,f) \in S_{ad}} J(u, v, f) \leq J(\overline{u}, \overline{v}, \overline{f}). \tag{5.31}$$

On the other hand, using the fact that the functional J is lower weakly semicontinuous, we also have

$$J(\overline{u}, \overline{v}, \overline{f}) \leq \inf_{(u,v,f) \in S_{ad}} J(u, v, f).$$

Thus, jointly to (5.31), one concludes that that $(\overline{u}, \overline{v}, \overline{f})$ is a global optimum.

5.4 First Order Necessary Conditions for a Local Optimal Solution

In the present section we derive the first order necessary optimality conditions for a local optimal solution $(\overline{u}, \overline{v}, \overline{f})$ of the optimal control problem (5.4). To this purpose, we use a Lagrange Multipliers theorem given by [66] in an abstract setting that we introduce in Subsection 5.4.1. Then, in Subsection 5.4.2 we prove that any local optimal solution is a regular point (see Definition 5.19 below) and in Subsection 5.4.3 we prove Theorem 5.6.

5.4.1 Abstract setting and a Lagrange multipliers theorem

Let us consider the following abstract optimization problem:

$$\min_{r\in\mathbb{M}} J(r) \ \text{ subject to } \ G(r)=0, \tag{5.32}$$

where $J:\mathbb{X}\to\mathbb{R}$ is a functional, $G:\mathbb{X}\to\mathbb{Y}$ is an operator, $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces and $\mathbb{M}\subset\mathbb{X}$ is a closed and convex subset. Note that the admissible set for problem (5.32) is

$$S=\{r\in\mathbb{M}\mid G(r)=0\}.$$

Next we define the Lagrangian functional, the Lagrange multipliers and the so called regular points.

Definition 5.17. (Lagrangian functional) The functional $\mathcal{L}:\mathbb{X}\times\mathbb{Y}'\to\mathbb{R}$, given by

$$\mathcal{L}(r,\xi)=J(r)-\langle\xi,G(r)\rangle_{\mathbb{Y}'}, \tag{5.33}$$

is called the Lagrangian functional related to problem (5.4). □

Definition 5.18. (Lagrange multipliers) Let $\overline{r}\in S$ be a local optimal solution of problem (5.32). Suppose that J and G are Fréchet differentiable in $\overline{r}$, the derivatives being denoted by $J'(\overline{r})$ and $G'(\overline{r})$, respectively. Then, $\xi\in\mathbb{Y}'$ is called a Lagrange multiplier for (5.32) at the point $\overline{r}$ if

$$\mathcal{L}'_r(\overline{r},\xi)[c]=J'(\overline{r})[c]-\langle\xi,G'(\overline{r})[c]\rangle_{\mathbb{Y}'}\geq 0,\ \forall c\in\mathcal{C}(\overline{r}), \tag{5.34}$$

where $\mathcal{C}(\overline{r})=\{\theta(r-\overline{r})\mid r\in\mathbb{M},\ \theta\geq 0\}$ is the conical hull of $\overline{r}\in\mathbb{M}$. □

Definition 5.19. (Regular point) Let $\overline{r}\in S$ be a local optimal solution of problem (5.32). The point $\overline{r}$ is called a regular point if

$$G'(\overline{r})[\mathcal{C}(\overline{r})]=\mathbb{Y}.$$

□

Finally, we state the theorem on the existence of Lagrange multipliers.

Theorem 5.20. ([66]) *Let $\overline{r}\in S$ be a local optimal solution of problem* (5.32)*. Suppose that J is Fréchet differentiable and G is continuously Fréchet differentiable. If $\overline{r}$ is a regular point, then the set of Lagrange multipliers for problem* (5.32) *at $\overline{r}$ is nonempty.*

5.4.2 Local optimal solutions are regular points

To apply the theory of Subsection 5.4.1 to our optimal control problem (5.4) and derive the first order necessary conditions for a local optimal solution, we will reformulate (5.4) using the abstract setting of (5.32). Since we want $\mathbb{X}$ and $\mathbb{Y}$ to be Banach spaces, let us define them as

$$\mathbb{X} = \tilde{X}_q \times \tilde{X}_q \times L^q(Q), \ \mathbb{Y} = L^q(Q) \times L^q(Q),$$

where $\tilde{X}_q = \{w \in X_q \mid \partial_{\boldsymbol{n}} w|_\Gamma = 0\}$. Next we define the operator $G = (G_1, G_2) : \mathbb{X} \to \mathbb{Y}$, where

$$G_1 : \mathbb{X} \to L^q(Q), \ G_2 : \mathbb{X} \to L^q(Q)$$

are defined for each $r = (u, v, f) \in \mathbb{X}$ as

$$\begin{cases} G_1(r) &= \partial_t u - \Delta u + \nabla \cdot (u \nabla v) \\ G_2(r) &= \partial_t v - \Delta v + u^s v - f v \ 1_{\Omega_c}. \end{cases}$$

Now, to consider the initial conditions (u^0, v^0), we introduce the space

$$\hat{X}_q = \{w \in \tilde{X}_q \mid w(0,x) = 0\}$$

and we define $\mathbb{M}$, the closed and convex subset of $\mathbb{X}$, as

$$\mathbb{M} = (\hat{u}, \hat{v}, \hat{f}) + \hat{X}_q \times \hat{X}_q \times (\mathcal{F} - \hat{f}),$$

where $(\hat{u}, \hat{v})$ is the strong solution of (2) given the control $\hat{f} \in L^q(Q)$. With the operator G and the set $\mathbb{M}$ defined, we rewrite the optimal control problem (5.4) as

$$\min_{r \in \mathbb{M}} J(r) \ \text{ subject to } \ G(r) = 0, \tag{5.35}$$

The admissible set for problem (5.35) is

$$S_{ad} = \{r \in \mathbb{M} \mid G(r) = 0\}.$$

We have the following results on the differenciability of the functional J and the operator G.

Lemma 5.21. *The functional $J : \mathbb{X} \to \mathbb{R}$ is Fréchet differentiable and the Fréchet derivative of J in $\overline{r} = (\overline{u}, \overline{v}, \overline{f}) \in \mathbb{X}$ in the direction $c = (U, V, F) \in \mathbb{X}$ is*

$$\begin{aligned} J'(\overline{r})[c] &= \gamma_u \int_0^T \int_\Omega sgn(\overline{u} - u_d)|\overline{u} - u_d|^{sq-1} U \ dx \ dt \\ &+ \gamma_v \int_0^T \int_\Omega (\overline{v} - v_d) \ V \ dx \ dt + \gamma_f \int_0^T \int_{\Omega_c} sgn(\overline{f})|\overline{f}|^{q-1} F \ dx \ dt. \end{aligned} \tag{5.36}$$

Lemma 5.22. *The operator $G : \mathbb{X} \to \mathbb{R}$ is continuously Fréchet differentiable and the Fréchet derivative of G in $\overline{r} = (\overline{u}, \overline{v}, \overline{f}) \in \mathbb{X}$ in the direction $c = (U, V, F) \in \mathbb{X}$ is the linear operator $G'(\overline{r})[c] = (G_1'(\overline{r})[c], G_2'(\overline{r})[c])$ given by*

$$\begin{cases} G_1'(\overline{r})[c] = \partial_t U - \Delta U + \nabla \cdot (U \nabla \overline{v}) + \nabla \cdot (\overline{u} \nabla V) \\ G_2'(\overline{r})[c] = \partial_t V - \Delta V + s\overline{u}^{s-1} U \overline{v} + \overline{u}^s V - \overline{f} V \, 1_{\Omega_c} - F \overline{v} \, 1_{\Omega_c}. \end{cases} \tag{5.37}$$

Next we prove the existence of Lagrange multipliers for the problem (5.35) associated to a local optimal solution $\overline{r} = (\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$. Accounting for Lemmas 5.21 and 5.22 and Theorem 5.20, now it suffices to prove that $\overline{r}$ is a regular point. From the definition of regular point and (5.37) we conclude that $\overline{r}$ is a regular point if, for each $(g_U, g_V) \in \mathbb{Y}$, there is $c = (U, V, F) \in \hat{X}_q \times \hat{X}_q \times \mathcal{C}(\overline{f})$ such that

$$\begin{cases} \partial_t U - \Delta U + \nabla \cdot (U \nabla \overline{v}) + \nabla \cdot (\overline{u} \nabla V) = g_U \\ \partial_t V - \Delta V + s\overline{u}^{s-1} U \overline{v} + \overline{u}^s V - \overline{f} V \, 1_{\Omega_c} - F \overline{v} \, 1_{\Omega_c} = g_V. \end{cases}$$

where $\mathcal{C}(\overline{f}) = \{\theta(f - \overline{f}) \mid f \in \mathcal{F}, \ \theta \geq 0\}$ is the conical hull of $\overline{f} \in \mathcal{F}$. Since $0 \in \mathcal{C}(\overline{f})$, we can take $F = 0$ and therefore, in order to prove that $\overline{r} = (\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$ is a regular point, it suffices to prove that, given $(g_U, g_V) \in \mathbb{Y}$, there is $(U, V) \in \hat{X}_q \times \hat{X}_q$ such that

$$\begin{cases} \partial_t U - \Delta U + \nabla \cdot (U \nabla \overline{v}) + \nabla \cdot (\overline{u} \nabla V) = g_U \\ \partial_t V - \Delta V + s\overline{u}^{s-1} U \overline{v} + \overline{u}^s V - \overline{f} V \, 1_{\Omega_c} = g_V. \end{cases} \tag{5.38}$$

