# On the Frobenius Number of Fibonacci Numerical Semigroups 

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#### Abstract

In this paper we compute the Frobenius number of certain Fi bonacci numerical semigroups, that is, numerical semigroups generated by a set of Fibonacci numbers, in terms of Fibonacci numbers.


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## 1 Introduction

Let $s_{1}, \ldots, s_{n}$ be positive integers such that their greatest commun divisor is one. Let $S=<s_{1}, \ldots, s_{n}>$ be the numerical semigroup ${ }^{1}$ generated by $s_{1}, \ldots, s_{n}$. A Fibonacci numerical semigroup is a numerical semigroup generated by a set of Fibonacci numbers $F_{i_{1}}, \ldots, F_{i_{r}}$, for some integers $3 \leq i_{1}<\cdots<i_{r}$ where g.c.d. $\left(F_{i_{1}}, \ldots, F_{i_{r}}\right)=1$.

The so-called Frobenius number, denoted by $g\left(s_{1}, \ldots, s_{n}\right)$, is defined as the largest integer not belonging to $S$, that is, the largest integer that is not representable as a nonnegative integer combination of $s_{1}, \ldots, s_{n}$. It is well known that $g\left(s_{1}, s_{2}\right)=s_{1} s_{2}-s_{1}-s_{2}$. In general, finding $g(S)$ is a difficult problem and so formulas and upper bounds for particular sequences are of interest. For instance, it is known [3] the value of $g(S)$ when $S$ is an arithmetical sequence

$$
\begin{equation*}
g(a, a+d, \ldots, a+k d)=a\left(\left\lfloor\frac{a-2}{k}\right\rfloor\right)+d(a-1) \tag{1}
\end{equation*}
$$

We refer the reader to [2] where a complete account on the Frobenius problem can be found.

In this paper, we investigate the value of $g\left(F_{i}, F_{j}, F_{l}\right)$ for some triples $3 \leq i<j<l$ (we always assume that g.c.d. $\left(F_{i}, F_{j}, F_{l}\right)=1$, recall that g.c.d. $\left(F_{i}, F_{i+l}\right)=1$ if $\left.i \nmid l\right)$.

We first notice that $g\left(F_{i}, F_{i+1}, F_{l}\right)=g\left(F_{i}, F_{i+1}\right)$ for any integer $l \geq i+$ 2. Indeed, since $F_{l}=F_{i+m}=F_{m} F_{i+1}+F_{m-1} F_{i}$ is a nonnegative integer combination of $F_{i}$ and $F_{i+1}$ then the semigroups $<F_{i}, F_{i+1}, F_{l}>$ and $<$ $F_{i}, F_{i+1}>$ generate the same set of elements and thus they have the same Frobenius number.

[^1]Let us consider then $g\left(F_{i}, F_{i+2}, F_{l}\right)$ with $l \geq i+3$. We notice that the case when $l=i+3$ is a consequence of equation (11) since the triple $\left\{F_{i}, F_{i+2}, F_{i+3}\right\}=\left\{F_{i}, F_{i}+F_{i+1}, F_{i}+2 F_{i+1}\right\}$ form an arithmetical sequence. However, it can be checked that $\left\{F_{i}, F_{i+2}, F_{i+k}\right\}$ do not form an arithmetical sequence when $k \geq 3$ and the calculation of $g\left(F_{i}, F_{i+2}, F_{i+k}\right)$ is more complicated.

We state our main result.
Theorem 1.1 Let $i, k \geq 3$ be integers and let $r=\left\lfloor\frac{F_{i}-1}{F_{k}}\right\rfloor$. Then,
$g\left(F_{i}, F_{i+2}, F_{i+k}\right)= \begin{cases}\left(F_{i}-1\right) F_{i+2}-F_{i}\left(r F_{k-2}+1\right) & \text { if } r=0 \text { or } r \geq 1 \text { and } \\ & F_{k-2} F_{i}<\left(F_{i}-r F_{k}\right) F_{i+2}, \\ \left(r F_{k}-1\right) F_{i+2}-F_{i}\left((r-1) F_{k-2}+1\right) & \text { otherwise. }\end{cases}$

Let $N\left(a_{1}, \ldots, a_{n}\right)$ be the number of positive integers with no representation by a nonnegative integer combination of $a_{1}, \ldots, a_{n}$. We refer the reader to [2, Chapter 5] for results related to $N\left(a_{1}, \ldots, a_{n}\right)$. Theorem 1.1] yields to the following result.

