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Lower bounds and exact values of the 2-color off-diagonal generalized weak Schur numbers $WS(2; k_1, k_2)$

(Brief Announcement)

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Abstract

In this study, we focus on the concept of the 2-color off-diagonal generalized weak Schur numbers, denoted as $WS(2; k_1, k_2)$. These numbers are defined for integers $k_i \geqslant 2$, where $i = 1, 2$, as the smallest integer M, such that any 2-coloring of the integer interval $[1, M]$ must contain a 2-colored solution to the equation $E_{k_j} : x_1 + x_2 + \ldots + x_{k_j} = x_{k_j+1}$ for $j = 1, 2$, with the condition that $x_i \neq x_j$ when $i \neq j$. Our objective is to determine lower bounds for these 2-color off-diagonal generalized weak Schur numbers and demonstrate that in several cases, these lower bounds match the exact values.

Keywords: Schur numbers; Weak Schur numbers; sum-free sets; weak sum-free-sets; off-diagonal weak Schur numbers.
2010 MSC: 05C55; 05D10; 05-04; 05A17.

1. Introduction

The *2-color off-diagonal generalized weak Schur numbers*, denoted as $WS(2; k_1, k_2)$. These numbers are defined for integers $k_i \geqslant 2$, where $i = 1, 2$, as the smallest integer M, such that any 2-coloring of the integer interval $[1, M]$ must contain a 2-colored solution to the equation $E_{k_j} : x_1 + x_2 + \ldots + x_{k_j} = x_{k_j+1}$ for $j = 1, 2$, with the condition that $x_i \neq x_j$ when $i \neq j$. For background and generalizations, see Schur [13], Rado [11], and Robertson and Schaal [12]. For various results, see Baumert and Golomb ([4], [8]), Heule [9], and Fredricksen and Sweet [7], Beutelspacher and Brestovansky [5], Znám [14], Ahmed and Schaal [3], and Boza et al. [6]). We explore the values and bounds of $WS(2; k_1, k_2)$. It is known (see Ahmed et al. [2]) that $WS(2; 2, 2) = 9$, $WS(2; 2, 3) = 16$, and for all $k \geqslant 4$,

$$WS(2; 2, k) = \begin{cases} 3k^2/2 - k/2 + 1, \text{ if } k \equiv 0, 3 \pmod 4; \\ 3k^2/2 - k/2 + 2, \text{ if } k \equiv 1, 2 \pmod 4. \end{cases}$$

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In this short paper, we prove several lower bounds and compute several exact values of $WS(2; k_1, k_2)$ for $k_2 \geqslant k_1 \geqslant 3$, as presented in the following sections.

2. Main Results

We establish lower bounds of $WS(2; k_1, k_2)$ for $k_2 \geqslant k_1 \geqslant 3$. We have also computed nineteen exact values of $WS(2; k_1, k_2)$ using SAT solvers (see Heule [10]; also see Ahmed [1] for an introduction to distributed SAT solving) to prove upper bounds.

Theorem 2.1. *$WS(2; 3, 3) = 24$, $WS(2; 4, 4) = 52$, and for positive integers k and p with $k \geqslant 3$ and $p \geqslant 0$, we have*

$$WS(2; k, k+p) \geqslant \begin{cases} k^3/2 + 2k^2 - 5k/2 + 1, & \text{if } p = 0 \text{ and } k \geqslant 5; \\ k^3/2 + 5k^2/2 + k, & \text{if } p = 1 \text{ and } 3 \leqslant k \leqslant 6; \\ k^3/2 + 3k^2 - 5k/2 + 2, & \text{if } p = 1 \text{ and } k \geqslant 7; \\ k^3/2 + 7k^2/2 + k + 1, & \text{if } p = 2 \text{ and } 3 \leqslant k \leqslant 7; \\ k^3/2 + 4k^2 - 5k/2 + 4, & \text{if } p = 2 \text{ and } k \geqslant 8; \\ k^3/2 + k^2(p+1) + k(p^2 + p - 1)/2, & \text{if } p \geqslant 3. \end{cases}$$

Theorem 2.2. *We have the following nineteen exact values:*
$WS(2; 3, 3) = 24$, $WS(2; 3, 4) = 39$, $WS(2; 3, 5) = 49$, $WS(2; 3, 6) = 66$, $WS(2; 3, 7) = 87$, $WS(2; 3, 8) = 111$, $WS(2; 3, 9) = 138$, $WS(2; 3, 10) = 168$, $WS(2; 4, 4) = 52$, $WS(2; 4, 5) = 76$, $WS(2; 4, 6) = 93$, $WS(2; 4, 7) = 118$, $WS(2; 4, 8) = 150$, $WS(2; 5, 5) = 101$, $WS(2; 5, 6) = 130$, $WS(2; 5, 7) = 156$, $WS(2; 6, 6) = 166$, $WS(2; 6, 7) = 204$, and $WS(2; 7, 7) = 253$.