Problem (5.38) is sometimes called the linearized problem related to (2). Now we prove that $\overline{r}$ is a regular point. For this, we will use the generic result Theorem C.1 given in the Appendix C. Here, we consider the Banach space for weak solutions

$$W_2 = \{v \in L^\infty(L^2) \cap L^2(H^1); \partial_t v \in L^2((H^1)')\}.$$

endowed with the norm

$$\|v\|_{W_p} = \|v\|_{L^\infty(L^p)} + \|v\|_{L^p(W^{1,p})} + \|\partial_t v\|_{L^{p'}((W^{1,p})')}.$$

Theorem 5.23. *Let $\overline{r} = (\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$. Then $\overline{r}$ is a regular point.*

Proof. As it was mentioned above, it suffices to prove that for each $(g_U, g_V) \in \mathbb{Y}$ there is $(U, V) \in X_q \times X_q$ satisfying

$$\begin{cases} \partial_t U - \Delta U = -\nabla \cdot (U \nabla \overline{v}) - \nabla \cdot (\overline{u} \nabla V) + g_U, \\ \partial_t V - \Delta V = -s\overline{u}^{s-1} U \overline{v} - \overline{u}^s V + \overline{f} V \, 1_{\Omega_c} + g_V, \\ \partial_{\boldsymbol{n}} U|_\Gamma = \partial_{\boldsymbol{n}} V|_\Gamma = 0, \ U(0, x) = V(0, x) = 0. \end{cases} \tag{5.39}$$

Using Theorem C.1, case $2a$, with $a_1 = b_1 = 0$, $\vec{c}_1 = \nabla \overline{v} \in L^{5q/(5-q)}$, $d = \overline{u} \in L^\infty(Q)$, $a_2 = \overline{u}^s + \overline{f} 1_{\Omega_c} \in L^q(Q)$, $b_2 = s\overline{u}^{s-1} \, \overline{v} \in L^\infty(Q)$ and $\vec{c}_2 = 0$, we conclude that there is

$$(U, V) \in W_2 \times X_2 \tag{5.40}$$

solution of (5.38). Therefore it suffices to prove that actually $(U, V) \in X_q \times X_q$. In fact, since $V \in X_2$, we have from Lemma 5.12 that $V \in L^{10}(Q)$. Let $Z_1 = -s\overline{u}^{s-1}U\overline{v} - \overline{u}^s V + \overline{f}V\, 1_{\Omega_c} + g_V$ be the right hand side of the V-equation of (5.38), then, accounting for the extra regularity of the coefficients (when compared to Theorem C.1) we conclude that $Z_1 \in L^{10q/(10+q)}(Q)$ and, from Lemma 1.13, we have

$$V \in X_{10q/(10+q)}. \tag{5.41}$$

Note that $10q/(10+q) < q$. We will enhance the regularity of V and prove that $V \in X_q$ by induction. In fact suppose that $Z_1 \in L^{10q/(10n+(5-4n)q)}(Q)$, with $10q/(10n + (5 - 4n)q) < q$. From Lemma 1.13 we have

$$V \in X_{10q/(10n+(5-4n)q)}.$$

Using Lemma 5.12 we have $V \in L^{10q/(10n+(5-4(n+1))q)}(Q)$. Applying this regularity to the less regular term of Z_1, $\overline{f}V\, 1_{\Omega_c}$, we conclude that

$$\overline{f}V\, 1_{\Omega_c} \in L^{10q/(10(n+1)+(5-4(n+1))q)}(Q).$$

Thus, if $10q/(10(n + 1) + (5 - 4(n + 1))q) < q$ then we conclude that

$$Z_1 \in L^{10q/(10(n+1)+(5-4(n+1))q)}(Q).$$

Therefore we have proved that, as long as $10q/(10n + (5 - 4n)q) < q$, if $Z_1 \in L^{10q/(10n+(5-4n)q)}(Q)$ then $Z_1 \in L^{10q/(10(n+1)+(5-4(n+1))q)}(Q)$. Recalling that $q > 5/2$, if we study the function $n \mapsto 10q/(10n + (5 - 4n)q)$, we conclude that there exists $n_0, n_1 \in \mathbb{N}$ such that, $10q/(10n_0 + (5 - 4n_0)q) < q$ and $10q/(10n_1 + (5 - 4n_1)q) \geq q$. Thus we proved that the right hand side of the V-equation of (5.38) belongs to $L^q(Q)$. Finally, from Lemma 1.13, we have

$$V \in X_q. \tag{5.42}$$

It remains to prove that $U \in X_q$. For this, we will analyze the right hand side of the U-equation of (5.38) and use (5.40) and (5.42). The right hand side of the U-equation is

$$g_U - U\Delta\overline{v} - \nabla U \cdot \nabla\overline{v} - \overline{u}\Delta V - \nabla\overline{u} \cdot \nabla V.$$

With the regularities obtained so far for U and V, we have

$$g_U - \overline{u}\Delta V - \nabla\overline{u} \cdot \nabla V \in L^q(Q)$$

and

$$Z_2 := U\Delta\overline{v} + \nabla U \cdot \nabla\overline{v} \in L^{10q/(10+q)}(Q). \tag{5.43}$$

Again, we can prove by induction that, as long as $10q/(10n + (5 - 4n)q) < p$, if $Z_2 \in$

$L^{10q/(10n+(5-4n)q)}(Q)$ then we have $Z_2 \in L^{10q/(10(n+1)+(5-4(n+1))q)}(Q)$. Recalling that $q > 5/2$, if we study the function $n \mapsto 10q/(10n + (5 - 4n)q)$, we conclude that there exists $n_0, n_1 \in \mathbb{N}$ such that, $10q/(10n_0+(5-4n_0)q) < q$ and $10q/(10n_1+(5-4n_1)q) \geq q$ and thus we proved that the right hand side of the U-equation of (5.38) belongs to $L^q(Q)$. From Lemma 1.13, we conclude that $U \in X_q$. ■

Now we are in position of proving Theorem 5.6, that is, we are able to prove that, for any $\overline{r} = (\overline{u}, \overline{v}, \overline{f}) \in S_{ad}$ local optimal solution of problem (5.35), there exists a unique Lagrange multiplier for (5.35) at $\overline{r}$.

5.4.3 Proof of Theorem 5.6

The proof is divided in two steps: the existence of Lagrange multiplier and the uniqueness.

Step 1: Existence

From Lemmas 5.21, 5.22 and Theorem 5.23 we have all the hypotheses of Theorem 5.20 fulfilled. Therefore there exists a Lagrange multiplier $\xi = (\lambda, \eta) \in L^{q'}(Q) \times L^{q'}(Q)$ satisfying, according to (5.34), the inequality

$$\mathcal{L}'_r(\overline{r}, \lambda, \eta)[c] = J'(\overline{r})[c] - \langle \lambda, G'_1(\overline{r})[c] \rangle_{L^{q'}(Q)} - \langle \eta, G'_2(\overline{r})[c] \rangle_{L^{q'}(Q)} \geq 0, \tag{5.44}$$

for all $c = (U, V, F) \in \hat{X}_q \times \hat{X}_q \times \mathcal{C}(\overline{f})$. Then, using (5.36) and (5.37) in (5.44) we conclude that there exists a Lagrange multiplier $\xi = (\lambda, \eta) \in L^{q'}(Q) \times L^{q'}(Q)$ such that, for all $(U, V, F) \in \hat{X}_q \times \hat{X}_q \times \mathcal{C}(\overline{f})$, we have

$$\begin{aligned}
&\gamma_u \int_0^T \int_\Omega sgn(\overline{u} - u_d)|\overline{u} - u_d|^{sq-1} U \, dx \, dt + \gamma_v \int_0^T \int_\Omega (\overline{v} - v_d) \, V \, dx \, dt \\
&+\gamma_f \int_0^T \int_{\Omega_c} sgn(\overline{f})|\overline{f}|^{q-1} F \, dx \, dt - \int_0^T \int_\Omega \Big(\partial_t U - \Delta U + \nabla \cdot (U \nabla \overline{v}) \\
&+\nabla \cdot (\overline{u} \nabla V) \Big) \lambda \, dx \, dt - \int_0^T \int_\Omega \Big(\partial_t V - \Delta V + s \overline{u}^{s-1} U \overline{v} + \overline{u}^s V \\
&-\overline{f} V \, 1_{\Omega_c} - F \overline{v} \, 1_{\Omega_c} \Big) \eta \, dx \, dt \geq 0
\end{aligned} \tag{5.45}$$