Corollary 1.2 Let $i, k \geq 3$ be integers and let $r=\left\lfloor\frac{F_{i}-1}{F_{k}}\right\rfloor$. Then,

$$
N\left(F_{i}, F_{i+2}, F_{i+k}\right)=\frac{\left(F_{i}-1\right)\left(F_{i+2}-1\right)-r F_{k-2}\left(2 F_{i}-F_{k}(1+r)\right)}{2} .
$$

## 2 Fibonacci semigroups

In order to prove Theorem 1.1 we need the following result due to Brauer and Shockley [1].

Lemma 2.1 Let $1<a_{1}<\cdots<a_{n}$ be integers with g.c.d. $\left(a_{1}, \ldots, a_{n}\right)=1$. Then,

$$
g\left(a_{1}, \ldots, a_{n}\right)=\max _{l \in\left\{1,2, \ldots, a_{n}-1\right\}}\left\{t_{l}\right\}-a_{1}
$$

where $t_{l}$ is the smallest positive integer congruent to $l$ modulo $a_{1}$, that is representable as a nonnegative integer combination of $a_{2}, \ldots, a_{n}$.

Proof. Let $L$ be a positive integer. If $L \equiv 0 \bmod a_{1}$ then $L$ is a nonnegative integer combination of $a_{1}$. If $L \equiv l \bmod a_{1}$ then $L$ is a nonnegative integer combination of $a_{1}, \ldots, a_{n}$ if and only if $L \geq t_{l}$.

Proof of Theorem 1.1. Let $T^{*}=\left\{t_{0}^{*}, \ldots, t_{F_{i}-1}^{*}\right\}$ where $t_{l}^{*}$ be the smallest positive integer congruent to $l$ modulo $F_{i}$, that is representable as a nonnegative integer combination of $F_{i+2}$ and $F_{i+k}$. We shall find $t_{l}^{*}$ for each $l=0,1, \ldots, F_{i}-1$. To this end, we consider all nonnegative integer combinations of $F_{i+2}$ and $F_{i+k}$. We construct the following table, denoted by $T_{1}$, having as entry $t_{x, y}$ the combination of the form $x F_{i+2}+y F_{i+k}$ with integers $x, y \geq 0$, see below.

| $x \backslash y$ | 0 | 1 | 2 | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | $F_{i+k}$ | $2 F_{i+k}$ | $\cdots$ |
| 1 | $F_{i+2}$ | $F_{i+k}+F_{i+2}$ | $2 F_{i+k}+F_{i+2}$ | $\cdots$ |
| 2 | $2 F_{i+2}$ | $F_{i+k}+2 F_{i+2}$ | $2 F_{i+k}+2 F_{i+2}$ | $\cdots$ |
| 3 | $3 F_{i+2}$ | $F_{i+k}+3 F_{i+2}$ | $2 F_{i+k}+3 F_{i+2}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  |
| $F_{k}-1$ | $\left(F_{k}-1\right) F_{i+2}$ | $F_{i+k}+\left(F_{k}-1\right) F_{i+2}$ | $2 F_{i+k}+\left(F_{k}-1\right) F_{i+2}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  |

We notice that

$$
F_{i+k}=F_{k-2} F_{i+1}+F_{k-1} F_{i+2}=F_{k-2}\left(F_{i+2}-F_{i}\right)+F_{k-1} F_{i+2}=F_{i+2} F_{k}-F_{k-2} F_{i}
$$

so, we obtain that
$x F_{i+2}+y F_{i+k}=x F_{i+2}+y\left(F_{i+2} F_{k}-F_{k-2} F_{i}\right)=\left(x+y F_{k}\right) F_{i+2}-y F_{k-2} F_{i}$.
Thus, $T_{1}$ can also be given by the following table, denoted by $T_{2}$,