3. Proof of lower bounds

Lemma 3.1. *For $k_2 \geqslant k_1 \geqslant 3$, we have $WS(2; k_1, k_2) \geqslant k_1 \left(\frac{k_1+1+(k_2+1)k_2}{2} - 1\right)$.*

Proof. Let $k_1 \left(\frac{k_1+1+(k_2+1)k_2}{2} - 1\right) = \ell$. Consider the coloring $\delta : [1, \ell - 1] \to [0, 1]$ defined as

$$\delta(x) = \begin{cases} 1, \text{ if } 1 \leqslant x \leqslant k_2(k_2+1)/2 - 1 = m \text{ (say)}; \\ 0, \text{ if } m + 1 \leqslant x \leqslant k_1 m + k_1(k_1+1)/2 - 1 = \ell - 1. \end{cases}$$

Suppose there exists a solution $a_1 + a_2 + \cdots + a_{k_2} = a_{k_2+1}$ with $a_1 < a_2 < \cdots < a_{k_2}$ to the equation E_{k_2} monochromatic in color 1. Now, $b_{k_2+1} \geqslant \sum_{i=1}^{k_2} i = k_2(k_2+1)/2 = m+1$, but $\delta(x) = 0$ for $m+1 \leqslant x \leqslant \ell-1$, which results a contradiction. Again, suppose there exists a solution $a_1 + a_2 + \cdots + a_{k_1} = a_{k_1+1} \leqslant \ell - 1$ with $a_1 < a_2 < \cdots < a_{k_1}$ to the equation E_{k_1} monochromatic in color 0. Now, $a_{k_1} \geqslant \sum_{i=1}^{k_1}(m + i) = k_1 m + k_1(k_1+1)/2 = \ell$, which is a contradiction. Hence, $WS(2; k_1, k_2) \geqslant \ell$. ■

Corollary 3.2. *For $k \geqslant 2$, $p \geqslant 0$, we have $WS(2; k, k+p) \geqslant k^3/2 + k^2(p+1) + k(p^2 + p - 1)/2$.*

Remark 3.3. *This lower bound coincides with the exact values of $WS(2; 3, 6)$, $WS(2; 3, 7)$, $WS(2; 3, 8)$, $WS(2; 3, 9)$, $WS(2; 3, 10)$, $WS(2; 4, 7)$, and $WS(2; 4, 8)$.*

The lower bound in Corollary 3.2 can be improved using a different coloring as follows:

Lemma 3.4. *For integers $k \geqslant 2$ and $p \geqslant 0$,*
$WS(2; k, k+p) \geqslant (-2k + 3k^2 + k^3 + (-1 + 3k + k^2)p + p^2 + (-k + k^2 + (-2 + 2k)p)z + (2 - 2k)z^2)/2$
where $z = \left\lceil (\alpha + \sqrt{\beta})/(4(k-1)) \right\rceil - 1$ with
$\alpha = 2(k-1)p - (k^2 + 3k - 2)$ *and* $\beta = 4(k^2 - 1)p^2 + 4(k^2 + 2k - 3)p + (k^4 + 22k^3 - 11k^2 - 12k + 4)$.

Proof. Let

$$\ell = (-2k + 3k^2 + k^3 + (-1 + 3k + k^2)p + p^2 + (-k + k^2 + (-2 + 2k)p)z + (2 - 2k)z^2)/2$$
$$u = (-2 - k + 2k^2 + k^3 + (-1 + 3k + k^2)p + p^2 + (2 - 3k + k^2 + (-2 + 2k)p)z + (2 - 2k)z^2)/2$$

Consider the coloring $\delta : [1, \ell - 1] \to [0, 1]$ defined as

$$\delta(x) = \begin{cases} 1, \text{ if } 1 \leqslant x \leqslant z; \\ 0, \text{ if } z+1 \leqslant x \leqslant k^2/2 + (2z+1)k/2 - 1 = m \text{ (say)}; \\ 1, \text{ if } m+1 = k^2/2 + (2z+1)k/2 \leqslant x \leqslant u; \\ 0, \text{ if } u+1 \leqslant x \leqslant \ell - 1; \end{cases}$$

We will prove that there is no solutions to the equation $E_k : x_1 + x_2 + \ldots + x_k = x_{k+1}$, monochromatic in color 0 and there is no solutions to the equation $E_{k+p} : x_1 + x_2 + \ldots + x_{k+p} = x_{k+p+1}$, monochromatic in color 1.