Since (5.45) is valid for all $(U, V, F) \in \hat{X}_q \times \hat{X}_q \times \mathcal{C}(\overline{f})$, we can deduce the optimality system (5.6) and the optimality condition (5.10). In fact, since $\hat{X}_q$ is a vectorial space, if $U, V \in \hat{X}_q$ then $-U, -V \in \hat{X}_q$. With this in mind, if we take $(V, F) = (0, 0)$ in (5.45) we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \Big(\partial_t U - \Delta U + \nabla \cdot (U \nabla \overline{v}) \Big) \lambda \, dx \, dt + \int_0^T \int_\Omega s \overline{u}^{s-1} U \overline{v} \eta \, dx \, dt \\
&= \gamma_u \int_0^T \int_\Omega sgn(\overline{u} - u_d)|\overline{u} - u_d|^{sq-1} U \, dx \, dt, \ \forall U \in \hat{X}_q.
\end{aligned} \tag{5.46}$$

On the other hand, if we take $(U, F) = (0, 0)$ in (5.45) we obtain

$$\int_0^T \int_\Omega \Big(\partial_t V - \Delta V + \overline{u}^s V - \overline{f} V 1_{\Omega_c}\Big)\eta \; dx \; dt + \int_0^T \int_\Omega \nabla \cdot (\overline{u}\nabla V)\lambda \; dx \; dt$$
$$= \gamma_v \int_0^T \int_\Omega (\overline{v} - v_d) \; V \; dx \; dt, \; \forall V \in \hat{X}_q. \tag{5.47}$$

Note that if, considering formal computations, we integrate by parts the terms of (5.46) and (5.47), passing the derivatives from (U, V) to (λ, η), we see that $(\lambda, \eta) \in L^{q'}(Q) \times L^{q'}(Q)$ satisfying (5.46) and (5.47) are actually very weak solutions of (5.6). Also, choosing $(U, V) = 0$ in (5.45) leads us to

$$\gamma_f \int_0^T \int_{\Omega_c} sgn(\overline{f})|\overline{f}|^{q-1} F \; dx \; dt + \int_0^T \int_{\Omega_c} \overline{v}\eta F \; dx \; dt \geq 0, \; \forall F \in \mathcal{C}(\overline{f}).$$

Hence, taking $F = \theta(f - \overline{f})$, with $\theta \geq 0$ and for all $f \in \mathcal{F}$ finally gives (5.10).

Step 2: Uniqueness

Now, to prove the uniqueness, we suppose that there are two Lagrange multipliers $(\lambda_1, \eta_1), (\lambda_2, \eta_2) \in L^{q'}(Q) \times L^{q'}(Q)$ satisfying (5.46) and (5.47). Let $(\tilde{\lambda}, \tilde{\eta}) = (\lambda_2, \eta_2) - (\lambda_1, \eta_1)$, we will prove that $\tilde{\lambda} = \tilde{\eta} = 0$. Subtracting the equation satisfied by (λ_1, η_1) from the respective equation that is satisfied by (λ_2, η_2) we conclude that $(\tilde{\lambda}, \tilde{\eta})$ satisfies

$$\int_0^T \int_\Omega \Big(\partial_t U - \Delta U + \nabla \cdot (U \nabla \overline{v})\Big)\tilde{\lambda} + s\overline{u}^{s-1} U \overline{v} \; \tilde{\eta} \; dx \; dt = 0, \; \forall U \in \hat{X}_q, \tag{5.48}$$

$$\int_0^T \int_\Omega \Big(\partial_t V - \Delta V + \overline{u}^s V - \overline{f} V 1_{\Omega_c}\Big)\tilde{\eta} + \nabla \cdot (\overline{u}\nabla V)\tilde{\lambda} \; dx \; dt = 0, \tag{5.49}$$

for all $V \in \hat{X}_q$. Summing (5.48) and (5.49) we obtain

$$\int_0^T \int_\Omega \Big(\partial_t U - \Delta U + \nabla \cdot (U \nabla \overline{v}) + \nabla \cdot (\overline{u}\nabla V)\Big)\tilde{\lambda} \; dx \; dt$$
$$+ \int_0^T \int_\Omega \Big(\partial_t V - \Delta V + \overline{u}^s V + s\overline{u}^{s-1} U \overline{v} - \overline{f} V 1_{\Omega_c}\Big)\tilde{\eta} \; dx \; dt = 0, \tag{5.50}$$

for all $(U, V) \in \hat{X}_q \times \hat{X}_q$. Now let $g_U = sgn(\tilde{\lambda})|\tilde{\lambda}|^{1/(q-1)}$ and $g_V = sgn(\tilde{\eta})|\tilde{\eta}|^{1/(q-1)}$. Since $(\tilde{\lambda}, \tilde{\eta}) \in L^{q'}(Q) \times L^{q'}(Q)$, with $q' = q/(q-1)$, we have $g_U, g_V \in L^q(Q)$. Take $(U, V) \in \hat{X}_q \times \hat{X}_q$ as the unique strong solution of (5.39) for this choice of g_U and g_V, therefore we have from (5.50)

$$\|\tilde{\lambda}\|^{q'}_{L^{q'}(Q)} + \|\tilde{\eta}\|^{q'}_{L^{q'}(Q)} = 0,$$

which implies that $\tilde{\lambda} = \tilde{\eta} = 0$.

5.4.4 Proof of Theorem 5.9

Case $g_\lambda \in L^p(Q)$, with $p \in [10/9, 10/7)$:

Let $\bar{t} = T - t$, then the backward problem (5.6) is equivalent to the forward one

$$\begin{cases} \partial_{\bar{t}}\lambda - \Delta\lambda = \nabla\overline{v} \cdot \nabla\lambda - s\overline{u}^{s-1}\overline{v}\eta + \gamma_u sgn(\overline{u} - u_d)|\overline{u} - u_d|^{sq-1}, \\ \partial_{\bar{t}}\eta - \Delta\eta = -\overline{u}^s\eta + \overline{f}\eta\, 1_{\Omega_c} - \nabla \cdot (\overline{u}\nabla\lambda) + \gamma_v(\overline{v} - v_d), \\ \partial_{\boldsymbol{n}}\lambda|_\Gamma = \partial_{\boldsymbol{n}}\eta|_\Gamma = 0, \;\; \lambda(0,x) = \eta(0,x) = 0. \end{cases} \tag{5.51}$$

Then, applying Theorem C.1, item 1b, with $U = \eta$, $V = \lambda$, $a_1 = \overline{u}^s - \overline{f}\, 1_{\Omega_c}$, $b_1 = 0$, $\vec{c}_1 = 0$, $d = \overline{u}$, $g_U = \gamma_v(\overline{v} - v_d)$, $a_2 = 0$, $b_2 = s\overline{u}^{s-1}\overline{v}$, $\vec{c}_2 = \nabla\overline{v}$ and $g_V = \gamma_u sgn(\overline{u} - u_d)|\overline{u} - u_d|^{sq-1}$ we conclude that there is a very weak solution $(\tilde{\lambda}, \tilde{\eta}) \in L^2(Q) \times L^2(Q)$ of (5.51) and, therefore, of (5.6). since $q > 5/2 > 2$ we have $q' < 2$ and hence $(\tilde{\lambda}, \tilde{\eta}) \in L^{q'}(Q) \times L^{q'}(Q)$. Then, from the uniqueness result of Theorem 5.6 we conclude that $(\tilde{\lambda}, \tilde{\eta})$ is equal to the Lagrange multiplier (λ, η) furnished by theorem 5.6 and $(\lambda, \eta) \in L^2(Q) \times L^2(Q)$.

Case $g_\lambda \in L^p(Q)$, with $p \in [10/7, 2)$:

Using the same argument of the previous case, this time applying Theorem C.1, item 1a, with $g_U = \gamma_v(\overline{v} - v_d) \in L^2(Q) \hookrightarrow L^{10/7}(Q)$ and $g_V = \gamma_u sgn(\overline{u} - u_d)|\overline{u} - u_d|^{sq-1} \in L^p(Q) \hookrightarrow L^{10/7}(Q)$, we conclude that the Lagrange multiplier (λ, η) furnished by Theorem 5.6 is a weak solution of (5.6) with regularity $(\lambda, \eta) \in W_2 \times W_2$. Now we enhance the regularity of (λ, η) by means of a bootstrap procedure analogous to the used in the proof of Theorem 5.23. We first enhance the regularity of λ. Since in the right hand side of the λ-equation we have $-s\overline{u}^{s-1}\overline{v}\eta + \gamma_u sgn(\overline{u} - u_d)|\overline{u} - u_d|^{sq-1} \in L^{10/3}(Q) + L^p(Q)$, with $p \le 2$, we apply the procedure and conclude that $\lambda \in X_p$.