| $x \backslash y$ | 0 | 1 | 2 | $\cdots$ | $r$ | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | $F_{k} F_{i+2}-F_{k-2} F_{i}$ | $2 F_{k} F_{i+2}-2 F_{k-2} F_{i}$ | $\cdots$ | $r F_{k} F_{i+2}-r F_{k-2} F_{i}$ | $\cdots$ |
| 1 | $F_{i+2}$ | $\left(1+F_{k}\right) F_{i+2}-F_{k-2} F_{i}$ | $\left(1+2 F_{k}\right) F_{i+2}-2 F_{k-2} F_{i}$ | $\cdots$ | $\left(1+r F_{k}\right) F_{i+2}-r F_{k-2} F_{i}$ | $\cdots$ |
| 2 | $2 F_{i+2}$ | $\left(2+F_{k}\right) F_{i+2}-F_{k-2} F_{i}$ | $\left(2+2 F_{k}\right) F_{i+2}-2 F_{k-2} F_{i}$ | $\cdots$ | $\left(2+r F_{k}\right) F_{i+2}-r F_{k-2} F_{i}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  | $\vdots$ |  |
| $l$ | $l F_{i+2}$ | $\left(l+F_{k}\right) F_{i+2}-F_{k-2} F_{i}$ | $\left(l+2 F_{k}\right) F_{i+2}-2 F_{k-2} F_{i}$ | $\cdots$ | $\left(l+r F_{k}\right) F_{i+2}-r F_{k-2} F_{i}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  | $\vdots$ |  |
| $F_{k}-1$ | $\left(F_{k}-1\right) F_{i+2}$ | $\left(2 F_{k}-1\right) F_{i+2}-F_{k-2} F_{i}$ | $\left(3 F_{k}-1\right) F_{i+2}-2 F_{k-2} F_{i}$ | $\cdots$ |  |  |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  | $\vdots$ |  |

Let $S$ be the set formed by the first $F_{k}-1$ entries of columns zero, one, two, and so on, that is,

$$
S=\left\{t_{0,0}, t_{1,0}, \ldots, t_{F_{k}-1,0}, t_{0,1}, t_{1,1}, \ldots, t_{F_{k}-1,1}, \ldots, t_{0, r}, t_{1, r}, \ldots, t_{F_{k}-1, r}, \ldots\right\}
$$

Remark 2.2 (a) Let $r=\left\lfloor\frac{F_{i}-1}{F_{k}}\right\rfloor$ and set $F_{i}-1=r F_{k}+l$ for some integer $0 \leq l \leq F_{k}-1$. Let

$$
S^{\prime}=\left\{t_{0,0}, t_{1,0}, \ldots, t_{F_{k}-1,0}, t_{0,1}, t_{1,1}, \ldots, t_{F_{k}-1,1}, \ldots, t_{2, r}, t_{1, r}, \ldots, t_{l, r}\right\}
$$

Then, for each $t_{x, y}=\left(x+y F_{k}\right) F_{i+2}-y F_{k-2} F_{i} \in S^{\prime}$ we have that $0 \leq x+y F_{k} \leq F_{i}-1$. Moreover, since g.c.d. $\left(F_{i+2}, F_{i}\right)=1$ then $S^{\prime}$ forms a complete system of rests modulo $F_{i}$.
(b) The elements of $S$ can be represented as $s_{x}=x F_{i+2}-\left\lfloor\frac{x}{F_{k}}\right\rfloor F_{k-2} F_{i}$ for $x=0,1, \ldots$. Indeed, it can be checked that $S=\bigcup_{q \geq 1} S_{q}$ where

$$
S_{q}=\left\{s_{q F_{k}}, s_{q F_{k}+1}, \ldots, s_{(q+1) F_{k}-1}\right\}=\left\{t_{0, q}, \ldots, t_{F_{k}-1, q}\right\}
$$

for each integer $q=0,1,2, \ldots$.
(c) By using table $T_{2}$ we have that $t_{i, j}<t_{k, l}$ for all $i \leq k$ and all $j \leq l$.

Lemma 2.3 Let $t_{u, v}$ be an entry of $T_{1}$ such that $t_{u, v} \notin S^{\prime}$. There exists $t_{x, y} \in S^{\prime}$ such that $t_{u, v} \equiv t_{x, y} \bmod F_{i}$ and $t_{u, v}>t_{x, y}$.