Suppose there is a solution $a_1 + a_2 + \cdots + a_k = a_{k+1}$ with $a_1 < a_2 < \cdots < a_k$ to E_k monochromatic in color 0. Then, $a_{k+1} = \sum_{i=1}^{k} a_i \geqslant \sum_{i=1}^{k} z + i = (2kz + k^2 + k)/2 = m + 1$, then $\sum_{i=1}^{k} a_i \notin [z+1, m]$. Now we verify that $\sum_{i=1}^{k} a_i$ also does not belong to the interval $[u+1, \ell-1]$. We consider two cases:

If $a_k \leqslant m$, then $\sum_{i=1}^{k} a_i \leqslant \sum_{i=0}^{k-1}(m-i) = (-k + k^3 + 2k^2 z)/2 < u + 1$.

If $a_k \geqslant u+1$, then $\sum_{i=1}^{k} a_i \geqslant (\sum_{i=1}^{k-1} z + i) + (u+1) > \ell - 1$.

In each of the above cases, we reach a contradiction.

Suppose there is a solution $a_1 + a_2 + \cdots + a_{k+p} = a_{k+p+1}$ with $a_1 < a_2 < \cdots < a_{k+p}$ to E_{k+p} monochromatic in color 1. Then $a_{k+p+1} = \sum_{i=1}^{k+p} a_i \geqslant \sum_{i=1}^{z} i + \sum_{i=0}^{k+p-z-1}(m+1) + i = u + 1$, which is a contradiction since a_{k+p+1} is colored in color 0. ■

Corollary 3.5. *Let $p = 0$. If $k \geqslant 5$ then $z = 1$ and $WS(2; k, k) \geqslant k^3/2 + 2k^2 - 5k/2 + 1$.*

Remark 3.6. *This lower bound coincides with the exact values of $WS(2; 5, 5)$, $WS(2; 6, 6)$ and $WS(2; 7, 7)$.*

Corollary 3.7. $WS(2; k, k+1) \geqslant \begin{cases} k^3/2 + 3k^2 - 5k/2 + 2, & \text{if } k \geqslant 7 \text{ (where } z = 2); \\ k^3/2 + 5k^2/2 + k = \ell \text{(say)}, & \text{if } k \leqslant 6. \end{cases}$

The $k \leqslant 6$ case can be proven using the coloring $\delta : [1, \ell-1] \to [0, 1]$ as defined below and showing that there is no solution to E_{k+1} in color 1 and no solution to E_k in color 0.

$$\delta(x) = \begin{cases} 1, \text{ if } 1 \leqslant x \leqslant k^2/2 + 3k/2 = m \text{ (say)}; \\ 0, \text{ if } m+1 = k^2/2 + 3k/2 + 1 \leqslant x \leqslant k^3/2 + 2k^2 + k/2 - 1 = n \text{ (say)}; \\ 1, \text{ if } n+1 = k^3/2 + 2k^2 + k/2 \leqslant x \leqslant k^3/2 + 5k^2/2 + k - 1 = \ell - 1. \end{cases}$$

Remark 3.8. *The lower bounds coincide with exact values of $WS(2; 3, 4)$, $WS(2; 4, 5)$, $WS(2; 5, 6)$ and $WS(2; 6, 7)$*

Corollary 3.9. $WS(2; k, k+2) \geqslant \begin{cases} k^3/2 + 7k^2/2 + k + 1, & \text{if } k \leqslant 7 \text{ (where } z = 2); \\ k^3/2 + 4k^2 - 5k/2 + 4, & \text{if } k \geqslant 8 \text{ (where } z = 3). \end{cases}$

Remark 3.10. *The lower bounds coincide with the exact values of $WS(2; 3, 5)$, $WS(2; 4, 6)$ and $WS(2; 5, 7)$.*

Proof of Theorem 2.1. The desired lower bound is obtained using Corollaries 3.2, 3.5, 3.7, and 3.9. ■

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