Next we apply the bootstrap argument to the η-equation. Since in the right of the η-equation we have $-\nabla \cdot (\overline{u}\nabla\lambda + \gamma_v(\overline{v} - v_d) \in L^p(Q) + L^2(Q)$, with $p \le 2$, we apply the procedure and conclude that $\eta \in X_p$, finishing the proof.

CONCLUSION

In this thesis we have focused on the chemotaxis-consumption models

$$\begin{cases} \partial_t u - \Delta u = -\nabla \cdot (u \nabla v), \quad \partial_t v - \Delta v = -u^s v, \\ \partial_{\mathbf{n}} u|_\Gamma = \partial_{\mathbf{n}} v|_\Gamma = 0, \quad u(0) = u^0, \quad v(0) = v^0, \end{cases} \tag{5.52}$$

where $\nabla \cdot (u\nabla v)$ is the chemotaxis term and $u^s v$ is the consumption term, with $s \geq 1$, and on optimal control problems subject to the controlled model

$$\begin{cases} \partial_t u - \Delta u = -\nabla \cdot (u \nabla v), \quad \partial_t v - \Delta v = -u^s v + f v 1_{\Omega_c}, \\ \partial_{\mathbf{n}} u|_\Gamma = \partial_{\mathbf{n}} v|_\Gamma = 0, \quad u(0) = u^0, \quad v(0) = v^0, \end{cases} \tag{5.53}$$

where $f : (0, T) \times \Omega \to \mathbb{R}$ is the control, being $T > 0$ a fixed and finite final time, $\Omega_c \subset \Omega$ is the control domain and 1_{Ω_c} is its characteristic function.

Reviewing the available literature about the models (5.52) we found works addressing only the case $s = 1$ that, in addition, were developed using classical in time solution tools and therefore considering smooth coefficients and smooth domains. This is not the most adequate framework to study the numerical approximation of (5.52) or optimal control problems subject to the controlled problem (5.53). In fact, when studying the numerical approximation of PDEs, one usually employs weak formulations of the problem posed in more general domains. Moreover, in the controlled problem (5.53), the control $f = f(t, x)$ is usually a L^q-function, acting as a nonsmooth coefficient in the chemical equation.

Therefore we identified the opportunity and the need of extending the existing theory about problem (5.52) and, in Chapter 2, we studied the existence and regularity of solutions of the models (1) in a weak setting, varying the power $s \geq 1$. We developed the results in terms of the regularity of the Neumann-Poisson problem (1.5) and, when necessary, of Hypothesis (H1). This allowed us to enlarge the class of the considered domains, when compared to the previous literature.

By means of a regularization procedure using adequate truncated models and the cancellation between the attraction and consumption effects (see Subsection 2.3.1), we established the existence of uniform in time weak solutions in $3D$ domains, and uniqueness and regularity in $2D$ (or $1D$) domains. The results of Chapter 2 were published in [11].

Another novelty, when compared to the available literature, was the study of the chemotaxis-consumption models considering all the powers $s \geq 1$. The regularity of the chemical concentration v does not depend on this power s, while the regularity of the density of cells u increases as $s \in [1,2]$ increases. For $s > 2$, only the L^p regularity of u increases with s. This was possible, in part, because of the dissipative term

$$\int_\Omega u^s |\nabla v|^2 \, dx$$

that appears in the formal energy inequality (2.22) and its rigorous version in Subsection 2.3.2. This dissipative term seems to be an interesting feature of the present chemotaxis-consumption models, specially when compared to the close related chemorepulsion-production model.

Indeed, concerning the chemorepulsion-production model, in [24, 21, 22], the authors also rely on the cancellation of the repulsion and production effects to show existence of weak solutions and introduce a potential production term u^p, for $p \in (1,2]$ (the case $p = 1$ was studied previously in [10]). On the other hand, it was not possible to control the effect of the production term u^p when $p > 2$ for large values of u and prove existence of weak solutions for the chemorepulsion model.

Based on the theory developed in Chapter 2, in Chapter 3, we designed a time discrete scheme for the chemotaxis-consumption models (5.52). Using the change of variables $z = \sqrt{v + \alpha^2}$ and a upper truncation of u in the nonlinear chemotaxis and consumption terms we proposed a Backward Euler scheme for the (u, z) problem and two different ways of retrieving an approximation for the function v. We proved the existence of solution to the time discrete scheme, uniform in time *a priori* estimates and convergence of the scheme towards a weak solution (u, v) of the chemotaxis-consumption model.

We remark that, although the existence of solution was proved in Chapter 2 (and published in [11]), the design of a convergent time discrete scheme was not straightforward. Indeed, in order to obtain a time discrete scheme satisfying an energy law, independently of the time step size, it was essential to propose the time discrete scheme in terms variable $z = \sqrt{v + \alpha^2}$ instead of the variable v. In addition, Lemma 3.10 was decisive to prove convergence in the case $s \in [1, 2)$. Indeed, regarding the backward Euler method, in the study of the convergence, we define the approximations u_m^{kr}, u_m^k, z_m^{kr} and z_m^k as in (3.24) and we have to prove that $u_m^{kr} - u_m^k$ and $z_m^{kr} - z_m^k$ go to zero in some norm as $(m, k) \to (\infty, 0)$. This is clear when $s \geq 2$, because, for $z_m^{kr} - z_m^k$, one can conclude (3.30) directly from (3.29) and, for $u_m^{kr} - u_m^k$, one can conclude (3.64) directly from (3.63). On the other hand, when $s \in [1, 2)$, we must rely on the term (3.32), where f'' is not strictly positive and therefore the desired conclusion is not immediate, hence the analysis carried out in Lemma 3.10 was necessary.

In Chapter 4, we studied optimal control problems related to weak solutions of (5.53). To do it, we introduced the concept of weak solutions of the controlled model (5.53) satisfying an energy inequality. To the best of our knowledge, this was the first time that the concept of weak solution with energy inequality was applied to this purpose. Next, we considered an optimal control problem for which we proved existence of global optimal solution and discussed its relation with two other related optimal control problems that might be of interest.

In Chapter 5, we studied an optimal control problem subject to strong solutions of (5.53). We introduced the appropriate concept of strong solution of the controlled problem (5.53), given the control f, and then proved a more generic and sharp regularity criterion, when compared to the available literature, that allows us to get existence and uniqueness of global-in-time strong solutions. Using this regularity criterion, we showed the existence of a global optimal solution, under the hypothesis that the admissible set is nonempty. Next, we established first order optimality conditions for any local optimal solution, proving existence, uniqueness and regularity of the associated Lagrange multipliers.

We also would like to remark that we have done a great effort in order to ensure that, along the whole work, all of our computations are rigorous. In what follows, based in what has been done in this thesis, we present some possible perspectives of future works.

Future works

In Chapter 2 we analyzed problem (5.52) using a regularization procedure and we wrote the results in terms of the regularity of the Poisson-Neumann problem (1.5) and Hypothesis (H1). Hypothesis (H1) is used in the proof of Lemma 2.10, in Appendix B, because it is sufficient to guarantee that we can apply Lemma B.3 to a certain boundary integral. In Appendix A we showed a large class of domains satisfy this hypothesis. Therefore, if one finds a less restrictive condition that makes the application of Lemma B.3 possible (the functions do not need to be pointwisely defined on the whole boundary, but only to be pointwisely defined over the regular components of the boundary, for example) or a less restrictive lemma to deal with the boundary integral term, then it is probable that one can enlarge even more the class of considered domains.

On the other hand, the regularity of the Poisson-Neumann problem (1.5) is considered because of the the self-diffusion operator and the boundary conditions that are present in problem (5.52): the Laplacian operator and the homogeneous Neumann boundary conditions. The effort of putting the results of Chapter 2 in terms of the regularity of an adequate boundary-value problem related to the self-diffusion operator is probably a useful structure to the extension of the analysis of Chapter 2 to chemotaxis-consumption models with more general self-diffusion operators and their corresponding boundary conditions. As an example we refer the reader to [23], where a chemorepulsion-production model with variable diffusion coefficients is studied.

In the previous literature we also find a considerable number of works about chemotaxis-consumption models coupled with models for fluids, namely, the (Navier-)Stokes equations. Hence, another interesting and open question is whether all the aforementioned technique used in Chapter 2 to study problem (5.52) can be extended to the chemotaxis-fluids models. It is probable that the present approach to (5.52) could be combined with the regularization for the fluid equations used in [62], for example.

In Chapter 3 we proposed a conservative, energy stable, positivity preserving and convergent time discrete scheme to approximate (5.52). Now, an interesting original work on the numerical approximation of (5.52) should propose a fully discrete scheme preserving properties of the solutions in the discrete level, such as conservation of the total population of cells, positivity and energy estimates, and which converges to a weak solution of (5.52).