Proof. We first notice that the set $S$ can be written as follows

$$
\begin{array}{cccccccccc}
\left\{s_{0},\right. & \ldots & , s_{F_{k}-1}, & s_{F_{k}}, & \ldots & , s_{2 F_{k}-1}, & \ldots & , s_{r F_{k}}, & \ldots & , s_{r F_{k}+l}=s_{F_{i}-1}, \\
s_{F_{i}}, & \ldots & , s_{F_{i}+F_{k}-1}, & s_{F_{i}+F_{k}}, & \ldots & , s_{F_{i}+2 F_{k}-1}, & \ldots & , s_{F_{i}+r F_{k}}, & \ldots & , s_{2 F_{i}-1} \\
s_{2 F_{i}}, & \ldots & , s_{2 F_{i}+F_{k}-1}, & s_{2 F_{i}+F_{k}}, & \ldots & , s_{2 F_{i}+2 F_{k}-1}, & \ldots & , s_{2 F_{i}+r F_{k}}, & \ldots & \left., s_{3 F_{i}-1}, \ldots\right\}
\end{array}
$$

where $S^{\prime}=\left\{s_{0}, \ldots, s_{F_{k}-1}, s_{F_{k}}, \ldots, s_{2 F_{k}-1}, \ldots, s_{r F_{k}}, \ldots, s_{F_{i}-1}\right\}$. We have two cases.

Case A) Suppose that $t_{u, v} \in S \backslash S^{\prime}$. Then $t_{u, v}$ is of the form $s_{p F_{i}+g}$ for some integers $p \geq 1$ and $0 \leq g \leq F_{i}-1$. It is clear that,
$s_{g}=g F_{i+2}-\left\lfloor\frac{g}{F_{k}}\right\rfloor F_{i} F_{k-2} \equiv\left(p F_{i}+g\right) F_{i+2}-\left\lfloor\frac{p F_{i}+g}{F_{k}}\right\rfloor F_{i} F_{k-2}=g_{p F_{i}+g} \bmod F_{i}$.

We will show that $s_{p F_{i}+g}>s_{g}$. To this end, it suffice to prove that $s_{F_{i}+g}>$ $s_{g}$ (since $s_{p F_{i}+g} \geq s_{F_{i}+g}$ ). Recall that $r=\left\lfloor\frac{F_{i}-1}{F_{k}}\right\rfloor$ and that $F_{i}-1=r F_{k}+l$ for some integer $0 \leq l \leq F_{k}-1$. We have two subcases.

Subcase a) If $r=0$ then $F_{k} \geq F_{i}$. If $F_{k}=F_{i}$ then $s_{F_{i}+g}=t_{g, 1}$ and, by Remark 2.2 (c), $t_{g, 0}<t_{g, 1}$. If $F_{k}>F_{i}$ then $s_{F_{i}+g}=t_{q, 0}$ for some integer $q \geq F_{i}$ and, by Remark 2.2 (c), $t_{g, 0}<t_{q, 0}$.

Subcase b) If $r \geq 1$ then $s_{F_{i}+g}>s_{g}$ holds if and only if

$$
\left(F_{i}+g\right) F_{i+2}-\left\lfloor\frac{F_{i}+g}{F_{k}}\right\rfloor F_{i} F_{k-2}>g F_{i+2}-\left\lfloor\frac{g}{F_{k}}\right\rfloor F_{i} F_{k-2}
$$

or equivalently if and only if

$$
F_{i+2}>F_{k-2}\left(\left\lfloor\frac{F_{i}+g}{F_{k}}\right\rfloor-\left\lfloor\frac{g}{F_{k}}\right\rfloor\right) .
$$

Let $g=m F_{k}+n$ with $0 \leq n \leq F_{k}-1$. Since $F_{i}-1=r F_{k}+l$ with $0 \leq l \leq F_{k}-1$ then

$$
\left\lfloor\frac{F_{i}-1+g+1}{F_{k}}\right\rfloor=\left\lfloor\frac{r F_{k}+l+m F_{k}+n+1}{F_{k}}\right\rfloor \leq r+m+1
$$

and thus

$$
\left\lfloor\frac{F_{i}+g}{F_{k}}\right\rfloor-\left\lfloor\frac{g}{F_{k}}\right\rfloor \leq r+m+1-m=r+1 .
$$