The convergence is probably one of the most important features of a discrete scheme, but in the case of the chemotaxis models, there is evidence that the preservation of the positivity in the discrete level is also relevant. Indeed, in [25], although the considered schemes are convergent, the authors show through numerical simulations that the scheme with "approximated" positivity of the discrete solutions prevents spurious oscillations. As far as we know, four techniques are used with the objective of preserving positivity of the discrete solutions of chemotaxis models:

1. the estimate of a singular functional and Finite Element Method (FEM) (implying approximate positivity);
2. "upwind" schemes in the context of Finite Volume Method (FVM), Discontinuous Galerkin method (DG) or FEM;
3. mass lumping combined with FEM in 2D domains;
4. and FEM schemes with stabilization terms used to preserve pointwise bounds.

Concerning the chemotaxis model (5.52), with $s = 1$, the singular functional approach is addressed in [27], where, among other schemes, the authors defined a fully discrete scheme possessing a kind of energy inequality and used a singular functional to obtain approximate positivity. The drawback of this scheme is that the energy inequality has a residual term whose estimation is not clear in 2D and 3D domains, yielding a decreasing energy only in the one dimensional case. Consequently, in the 2D or 3D cases, convergence is not clear.

The upwind approaches are possibly the better suited to preserve (exact) positivity of the discrete solutions and in our opinion, their application to the chemotaxis-consumption models (5.52) may lead to original works. For an example of these approaches in biology related models, we refer the reader to [1]. On the other hand, accounting for the complex procedures involved in obtaining energy estimates that can be observed in Chapters 2 and 3, we do not expect that these upwind approaches will lead to fully discrete energy stable schemes. Since energy stability is a key ingredient to prove convergence, the design of a convergent upwind fully discrete scheme may also remain as an open question in this case.

In [32] the authors design a fully discrete scheme for a Keller-Segel model applying the mass lumping with FEM approach to obtain conditional positivity of the discrete solutions in 2D domains. The drawback of this approach is that the positivity is attained provided that the space mesh size is small enough. This condition may lead to the need of very small mesh sizes, increasing the computational cost. In a future work, we intend to combine the regularization procedure used in Chapters 2 and 3 with this mass lumping approach to design an energy stable and convergent fully discrete scheme to approximate (5.52), preserving positivity and total population of cells.

For a FEM scheme with stabilization terms we cite [2]. The authors propose two schemes to approximate the Keller-Segel model which preserve the lower bounds of the solutions, yielding positivity in the discrete level. In addition, they prove a discrete energy law that is satisfied by the discrete solutions.

In Chapter 4 we studied an optimal control problem subject to problem (5.53) in the weak setting. We defined the adequate concept of weak solution of (5.53) satisfying the energy inequality (4.3) and used it to define control problems with bounded controls for which we proved existence of optimal solution. Nevertheless, the existence of optimal solution for the optimal control problem subject to weak solutions (5.53) which satisfy (4.3), but without bounded controls, that is, the minimization problem (4.5), remains as an open question. In addition, in Remark 4.7 we have already pointed out some other related questions which have the potential of being the focus of future research.

To the best of our knowledge, there is not any study of a optimal control problem related to chemotaxis-consumption-fluid models in a weak setting. Then, a future work could deal with this problem, possibly extending the ideas of Chapter 4 to chemotaxis-consumption-fluid models.

As it was mentioned in the introduction, it was not clear how to deduce some type of optimality system associated to local optimal solutions using only the weak regularity that is available in 3D domains. On the other hand, analogously to the studies about optimal control problems related to chemotaxis models in 2D (and 1D) domains [51, 29, 31, 5, 65, 54], it is probable that, in 2D (and 1D) domains, using the ideas of these cited previous works, one has more regularity, which will possibly allow one to prove the existence of global optimal solution and to derive an optimality system, establishing existence and regularity of Lagrange multipliers for any local optimum.

In Chapter 5, in order to study an optimal control problem related to strong solutions of (5.53) in 3D domains, we proved a regularity criterion that allows us to get existence and uniqueness of global-in-time strong solutions. The uniqueness of solution (u, v), given the control $f = f(t, x)$, allows us to define the state $(u, v) = (u(f), v(f))$ in terms of the control f. Then the functional (5.2) can be written in terms of f as

$$\begin{aligned} J(f) &:= \frac{\gamma_u}{sq}\int_0^T \|u(f)(t) - u_d(t)\|_{L^{sq}}^{sq}\, dt \\ &+\frac{\gamma_v}{2}\int_0^T \|v(f)(t) - v_d(t)\|_{L^2}^2\, dt + \frac{\gamma_f}{q}\int_0^T \|f(t)\|_{L^q(\Omega_c)}^q\, dt. \end{aligned} \tag{5.54}$$

An interesting work should investigate under which conditions we can calculate the derivative of J with respect to f, exhibit its expression and use it to propose descent methods. For an example of this kind of work we refer the reader to [9].

Once one computes the derivative of J with respect to f, we expect that it is possible to use it to obtain the optimality conditions of Chapter 5, similarly to the procedure adopted in [9, 36] and, in the context of chemotaxis-Navier-Stokes-consumption models, in [44]. In our opinion, this should be the subject of further study and, if this is indeed possible, another question that arises is whether, using this procedure, one can establish the same optimality conditions under the same hypothesis of Chapter 5.

Finally, we would like to remark that, even when compared to other chemotaxis models, the chemotaxis-consumption models (with attraction and consumption) studied in this thesis showed to be very challenging due to the complex procedures needed to obtain of energy estimates that can be observed in Chapter 2, probably mainly because of the test functions and the treatment of a boundary integral that are involved in these procedures. This is why we expect that the treatment given to the present chemotaxis-consumption models can be useful to the approach of other chemotaxis models.

Appendix A

Hypothesis (H1)

In the proof of Lemma 2.10 we will need Hypothesis (H1) (see page 25). Therefore, in order to show that this hypothesis is not too restrictive, we show that Hypothesis (H1) holds if the Poisson-Neumann problem (1.5) has the $W^{3,p}$-regularity (see definition 1.6 in page 18), for $p > N$. According to [20], this is true if Γ is at least $C^{2,1}$, for example. Up to our knowledge, the validity of Hypothesis (H1) in other domains of practical interest, such as polyhedra and polygons, is an open question.

Lemma A.1. *Suppose that the Poisson-Neumann problem* (1.5) *has the $W^{3,p}$-regularity, for some $p > N$, and let $z \in H^2(\Omega)$ such that $\partial_\eta z\Big|_\Gamma = 0$. Then there is a sequence $\{\rho_n\} \subset C^2(\overline{\Omega})$, with $\partial_\eta \rho_n\Big|_\Gamma = 0$, which converges to z in $H^2(\Omega)$.*

Proof. For any fixed $z \in H^2(\Omega)$ such that $\partial_\eta z\Big|_\Gamma = 0$, define $f = -\Delta z + z$. Note that $f \in L^2(\Omega)$ and $z \in H^2(\Omega)$ is the solution of

$$\begin{cases} -\Delta z + z &= f \\ \partial_\eta z\Big|_\Gamma &= 0. \end{cases} \tag{A.1}$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of mollifiers of f, that is, $f_n \in C_c^\infty(\Omega)$ and $f_n \to f$ in $L^2(\Omega)$ as $n \to \infty$. Then, for each fixed $n \in \mathbb{N}$, consider the following regularized problem: *Find $\rho_n : \Omega \to \mathbb{R}$ such that*

$$\begin{cases} -\Delta \rho_n + \rho_n &= f_n \\ \partial_\eta \rho_n\Big|_\Gamma &= 0. \end{cases} \tag{A.2}$$

Considering the hypothesis that the Poisson-Neumann problem (1.5) has the $W^{3,p}$-regularity, for some $p > N$, we can conclude that, for each $n \in \mathbb{N}$, there is one, and only one, function $\rho_n \in W^{3,p}(\Omega) \subset C^2(\overline{\Omega})$ such that $\partial_\eta \rho_n\Big|_\Gamma = 0$ which solves problem (A.2).

Since the functions z and ρ_n solve the problems (A.1) and (A.2), respectively, the functions $z - \rho_n$ solve the problem

$$\begin{cases} -\Delta(z-\rho_n) + (z-\rho_n) &= (f - f_n) \\ \partial_\eta (z-\rho_n)\Big|_\Gamma &= 0. \end{cases} \tag{A.3}$$

Then, using $(z - \rho_n)$ and $-\Delta(z - \rho_n)$ as test functions in (A.3) we can conclude that

$$\|z - \rho_n\|_{L^2(\Omega)}, \|\nabla(z - \rho_n)\|_{L^2(\Omega)}, \|\Delta(z - \rho_n)\|_{L^2(\Omega)} \leq C\|f - f_n\|_{L^2(\Omega)}.$$

Since $f_n \to f$ in $L^2(\Omega)$ as $n \to \infty$, the latter implies that $\rho_n \to z$ in $H^2(\Omega)$ as $n \to \infty$, finishing the proof. ∎

Appendix B

Proof of Lemma 2.10

Before proving Lemma 2.10, we must present some technical results.

Lemma B.1. *Let $z : \Omega \to \mathbb{R}$ be a $C^2(\overline{\Omega})$ function such that $\partial_\eta z\Big|_\Gamma = 0$. Then*

$$\int_\Omega |\Delta z|^2 \, dx = \int_\Omega |D^2 z|^2 \, dx - \frac{1}{2}\int_\Gamma \nabla(|\nabla z|^2) \cdot \eta \, d\Gamma.$$

Proof. It suffices to prove the inequality for sufficiently regular functions and then pass to the limit. Integrating by parts we have

$$\begin{aligned}\int_\Omega |\Delta z|^2 \, dx = -\int_\Omega \nabla z \cdot \nabla \Delta z \, dx &= \int_\Omega |D^2 z|^2 \, dx - \int_\Gamma [(\nabla z)^T D^2 z] \cdot \eta \, d\Gamma \\ &= \int_\Omega |D^2 z|^2 \, dx - \frac{1}{2}\int_\Gamma \nabla |\nabla z|^2 \cdot \eta \, d\Gamma.\end{aligned}$$

■

Lemma B.2. *There is a constant $C > 0$ such that*

1. $$\int_\Omega |D^2 z|^2 \, dx + \int_\Omega \frac{|\nabla z|^2}{z} \Delta z \, dx = 4\int_\Omega z |D^2 \sqrt{z}|^2 \, dx + \frac{3}{4}\int_\Omega \frac{|\nabla z|^4}{z^2} \, dx,$$

2. $$\int_\Omega |D^2 z|^2 \, dx \le C\Big(\int_\Omega z|D^2\sqrt{z}|^2 \, dx + \int_\Omega \frac{|\nabla z|^4}{z^2} \, dx\Big),$$

for all $z \in H^2(\Omega)$ such that $\partial_\eta z\Big|_\Gamma = 0$ and $z \ge \alpha$, for some $\alpha > 0$.

Proof. See lemma 3.3 of [60] for item 1. The inequality in item 2 is a direct consequence of the identity in item 1. ■

The next two results will allow us to estimate the boundary integral.

Lemma B.3. *Let $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$, each Γ_i defined through a parametrization of one variable of $\mathbb{R}^3$ by the other two. Then there is $C > 0$ such that, for all i, one has*

$$\Big| \int_{\Gamma_i} \nabla |\nabla z|^2 \cdot \eta \, d\Gamma_i \Big| \le C \int_{\Gamma_i} |\nabla z|^2 d\Gamma_i,$$

for all $z \in C^2(\overline{\Omega})$ *such that* $\partial_\eta z\Big|_\Gamma = 0$.

Proof. See [26]. ■

Lemma B.4. *Let* Ω *be a Lipschitz domain. Then, for each* $\delta > 0$*, there is a* $C(\delta) > 0$ *such that*

$$\|\nabla z\|_{L^2(\Gamma)} \leq C(\delta)\|z\|_{L^2(\Omega)} + \delta\|D^2 z\|_{L^2(\Omega)}, \ \forall \ z \in H^2(\Omega).$$

Proof. This is Lemma 2.4 in [49]. ■

Now we are in position of proving Lemma 2.10.

Proof of Lemma 2.10. We recall that, by hypothesis, $z(x) \geq \alpha > 0, \ a.e. \ x \in \Omega$. Now we divide the proof in two main steps.

STEP 1: First of all, we are going to obtain the inequality

$$\begin{aligned} 2\int_\Omega |\Delta z|^2 \ dx + 2\int_\Omega \frac{|\nabla z|^2}{z}\Delta z \ dx \ &\geq 8\int_\Omega z|D^2\sqrt{z}|^2 \ dx + \frac{3}{2}\int_\Omega \frac{|\nabla z|^4}{z^2} \\ &-C\delta\|D^2 z\|^2_{L^2(\Omega)} - C(\delta)\|\nabla z\|^2_{L^2(\Omega)}. \end{aligned} \tag{B.1}$$

In fact, from Hypothesis (H1), we have the existence of a sequence $\{z_j\}$ such that $z_j \in C^2(\overline{\Omega})$, the trace of the normal derivative of z_j is zero and ; $\|z - z_j\|_{H^2(\Omega)} \to 0$. We can choose the sequence $\{z_j\}$ such that

$$\frac{\alpha}{2} \leq z_j(x) \ a.e. \ x \in \Omega, \forall j \geq j_0.$$

Now, applying Lemmas B.1 and B.2-1, we have

$$\begin{aligned} &2\int_\Omega |\Delta z_j|^2 \ dx + 2\int_\Omega \frac{|\nabla z_j|^2}{z_j}\Delta z_j \ dx \\ &= 2\int_\Omega |D^2 z_j|^2 \ dx + 2\int_\Omega \frac{|\nabla z_j|^2}{z_j}\Delta z_j \ dx - \int_\Gamma \nabla|\nabla z_j|^2 \cdot \eta \ d\Gamma \\ &= 8\int_\Omega z_j|D^2\sqrt{z_j}|^2 \ dx + \frac{3}{2}\int_\Omega \frac{|\nabla z_j|^4}{z_j^2} - \int_\Gamma \nabla|\nabla z_j|^2 \cdot \eta \ d\Gamma. \end{aligned}$$

Now we apply Lemma B.3 and Lemma B.4 (in this order) to obtain

$$\begin{aligned} &2\int_\Omega |\Delta z_j|^2 \ dx + 2\int_\Omega \frac{|\nabla z_j|^2}{z_j}\Delta z_j \ dx \\ &\geq 8\int_\Omega z_j|D^2\sqrt{z_j}|^2 \ dx + \frac{3}{2}\int_\Omega \frac{|\nabla z_j|^4}{z_j^2} - C\|\nabla z_j\|^2_{L^2(\Gamma)} \\ &\geq 8\int_\Omega z_j|D^2\sqrt{z_j}|^2 \ dx + \frac{3}{2}\int_\Omega \frac{|\nabla z_j|^4}{z_j^2} \\ &- C\delta\|D^2 z_j\|^2_{L^2(\Omega)} - C(\delta)\|\nabla z_j\|^2_{L^2(\Omega)}, \end{aligned}$$

where $\delta > 0$ is supposed to be a sufficiently small number to be chosen later. Now we use the fact that $z_j \to z$ in $H^2(\Omega)$ as $j \to \infty$ and thus we obtain inequality (B.1), finishing the first step of the proof.

STEP 2: Next, we apply Lemma B.2-2 to the right hand side of (B.1) and choose $\delta > 0$ small enough, then there exist two constants $C_1, C_2 > 0$ such that

$$2\int_\Omega |\Delta z|^2 \, dx + 2\int_\Omega \frac{|\nabla z|^2}{z} \Delta z \, dx \geq C_1 \Big(\int_\Omega |D^2 z|^2 \, dx + \int_\Omega \frac{|\nabla z|^4}{z^2} \, dx \Big) - C_2 \|\nabla z\|^2_{L^2(\Omega)}$$

and the proof of Lemma 2.10 is finished. ∎

Appendix C

Existence of solution for a generic linear system

We introduce the following general prototype of a linearized problem related to chemotaxis models,

$$\begin{cases} \partial_t U - \Delta U + a_1 U + b_1 V + \nabla \cdot (U \vec{c}_1) + \nabla \cdot (d \nabla V) = g_U, \\ \partial_t V - \Delta V + a_2 V + b_2 U + \vec{c}_2 \cdot \nabla V = g_V, \\ \partial_{\boldsymbol{n}} U|_\Gamma = \partial_{\boldsymbol{n}} V|_\Gamma = 0, \ U(0,x) = V(0,x) = 0, \end{cases} \tag{C.1}$$

where the coefficients $a_i, b_i, \vec{c}_i$ and d are rather singular functions. Next we state and prove the theorem of existence of problem (C.1).