So, it is enough to show that $F_{i+2}>(r+1) F_{k-2}$ or equivalently to show that $F_{i}+F_{i+1}>(r+1) F_{k-2}$. Since $F_{i}=r F_{k}+l+1$ then the latter inequality holds if and only if $r F_{k}+l+1+F_{i+1}>r F_{k-2}+F_{k-2}$, that is, if and only if

$$
r\left(F_{k}-F_{k-2}\right)+l+1+F_{i+1}=r\left(F_{k-1}\right)+l+1+F_{i+1}>F_{k-2}
$$

which is true since $r \geq 1$.
Case B) Suppose that $t_{u, v} \notin S$. Then we have that $0 \leq x \leq F_{k}-1<u$. If $v \geq y$ then, by Remark 2.2 (c), $t_{x, y}<t_{x, v}<t_{u, v}$. So, we suppose that $v<y$. Since, $t_{u, v} \equiv t_{x, y} \bmod F_{i}$ then $u+v F_{k} \equiv x+y F_{k} \bmod F_{i}$ but, by Remark 2.2 (a), $0 \leq x+y F_{k} \leq F_{i}-1$ so $u+v F_{k}=d\left(x+y F_{k}\right)$ for some integer $d \geq 1$ and thus $u+v F_{k} \geq x+y F_{k}$. Also, since $v<y$, then $-v F_{k-2} F_{i}>-y F_{k-2} F_{i}$. So, combining the last two inequalities we have that

$$
t_{u, v}=\left(u+v F_{k}\right) F_{i+2}-v F_{k-2} F_{i}>\left(x+y F_{k}\right) F_{i+2}-y F_{k-2} F_{i}=t_{x, y}
$$

Therefore, by the above lemma, we have that for each $x=0, \ldots, F_{i}-1$, $s_{x}$ is the smallest positive integer congruent to $l$ modulo $F_{i}$, for some integer $0 \leq l \leq F_{i}-1$, that is representable as a nonnegative integer combination of $F_{i+2}$ and $F_{i+k}$, that is, $S^{\prime}=T^{*}$.

Now, by Remark 2.2 (c), if $r \geq 1$ then

$$
\begin{gathered}
t_{F_{k}-1, i}=\max _{0 \leq x \leq F_{k}-1}\left\{t_{x, i} \mid t_{x, i} \in S^{\prime}\right\} \text { for each } i=0, \ldots, r-1, \\
t_{F_{k}-1, r-1}=\max _{0 \leq i \leq r-1}\left\{t_{F_{k}-1, i} \mid t_{F_{k}-1, i} \in S^{\prime}\right\},
\end{gathered}
$$

and

$$
t_{l, r}=\max _{0 \leq x \leq l}\left\{t_{x, r} \mid t_{x, r} \in S^{\prime}\right\}
$$

Thus,

$$
\max \left\{s \mid s \in S^{\prime}\right\}= \begin{cases}t_{l, r} & \text { if } r=0 \\ \max \left\{t_{F_{k}-1, r-1}, t_{l, r}\right\} & \text { otherwise }\end{cases}
$$

The result follows since $t_{l, r}>t_{F_{k}-1, r-1}$ if and only if
$\left(r F_{k}+l\right) F_{i+2}-r F_{k-2} F_{i}=\left(F_{i}-1\right) F_{i+2}-r F_{k-2} F_{i}>\left(r F_{k}-1\right) F_{i+2}-(r-1) F_{k-2} F_{i}$
or equivalently, if and only if $F_{i+2}\left(F_{i}-r F_{k}\right)>F_{k-2} F_{i}$.
We will use the following result due to Selmer [4] to show Corollary 1.2 ,
Lemma 2.4 Let $1<a_{1}<\cdots<a_{n}$ be integers with g.c.d. $\left(a_{1}, \ldots, a_{n}\right)=1$. If $L=\left\{1, \ldots, a_{1}-1\right\}$ then

$$
N\left(a_{1}, \ldots, a_{n}\right)=\frac{1}{a_{1}} \sum_{l \in L} t_{l}-\frac{a_{1}-1}{2} .
$$

where $t_{l}$ is the smallest positive integer congruent to $l$ modulo $a_{1}$, that is representable as a nonnegative integer combination of $a_{2}, \ldots, a_{n}$.