Theorem C.1. *For $i = 1, 2$, let $a_i \in L^{5/2}(Q)$ and $\vec{c}_i \in L^5(Q)$ with $\nabla \cdot \vec{c}_i \in L^1(Q)$ and $\vec{c}_1 \cdot \vec{n}|_\Gamma = 0$.*

1. *If $b_i \in L^{5/2}(Q)$ and $d \in L^\infty(Q)$ we have:*
   - *(a) if $g_U, g_V \in L^{10/7}(Q)$ then there is a weak solution $(U, V) \in W_2 \times W_2$ to (C.1);*
   - *(b) if $g_U, g_V \in L^{10/9}(Q)$ and $\nabla d \in L^5(Q)$ then there is a very weak solution $(U, V) \in L^2(Q) \times L^2(Q)$ to (C.1);*
2. *if $b_1 \in L^{5/3}(Q)$ and $b_2, d \in L^5(Q)$ we have:*
   - *(a) if $g_U \in L^{10/7}(Q)$ and $g_V \in L^2(Q)$ then there is a weak-strong solution $(U, V) \in W_2 \times X_2$ to (C.1);*
   - *(b) if $g_U \in L^{10/9}(Q)$ and $g_V \in L^{10/7}(Q)$ then there is a very weak-weak solution $(U, V) \in L^2(Q) \times W_2$ to (C.1).*

Proof. We will prove this result by means of the Galerkin method. Let $\{\varphi_m\}$ be a basis of $H^1(\Omega)$ of functions satisfying

$$-\Delta \varphi_m + \varphi_m = \lambda_m \varphi_m, \ \partial_{\boldsymbol{n}} \varphi_m|_\Gamma = 0,$$

for each $m \in \mathbb{N}$, and define X^n as the n-dimensional space generated by the first n elements of $\{\varphi_m\}$. Also, for $i = 1, 2$, let $a_i^n, b_i^n, d^n \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3)$ and $\vec{c}_i^n \in$

$(C_c^\infty(\mathbb{R}\times\mathbb{R}^3))^3$ be mollifier regularizations of a_i^n, b_i^n, d^n and $\vec{c}_i^n$ such that

$$a_i^n \to a_i \text{ strongly in } L^{5/2}(Q),\ \text{ for } i=1,2,$$

$$\vec{c}_i^n \to \vec{c}_i \text{ strongly in } (L^5(Q))^3,\ \text{ for } i=1,2,$$

where in the case of item 1 we have

$$b_i^n \to b_i \text{ strongly in } L^{5/2}(Q),\ \text{ for } i=1,2,$$

$$d^n \text{ is bounded in } L^\infty(Q) \text{ and converges to } d \text{ strongly in } L^p(Q),\ \text{ for any } p\in[1,\infty),$$

with

$$d^n \to d \text{ strongly in } L^5(W^{1,5})$$

in the case of item 1b, and in the case of item 2

$$b_1^n \to b_1 \text{ strongly in } L^{5/3}(Q),$$

$$b_2^n \to b_2 \text{ strongly in } L^5(Q),$$

$$d^n \to d \text{ strongly in } L^5(Q).$$

We look for Galerkin solutions (U_n, V_n) of the form

$$U_n(t,x)=\sum_{j=1}^n g_j^n(t)\varphi_j(x) \qquad \text{and} \qquad V_n(t,x)=\sum_{j=1}^n h_j^n(t)\varphi_j(x)$$

such that

$$\begin{aligned}(\partial_t U_n,\varphi)+(\nabla U_n,\nabla\varphi)+(a_1^n U_n,\varphi)+(b_1^n V_n,\varphi)& \qquad (C.2)\\ -(U_n\vec{c}_1^n,\nabla\varphi)-(d^n\nabla V_n,\nabla\varphi)=(g_U,\varphi)\,,&\\ (\partial_t V_n,\varphi)-(\Delta V_n,\varphi)+(a_2^n V_n,\varphi)+(b_2^n U_n,\varphi)+(\vec{c}_2^n\cdot\nabla V,\varphi)=(g_V,\varphi)\,,& \qquad (C.3)\\ U_n(0,x)=V_n(0,x)=0,& \qquad (C.4)\end{aligned}$$

for all $\varphi\in X^n$. From the results on linear ordinary differential systems with smooth coefficients we have the existence and uniqueness of global classical solution $(U_n,V_n)\in C^1([0,T];X^n\times X^n)$ satisfying (C.2)-(C.4), for each $n\in\mathbb{N}$. Next we obtain *a priori* estimates for (U_n,V_n) that we will use to pass to the limit as $n\to\infty$. Now deal with each case of the theorem.

Case 1*a*: We begin by taking $\varphi=U_n\in X^n$ in (C.2) and obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|U_n\|_{L^2}^2+\|\nabla U_n\|_{L^2}^2\le\|a_1^n\|_{L^{5/2}}\|U_n\|_{L^{10/3}}^2\\ &+\|b_1^n\|_{L^{5/2}}\|V_n\|_{L^{10/3}}\|U_n\|_{L^{10/3}}+\|U_n\|_{L^{10/3}}\|\vec{c}_1^n\|_{L^5}\|\nabla U_n\|_{L^2}\\ &+\|d^n\|_{L^\infty}\|\nabla V_n\|_{L^2}\|\nabla U_n\|_{L^2}+\|g_U\|_{L^{10/7}}\|U_n\|_{L^{10/3}}\end{aligned}$$

Next, among other things, we must use Young's inequality and we highlight the following estimation:

$$\begin{aligned}&\|g_U\|_{L^{10/7}}\|U_n\|_{L^{10/3}} \leq C_1\|g_U\|_{L^{10/7}}\|U_n\|_{L^2}^{2/5}\|U_n\|_{H^1}^{3/5}\\ &\leq C_1\|g_U\|_{L^{10/7}}^{2/7}\|U_n\|_{L^2}^{2/5}\|g_U\|_{L^{10/7}}^{5/7}\|U_n\|_{H^1}^{3/5}\\ &\leq C_2\|g_U\|_{L^{10/7}}^{10/7}\|U_n\|_{L^2}^{2} + C_3\|g_U\|_{L^{10/7}}^{25/28}\|U_n\|_{H^1}^{15/20}\\ &\leq C_2\|g_U\|_{L^{10/7}}^{10/7}\|U_n\|_{L^2}^{2} + C_4\|g_U\|_{L^{10/7}}^{10/7} + C_5\|U_n\|_{H^1}^{2}.\end{aligned}$$

Then, applying the properties of the mollified sequences, the interpolation inequality (1.2) and Young's inequality with the appropriate weights we conclude that for any $\delta > 0$ there are $C, \tilde{\beta} > 0$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|U_n\|_{L^2}^2 + \tilde{\beta}\|\nabla U_n\|_{L^2}^2 \leq C(\|a_1\|_{L^{5/2}}^{5/2} + \|b_1\|_{L^{5/2}}^{5/2}\\ &+\|\vec{c}_1\|_{L^5}^5 + \|g_U\|_{L^{10/7}}^{10/7})\|U_n\|_{L^2}^2 + C\|V_n\|_{L^2}^2\\ &+C\|g_U\|_{L^{10/7}}^{10/7} + C(\|d\|_{L^\infty}^2 + 1)\|\nabla V_n\|_{L^2}^2.\end{aligned} \tag{C.5}$$

Now we take $\varphi = V_n \in X^n$ in (C.3), which gives us

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|V_n\|_{L^2}^2 + \|\nabla V_n\|_{L^2}^2 \leq \|a_2^n\|_{L^{5/2}}\|V_n\|_{L^{10/3}}^2\\ &+ \|b_2^n\|_{L^{5/2}}\|U_n\|_{L^{10/3}}\|V_n\|_{L^{10/3}}\\ &+ \|\vec{c}_2^{\,n}\|_{L^5}\|\nabla V_n\|_{L^2}\|V_n\|_{L^{10/3}} + \|g_V\|_{L^{10/7}}\|V_n\|_{L^{10/3}}.\end{aligned}$$

Applying the properties of the mollified sequences, the interpolation inequalities (1.2) and Young's inequality with the appropriate weights we conclude that for any $\delta > 0$ there are $C, \tilde{\beta} > 0$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|V_n\|_{L^2}^2 + \tilde{\beta}\|\nabla V_n\|_{L^2}^2 \leq C\|U_n\|_{L^2}^2\\ &+C(\|a_2\|_{L^{5/2}}^{5/2} + \|b_2\|_{L^{5/2}}^{5/2} + \|\vec{c}_2\|_{L^5}^5 + \|g_V\|_{L^{10/7}}^{10/7})\|V_n\|_{L^2}^2\\ &+C\|g_V\|_{L^{10/7}}^{10/7} + \delta\|\nabla U_n\|_{L^2}^2.\end{aligned} \tag{C.6}$$

Let $C_0 = 2C(\|d\|_{L^\infty}^2 + 1)/\tilde{\beta}$, summing (C.5) and C_0 times (C.6) and choosing $\delta > 0$ small enough then the terms $\delta\|\nabla U_n\|_{L^2}^2$ and $C(\|d\|_{L^\infty}^2 + 1)\|\nabla V_n\|_{L^2}^2$ can be absorbed and we conclude that there is $\beta > 0$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|U_n\|_{L^2}^2 + C_0\|V_n\|_{L^2}^2) + \beta(\|\nabla U_n\|_{L^2}^2 + \|\nabla V_n\|_{L^2}^2)\\ &\leq C(\|a_1\|_{L^{5/2}}^{5/2} + \|b_1\|_{L^{5/2}}^{5/2} + \|\vec{c}_1\|_{L^5}^5\\ &+\|g_U\|_{L^{10/7}}^{10/7} + 1)\|U_n\|_{L^2}^2 + C(\|a_2\|_{L^{5/2}}^{5/2} + \|b_2\|_{L^{5/2}}^{5/2}\\ &+\|\vec{c}_2\|_{L^5}^5 + \|g_V\|_{L^{10/7}}^{10/7} + 1)\|V_n\|_{L^2}^2\\ &+C\|g_U\|_{L^{10/7}}^{10/7} + C\|g_V\|_{L^{10/7}}^{10/7}.\end{aligned} \tag{C.7}$$

Since $\|a_1\|^{5/2}_{L^{5/2}}$, $\|b_1\|^{5/2}_{L^{5/2}}$, $\|\vec{c}_1\|^5_{L^5}$, $\|g_U\|^{10/7}_{L^{10/7}}$, $\|a_2\|^{5/2}_{L^{5/2}}$, $\|b_2\|^5_{L^5}$, $\|\vec{c}_2\|^5_{L^5}$, $\|g_V\|^{10/7}_{L^{10/7}} \in L^1(0,T)$, we are able to apply Gronwall's Lemma to (C.7) and conclude that

$$(U_n, V_n) \text{ is bounded in } L^\infty(L^2) \times L^\infty(L^2) \cap L^2(H^1) \times L^2(H^1).$$

Using this bound in the equations (C.2) and (C.3) we also obtain n-independent bounds for $\partial_t U_n$ and $\partial_t V_n$, which leads us to

$$(U_n, V_n) \text{ is bounded in } W_2 \times W_2. \tag{C.8}$$

Next, we skip the standard procedures of the application of the Galerkin's method to linear equations and state that with (C.8) we are able to pass to the limit as $n \to \infty$ in (C.2) and (C.3), concluding that there is $(U,V) \in W_2 \times W_2$ solution of problem (C.1).

Case 1*b*: The n-independent *a priori* estimates for this case are similar to those of the case 1a, but now, instead of choosing $\varphi = U_n \in X^n$ in (C.2) and $\varphi = V_n \in X^n$ in (C.3), we take $\varphi = (-\Delta + I)^{-1} U_n \in X^n$ in (C.2) and $\varphi = (-\Delta + I)^{-1} V_n \in X^n$ in (C.3), where $\Phi = (-\Delta + I)^{-1} U_n$ is well defined as the function

$$-\Delta\Phi + \Phi = U_n, \ \partial_{\boldsymbol{n}} \Phi|_\Gamma = 0.$$

We also use the fact that there is a constant $C > 0$ such that

$$\|\phi\|_{H^2} \le C\|(-\Delta + I)\phi\|_{L^2}, \ \forall\, \phi \in H^2(\Omega).$$

Another relevant change is that we integrate by parts to reduce the order of the space derivatives of U_n and V_n in (C.2) and (C.3) and we highlight the term $(d^n \nabla V_n, \nabla\varphi)$ of the (C.2) that, in this very weak solution setting, is written as $-(V_n \nabla d^n, \nabla\varphi) - (V_n d^n, \Delta\varphi)$.

Case 2*a*: We take $\varphi = V_n - \Delta V_n \in X^n$ in (C.3), which gives us

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V_n\|^2_{H^1} + \|\nabla V_n\|^2_{L^2} + \|\Delta V_n\|^2_{L^2} \le \|a_2^n\|_{L^{5/2}}\|V_n\|^2_{L^{10/3}} \\
&+ \|a_2^n\|_{L^{5/2}}\|V_n\|_{L^{10}}\|\Delta V_n\|_{L^2} + \|b_2^n\|_{L^5}\|U_n\|_{L^{10/3}}\|V_n\|_{L^2} \\
&+ \|b_2^n\|_{L^5}\|U_n\|_{L^{10/3}}\|\Delta V_n\|_{L^2} + C\|\vec{c}_2^{\,n}\|_{L^5}\|\nabla V_n\|_{L^{10/3}}\|V_n\|_{L^2} \\
&+ C\|\vec{c}_2^{\,n}\|_{L^5}\|\nabla V_n\|_{L^{10/3}}\|\Delta V_n\|_{L^2} + \|g_V\|_{L^2}(\|V_n\|_{L^2} + \|\Delta V_n\|_{L^2}).
\end{aligned}$$

Applying the properties of the mollified sequences, the interpolation inequalities (1.2) and (1.3) and Young's inequality with the appropriate weights we conclude that for any $\delta > 0$ there are $C, \tilde{\beta} > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V_n\|_{H^1}^2 + \tilde{\beta}(\|\nabla V_n\|_{L^2}^2 + \|\Delta V_n\|_{L^2}^2) \le C(\|a_2\|_{L^{5/2}}^{5/2} \\
&+\|b_2\|_{L^5}^5 + \|\vec{c}_2\|_{L^5}^5 + \|g_V\|_{L^2}^2 + 1)\|V_n\|_{H^1}^2 + C\|U_n\|_{L^2}^2 \\
&+C\|g_V\|_{L^2}^2 + \delta\|\nabla U_n\|_{L^2}^2.
\end{aligned} \tag{C.9}$$

Now we take $\varphi = U_n \in X^n$ in (C.2) and obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_n\|_{L^2}^2 + \|\nabla U_n\|_{L^2}^2 \le \|a_1^n\|_{L^{5/2}}\|U_n\|_{L^{10/3}}^2 \\
&+ \|b_1^n\|_{L^{5/3}}\|V_n\|_{L^{10}}\|U_n\|_{L^{10/3}} + \|U_n\|_{L^{10/3}}\|\vec{c}_1^n\|_{L^5}\|\nabla U_n\|_{L^2} \\
&+ \|d^n\|_{L^5}\|\nabla V_n\|_{L^{10/3}}\|\nabla U_n\|_{L^2} + \|g_U\|_{L^{10/7}}\|U_n\|_{L^{10/3}}.
\end{aligned}$$

Applying the properties of the mollified sequences, the interpolation inequality (1.2) and Young's inequality with the appropriate weights we conclude again that for any $\delta > 0$ there are $C, \tilde{\beta} > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_n\|_{L^2}^2 + \tilde{\beta}\|\nabla U_n\|_{L^2}^2 \le C(\|a_1\|_{L^{5/2}}^{5/2} + \|b_1\|_{L^{5/3}}^{5/3} \\
&+\|\vec{c}_1\|_{L^5}^5 + \|d\|_{L^5}^5 + \|g_U\|_{L^{10/7}}^{10/7} + 1)\|U_n\|_{L^2}^2 \\
&+C\|V_n\|_{H^1}^2 + C\|g_U\|_{L^{10/7}}^{10/7} + \delta\|\Delta V_n\|_{L^2}^2.
\end{aligned} \tag{C.10}$$

Summing (C.9) and (C.10) and choosing $\delta > 0$ small enough so that the terms $\delta\|\nabla U_n\|_{L^2}^2 + \delta\|\Delta V_n\|_{L^2}^2$ on the right hand side can be absorbed by the corresponding terms on the left hand side, we conclude that there is $\beta > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|U_n\|_{L^2}^2 + \|V_n\|_{H^1}^2) + \beta(\|\nabla U_n\|_{L^2}^2 + \|\nabla V_n\|_{L^2}^2) \\
&+\beta\|\Delta V_n\|_{L^2}^2 \le C(\|a_1\|_{L^{5/2}}^{5/2} + \|b_1\|_{L^{5/3}}^{5/3} + \|\vec{c}_1\|_{L^5}^5 \\
&+\|d\|_{L^5}^5 + \|g_U\|_{L^{10/7}}^{10/7} + 1)\|U_n\|_{L^2}^2 \\
&+C(\|a_2\|_{L^{5/2}}^{5/2} + \|b_2\|_{L^5}^5 + \|\vec{c}_2\|_{L^5}^5 + \|g_V\|_{L^2}^2 + 1)\|V_n\|_{H^1}^2 \\
&+C\|g_U\|_{L^{10/7}}^{10/7} + C\|g_V\|_{L^2}^2.
\end{aligned} \tag{C.11}$$

Since $\|a_1\|_{L^{5/2}}^{5/2}$, $\|b_1\|_{L^{5/3}}^{5/3}$, $\|\vec{c}_1\|_{L^5}^5$, $\|d\|_{L^5}^5$, $\|g_U\|_{L^{10/7}}^{10/7}$, $\|a_2\|_{L^{5/2}}^{5/2}$, $\|b_2\|_{L^5}^5$, $\|\vec{c}_2\|_{L^5}^5$, $\|g_V\|_{L^2}^2 \in L^1(0,T)$, we are able to apply Gronwall's Lemma to (C.11) and conclude that

$$(U_n, V_n) \text{ is bounded in } L^\infty(L^2) \times L^\infty(H^1) \cap L^2(H^1) \times L^2(H^2).$$

Using this bound in the equations (C.2) and (C.3) we also obtain n-independent bounds for $\partial_t U_n$ and $\partial_t V_n$, which leads us to

$$(U_n, V_n) \text{ is bounded in } W_2 \times X_2. \tag{C.12}$$

Again, we skip the standard procedures of the application of the Galerkin's method to linear equations and state that with (C.12) we are able to pass to the limit as $n \to \infty$ in (C.2) and (C.3), concluding that there is $(U, V) \in W_2 \times X_2$ solution of problem (C.1).

Case 2*b*: The n-independent *a priori* estimates for this case are similar to those of the case $2a$ and one can obtain them based on the previous cases. ■

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