Proof. The number of $M \equiv l \not \equiv 0 \bmod a_{1}$ with $0<M<t_{l}$ is given by $\left\lfloor\frac{t_{1}}{a_{1}}\right\rfloor$. By assuming that $0<l<a_{1}$, we have $\left\lfloor\frac{t_{l}}{a_{1}}\right\rfloor=\frac{t_{l}-l}{a_{1}}$. The result follows by summing over $l \in L$.

Proof of Corollary 1.2. Let $r=\left\lfloor\frac{F_{i}-1}{F_{k}}\right\rfloor$ and set $F_{i}-1=r F_{k}+l$ for some
integer $0 \leq l \leq F_{k}-1$. By Lemma 2.4 and Remark 2.2 (b), we have

$$
\begin{aligned}
N\left(F_{i}, F_{i+2}, F_{i+k}\right) & =\frac{1}{F_{i}} \sum_{s \in S^{\prime}} s-\frac{F_{i}-1}{2} \\
& =\frac{1}{F_{i}} \sum_{j=0}^{F_{i}-1}\left(j F_{i+2}-F_{k-2}\left\lfloor\frac{j}{F_{k}}\right\rfloor F_{i}\right)-\frac{F_{i}-1}{2} \\
& =\frac{1}{F_{i}}\left(F_{i+2} \frac{\left(F_{i}-1\right) F_{i}}{2}\right)-\frac{1}{F_{i}}\left(F_{k-2} F_{i}\right) \sum_{j=0}^{F_{i}-1}\left\lfloor\frac{j}{F_{k}}\right\rfloor-\frac{F_{i}-1}{2} .
\end{aligned}
$$

By using the table $T_{1}$, it is easy to verify that

$$
\sum_{j=0}^{F_{i}-1}\left\lfloor\frac{j}{F_{k}}\right\rfloor=0+F_{k}+2 F_{k}+\cdots+(r-1) F_{k}+r(l+1)=\frac{F_{k}(r-1) r}{2}+r(l+1)
$$

and, since $l+1=F_{i}-r F_{k}$, then

$$
\begin{aligned}
N\left(F_{i}, F_{i+2}, F_{i+k}\right) & =\frac{F_{i+2}\left(F_{i}-1\right)}{2}-F_{k-2}\left(\frac{F_{k}(r-1) r}{2}+r\left(F_{i}-r F_{k}\right)\right)-\frac{F_{i}-1}{2} \\
& =\frac{\left(F_{i}-1\right)\left(F_{i+2}-1\right)}{2}-F_{k-2}\left(\frac{F_{k} r^{2}-F_{k} r+2 F_{i} r-2 r^{2} F_{k}}{2}\right) \\
& =\frac{\left(F_{i}-1\right)\left(F_{i+2}-1\right)-r F_{k-2}\left(2 F_{i}-F_{k}(1+r)\right)}{2}
\end{aligned}
$$

We end with the following problem.
Problem 2.5 Find upper (and lower) bounds (or formulas) for $g\left(F_{i}, F_{j}, F_{k}\right)$ for further triples $3 \leq i<j<k$.

## References

[1] A. Brauer and J.E. Shockley, On a problem of Frobenius, Journal für Reine und Angewandte Mathematik 211 (1962), 215-220.
[2] J.L. Ramírez Alfonsín, The Diophantine Frobenius Problem, Oxford Lectures Series in Mathematics and its Applications 30, Oxford University Press, (2005).
[3] J.B. Roberts, Note on linear forms, Proc. Amer. Math. Soc. 7 (1956), 465-469.
[4] E.S. Selmer, On the linear diophantine Problem of Frobenius, Journal für Reine und Angewandte Mathematik 293/294(1) (1977), 1-17.

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[^1]:    ${ }^{1}$ Recall that a semigroup $(S, *)$ consists of a nonempty set $S$ and an associative binary operation $*$ on $S$. If, in addition, there exists an element, which is usually denoted by 0 , in $S$ such that $a+0=0+a=a$ for all $a \in S$, we say that $(S, *)$ is a monoid. A numerical semigroup is a submonoid of $\mathbb{N}$ such that the greatest common divisor of its elements is equal to one.

